Essential spectra of quasi-parabolic composition operators on Hardy spaces of analytic functions

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A B S T R A C T

In this work we study the essential spectra of composition operators on Hardy spaces of analytic functions which might be termed as "quasi-parabolic." This is the class of composition operators on H^2 with symbols whose conjugate with the Cayley transform on the upper half-plane are of the form $\varphi(z) = z + \psi(z)$, where $\psi \in H^\infty(\mathbb{H})$ and $\Im(\psi(z)) > \epsilon > 0$. We especially examine the case where ψ is discontinuous at infinity. A new method is devised to show that this type of composition operator fall in a C*-algebra of Toeplitz operators and Fourier multipliers. This method enables us to provide new examples of essentially normal composition operators and to calculate their essential spectra.

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1. Introduction

This work is motivated by the results of Cowen (see [5]) on the spectra of composition operators on $H^2(\mathbb{D})$ induced by parabolic linear fractional non-automorphisms that fix a point ξ on the boundary. These composition operators are precisely the essentially normal linear fractional composition operators [3]. These linear fractional transformations for $\xi = 1$ take the form

$$\varphi_a(z) = \frac{2iz + a(1-z)}{2i + a(1-z)}$$

with $\Im(a) > 0$. Their upper half-plane re-incarnations via the Cayley transform $\mathfrak{C}$ (see p. 3) are the translations

$$\mathfrak{C}^{-1} \circ \varphi_a \circ \mathfrak{C}(w) = w + a$$

acting on the upper half-plane.

Cowen [5] has proved that

$$\sigma(C_{\varphi_a}) = \sigma_e(C_{\varphi_a}) = \left\{ e^{iat} : t \in [0, \infty) \right\} \cup \{0\}.$$

Bourdon, Levi, Narayan, Shapiro [3] dealt with composition operators with symbols φ such that the upper half-plane re-incarnation of φ satisfies

$$\mathfrak{C}^{-1} \circ \varphi \circ \mathfrak{C}(z) = pz + \psi(z),$$

where $p > 0$, $\Im(\psi(z)) > \epsilon > 0$ for all $z \in \mathbb{H}$ and $\lim_{z \to \infty} \psi(z) = \psi_0 \in \mathbb{H}$ exist. Their results imply that the essential spectrum of such a composition operator with $p = 1$ is

$$\left\{ e^{i\psi_0 t} : t \in [0, \infty) \right\} \cup \{0\}.$$

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In this work we are interested in composition operators whose symbols φ have upper half-plane re-incarnation

$$\mathfrak{C}^{-1} \circ \varphi \circ \mathfrak{C}(z) = z + \psi(z)$$

for a bounded analytic function ψ satisfying $\Im(\psi(z)) > \epsilon > 0$ for all $z \in \mathbb{H}$. This class will obviously include those studied in [3] with $p = 1$. However we will be particularly interested in the case where ψ does not have a limit at infinity. We call such composition operators "quasi-parabolic." Our most precise result is obtained when the boundary values of ψ lie in QC, the space of quasi-continuous functions on $\mathbb{T}$, which is defined as

$$QC = \left[H^\infty + C(\mathbb{T})\right] \cap \left[\overline{H^\infty + C(\mathbb{T})}\right].$$

We recall that the set of cluster points $\mathcal{C}_\xi(\psi)$ of $\psi \in H^\infty$ is defined to be the set of points $z \in \mathbb{C}$ for which there is a sequence $\{z_n\} \subset \mathbb{D}$ so that $z_n \to \xi$ and $\psi(z_n) \to z$.

In particular we prove the following theorem.

Theorem B. *Let* $\varphi : \mathbb{D} \to \mathbb{D}$ *be an analytic self-map of* $\mathbb{D}$ *such that*

$$\varphi(z) = \frac{2iz + \eta(z)(1 - z)}{2i + \eta(z)(1 - z)},$$

where $\eta \in H^\infty(\mathbb{D})$ *with* $\Im(\eta(z)) > \epsilon > 0$ *for all* $z \in \mathbb{D}$. *If* $\eta \in QC \cap H^\infty$ *then we have*

(1) $C_\varphi : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ *is essentially normal,*
(2) $\sigma_e(C_\varphi) = \{e^{izt} : t \in [0, \infty], \ z \in \mathcal{C}_1(\eta)\} \cup \{0\}$,

where $\mathcal{C}_1(\eta)$ *is the set of cluster points of* η *at* 1.

Moreover, for general $\eta \in H^\infty$ with $\Im(\eta(z)) > \epsilon > 0$ (but no requirement that $\eta \in QC$), we have

$$\sigma_e(C_\varphi) \supseteq \left\{e^{izt} : t \in [0, \infty), \ z \in \mathcal{R}_1(\eta)\right\} \cup \{0\},$$

where the local essential range $\mathcal{R}_\xi(\eta)$ of an $\eta \in L^\infty(\mathbb{T})$ at $\xi \in \mathbb{T}$ is defined to be the set of points $z \in \mathbb{C}$ so that, for all $\varepsilon > 0$ and $\delta > 0$, the set

$$\eta^{-1}\left(B(z, \varepsilon)\right) \cap \left\{e^{it} : |t - t_0| \leqslant \delta\right\}$$

has positive Lebesgue measure, where $e^{it_0} = \xi$. We note that ([25]) for functions $\eta \in QC \cap H^\infty$,

$$\mathcal{R}_\xi(\eta) = \mathcal{C}_\xi(\eta).$$

The local essential range $\mathcal{R}_\infty(\psi)$ of $\psi \in L^\infty(\mathbb{R})$ at ∞ is defined as the set of points $z \in \mathbb{C}$ so that, for all $\varepsilon > 0$ and $n > 0$, we have

$$\lambda\left(\psi^{-1}\left(B(z, \varepsilon)\right) \cap \left(\mathbb{R} - [-n, n]\right)\right) > 0,$$

where λ is the Lebesgue measure on $\mathbb{R}$.

The Cayley transform induces a natural isometric isomorphism between $H^2(\mathbb{D})$ and $H^2(\mathbb{H})$. Under this identification "quasi-parabolic" composition operators correspond to operators of the form

$$T_{\frac{\varphi(z)+i}{z+i}} C_\varphi = C_\varphi + T_{\frac{\psi(z)}{z+i}} C_\varphi,$$

where $\varphi(z) = z + \psi(z)$ with $\psi \in H^\infty$ on the upper half-plane, and T_a is the multiplication operator by a.

We work on the upper half-plane and use Banach algebra techniques to compute the essential spectra of operators that correspond to "quasi-parabolic" operators. Our treatment is motivated by [9] where the translation operators are considered as Fourier multipliers on H^2 (we refer the reader to [17] for the definition and properties of Fourier multipliers). Throughout the present work, $H^2(\mathbb{H})$ will be considered as a closed subspace of $L^2(\mathbb{R})$ via the boundary values. With the help of Cauchy integral formula we prove an integral formula that gives composition operators as integral operators. Using this integral formula we show that operators that correspond to "quasi-parabolic" operators fall in a C*-algebra generated by Toeplitz operators and Fourier multipliers.

The remainder of this paper is organized as follows. In Section 2 we give the basic definitions and preliminary material that we will use throughout. For the benefit of the reader we explicitly recall some facts from Banach algebras and Toeplitz operators. In Section 3 we first prove an integral representation formula for composition operators on H^2 of the upper half-plane. Then we use this integral formula to prove that a "quasi-parabolic" composition operator is written as a series of Toeplitz operators and Fourier multipliers which converges in operator norm. In Section 4 we analyze the C*-algebra generated by Toeplitz operators with $QC(\mathbb{R})$ symbols and Fourier multipliers modulo compact operators. We show that this C*-algebra is commutative and we identify its maximal ideal space using a related theorem of Power (see [18]). In Section 5, using the machinery developed in Sections 3 and 4, we determine the essential spectra of "quasi-parabolic" composition

operators. We also give an example of a "quasi-parabolic" composition operator C_φ for which $\psi \in QC(\mathbb{R})$ but does not have a limit at infinity and compute its essential spectrum.

In the last section we examine the case of C_φ with

$$\varphi(z) = z + \psi(z),$$

where $\psi \in H^\infty(\mathbb{H})$, $\Im(\psi(z)) > \epsilon > 0$ but ψ is not necessarily in $QC(\mathbb{R})$. Using Power's theorem on the C*-algebra generated by Toeplitz operators with $L^\infty(\mathbb{R})$ symbols and Fourier multipliers, we prove the result

$$\sigma_e(C_\varphi) \supseteq \left\{ e^{izt} : z \in \mathcal{R}_\infty(\psi), \ t \in [0, \infty) \right\} \cup \{0\},$$

where $\varphi(z) = z + \psi(z)$, $\psi \in H^\infty$ with $\Im(\psi(z)) > \epsilon > 0$.

2. Notation and preliminaries

In this section we fix the notation that we will use throughout and recall some preliminary facts that will be used in the sequel.

Let S be a compact Hausdorff topological space. The space of all complex valued continuous functions on S will be denoted by $C(S)$. For any $f \in C(S)$, $\|f\|_\infty$ will denote the sup-norm of f, i.e.

$$\|f\|_\infty = \sup \left\{ |f(s)| : s \in S \right\}.$$

For a Banach space X, $K(X)$ will denote the space of all compact operators on X and $\mathcal{B}(X)$ will denote the space of all bounded linear operators on X. The open unit disc will be denoted by $\mathbb{D}$, the open upper half-plane will be denoted by $\mathbb{H}$, the real line will be denoted by $\mathbb{R}$ and the complex plane will be denoted by $\mathbb{C}$. The one point compactification of $\mathbb{R}$ will be denoted by $\dot{\mathbb{R}}$ which is homeomorphic to $\mathbb{T}$. For any $z \in \mathbb{C}$, $\Re(z)$ will denote the real part, and $\Im(z)$ will denote the imaginary part of z, respectively. For any subset $S \subset B(H)$, where H is a Hilbert space, the C*-algebra generated by S will be denoted by $C^*(S)$. The Cayley transform $\mathfrak{C}$ will be defined by

$$\mathfrak{C}(z) = \frac{z - i}{z + i}.$$

For any $a \in L^\infty(\mathbb{R})$ (or $a \in L^\infty(\mathbb{T})$), M_a will be the multiplication operator on $L^2(\mathbb{R})$ (or $L^2(\mathbb{T})$) defined as

$$M_a(f)(x) = a(x)f(x).$$

For convenience, we remind the reader of the rudiments of Gelfand theory of commutative Banach algebras and Toeplitz operators.

Let A be a commutative Banach algebra. Then its maximal ideal space $M(A)$ is defined as

$$M(A) = \left\{ x \in A^* : x(ab) = x(a)x(b) \ \forall a, b \in A \right\}$$

where A^* is the dual space of A. If A has identity then $M(A)$ is a compact Hausdorff topological space with the weak* topology. The Gelfand transform $\Gamma : A \to C(M(A))$ is defined as

$$\Gamma(a)(x) = x(a).$$

If A is a commutative C*-algebra with identity, then Γ is an isometric *-isomorphism between A and $C(M(A))$. If A is a C*-algebra and I is a two-sided closed ideal of A, then the quotient algebra A/I is also a C*-algebra (see [1] and [7]). For $a \in A$ the spectrum $\sigma_A(a)$ of a on A is defined as

$$\sigma_A(a) = \{ \lambda \in \mathbb{C} : \lambda e - a \text{ is not invertible in } A \},$$

where e is the identity of A. We will use the spectral permanency property of C*-algebras (see [20, p. 283] and [7, p. 15]); i.e. if A is a C*-algebra with identity and B is a closed *-subalgebra of A, then for any $b \in B$ we have

$$\sigma_B(b) = \sigma_A(b). \tag{1}$$

To compute essential spectra we employ the following important fact (see [20, p. 268] and [7, pp. 6, 7]): If A is a commutative Banach algebra with identity then for any $a \in A$ we have

$$\sigma_A(a) = \left\{ \Gamma(a)(x) = x(a) : x \in M(A) \right\}. \tag{2}$$

In general (for A not necessarily commutative), we have

$$\sigma_A(a) \supseteq \left\{ x(a) : x \in M(A) \right\}. \tag{3}$$

For a Banach algebra A, we denote by $com(A)$ the closed ideal in A generated by the commutators $\{a_1 a_2 - a_2 a_1 : a_1, a_2 \in A\}$. It is an algebraic fact that the quotient algebra $A/com(A)$ is a commutative Banach algebra. The reader can find detailed information about Banach and C*-algebras in [20] and [7] related to what we have reviewed so far.

The essential spectrum $\sigma_e(T)$ of an operator T acting on a Banach space X is the spectrum of the coset of T in the Calkin algebra $\mathcal{B}(X)/K(X)$, the algebra of bounded linear operators modulo compact operators. The well-known Atkinson's theorem identifies the essential spectrum of T as the set of all $\lambda \in \mathbb{C}$ for which $\lambda I - T$ is not a Fredholm operator. The essential norm of T will be denoted by $\|T\|_e$ which is defined as

$$\|T\|_e = \inf\{\|T + K\| : K \in K(X)\}.$$

The bracket $[\cdot]$ will denote the equivalence class modulo $K(X)$. An operator $T \in \mathcal{B}(H)$ is called essentially normal if $T^*T - TT^* \in K(H)$ where H is a Hilbert space and T^* denotes the Hilbert space adjoint of T.

The Hardy space of the unit disc will be denoted by $H^2(\mathbb{D})$ and the Hardy space of the upper half-plane will be denoted by $H^2(\mathbb{H})$.

The two Hardy spaces $H^2(\mathbb{D})$ and $H^2(\mathbb{H})$ are isometrically isomorphic. An isometric isomorphism $\Phi : H^2(\mathbb{D}) \to H^2(\mathbb{H})$ is given by

$$\Phi(g)(z) = \left(\frac{1}{\sqrt{\pi}(z+i)}\right) g\left(\frac{z-i}{z+i}\right). \tag{4}$$

The mapping Φ has an inverse $\Phi^{-1} : H^2(\mathbb{H}) \to H^2(\mathbb{D})$ given by

$$\Phi^{-1}(f)(z) = \frac{e^{\frac{i\pi}{2}}(4\pi)^{\frac{1}{2}}}{(1-z)} f\left(\frac{i(1+z)}{1-z}\right).$$

For more details see [11, pp. 128–131] and [14].

Using the isometric isomorphism Φ, one may transfer Fatou's theorem in the unit disc case to upper half-plane and may embed $H^2(\mathbb{H})$ in $L^2(\mathbb{R})$ via $f \to f^*$ where $f^*(x) = \lim_{y \to 0} f(x + iy)$. This embedding is an isometry.

Throughout the paper, using Φ, we will go back and forth between $H^2(\mathbb{D})$ and $H^2(\mathbb{H})$. We use the property that Φ preserves spectra, compactness and essential spectra i.e. if $T \in \mathcal{B}(H^2(\mathbb{D}))$ then

$$\sigma_{\mathcal{B}(H^2(\mathbb{D}))}(T) = \sigma_{\mathcal{B}(H^2(\mathbb{H}))}(\Phi \circ T \circ \Phi^{-1}),$$

$K \in K(H^2(\mathbb{D}))$ if and only if $\Phi \circ K \circ \Phi^{-1} \in K(H^2(\mathbb{H}))$ and hence we have

$$\sigma_e(T) = \sigma_e(\Phi \circ T \circ \Phi^{-1}). \tag{5}$$

We also note that $T \in \mathcal{B}(H^2(\mathbb{D}))$ is essentially normal if and only if $\Phi \circ T \circ \Phi^{-1} \in \mathcal{B}(H^2(\mathbb{H}))$ is essentially normal.

The Toeplitz operator with symbol a is defined as

$$T_a = PM_a|_{H^2},$$

where P denotes the orthogonal projection of L^2 onto H^2. A good reference about Toeplitz operators on H^2 is Douglas' treatise [8]. Although the Toeplitz operators treated in [8] act on the Hardy space of the unit disc, the results can be transfered to the upper half-plane case using the isometric isomorphism Φ introduced by Eq. (4). In the sequel the following identity will be used:

$$\Phi^{-1} \circ T_a \circ \Phi = T_{a \circ \mathfrak{C}^{-1}}, \tag{6}$$

where $a \in L^\infty(\mathbb{R})$. We also employ the fact

$$\|T_a\|_e = \|T_a\| = \|a\|_\infty \tag{7}$$

for any $a \in L^\infty(\mathbb{R})$, which is a consequence of Theorem 7.11 of [8, pp. 160–161] and Eq. (6). For any subalgebra $A \subseteq L^\infty(\mathbb{R})$ the Toeplitz C*-algebra generated by symbols in A is defined to be

$$\mathcal{T}(A) = C^*(\{T_a : a \in A\}).$$

It is a well-known result of Sarason (see [21,23] and also [19]) that the set of functions

$$H^\infty + C = \{f_1 + f_2 : f_1 \in H^\infty(\mathbb{D}), \ f_2 \in C(\mathbb{T})\}$$

is a closed subalgebra of $L^\infty(\mathbb{T})$. The following theorem of Douglas [8] will be used in the sequel.

Theorem 1 (DOUGLAS' THEOREM). *Let $a,b \in H^\infty + C$ then the semi-commutators*

$$T_{ab} - T_a T_b \in K(H^2(\mathbb{D})), \qquad T_{ab} - T_b T_a \in K(H^2(\mathbb{D})),$$

and hence the commutator

$$[T_a, T_b] = T_a T_b - T_b T_a \in K(H^2(\mathbb{D})).$$

Let QC be the C*-algebra of functions in $H^\infty + C$ whose complex conjugates also belong to $H^\infty + C$. Let us also define the upper half-plane version of QC as the following:

$$QC(\mathbb{R}) = \left\{ a \in L^\infty(\mathbb{R}) : a \circ \mathfrak{C}^{-1} \in QC \right\}.$$

Going back and forth with Cayley transform one can deduce that $QC(\mathbb{R})$ is a closed subalgebra of $L^\infty(\mathbb{R})$.

By Douglas' theorem and Eq. (6), if $a, b \in QC(\mathbb{R})$, then

$$T_a T_b - T_{ab} \in K\big(H^2(\mathbb{H})\big).$$

Let $scom(QC(\mathbb{R}))$ be the closed ideal in $\mathcal{T}(QC(\mathbb{R}))$ generated by the semi-commutators $\{T_a T_b - T_{ab}: a, b \in QC(\mathbb{R})\}$. Then we have

$$com\big(\mathcal{T}\big(QC(\mathbb{R})\big)\big) \subseteq scom\big(QC(\mathbb{R})\big) \subseteq K\big(H^2(\mathbb{H})\big).$$

By Proposition 7.12 of [8] and Eq. (6) we have

$$com\big(\mathcal{T}\big(QC(\mathbb{R})\big)\big) = scom\big(QC(\mathbb{R})\big) = K\big(H^2(\mathbb{H})\big). \tag{8}$$

Now consider the symbol map

$$\Sigma : QC(\mathbb{R}) \to \mathcal{T}\big(QC(\mathbb{R})\big)$$

defined as $\Sigma(a) = T_a$. This map is linear but not necessarily multiplicative; however if we let q be the quotient map

$$q : \mathcal{T}\big(QC(\mathbb{R})\big) \to \mathcal{T}\big(QC(\mathbb{R})\big)/scom\big(QC(\mathbb{R})\big),$$

then $q \circ \Sigma$ is multiplicative; moreover by Eqs. (7) and (8), we conclude that $q \circ \Sigma$ is an isometric *-isomorphism from $QC(\mathbb{R})$ onto $\mathcal{T}(QC(\mathbb{R}))/K(H^2(\mathbb{H}))$.

Definition 2. Let $\varphi : \mathbb{D} \to \mathbb{D}$ or $\varphi : \mathbb{H} \to \mathbb{H}$ be a holomorphic self-map of the unit disc or the upper half-plane. The *composition operator* C_φ on $H^p(\mathbb{D})$ or $H^p(\mathbb{H})$ with symbol φ is defined by

$$C_\varphi(g)(z) = g\big(\varphi(z)\big), \quad z \in \mathbb{D} \text{ or } z \in \mathbb{H}.$$

Composition operators of the unit disc are always bounded [6] whereas composition operators of the upper half-plane are not always bounded. For the boundedness problem of composition operators of the upper half-plane see [14].

The composition operator C_φ on $H^2(\mathbb{D})$ is carried over to $(\frac{\tilde{\varphi}(z)+i}{z+i})C_{\tilde{\varphi}}$ on $H^2(\mathbb{H})$ through Φ, where $\tilde{\varphi} = \mathfrak{C} \circ \varphi \circ \mathfrak{C}^{-1}$, i.e. we have

$$\Phi C_\varphi \Phi^{-1} = T_{\left(\frac{\tilde{\varphi}(z)+i}{z+i}\right)} C_{\tilde{\varphi}}. \tag{9}$$

However this gives us the boundedness of $C_\varphi : H^2(\mathbb{H}) \to H^2(\mathbb{H})$ for

$$\varphi(z) = pz + \psi(z),$$

where $p > 0$, $\psi \in H^\infty$ and $\Im(\psi(z)) > \epsilon > 0$ for all $z \in \mathbb{H}$:

Let $\tilde{\varphi} : \mathbb{D} \to \mathbb{D}$ be an analytic self-map of $\mathbb{D}$ such that $\varphi = \mathfrak{C}^{-1} \circ \tilde{\varphi} \circ \mathfrak{C}$, then we have

$$\Phi C_{\tilde{\varphi}} \Phi^{-1} = T_\tau C_\varphi$$

where

$$\tau(z) = \frac{\varphi(z) + i}{z + i}.$$

If

$$\varphi(z) = pz + \psi(z)$$

with $p > 0$, $\psi \in H^\infty$ and $\Im(\psi(z)) > \epsilon > 0$, then $T_{\frac{1}{\tau}}$ is a bounded operator. Since $\Phi C_{\tilde{\varphi}} \Phi^{-1}$ is always bounded we conclude that C_φ is bounded on $H^2(\mathbb{H})$.

We recall that any function in $H^2(\mathbb{H})$ can be recovered from its boundary values by means of the Cauchy integral. In fact we have [12, pp. 112–116] if $f \in H^2(\mathbb{H})$ and if f^* is its non-tangential boundary value function on $\mathbb{R}$, then

$$f(z) = \frac{1}{2\pi i} \int\limits_{-\infty}^{+\infty} \frac{f^*(x)\,dx}{x - z}, \quad z \in \mathbb{H}. \tag{10}$$

The Fourier transform $\mathcal{F}f$ of $f \in \mathcal{S}(\mathbb{R})$ (the Schwartz space, for a definition see [20, Section 7.3, pp. 168] and [27, p. 134]) is defined by

$$(\mathcal{F}f)(t) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} e^{-itx} f(x)\, dx.$$

The Fourier transform extends to an invertible isometry from $L^2(\mathbb{R})$ onto itself with inverse

$$(\mathcal{F}^{-1}f)(t) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} e^{itx} f(x)\, dx.$$

The following is a consequence of a theorem due to Paley and Wiener [12, pp. 110–111]. Let $1 < p < \infty$. For $f \in L^p(\mathbb{R})$, the following assertions are equivalent:

(i) $f \in H^p$,
(ii) $\operatorname{supp}(\hat{f}) \subseteq [0, \infty)$.

A reformulation of the Paley–Wiener theorem says that the image of $H^2(\mathbb{H})$ under the Fourier transform is $L^2([0, \infty))$. By the Paley–Wiener theorem we observe that the operator

$$D_\vartheta = \mathcal{F}^{-1} M_\vartheta \mathcal{F}$$

for $\vartheta \in C([0, \infty))$ maps $H^2(\mathbb{H})$ into itself, where $C([0, \infty))$ denotes the set of continuous functions on $[0, \infty)$ which have limits at infinity. Since $\mathcal{F}$ is unitary we also observe that

$$\|D_\vartheta\| = \|M_\vartheta\| = \|\vartheta\|_\infty. \tag{11}$$

Let F be defined as

$$F = \left\{ D_\vartheta \in B\big(H^2(\mathbb{H})\big) \colon \vartheta \in C\big([0, \infty)\big) \right\}. \tag{12}$$

We observe that F is a commutative C*-algebra with identity and the map $D : C([0, \infty)) \to F$ given by

$$D(\vartheta) = D_\vartheta$$

is an isometric *-isomorphism by Eq. (11). Hence F is isometrically *-isomorphic to $C([0, \infty))$. The operator D_ϑ is usually called a "Fourier Multiplier."

An important example of a Fourier multiplier is the translation operator $S_w : H^2(\mathbb{H}) \to H^2(\mathbb{H})$ defined as

$$S_w f(z) = f(z + w)$$

where $w \in \mathbb{H}$. We recall that

$$S_w = D_\vartheta$$

where $\vartheta(t) = e^{iwt}$ (see [9] and [10]). Other examples of Fourier multipliers that we will need come from convolution operators defined in the following way:

$$K_n f(x) = \frac{1}{2\pi i} \int\limits_{-\infty}^{\infty} \frac{-f(w)\, dw}{(x - w + i\alpha)^{n+1}}, \tag{13}$$

where $\alpha \in \mathbb{R}^+$. We observe that

$$\begin{aligned}
\mathcal{F}K_n f(x) &= \int\limits_{-\infty}^{\infty} e^{-itx} \left(\int\limits_{-\infty}^{\infty} \frac{-f(w)\, dw}{(t - w + i\alpha)^{n+1}} \right) dt \\
&= \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \frac{e^{-i(t-w)} e^{-iwx} (-f(w))}{(t - w + i\alpha)^{n+1}}\, dw\, dt \\
&= \left(\int\limits_{-\infty}^{\infty} \frac{-e^{-ivx}\, dv}{(v + i\alpha)^{n+1}} \right) \left(\int\limits_{-\infty}^{\infty} e^{-iwx} f(w)\, dw \right).
\end{aligned}$$

Since

$$\int\limits_{-\infty}^{\infty} \frac{-e^{-ivx}\,dv}{(v+i\alpha)^{n+1}} = \frac{(-ix)^n e^{-\alpha x}}{n!},$$

this implies that

$$K_n = D_{\vartheta_n} \tag{14}$$

where

$$\vartheta_n(t) = \frac{(-it)^n e^{-\alpha t}}{n!}.$$

For $p > 0$ the dilation operator $V_p \in \mathcal{B}(H^2(\mathbb{H}))$ is defined as

$$V_p f(z) = f(pz). \tag{15}$$

3. An approximation scheme for composition operators on Hardy spaces of the upper half-plane

In this section we devise an integral representation formula for composition operators and using this integral formula we develop an approximation scheme for composition operators induced by maps of the form

$$\varphi(z) = pz + \psi(z),$$

where $p > 0$ and $\psi \in H^\infty$ such that $\Im(\psi(z)) > \epsilon > 0$ for all $z \in \mathbb{H}$. By the preceding section we know that these maps induce bounded composition operators on $H^2(\mathbb{H})$. We approximate these operators by linear combinations of Toeplitz operators and Fourier multipliers. In establishing this approximation scheme our main tool is the integral representation formula that we prove below.

One can use Eq. (10) to represent composition operators with an integral kernel under some conditions on the analytic symbol $\varphi : \mathbb{H} \to \mathbb{H}$. One may apply the argument (using the Cayley transform) done after Eq. (4) to $H^\infty(\mathbb{H})$ to show that

$$\lim_{t \to 0} \varphi(x+it) = \varphi^*(x)$$

exists for almost every $x \in \mathbb{R}$. The most important condition that we will impose on φ is $\Im(\varphi^*(x)) > 0$ for almost every $x \in \mathbb{R}$. We have the following proposition.

Proposition 3. *Let $\varphi : \mathbb{H} \to \mathbb{H}$ be an analytic function such that the non-tangential boundary value function φ^* of φ satisfies $\Im(\varphi^*(x)) > 0$ for almost every $x \in \mathbb{R}$. Then the composition operator C_φ on $H^2(\mathbb{H})$ is given by*

$$(C_\varphi f)^*(x) = \frac{1}{2\pi i} \int\limits_{-\infty}^{\infty} \frac{f^*(\xi)\,d\xi}{\xi - \varphi^*(x)} \quad \text{for almost every } x \in \mathbb{R}.$$

Proof. By Eq. (10) above one has

$$C_\varphi(f)(x+it) = \frac{1}{2\pi i} \int\limits_{-\infty}^{\infty} \frac{f^*(\xi)\,d\xi}{\xi - \varphi(x+it)}.$$

Let $x \in \mathbb{R}$ be such that $\lim_{t \to 0} \varphi(x+it) = \varphi^*(x)$ exists and $\Im(\varphi^*(x)) > 0$. We have

$$\left| C_\varphi(f)(x+it) - \frac{1}{2\pi i} \int\limits_{-\infty}^{\infty} \frac{f^*(\xi)\,d\xi}{\xi - \varphi^*(x)} \right|$$

$$= \left| \frac{1}{2\pi i} \int\limits_{-\infty}^{\infty} \frac{f^*(\xi)\,d\xi}{\xi - \varphi(x+it)} - \frac{1}{2\pi i} \int\limits_{-\infty}^{\infty} \frac{f^*(\xi)\,d\xi}{\xi - \varphi^*(x)} \right|$$

$$= \frac{1}{2\pi} \left| \varphi(x+it) - \varphi^*(x) \right| \left| \int\limits_{-\infty}^{\infty} \frac{f^*(\xi)\,d\xi}{(\xi - \varphi(x+it))(\xi - \varphi^*(x))} \right|$$

$$\leqslant \frac{|\varphi(x+it) - \varphi^*(x)|}{2\pi} \|f\|_2 \left(\int\limits_{-\infty}^{\infty} \frac{d\xi}{(|(\xi - \varphi(x+it))(\xi - \varphi^*(x))|)^2} \right)^{\frac{1}{2}}, \tag{16}$$

by Cauchy–Schwarz inequality. When $|\varphi(x+it) - \varphi^*(x)| < \varepsilon$, by triangle inequality, we have

$$\big|\xi - \varphi(x+it)\big| \geqslant \big|\xi - \varphi^*(x)\big| - \varepsilon. \tag{17}$$

Fix $\varepsilon_0 > 0$ such that

$$\varepsilon_0 = \frac{\inf\{|\xi - \varphi^*(x)|\colon \xi \in \mathbb{R}\}}{2}.$$

This is possible since $\Im(\varphi^*(x)) > 0$.

Choose $\varepsilon > 0$ such that $\varepsilon_0 > \varepsilon$. Since $\lim_{t\to 0} \varphi(x+it) = \varphi^*(x)$ exists, there exists $\delta > 0$ such that for all $0 < t < \delta$ we have

$$\big|\varphi(x+it) - \varphi^*(x)\big| < \varepsilon < \varepsilon_0.$$

So by Eq. (17) one has

$$\big|\xi - \varphi(x+it)\big| \geqslant \big|\xi - \varphi^*(x)\big| - \varepsilon_0 \geqslant \varepsilon_0 \tag{18}$$

for all t such that $0 < t < \delta$. By Eq. (18) we have

$$\frac{1}{|\xi - \varphi(x+it)|} \leqslant \frac{1}{|\xi - \varphi^*(x)| - \varepsilon_0}$$

which implies that

$$\int\limits_{-\infty}^{\infty} \frac{d\xi}{(|(\xi - \varphi(x+it))(\xi - \varphi^*(x))|)^2} \leqslant \int\limits_{-\infty}^{\infty} \frac{d\xi}{|\xi - \varphi^*(x)|^4 - \varepsilon_0|\xi - \varphi^*(x)|^2}. \tag{19}$$

By the right-hand side inequality of Eq. (18), the integral on the right-hand side of Eq. (19) converges and its value only depends on x and ε_0. Let $M_{\varepsilon_0,x}$ be the value of that integral, then by Eqs. (16) and (19) we have

$$\left| C_\varphi(f)(x+it) - \frac{1}{2\pi i} \int\limits_{-\infty}^{\infty} \frac{f^*(\xi)\,d\xi}{\xi - \varphi^*(x)} \right|$$

$$\leqslant \frac{|\varphi(x+it) - \varphi^*(x)|}{2\pi} \|f\|_2 \left(\int\limits_{-\infty}^{\infty} \frac{d\xi}{|\xi - \varphi^*(x)|^4 - \varepsilon_0|\xi - \varphi^*(x)|^2} \right)^{\frac{1}{2}}$$

$$= \frac{|\varphi(x+it) - \varphi^*(x)|}{2\pi} \|f\|_2 (M_{\varepsilon_0,x})^{\frac{1}{2}} \leqslant \frac{\varepsilon}{2\pi} \|f\|_2 (M_{\varepsilon_0,x})^{\frac{1}{2}}.$$

Hence we have

$$\lim_{t\to 0} C_\varphi(f)(x+it) = C_\varphi(f)^*(x) = \frac{1}{2\pi i} \int\limits_{-\infty}^{\infty} \frac{f^*(\xi)\,d\xi}{\xi - \varphi^*(x)}$$

for $x \in \mathbb{R}$ almost everywhere. $\quad\square$

Throughout the rest of the paper we will identify a function f in H^2 or H^∞ with its boundary function f^*. We continue with the following simple geometric lemma that will be helpful in our task.

Lemma 4. *Let $K \subset \mathbb{H}$ be a compact subset of $\mathbb{H}$. Then there is an $\alpha \in \mathbb{R}^+$ such that $\sup\{|\frac{\alpha i - z}{\alpha}|\colon z \in K\} < \delta < 1$ for some $\delta \in (0,1)$.*

Proof. Let $\varepsilon = \inf\{\Im(z)\colon z \in K\}$, $R_1 = \sup\{\Im(z)\colon z \in K\}$, $R_2 = \sup\{\Re(z)\colon z \in K\}$, $R_3 = \inf\{\Re(z)\colon z \in K\}$ and $R = \max\{|R_2|, |R_3|\}$. Since K is compact $\varepsilon \neq 0$, $R_1 < +\infty$ and also $R < +\infty$. Let C be the center of the circle passing through the points $\frac{\varepsilon}{2}i$, $-R - R_1 + i\varepsilon$ and $R + R_1 + i\varepsilon$. Then C will be on the imaginary axis, hence $C = \alpha i$ for some $\alpha \in \mathbb{R}^+$ and this α satisfies what we want. $\quad\square$

We formulate and prove our approximation scheme as the following proposition.

Proposition 5. *Let $\varphi : \mathbb{H} \to \mathbb{H}$ be an analytic self-map of $\mathbb{H}$ such that*

$$\varphi(z) = pz + \psi(z),$$

$p > 0$ and $\psi \in H^\infty$ is such that $\Im(\psi(z)) > \epsilon > 0$ for all $z \in \mathbb{H}$. Then there is an $\alpha \in \mathbb{R}^+$ such that for $C_\varphi : H^2 \to H^2$ we have

$$C_\varphi = V_p \sum_{n=0}^{\infty} T_{\tau^n} D_{\vartheta_n},$$

where the convergence of the series is in operator norm, T_{τ^n} is the Toeplitz operator with symbol τ^n,

$$\tau(x) = i\alpha - \tilde{\psi}(x), \quad \tilde{\psi}(x) = \psi\left(\frac{x}{p}\right),$$

V_p is the dilation operator defined in Eq. (15) and D_{ϑ_n} is the Fourier multiplier with $\vartheta_n(t) = \frac{(-it)^n e^{-\alpha t}}{n!}$.

Proof. Since for $\varphi(z) = pz + \psi(z)$ where $\psi \in H^\infty$ with $\Im(\psi(z)) > \epsilon > 0$ for all $z \in \mathbb{H}$ and $p > 0$, we have

$$\Im(\varphi^*(x)) \geqslant \epsilon > 0 \quad \text{for almost every } x \in \mathbb{R}.$$

We can use Proposition 3 for $C_\varphi : H^2 \to H^2$ to have

$$(C_\varphi f)(x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(w)\,dw}{w - \varphi(x)} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(w)\,dw}{w - px - \psi(x)}.$$

Without loss of generality, we take $p = 1$, since if $p \neq 1$ then we have

$$(V_{\frac{1}{p}} C_\varphi)(f)(x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(w)\,dw}{w - x - \tilde{\psi}(x)}, \tag{20}$$

where $\tilde{\psi}(x) = \psi(\frac{x}{p})$ and $V_\beta f(z) = f(\beta z)$ $(\beta > 0)$ is the dilation operator. We observe that

$$\frac{-1}{x - w + \psi(x)} = \frac{-1}{x - w + i\alpha - (i\alpha - \psi(x))} = \frac{-1}{(x - w + i\alpha)(1 - (\frac{i\alpha - \psi(x)}{x - w + i\alpha}))}. \tag{21}$$

Since $\Im(\psi(z)) > \epsilon > 0$ for all $z \in \mathbb{H}$ and $\psi \in H^\infty$, we have $\overline{\psi(\mathbb{H})}$ is compact in $\mathbb{H}$, and then by Lemma 4 there is an $\alpha > 0$ such that

$$\left| \frac{i\alpha - \psi(x)}{x - w + i\alpha} \right| < \delta < 1$$

for all $x, w \in \mathbb{R}$, so we have

$$\frac{1}{1 - (\frac{i\alpha - \psi(x)}{x - w + i\alpha})} = \sum_{n=0}^{\infty} \left(\frac{i\alpha - \psi(x)}{x - w + i\alpha} \right)^n.$$

Inserting this into Eq. (21) and then into Eq. (20), we have

$$(C_\varphi f)(x) = \sum_{n=0}^{M-1} T_{\tau^n} K_n f(x) + R_M f(x),$$

where $T_{\tau^n} f(x) = \tau^n(x) f(x)$, $\tau(x) = i\alpha - \psi(x)$, K_n is as in Eq. (13) and

$$R_M f(x) = \frac{1}{2\pi i} T_{\tau^{M+1}} \int_{-\infty}^{\infty} \frac{f(w)\,dw}{(x - w + i\alpha)^M (w - x - \psi(x))}.$$

By Eq. (14) we have

$$K_n f(x) = D_{\vartheta_n} f(x) \quad \text{and} \quad \vartheta_n(t) = \frac{(-it)^n e^{-\alpha t}}{n!}.$$

Since C_φ is bounded it is not difficult to see that

$$\|R_M\| \leqslant \|T_\tau\| \|C_\varphi\| \delta^M$$

which implies that $\|R_M\| \to 0$ as $M \to \infty$. Hence we have

$$C_\varphi = \sum_{n=0}^{\infty} T_{\tau^n} D_{\vartheta_n},$$

where the convergence is in operator norm. $\quad\square$

4. A Ψ–C^*-algebra of operators on Hardy spaces of analytic functions

In the preceding section we have shown that "quasi-parabolic" composition operators on the upper half-plane lie in the C^*-algebra generated by certain Toeplitz operators and Fourier multipliers. In this section we will identify the maximal ideal space of the C^*-algebra generated by Toeplitz operators with a class of symbols and Fourier multipliers. The C^*-algebras generated by multiplication operators and Fourier multipliers on $L^2(\mathbb{R})$ are called "pseudo-differential C^*-algebras" and they have been studied in a series of papers by Power (see [17,18]) and by Cordes and Herman (see [4]). Our C^*-algebra is an analogue of "pseudo-differential C^*-algebras" introduced in [17] and [18]; however our C^*-algebra acts on H^2 instead of L^2. Our "Ψ–C^*-algebra" will be denoted by $\Psi(A, C([0, \infty]))$ and is defined as

$$\Psi\big(A, C\big([0, \infty]\big)\big) = C^*\big(\mathcal{T}(A) \cup F\big),$$

where $A \subseteq L^\infty(\mathbb{R})$ is a closed subalgebra of $L^\infty(\mathbb{R})$ and F is as defined by Eq. (12).

We will now show that if $a \in QC(\mathbb{R})$ and $\vartheta \in C([0, \infty])$, the commutator $[T_a, D_\vartheta]$ is compact on $H^2(\mathbb{H})$. But before that, we state the following fact from [16, p. 215] which implies that

$$PM_a - M_aP \in K\big(L^2\big)$$

for all $a \in QC$, where P denotes the orthogonal projection of L^2 onto H^2:

Lemma 6. *Let $a \in L^\infty(\mathbb{T})$ and P be the orthogonal projection of $L^2(\mathbb{T})$ onto $H^2(\mathbb{D})$ then the commutator $[P, M_a] = PM_a - M_aP$ is compact on $L^2(\mathbb{T})$ if and only if $a \in QC$.*

The following lemma and its proof is a slight modification of Lemma 2.0.15 of [24].

Lemma 7. *Let $a \in QC(\mathbb{R})$ and $\vartheta \in C([0, \infty])$. Then we have*

$$[T_a, D_\vartheta] = T_aD_\vartheta - D_\vartheta T_a \in K\big(H^2(\mathbb{H})\big).$$

Proof. Let $\tilde{P} : L^2(\mathbb{R}) \to H^2(\mathbb{H})$ be the orthogonal projection of L^2 onto H^2 and let $a \in QC(\mathbb{R})$. Observe that

$$D_{\chi_{[0,\infty)}} = \tilde{P}$$

where $\chi_{[0,\infty)}$ is the characteristic function of $[0, \infty)$. Let $P : L^2(\mathbb{T}) \to H^2(\mathbb{D})$ be the orthogonal projection of L^2 onto H^2 on the unit disc. By Lemma 6 and by the use of Φ defined as in Eq. (4) (observe that Φ extends to be an isometric isomorphism from $L^2(\mathbb{T})$ onto $L^2(\mathbb{R})$) we have

$$[M_a, D_{\chi_{[0,\infty)}}] = [M_a, \tilde{P}] = \Phi \circ [M_{a \circ \mathfrak{C}^{-1}}, P] \circ \Phi^{-1} \in K\big(L^2(\mathbb{R})\big).$$

Consider $D_{\chi_{[t,\infty)}}$ for $t > 0$ on L^2:

$$\begin{aligned} D_{\chi_{[t,\infty)}} &= \mathcal{F}^{-1}M_{\chi_{[t,\infty)}}\mathcal{F} = \mathcal{F}^{-1}S_{-t}M_{\chi_{[0,\infty)}}S_t\mathcal{F} \\ &= \mathcal{F}^{-1}S_{-t}\mathcal{F}\mathcal{F}^{-1}M_{\chi_{[0,\infty)}}\mathcal{F}\mathcal{F}^{-1}S_t\mathcal{F} = M_{e^{-itw}}D_{\chi_{[0,\infty)}}M_{e^{itw}}, \end{aligned}$$

where $S_t : L^2 \to L^2$ is the translation operator

$$S_t f(x) = f(x + t).$$

Hence we have

$$[M_a, D_{\chi_{[t,\infty)}}] = M_{e^{-itw}}[M_a, D_{\chi_{[0,\infty)}}]M_{e^{itw}} \in K\big(L^2(\mathbb{R})\big).$$

Since the algebra of compact operators is an ideal. So we have

$$[T_a, D_{\chi_{[t,\infty)}}] = \tilde{P}[M_a, D_{\chi_{[t,\infty)}}]|_{H^2} \in K\big(H^2(\mathbb{H})\big).$$

Consider the characteristic function $\chi_{[t,r)}$ of some interval $[t, r)$ where $0 < t < r$. Since

$$\chi_{[t,r)} = \chi_{[t,\infty)} - \chi_{[r,\infty)}$$

we have

$$D_{\chi_{[t,r)}} = D_{\chi_{[t,\infty)}} - D_{\chi_{[r,\infty)}}.$$

So

$$[T_a, D_{\chi_{[t,r)}}] = [T_a, D_{\chi_{[t,\infty)}}] - [T_a, D_{\chi_{[r,\infty)}}] \in K\big(H^2(\mathbb{H})\big).$$

Let $\vartheta \in C([0, \infty])$ then for all $\varepsilon > 0$ there are $t_0 = 0 < t_1 < \cdots < t_n \in \mathbb{R}^+$ and $c_1, c_2, \ldots, c_n, c_{n+1} \in \mathbb{C}$ such that

$$\left\| \vartheta - \left(\sum_{j=1}^n c_j \chi_{[t_{j-1}, t_j)} \right) - c_{n+1} \chi_{[t_n, \infty)} \right\|_\infty < \frac{\varepsilon}{2\|T_a\|}.$$

Hence we have

$$\left\| [T_a, D_\vartheta] - \left[T_a, \sum_{j=1}^n c_j D_{\chi_{[t_{j-1}, t_j)}} + c_{n+1} D_{\chi_{[t_{n+1}, \infty)}} \right] \right\|$$

$$= \left\| [T_a, D_{\vartheta - (\sum_{j=1}^n c_j \chi_{[t_{j-1}, t_j]}) - c_{n+1} \chi_{[t_n, \infty)}}] \right\| \leqslant 2\|T_a\| \frac{\varepsilon}{2\|T_a\|} = \varepsilon.$$

Since

$$\left[T_a, \sum_{j=1}^n c_j D_{\chi_{[t_{j-1}, t_j)}} + c_{n+1} D_{\chi_{[t_{n+1}, \infty)}} \right] \in K\left(H^2(\mathbb{H})\right),$$

letting $\varepsilon \to 0$ we have $[T_a, D_\vartheta] \in K(H^2(\mathbb{H}))$. $\square$

Now consider the C*-algebra $\Psi(QC(\mathbb{R}), C([0, \infty]))$. By Douglas' theorem and Lemma 7, the commutator ideal $com(\Psi(QC(\mathbb{R}), C([0, \infty]))$ falls inside the ideal of compact operators $K(H^2(\mathbb{H}))$. Since $\mathcal{T}(C(\dot{\mathbb{R}})) \subset \Psi(QC(\mathbb{R}), C([0, \infty]))$ as in Eq. (9) we conclude that

$$com\left(\Psi\left(QC(\mathbb{R}), C([0, \infty])\right)\right) = K\left(H^2(\mathbb{H})\right).$$

Therefore we have

$$\Psi\left(QC(\mathbb{R}), C([0, \infty])\right)/K\left(H^2(\mathbb{H})\right) = \Psi\left(QC(\mathbb{R}), C([0, \infty])\right)/com\left(\Psi\left(QC(\mathbb{R}), C([0, \infty])\right)\right) \tag{22}$$

and $\Psi(QC(\mathbb{R}), C([0, \infty]))/K(H^2(\mathbb{H}))$ is a commutative C*-algebra with identity. So it is natural to ask for its maximal ideal space and its Gelfand transform. We will use the following theorem of Power (see [18]) to characterize its maximal ideal space:

Theorem 8 (Power's theorem). *Let C_1, C_2 be two C*-subalgebras of $B(H)$ with identity, where H is a separable Hilbert space, such that $M(C_i) \neq \emptyset$, where $M(C_i)$ is the space of multiplicative linear functionals of C_i, $i = 1, 2$, and let C be the C*-algebra that they generate. Then for the commutative C*-algebra $\tilde{C} = C/com(C)$ we have $M(\tilde{C}) = P(C_1, C_2) \subset M(C_1) \times M(C_2)$, where $P(C_1, C_2)$ is defined to be the set of points $(x_1, x_2) \in M(C_1) \times M(C_2)$ satisfying the condition:*
Given $0 \leqslant a_1 \leqslant 1$, $0 \leqslant a_2 \leqslant 1$, $a_1 \in C_1$, $a_2 \in C_2$

$$x_i(a_i) = 1 \quad \text{with } i = 1, 2 \quad \Rightarrow \quad \|a_1 a_2\| = 1.$$

Proof of this theorem can be found in [18]. Using Power's theorem we prove the following result.

Theorem 9. *Let*

$$\Psi\left(QC(\mathbb{R}), C([0, \infty])\right) = C^*\left(\mathcal{T}\left(QC(\mathbb{R})\right) \cup F\right).$$

Then the C-algebra $\Psi(QC(\mathbb{R}), C([0, \infty]))/K(H^2(\mathbb{H}))$ is a commutative C*-algebra and its maximal ideal space is*

$$M\left(\Psi\left(QC(\mathbb{R}), C([0, \infty])\right)\right) \cong \left(M_\infty\left(QC(\mathbb{R})\right) \times [0, \infty]\right) \cup \left(M\left(QC(\mathbb{R})\right) \times \{\infty\}\right),$$

where

$$M_\infty\left(QC(\mathbb{R})\right) = \left\{ x \in M\left(QC(\mathbb{R})\right) : x|_{C(\dot{\mathbb{R}})} = \delta_\infty \text{ with } \delta_\infty(f) = \lim_{t \to \infty} f(t) \right\}$$

is the fiber of $M(QC(\mathbb{R}))$ at ∞.

Proof. By Eq. (22) we already know that $\Psi(QC(\mathbb{R}), C([0, \infty]))/K(H^2(\mathbb{H}))$ is a commutative C*-algebra. Since any $x \in M(A)$ vanishes on $com(A)$ we have

$$M(A) = M\left(A/com(A)\right).$$

By Eq. (8)

$$\mathcal{T}\left(QC(\mathbb{R})\right)/com\left(\mathcal{T}\left(QC(\mathbb{R})\right)\right) = \mathcal{T}\left(QC(\mathbb{R})\right)/K\left(H^2(\mathbb{H})\right)$$

is isometrically *-isomorphic to $QC(\mathbb{R})$, hence we have

$$M\big(\mathcal{T}\big(QC(\mathbb{R})\big)\big) = M\big(QC(\mathbb{R})\big).$$

Now we are ready to use Power's theorem. In our case,

$$H = H^2, \qquad C_1 = \mathcal{T}\big(QC(\mathbb{R})\big), \qquad C_2 = F \quad \text{and} \quad \tilde{C} = \Psi\big(QC(\mathbb{R}), C\big([0, \infty]\big)\big)/K\big(H^2(\mathbb{H})\big).$$

We have

$$M(C_1) = M\big(QC(\mathbb{R})\big) \quad \text{and} \quad M(C_2) = [0, \infty].$$

So we need to determine $(x, y) \in M(QC(\mathbb{R})) \times [0, \infty]$ so that for all $a \in QC(\mathbb{R})$ and $\vartheta \in C([0, \infty])$ with $0 < a, \vartheta \leqslant 1$, we have

$$\hat{a}(x) = \vartheta(y) = 1 \quad \Rightarrow \quad \|T_a D_\vartheta\| = 1 \quad \text{or} \quad \|D_\vartheta T_a\| = 1.$$

For any $x \in M(QC(\mathbb{R}))$ consider $\tilde{x} = x|_{C(\dot{\mathbb{R}})}$ then $\tilde{x} \in M(C(\dot{\mathbb{R}})) = \dot{\mathbb{R}}$. Hence $M(QC(\mathbb{R}))$ is fibered over $\dot{\mathbb{R}}$, i.e.

$$M\big(QC(\mathbb{R})\big) = \bigcup_{t \in \dot{\mathbb{R}}} M_t,$$

where

$$M_t = \big\{x \in M\big(QC(\mathbb{R})\big)\colon \tilde{x} = x|_{C(\dot{\mathbb{R}})} = \delta_t\big\}.$$

Let $x \in M(QC(\mathbb{R}))$ such that $x \in M_t$ with $t \neq \infty$ and $y \in [0, \infty)$. Choose $a \in C(\dot{\mathbb{R}})$ and $\vartheta \in C([0, \infty])$ such that

$$\hat{a}(x) = a(t) = \vartheta(y) = 1, \quad 0 \leqslant a \leqslant 1, \quad 0 \leqslant \vartheta \leqslant 1, \quad a(z) < 1$$

for all $z \in \mathbb{R}\setminus\{t\}$ and $\vartheta(w) < 1$ for all $w \in [0, \infty]\setminus\{y\}$, where both a and ϑ have compact supports. Consider $\|T_a D_\vartheta\|_{H^2}$. Let $\tilde{\vartheta}$ be

$$\tilde{\vartheta}(w) = \begin{cases} \vartheta(w) & \text{if } w \geqslant 0, \\ \vartheta(-w) & \text{if } w < 0, \end{cases}$$

then

$$P M_a D_{\tilde{\vartheta}}|_{H^2} = T_a D_\vartheta,$$

where $P : L^2 \to H^2$ is the orthogonal projection of L^2 onto H^2. So we have

$$\|T_a D_\vartheta\|_{H^2} \leqslant \|M_a D_{\tilde{\vartheta}}\|_{L^2}.$$

By a result of Power (see [17] and also [24]) under these conditions we have

$$\|M_a D_{\tilde{\vartheta}}\|_{L^2} < 1 \quad \Rightarrow \quad \|T_a D_\vartheta\|_{H^2} < 1 \quad \Rightarrow \quad (x, y) \notin M(\tilde{C}), \qquad (23)$$

so if $(x, y) \in M(\tilde{C})$, then either $y = \infty$ or $x \in M_\infty(QC(\mathbb{R}))$.

Let $y = \infty$ and $x \in M(QC(\mathbb{R}))$. Let $a \in QC(\mathbb{R})$ and $\vartheta \in C([0, \infty])$ such that

$$0 \leqslant a, \vartheta \leqslant 1 \quad \text{and} \quad \hat{a}(x) = \vartheta(y) = 1.$$

Consider

$$\begin{aligned} \|D_\vartheta T_a\|_{H^2} &= \big\|\mathcal{F}D_\vartheta T_a \mathcal{F}^{-1}\big\|_{L^2([0,\infty))} = \big\|M_\vartheta \mathcal{F}T_a \mathcal{F}^{-1}\big\|_{L^2([0,\infty))} \\ &= \big\|M_\vartheta \mathcal{F}\big(\mathcal{F}^{-1}M_{\chi_{[0,\infty)}}\mathcal{F}\big)M_a \mathcal{F}^{-1}\big\|_{L^2([0,\infty))} = \big\|M_\vartheta \mathcal{F}M_a \mathcal{F}^{-1}\big\|_{L^2([0,\infty))}. \end{aligned} \qquad (24)$$

Choose $f \in L^2([0, \infty))$ with $\|f\|_{L^2([0,\infty))} = 1$ such that

$$\big\|\big(\mathcal{F}M_a \mathcal{F}^{-1}\big)f\big\| \geqslant 1 - \varepsilon$$

for given $\varepsilon > 0$. Since $\vartheta(\infty) = 1$ there exists $w_0 > 0$ so that

$$1 - \varepsilon \leqslant \vartheta(w) \leqslant 1 \quad \forall w \geqslant w_0.$$

Let $t_0 \geqslant w_0$. Since the support of $(S_{-t_0}\mathcal{F}M_a \mathcal{F}^{-1})f$ lies in $[t_0, \infty)$ where S_t is the translation by t, we have

$$\begin{aligned} &\big\|M_\vartheta\big(S_{-t_0}\mathcal{F}M_a \mathcal{F}^{-1}\big)f\big\|_2 \\ &\quad \geqslant \inf\big\{\vartheta(w)\colon w \in (w_0, \infty)\big\}\big\|\big(\mathcal{F}M_a \mathcal{F}^{-1}\big)f\big\|_2 \geqslant (1 - \varepsilon)^2. \end{aligned} \qquad (25)$$

Since

$$S_{-t_0}\mathcal{F}M_a \mathcal{F}^{-1} = \mathcal{F}M_a \mathcal{F}^{-1}S_{-t_0}$$

and S_{-t_0} is an isometry on $L^2([0,\infty))$ by Eqs. (24) and (25), we conclude that

$$\left\| M_\vartheta \mathcal{F} M_a \mathcal{F}^{-1} \right\|_{L^2([0,\infty))} = \| D_\vartheta T_a \|_{H^2} = 1 \quad \Rightarrow \quad (x,\infty) \in M(\tilde{C}) \quad \forall x \in M(C_1).$$

Now let $x \in M_\infty(QC(\mathbb{R}))$ and $y \in [0,\infty]$. Let $a \in QC(\mathbb{R})$ and $\vartheta \in C([0,\infty])$ such that

$$\hat{a}(x) = \vartheta(y) = 1 \quad \text{and} \quad 0 \leqslant \phi, \vartheta \leqslant 1.$$

By a result of Sarason (see [22, Lemmas 5 and 7]) for a given $\varepsilon > 0$ there is a $\delta > 0$ so that

$$\left| \hat{a}(x) - \frac{1}{2\delta} \int_{-\delta}^{\delta} a \circ \mathfrak{C}^{-1}(e^{i\theta}) \, d\theta \right| \leqslant \varepsilon. \tag{26}$$

Since $\hat{a}(x) = 1$ and $0 \leqslant a \leqslant 1$, this implies that for all $\varepsilon > 0$ there exists $w_0 > 0$ such that $1 - \varepsilon \leqslant a(w) \leqslant 1$ for a.e. w with $|w| > w_0$. Let $\tilde{\vartheta}$ be

$$\tilde{\vartheta}(w) = \begin{cases} \vartheta(w) & \text{if } w \geqslant 0, \\ 0 & \text{if } w < 0. \end{cases}$$

Then we have

$$D_\vartheta T_a = D_{\tilde{\vartheta}} M_a.$$

Let $\varepsilon > 0$ be given. Let $g \in H^2$ so that $\|g\|_2 = 1$ and $\|D_{\tilde{\vartheta}} g\|_2 \geqslant 1 - \varepsilon$. Since $1 - \varepsilon \leqslant a(w) \leqslant 1$ for a.e. w with $|w| > w_0$, there is a $w_1 > 2w_0$ so that

$$\|S_{w_1} g - M_a S_{w_1} g\|_2 \leqslant 2\varepsilon.$$

We have $\|D_{\tilde{\vartheta}}\| = 1$ and this implies that

$$\|D_{\tilde{\vartheta}} S_{w_1} g - D_{\tilde{\vartheta}} M_a S_{w_1} g\|_2 \leqslant 2\varepsilon. \tag{27}$$

Since $S_w D_{\tilde{\vartheta}} = D_{\tilde{\vartheta}} S_w$ and S_w is unitary for all $w \in \mathbb{R}$, we have

$$\|D_{\tilde{\vartheta}} M_a S_{w_1} g\|_2 \geqslant 1 - 3\varepsilon$$

and $(x, y) \in M(\tilde{C})$ for all $x \in M_\infty(C_1)$. $\square$

The Gelfand transform Γ of $\Psi(QC(\mathbb{R}), C([0,\infty]))/K(H^2(\mathbb{H}))$ looks like

$$\Gamma\left(\left[\sum_{j=1}^{\infty} T_{a_j} D_{\vartheta_j} \right]\right)(x,t) = \begin{cases} \sum_{j=1}^{\infty} \hat{a}_j(x) \hat{\vartheta}_j(t) & \text{if } x \in M_\infty(QC(\mathbb{R})), \\ \sum_{j=1}^{\infty} \hat{a}_j(x) \hat{\vartheta}_j(\infty) & \text{if } t = \infty. \end{cases} \tag{28}$$

5. Main results

In this section we characterize the essential spectra of quasi-parabolic composition operators with translation functions in QC class which is the main aim of the paper. In doing this we will heavily use Banach algebraic methods. We start with the following proposition from Hoffman's book (see [11, p. 171]):

Proposition 10. *Let f be a function in $A \subseteq L^\infty(\mathbb{T})$ where A is a closed *-subalgebra of $L^\infty(\mathbb{T})$ which contains $C(\mathbb{T})$. The range of $\hat{f}$ on the fiber $M_\alpha(A)$ consists of all complex numbers ζ with this property: for each neighborhood N of α and each $\varepsilon > 0$, the set*

$$\{|f - \zeta| < \varepsilon\} \cap N$$

has positive Lebesgue measure.

Hoffman states and proves Proposition 10 for $A = L^\infty(\mathbb{T})$ but in fact his proof works for a general C*-subalgebra of $L^\infty(\mathbb{T})$ that contains $C(\mathbb{T})$.

Using a result of Shapiro [25] we deduce the following lemma that might be regarded as the upper half-plane version of that result:

Lemma 11. *If $\psi \in QC(\mathbb{R}) \cap H^\infty(\mathbb{H})$ we have*

$$\mathcal{R}_\infty(\psi) = \mathcal{C}_\infty(\psi)$$

where $\mathcal{C}_\infty(\psi)$ is the cluster set of ψ at infinity which is defined as the set of points $z \in \mathbb{C}$ for which there is a sequence $\{z_n\} \subset \mathbb{H}$ so that $z_n \to \infty$ and $\psi(z_n) \to z$.

Proof. Since the pullback measure $\lambda_0(E) = |\mathfrak{C}(E)|$ is absolutely continuous with respect to the Lebesgue measure λ on $\mathbb{R}$ where $|\cdot|$ denotes the Lebesgue measure on $\mathbb{T}$ and E denotes a Borel subset of $\mathbb{R}$, we have if $\psi \in L^\infty(\mathbb{R})$ then

$$\mathcal{R}_\infty(\psi) = \mathcal{R}_1(\psi \circ \mathfrak{C}^{-1}). \tag{29}$$

By the result of Shapiro (see [25]) if $\psi \in QC(\mathbb{R}) \cap H^\infty(\mathbb{H})$ then we have

$$\mathcal{R}_1(\psi \circ \mathfrak{C}^{-1}) = \mathcal{C}_1(\psi \circ \mathfrak{C}^{-1}).$$

Since

$$\mathcal{C}_1(\psi \circ \mathfrak{C}^{-1}) = \mathcal{C}_\infty(\psi)$$

we have

$$\mathcal{R}_\infty(\psi) = \mathcal{C}_\infty(\psi). \quad \square$$

Firstly we have the following result on the upper half-plane:

Theorem A. Let $\psi \in QC(\mathbb{R}) \cap H^\infty(\mathbb{H})$ such that $\Im(\psi(z)) > \epsilon > 0$ for all $z \in \mathbb{H}$ then for $\varphi(z) = z + \psi(z)$ we have

(1) $C_\varphi : H^2(\mathbb{H}) \to H^2(\mathbb{H})$ is essentially normal,
(2) $\sigma_e(C_\varphi) = \{e^{izt} : t \in [0, \infty], \ z \in \mathcal{C}_\infty(\psi) = \mathcal{R}_\infty(\psi)\} \cup \{0\}$

where $\mathcal{C}_\infty(\psi)$ and $\mathcal{R}_\infty(\psi)$ are the set of cluster points and the local essential range of ψ at ∞ respectively.

Proof. By Proposition 5 we have the following series expansion for C_φ:

$$C_\varphi = \sum_{j=0}^{\infty} \frac{1}{j!} (T_\tau)^j D_{(-it)^j e^{-\alpha t}} \tag{30}$$

where $\tau(z) = i\alpha - \psi(z)$. So we conclude that if $\psi \in QC(\mathbb{R}) \cap H^\infty(\mathbb{H})$ with $\Im(\psi(z)) > \epsilon > 0$ then

$$C_\varphi \in \Psi(QC(\mathbb{R}), C([0, \infty]))$$

where $\varphi(z) = z + \psi(z)$. Since $\Psi(QC(\mathbb{R}), C([0, \infty]))/K(H^2(\mathbb{H}))$ is commutative, for any $T \in \Psi(QC(\mathbb{R}), C([0, \infty]))$ we have $T^* \in \Psi(QC(\mathbb{R}), C([0, \infty]))$ and

$$[TT^*] = [T][T^*] = [T^*][T] = [T^*T]. \tag{31}$$

This implies that $(TT^* - T^*T) \in K(H^2(\mathbb{H}))$. Since $C_\varphi \in \Psi(QC(\mathbb{R}), C([0, \infty]))$ we also have

$$(C_\varphi^* C_\varphi - C_\varphi C_\varphi^*) \in K(H^2(\mathbb{H})).$$

This proves (1).

For (2) we look at the values of $\Gamma[C_\varphi]$ at $M(\Psi(QC(\mathbb{R}), C([0, \infty]))/K(H^2(\mathbb{H})))$ where Γ is the Gelfand transform of $\Psi(QC(\mathbb{R}), C([0, \infty]))/K(H^2(\mathbb{H}))$. By Theorem 9 we have

$$M(\Psi(QC(\mathbb{R}), C([0, \infty]))/K(H^2(\mathbb{H}))) = (M(QC(\mathbb{R})) \times \{\infty\}) \cup (M_\infty(QC(\mathbb{R})) \times [0, \infty]).$$

By Eqs. (28) and (30) we have the Gelfand transform $\Gamma[C_\varphi]$ of C_φ at $t = \infty$ as

$$(\Gamma[C_\varphi])(x, \infty) = \sum_{j=0}^{\infty} \frac{1}{j!} \hat{\tau}(x) \vartheta_j(\infty) = 0 \quad \forall x \in M(QC(\mathbb{R})) \tag{32}$$

since $\vartheta_j(\infty) = 0$ for all $j \in \mathbb{N}$ where $\vartheta_j(t) = (-it)^j e^{-\alpha t}$. We calculate $\Gamma[C_\varphi]$ of C_φ for $x \in M_\infty(QC(\mathbb{R}))$ as

$$(\Gamma[C_\varphi])(x, t) = \left(\Gamma\left[\sum_{j=0}^{\infty} \frac{1}{j!} (T_\tau)^j D_{(-it)^j e^{-\alpha t}} \right] \right)(x, t)$$

$$= \sum_{j=0}^{\infty} \frac{1}{j!} \hat{\tau}(x)^j (-it)^j e^{-\alpha t} = e^{i\hat{\psi}(x)t} \tag{33}$$

for all $x \in M_\infty(QC(\mathbb{R}))$ and $t \in [0, \infty]$. So we have $\Gamma[C_\varphi]$ as the following:

$$\Gamma\big([C_\varphi]\big)(x,t) = \begin{cases} e^{i\hat{\psi}(x)t} & \text{if } x \in M_\infty(QC(\mathbb{R})), \\ 0 & \text{if } t = \infty. \end{cases} \tag{34}$$

Since $\Psi = \Psi(QC(\mathbb{R}), C([0,\infty]))/K(H^2(\mathbb{H}))$ is a commutative Banach algebra with identity, by Eqs. (2) and (34) we have

$$\begin{aligned} \sigma_\Psi\big([C_\varphi]\big) &= \big\{\Gamma[C_\varphi](x,t): (x,t) \in M\big(\Psi\big(QC(\mathbb{R}), C([0,\infty])\big)/K\big(H^2\big)\big)\big\} \\ &= \big\{e^{ix(\psi)t}: x \in M_\infty\big(QC(\mathbb{R})\big), \ t \in [0,\infty)\big\} \cup \{0\}. \end{aligned} \tag{35}$$

Since Ψ is a closed *-subalgebra of the Calkin algebra $\mathcal{B}(H^2(\mathbb{H}))/K(H^2(\mathbb{H}))$ which is also a C*-algebra, by Eq. (1) we have

$$\sigma_\Psi\big([C_\varphi]\big) = \sigma_{\mathcal{B}(H^2)/K(H^2)}\big([C_\varphi]\big). \tag{36}$$

But by definition $\sigma_{\mathcal{B}(H^2)/K(H^2)}([C_\varphi])$ is the essential spectrum of C_φ. Hence we have

$$\sigma_e(C_\varphi) = \big\{e^{i\hat{\psi}(x)t}: x \in M_\infty\big(QC(\mathbb{R})\big), \ t \in [0,\infty)\big\} \cup \{0\}. \tag{37}$$

Now it only remains for us to understand what the set $\{\hat{\psi}(x) = x(\psi): x \in M_\infty(QC(\mathbb{R}))\}$ looks like, where $M_\infty(QC(\mathbb{R}))$ is as defined in Theorem 9. By Proposition 10 and Eq. (29) we have

$$\big\{\hat{\psi}(x): x \in M_\infty\big(QC(\mathbb{R})\big)\big\} = \big\{\psi \circ \hat{\mathfrak{C}}^{-1}(x): x \in M_1(QC)\big\} = \mathcal{R}_1\big(\psi \circ \mathfrak{C}^{-1}\big) = \mathcal{R}_\infty(\psi).$$

By Lemma 11 we have

$$\begin{aligned} \sigma_e(C_\varphi) &= \big\{(\Gamma[C_\varphi])(x,t): (x,t) \in M\big(\Psi\big(QC(\mathbb{R}), C([0,\infty])\big)/K\big(H^2(\mathbb{H})\big)\big)\big\} \\ &= \big\{e^{izt}: t \in [0,\infty), \ z \in \mathcal{C}_\infty(\psi) = \mathcal{R}_\infty(\psi)\big\} \cup \{0\}. \quad \square \end{aligned}$$

Theorem B. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic self-map of $\mathbb{D}$ such that*

$$\varphi(z) = \frac{2iz + \eta(z)(1-z)}{2i + \eta(z)(1-z)}$$

where $\eta \in H^\infty(\mathbb{D})$ with $\Im(\eta(z)) > \epsilon > 0$ for all $z \in \mathbb{D}$. If $\eta \in QC \cap H^\infty$ then we have

(1) $C_\varphi : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ *is essentially normal,*
(2) $\sigma_e(C_\varphi) = \{e^{izt}: t \in [0,\infty], \ z \in \mathcal{C}_1(\eta) = \mathcal{R}_1(\eta)\} \cup \{0\}$

where $\mathcal{C}_1(\eta)$ and $\mathcal{R}_1(\eta)$ are the set of cluster points and the local essential range of η at 1 respectively.

Proof. Using the isometric isomorphism $\Phi : H^2(\mathbb{D}) \to H^2(\mathbb{H})$ introduced in Section 2, if $\varphi : \mathbb{D} \to \mathbb{D}$ is of the form

$$\varphi(z) = \frac{2iz + \eta(z)(1-z)}{2i + \eta(z)(1-z)}$$

where $\eta \in H^\infty(\mathbb{D})$ satisfies $\Im(\eta(z)) > \delta > 0$ then, by Eq. (9), for $\tilde{\varphi} = \mathfrak{C}^{-1} \circ \varphi \circ \mathfrak{C}$ we have $\tilde{\varphi}(z) = z + \eta \circ \mathfrak{C}(z)$ and

$$\Phi \circ C_\varphi \circ \Phi^{-1} = C_{\tilde{\varphi}} + T_{\frac{\eta \circ \mathfrak{C}(z)}{z+i}} C_{\tilde{\varphi}}. \tag{38}$$

For $\eta \in QC$ we have both

$$C_{\tilde{\varphi}} \in \Psi\big(QC(\mathbb{R}), C([0,\infty])\big) \quad \text{and} \quad T_{\frac{\eta \circ \mathfrak{C}(z)}{z+i}} \in \Psi\big(QC(\mathbb{R}), C([0,\infty])\big)$$

and hence

$$\Phi \circ C_\varphi \circ \Phi^{-1} \in \Psi\big(QC(\mathbb{R}), C([0,\infty])\big).$$

Since $\Psi(QC(\mathbb{R}), C([0,\infty]))/K(H^2)$ is commutative and Φ is an isometric isomorphism, (1) follows from the argument following Eq. (5) (C_φ is essentially normal if and only if $\Phi \circ C_\varphi \circ \Phi^{-1}$ is essentially normal) and by Eq. (31).

For (2) we look at the values of $\Gamma[\Phi \circ C_\varphi \circ \Phi^{-1}]$ at $M(\Psi(QC(\mathbb{R}), C([0,\infty]))/K(H^2))$ where Γ is the Gelfand transform of $\Psi(QC(\mathbb{R}), C([0,\infty]))/K(H^2)$. Again applying the Gelfand transform for

$$(x,\infty) \in M\big(\Psi\big(QC(\mathbb{R}), C([0,\infty])\big)/K\big(H^2\big)\big) \subset M\big(QC(\mathbb{R})\big) \times [0,\infty]$$

we have

$$\begin{aligned} &\big(\Gamma\big[\Phi \circ C_\varphi \circ \Phi^{-1}\big]\big)(x,\infty) \\ &\quad = \big(\Gamma[C_{\tilde{\varphi}}]\big)(x,\infty) + \big(\big(\Gamma[T_{\eta \circ \mathfrak{C}}]\big)(x,\infty)\big)\big(\big(\Gamma[T_{\frac{1}{i+z}}]\big)(x,\infty)\big)\big(\big(\Gamma[C_{\tilde{\varphi}}]\big)(x,\infty)\big). \end{aligned}$$

Appealing to Eq. (32) we have $(\Gamma[C_{\tilde{\varphi}}])(x, \infty) = 0$ for all $x \in M(QC(\mathbb{R}))$ hence we have

$$\left(\Gamma\left[\Phi \circ C_{\varphi} \circ \Phi^{-1}\right]\right)(x, \infty) = 0$$

for all $x \in M(QC(\mathbb{R}))$. Applying the Gelfand transform for

$$(x, t) \in M_{\infty}(QC) \times [0, \infty] \subset M\left(\Psi\left(QC(\mathbb{R}), C([0, \infty])\right)/K(H^2)\right)$$

we have

$$\left(\Gamma\left[\Phi \circ C_{\varphi} \circ \Phi^{-1}\right]\right)(x, t) = \left(\Gamma[C_{\tilde{\varphi}}]\right)(x, t) + \left(\left(\Gamma[T_{\eta \circ \mathfrak{C}}]\right)(x, t)\right)\left(\left(\Gamma[T_{\frac{1}{i+z}}]\right)(x, t)\right)\left(\left(\Gamma[C_{\tilde{\varphi}}]\right)(x, t)\right).$$

Since $x \in M_{\infty}(QC(\mathbb{R}))$ we have

$$\left(\Gamma[T_{\frac{1}{i+z}}]\right)(x, t) = \left(\widehat{\frac{1}{i+z}}\right)(x) = x\left(\frac{1}{i+z}\right) = 0.$$

Hence we have

$$\left(\Gamma\left[\Phi \circ C_{\varphi} \circ \Phi^{-1}\right]\right)(x, t) = \left(\Gamma[C_{\tilde{\varphi}}]\right)(x, t)$$

for all $(x, t) \in M_{\infty}(QC(\mathbb{R})) \times [0, \infty]$. Moreover we have

$$\left(\Gamma\left[\Phi \circ C_{\varphi} \circ \Phi^{-1}\right]\right)(x, t) = \left(\Gamma[C_{\tilde{\varphi}}]\right)(x, t)$$

for all $(x, t) \in M(\Psi(QC(\mathbb{R}), C([0, \infty]))/K(H^2))$. Therefore by similar arguments in Theorem A (Eqs. (35) and (36)) we have

$$\sigma_e\left(\Phi \circ C_{\varphi} \circ \Phi^{-1}\right) = \sigma_e(C_{\tilde{\varphi}}).$$

By Theorem A (together with Eq. (37)) and Eq. (5) we have

$$\sigma_e(C_{\varphi}) = \sigma_e(C_{\tilde{\varphi}}) = \left\{e^{izt} : z \in \mathcal{R}_{\infty}(\eta \circ \mathfrak{C}) = \mathcal{R}_1(\eta), \ t \in [0, \infty]\right\}. \qquad \Box$$

We add a few remarks on these theorems:

1. In case $\psi \in H^{\infty}(\mathbb{H}) \cap C(\dot{\mathbb{R}}) \subset QC(\mathbb{R})$ we recall that

$$\sigma_e(C_{\varphi}) = \left\{e^{iz_0 t} : t \in [0, \infty), \ z_0 = \lim_{x \to \infty} \psi(x)\right\} \cup \{0\}$$

where $\varphi(z) = z + \psi(z)$ with $\Im(\psi(z)) > \delta > 0$ for all $z \in \mathbb{H}$. Hence we recapture a result from the work of Kriete and Moorhouse [13] and also from the work of Bourdon, Levi, Narayan and Shapiro [3]. However Theorems A and B allow us to compute the essential spectra of operators not considered by [3] or [13]. To illustrate this point we give an example of $\psi \in QC(\mathbb{R}) \cap H^{\infty}(\mathbb{H})$ so that $\psi \notin C(\dot{\mathbb{R}})$. Recall that an analytic function f on the unit disc is in the Dirichlet space if and only if $\int_{\mathbb{D}} |f'(z)|^2 \, dA(z) < \infty$. We will use the following proposition (see also [26]):

Proposition 12. [1] *Every bounded analytic function of the unit disc that is in the Dirichlet space is in QC.*

Proof. By the VMO version of Fefferman's theorem on BMO (see Chapter 5 of [22]): $f \in VMOA$ if and only if $|f'(z)|^2(1 - |z|) \, dA(z)$ is a vanishing Carleson measure. And by a result of Sarason (see [20] and [22]) we have $VMOA \cap L^{\infty} = ([H^{\infty} + C(\mathbb{T})] \cap [\overline{H^{\infty} + C(\mathbb{T})}]) \cap H^{\infty}$. So we only need to show that if f is in the Dirichlet space then $|f'(z)|^2(1 - |z|) \, dA(z)$ is a vanishing Carleson measure: Let $S(I) = \{re^{i\theta} : e^{i\theta} \in I \text{ and } 1 - \frac{|I|}{2} < r < 1\}$ be the Carleson window associated to the arc $I \subseteq \mathbb{T}$ where $|I|$ denotes the length of the arc I. Then for any $z \in S(I)$ we have $1 - |z| < |I|$. So we have

$$\int_{S(I)} |f'(z)|^2(1 - |z|) \, dA(z) \leqslant |I| \int_{S(I)} |f'(z)|^2 \, dA(z).$$

Let I_n be a sequence of decreasing arcs so that $|I_n| \to 0$ and let $\chi_{S(I_n)}$ be the characteristic function of $S(I_n)$. Then $\chi_{S(I_n)}$ is in the unit ball of $L^{\infty}(\mathbb{D}, dA)$ which is the dual of $L^1(\mathbb{D}, dA)$. Hence by Banach–Alaoglu theorem, there is a subsequence $\chi_{S(I_{n_k})}$ so that

$$\int_{\mathbb{D}} \chi_{S(I_{n_k})} g \, dA = \int_{S(I_{n_k})} g \, dA \to \int_{\mathbb{D}} \phi g \, dA$$

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[1] The author is indebted to Professor Joe Cima for pointing out this proposition.
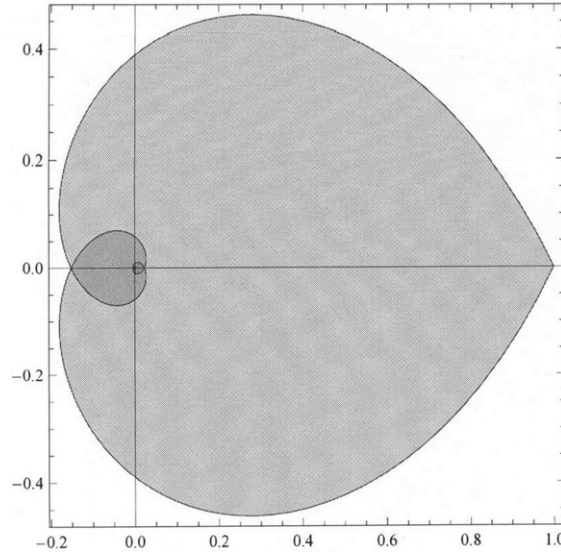
Fig. 1. Essential spectra of C_φ for $a = -5$ and $b = 5$.

as $k \to \infty$ for all $g \in L^1(\mathbb{D}, dA)$ for some $\phi \in L^\infty(\mathbb{D}, dA)$. In particular if we take $g \in L^\infty(\mathbb{D}, dA) \subset L^1(\mathbb{D}, dA)$ we observe that

$$\left| \int\limits_{S(I_{n_k})} g \, dA \right| \leqslant A\big(S(I_{n_k})\big) \|g\|_\infty$$

so we have $|\int_{S(I_{n_k})} g \, dA| \to 0$ as $k \to \infty$ if $g \in L^\infty(\mathbb{D}, dA)$. Since $L^\infty(\mathbb{D}, dA)$ is dense in $L^1(\mathbb{D}, dA)$ we have $\int_\mathbb{D} \phi g \, dA = 0$ for all $g \in L^1(\mathbb{D}, dA)$ and hence $\phi \equiv 0$. Since $|f'|^2 \in L^1(\mathbb{D}, dA)$ we have $\int_{S(I_n)} |f'|^2 \, dA \to 0$ as $n \to \infty$. Since $L^1(\mathbb{D}, dA)$ is separable the weak* topology of the unit ball of $L^\infty(\mathbb{D}, dA)$ is metrizable. This implies that

$$\lim_{|I| \to 0} \int\limits_{S(I)} |f'|^2 \, dA = 0.$$

This proves the proposition. □

We are ready to construct our example of a "quasi-parabolic" composition operator which has thick essential spectrum: Let D be the simply connected region bounded by the curve $\alpha : [-\pi, \pi] \to \mathbb{C}$ such that α is continuous on $[-\pi, 0) \cup (0, \pi]$ with

$$\alpha(t) = \begin{cases} 3i + i(t + i\frac{(b-a)}{2}\sin(\frac{3\pi^2}{4t})) + \frac{(a+b)}{2} & \text{if } t \in (0, \frac{\pi}{2}], \\ i(3 + t) + \frac{(a+b)}{2} & \text{if } t \in [-\frac{\pi}{2}, 0] \end{cases}$$

and $\alpha(\pi) = \alpha(-\pi) = (1 + \frac{\pi}{2}) + i(3 + \frac{\pi}{2}) + \frac{(a+b)}{2}$, $a < b$. By the Riemann mapping theorem there is a conformal mapping $\tilde\psi : \mathbb{D} \to D$ that is bi-holomorphic. One can choose $\tilde\psi$ to satisfy $\lim_{\theta \to 0^-} \tilde\psi(e^{i\theta}) = 3i$. Since D has finite area and $\tilde\psi$ is one-to-one and onto, $\tilde\psi$ is in the Dirichlet space and hence, by Proposition 12, $\tilde\psi \in QC$. Let $\psi = \tilde\psi \circ \mathfrak{C}$. Then $\psi \in QC(\mathbb{R})$ and $\psi \notin H^\infty(\mathbb{H}) \cap C(\dot{\mathbb{R}})$. We observe that

$$\mathcal{C}_\infty(\psi) = \big\{3i + x : x \in [a, b]\big\}.$$

So for $C_\varphi : H^2(\mathbb{H}) \to H^2(\mathbb{H})$ and for $C_{\tilde\varphi} : H^2(\mathbb{D}) \to H^2(\mathbb{D})$, we have (see Figs. 1, 2)

$$\sigma_e(C_\varphi) = \sigma_e(C_{\tilde\varphi}) = \big\{e^{i(3i+x)t} : t \in [0, \infty), \ x \in [a, b]\big\} \cup \{0\}$$

where

$$\varphi(z) = z + \psi(z) \quad \text{and} \quad \tilde\varphi = \mathfrak{C} \circ \varphi \circ \mathfrak{C}^{-1}.$$
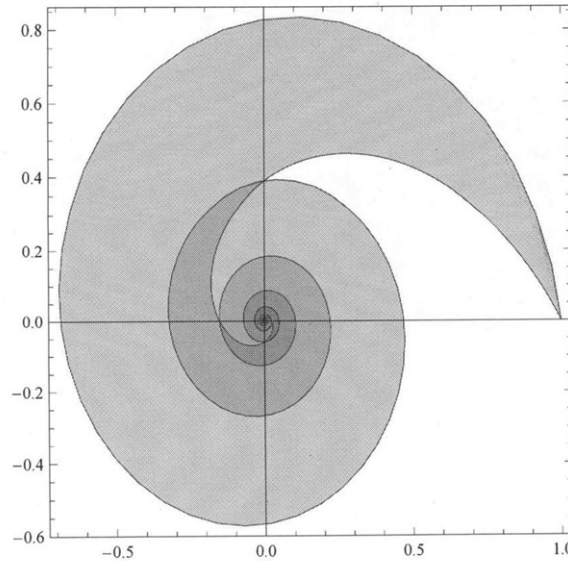
Fig. 2. Essential spectra of C_φ for $a = 5$ and $b = 25$.

2. We recall that for any $a, \vartheta \in C(\dot{\mathbb{R}})$ the operator $M_a D_\vartheta$ is Hilbert Schmidt on $L^2(\mathbb{R})$ (see pp. 482–484 of [17]). Hence for $a \in C(\dot{\mathbb{R}})$ and $\vartheta \in C([0, \infty])$ with $a(\infty) = \vartheta(\infty) = 0$ the operator $T_a D_\vartheta$ is compact on $H^2(\mathbb{H})$. Combining this fact with Eqs. (30) and (38) we conclude that for $\varphi : \mathbb{D} \to \mathbb{D}$ with

$$\tilde{\varphi}(z) = \mathfrak{C}^{-1} \circ \varphi \circ \mathfrak{C}(z) = z + \psi(z),$$

$\psi \in H^\infty$ and $\Im(\psi(z)) > \epsilon > 0$, the operator $\Phi \circ C_\varphi \circ \Phi^{-1} - C_{\tilde{\varphi}}$ is compact on $H^2(\mathbb{H})$.

6. Further results

In this last part of the paper we will prove a more general result about C_φ with $\varphi(z) = z + \psi(z)$, $\psi \in H^\infty$: We will show that if $\varphi(z) = z + \psi(z)$ with $\psi \in H^\infty$ and $\Im(\psi(z)) > \epsilon > 0$ for all $z \in \mathbb{H}$ then

$$\sigma_e(C_\varphi) \supseteq \left\{ e^{izt} : t \in [0, \infty), \ z \in \mathcal{R}_\infty(\psi) \right\} \cup \{0\}$$

where $\mathcal{R}_\infty(\psi)$ is the local essential range of ψ at infinity. We use the following Theorems 13 and 14 to prove the above result. Theorem 13 is due to Axler [2] and we give a full proof of Theorem 14 using Power's theorem and Axler's theorem.

Theorem 13 (AXLER'S THEOREM). *Let* $f \in L^\infty$, *then there is a Blaschke product* B *and* $b \in H^\infty + C$ *so that* $f = \bar{B}b$.

Theorem 14. *Let*

$$\Psi\left(L^\infty(\mathbb{R}), C([0, \infty])\right) = C^*\left(\mathcal{T}\left(L^\infty(\mathbb{R})\right) \cup F\right)$$

be the C^*-algebra generated by Toeplitz operators with L^∞ symbols and Fourier multipliers on $H^2(\mathbb{H})$ and let $M(\Psi)$ be the space of multiplicative linear functionals on $\Psi(L^\infty(\mathbb{R}), C([0, \infty]))/K(H^2(\mathbb{H}))$. Then we have

$$M(\Psi) \cong \left(M_\infty\left(L^\infty(\mathbb{R})\right) \times [0, \infty]\right) \cup \left(M\left(L^\infty(\mathbb{R})\right) \times \{\infty\}\right)$$

where

$$M_\infty\left(L^\infty(\mathbb{R})\right) = \left\{x \in M\left(L^\infty(\mathbb{R})\right) : x|_{C(\dot{\mathbb{R}})} = \delta_\infty\right\}$$

is the fiber of $M(L^\infty(\mathbb{R}))$ at infinity.

Proof. Consider the symbol map

$$\Sigma : L^\infty(\mathbb{R}) \to \mathcal{T}\left(L^\infty(\mathbb{R})\right)$$

defined by $\Sigma(\psi) = T_\psi$. Clearly Σ is injective. Let $\psi_1 = \varphi_1 \bar{\varphi}_2$ and $\psi_2 = \varphi_3 \bar{\varphi}_4$ with $\varphi_j \in H^\infty$, $j \in \{1, 2, 3, 4\}$. Then $T_{\overline{\varphi_2 \varphi_4}} = T_{\bar{\varphi}_2} T_{\bar{\varphi}_4}$. This implies that

$$T_{\psi_1 \psi_2} - T_{\psi_1} T_{\psi_2} = T_{\bar{\varphi}_2}[T_{\bar{\varphi}_4}, T_{\varphi_1}]T_{\varphi_3}.$$

Since $L^\infty(\mathbb{R})$ is spanned by such ψ_1 and ψ_2's we have

$$T_{\psi_1 \psi_2} - T_{\psi_1} T_{\psi_2} \in com\big(\mathcal{T}\big(L^\infty(\mathbb{R})\big)\big)$$

for all $\psi_1, \psi_2 \in L^\infty(\mathbb{R})$ (for more details see [15, p. 345]). Since Σ is injective, $q \circ \Sigma$ is a C*-algebra isomorphism from $L^\infty(\mathbb{R})$ onto $\mathcal{T}(L^\infty(\mathbb{R}))/com(\mathcal{T}(L^\infty(\mathbb{R})))$ where

$$q : \mathcal{T}\big(L^\infty(\mathbb{R})\big) \to \mathcal{T}\big(L^\infty(\mathbb{R})\big)/com\big(\mathcal{T}\big(L^\infty(\mathbb{R})\big)\big)$$

is the quotient map. So $M(\mathcal{T}(L^\infty(\mathbb{R}))) = M(L^\infty(\mathbb{R}))$. Since

$$K\big(H^2\big) \subset com\big(\mathcal{T}\big(L^\infty(\mathbb{R})\big)\big)$$

we also have

$$M\big(\mathcal{T}\big(L^\infty(\mathbb{R})\big)/K\big(H^2\big)\big) \cong M\big(L^\infty(\mathbb{R})\big).$$

Hence using Power's theorem we can identify

$$M(\Psi) \cong \big(M_\infty\big(L^\infty(\mathbb{R})\big) \times [0,\infty]\big) \cup \big(M\big(L^\infty(\mathbb{R})\big) \times \{\infty\}\big)$$

where

$$M_\infty\big(L^\infty(\mathbb{R})\big) = \big\{x \in M\big(L^\infty(\mathbb{R})\big)\colon x|_{C(\dot{\mathbb{R}})} = \delta_\infty\big\}$$

is the fiber of $M(L^\infty(\mathbb{R}))$ at infinity:

Let $(x, y) \in M(L^\infty(\mathbb{R})) \times [0,\infty)$ so that $x|_{C(\dot{\mathbb{R}})} = \delta_t$ with $t \neq \infty$. Choose $a \in C(\dot{\mathbb{R}})$ and $\vartheta \in C([0,\infty])$ having compact supports such that

$$a(t) = \vartheta(y) = 1, \quad 0 \leqslant a \leqslant 1,\ 0 \leqslant \vartheta \leqslant 1,\ a(z) < 1$$

for all $z \in \mathbb{R}\setminus\{t\}$ and $\vartheta(w) < 1$ for all $w \in [0,\infty]\setminus\{y\}$. Using the same arguments as in Eq. (23) we have

$$\|T_a D_\vartheta\|_{H^2} < 1$$

which implies that

$$(x, y) \notin M\big(\Psi\big(\mathcal{T}\big(L^\infty(\mathbb{R})\big), C\big([0,\infty]\big)\big)/K\big(H^2(\mathbb{H})\big)\big).$$

So if $(x, y) \in M(\Psi(L^\infty(\mathbb{R}), C([0,\infty]))/K(H^2(\mathbb{H})))$ then either $y = \infty$ or $x \in M_\infty(L^\infty(\mathbb{R}))$.

Let $y = \infty$ and $x \in M(L^\infty(\mathbb{R}))$. Let $a \in L^\infty(\mathbb{R})$ and $\vartheta \in C([0,\infty])$ such that

$$0 \leqslant \vartheta \leqslant 1,\ 0 \leqslant a \leqslant 1 \quad \text{and} \quad \hat{a}(x) = \vartheta(y) = 1.$$

Using the same arguments as in Eqs. (24) and (25) together with the fact that

$$\lim_{w \to \infty} \|K S_w f\|_2 = 0 \tag{39}$$

for any $K \in K(H^2(\mathbb{H}))$ and for all $f \in H^2(\mathbb{H})$, we have

$$\|D_\vartheta T_a\|_e = \|D_\vartheta T_a\|_{H^2} = 1.$$

This implies that

$$(x, \infty) \in M\big(\Psi\big(L^\infty(\mathbb{R}), C\big([0,\infty]\big)\big)/K\big(H^2(\mathbb{H})\big)\big)$$

for all $x \in M(L^\infty(\mathbb{R}))$.

Now let $x \in M_\infty(L^\infty(\mathbb{R}))$ and $y \in [0,\infty]$. Let $a \in L^\infty(\mathbb{R})$ and $\vartheta \in C([0,\infty])$ such that

$$\hat{a}(x) = \vartheta(y) = 1 \quad \text{and} \quad 0 \leqslant \vartheta \leqslant 1,\ 0 \leqslant a \leqslant 1.$$

Since $a \in L^\infty$, by Axler's theorem there is a Blaschke product B and $b \in H^\infty + C$ so that $a = \bar{B}b$. Since $|\hat{\bar{B}}|(x) = |\hat{\bar{B}}(x)| = 1$ we have $|\hat{b}(x)| = 1$. We have $M(H^\infty + C) \cong M(H^\infty)/\mathbb{H}$ (see Corollary 6.42 of [8]), the Poisson kernel is also asymptotically multiplicative on $H^\infty + C$ (see Lemma 6.44 of [8]) and by Carleson's Corona theorem we observe that Eq. (26) is also valid for b for any $\varepsilon > 0$. Since $0 \leqslant |b| \leqslant 1$ this implies that there is a $w_0 > 0$ so that $1 - \varepsilon < |b(w)| < 1$ for a.e. w with $|w| > w_0$. After that we use the same arguments as in Eqs. (27) and (39) and since $|\bar{B}| = 1$ a.e., we have

$$\|D_\vartheta T_{\bar{B}b}\|_e = \|D_\vartheta M_{\bar{B}} M_b\|_{H^2} = 1.$$

This implies that

$$(x, y) \in M\big(\Psi/K\big(H^2(\mathbb{H})\big)\big)$$

for all $x \in M_\infty(L^\infty(\mathbb{R}))$. $\square$

Now we state and prove the main result of this section:

Theorem C. *Let $\varphi : \mathbb{H} \to \mathbb{H}$ be an analytic self-map of $\mathbb{H}$ such that $\varphi(z) = z + \psi(z)$ with $\psi \in H^\infty(\mathbb{H})$ and $\Im(\psi(z)) > \epsilon > 0$ for all $z \in \mathbb{H}$. Then for $C_\varphi : H^2(\mathbb{H}) \to H^2(\mathbb{H})$ we have*

$$\sigma_e(C_\varphi) \supseteq \left\{ e^{izt} : z \in \mathcal{R}_\infty(\psi), \ t \in [0, \infty) \right\} \cup \{0\}$$

where $\mathcal{R}_\infty(\psi)$ is the local essential range of ψ at infinity.

Proof. By Proposition 5 we have if $\varphi(z) = z + \psi(z)$ with $\psi \in H^\infty(\mathbb{H})$ and $\Im(\psi(z)) > \delta > 0$ for all $z \in \mathbb{H}$ then

$$C_\varphi \in \Psi\left(L^\infty(\mathbb{R}), C([0, \infty])\right).$$

By Theorem 14 we have

$$M\left(\Psi\left(L^\infty(\mathbb{R}), C([0, \infty])\right) / K\left(H^2(\mathbb{H})\right)\right) \cong \left(M_\infty\left(L^\infty(\mathbb{R})\right) \times [0, \infty]\right) \cup \left(M\left(L^\infty(\mathbb{R})\right) \times \{\infty\}\right).$$

If $y = \infty$ then for $\varphi(z) = z + \psi(z)$ with $\psi \in H^\infty$ and $\Im(\psi(z)) > \delta > 0$ we have

$$(x, \infty)\left([C_\varphi]\right) = \sum_{j=0}^\infty \frac{1}{j!} \left(\hat{\tau}(x)\right)^j \hat{\vartheta}_j(\infty) = 0$$

since $\hat{\vartheta}_j(\infty) = 0$ for all j where τ and ϑ_j are as in Eq. (32). If $x \in M_\infty(L^\infty(\mathbb{R}))$ then as in Eq. (33), we have $(x, y)([C_\varphi]) = e^{i\hat{\psi}(x)y}$. By Eqs. (34) and (3) we have

$$\sigma_e(C_\varphi) \supseteq \left\{ x([C_\varphi]) : x \in M\left(\Psi\left(L^\infty(\mathbb{R}), C([0, \infty])\right) / K\left(H^2(\mathbb{H})\right)\right) \right\}$$

and this implies that

$$\sigma_e(C_\varphi) \supseteq \left\{ e^{i\hat{\psi}(x)t} : x \in M_\infty\left(L^\infty(\mathbb{R})\right), \ t \in [0, \infty) \right\} \cup \{0\}.$$

By Proposition 10 we have

$$\left\{ \hat{\psi}(x) : x \in M_\infty\left(L^\infty(\mathbb{R})\right) \right\} = \mathcal{R}_\infty(\psi)$$

where $\mathcal{R}_\infty(\psi)$ is the local essential range of ψ at infinity. Hence we have

$$\sigma_e(C_\varphi) \supseteq \left\{ e^{izt} : z \in \mathcal{R}_\infty(\psi), \ t \in [0, \infty) \right\} \cup \{0\}. \qquad \square$$

In the above theorem we do not have in general equality since $\Psi(L^\infty(\mathbb{R}), C([0, \infty]))$ is not commutative. And we also have the corresponding result for the unit disc:

Theorem D. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic self-map of $\mathbb{D}$ such that*

$$\varphi(z) = \frac{2iz + \eta(z)(1 - z)}{2i + \eta(z)(1 - z)}$$

where $\eta \in H^\infty(\mathbb{D})$ with $\Im(\eta(z)) > \epsilon > 0$ for all $z \in \mathbb{D}$. Then for $C_\varphi : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ we have

$$\sigma_e(C_\varphi) \supseteq \left\{ e^{izt} : t \in [0, \infty), \ z \in \mathcal{R}_1(\eta) \right\} \cup \{0\}$$

where $\mathcal{R}_1(\eta)$ is the local essential range of η at 1.

Proof. Repeating the same arguments as in the proof of Theorem B, we have

$$\Phi \circ C_\varphi \circ \Phi^{-1} \in \Psi\left(L^\infty(\mathbb{R}), C([0, \infty])\right) / K\left(H^2(\mathbb{H})\right).$$

Take $(x, \infty) \in M(\Psi(L^\infty(\mathbb{R}), C([0, \infty])) / K(H^2(\mathbb{H})))$, since $(x, \infty)([C_{\tilde{\varphi}}]) = 0$ we have

$$(x, \infty)\left(\left[\Phi \circ C_\varphi \circ \Phi^{-1}\right]\right) = 0.$$

For $(x, y) \in M_\infty(L^\infty(\mathbb{R})) \times [0, \infty]$ we have $(x, y)([T_{\frac{1}{i+z}}]) = 0$ and hence we have

$$(x, y)\left(\left[\Phi \circ C_\varphi \circ \Phi^{-1}\right]\right) = (x, y)\left([C_{\tilde{\varphi}}]\right)$$

for all $x \in M_\infty(L^\infty(\mathbb{R}))$. Therefore by Eq. (5) and Theorem C we have

$$\sigma_e(C_\varphi) = \sigma_e\left(\Phi \circ C_\varphi \circ \Phi^{-1}\right) \supseteq \left\{ e^{izt} : t \in [0, \infty), z \in \mathcal{R}_1(\eta) = \mathcal{R}_\infty(\eta \circ \mathfrak{C}) \right\} \cup \{0\}. \qquad \square$$

Acknowledgments

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