# APPROXIMATION AND POLYNOMIAL CONVEXITY IN SEVERAL COMPLEX VARIABLES

by

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## Abstract

This thesis is a survey on selected topics in approximation theory. The topics use either the techniques from the theory of several complex variables or those that arise in the study of the subject. We also go through elementary theory of polynomially convex sets in complex analysis.

ÇOK DEĞİŞKENLİ KARMAŞIK ANALİZDE YAKLAŞIMLAR VE POLİNOMSAL KONVEKSLİK

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## Özet

Bu tez, yaklaşımlar teorisinde seçilmiş konular üzerine bir araştırmadır. Bu konular incelenirken ya çok değişkenli karmaşık analiz teorisinde mevcut olan, ya da inceleme sırasında ortaya çıkan teknikler kullanılıyor. Araştırmamız aynı zamanda karmaşık analizde polinomsal konveks kümelerin temel teorisini de inceliyor.

*Anneme, babama*

*ve*

*sevgili Berke'ye*

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# Contents

# CHAPTER 1

# INTRODUCTION

## 1.1 Introduction

In this thesis basic theory of approximation in $\mathbb{C}^n$ when $n \geq 1$ is surveyed. The main object of study in complex analysis is the holomorphic function spaces. Let $K$ be a compact set in $\mathbb{C}^n$. It is possible to consider various classes of function spaces, such as $H(K)$, $A(K)$, $R(K)$, and $P(K)$ (see section 1.2 for their definition) all of which are mostly interest of study in the theory of functions. From their definitions it follows immediately that

$$P(K) \subset R(K) \subset H(K) \subset A(K).$$

In Chapter 2 we first take a glance at the classical approximation results in $\mathbb{C}$ such as Runge Theorem, Lavrentiev's Theorem and Mergelyan Theorem that can be taught in a standard graduate complex analysis course. We follow the book of Rudin for this chapter. Chapter 4 is devoted to an exposition of the theory of polynomially convex sets. A compact subset of $\mathbb{C}^n$ is polynomially convex if it is defined by a family, finite or infinite, of polynomial inequalities. These sets play an important role in the theory of functions of several complex variables, especially in questions concerning approximation. Chapter 3 is devoted to the generalization of Runge Theorem to $\mathbb{C}^n$, so called the Oka-Weil Theorem.

Not every compact subset of $\mathbb{C}^n$ is polynomially convex. Generally it is a difficult task to find criteria for checking whether a given compact set is polynomially convex. In Chapter 4 we give a few well-known examples of polynomially convex sets. In section 4.2 we present two results that are full characterizations of polynomially convex sets. A fundamental connection between polynomial convexity and plurisubharmonic functions is presented by Theorem 4.2.1. The second (Theorem 4.2.4) characterization of polynomial convexity is a recent result of Duval and Sibony which uses the concept of

Jensen measures. These measures recently attracted the attention of quite a number of mathematicians and proved to have important applications in complex analysis.

Basic definitions are given in Chapter 1. Classical results in several complex variables and pluripotential theory are also included in section 1.2. In section 1.3 the class of Jensen measures is introduced. Elementary theory of differential forms, $\overline{\partial}$-operator and currents that we will need are given in sections 1.4, 1.5 and 1.6, respectively. Section 1.7 is devoted to the solution of the $\overline{\partial}$-equation in the polydisk. One of the important tasks to do in complex analysis is to solve the $\overline{\partial}$-equation. This concept is revisited in Theorem 3.1.8 for the case of polynomial polyhedra. This result is the essence of the proof of the Oka-Weil theorem.

## 1.2 Basic Definitions and Theorems

We refer to [Hör73] for detailed information on the content of this section. Let $\Omega$ be an open set in $\mathbb{C}^n$. For any $k = 0, 1, \ldots$, we will denote by $C^k(\Omega)$ the space of all $k$ times continuously differentiable complex-valued functions in $\Omega$. $C_0^k(A)$, where $A$ is a subset of $\Omega$, will denote the set of functions in $C^k(\Omega)$ vanishing outside a compact subset of $A$. $C^\infty(\Omega)$ will denote the algebra of all finitely differentiable complex-valued functions on $\Omega$. We write $C^\infty$ for $C^\infty(\Omega)$. We will define the operators on $C^\infty$ as follows:

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \qquad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right) \tag{1.1}$$

**Definition 1.2.1.** A function $f \in C^1$ is said to be *holomorphic* on $\Omega$ if

$$\frac{\partial f}{\partial \bar{z}_j} = 0 \tag{1.2}$$

for every $j = 1, \ldots, n$. The set of all holomorphic functions defined on $\Omega$ will be denoted by $\mathrm{Hol}(\Omega)$.

We will think of the set $\mathrm{Hol}(\Omega)$ together with the compact-open topology. A sequence of functions $f_j \in \mathrm{Hol}(\Omega)$ converges to a function $f$ in $\mathrm{Hol}(\Omega)$ if $f_j$ converges uniformly to $f$ on every compact subset of $\Omega$.

Any function which satisfies (1.2), satisfies the Cauchy-Riemann equations in the $z_j$th coordinate for any $j$. Hence a holomorphic function is holomorphic in each variable. The converse of this statement is known as Hartogs theorem.

**Theorem 1.2.2.** [Hartogs Theorem] *Let $f$ be a complex valued function defined in an open set $\Omega \subset \mathbb{C}^n$. Suppose that $f$ is holomorphic in each variable $z_j$ when the other coordinates $z_k$ for $k \neq j$ are fixed. Then $f$ is holomorphic in $\Omega$.*

**Theorem 1.2.3.** [Hartogs Extension Theorem] *Suppose $\Omega$ is a bounded nonempty open connected subset of $\mathbb{C}^n$ with connected boundary $b\Omega$. If $n \geq 2$, then each function $f$ that is holomorphic in some connected neighborhood of the boundary of $\Omega$, $b\Omega$, has a holomorphic extension to $\Omega$.*

The (open) ball of radius $r$ centered at $z^0 \in \mathbb{C}^n$ is the set

$$\mathbb{B}_n\left(z^0, r\right) = \left\{ z \in \mathbb{C}^n : \left|z - z^0\right| = \left( \sum_{j=1}^{n} \left|z_j - z_j^0\right|^2 \right)^{1/2} < r \right\}.$$

Similarly, the (open) polydisk of polyradius $r = (r_1, \cdots, r_n)$ centered at $z^0$ is the set

$$\mathbb{U}^n(z^0, r) = \left\{ z \in \mathbb{C}^n : \left|z_1 - z_1^0\right| < r_1, \cdots, \left|z_n - z_n^0\right| < r_n \right\}.$$

$\mathbb{B}_n$ and $\mathbb{U}^n$ denoting, respectively, the open ball of center 0 and radius one and open polydisk of polyradius $(1, \cdots, 1)$ and center 0 will be used consistently throughout the text.

**Definition 1.2.4.** An open set $\Omega \subset \mathbb{C}^n$ is called a *domain of holomorphy* if there are no open sets $\Omega_1$ and $\Omega_2$ in $\mathbb{C}^n$ with the following properties:

(i) $\emptyset \neq \Omega_1 \subset \Omega_2 \cap \Omega$.

(ii) $\Omega_2$ is connected and not contained in $\Omega$.

(iii) For every $f \in \mathrm{Hol}(\Omega)$ there is a function $f_2 \in \mathrm{Hol}(\Omega_2)$ (necessarily uniquely determined) such that $f = f_2$ in $\Omega_1$.

So we can say that a domain of holomorphy is a set which is maximal in the sense that there exists a holomorphic function on this set which cannot be extended to a bigger set.

**Definition 1.2.5.** If $K$ is a compact subset of $\Omega$, we define the *holomorphic hull*, $\widehat{K}^{\mathrm{Hol}}$ of $K$ with respect to $\Omega$ by

$$\widehat{K}_{\Omega}^{\mathrm{Hol}} = \{z \in \Omega : |f(z)| \leq \|f\|_K \text{ if } f \in \mathrm{Hol}(\Omega)\}$$

where $\|f\|_K = \max_K |f|$.

**Theorem 1.2.6.** *The following conditions are equivalent:*

*(i)* $\Omega$ *is a domain of holomorphy,*

*(ii)* $\widehat{K}_{\Omega}^{\mathrm{Hol}} \subset\subset \Omega$ *if* $K \subset\subset \Omega$

*where* $K \subset\subset \Omega$ *means that* $K$ *is relatively compact in* $\Omega$*, that is,* $K$ *is contained in a compact subset of* $\Omega$.

**Definition 1.2.7.** Let $X$ be a topological space. We say that a function $f : X \to [-\infty, \infty)$ is *upper-semicontinuous* if the set $\{x \in X : f(x) < a\}$ is open in $X$ for each $a \in \mathbb{R}$.

**Definition 1.2.8.** Let $U$ be an open set of $\mathbb{C}$. A function $f : U \to [-\infty, \infty)$ is called *subharmonic* if it is upper-semicontinuous and satisfies the local submean inequality, i.e. given $w \in U$, there exists $\rho > 0$ such that

$$f(w) \leq \frac{1}{2\pi} \int_0^{2\pi} f(w + re^{it})dt \qquad (0 \leq r < \rho). \tag{1.3}$$

**Definition 1.2.9.** A function $f$ defined in an open set $\Omega \subset \mathbb{C}^n$ with values in $[-\infty, +\infty)$ is called *plurisubharmonic* if

(i) $f$ is upper-semicontinuous,

(ii) For arbitrary $z$ and $w \in \mathbb{C}^n$, the function $\tau \to f(z + \tau w)$ is subharmonic in the part of $\mathbb{C}$ where it is defined.

We will denote the set of such functions by $\mathrm{PSH}(\Omega)$.

**Definition 1.2.10.** A smooth function $u$ on $\Omega$ is said to be *strictly plurisubharmonic* on $\Omega$ if for every relatively compact open set $V \subset \Omega$ there exists a number $\varepsilon > 0$ so that $u(z) - \varepsilon |z|^2$ is plurisubharmonic on $V$.

The elementary theory of plurisubharmonic functions parallels that of subharmonic functions rather closely. In particular, plurisubharmonic functions enjoy the following properties. We refer to [Kli91] for more about plurisubharmonic functions.

**Proposition 1.2.11.**

*(a) If $\{u_j\}_{j=1,\dots}$ is a monotonically decreasing sequence of plurisubharmonic functions defined on a domain $\Omega$, then the function $u$ defined by $u(z) = \lim_{j\to\infty} u_j(z)$ is also plurisubharmonic.*

*(b) If $\{u_\alpha\}_{\alpha\in A}$ is an arbitrary collection of plurisubharmonic functions on a domain $\Omega$, and if $u(z) = \sup_\alpha(z)$ then the upper regularization of $u$ defined by*

$$u^*(w) = \lim_{\varepsilon\to 0^+}\left(\sup_{|w-z|<\varepsilon} u(z)\right)$$

*is plurisubharmonic or else identically $+\infty$.*

*(c) A plurisubharmonic function on a connected open set in $\mathbb{C}^n$ is either identically $-\infty$ or else is locally integrable with respect to Lebesgue measure on $\mathbb{C}^n$.*

*(d) If $u$ is a plurisubharmonic function on the domain $\Omega$, there is a decreasing sequence $\{u_j\}_{j=1,\dots}$ of functions of class $C^\infty$ on $\Omega$ with $u(z) = \lim_{j\to\infty} u_j(z)$ for all $z$ and with the property that if $K$ is a compact subset of $\Omega$, then all but finitely many of the functions $u_j$ are plurisubharmonic on a neighborhood of $K$.*

*(e) If $u$ is a plurisubharmonic function on a domain $\Omega$ and if $\chi : \mathbb{R} \to \mathbb{R}$ satisfies $\chi'$, $\chi'' \geq 0$, then $\chi \circ u$ is plurisubharmonic on $\Omega$.*

*(f) Plurisubharmonic functions are subharmonic in the sense of potential theory. Therefore whenever $\mathbb{B}_n(z_0, r) \subset \Omega$*

$$u(z_0) \leq \frac{n!}{\pi^n r^{2n}} \int_{\mathbb{B}_n(z_0,r)} u(z)\, d\mathcal{L}(z)$$

*for every $u \in \mathrm{PSH}(\Omega)$.*

*(g) Let $u \in \mathrm{PSH}(\Omega)$. Let $\Omega_j$ be domains so that $\Omega_j \subset \Omega_{j+1}$ is relatively compact for every $j$ and $\Omega = \cup \Omega_j$. There exist functions $u_j \in C^\infty(\Omega) \cap \mathrm{PSH}(\Omega_j)$ such that $u_j(z)$ eventually decreases to $u(z)$ for every $z \in \Omega$.*

**Theorem 1.2.12.** *[Hör73] Let $\Omega$ be a domain in $\mathbb{C}^n$, let $K$ be a compact subset of $\Omega$, and let $g$ be a continuous function on $\Omega$. If $\{u_k\}_{k=1,\dots}$ is a sequence of plurisubharmonic functions that is locally uniformly bounded on $\Omega$ and that satisfies*

$$\limsup_{k\to\infty} u_k(z) \le g(z) \text{ for all } z \in \Omega,$$

*then for each $\varepsilon > 0$ there is a $k_\varepsilon$ such that for $k > k_\varepsilon$, $u_k(z) < g(z) + \varepsilon$ for every $z \in K$.*

*Proof.* We first suppose that the function $g$ is constant. Without loss of generality, we can suppose that the sequence is uniformly bounded on $\Omega$. It can then be supposed that $g = C$ with $C < 0$ and that each $u_k$ is negative on $\Omega$.

Choose a $\delta < \frac{1}{3} \operatorname{dist}(K, \mathbb{C}^n \backslash \Omega)$. If $z_0 \in K$, then by Proposition 1.2.11 (f)

$$u_k(z_0) \le \frac{n!}{\pi^n \delta^{2n}} \int_{\mathbb{B}_n(z_0,\delta)} u_k(z) d\mathcal{L}(z).$$

Fatou′s lemma implies that

$$\limsup_{k\to\infty} \int_{\mathbb{B}_n(z_0,\delta)} u_k(z) d\mathcal{L}(z) \le \int_{\mathbb{B}_n(z_0,\delta)} C d\mathcal{L}(z) = C\frac{\pi^n \delta^{2n}}{n!}.$$

Thus there is $k(z_0)$ large enough that for $k > k(z_0)$,

$$\int_{\mathbb{B}_n(z_0,\delta)} u_k(z) d\mathcal{L}(z) < \frac{\pi^n \delta^{2n}}{n!} (C + \varepsilon/2).$$

If $|w - z_0| < r$ for an $r < \delta$, then, becouse the $u'$s are negative, we have, for large $k$,

$$\begin{aligned} u_k(w) &\le \frac{n!}{\pi^n (\delta + r)^{2n}} \int_{\mathbb{B}_n(w,\delta+r)} u_k(z) d\mathcal{L}(z) \\ &\le \frac{n!}{\pi^n (\delta + r)^{2n}} \int_{\mathbb{B}_n(z_0,\delta)} u_k(z) d\mathcal{L}(z) \\ &\le \frac{\delta^{2n}}{(\delta + r)^{2n}} (C + \varepsilon/2) \\ &\le C + \varepsilon/2 \end{aligned}$$

since $\mathbb{B}_n(w, \delta + r) \supset \mathbb{B}_n(z_0, \delta)$. Thus, for each $z_0 \in K$, we have found a neighborhood of $z_0$ on which $u_k \le C + \varepsilon$ provided $k$ is big enough. Compactness now implies the result.

Having established the result when the function $g$ is constant, we derive the general case. Let $K$, $\varepsilon$, and $g$ be as given in the theorem. Because the function $g$ is continuous, compactness yields finitely many compact sets $E_1, \cdots, E_q$ with union $K$ and corresponding constants $c_1, \cdots, c_q$ such that for each $j$ and all $x \in E_j$ , $g(x) < c_j < g(x) + \varepsilon/2$. By the special case of the result that we have proved, there is an integer $k_j$ such that $u_k(x) < c_j + \varepsilon/2$ if $j > k_j$, $x \in \overline{E}_j$ . With $k > \max\{k_1, \cdots, k_q\}$, we have that for all $x \in K$, $u_k(x) < g(x) + \varepsilon$. The theorem is proved.

□

**Definition 1.2.13.** If $K$ is a compact subset of the open set $\Omega \subset \mathbb{C}^n$ we define the *plurisubharmonic hull* $\widehat{K}_\Omega^{Psh}$ of $K$ with respect to $\Omega$ by

$$\widehat{K}_\Omega^{Psh} = \{z \in \Omega : f(z) \leq \|f\|_K \text{ for all } f \in \mathrm{PSH}(\Omega)\}.$$

Let $\delta$ be an arbitrary continuous function on $\mathbb{C}^n$ such that $\delta > 0$ except at 0 and

$$\delta(tz) = |t|\,\delta(z), \quad t \in \mathbb{C}, \quad z \in \mathbb{C}^n.$$

Set $\delta(z, b\Omega) = \inf_{w \notin \Omega} \delta(z - w)$. It's clear that $\delta(z, b\Omega)$ is a continuous function of $z$.

**Definition 1.2.14.** The open set $\Omega \in \mathbb{C}^n$ is called *pseudoconvex* if the following equivalent conditions are satisfied:

(i) $-log\ \delta(z, b\Omega)$ is plurisubharmonic in $\Omega$.

(ii) There exists a continuous plurisubharmonic function $f$ in $\Omega$ such that

$$\Omega_c = \{z \in \Omega : f(z) < c\} \subset\subset \Omega$$

for every $c \in \mathbb{R}$.

(iii) $\widehat{K}_\Omega^{Psh} \subset\subset \Omega$ if $K \subset\subset \Omega$.

**Definition 1.2.15.** A domain of holomorphy $\Omega \subset \mathbb{C}^n$ is called a *Runge domain* if polynomials are dense in $\mathrm{Hol}(\Omega)$, that is, if every $f \in \mathrm{Hol}(\Omega)$ can be uniformly approximated on an arbitrary compact set in $\Omega$ by polynomials.

Here and throughout this work polynomials are understood to be holomorphic polynomials.

**Definition 1.2.16.** Let $K$ be a compact subset of $\mathbb{C}^n$. We define the *polynomial hull* of $K$, denoted by $\widehat{K}$, by

$$\widehat{K} = \{z \in \mathbb{C}^n : |p(z)| \leq \|p\|_K \text{ for all polynomials } p\}.$$

$K$ is called polynomially convex if $\widehat{K} = K$.

The definition can also be stated as follows:

*A compact subset $K$ of $\mathbb{C}^n$ is polynomially convex if and only if for every $z_0$ in $\mathbb{C}^n \backslash K$ we can find a polynomial $p$ with*

$$|p(z_0)| > \|p\|_K.$$

**Proposition 1.2.17.**

$$K \subset \widehat{K}_\Omega^{Psh} \subset \widehat{K}_\Omega^{\mathrm{Hol}} \subset \widehat{K}$$

*Proof.* Clearly $K \subset \widehat{K}_\Omega^{Psh}$. Let $z \in \widehat{K}_\Omega^{Psh}$ and $f \in \mathrm{Hol}(\Omega)$. Then $|f| \in \mathrm{PSH}(\Omega)$. Hence $|f(z)| \leq \|f\|_K$ and therefore $z \in \widehat{K}_\Omega^{\mathrm{Hol}}$ which means $\widehat{K}_\Omega^{Psh} \subset \widehat{K}_\Omega^{Psh}$. Let $z \in \widehat{K}_\Omega^{\mathrm{Hol}}$ and $p$ be a polynomial in $\mathbb{C}^n$. Then $p|_\Omega \in \mathrm{Hol}(\Omega)$ and $|p(z)| \leq \|p\|_K$. Thus $z \in \widehat{K}$ and $\widehat{K}_\Omega^{\mathrm{Hol}} \subset \widehat{K}$. $\square$

**Theorem 1.2.18.** *The following conditions on a domain of holomorphy $\Omega \subset \mathbb{C}^n$ are equivalent:*

*(i)* *$\Omega$ is a Runge domain.*

*(ii)* *$\widehat{K} = \widehat{K}_\Omega^{\mathrm{Hol}}$ if $K \subset \Omega$ is compact.*

*(iii)* *$\widehat{K} \cap \Omega \subset\subset \Omega$ if $K \subset \Omega$ is compact.*

On domains of holomorphy, plurisubharmonic functions can be approximated by plurisubharmonic functions of particularly simple form. The following result was stated by Bremermann [Bre58]

**Theorem 1.2.19.** *If $\Omega$ is a domain of holomorphy in $\mathbb{C}^n$, and if $u$ is a continuous plurisubharmonic function on $\Omega$, then for each compact subset $K$ of $\Omega$ and for every $\varepsilon > 0$, there are finitely many holomorphic functions $f_1, \cdots, f_r$ on $\Omega$ such that for suitable positive constants $c_j$,*

$$u(z) \leq \max_{j=1,\ldots,r} c_j \log\left|f_j(z)\right| \leq u(z) + \varepsilon.$$

*In the event that $\Omega$ is a Runge domain in $\mathbb{C}^n$, the holomorphic functions $f_j$ can be taken to be polynomials.*

*Proof.* Introduce the domain $\Omega^*$ in $\mathbb{C}^{n+1}$ defined by

$$\Omega^* = \left\{(z,w) \in \mathbb{C}^n \times \mathbb{C} : |w| < e^{-u(z)}\right\}.$$

This domain is pseudoconvex and so a domain of holomorphy.

For $z_0 \in \Omega$, define the function $f_{z_0}$ by $f_{z_0}(w) = \sum_{k=0}^{\infty} e^{ku(z_0)} w^k$, which is defined and holomorphic in the planar domain $\left\{w \in \mathbb{C} : |w| < e^{-u(z_0)}\right\}$. The domain $\Omega^*$ is a domain of holomorphy, so there is a function $F \in \mathrm{Hol}(\Omega^*)$ with $F(z_0, w) = f_{z_0}(w)$ for all $w \in \mathbb{C}$ with $|w| < e^{-u(z_0)}$. The function $F$ admits an expansion $F(z,w) = \sum_{k=0}^{\infty} a_k(z) w^k$ with coefficients $a_k \in \mathrm{Hol}(\Omega)$ that satisfy

$$\limsup_{k\to\infty} \sum \frac{\log|a_k(z)|}{k} \leq u(z)$$

for all $z \in \Omega$ by Hadamard′s formula for the radius of convergence of a power series.

Theorem 1.2.12 implies that for $\varepsilon > 0$, there is $k_0$ large enough that for $k > k_0$, $\frac{\log|a_k(z)|}{k} \leq u(z) + \varepsilon$ for all $z \in K$. By the choice of $F$, $\limsup_{k\to\infty} \frac{\log|a_k(z_0)|}{k} = u(z_0)$, whence by continuity, $\limsup_{k\to\infty} \frac{\log|a_k(z)|}{k} > u(z_0) - \varepsilon$ for all $z$ in a neighborhood of $z_0$. By compactness, a finite number of choices of the point $z_0$ will yield a cover of $K$ by the corresponding neighborhoods. The theorem follows. $\square$

**Definition 1.2.20.** Let $X$ be a compact Hausdorff space. A *uniform algebra* on $X$ is an algebra $U$ of continuous complex-valued functions on $X$ satisfying

(i) $U$ is closed under uniform convergence on $X$,

(ii) $U$ contains the constants,

(iii) $U$ separates the points of $X$.

**Theorem 1.2.21.** [Stone-Weierstrass Theorem] *$U$ is a subalgebra of $C(X)$ containing the constants and separating points. If*

$$f \in U \Rightarrow \bar{f} \in U,$$

*then $U$ is dense in $C(X)$.*

Let $K \subset \mathbb{C}^n$ be compact. We will use the following notations. $C(K)$ is the class of all continuous complex-valued functions with supremum norm on $K$. $A(K)$ is the uniform limits of continuous complex-valued functions holomorphic in some neigborhood of $K$. $H(K)$ is the class of continuous complex-valued functions on $K$ which are holomorphic on $K^\circ$, the interior of $K$. $P(K)$ is the class of functions consisting of uniform limits of polynomials restricted to $K$. $R(K)$ is the uniform closure in $C(K)$ of rational functions $r = p/q$ where $p$ and $q$ are polynomials and $q(z) \neq 0$ for $z \in K$.

Evidently,

$$P(K) \subset R(K) \subset H(K) \subset A(K) \subset C(K).$$

One of the major problems is to determine when equality holds between these spaces.

## 1.3 Jensen Measures

We will now introduce and mention a few basic facts about the class of Jensen measures in complex analysis. Let $\Omega \subset \mathbb{C}^n$ be an open set. Let $C_0(\Omega)$ be the space of all compactly supported continuous functions on $\Omega$. By the Riesz representation theorem (see Theorem 6.19 in [Rud87]) the dual space $C_0^*(\Omega)$ of $C_0(\Omega)$ can be considered as the class of all compactly supported Borel measures on $\Omega$. Let $M_0(\Omega)$ be the class of all positive Borel probability measures in $C_0^*(\Omega)$. We consider the set $M_0(\Omega)$ together with the induced weak-$*$ topology on it from $C_0^*(\Omega)$. A sequence $\mu_j$ from $M_0(\Omega)$ converges to a measure $\mu \in M_0(\Omega)$ if the supports $\operatorname{supp}\mu_j$ are contained in some fixed compact set $K \subset \Omega$ and $\mu_j(\varphi)$ converges to $\mu(\varphi)$ for every function $\varphi \in C_0(\Omega)$. Let $z \in \Omega$ be a point. A measure $\mu \in M_0(\Omega)$ is called a Jensen measure with barycenter $z$ on $\Omega$ if

$$u(z) \leq \int u \, d\mu$$

for every $u \in \mathrm{PSH}(\Omega)$. We denote by $J_z$ the class of all Jensen measures $\mu$ with barycenter $z$ on $\Omega$.

The class $J_z$ is evidently convex. We will also show that it is weak-$*$ closed. First we need a Lemma:

**Lemma 1.3.1.** *Let $s$ be an upper bounded upper semicontinuous function on a compact metric space $X$ and $\{\mu_j\} \subset C^*(X)$ be a sequence of measures converging weak-$*$ to a measure $\mu \in C^*(X)$. Then*

$$\limsup_j \int s\, d\mu_j \leq \int s\, d\mu.$$

*Proof.* There exist functions $\varphi_k \in C(X)$ so that $\varphi_k \downarrow s$ on $X$. Then

$$\limsup_j \int s\, d\mu_j \leq \limsup_j \int \varphi_k\, d\mu_j = \int \varphi_k\, d\mu$$

for all $k$. Finally by the monotone convergence theorem

$$\limsup_j \int s\, d\mu_j \leq \int s\, d\mu.$$

□

**Corollary 1.3.2.** *$J_z$ is weak-$*$ closed.*

*Proof.* Suppose $\mu_j$ is a sequence in $J_z$ that converges weak-$*$ to $\mu \in M_0(\Omega)$. There exists a compact set $K$ in $\Omega$ so that $\operatorname{supp}\mu_j$ and $\operatorname{supp}\mu$ is contained in $K$ for every $j$. Take any function $u$ in $\mathrm{PSH}(\Omega)$. Then by Lemma 1.3.1,

$$u(z) \leq \limsup_j \int u\, d\mu_j \leq \int u\, d\mu.$$

Therefore, $\mu \in J_z$ and $J_z$ is compact. □

When $\Omega$ is a Runge domain, a measure $\mu$ is in $J_z$ if and only if

$$|p(z)| \leq \int |p|\, d\mu$$

for every polynomial $p$. This a simple consequence of Theorem 1.2.19.

## 1.4 Differential Forms

We will study differential forms on an open subset $\Omega$ of real Euclidean $n$-space $\mathbb{R}^n$. For sections 1.4 and 1.5 we refer to [AleWer98].

**Definition 1.4.1.** Let $\Omega$ be an open subset of $\mathbb{R}^n$. For any $x \in \Omega$ we define the *tangent space* at $x$, $T_x$, as the collection of all maps $v : C^\infty \to \mathbb{C}$ for which

(a) $v$ is linear.

(b) $v(f \cdot g) = f(x) \cdot v(g) + g(x) \cdot v(f)$, where f, g $\in C^\infty$.

The elements of $T_x$ are called tangent vectors at $x$, and the dual space to $T_x$ is denoted by $T_x^*$.

**Definition 1.4.2.** A $1$*-form* $\omega$ on $\Omega$ is a map $\omega$ assigning to each $x$ in $\Omega$ an element of $T_x^*$.

$dx_1, \dots, dx_n$ are particular 1-forms.

**Lemma 1.4.3.** *Every 1-form* $\omega$ *admits a unique representation*

$$\omega = \sum_1^n C_j dx_j,$$

*the* $C_j$ *being scalar functions on* $\Omega$*.*

Let $V$ be an $n$-dimensional vector space over $\mathbb{C}$. Denote by $\wedge^p(V)$ the vector space of $p$-linear alternating maps of $V \times \cdots \times V \to \mathbb{C}$, where alternating means that the value of the function changes sign if two of the variables are interchanged.

Define $\mathcal{G}(V)$ as the direct sum

$$\mathcal{G}(V) = \wedge^0(V) \oplus \wedge^1(V) \oplus \cdots \oplus \wedge^n(V).$$

Here $\wedge^0(V) = \mathbb{C}$ and $\wedge^1(V)$ is the dual space of $V$. Put $\wedge^j(V) = 0$ for $j > n$.

We now introduce a multiplication in the vector space $\mathcal{G}(V)$. Fix $\tau \in \wedge^p(V)$, $\sigma \in \wedge^q(V)$. The map

$$(\xi_1, \dots, \xi_p, \xi_{p+1}, \dots, \xi_{p+q}) \to \tau(\xi_1, \dots, \xi_p)\,\sigma(\xi_{p+1}, \dots, \xi_{p+q})$$

is a $(p+q)$-linear map from $V \times \cdots \times V$ ($p+q$ factors)$\to \mathbb{C}$. It is, however, not alternating. To obtain an alternating map, we use

**Definition 1.4.4.** Let $\tau \in \wedge^p(V)$, $\sigma \in \wedge^q(V)$, $p, q \geq 1$.

$$\tau \wedge \sigma\left(\xi_1, \ldots, \xi_{p+q}\right) = \frac{1}{(p+q)!} \sum_{\pi} (-1)^{\pi} \tau\left(\xi_{\pi(1)}, \ldots, \xi_{\pi(p)}\right) \cdot \sigma\left(\xi_{\pi(p+1)}, \cdots, \xi_{\pi(p+q)}\right),$$

the sum being taken over all permutations $\pi$ of the set $\{1, 2, \ldots, p+q\}$, and $(-1)^{\pi}$ denoting the sign of the permutation $\pi$.

**Lemma 1.4.5.** $\tau \wedge \sigma$ *as defined is* $(p+q)$*-linear and alternating and so* $\tau \wedge \sigma \in \wedge^{p+q}(V)$.

The operation $\wedge$ defines a product for pairs of elements, one in $\wedge^p(V)$ and one in $\wedge^q(V)$, the value lying in $\wedge^{p+q}(V)$, hence in $\mathcal{G}(V)$. By linearity, $\wedge$ extends to a product on arbitrary pairs of elements of $\mathcal{G}(V)$ with value in $\mathcal{G}(V)$. For $\tau \in \wedge^0(V)$, $\sigma \in \mathcal{G}(V)$, define $\tau \wedge \sigma$ as a scalar multiplication by $\tau$.

**Lemma 1.4.6.** *If* $\tau \in \wedge^p(V)$, $\sigma \in \wedge^q(V)$, *then* $\tau \wedge \sigma = (-1)^{pq} \sigma \wedge \tau$.

Let $e_1, \ldots, e_n$ form a basis for $\wedge^1(V)$.

**Lemma 1.4.7.** *Fix p. The set of elements*

$$e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_p}, \qquad 1 \leq i_1 < i_2 < \cdots < i_p \leq n,$$

*forms a basis for* $\wedge^p(V)$

We now apply preceding to the case when $V = T_x$, $x \in \Omega$. Then $\wedge^p(T_x)$ is the space of all $p$-linear alternating functions on $T_x$, and so, for $p = 1$, coincides with $T_x^*$. Thus the following extends our definition of a 1-form.

**Definition 1.4.8.** A *p-form* $\omega^p$ on $\Omega$ is a map $\omega^p$ assigning to each $x$ in $\Omega$ an element of $\wedge^p(T_x)$.

Let $\tau^p$ and $\sigma^q$ be, respectively, a $p$-form and $q$-form. For $x \in \Omega$, put

$$\tau^p \wedge \sigma^q(x) = \tau^p(x)\sigma^q(x) \in \wedge^{p+q}(T_x).$$

In particular, since $dx_1, \ldots, dx_n$ are 1-forms,

$$dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_p}$$

is a $p$-form for each choice of $(i_1, \ldots i_p)$.

Because of Lemma 1.4.6,

$$dx_j \wedge dx_j = 0 \ \ for\ each\ j.$$

Hence $dx_{i_1} \wedge \cdots \wedge dx_{i_p} = 0$ unless the $i_\nu$ are distinct.

**Lemma 1.4.9.** *Let $\omega^p$ be any $p$-form on $\Omega$. Then there exist (unique) scalar functions $C_{i_1}, \cdots, C_{i_p}$ on $\Omega$ such that*

$$\omega^p = \sum_{i_1 < i_2 < \cdots < i_p} C_{i_1 \cdots i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_p}.$$

**Definition 1.4.10.** $\wedge^p(\Omega)$ consists of all $p$-forms $\omega^p$ such that the functions $C_{i_1 \cdots i_p}$ occurring in Lemma 1.4.9 lie in $C^\infty(\Omega)$. We set $\wedge^0(\Omega) = C^\infty$.

Consider the map $f \to df$ from $C^\infty \to \wedge^1(\Omega)$. We wish to extend $d$ to a linear map $\wedge^p(\Omega) \to \wedge^{p+1}(\Omega)$, for all $p$.

**Definition 1.4.11.** Let $\omega^p \in \wedge^p(\Omega)$, $p = 0, 1, 2, \ldots$. Then

$$\omega^p = \sum_{i_1 < i_2 < \cdots < i_p} C_{i_1 \cdots i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_p}.$$

Define

$$d\omega^p = \sum_{i_1 < i_2 < \cdots < i_p} dC_{i_1 \cdots i_p} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p}.$$

Note that $d$ maps $\wedge^p(\Omega) \to \wedge^{p+1}(\Omega)$. We call $d\omega^p$ the exterior derivative of $\omega^p$.

## 1.5 The $\bar{\partial}$-Operator

Let $\Omega$ be an open subset of $\mathbb{C}^n$. The complex coordinate functions $z_1, \ldots, z_n$ as well as their conjugates $\bar{z}_1, \ldots, \bar{z}_n$ lie in $C^\infty(\Omega)$. Hence the forms

$$dz_1, \ldots, dz_n, \qquad d\bar{z}_1, \ldots, d\bar{z}_n$$

all belong to $\wedge^1(\Omega)$. Fix $x \in \Omega$. Note that $\wedge^1(T_x) = T_x^*$ has dimension $2n$ over $\mathbb{R}$, since $\mathbb{C}^n = \mathbb{R}^{2n}$. If $z_j = x_j + iy_j$, then

$$(dx_1)_x, \ldots, (dx_n)_x, \qquad (dy_1)_x, \ldots, (dy_n)_x$$

form a basis for $T_x^*$. Since $dx_j = 1/2\left(dz_j + d\bar{z}_j\right)$ and $dy_j = 1/2i\left(dz_j - d\bar{z}_j\right)$,

$$(dz_1)_x, \ldots, (dz_n)_x, \qquad (d\bar{z}_1)_x, \ldots, (d\bar{z}_n)_x$$

also form a basis for $T_x^*$. In fact,

**Lemma 1.5.1.** *If* $\omega \in \wedge^1(\Omega)$, *then*

$$\omega = \sum_{j=1}^{n} a_j dz_j + b_j d\bar{z}_j,$$

*where* $a_j, b_j \in C^\infty(\Omega)$.

Fix $f \in C^\infty(\Omega)$. Since $(x_1, \cdots, x_n, y_1, \cdots, y_n)$ are real coordinates in $\mathbb{C}^n$,

$$\begin{aligned} df &= \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j + \frac{\partial f}{\partial y_j} dy_j \\ &= \sum_{j=1}^{n} \left(\frac{1}{2}\frac{\partial f}{\partial x_j} + \frac{1}{2i}\frac{\partial f}{\partial y_j}\right) dz_j + \left(\frac{1}{2}\frac{\partial f}{\partial x_j} - \frac{1}{2i}\frac{\partial f}{\partial y_j}\right) d\bar{z}_j. \end{aligned}$$

Then from 1.1,

$$df = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j} dz_j + \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j.$$

We define

$$\partial f = \sum_{1}^{n} \frac{\partial f}{\partial z_j} dz_j, \quad \bar{\partial} f = \sum_{1}^{n} \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j.$$

Note that $\partial f + \bar{\partial} f = df$, if $f \in C^\infty$.

Let $I$ be any $p$-tuple of integers, $I = (i_1, i_2, \ldots, i_p)$, $1 \leq i_j \leq n$, all $j$. Put

$$dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_p}.$$

Thus $dz_I \in \wedge^p(\Omega)$.

Let $J$ be any $q$-tuple $\left(j_1, \ldots, j_q\right)$, $1 \leq j_k \leq n$, all $k$, and put

$$d\bar{z}_J = d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}.$$

So $d\bar{z}_J \in \wedge^q(\Omega)$. Then

$$dz_I \wedge d\bar{z}_J \in \wedge^{p+q}(\Omega).$$

For $I$ as above, put $|I| = p$ and $|J| = q$.

**Definition 1.5.2.** Fix integers $p, q \geq 0$. $\wedge^{p,q}(\Omega)$ is the space of all $\omega \in \wedge^{p+q}(\Omega)$ such that

$$\omega = \sum_{I,J} a_{IJ} dz_I \wedge d\bar{z}_J,$$

the sum being extended over all $I, J$ with $|I| = p$, $|J| = q$, and with each $a_{I,J} \in C^\infty$.

An element of $\wedge^{p,q}(\Omega)$ is called a *form of bidegree* $(p,q)$. We now have a direct sum decomposition of each $\wedge^p(\Omega)$:

**Lemma 1.5.3.**

$$\wedge^p(\Omega) = \wedge^{0,p}(\Omega) \oplus \wedge^{1,p-1}(\Omega) \oplus \wedge^{2,p-2}(\Omega) \oplus \cdots \oplus \wedge^{p,0}(\Omega).$$

We extend the definition of $\partial$ and $\bar{\partial}$ to maps from $\wedge^p(\Omega) \to \wedge^{p+1}(\Omega)$ for $p$, as follows:

**Definition 1.5.4.** Choose $\omega^p$ in $\wedge^p(\Omega)$,

$$\omega^p = \sum_{I,J} a_{I,J} dz_I \wedge d\bar{z}_J,$$

$$\partial\omega^p = \sum_{I,J} \partial a_{I,J} \wedge dz_I \wedge d\bar{z}_J,$$

and

$$\bar{\partial}\omega^p = \sum_{I,J} \bar{\partial} a_{I,J} \wedge dz_I \wedge d\bar{z}_J.$$

Since $a_{I,J} \in C^\infty$,

$$\bar{\partial}\omega^p + \partial\omega^p = \sum_{I,J} da_{I,J} \wedge dz_I \wedge d\bar{z}_J = d\omega^p$$

so we have

$$\bar{\partial} + \partial = d$$

as operators from $\wedge^p(\Omega) \to \wedge^{p+1}(\Omega)$. Note that if $\omega \in \wedge^{p,q}$, $\partial\omega \in \wedge^{p+1,q}$ and $\bar{\partial}\omega \in \wedge^{p,q+1}$.

**Lemma 1.5.5.** $\overline{\partial}^2 = 0$, $\partial^2 = 0$, *and* $\partial\overline{\partial} = \overline{\partial}\partial = 0$.

**Lemma 1.5.6.** *If* $\omega^p \in \wedge^p(\Omega)$ *and* $\omega^q \in \wedge^q(\Omega)$, *then*

$$\overline{\partial}(\omega^p \wedge \omega^q) = \overline{\partial}\omega^p \wedge \omega^q + (-1)^p \omega^p \wedge \overline{\partial}\omega^q.$$

## 1.6 The Currents

We will need elementary theory of currents. We refer to [Sto07] for this section. If $\Omega \subset \mathbb{R}^n$, then $\mathcal{D}(\Omega)$ is the subspace of $C^\infty(\Omega)$ that consists of the functions with compact support. The space $\wedge^p(\Omega)$ contains the subspace $\mathcal{D}^p(\Omega)$ of the compactly supported $p$-forms on $\Omega$. Thus $\mathcal{D}(\Omega) = \mathcal{D}^0(\Omega)$. If $\Omega$ is a domain in $\mathbb{C}^n$, $\mathcal{D}^{p,q}(\Omega)$ is the space of compactly supported forms of bidegree $(p,q)$. $\mathcal{D}^{p,q}(\Omega)$ is again a subspace of $\wedge^{p,q}(\Omega)$. A natural way of defining a topology on $\mathcal{D}^p(\Omega)$ is as follows: A sequence $\alpha_j \in \mathcal{D}^p(\Omega)$ converges to 0 if the sequences of coefficients of $\alpha_j$, as well as the sequences of the derivatives of all fixed orders of these coefficients, converge to 0 uniformly on compact subsets of $\Omega$.

**Definition 1.6.1.** A *current* of dimension $p$ and of degree $n-p$ on $\Omega$ is a $\mathbb{C}$-linear functional $T$ on the space $\mathcal{D}^p(\Omega)$ that has the following continuity property: If $\{\alpha_j\}_{j=1,\ldots}$ is a sequence in $\mathcal{D}^p(\Omega)$ such that for some fixed compact set $K \subset \Omega$, supp $\alpha_j \subset K$ for all $j$ and if, in addition, $\alpha_j$ converges to 0, then the sequence $\{T(\alpha_j)\}_{j=1,\ldots}$ converges to zero.

The space of currents of dimension $p$ (and degree $n-p$) on $\Omega$ is denoted by $\mathcal{D}'^p(\Omega)$.

A current of bidimension $(p,q)$ and bidegree $(n-p,n-q)$ is a $\mathbb{C}$-linear functional on $\mathcal{D}^{p,q}(\Omega)$ with the indicated continuity property. If $T$ is a current of dimension $p$, then the support of $T$ is the smallest closed subset $K$ of $\Omega$ with the property that $T(\alpha_j) = 0$ for all $\alpha \in \mathcal{D}^p(\Omega)$ that vanish on a neighborhood of $K$.

**Definition 1.6.2.** If $\Omega$ is a domain in $\mathbb{C}^n$, an element $\varphi \in \wedge^{p,p}(\Omega)$ is said to be *positive* if whenever $\alpha_j$, $j = 1,\cdots,n-p$, are $(1,0)$-forms defined in $\Omega$ with continuous compactly supported coefficients, then

$$\int_\Omega \varphi \wedge i\alpha_1 \wedge \overline{\alpha}_1 \wedge \cdots \wedge i\alpha_{n-p} \wedge \overline{\alpha}_{n-p} \geq 0.$$

**Definition 1.6.3.** A current $T$ in $\mathcal{D}_{n-p,n-p}(\Omega)$ is said to be *positive* if for all nonnegative functions $f \in \mathcal{D}(\Omega)$ and for all forms $\alpha_1, \cdots, \alpha_p \in \mathcal{D}^{1,0}(\Omega)$, the quantity $T\left(fi\alpha_1 \wedge \overline{\alpha}_1 \wedge \cdots \wedge i\alpha_p \wedge \overline{\alpha}_p\right)$ is nonnegative.

Perhaps the simplest example of a positive current is the current $[\lambda]$ of integration over the complex line $\lambda$ in $\mathbb{C}^n$.

If $T \in \mathcal{D}_{p,p}(\Omega)$ is positive, and $\varphi \in \wedge^{q,q}(\Omega)$ is a positive form, then the exterior product $T \wedge \varphi \in \mathcal{D}_{p+q,p+q}(\Omega)$, which is defined by $T \wedge \varphi(\alpha) = T(\varphi \wedge \alpha)$ is also positive, for any $\alpha \in \wedge^{n-p-q}(\Omega)$.

## 1.7 The Equation $\bar{\partial} u = f$

As before, fix an open set $\Omega \subset \mathbb{C}^n$. Given $f \in \wedge^{p,q+1}(\Omega)$, we seek $u \in \wedge^{p,q}$ such that

$$\overline{\partial} u = f. \tag{1.4}$$

Since $\overline{\partial}^2 = 0$ from Lemma 1.5.5, a necessary condition on $f$ is

$$\overline{\partial} f = 0. \tag{1.5}$$

If (1.5) holds, we say that $f$ is $\overline{\partial}$-*closed*. What is a sufficient condition on $f$ to solve $\overline{\partial} u = f$? It turns out that this will depend on the domain $\Omega$. We refer to [AleWer98] for this section.

Recall the analogous problem for the operator $d$ on a domain $\Omega \subset \mathbb{R}^n$. If $\omega^p$ is a $p$-form in $\wedge^p(\Omega)$, the condition

$$d\omega^p = 0 \qquad (\omega \text{ is closed}) \tag{1.6}$$

is necessary in order that we can find some $\tau^{p-1}$ in $\wedge^{p-1}(\Omega)$ with

$$d\tau^{p-1} = \omega^p. \tag{1.7}$$

However, (1.6) is, in general, not sufficient. (Think of an example when $p = 1$ and $\Omega$ is an annulus in $\mathbb{R}^2$.) If $\Omega$ is simply connected, then (1.6) is sufficient in order that (1.7) admit a solution.

For the $\bar{\partial}$-operator, a purely topological condition on $\Omega$ is inadequate. We shall find various conditions in order that (1.4) will have a solution. Denote by $\overline{\mathbb{U}}^n$ the closed unit polydisk in $\mathbb{C}^n$ : $\{z \in \mathbb{C}^n : |z_j| \leq 1, j = 1, \ldots, n\}$.

**Theorem 1.7.1.** [Complex Poincare Lemma] *Let $\Omega$ be a neighborhood of $\overline{\mathbb{U}}^n$. Fix $\omega \in \wedge^{p,q}(\Omega)$, $q > 0$, with $\bar{\partial} = 0$. Then there exists a neighborhood $\Omega^*$ of $\overline{\mathbb{U}}^n$ and there exists $\omega^* \in \wedge^{p,q-1}(\Omega^*)$ such that*

$$\bar{\partial}\omega^* = \omega \text{ in } \Omega^*.$$

We need some preliminary work in order to prove Theorem 1.7.1.

**Lemma 1.7.2.** *Let $F \in C_0^1(\mathbb{C})$. Then*

$$F(\zeta) = -\frac{1}{\pi} \int_{\mathbb{C}} \int \frac{\partial F}{\partial \bar{z}} \frac{dxdy}{z - \zeta}, \quad \text{all } \zeta \in \mathbb{C}. \tag{1.8}$$

*Proof.* Fix $\zeta$ and choose $R > |\zeta|$ with $\operatorname{supp} F \subset \{z : |z| < R\}$. Fix $\varepsilon > 0$ and small. Put $\Omega_\varepsilon = \{z : |z| < R \text{ and } |z - \zeta| > \varepsilon\}$.

The 1-form $F\ dz/z - \zeta$ is smooth on $\Omega_\varepsilon$ and

$$d\left(\frac{F\ dz}{z-\zeta}\right) = \frac{\partial}{\partial \bar{z}}\left(\frac{F}{z-\zeta}\right) d\bar{z}\, dz = \frac{\partial F}{\partial \bar{z}} \frac{d\bar{z}\, dz}{z-\zeta}.$$

By Stokes's Theorem

$$\int_{\Omega_\varepsilon} d\left(\frac{F\ dz}{z-\zeta}\right) = \int_{\partial\Omega_\varepsilon} \frac{F\ dz}{z-\zeta}.$$

Since $F = 0$ on $\{z : |z| = R\}$, the right side is

$$\int_{|z-\zeta|=\varepsilon} \frac{F\ dz}{z-\zeta} = -\int_0^{2\pi} F(\zeta + \varepsilon e^{i\theta}) i d\theta,$$

so

$$\int_{\Omega_\varepsilon} \frac{\partial F}{\partial \bar{z}} \frac{d\bar{z}\, dz}{z-\zeta} = -\int_0^{2\pi} F(\zeta + \varepsilon e^{i\theta}) i d\theta.$$

Letting $\varepsilon \to 0$ we get

$$\int_{|z|<R} \frac{\partial F}{\partial \bar{z}} \frac{d\bar{z}\, dz}{z-\zeta} = -2\pi i F(\zeta).$$

Since $\partial F/\partial \bar{z}$ for $|z| > R$ and since $d\bar{z}\, dz = 2i\ dx\ dy$, this gives

$$\int \frac{\partial F}{\partial \bar{z}} \frac{dx\ dy}{z-\zeta} = -\pi F(\zeta),$$

i.e., (1.8).

□

**Lemma 1.7.3.** *Let $G \in C_0^2(\mathbb{C})$. Then*

$$G(\zeta) = -\frac{1}{2\pi}\int_C\int \Delta G(z) log \frac{1}{|z-\zeta|} dxdy, \quad all\ \zeta \in \mathbb{C}. \tag{1.9}$$

*Proof.* The proof is very much like that of Lemma 1.7.2. With $\Omega_\varepsilon$ as in that proof, start with Green′s formula

$$\int_{\Omega_\varepsilon}\int (u\Delta v - v\Delta u)\, dxdy = \int_{\partial\Omega_\varepsilon}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) ds$$

and take $u = G$, $v = log|z-\zeta|$. We leave the details to the reader. □

**Lemma 1.7.4.** *Let $\phi \in C^1(\mathbb{R}^2)$ and assume that $\phi$ has compact support. Put*

$$\Phi(\zeta) = -\frac{1}{\pi}\int_{\mathbb{R}^2}\phi(z)\frac{dxdy}{z-\zeta}.$$

*Then $\Phi \in C^1\left(\mathbb{R}^2\right)$ and $\partial\Phi/\partial\overline{\zeta} = \phi(\zeta)$, all $\zeta$.*

*Proof.* Choose $R$ with supp $\phi \subset \{z : |z| \leq R\}$.

$$\begin{aligned}
\pi\Phi(\zeta) &= \int_{|z|\leq R}\phi(z)\frac{1}{\zeta - z}dxdy \\
&= \int_{|z'-\zeta|\leq R}\phi\left(\zeta - z'\right)\frac{dx'dy'}{z'} \\
&= \int_{\mathbb{R}^2}\phi\left(\zeta - z'\right)\frac{dx'dy'}{z'}.
\end{aligned}$$

Since $1/z' \in L^1(dx'dy')$ on compact sets, it is legal to differentiate the last integral under the integral sign. We get

$$\begin{aligned}
\pi\frac{\partial\Phi}{\partial\overline{\zeta}}(\zeta) &= \int_{\mathbb{R}^2}\frac{\partial}{\partial\overline{\zeta}}\left[\phi\left(\zeta - z'\right)\right]\frac{dx'dy'}{z'} \\
&= \int_{\mathbb{R}^2}\frac{\partial\phi}{\partial\overline{\zeta}}\left(\zeta - z'\right)\frac{dx'dy'}{z'} \\
&= \int_{\mathbb{R}^2}\frac{\partial\phi}{\partial\overline{\zeta}}(z)\frac{dxdy}{\zeta - z}.
\end{aligned}$$

On the other hand, Lemma 1.7.2 gives that

$$-\pi\phi(\zeta) = \int_{\mathbb{R}^2}\frac{\partial\phi}{\partial\overline{\zeta}}(z)\frac{dxdy}{z-\zeta}.$$

Hence $\partial\Phi/\partial\overline{\zeta} = \phi$. □

**Lemma 1.7.5.** *Let $\Omega$ be a neigborhood of $\overline{\mathbb{U}}^n$ and let $f$ be a function in $C^\infty(\Omega)$. Fix $j, 1 \leq j \leq n$. Assume that*

$$\frac{\partial f}{\partial \bar{z}_k} = 0 \text{ in } \Omega,\ k = k_1, \ldots, k_s,\ \text{each } k_i \neq j. \tag{1.10}$$

*Then we can find a neigborhood $\Omega_1$ of $\overline{\mathbb{U}}^n$ and $F$ in $C^\infty(\Omega_1)$ such that*

*(a)* $\partial F/\partial \overline{\zeta}_j = f$ *in* $\Omega_1$.

*(b)* $\partial F/\partial \overline{\zeta}_k = 0$ *in* $\Omega_1$, $k = k_1 \ldots, k_s$.

*Proof.* Choose $\varepsilon > 0$ so that if $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $|z_v| < 1 + 2\varepsilon$ for all $v$, then $z \in \Omega$. Choose $\psi \in C^\infty(\mathbb{R}^2)$, having support contained in $\{z : |z| < 1 + 2\varepsilon\}$, with $\psi(z) = 1$ for $|z| < 1 + \varepsilon$. Put

$$F(\zeta_1, \cdots, \zeta_j, \cdots, \zeta_n) = -\frac{1}{\pi} \int_{\mathbb{R}^2} \psi(z) f(\zeta_1, \cdots \zeta_{j-1}, z, \zeta_{j+1}, \cdots \zeta_n) \frac{dxdy}{z - \zeta_j}.$$

For fixed $\zeta_1, \cdots, \zeta_{j-1}, \zeta_{j+1}, \cdots, \zeta_n$ with $|\zeta_v| < 1 + \varepsilon$, all $v$, we now apply Lemma 1.7.4 with

$$\begin{aligned} \phi(z) &= \psi(z) f(\zeta_1, \cdots, \zeta_{j-1}, z, \zeta_{j+1}, \cdots, \zeta_n), \quad |z| < 1 + 2\varepsilon \\ &= 0 \quad \text{outside supp } \psi. \end{aligned}$$

We obtain

$$\frac{\partial F}{\partial \overline{\zeta}_j}(\zeta_1, \cdots, \zeta_j, \cdots, \zeta_n) = \phi(\zeta_j) = f(\zeta_1, \cdots, \zeta_{j-1}, \zeta_j, \zeta_{j+1}, \cdots, \zeta_n),$$

if $|\zeta_j| < 1 + \varepsilon$, and so (a) holds with

$$\Omega_1 = \{\zeta \in \mathbb{C}^n : |\zeta_n| < 1 + \varepsilon,\ \text{all } v\}.$$

Part (b) now follows directly from (1.10) by differentiation under the integral sign. □

*Proof of Theorem 1.7.1.* We call a form

$$\sum_{I,J} C_{IJ} dz_I \wedge d\bar{z}_J$$

of level $v$, if for some $I$ and $J$ with $J = (j_1, j_2, \cdots, v)$, where $j_1 < j_2 < \cdots < v$, we have $C_{I,J} \neq 0$; while for each $I$ and $J$ with $J = (j_1, \cdots, j_s)$ where $j_1 < \cdots < j_s$ and $j_s > v$, we have $C_{I,J} = 0$. Consider first a form $\omega$ of level 1 such that $\bar{\partial}\omega = 0$. Then $\omega \in \wedge^{p,1}(\Omega)$ for some $p$ and we have

$$\begin{aligned} \omega &= \sum_I a_I d\bar{z}_1 \wedge dz_I \quad a_I \in C^\infty(\Omega) \quad \text{for each } I. \\ 0 = \bar{\partial}\omega &= \sum_{I,k} \frac{\partial a_I}{\partial \bar{z}_k} d\bar{z}_k \wedge d\bar{z}_I \wedge dz_I. \end{aligned}$$

Hence $(\partial_I a/\partial \bar{z}_k)\, d\bar{z}_k \wedge d\bar{z}_1 \wedge dz_I = 0$ for each $k$ and $I$. It follows that

$$\frac{\partial a_I}{\partial \bar{z}_k} = 0, \quad k \geq 2, \text{ all } I.$$

By Lemma 1.7.5 there exists for every $I$, $A_I$ in $C^\infty(\Omega_1)$, $\Omega_1$ being some neighborhood of $\Delta^n$, such that

$$\frac{\partial A_I}{\partial \bar{z}_1} = a_I \text{ and } \frac{\partial A_I}{\partial \bar{z}_k} = 0, \quad k = 2, \ldots, n.$$

Put $\widetilde{\omega} = \sum_I A_I dz_I \in \wedge^{p,0}(\Omega_1)$.

$$\bar{\partial}\widetilde{\omega} = \sum_{I,k} \frac{\partial A_I}{\partial \bar{z}_k} d\bar{z}_k \wedge dz_I = \omega.$$

We proceed by induction. Assume that the assertion of the theorem holds whenever $\omega$ is of level $\leq v-1$ and consider $\omega$ of level $v$. By hypothesis $\omega \in \wedge^{p,q}(\Omega)$ and $\bar{\partial}\omega = 0$.

We can find forms $\alpha$ and $\beta$ of level $\leq v-1$ so that

$$\begin{aligned} \omega &= d\bar{z}_v \wedge \alpha + \beta \\ 0 = \bar{\partial}\omega &= -d\bar{z}_v \wedge \bar{\partial}\alpha + \bar{\partial}\beta, \end{aligned}$$

where we have used Lemma 1.5.6. So

$$0 = d\bar{z}_v \wedge \bar{\partial}\alpha - \bar{\partial}\beta \tag{1.11}$$

Put

$$\alpha = \sum_{I,J} a_{I,J} dz_I \wedge d\bar{z}_J, \qquad \beta = \sum_{I,J} b_{I,J} dz_I \wedge d\bar{z}_J.$$

Equation (1.11) gives

$$0 = d\bar{z}_v \wedge \sum_{I,J,k} \frac{\partial a_{IJ}}{\partial \bar{z}_k} d\bar{z}_k \wedge dz_I \wedge d\bar{z}_J - \sum_{I,J,k} \frac{\partial b_{IJ}}{\partial \bar{z}_k} d\bar{z}_k \wedge dz_I \wedge d\bar{z}_J. \tag{1.12}$$

Fix $k > v$, and look at the terms on the right side of (1.12) containing $d\bar{z}_v \wedge d\bar{z}_k$. Becouse $\alpha$ and $\beta$ are the level $\leq v-1$, these are the terms:

$$d\bar{z}_v \wedge \frac{\partial a_{IJ}}{\partial \bar{z}_k} d\bar{z}_k \wedge dz_I \wedge d\bar{z}_J.$$

It follows that for each $I$ and $J$, $A_{IJ} \in C^\infty(\Omega_1)$ with

$$\frac{\partial A_{IJ}}{\partial \bar{z}_v} = a_{IJ}, \qquad \frac{\partial A_{IJ}}{\partial \bar{z}_k} = 0, \qquad k > v$$

Put

$$\begin{aligned} \omega_1 &= \sum_{I,J} A_{IJ} dz_I \wedge d\bar{z}_J \in \wedge^{p,q-1}(\Omega_1), \\ \bar{\partial}\omega_1 &= \sum_{I,J,k} \frac{\partial A_{IJ}}{\partial \bar{z}_k} d\bar{z}_k \wedge dz_I \wedge d\bar{z}_J \\ &= \sum_{I,J} a_{IJ} d\bar{z}_v \wedge dz_I \wedge d\bar{z}_J + \gamma, \end{aligned}$$

where $\gamma$ is a form of level $\leq v-1$. Thus

$$\bar{\partial}\omega_1 = d\bar{z}_v \wedge \alpha + \gamma$$

Hence

$$\bar{\partial}\omega_1 - \omega = \gamma - \beta$$

is a form of level $\leq v-1$. Also

$$\bar{\partial}(\gamma - \beta) = \bar{\partial}\left(\bar{\partial}\omega_1 - \omega\right) = 0.$$

By induction hypothesis, we can choose a neigborhood $\Omega_2$ of $\overline{\mathbb{U}}^n$ and $\tau \in \wedge^{p,q-1}(\Omega_2)$ with $\bar{\partial}\tau = \gamma - \beta$. Then

$$\bar{\partial}(\omega_1 - \tau) = \bar{\partial}\omega_1 - \bar{\partial}\tau = \omega + (\gamma - \beta) - (\gamma - \beta) = \omega.$$

$\omega_1 - \tau$ is now the desired $\omega^*$. $\square$

# CHAPTER 2

# APPROXIMATION IN $\mathbb{C}$

## 2.1 Runge Theorem

For this section, see [Rud87] for detailed information. First we will work on a result of Hahn-Banach Theorem.

**Theorem 2.1.1.** *Let $M$ be a linear subspace of a normed linear space $X$, $x_0 \in X$. Then $x_0 \in \overline{M}$ if and only if for any bounded linear functional on $X$ such that $f(x) = 0$ for all $x \in M$ we have $f(x_0) = 0$.*

*Proof.* Let $x_0 \in \overline{M}$ and $f$ be bounded linear functional on $X$, $f(x) = 0$ for all $x \in M$. The continuity shows $f(x_0) = 0$.

Let $x_0 \notin \overline{M}$. Then there exists $\delta > 0$ such that $\|x - x_0\| > \delta$ for $x \in M$. Let $M'$ be the subspace generated by $M$ and $x_0$, and define $f(x + \lambda x_0) := \lambda$ if $x \in M$ and $\lambda$ is a scalar. We can get

$$\delta |\lambda| \leq |\lambda| \left\|\lambda^{-1} x + x_0\right\| = \|x + \lambda x_0\|.$$

Hence

$$\frac{|\lambda|}{\|\lambda x_0 + x\|} \leq \frac{1}{\delta}.$$

Since

$$\|f\| = \sup_{\substack{x \in M \\ \lambda \text{ scalar}}} \frac{\|f(x + \lambda x_0)\|}{\|x + \lambda x_0\|} = \sup_{\substack{x \in M \\ \lambda \text{ scalar}}} \frac{|\lambda|}{\|x + \lambda x_0\|} \leq \frac{1}{\delta},$$

we see that $f$ is a linear functional on $M'$ whose norm is at most $\frac{1}{\delta}$. Also $f(x) = 0$ on $M$, $f(x_0) = 1$. From Hahn Banach Theorem we can extend this to $X$. We founded a bounded linear functional on $X$ such that $f(x) = 0$, for all $x \in M$ but $f(x_0) \neq 0$.

$\square$

Denote $\overline{\mathbb{C}}$ as the union of the complex plane and the point $\infty$.

**Theorem 2.1.2.** [Runge Theorem] *Let $K \subset \overline{\mathbb{C}}$ be a compact set and $\{\alpha_j\}$ be a set which contains one point in each component of $\overline{\mathbb{C}} \backslash K$. If $f \in \mathrm{Hol}(\Omega)$ where $\Omega$ is an open set containing $K$ and $\varepsilon > 0$, then there exists a rational function $R$, whose poles lie in $\{\alpha_j\}$ such that*

$$|f(z) - R(z)| < \varepsilon$$

*for every $z \in K$.*

*Proof.* Let $M \subset C(K)$ consisting of the restrictions to $K$ of all rational functions which have all their poles in $\{\alpha_j\}$. If we can show that $f \in \overline{M}$, the proof is done. From Theorem 2.1.1 we get $f \in \overline{M}$ if and only if every bounded linear functional on $C(K)$ which vanishes on $M$ also vanishes at $f$. From Riesz Representation Theorem it is enough to prove the following claim.

*Claim:* If $\mu$ is a complex Borel measure on $K$ such that

$$\int_K R d\mu = 0$$

for every rational function $R$ with poles only in the set $\{\alpha_j\}$, and if $f \in \mathrm{Hol}(\Omega)$, then we also have

$$\int_K f d\mu = 0.$$

*Proof of Claim:* Assume $\int_K R d\mu = 0$. Define

$$h(z) := \int_K \frac{d\mu(\zeta)}{\zeta - z}$$

where $z \in \overline{\mathbb{C}} \backslash K$. $h \in \mathrm{Hol}(\overline{\mathbb{C}} \backslash K)$. Let $V_j$ be the component of $\overline{\mathbb{C}} \backslash K$ which contains $\alpha_j$, and suppose $D(\alpha_j; r) \subset V_j$.

First let $\alpha_j \neq \infty$. Fix $z \in D(\alpha_j; r)$, then

$$\frac{1}{\zeta - z} = \lim_{N \to \infty} \sum_{n=0}^{N} \frac{(z - \alpha_j)^n}{(\zeta - \alpha_j)^{n+1}} \tag{2.1}$$

uniformly for $\zeta \in K$. Since the right side of the equation (2.1) is a rational function with poles only in the set $\{\alpha_j\}$, the integral over $K$ is 0. So

$$\int_K \frac{1}{\zeta - z} d\mu = \lim_{N \to \infty} \sum_{n=0}^{N} \int_K \frac{(z - \alpha_j)^n}{(\zeta - \alpha_j)^{n+1}} d\mu = 0$$

Hence $h(z) = 0$ for all $z \in D(\alpha_j; r)$, and hence for all $z \in V_j$.

Now let $\alpha_j = \infty$.

$$\frac{1}{\zeta - z} = \lim_{N\to\infty} \sum_{n=0}^{N} z^{-n-1}\zeta^n$$

for $\zeta \in K$ and $|z| > r$. So $h(z) = 0$ in $D(\infty; r)$, hence in $V_j$. We get that $h(z) = 0$ in $\overline{\mathbb{C}} \backslash K$.

Now choose a cycle $\Gamma$ in $\Omega \backslash K$ such that the Cauchy formula:

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z} d\zeta$$

holds for every $f \in \mathrm{Hol}(\Omega)$ and for every $z \in K$. Then

$$\begin{aligned} \int_K f d\mu &= \int_K d\mu(\zeta) \left[ \frac{1}{2\pi i} \int_\Gamma \frac{f(\omega)}{\omega - \zeta} d\omega \right] \\ &= \frac{1}{2\pi i} \int_\Gamma f(\omega) d\omega \int_K \frac{d\mu(\zeta)}{\omega - \zeta} \\ &= -\frac{1}{2\pi i} \int_\Gamma f(\omega) h(\omega) d\omega \\ &= 0 \end{aligned}$$

In second equality Fubuni's Theorem is legitimate since we are dealing with Borel measures and continuous functions on compact spaces and the last equality depends on the fact that $\Gamma^* \subset \Omega \backslash K$, where $h(\omega) = 0$. We proved the claim and so the proof is done.

□

**Theorem 2.1.3.** [Runge Theorem] *Let $K \subset \overline{\mathbb{C}}$ be compact with $\overline{\mathbb{C}} \backslash K$ connected. Then $A(K) = P(K)$.*

*Proof.* $\overline{\mathbb{C}} \backslash K$ is connected hence has only one component. Take $\alpha_j = \infty$ in Theorem 2.1.2, so for $\varepsilon > 0$ there exists a rational function which has a pole at $\infty$, hence a polynomial $p$, such that

$$|f(z) - p(z)| < \varepsilon.$$

□

## 2.2 Lavrentiev's Theorem

Let $\mu$ be a measure of compact support $\subset \mathbb{C}$. We define the logarithmic potential $\mu^*$ of $\mu$ by

$$\mu^* = \int log \left| \frac{1}{z-\zeta} \right| d\mu(\zeta).$$

We define the Cauchy transform $\widehat{\mu}$ of $\mu$ by

$$\widehat{\mu} = \int \frac{1}{\zeta - z} d\mu(\zeta).$$

**Lemma 2.2.1.** [Carleson] *Let $E$ be a compact plane set with $\mathbb{C} \backslash E$ connected and fix $z_0 \in bE$. Then*

*(a) there exist probability measures $\sigma_t$ for each $t > 0$ with $\sigma_t$ carried on $\mathbb{C} \backslash E$ such that:*

*Let $\alpha$ be a real measure on $E$ satisfying*

$$\int_E \left| log \left| \frac{1}{z_0 - \zeta} \right| \right| d\left|\alpha\right|(\zeta) < \infty \tag{2.2}$$

*and*

*(b)*

$$\lim_{t \to 0} \int \alpha^* d\sigma_t(z) = \alpha^*(z_0).$$

*Proof.* We may assume that $z_0 = 0$. Fix $t > 0$. Since $0 \in bE$ and $\mathbb{C} \backslash E$ is connected, there exists a probability measure $\sigma_t$ carried on $\mathbb{C} \backslash E$ such that

$$\sigma_t \{z : r_1 < |z| < r_2\} = \frac{1}{t}(r_2 - r_1) \quad \text{for } 0 < r_1 < r_2 \leq t$$

and $\sigma_t = 0$ outside $|z| \leq t$.

If some line segment, with 0 as one end point and length $t$, happens to lie in $\mathbb{C} \backslash E$, we may of course take $\sigma_t$ as $1/t \cdot$ linear measure on that segment.

Then for all $\zeta \in \mathbb{C}$ we have

$$\begin{aligned}
\int \log \left| \frac{1}{z-\zeta} \right| d\sigma_t(z) &\leq \int \log \left| \frac{1}{|z|-|\zeta|} \right| d\sigma_t(z) \\
&\leq \frac{1}{t} \int_0^t \log \frac{1}{||r|-|\zeta||} dr \\
&\leq \frac{1}{t} \int_0^t \log \frac{1}{|\zeta|} \frac{1}{\left| \frac{r}{|\zeta|} - 1 \right|} dr \\
&\leq \frac{1}{t} \int_0^t \log \frac{1}{|\zeta|} dr + \frac{1}{t} \int_0^t \log \frac{1}{\left| \frac{r}{|\zeta|} - 1 \right|} dr \\
&\leq \log \frac{1}{|\zeta|} + \frac{1}{t} \int_0^t \log \frac{1}{|1 - r/|\zeta||} dr.
\end{aligned}$$

The last term is bounded above by a constant $A$ independent of $t$ and $|\zeta|$. Hence we have

$$\int \log \left| \frac{1}{z-\zeta} \right| d\sigma_t(z) \leq \log \frac{1}{|\zeta|} + A, \quad \text{all } \zeta, \text{ all } t > 0. \tag{2.3}$$

Also, as $t \to 0$, $\sigma_t \to$ point mass at 0. Hence for each fixed $\zeta \neq 0$.

$$\lim_{t \to 0} \int \log \left| \frac{1}{z-\zeta} \right| d\sigma_t(z) = \log \frac{1}{|\zeta|}. \tag{2.4}$$

Now for fixed $t$, Fubini$'$s theorem gives

$$\int \alpha^*(z) d\sigma_t(z) = \int \left\{ \int \log \left| \frac{1}{z-\zeta} \right| d\sigma_t(z) \right\} d\alpha(\zeta).$$

By (2.3), (2.4), and (2.2), the integrand on the right tends to $\log 1/|\zeta|$ dominantly with respect to $|\alpha|$. Hence

$$\lim_{t \to 0} \int \alpha^*(z) d\sigma_t(z) = \lim_{t \to 0} \int \log \frac{1}{|\zeta|} d\alpha(\zeta) = \alpha^*(0)$$

□

**Lemma 2.2.2.** *The functions*

$$\int \left| \log \left| \frac{1}{z-\zeta} \right| \right| d|\mu|(\zeta) \quad \textit{and} \quad \int \left| \frac{1}{\zeta - z} \right| d|\mu|(\zeta)$$

*are summable $-dxdy$ over compact sets in $\mathbb{C}$. It follows that these functions are finite a.e. $-dxdy$ and hence that $\mu^*$ and $\widehat{\mu}$ are defined a.e. "$-dxdy$.*

*Proof.* Since $1/r \geq |\log r|$ for small $r > 0$, we need only consider the second integral. Fix $R > 0$ with $\operatorname{supp}|\mu| \subset \{z : |z| < R\}$.

$$\gamma = \int_{|z|\leq R} dx\, dy \left\{ \int \left| \frac{1}{\zeta - z} \right| d\,|\mu|\,(\zeta) \right\} = \int d\,|\mu|\,(\zeta) \int_{|z|\leq R} \frac{dx\, dy}{|z-\zeta|}.$$

For $\zeta \in \operatorname{supp}|\mu|$ and $|z| \leq R$, $|z-\zeta| \leq 2R$.

$$\int_{|z|\leq R} \frac{dx\, dy}{|z-\zeta|} \leq \int_{|z'|\leq 2R} \frac{dx'\, dy'}{|z'|} = \int_0^{2R} r dr \int_0^{2\pi} \frac{d\theta}{r} = 4\pi R.$$

Hence $\gamma \leq 4\pi R \cdot \|\mu\|$.

$\square$

**Lemma 2.2.3.** *If $\mu$ is a measure with compact support in $C$, and if $\hat{\mu}(z) = 0$ a.e. $-dx\, dy$, then $\mu = 0$. Also, if $\mu^*(z) = 0$ a.e. $-dx\, dy$, then $\mu = 0$.*

*Proof.* Fix $g \in C_0^1(\mathbb{C})$. by Lemma 1.7.2

$$\int g(\zeta) d\mu(\zeta) = \int d\mu(\zeta) \left[ -\frac{1}{\pi} \int \frac{\partial g}{\partial \bar{z}}(z) \frac{dxdy}{z-\zeta} \right].$$

Fubini's theorem now gives

$$\frac{1}{\pi} \int \frac{\partial g}{\partial \bar{z}}(z) \widehat{\mu}(z) dxdy = \int g d\mu. \tag{2.5}$$

Since $\widehat{\mu} = 0$ a.e., we deduce that

$$\int g d\mu = 0.$$

But the class of functions obtained by restricting to supp $\mu$ the functions in $C_0^1(\mathbb{C})$ is dense in $C(\operatorname{supp}\mu)$ by the Stone-Weierstrass theorem. Hence $\mu = 0$.

Using (1.9), we get similarly for $g \in C_0^2(\mathbb{C})$,

$$-\int g d\mu = \frac{1}{2\pi} \int \Delta g(z) \cdot \mu^*(z) dxdy$$

and conclude that $\mu = 0$ if $\mu^* = 0$ a.e.

$\square$

**Theorem 2.2.4.** [Lavrentiev's Theorem] *Let $K \subset \mathbb{C}$ be compact with $\mathbb{C} \backslash K$ connected. Then $P(K) = C(K)$ if and only if $K^\circ = \emptyset$*

*Proof.* Let $\alpha$ be a real measure on $X$ with $\alpha \perp Re(P(X))$. Then

$$\int Re\,\zeta^n\, d\alpha(\zeta) = 0, \quad n \geq 0$$

and

$$\int Im\,\zeta^n\, d\alpha = \int Re(-i\zeta^n) d\alpha = 0, \quad n \geq 0,$$

so that

$$\int \zeta^n\, d\alpha = 0, \quad n \geq 0.$$

For $|z|$ large,

$$\log\left(1 - \frac{\zeta}{z}\right) = \sum_0^\infty c_n(z)\zeta^n,$$

the series converging uniformly for $\zeta \in X$. Hence

$$\int \log\left(1 - \frac{\zeta}{z}\right) d\alpha(\zeta) = \sum_0^\infty c_n(z) \int \zeta^n d\alpha(\zeta) = 0,$$

whence

$$\int Re\left(\log\left(1 - \frac{\zeta}{z}\right)\right) d\alpha(\zeta) = 0$$

or

$$\int \log|z - \zeta|\, d\alpha(\zeta) - \int \log|z|\, d\alpha(\zeta) = 0,$$

whence

$$\int \log|z - \zeta|\, d\alpha(\zeta) = 0,$$

since $\alpha \perp 1$. Since

$$\int \log|z - \zeta|\, d\alpha(\zeta) = 0$$

is harmonic in $\mathbb{C}\backslash X$, the function vanishes not only for large $|z|$, but in fact for all $z$ in $\mathbb{C}\backslash X$, and so

$$\alpha^*(z) = 0, \quad z \in \mathbb{C}\backslash X.$$

By (2.2.1) it follows that we also have

$$\alpha^*(z_0) = 0, \qquad z_0 \in X,$$

provided (2.2) holds at $z_0$. By (2.2.2) this implies that

$$\alpha^* = 0 \text{ a.e. } - dxdy.$$

By Lemma 2.2.3 this implies that $\alpha = 0$. Hence

$$Re\ P(X) \text{ is dense in } C_R(X). \tag{2.6}$$

Now choose $\mu \in P(X)^{\perp}$. Fix $z_0 \in X$ with

$$\int \left| \frac{1}{z - z_0} \right| d\left|\mu\right|(z) < \infty. \tag{2.7}$$

Because of (2.6) we can find for each positive integer $k$ a polynomial $P_k$ such that

$$|Re\ P_k(z) - |z - z_0|| \le \frac{1}{k}, \quad z \in X \tag{2.8}$$

and

$$\begin{aligned} P_k(z_0) &= 0. \\ f_k(z) &= \frac{e^{-kP_k(z)} - 1}{z - z_0} \end{aligned}$$

is an entire function and hence its restriction to $X$ lies in $P(X)$. Hence

$$\int f_k d\mu = 0. \tag{2.9}$$

Equation (2.8) gives

$$Re\ kP_k(z) - k\left|z - z_0\right| \ge -1,$$

whence

$$e^{-kP_k(z)} \le e^{-k|z - z_0| + 1}, \quad z \in X.$$

It follows that $f_k(z) \to -1/z - z_0$ for all $z \in X\ z_0$, as $k \to \infty$, and also

$$|f_k(z)| \le \frac{4}{|z - z_0|}, \quad z \in X.$$

Since by (2.7) $1/\left|z - z_0\right|$ is summable with respect to $|\mu|$, this implies that

$$\int f_k d\mu \to -\int \frac{d\mu(z)}{z - z_0}$$

by dominated convergence. Equation (2.9) then gives that

$$\int \frac{d\mu(z)}{z - z_0} = 0.$$

Since (2.7) holds a.e. on $X$ by Lemma 2.2.2, and since certainly

$$\int \frac{d\mu(z)}{z - z_0} = 0 \quad \text{for } z_0 \in \mathbb{C} \backslash X$$

we conclude that $\widehat{\mu} = 0$ a.e., so $\mu = 0$ by Lemma 2.2.3. Thus $\mu \perp P(X)$ implies that $\mu = 0$. So $P(X) = C(X)$. □

## 2.3 Mergelyan's Theorem

**Definition 2.3.1.** Suppose $\gamma_0$ and $\gamma_1$ are closed curves in a topological space $X$, both with parameter interval $I = [0,1]$. We say that $\gamma_0$ and $\gamma_1$ are *$X$-homotopic* if there is a continuous mapping $H$ of the unit square $I^2 = I \times I$ into $X$ such that

$$H(s,0) = \gamma_0(s), \qquad H(s,1) = \gamma_1(s), \qquad H(0,t) = H(1,t) \tag{2.10}$$

for all $s \in I$ and $t \in I$. Put $\gamma_t(s) = H(s,t)$. Then (2.10) defines a one-parameter family of closed curves $\gamma_t$ in $X$, which connects $\gamma_0$ and $\gamma_1$. Intuitively, this means that $\gamma_0$ can be continuously deformed to $\gamma_1$, within $X$.

If $\gamma_0$ is $X$-homotopic to a constant mapping $\gamma_1$ (i.e, if $\gamma_1^*$ consists of just one point), we say that $\gamma_0$ is *null-homotopic* in $X$.

**Definition 2.3.2.** If $X$ is connected and if every closed curve in $X$ is null-homotopic, $X$ is said to be *simply connected*.

Now let's consider the polynomial convexity in $\mathbb{C}$.

**Lemma 2.3.3.** *Let $K \subset \mathbb{C}$ be compact. Then*

$$\widehat{K} = K \cup \{\textit{bounded components of } \mathbb{C} \backslash K\}.$$

*Proof.* Name all the bounded connected components of $\mathbb{C} \backslash K$ as $G_1, G_2, \cdots$. By maximum modulus principle, for all $x \in G_i$ and for every polynomial $p$; $|p(x)| \le \|p\|_K$. Hence for all $i$, $G_i \subset \hat{K}$. □

From Lemma 2.3.3 we have the following trivial result:

**Lemma 2.3.4.** *Let $K$ be a compact set in $\mathbb{C}$. $\mathbb{C} \backslash K$ is connected if and only if for each $x_0$ in $\mathbb{C} \backslash K$ we can find a polynomial $p$ such that*

$$|p(x_0)| > \|p\|_K.$$

**Theorem 2.3.5** ([Rud87])**.** *For a plane region $\Omega$, each of the following conditions implies all others.*

*(a)* $\Omega$ *is homeomorphic to the open unit disc* $U$.

*(b)* $\Omega$ *is simply connected.*

*(c)* $\overline{C}\backslash\Omega$ *is connected.*

*(d) Every* $f \in \mathrm{Hol}(\Omega)$ *can be approximated by polynomials, uniformly on compact subsets of* $\Omega$.

*(e) For every* $f \in \mathrm{Hol}(\Omega)$ *and every closed path* $\gamma$ *in* $\Omega$,

$$\int_{\gamma} f(z)dz = 0.$$

*(f) To every* $f \in \mathrm{Hol}(\Omega)$ *corresponds an* $F \in \mathrm{Hol}(\Omega)$ *such that* $F' = f$.

*(g) If* $f \in \mathrm{Hol}(\Omega)$ *and* $1/f \in \mathrm{Hol}(\Omega)$, *there exists a* $g \in \mathrm{Hol}(\Omega)$ *such that* $f = exp(g)$.

*(h) If* $f \in \mathrm{Hol}(\Omega)$ *and* $1/f \in \mathrm{Hol}(\Omega)$, *there exists a* $\varphi \in \mathrm{Hol}(\Omega)$ *such that* $f = \varphi^2$.

**Lemma 2.3.6.** *[[Rud87]] Suppose* $f \in C_c'(R^2)$, *the space of all continuously real differentiable functions in the plane, with compact support. Put*

$$\bar{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right). \tag{2.11}$$

*Then the following "Cauchy formula" holds:*

$$f(z) = -\frac{1}{\pi}\int_{R^2}\int \frac{(\bar{\partial} f)(\zeta)}{\zeta - z} d\xi d\eta \quad (\zeta = \xi + i\eta). \tag{2.12}$$

**Theorem 2.3.7.** [Tietze's Extension Theorem] *[Rud87] Suppose* $K$ *is a compact subset of a locally compact Hausdorff space* $X$, *and* $f \in C(K)$. *Then there exists an* $F \in C_c(X)$ *such that* $F(x) = f(x)$ *for all* $x \in K$.

The following fact will be useful in the proof of next Lemma. We refer to [[Rud87] Th. 14.15] for its proof.

**Proposition 2.3.8.** *Suppose* $F \in \mathrm{Hol}(U\backslash 0)$, $F$ *is one-to-one in* $U$, $F$ *has a pole of order* 1 *at* $z = 0$, *with residue* 1 *and neither* $w_1$ *nor* $w_2$ *are in* $F(U)$. *Then* $|w_1 - w_2| \leq 4$.

Let $E$ be a compact subset of $\overline{\mathbb{C}}$. By diameter of $E$ we mean the supremum of the numbers $|z_1 - z_2|$, where $z_1 \in E$ and $z_2 \in E$.

**Lemma 2.3.9.** *Suppose $D$ is an open disc of radius $r > 0$, $E \subset D$, $E$ is compact and connected, $\Omega = \overline{\mathbb{C}} \backslash E$ is connected, and the diameter of $E$ is at least $r$. Then there is a function $g \in \mathrm{Hol}(\Omega)$ and a constant $b$, with the following property: If*

$$Q(\zeta, z) = g(z) + (\zeta - b) g^2(z), \tag{2.13}$$

*the inequalities*

$$|Q(\zeta, z)| < \frac{100}{r} \tag{2.14}$$

$$\left| Q(\zeta, z) - \frac{1}{z - \zeta} \right| < \frac{1000 r^2}{|z - \zeta|^3} \tag{2.15}$$

*hold for all $z \in \Omega$ and for all $\zeta \in D$.*

*Proof.* We assume, without loss of generality, that the center of $D$ is at the origin. So $D = D(0; r)$. Since $\Omega$ is simply connected, the Riemann mapping theorem shows that there is a conformal mapping $F$ from unit disc $U$ onto $\Omega$. Again without loss of generality, we can choose $F$ such that $F(0) = \infty$. F has an expansion of the form

$$F(w) = \frac{a}{w} + \sum_{n=0}^{\infty} c_n w^n \quad (w \in U) \tag{2.16}$$

for some $a \neq 0$.

We define

$$g(z) := \frac{1}{a} F^{-1}(z) \quad (z \in \Omega), \tag{2.17}$$

where $F^{-1}$ is the mapping inverse from $\Omega$ onto $U$. We put

$$b = \frac{1}{2\pi i} \int_{\Gamma} z g(z) dz, \tag{2.18}$$

where $\Gamma$ is the positively oriented circle with center 0 and radius $r$.

By (2.16) , Theorem 2.3.8 can be applied to $F/a$. It asserts that the diameter of the complement of $(F/a)(U)$ is at most 4. Note that $\overline{\mathbb{C}}\backslash(F/a)(U) = \{z/a : z \in E\}$. So $diam\, E \leq 4|a|$. Since $diam\, E \geq r$, it follows that

$$|a| \geq \frac{r}{4}. \tag{2.19}$$

Since $g$ is a conformal mapping of $\Omega$ onto $D(0;1/|a|)$, (2.19) shows that

$$|g(z)| < \frac{1}{|a|} \leq \frac{4}{r}, \quad (z \in \Omega). \tag{2.20}$$

Recall that $\Gamma = \{|z| = r\}$. Hence $\Gamma$ has length $2\pi r$, (2.18) gives

$$\begin{aligned}
|b| &\leq \left|\frac{1}{2\pi i}\right| \int_\Gamma |z|\,|g(z)|\,|dz| && (2.21)\\
&< \frac{1}{2\pi} r \frac{4}{r} \int_\Gamma |dz| && (2.22)\\
&= 4r. && (2.23)
\end{aligned}$$

If $\zeta \in D$, then $|\zeta| < r$, so (2.13), (2.20) and (2.23) imply

$$\begin{aligned}
|Q| &\leq |g| + (|\zeta| + |b|)\left|g^2\right|\\
&\leq \frac{4}{r} + 5r\left(\frac{16}{r^2}\right)\\
&< \frac{100}{r}.
\end{aligned}$$

This proves (2.14).

Fix $\zeta \in D$. If $z = F(w)$, then $zg(z) = wF(w)/a$; and since $wF(w) \to a$ as $w \to 0$, we have $zg(z) \to 1$ as $z \to \infty$. Hence $g$ has an expansion of the form

$$g(z) = \frac{1}{z-\zeta} + \frac{\lambda_2(\zeta)}{(z-\zeta)^2} + \frac{\lambda_3(\zeta)}{(z-\zeta)^3} + \cdots \qquad (|z-\zeta| > 2r). \tag{2.24}$$

Let $\Gamma_0$ be a large circle with center at 0; (2.24) gives (by (2.18) and Cauchy's theorem) that

$$\begin{aligned}
\lambda_2(\zeta) &= \frac{1}{2\pi i}\int_{\Gamma_0} (z-\zeta)g(z)dz && (2.25)\\
&= \frac{1}{2\pi i}\int_{\Gamma_0} zg(z)dz - \frac{1}{2\pi i}\zeta\int_{\Gamma_0} g(z)dz && (2.26)\\
&= b - \zeta. && (2.27)
\end{aligned}$$

Substitute this value of $\lambda_2(\zeta)$ into (2.24). Then (2.13) shows that the function

$$\varphi(z) = \left[ Q(\zeta, z) - \frac{1}{z - \zeta} \right] (z - \zeta)^3 \tag{2.28}$$

is bounded as $z \to \infty$. Hence $\varphi$ has a removable singularity at $\infty$. If $z \in \Omega \cap D$, then $|z - \zeta| < 2r$, so (2.14) and (2.28) give

$$|\varphi(z)| < 8r^3 \, |Q(\zeta, z)| + 4r^2 < 1000r^2. \tag{2.29}$$

By the maximum modulus theorem, (2.29) holds for all $z \in \Omega$. This proves (2.15).

□

Runge's theorem is a special case of the following theorem.

**Theorem 2.3.10.** [Mergelyan's Theorem] *Let $K \subset \mathbb{C}$ be compact with $\mathbb{C} \backslash K$ connected. Then $H(K) = P(K)$.*

*Proof.* Let $f \in H(K)$. By theorem 2.3.7, $f$ can be extended to a continuous function in the plane, with compact support. We fix one such extension, and denote it again by $f$. Define

$$\omega(\delta) := \sup |f(z_2) - f(z_1)|$$

for any $\delta > 0$, where $|z_2 - z_1| \leq \delta$. Since $f$ is uniformly continuous, we have

$$\lim_{\delta \to 0} \omega(\delta) = 0. \tag{2.30}$$

From now on, $\delta$ will be fixed. We shall prove that there is a polynomial $p$ such that

$$|f(z) - p(z)| < 10000\omega(\delta) \quad (z \in K). \tag{2.31}$$

By (2.30), this proves the theorem.

Our first objective is the construction of a function $\Phi \in C_c'(R^2)$, such that for all $z$

$$|f(z) - \Phi(z)| \leq \omega(\delta), \tag{2.32}$$

$$\left|(\bar{\delta}\Phi)(z)\right| < \frac{2\omega(\delta)}{\delta}, \tag{2.33}$$

and

$$\Phi(z) = -\frac{1}{\pi}\int_X\int \frac{(\bar{\delta}\Phi)(\zeta)}{\zeta - z} d\xi d\eta \quad (\zeta = \xi + i\eta), \tag{2.34}$$

where $X$ is the set of all points in the support of $\Phi$ whose distance from the complement of $K$ does not exceed $\delta$. (Thus $X$ contains no point which is "far within" $K$.)

We construct $\Phi$ as the convolution of $f$ with a smoothing function $A$. Put $a(r) = 0$ if $r > \delta$, ,put

$$a(r) = \frac{3}{\pi\delta^2}\left(1 - \frac{r^2}{\delta^2}\right)^2 \quad (0 \le r \le \delta), \tag{2.35}$$

and define

$$A(z) = a(|z|) \tag{2.36}$$

for all complex $z$. It is clear that $A \in C'_c(R^2)$.

$$\begin{aligned}
\int_{R^2}\int A &= \int_0^\delta \int_0^{2\pi} \frac{3}{\pi\delta^2}\left(\frac{\delta^2 - r^2}{\delta^2}\right)^2 r d\theta dr \\
&= 2\pi\int_0^\delta \frac{3}{\pi\delta^2}\left(\delta^2 - r^2\right)^2 r dr \\
&= 1.
\end{aligned}$$

and since $A$ has a compact support, from Stoke's theorem

$$\int_{R^2}\int \bar{\delta}A = \int_{bR^2}\int A = 0.$$

$$\begin{aligned}
\int_{R^2}\int |\bar{\delta}A| &= \int_0^\delta \int_0^{2\pi} \frac{1}{2}\left|e^{i\theta}\right|\left|\frac{\partial}{\partial r}A + \frac{i}{r}\frac{\partial A}{\partial\theta}\right| r d\theta dr \\
&= \int_0^\delta \int_0^{2\pi} -\frac{r}{2}\frac{\partial a}{\partial r} d\theta dr \\
&= \int_0^\delta \int_0^{2\pi} -\frac{r}{2}\left(\frac{3}{\pi\delta^2}2\left(1 - \frac{r^2}{\delta^2}\right)\left(-\frac{2r}{\delta^2}\right)\right) d\theta dr \\
&= \int_0^\delta \int_0^{2\pi} \frac{6\left(\delta^2 - r^2\right)}{\pi\delta^6} d\theta dr \\
&= \frac{24}{15\delta} \\
&< \frac{2}{\delta}.
\end{aligned}$$

Hence we get

$$\int_{R^2}\int A = 1, \tag{2.37}$$

$$\int_{R^2}\int \bar{\delta}A = 0, \tag{2.38}$$

$$\int_{R^2}\int |\bar{\delta}A| < \frac{2}{\delta}. \tag{2.39}$$

Now define

$$\Phi(z) = \int_{R^2}\int f(z-\zeta)A(\zeta)d\xi d\eta = \int_{R^2}\int A(z-\zeta)f(\zeta)d\xi d\eta. \tag{2.40}$$

Since $f$ and $A$ have compact support, so does $\Phi$. Since

$$\Phi(z) - f(z) = \int_{R^2}\int [f(z-\zeta) - f(z)]A(\zeta)d\xi d\eta \tag{2.41}$$

and $A(\zeta) = 0$ if $|\zeta| > \delta$,

$$|\Phi(z) - f(z)| \leq \int_{|\zeta|\leq\delta}\int |f(z-\zeta) - f(z)|\, d\xi d\eta \int_{|\zeta|\leq\delta}\int A(\zeta)d\xi d\eta \;\leq\; \omega(\delta).$$

So we have shown (2.32).

The difference quotients of $A$ converge boundedly to the corresponding partial derivatives of $A$, since $A \in C_c'(R^2)$. Hence the last expression in $\Phi(z)$ may be differentiated under the integral sign,

$$\begin{aligned}
(\bar{\delta}\Phi)(z) &= \int_{R^2}\int (\bar{\delta}A)(z-\zeta)f(\zeta)d\xi d\eta && (2.42)\\
&= \int_{R^2}\int f(z-\zeta)(\bar{\delta}A)(\zeta)d\xi d\eta && (2.43)\\
&= \int_{R^2}\int [f(z-\zeta) - f(z)]\,(\bar{\delta}A)(\zeta)d\xi d\eta && (2.44)
\end{aligned}$$

The last equality depends on (2.38). Now (2.39) and (2.44) give (2.33). If we write (2.44) with $\Phi_x$, and $\Phi_y$, in place of $\bar{\delta}\Phi$, we see that $\Phi$ has continuous partial derivatives. Hence Lemma 2.3.6 applies to $\Phi$, and (2.34) will follow if we can show that $\bar{\delta}\Phi = 0$ in

$G$, where $G$ is the set of all $z \in K$ whose distance from the complement of $K$ exceeds $\delta$. We shall do this by showing that

$$\Phi(z) = f(z) \quad (z \in G); \tag{2.45}$$

note that $\bar{\delta} f = 0$ in $G$, since $f$ is holomorphic there. Now if $z \in G$, then $z - \zeta$ is in the interior of $K$ for all $\zeta$ with $|\zeta| < \delta$. The mean value property for harmonic functions therefore gives, by the first equation in (2.40),

$$\begin{aligned}
\Phi(z) &= \int_0^{\delta} a(r) r dr \int_0^{2\pi} f(z - re^{i\theta}) d\theta && (2.46) \\
&= 2\pi f(z) \int_0^{\delta} a(r) r dr && (2.47) \\
&= f(z) \int_{R^2} \int A = f(z) && (2.48)
\end{aligned}$$

for all $z \in G$.

We have now proved (2.32), (2.33), and (2.34).

The definition of $X$ shows that $X$ is compact and that $X$ can be covered by finitely many open discs $D_l, \ldots, D_n$ of radius $2\delta$, whose centers are not in $K$. Since $\overline{\mathbb{C}} \backslash K$ is connected, the center of each $D_j$ can be joined to $\infty$ by a polygonal path in $\overline{\mathbb{C}} \backslash K$. It follows that each $D_j$ contains a compact connected set $E_j$, of diameter at least $2\delta$, so that $\overline{\mathbb{C}} \backslash E_j$ is connected and so that $K \cap E_j = \emptyset$.

We now apply Lemma 2.3.9, with $r = 2\delta$. There exists functions $g_j \in \mathrm{Hol}(\overline{\mathbb{C}} \backslash E_j)$ and constants $b_j$ so that the inequalities

$$\left| Q_j(\zeta, z) \right| < \frac{50}{\delta}, \tag{2.49}$$

$$\left| Q_j(\zeta, z) - \frac{1}{z - \zeta} \right| < \frac{4000\delta^2}{|z - \zeta|^3} \tag{2.50}$$

hold for $z \notin E_j$ and $\zeta \in D_j$, if

$$Q_j(\zeta, z) = g_j(z) + (\zeta - b_j) g_j^2(z). \tag{2.51}$$

Let $\Omega$ be the complement of $E_l \cup \cdots \cup E_n$. Then $\Omega$ is an open set which contains $K$. Put $X_1 = X \cap D_1$ and $X_j = (X \cap D_j) - (X_1 \cup \cdots \cup X_{j-1})$, for $2 \leq j \leq n$. Define

$$R(\zeta, z) = Q_j(\zeta, z) \quad (\zeta \in X_j,\ z \in \Omega) \tag{2.52}$$

and

$$F(z) = \frac{1}{\pi}\int_X\int (\bar{\partial}\Phi)(\zeta)R(\zeta,z)d\xi d\eta \quad (z \in \Omega) \tag{2.53}$$

Since

$$F(z) = \sum_{j=1}^{n} \frac{1}{\pi}\int_{X_j}\int (\bar{\partial}\Phi)(\zeta)Q_j(\zeta,z)d\xi d\eta, \tag{2.54}$$

(2.51) shows that $F$ is a finite linear combination of the functions $g_j$ and $g_j^2$ Hence $F \in \mathrm{Hol}(\Omega)$.

By (2.53), (2.33), and (2.34) we have

$$\begin{aligned} |F(z) - \Phi(z)| &< \frac{1}{\pi}\int_X\int \left|\bar{\partial}\Phi(\zeta)R(\zeta,z) + \frac{\bar{\partial}\Phi(\zeta)}{\zeta - z}\right| \quad (z \in \Omega) &(2.55)\\ &< \frac{2\omega}{\pi\delta}\int_X\int \left|R(\zeta,z) - \frac{1}{z-\zeta}\right| d\xi d\eta \quad (z \in \Omega). &(2.56)\end{aligned}$$

Observe that the inequalities (2.49) and (2.50) are valid with $R$ in place of $Q_j$ if $\zeta \in X$ and $z \in \Omega$. For if $\zeta \in X$ then $\zeta \in X_j$ for some $j$,and then $R(\zeta,z) = Q_j(\zeta,z)$ for all $z \in \Omega$.

Now fix $z \in \Omega$, put $\zeta = z + \rho e^{i\theta}$, and estimate the integrand in (2.56) by (2.49) if $\rho < 4\delta$, by (2.50) if $4\delta \leq \rho$. Then the integral in (2.56) is less than

$$2\pi\int_0^{4\delta}\left(\frac{50}{\delta} + \frac{1}{\rho}\right)\rho d\rho + 2\pi\int_{4\delta}^{\infty}\frac{4000\delta^2}{\rho^3}\rho d\rho = 2808\pi\delta.$$

Hence (2.56) yields

$$|F(z) - \Phi(z)| < 6000\omega(\delta) \quad (z \in \Omega). \tag{2.57}$$

Since $F \in \mathrm{Hol}(\Omega)$, $K \subset \Omega$, and $\overline{\mathbb{C}}\backslash K$ is connected, Runge's theorem shows that $F$ can be uniformly approximated on $K$ by polynomials. Hence (2.32) and (2.57) show that (2.31) can be satisfied. This completes the proof.

□

# CHAPTER 3

# APPROXIMATION IN $\mathbb{C}^n$

## 3.1 Oka-Weil Theorem

Now we want to generalize the Runge Approximation Theorem to $\mathbb{C}^n$, for $n > 1$. The condition '$\mathbb{C} \backslash K$ is connected' is a purely topological restriction on $K$. No such purely topological restriction can suffice when $n > 1$.

*Example* 3.1.1. Consider the two sets in $\mathbb{C}^2 = \{(z,w) : z,w \in \mathbb{C}\}$ defined as follows:

$$\begin{aligned} K_1 &= \left\{(x_z, x_w) \in \mathbb{R}^2 : x_z^2 + x_w^2 \leq 1\right\}, \\ K_2 &= \{(z,0) : |z| \leq 1\}. \end{aligned}$$

Both of these sets are polynomially convex in $\mathbb{C}^2$; i.e., $K_1 = \widehat{K}_1$ and $K_2 = \widehat{K}_2$; thus each set satisfies the obvious necessary condition for holomorphic polynomials to be dense in the space of continuous functions on the set. However, $K_2$ lies in the complex $z$-plane and $P(K_2)$ can be identified with $P(K)$ where $K$ is the closed unit disk in one complex variable; since $K^\circ \neq \emptyset$, the observation made regarding Lavrentiev$'$s theorem shows that $P(K_2) \neq C(K_2)$.

So the question is: What condition on $K$ will assure $A(K) = P(K)$ for a compact subset $K$ of $\mathbb{C}^n$?

For $K \subset \mathbb{C}$ Lemma 2.3.4 gives that $A(K) = P(K)$ if $K$ is polynomially convex. Formulated in this way the Runge Approximation Theorem admits a generalization to $\mathbb{C}^n$ for $n > 1$.

*Remark* 3.1.2. Let $K = \overline{\mathbb{B}} - \mathbb{B}(0,r),\ 0 < r < 1$. $A(K) = P(K)$ from Theorem 1.2.3. However; since $\widehat{K} = \overline{\mathbb{B}} \neq K$, $K$ is not polynomially convex.

Now it is time to state our result as a theorem.

**Theorem 3.1.3.** [The Oka-Weil Theorem] *Let $K$ be a compact polynomially convex set in $\mathbb{C}^n$. Then $A(K) = P(K)$.*

The rest of this section is devoted to the proof of the Oka-Weil Theorem.

**Definition 3.1.4.** A subset $\Pi$ of $\mathbb{C}^n$ is a *polynomial polyhedra* if there exist polynomials $p_1, ..., p_s$ such that

$$\Pi = \left\{ z \in \mathbb{C}^n : \left| z_j \right| \leq 1, \text{ all } j, \text{ and } |p_k(z)| \leq 1, k = 1, 2, \ldots, s \right\}.$$

**Lemma 3.1.5.** *Let $K$ be a compact polynomially convex subset of $\overline{\overline{\mathbb{U}}}^n$. Let $V$ be an open set containing $K$. Then there exists a polynomial polyhedra $\Pi$ with $K \subset \Pi \subset V$.*

*Proof.* For each $x \in \overline{\mathbb{U}}^n \backslash V$ there exists a polynomial $p_x$ with $|p_x(x)| > 1$ and $|p_x| \leq 1$ on $K$.

Then $|p_x| > 1$ in some neighborhood $N_x$ of $x$. By compactness of $\overline{\mathbb{U}}^n \backslash V$, a finite collection $N_{x_1}, \ldots, N_{x_r}$ covers $\overline{\mathbb{U}}^n \backslash V$. Put

$$\Pi = \left\{ z \in \overline{\mathbb{U}}^n : |p_{x_1}(z)| \leq 1, \ldots, |p_{x_r}(z)| \leq 1 \right\}.$$

If $z \in K$, then $z \in \Pi$, so $K \subset \Pi$. Suppose that $z \notin V$. If $z \notin \overline{\mathbb{U}}^n$, then $z \notin \Pi$. If $z \in \overline{\mathbb{U}}^n$, then $z \in \overline{\mathbb{U}}^n \backslash V$. Hence $z \in N_{x_j}$ for some $j$. Hence $\left| p_{x_j}(z) \right| > 1$. Thus $z \notin \Pi$. Hence $\Pi \subset V$. □

Let now $\Pi$ be a polynomial polyhedra in $\mathbb{C}^n$,

$$\Pi = \left\{ z \in \overline{\mathbb{U}}^n : \left| p_j(z) \right| \leq 1, j = 1, \ldots, r \right\}.$$

We can embed $\Pi$ in $\mathbb{C}^n$ by the map

$$\Phi : z \to (z, p_1(z), \ldots, p_r(z)). \tag{3.1}$$

$\Phi$ maps $\Pi$ homeomorphically onto the subset of $\overline{\mathbb{U}}^{n+r}$ defined by the equations

$$z_{n+1} - p_1(z) = 0, \ldots, z_{n+r} - p_r(z) = 0.$$

**Definition 3.1.6.** Let $\Omega$ be an open set in $\mathbb{C}^n$ and $W$ be an open set in $\mathbb{C}^k$. Let $u = (u_1, \ldots, u_n)$ be a map of $W$ into $\Omega$. Assume that each $u_j \in \mathrm{Hol}(W)$. For each $I = (i_1, \ldots, i_r)$, $J = (j_1, \ldots, j_s)$ put

$$du_I = du_{i_1} \wedge du_{i_2} \wedge \cdots \wedge du_{i_r}$$

and define $d\bar{u}_J$ similarly. Thus $du_I \wedge d\bar{u}_J \in \wedge^{r,s}(W)$. Fix $\omega \in \wedge^{r,s}(\Omega)$,

$$\omega = \sum_{I,J} a_{IJ} dz_I \wedge d\bar{z}_J.$$

Define

$$\omega(u) = \sum_{I,J} a_{I,J}(u) du_I \wedge d\bar{u}_J \in \wedge^{r,s}(W).$$

*Example* 3.1.7. Assume that each $u_j$ is holomorphic. $d(\omega(u)) = (d\omega)(u)$ and $\bar{\partial}(\omega(u)) = (\bar{\partial}\omega)(u)$.

**Theorem 3.1.8.** *Let $\Pi$ be a polynomial polyhedra in $\mathbb{C}^n$ and $\Omega$ a neigborhood of $\Pi$. Given that $\phi \in \wedge^{p,q}(\Omega)$, $q > 0$, with $\bar{\partial}\phi = 0$, then there exists a neighborhood $\Omega_1$ of $\Pi$ and $\psi \in \wedge^{p,q-1}(\Omega_1)$ with $\bar{\partial}\psi = \phi$.*

*Proof.* We denote

$$P^k(q_1, \ldots, q_r) = \left\{ z \in \overline{\mathbb{U}}^k : \left|q_j(z)\right| \leq 1, j = 1, \ldots, r \right\},$$

the $q_j$ being polynomials in $z_1, \ldots, z_k$. Every polynomial polyhedra is of this form.

We shall prove our theorem by induction on $r$. The case $r = 0$ corresponds to the polynomial polyhedra $\overline{\mathbb{U}}^k$ and the assertion holds, for all $k$, by Theorem 1.7.1. Fix $r$ now and suppose that the assertion holds for this $r$ and all $k$ and all $(p,q), q > 0$. Fix $n$ and polynomials $p_1, \ldots, p_{r+1}$ in $\mathbb{C}^n$ and consider $\phi \in \wedge^{p,q}(\Omega)$, $\Omega$ some neighborhood of $P^n(p_1, \ldots, p_{r+1})$. We first sketch the argument.

*Step 1.* Embed $P^n(p_1, \ldots, p_r)$ by the map $u : z \to (z, p_{r+1}(z))$. Note that $p_1, \ldots, p_r$ are polynomials in $z_1, \ldots, z_{n+1}$ which do not involve $z_{n+1}$. Let $\sum$ denote the image of $P^n(p_1, \ldots, p_{r+1})$ under $u$. $\pi$ denotes the projection $(z, z_{n+1}) \to z$ from $\mathbb{C}^{n+1} \to \mathbb{C}^n$. Note $\pi \circ u =$ identity.

*Step 2.* Find a $\bar{\partial}$-closed form $\Phi_1$ defined in a neighborhood of

$$P^{n+1}(p_1, \ldots, p_r)$$

with $\Phi_1 = \phi(\pi)$ on $\Sigma$.

*Step 3.* By induction hypothesis, there exists $\Psi$ in a neighborhood of $P^n(p_1,\dots,p_{r+1})$ with $\bar\partial\Psi = \Phi_1$. Put $\psi = \Psi(u)$. Then

$$\bar\partial\psi = (\bar\partial\Psi)(u) = \Phi_1(u) = \phi.$$

As to the details, choose a neighborhood $\Omega_1$ of $P^n(p_1,\dots,p_{r+1})$ with $\overline{\Omega}_1 \subset \Omega$. Choose $\lambda \in C^\infty(\mathbb{C}^n)$, $\lambda = 1$ on $\overline{\Omega}_1$, $\lambda = 0$ outside $\Omega$. Put $\Phi = (\lambda\cdot\phi)(\pi)$, defined $= 0$ outside $\pi^{-1}(\Omega)$.

Let $\chi$ be a form of type $(p,q)$ defined in a neigborhood of $P^{n+1}(p_1,\dots,p_r)$. Put

$$\Phi_1 = \Phi - (z_{n+1} - p_{r+1}(z))\cdot\chi. \tag{3.2}$$

Then $\Phi_1 = \Phi = \phi(\pi)$ on $\Sigma$.

We want to choose $\chi$ such that $\Phi_1$ is $\bar\partial$-closed. This means that

$$\bar\partial\Phi = (z_{n+1} - p_{r+1}(z))\bar\partial\chi$$

or

$$\bar\partial\chi = \frac{\bar\partial\Phi}{(z_{n+1} - p_{r+1}(z))}. \tag{3.3}$$

Observe that $\bar\partial\Phi = \bar\partial\phi(\pi) = 0$ in a neighborhood of $\Sigma$, whence the right-hand side in (3.3) can be taken to be 0 in a neighborhood of $\Sigma$ and is then in $C^\infty$ in a neigborhood of $P^{n+1}(p_1,\dots,p_r)$. Also

$$\bar\partial\left\{\frac{\bar\partial\Phi}{(z_{n+1} - p_{r+1}(z))}\right\} = 0.$$

By induction hypothesis, now, there exists $\chi$ satisfying (3.3). The corresponding $\Phi_1$ in (3.2) is then $\bar\partial$-closed in some neighborhood of $P^{n+1}(p_1,\dots,p_r)$. By induction hypothesis again, there exists *a* $(p,q-1)$ form $\Psi$ in a neighborhood of $P^{n+1}(p_1,\dots,p_r)$ with $\bar\partial\Psi = \Phi_1$. As in step 3, then, making use of Exercise 3.1.7, we obtain a $(p,q-1)$ form $\psi$ in a neighborhood of $P^n(p_1,\dots,p_{r+1})$ with $\bar\partial\psi = \phi$. $\square$

We keep the notations introduced in the last proof.

**Lemma 3.1.9.** *Fix $k$ and polynomials $q_1,\ldots,q_r$ in $z=(z_1,\ldots,z_k)$. Let $f$ be holomorphic in a neighborhood $W$ of $\Pi = P^k(q_1,\ldots,q_r)$. Then there exists an $F$ which is holomorphic in a neighborhood of $\Pi' = P^{k+1}(q_2,\ldots,q_r)$ such that*

$$F(z,q_1(z)) = f(z), \quad \textit{all } z \in \Pi.$$

*Proof.* Note that if $z \in \Pi$, then $(z,q_1(z)) \in \Pi'$. Let $\sum$ be the subset of $\Pi'$ defined by $z_{k+1} - q_1(z) = 0$. Choose $\phi \in C_0^\infty(\pi^{-1}(W))$ with $\phi = 1$ in a neighborhood of $\sum$.

We seek a function $G$ defined in a neighborhood of $\Pi'$ so that with

$$F(z,z_{k+1}) = \phi(z,z_{k+1})f(z) - (z_{k+1} - q_1(z))G(z,z_{k+1}),$$

$F$ is holomorphic in a neighborhood of $\Pi'$. We define $\phi \cdot f = 0$ outside $\pi^{-1}(W)$. We need $\bar\partial F = 0$ and so

$$f\bar\partial\phi = (z_{k+1} - q_1(z))\bar\partial G$$

or

$$\bar\partial G = \frac{f\bar\partial\phi}{(z_{k+1} - q_1(z))} = \omega. \tag{3.4}$$

Note that the numerator vanishes in a neighborhood of $\sum$, so $\omega$ is a smooth form in some neighborhood of $\Pi'$. Also $\bar\partial\omega = 0$. By Theorem 3.1.8, we can thus find $G$ satisfying (3.4) in some neighborhood or $\Pi'$. The corresponding $F$ now has the required properties. □

**Theorem 3.1.10.** [Oka Extension Theorem] *Given $f$ holomorphic in some neighborhood of $\Pi$; then there exists $F$ holomorphic in a neighborhood of $\overline{\mathbb{U}}^{n+r}$ such that*

$$F\left(z,p_1(z),\ldots,p_r(z)\right) = f(z), \quad z \in \Pi.$$

*Proof.* $p_1,\ldots,p_r$ are given polynomials in $z_1,\ldots,z_n$ and $\Pi = P^n(p_1,\ldots,p_r)$. $f$ is holomorphic in a neigborhood of $\Pi$. For $j = 1,2,\ldots,r$ we consider the assertion

$A(j)$ : there exists $F_j$ holomorphic in a neighborhood of $P^{n+j}(p_{j+1},\ldots,p_r)$

such that $F_j(z,p_1(z),\ldots,p_j(z)) = f(z)$, all $z \in \Pi$.

$A(1)$ holds by Lemma 3.1.9. Assume that $A(j)$ holds for some $j$ . Thus $F_j$ is holomorphic in a neighborhood of $P^{n+j}(p_{j+1},\ldots,p_r)$. By Lemma 3.1.9, there exists $F_{j+1}$ is holomorphic in a neighborhood of $P^{n+j+1}(p_{j+2},\ldots,p_r)$ with $F_{j+1}(\zeta,p_{j+1}(z)) = F_j(\zeta)$, $\zeta \in P^{n+j}(p_{j+1},\ldots,p_r)$ and $\zeta = (z,z_{n+1},\ldots,z_{n+j})$. By choice of $F_j$.

$$F_j(z,p_1(z),\ldots,p_j(z)) = f(z), \quad \text{all } z \in \Pi.$$

Hence

$$F_{j+1}(z,p_1(z),\ldots,p_j(z),p_{j+1}(z)) = f(z), \quad \text{all } z \in \Pi.$$

Thus $A(j+1)$ holds. Hence $A(1)$,$A(2)$,$\ldots$,$A(r)$ all hold. But $A(r)$ provides $F$ holomorphic in a neighborhood of $\overline{\mathbb{U}}^{n+r}$ with

$$F(z,p_1(z),\ldots,p_r(z)) = f(z), \quad \text{all } z \in \Pi.$$

□

*Proof of the Oka-Weil Theorem.* Without loss of generality we may assume that $K \subset \overline{\mathbb{U}}^n$. The function $f$ is holomorphic in a neighborhood $V$ of $K$. By Lemma 3.1.5 there exists a polynomial polyhedra $\Pi$ with $K \subset \Pi \subset V$. Then $f$ is holomorphic in a neighborhood of $\Pi$. By Theorem 3.1.10 we can find $F$ satisfying

$$F(z,p_1(z),\ldots,p_r(z)) = f(z), \quad z \in \Pi, \tag{3.5}$$

where $F$ is holomorphic in a neighborhood of $\overline{\mathbb{U}}^{n+r}$. Expand $F$ in a Taylor series around 0,

$$F(z,z_{n+1},\cdots,z_{n+r}) = \sum_{v} a_v z_1^{v_1}\cdots z_n^{v_n} z_{n+1}^{v_{n+1}}\cdots z_{n+r}^{v_{n+r}}.$$

The series converges uniformly in $\overline{\mathbb{U}}^{n+r}$. Thus a sequence $S_j$ of partial sums of this series converges uniformly to $F$ on $\overline{\mathbb{U}}^{n+r}$, and hence in particular on $\Phi(\Pi)$, where $\Phi$ is the embedding defined by (3.1). Thus

$$S_j\,(z,p_1(z),\ldots,p_r(z))$$

converges uniformly to $F\,(z,p_1(z),\ldots,p_r(z))$ for $z \in \Pi$, or, in other words, converges to $f(z)$, by (3.5). Since $S_j\,(z,p_1(z),\ldots,p_r(z))$ is a polynomial in $z$ for each $j$, we are done. □

# CHAPTER 4

# POLYNOMIAL CONVEXITY

## 4.1 Elementary Properties of Polynomially Convex Sets

In this section we provide some examples of polynomially convex sets in $\mathbb{C}^n$. In general it is not an easy task to decide whether a compact set in $\mathbb{C}^n$ is polynomially convex or not. Theorem 4.1.3 provides set to be polynomially convex.

**Proposition 4.1.1.** *Every compact convex set in $\mathbb{C}^n$ is polynomially convex.*

*Proof.* If $K \subset \mathbb{C}^n$ is a compact convex set then for each point $z \in \mathbb{C}^n \backslash K$ there is a real-valued real-linear functional $\ell$ on $\mathbb{C}^n = \mathbb{R}^{2n}$ with

$$\begin{aligned} \ell &< 1 \text{ on } K \\ \ell(z) &= 1. \end{aligned}$$

Say $\ell$ is the real part of a complex-linear functional $\mathcal{L}$ on $\mathbb{C}^n$. Then the entire function $F = e^{\mathcal{L}}$ satisfies

$$|F(z)| = \left|e^{\mathcal{L}(z)}\right| = e^{\ell(z)} = e > \|F\|_K$$

Hence $K$ is polynomially convex. $\square$

**Proposition 4.1.2.** *Every compact subset of $\mathbb{R}^n$ is a polynomially convex subset of $\mathbb{C}^n$.*

*Proof.* Let $K \subset \mathbb{R}^n$ be a compact set. The Weierstrass approximation theorem implies that if $x \in \mathbb{R}^n \backslash K$, then there is a polynomial $p$ with

$$p(x) = 1 > \|p\|_K .$$

Consequently, $\widehat{K} \cap \mathbb{R}^n = K$. If $w = u + iv \in \mathbb{C}^n$ with $u, v \in \mathbb{R}^n$, $v \neq 0$, then the entire function $F$ defined by

$$F(z) = \prod_{j=1}^{n} e^{-(z_j - u_j)^2}$$

satisfies $|F| \leq 1$ on $\mathbb{R}^n$ and $|F(w)| = e^{v_2^1 + \cdots + v_n^2} > 1$. So $w \notin \widehat{K}$. Thus $K = \widehat{K}$. □

Important examples among the convex sets are the closed balls and polydisks.

Certain formal properties of polynomially convex sets are evident. For example, the intersection of an arbitrary family of polynomially convex sets is polynomially convex, whereas the union is generally not.

For a compact subset $X$ of $\mathbb{C}^n$, $\widehat{X}$ is the smallest polynomially compact convex set containing $X$.

A polynomially convex subset $X$ of $\mathbb{C}^n$ can be written as the intersection $\cap_p p^{-1}\left(\overline{\mathbb{U}}\right)$, $\mathbb{U}$ the open unit disk in $\mathbb{C}$, where the intersection extends over all the polynomials $p$ that are bounded by one in modulus on $X$. A consequence of this simple observation is that if $\Omega$ is a neighborhood of $X$, then there is a polynomial polyhedra $\Pi$ such that $X \subset \Pi \subset \Omega$ (see Lemma 3.1.5). This is not unlike the process of approximating arbitrary compact convex sets in $\mathbb{R}^n$ by compact convex polyhedra.

There is a natural way to identify $P(X)$ with $P(\widehat{X})$. Consider first the case of $P(X)$. If $X$ is a compact subset of $\mathbb{C}^n$, there is a natural extension of each function $f \in P(X)$ to a function $\widehat{f} \in C(\widehat{X})$. To construct $\widehat{f}$ note that because $f \in P(X)$, there is a sequence $\{p_j\}_{j=1,\ldots}$ of polynomials that converges uniformly on $X$ to $f$. If $y$ is any point of $\widehat{X}$, then the sequence $p_j(y)_{j=1,\ldots}$ is a Cauchy sequence and so converges. The limit of this sequence is defined to be $\widehat{f}(y)$. The value $\widehat{f}(y)$ is independent of the choice of the sequence of polynomials. This construction gives an extension of $f \in P(X)$ to a function $\widehat{f}$ defined on all of $\widehat{X}$. By uniform convergence, $\widehat{f}$ is continuous and lies in $P(\widehat{X})$. By way of the identification of $f$ with $\widehat{f}$, the algebra $P(X)$ can be identified naturally with the algebra $P(\widehat{X})$.

**Theorem 4.1.3.** *If $X$ is a compact polynomially convex subset of $\mathbb{C}^n$, and if $f \in P(X)$ then the graph of $f$ is a polynomially convex subset of $\mathbb{C}^{n+1}$.*

In particular, if $f$ is a function continuous on the closed unit disk and holomorphic on its interior, then the graph of $f$ is a polynomially convex subset of $\mathbb{C}^2$.

*Proof.* Denote by $\Gamma$ the graph $\{(z, f(z)) : z \in X\}$ of $f$. The set $\Gamma$ is compact. Let $(z_0, \zeta_0) \in C^{n+1} \backslash \Gamma$. This means either $z_0 \notin X$ or $f(z_0) \neq \zeta_0$. If $z_0 \notin X$, then there is a polynomial $p$ on $\mathbb{C}^n$ such that

$$|p(z_0)| > \|p\|_X \,.$$

So if we consider $p$ as a function on $\mathbb{C}^{n+1}$, it shows that $(z_0, \zeta_0) \notin \widehat{\Gamma}$.

If $z_0 \in X$, then $\zeta_0 \neq f(z_0)$. Let $c = |\zeta_0 - f(z_0)|$. There is a polynomial $Q$ such that $\|Q - f\|_X < \frac{c}{4}$. Let

$$\Delta = \left\{(z, \zeta) \in \mathbb{C}^n \times \mathbb{C} : z \in X \text{ and } |\zeta - Q(z)| \leq \frac{c}{2}\right\}$$

This is a compact polynomially convex subset of $\mathbb{C}^n \times \mathbb{C}$, and it contains $\Gamma$, for if $(z, \zeta) \in \Gamma$, then

$$|\zeta - Q(z)| = |f(z) - Q(z)| < \frac{c}{4}.$$

Also $(z_0, \zeta_0) \notin \Delta$, since

$$|\zeta_0 - Q(z_0)| \geq |\zeta_0 - f(z_0)| - |f(z_0) - Q(z_0)| > \frac{3c}{4}.$$

That is a compact polynomially convex subset $\Delta$ contains $\Gamma$ and not the point $(z_0, \zeta_0)$, so $(z_0, \zeta_0) \notin \widehat{\Gamma}$. □

### 4.2 A Characterization of Polynomially Convex Sets

In the present section we establish a full characterization of polynomially convex set which is folklore and a recent characterization of polynomially convex hulls, obtained by Duval and Sibony [DuvSib95].

**Theorem 4.2.1.** *If $X$ is a compact, polynomially convex subset of $\mathbb{C}^n$, then there is a non-negative plurisubharmonic function, $v$, on $\mathbb{C}^n$ with $\lim_{z \to \infty} v(z) = \infty$, with $X = v^{-1}(0)$, and with the additional properties that $v$ is of class $C^\infty$ on $\mathbb{C}^n$ and strictly plurisubharmonic on $\mathbb{C}^n \backslash X$. The function $v$ can be chosen to satisfy $v(z) = |z|^2$ for $z$ near* *infinity. Conversely, if $v$ is a nonnegative plurisubharmonic function on $\mathbb{C}^n$ such that $\lim_{z\to\infty} v(z) = \infty$, then the set $v^{-1}(0)$ is polynomially convex.*

*Proof.* Fix a nonnegative function $\chi$ of class $C^\infty$ on $\mathbb{R}$ with the properties that $\chi(t) = 0$ if $t < \frac{1}{2}$ and $\chi(1) = 1$ respectively. Require also that $\chi'$ and $\chi''$ be nonnegative and strictly positive on $t > \frac{1}{2}$. Given a point $z \in \mathbb{C}^n \setminus X$, there is a polynomial $p_z$ such that $p_z(z) = 1$ and $|p_z| < \frac{1}{4}$ on $X$. The function $|p_z|^2$ is of class $C^\infty$ and is plurisubharmonic. If $\varepsilon_z > 0$ is sufficiently small, then the function $\eta_z$ defined by $\eta_z(w) = \chi\left(|p_z(w)|^2 + \varepsilon_z |w|^2\right)$ is plurisubharmonic and of class $C^\infty$ on $\mathbb{C}^n$. It vanishes on a neighborhood of $X$, and is strictly plurisubharmonic on a neighborhood $W_z$ of the point $z$. A countable number of the neighborhoods $W_z$, say $W_1, \cdots$, cover $\mathbb{C}^n \setminus X$. Let $\eta_1, \ldots$ be the associated functions. If $\{\delta_j\}_{j=1,\cdots}$ is a sequence of positive numbers that decrease sufficiently rapidly to zero, then the function $u$ defined by $u = \sum_{j=1,\cdots} \delta_j \eta_j$ is a nonnegative plurisubharmonic function of class $C^\infty$ with $X$ as its zero set that is strictly plurisubharmonic on $\mathbb{C}^n \setminus X$. It satisfies $\lim_{w\to\infty} u(w) = \infty$.

To obtain the function $v$ of the statement of the theorem, fix an $R > 0$ so large that the set $X$ is contained in the ball $\mathbb{B}_n(R)$. Let $\eta : \mathbb{R} \to [0, \infty)$ be a smooth function with $\eta(t) = 0$ on $[0, R)$and with $\eta(t) = t^2$ when $t > 3R$. Require also that $\eta'$ and $\eta''$ be nonnegative. Let $\rho : \mathbb{R} \to [0, 1]$ satisfy $\rho(t) = 0$ if $t > 3R$ and $\rho(t) = t$ when $t \in [0, 2R)$. The function $v$ we desired can be defined by $v(w) = \eta(|w|) + \varepsilon\rho(|w|)u(w)$ for sufficiently small positive $\varepsilon$.

This completes the proof of one implication of the theorem. We postpone the proof of the final statement of the theorem for the moment; it will be contained in a more general result, Theorem 4.2.3, below. $\square$

**Corollary 4.2.2.** *If $X$ is a compact subset of $\mathbb{C}^n$ and $x_0 \in \widehat{X}$, then for each Jensen measure $\mu$ for $x_0$ carried by $X$ and for each plurisubharmonic function $u$ defined on a neighborhood of $\widehat{X}$,*

$$u(x_0) \leq \int u(z) d\mu(z).$$

*Proof.* By the monotone convergence theorem, it suffices to prove that the desired inequality holds when $u$ is a continuous plurisubharmonic function. Accordingly, let $u$ be such a function, and let $\varepsilon > 0$ be given. By the preceding theorem, there are polynomials $P_1, \cdots, P_r$ and positive constants $c_1, \cdots, c_r$ such that on a neighborhood of $\widehat{X}$ the inequalities

$$u - \varepsilon < \max_{j=1,\ldots,r} c_j \log |P_j| < u$$

are satisfied. Then for each $k$,

$$\int u(z)d\mu(z) \geq \int \max_{j=1,\ldots,r} c_j \log |P_j| \, d\mu \geq c_k \log |P_k(x_0)| \, .$$

It follows that, as desired, $u(x_0) \leq \int u(z)d\mu(z)$. $\square$

We can now complete the proof of Theorem 4.2.1. What remains to be proved is the final assertion. It is a consequence of a more general fact:

**Theorem 4.2.3.** *If $X$ is a compact subset of $\mathbb{C}^n$, then $\widehat{X}$ coincides with $\widehat{X}^{Psh}$.*

*Proof.* For every polynomial $P$ the function $|P|$ is plurisubharmonic on $\mathbb{C}^n$, whence the inclusion $\widehat{X} \supset \widehat{X}^{Psh}$.

For the reverse inclusion, let $p$ be a point of $\widehat{X}$. There is a Jensen measure $\mu$ for $p$ supported by $X$. The corollary just proved shows that for every plurisubharmonic function $u$ on $\mathbb{C}^n$, $u(p) \leq \int_X u(x)d\mu(x)$, which implies the inequality $u(p) \leq \sup_X u(x)$, whence $p \in \widehat{X}^{Psh}$. The theorem is proved. $\square$

We now turn to the characterization of polynomially convex sets found by Duval and Sibony. Denote by $\delta_x$ the positive measure of unit mass with support the singleton $x$.

**Theorem 4.2.4.** *For a compact set $X$ in $\mathbb{C}^n$ and a point $x \in \mathbb{C}^n$, the following are equivalent:*

*(a) $x \in \widehat{X}$;*

*(b) There is a positive current $T \in \mathcal{D}_{n-1,n-1}(\mathbb{C}^n)$ such that $dd^cT = \mu - \delta_x$ for a probability measure $\mu$ supported in $X$.*

The conclusion in part (b) is that for each $C^\infty$ function $\varphi$ on $\mathbb{C}^n$,

$$\int \varphi d\mu - \varphi(x) = T(dd^c\varphi).$$

That (b) implies (a) is a consequence of a more general result:

**Theorem 4.2.5.** *If $X$ is a compact subset of $\mathbb{C}^n$, if $T \in \mathcal{D}_{n-1,n-1}(\mathbb{C}^n \backslash X)$ is positive and has bounded support, and if $dd^cT$ is negative in $\mathbb{C}^n \backslash X$, then the support of $T$ is contained in $\widehat{X}$.*

*Proof.* If $x \in \text{supp } T \backslash \widehat{X}$, then by Theorem 4.2.1, there is a nonnegative smooth plurisubharmonic function $u$ on $\mathbb{C}^n$ that vanishes on a neighborhood of $\widehat{X}$ and that is strictly plurisubharmonic where it is positive, which includes a neighborhood of the point $x$. We then have that

$$0 < T(dd^c u) = (dd^c T)(u) \leq 0,$$

which is impossible. $\square$

We have supp $dd^cT \subset$ supp $T$, so this result yields that (b) implies (a).

That (a) implies (b) is a consequence of a more precise statement:

**Theorem 4.2.6.** *Let $X$ be a compact subset of $\mathbb{C}^n$, let $x_0 \in \widehat{X}$, and let $\mu$ be a Jensen measure for $x_0$ supported in $X$. There is a positive current $T$ of bidimension $(1,1)$ and with bounded support such that $dd^cT = \mu - \delta_{x_0}$.*

*Proof.* Fix an $R > 0$ large enough that $\widehat{X} \subset \mathbb{B}_n(R)$.

By a flat disk contained in $\mathbb{B}_n(R)$ we shall understand a disk that is contained in the intersection of $\mathbb{B}_n(R)$ with a complex line in $\mathbb{C}^n$.

Introduce the class $\mathcal{K}_0$ of currents of bidimension $(1,1)$ of the form $S = g_D[D]$, where $D$ is a flat disk contained in $\mathbb{B}_n(R)$ and where $g_D$ is the Green function for $D$, so that if $c_D$ is the center of $D$, then $g_D$ is nonnegative and harmonic on $D \backslash c_D$, $g_D$ vanishes on $bD$, and, with $\Delta$ denoting the Laplacian in the complex line that contains $D$, $\Delta g_D = \delta_{c_D}$. (On the unit disk $\mathbb{U}$ in $\mathbb{C}$, the Green function is $-log|z|$.) Thus, for a smooth two-form $\alpha$ on $\mathbb{C}^n$, $S(\alpha) = \int_D g_D \alpha$. This integral exists, for $g_D$ has a logarithmic singularity at $c_D$.

Let $\mathcal{K}$ denote the cone generated by the set $\mathcal{K}_0$.We shall show that $\mu - \delta_{x_0}$ lies in the weak$^*$ closure of the cone $dd^c\mathcal{K} = \{dd^cS : S \in \mathcal{K}\}$ in the dual space of the space $\wedge^{1,1}(\mathbb{C}^n)$. In the contrary case, there is a weak$^*$ continuous linear functional on the dual space of $\wedge^{1,1}(\mathbb{C}^n)$ that separates $\mu - \delta_{x_0}$ from the cone $dd^c\mathcal{K}$. Weak$^*$ continuous linear functionals are point evaluations, so there is a function $\varphi \in C^\infty(\mathbb{C}^n)$ such that $\int \varphi d\mu - \varphi(x_0) < 0 \leq T(dd^c\varphi)$ for all $T \in \mathcal{K}$.

This condition implies that if $D$ is a flat disk in $\mathbb{B}_n(R)$, then $\int_D g_D dd^c\varphi$ is nonnegative. Because this happens for all disks $\widetilde{D}$ contained in the line $\lambda$ that contains $D$ and that are contained in $\mathbb{B}_n(R)$, it follows that the Laplacian of $\varphi$ on $\lambda\cap\mathbb{B}_n(R)$ is nonnegative. Thus, $\varphi$ is subharmonic on $\lambda\cap\mathbb{B}_n(R)$, and $\varphi$ is plurisubharmonic on $\mathbb{B}_n(R)$. It satisfies

$$\int \widetilde{\varphi} d\mu < \widetilde{\varphi}(x_0),$$

which is impossible by Corollary 4.2.2, for $\mu$ is a Jensen measure for $x_0$. Thus, as claimed, $\mu-\delta_{x_0}$ lies in the weak* closure of the cone $dd^c\mathcal{K}$.

Consequently, there is a net $\{dd^cT_\gamma\}_{\gamma\,\Gamma}$ in $dd^c\mathcal{K}$ that converges in the weak* sense to $\mu-\delta_{x_0}$. For each $\varphi\in C^\infty(\mathbb{C})^n$, there are $\gamma_0$ and $M>0$ such that $\left|dd^cT_\gamma(\varphi)\right|\leq M$ for $\gamma>\gamma_0$. Apply this to the function $|z|^2$. Each $T_\gamma$ is of the form

$$T_\gamma = \sum_{j=1,\cdots} \lambda_j^\gamma g_j^\gamma \left[D_j^\gamma\right]$$

for some choice of positive numbers $\lambda_j^\gamma$ and some choice of flat disks $D_j^\gamma$ contained in $\mathbb{B}_n(R)$. For each $\gamma$ and $j$ , $g_j^\gamma$ denotes the Green function associated with the disk $D_j^\gamma$. Thus,

$$dd^cT_\gamma\left(|z|^2\right) = \sum_{j=1,\cdots}\lambda_j^\gamma \sum_{r=1,\cdots,n}\frac{i}{2}\int_{D_j^\gamma} g_j^\gamma dz_r\wedge d\bar{z}_r.$$

It follows that if $v_r^\gamma$ is the positive measure defined by

$$\int f dv_r^\gamma = \sum_{j=1,\cdots}\lambda_j^\gamma\frac{i}{2}\int_{D_j^\gamma} f g_j^\gamma dz_r\wedge d\bar{z}_r,$$

then for $\gamma>\gamma_0$, the measures $v_r^\gamma$ are uniformly bounded in norm. They are supported in $\mathbb{B}_n(R)$. By passing to a suitable subnet, we can suppose that each of the nets $\{v_r^\gamma\}_{\gamma\in\Gamma}$ converges in the weak* topology on the space of measures on $\overline{\mathbb{B}_n(R)}$, viewed as the dual space of the space $C(\overline{\mathbb{B}_n(R)})$, to a measure $v_r$. The measures $v_r$ are nonnegative.

We now have that the current $T$ of bidimension $(1,1)$ given by

$$T\left(\sum_{j,k=1,\cdots,n}\alpha_{j,k}dz_j\wedge d\bar{z}_k\right) = \sum_{r=1,\cdots,n}\int\alpha_{r,r}dv_r$$

has support in $\mathbb{B}_n(R)$, satisfies $T(dd^c\varphi)=\int\varphi d\mu-\varphi(x_0)$, and is positive.

This completes the proof of the theorem and with it the proof of Theorem 4.2.4. $\square$

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