Presented at ISAB 2010
Fukushima July 22-24, 2010

Motion Control-A SMC Approach

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1. Abstract

Motion control involves many diversified control problems of complex nonlinear systems. In this paper we will be addressing the SMC approach for multi-body mechanical systems control. The main feature of the SMC is constraint of the system motion into manifold in system state space. It will be shown that usage of the SMC methods is a natural way of addressing problems in motion control including constrained systems, redundant systems and functionally related systems to name some. The consistent application of the SMC methods leads to natural decomposition of system motion for redundant tasks and allows simple, straight forward dynamical decoupling of the multiple tasks.

2. Introduction

Both theory of SMC systems and the control of multi-body systems are well developed and vast literature is available for both. SMC has been originally developed for dynamical systems [1] with discontinuous control and as such did not gain easy acceptance in the control of mechanical systems. Early works attempts of SMC application to mechanical systems [2] resulted in the chattering phenomena and triggered different approaches to control smoothing techniques. Later works [3] in SMC systems demonstrated applicability of these methods to electromechanical and mechanical systems and illustrated ways of eliminating or reducing chattering problems. The core idea of SMC is to constraint system motion in a manifold in system state space and reaching that manifold in the final time. Dynamics of constrained motion is of lower dimension then the original system and methods had been developed to write equations of motion of the constrained system, along with the efficient design procedures [2].

The body of literature on multi-body systems control is vast and includes solutions in many different frameworks. A comprehensive, consistent and well presented, different aspects of robotic systems control can be found in [4]. Basic feature of robotic multi-body systems is nonlinear dynamics with complex interconnecting terms and linearity in control input. The control problems in both configuration space and in the operational space of multi-body systems can be formulated as restriction of the motion to stay in some hyper-surface defined by the desired changes of the state variables. This formulation includes the constrained systems also. Such a structure of multi-body systems and the mathematical formulation of the control goal is consistent with SMC methods.

Paper is organized in the following way. In the third section the configuration space description of multi-body systems and constraint/task formulation will be given. Additionally basics of SMC methods will be shown. In section four constraints in configuration and in the operational space will be discussed. It the fifth section the operational space control of redundant multi-body systems in SMC framework will be discussed.

3. System Description and Basics of SMC

3.1. Basics of SMC

Let dynamic system

$$\mathbf{E}\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},t) + \mathbf{B}(\mathbf{x},t)\mathbf{u} + \mathbf{d}(\mathbf{x},t) \tag{1}$$

where $\mathbf{E} \in \Re^{n\times n}$ is full rank matrix, $\mathbf{x} \in \Re^{n\times 1}$ stands for the state vector, $\mathbf{f}(\mathbf{x},t) \in \Re^{n\times 1}$ is vector function of state and time, $\mathbf{B}(\mathbf{x},t) \in \Re^{n\times m}$ is full column rank control distribution matrix, $\mathbf{u} \in \Re^{m\times 1}$ is the control vector and $\mathbf{d}(\mathbf{x},t) \in \Re^{n\times 1}$ is disturbance.

Let control system requirements are satisfied if system (1) is enforced to exhibit motion in manifold (so-called sliding mode)

$$\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{0} \in \Re^{m\times 1} \tag{2}$$

Here $\boldsymbol{\sigma}^T(\mathbf{x}) = [\sigma_1 \quad \dots \quad \sigma_m]$ is assumed continuous. The equations of motion are derived using so-called equivalent control method [1]. In this method control is found as solution of the algebraic equation $\dot{\boldsymbol{\sigma}}(\mathbf{x},\mathbf{u}_{eq}) = 0$ and substituted into (1). Having $(\partial\boldsymbol{\sigma}(\mathbf{x})/\partial\mathbf{x}) = \mathbf{G}$ as a full row rank matrix the equivalent control can be determined as

$$\mathbf{u}_{eq} = -\left(\mathbf{G}\mathbf{E}^{-1}\mathbf{B}\right)^{-1}\mathbf{G}\mathbf{E}^{-1}\left(\mathbf{f}(\mathbf{x},t) + \mathbf{d}(\mathbf{x},t)\right) \tag{3}$$

Control (3) has a specific meaning – if applied to system (1) it will enforce motion satisfying $\dot{\boldsymbol{\sigma}}(\mathbf{x},\mathbf{u}_{eq}) = 0$ - thus generating no change in the distance from the manifold (2). Assuming manifold (2) consistent initial conditions $\boldsymbol{\sigma}(\mathbf{0}) = \mathbf{0}$ application of the equivalent control (3) to system (1) gives equations of motion as

$$\begin{aligned} &\mathbf{E}\dot{\mathbf{x}} = \left[\mathbf{I} - \mathbf{B}\left(\mathbf{G}\mathbf{E}^{-1}\mathbf{B}\right)^{-1}\mathbf{G}\mathbf{E}^{-1}\right]\left(\mathbf{f}(\mathbf{x},t) + \mathbf{d}(\mathbf{x},t)\right) \\ &\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{0} \end{aligned} \tag{4}$$

System (4) describes $(n-m)$ order dynamics. The projection matrix $\mathbf{P} = \mathbf{I} - \mathbf{B}\left(\mathbf{G}\mathbf{E}^{-1}\mathbf{B}\right)^{-1}\mathbf{G}\mathbf{E}^{-1}$ satisfies $\mathbf{P}\mathbf{B} = \mathbf{0}$ and $\mathbf{G}\mathbf{E}^{-1}\mathbf{P} = \mathbf{0}$. From these conditions it is easy to determine that if $\left(\mathbf{f}(\mathbf{x},t) + \mathbf{d}(\mathbf{x},t)\right) = \mathbf{B}\boldsymbol{\xi} + \boldsymbol{\upsilon}$ then dynamics (4) will be reduced to

$$\mathbf{E}\dot{\mathbf{x}} = \mathbf{P}\boldsymbol{\upsilon}, \quad \boldsymbol{\sigma}(\mathbf{x}) = \mathbf{0} \tag{5}$$

Thus, sliding mode motion will not depend on the component $\mathbf{B}\boldsymbol{\xi}$ of the so-called matching system dynamics and the matching disturbances.

Knowing equations of motion in sliding mode allows selection of manifold (2) to satisfy closed loop specification. Note that this can be realized prior to control input selection. Selection of the control input is now related to enforcing the stability of equilibrium $\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{0}$, or in other words enforcing sliding mode in manifold (2). Let Lyapunov function candidate be $V = \boldsymbol{\sigma}^T \boldsymbol{\sigma} / 2$. Stability conditions require derivative of the Lyapunov function candidate to be negative definite. Let $\dot{V} = \boldsymbol{\sigma}^T \dot{\boldsymbol{\sigma}} = -\rho \boldsymbol{\sigma}^T \boldsymbol{\Psi}(\boldsymbol{\sigma}) < 0$ where function $\boldsymbol{\Psi}(\boldsymbol{\sigma})$ satisfies component-vise conditions $sign(\Psi_i(\boldsymbol{\sigma})) = sign(\sigma_i)$ and let $\rho > 0$. Then control input

$$\mathbf{u} = \mathbf{u}_{eq} - \rho \left(\mathbf{G}\mathbf{E}^{-1}\mathbf{B} \right)^{-1} \boldsymbol{\Psi}(\boldsymbol{\sigma}) \tag{6}$$

will enforce stability of the equilibrium $\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{0}$. Strictly speaking sliding mode will be enforced in manifold (2) and is reached in finite time. In continuous time implementation function $\boldsymbol{\Psi}(\boldsymbol{\sigma})$ should be discontinuous in order to guaranty the existence of the sliding mode. In the discrete-time implementation $\boldsymbol{\Psi}(\boldsymbol{\sigma})$ can be selected continuous (in the discrete-time sense) [3].

For known $(\mathbf{u}, \boldsymbol{\sigma})$ and matrix $\left(\mathbf{G}\mathbf{E}^{-1}\mathbf{B} \right)$ equivalent control can be estimated from projection of the system dynamics into manifold (2)

$$\dot{\boldsymbol{\sigma}} = \mathbf{G}\dot{\mathbf{x}} = \left(\mathbf{G}\mathbf{E}^{-1}\mathbf{B} \right)\left(\mathbf{u} - \mathbf{u}_{eq} \right) \tag{7}$$

Using estimated equivalent acceleration instead of the exact one allows control input evaluation based on the measured function $(\boldsymbol{\sigma})$ and matrix $\left(\mathbf{G}\mathbf{E}^{-1}\mathbf{B} \right)$.

3.2. *Dynamics of Multi-body Systems*

Configuration space dynamics of multi-body, rigid, fully actuated $n-$dof system can be expressed as

$$\mathbf{A}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{b}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{g}(\mathbf{q}) = \boldsymbol{\tau} \tag{8}$$

Where $\mathbf{q} \in \Re^{n \times 1}$ denotes the configuration vector; $\mathbf{A}(\mathbf{q}) \in \Re^{n \times n}$ stands for positive definite kinetic energy matrix (sometimes termed inertia matrix) with bounded strictly positive elements $0 < a_{ij}^- \le a_{ij}(\mathbf{q}) \le a_{ij}^+$ hence $A^- \le \|\mathbf{A}(\mathbf{q})\| \le A^+$, where A^-, A^+ are two known scalars with bounds $0 < A^- \le A^+$; $\mathbf{b}(\mathbf{q}, \dot{\mathbf{q}}) \in \Re^{n \times 1}$ stands for vector Coriolis forces, viscous friction and centripetal forces and is bounded by $\|\mathbf{b}(\mathbf{q}, \dot{\mathbf{q}})\| \le b^+$; $\mathbf{g}(\mathbf{q}) \in \Re^{n \times 1}$ stands for vector of gravity terms bounded by $\|\mathbf{g}(\mathbf{q})\| \le g^+$; $\boldsymbol{\tau} \in \Re^{nx1}$ stands for vector of generalized joint forces bounded by $\|\boldsymbol{\tau}\| \le \tau^+$ (In further text we will sometimes refer to $\boldsymbol{\tau}$ as the control vector or input force vector). Positive scalars A^-, A^+ , b^+ , τ^+ are assumed known where any induced matrix or vector norm may be used in their definition. The kinetic energy matrix depends on the current system configuration thus it reflects current system configuration.

4. *Control of Multi-body Systems*

4.1. *Constrained Systems*

Let us analyze behavior of system (8) under assumption that motion is required to satisfy $m < n$ hard holonomic constraints $\phi(\mathbf{q}) = \mathbf{0} \in \Re^{m \times 1}, m < n$. Jacobian associated with constraints is defined as $\boldsymbol{\Phi} = \left(\partial \phi(\mathbf{q}) / \partial \mathbf{q} \right) \in \Re^{m \times n}, m < n$ and is assumed to have full row rank. Dynamics of system (8) in contact with constraint manifold can be described by:

$$\mathbf{A}\ddot{\mathbf{q}} + \mathbf{b}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{g}(\mathbf{q}) - \boldsymbol{\Phi}^T \boldsymbol{\lambda} = \boldsymbol{\tau} \tag{9}$$

Here $\boldsymbol{\lambda} \in \Re^{m \times 1}$ stands for vector of Lagrange multipliers. Lagrange multipliers stand for the force vector needed to maintain system (9) in constraint manifold. There are many methods to determine Lagrange multipliers [5]. Here Lagrange multiplier $\boldsymbol{\lambda}$ will be taken as virtual control input in system (9).

Satisfying constraints $\phi(\mathbf{q}) = \mathbf{0}$ can be interpreted as enforcing zero velocity in constrained directions, or equivalently selecting Lagrange multipliers that enforce stability in manifold $\boldsymbol{\sigma}(\mathbf{q}, \dot{\mathbf{q}}) = \dot{\phi}(\mathbf{q}) = \boldsymbol{\Phi}\dot{\mathbf{q}} = \mathbf{0}$. (Note that here matrix $\boldsymbol{\Phi}$ plays the same role as matrix $\mathbf{G}$ in (7). If Lagrange multipliers are taken as control in (9), matrix $\boldsymbol{\Phi}^T$ can be interpreted as control distribution matrix. The acceleration in the constrained direction is

$$\ddot{\phi}(\mathbf{q}) = \boldsymbol{\Phi}\mathbf{A}^{-1}(\boldsymbol{\tau} - \mathbf{b} - \mathbf{g}) + \boldsymbol{\Phi}\mathbf{A}^{-1}\boldsymbol{\Phi}^T \boldsymbol{\lambda} + \dot{\boldsymbol{\Phi}}\dot{\mathbf{q}} \tag{10}$$

Lagrange multipliers satisfying $\ddot{\phi}(\mathbf{q}(t)) = \mathbf{0}$ can be determined as

$$\boldsymbol{\lambda} = -\left(\boldsymbol{\Phi}\mathbf{A}^{-1}\boldsymbol{\Phi}^T \right)^{-1} \boldsymbol{\Phi}\mathbf{A}^{-1}\left((\boldsymbol{\tau} - \mathbf{b} - \mathbf{g}) + \dot{\boldsymbol{\Phi}}\dot{\mathbf{q}} \right) \tag{11}$$

Here $\boldsymbol{\Phi}^{\#\mathrm{T}} = \left(\boldsymbol{\Phi}\mathbf{A}^{-1}\boldsymbol{\Phi}^T \right)^{-1} \boldsymbol{\Phi}\mathbf{A}^{-1}$ stands for the transpose of the generalized inverse of constraint Jacobian. It has the same structure as matrix $\left(\mathbf{G}\mathbf{E}^{-1}\mathbf{B} \right)^{-1}\mathbf{G}\mathbf{E}^{-1}$ which appears in expression for the equivalent control (3).

By inserting (11) into (9) equations of motion in manifold (7.58) can be obtained in the following form

$$\begin{aligned}
&\mathbf{A}\ddot{\mathbf{q}} = \boldsymbol{\Gamma}^T(\boldsymbol{\tau} - \mathbf{b} + \mathbf{g}) - \boldsymbol{\Phi}^T \boldsymbol{\Lambda}_{\Phi} \dot{\boldsymbol{\Phi}}\dot{\mathbf{q}} \\
&\phi(\mathbf{q}) = \mathbf{0} \in \Re^{m \times 1} \\
&\left(\mathbf{I} - \boldsymbol{\Phi}^T \left(\boldsymbol{\Phi}\mathbf{A}^{-1}\boldsymbol{\Phi}^T \right)^{-1} \boldsymbol{\Phi}\mathbf{A}^{-1} \right) = \boldsymbol{\Gamma}^T; \\
&\boldsymbol{\Lambda}_{\Phi} = \left(\boldsymbol{\Phi}\mathbf{A}^{-1}\boldsymbol{\Phi}^T \right)^{-1}
\end{aligned} \tag{12}$$

The structure of the projection matrix in (12) and (4) is the same – it is dynamically consistent null space projection matrix. This illustrates similarities in the description of the dynamics of constrained systems and systems with sliding modes – an expected result having in mind that in both cases motion of the system is forced to satisfy functional relationship defined by constraint manifold or the sliding mode manifold. Matrix $\boldsymbol{\Gamma}$ satisfies conditions $\boldsymbol{\Gamma}^T \boldsymbol{\Phi}^T = 0$ and $\boldsymbol{\Phi}\mathbf{A}^{-1}\boldsymbol{\Gamma}^T = 0$. Constrained system (12) is describing $(n-m)$ order dynamics. That is easy to verify. By expressing velocity in constrained direction as $\dot{\phi} = \boldsymbol{\Phi}\dot{\mathbf{q}} = \left[\boldsymbol{\Phi}_{\Phi 1} \vdots \boldsymbol{\Phi}_{\Phi 2} \right]\dot{\mathbf{q}}$ we can determine m components $\dot{\mathbf{q}}_1 = \boldsymbol{\Phi}_{\Phi 1}^{-1}\left(\dot{\phi} - \boldsymbol{\Phi}_{\Phi 2}\dot{\mathbf{q}}_2 \right)$ of the configuration space velocity vector. The remaining $(n-m)$ components of the configuration vector are then describing motion in the unconstrained direction. Our aim is to find transformations of variables such that in the new set of variables dynamics in constrained and in unconstrained directions are dynamically decoupled. Let motion in unconstrained direction be described by a velocity vector $\dot{\varphi} = \mathbf{\Phi}_1 \mathbf{\Gamma} \dot{\mathbf{q}} \in \Re^{(n-m)\times 1}$ with full row rank matrix $\mathbf{\Phi}_1 \in \Re^{(n-m)\times n}$ yet to be determined and $\mathbf{\Gamma} \in \Re^{n\times n}$ null space projection matrix defined in (12).

Let forces $\mathbf{f}_\Phi \in \Re^{m\times 1}$ and $\mathbf{f}_\Gamma \in \Re^{(n-m)\times 1}$ acting in the corresponding subspaces are projected into configuration space by the transpose of corresponding matrices $\mathbf{\Phi}^T$ and $(\mathbf{\Phi}_1 \mathbf{\Gamma})^T$.

New set of velocities $(\dot{\phi}, \dot{\varphi})$ and the corresponding forces can be expressed as

$$\dot{\mathbf{q}}_{\Phi\Gamma} = \begin{bmatrix} \dot{\phi} \\ \dot{\varphi} \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi} \\ \mathbf{\Gamma}_1 \end{bmatrix} \dot{\mathbf{q}} = \mathbf{J}_{\Phi\Gamma} \dot{\mathbf{q}} \tag{13}$$

Here $\mathbf{J}_{\Phi\Gamma} \in \Re^{n\times n}$ stands for the Jacobian matrix. In order to have regular transformation Jacobian $\mathbf{J}_{\Phi\Gamma}$ has to have full rank. That defines the selection of the matrix $\mathbf{\Gamma}_1 = \mathbf{\Phi}_1 \mathbf{\Gamma}$ consistent with constraint Jacobian $\mathbf{\Phi}$. The acceleration $\ddot{\mathbf{q}}_{\Phi\Gamma}$ can be expressed as

$$\ddot{\mathbf{q}}_{\Phi\Gamma} = \mathbf{J}_{\Phi\Gamma} \ddot{\mathbf{q}} + \dot{\mathbf{J}}_{\Phi\Gamma} \dot{\mathbf{q}} \tag{14}$$

Insertion (8) and (13) into (14) yields system dynamics

$$\begin{bmatrix} \ddot{\phi} \\ \ddot{\varphi} \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi A}^{-1} \mathbf{\Phi}^T & \mathbf{\Phi A}^{-1} \mathbf{\Gamma}_1^T \\ \mathbf{\Gamma}_1 \mathbf{A}^{-1} \mathbf{\Phi}^T & \mathbf{\Gamma}_1 \mathbf{A}^{-1} \mathbf{\Gamma}_1^T \end{bmatrix} \begin{bmatrix} \mathbf{f}_\Phi \\ \mathbf{f}_\Gamma \end{bmatrix} - \begin{bmatrix} \mathbf{\Phi A}^{-1}(\mathbf{b}+\mathbf{g}) \\ \mathbf{\Gamma}_1 \mathbf{A}^{-1}(\mathbf{b}+\mathbf{g}) \end{bmatrix} + \begin{bmatrix} \dot{\mathbf{\Phi}} \dot{\mathbf{q}} \\ \dot{\mathbf{\Gamma}}_1 \dot{\mathbf{q}} \end{bmatrix} \tag{15}$$

The dynamical decoupling in (15) can be verified just by analyzing extradiagonal elements of the control distribution matrix. These terms are $\mathbf{\Phi A}^{-1} \mathbf{\Gamma}_1^T$ and $\mathbf{\Gamma}_1 \mathbf{A}^{-1} \mathbf{\Phi}^T$. By applying matrix inversion in block form and equalities $\mathbf{\Phi A}^{-1} \mathbf{\Gamma}_1^T = \mathbf{0}$ and $\mathbf{\Gamma}_1 \mathbf{A}^{-1} \mathbf{\Phi}^T = \mathbf{0}$ the dynamics (15) can be rearranged into two dynamically decoupled sub-systems

$$\begin{aligned} &\mathbf{\Lambda}_\Phi \ddot{\phi} + \mathbf{\Lambda}_\Phi \mathbf{\Phi A}^{-1}(\mathbf{b}+\mathbf{g}) - \mathbf{\Lambda}_\Phi \dot{\mathbf{\Phi}} \dot{\mathbf{q}} = \mathbf{f}_\Phi \\ &\mathbf{\Lambda}_{\Gamma 1} \ddot{\varphi} + \mathbf{\Lambda}_{\Gamma 1} \mathbf{\Gamma}_1 \mathbf{A}^{-1}(\mathbf{b}+\mathbf{g}) - \mathbf{\Lambda}_{\Gamma 1} \dot{\mathbf{\Gamma}}_1 \dot{\mathbf{q}} = \mathbf{f}_\Gamma \\ &\mathbf{\Lambda}_\Phi = \left(\mathbf{\Phi A}^{-1} \mathbf{\Phi}^T\right)^{-1}; \quad \mathbf{\Lambda}_{\Gamma 1} = \left(\mathbf{\Gamma}_1 \mathbf{A}^{-1} \mathbf{\Gamma}_1^T\right)^{-1} \end{aligned} \tag{16}$$

4.2. *Operational Space Dynamics and Control*

In general operational space coordinates may represent any set of coordinates defining kinematic mapping between configuration space and the operational space. Assume task vector is given by

$$\mathbf{x} = \mathbf{x}(\mathbf{q}), \quad \mathbf{x} \in \Re^{p\times 1}, \ \mathbf{q} \in \Re^{n\times 1}, \qquad p \le n \tag{17}$$

By taking derivative of $\mathbf{x}(\mathbf{q}) \in \Re^{p\times 1}$ the following relationship is obtained

$$\dot{\mathbf{x}} = \left(\frac{\partial \mathbf{x}(\mathbf{q})}{\partial \mathbf{q}}\right) \dot{\mathbf{q}} = \mathbf{J}(\mathbf{q}) \dot{\mathbf{q}}; \ \ddot{\mathbf{x}} = \mathbf{J} \ddot{\mathbf{q}} + \dot{\mathbf{J}} \dot{\mathbf{q}} \tag{18}$$

Here $\ddot{\mathbf{x}} \in \Re^{p\times 1}$ stands for the vector of operational space accelerations, $\mathbf{J}(\mathbf{q}) \in \Re^{p\times n}$ is task associated Jacobian matrix. Jacobian has full row rank for all permissible values of the task vector. The singularities are defined as configurations for which $\det(\mathbf{J}(\mathbf{q})) = 0$ [4].

Let, similarly to the procedure applied in the analysis of constrained systems, internal configuration of system (8) (we will also use term posture) consistent with given task be described by a minimal set of independent coordinates $\mathbf{x}_P \in \Re^{(n-p)\times 1}$. Then we can decompose the configuration space velocity vector into the task velocity vector $\dot{\mathbf{x}} = \mathbf{J} \dot{\mathbf{q}}$ and the task consistent posture vector $\dot{\mathbf{x}}_P = (\mathbf{J}_P \mathbf{\Gamma}) \dot{\mathbf{q}}$, where $\mathbf{J}_P = (\partial \mathbf{x}_P / \partial \mathbf{q}) \in \Re^{(n-p)\times n}$ stands for posture Jacobian and $\mathbf{\Gamma}$ stands for the task consistent null space projection matrix. Let operational space force be $\mathbf{f}_x \in \Re^{p\times 1}$, then vector of the configuration space forces induced by the operational space forces is $\boldsymbol{\tau}_x = \mathbf{J}^T \mathbf{f}_x$. Similarly $\mathbf{f}_P \in \Re^{(n-p)\times 1}$ is the posture related force vector and corresponding configuration space force can be expressed as $\boldsymbol{\tau}_P = (\mathbf{J}_P \mathbf{\Gamma})^T \mathbf{f}_P$.

By concatenating the task and posture velocities the operational space velocity can be expressed as

$$\dot{\mathbf{x}}_{JP} = \begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{x}}_P \end{bmatrix} = \begin{bmatrix} \mathbf{J} \\ (\mathbf{J}_P \mathbf{\Gamma}) \end{bmatrix} \dot{\mathbf{q}} = \begin{bmatrix} \mathbf{J} \\ \mathbf{\Gamma}_P \end{bmatrix} \dot{\mathbf{q}} = \mathbf{J}_{JP} \dot{\mathbf{q}} \tag{19}$$

Here $\mathbf{J}_{JP} \in \Re^{n\times n}$ is, due to the selection of the task and posture Jacobians, full rank Jacobian matrix. The acceleration can be written in the following form

$$\begin{bmatrix} \ddot{\mathbf{x}} \\ \ddot{\mathbf{x}}_P \end{bmatrix} = \begin{bmatrix} \mathbf{J A}^{-1} \mathbf{J}^T & \mathbf{J A}^{-1} \mathbf{\Gamma}_P^T \\ \mathbf{\Gamma}_P \mathbf{A}^{-1} \mathbf{J}^T & \mathbf{\Gamma}_P \mathbf{A}^{-1} \mathbf{\Gamma}_P^T \end{bmatrix} \begin{bmatrix} \mathbf{f}_x \\ \mathbf{f}_P \end{bmatrix} - \begin{bmatrix} \mathbf{J A}^{-1}(\mathbf{b}+\mathbf{g}) \\ \mathbf{\Gamma}_P \mathbf{A}^{-1}(\mathbf{b}+\mathbf{g}) \end{bmatrix} + \begin{bmatrix} \dot{\mathbf{J}} \dot{\mathbf{q}} \\ \dot{\mathbf{\Gamma}}_P \dot{\mathbf{q}} \end{bmatrix} \tag{20}$$

The control distribution matrix in (20) is not diagonal thus task and posture are dynamically coupled. Since task Jacobian is defined, and consequently the posture Jacobian is selected to be consistent with task, the decoupling can be then achieved due to selection of the matrix $\mathbf{\Gamma}$. With $\mathbf{\Gamma} = (\mathbf{I} - \mathbf{J}^\# \mathbf{J})$ and generalized inverse $\mathbf{J}^\# = \mathbf{A}^{-1} \mathbf{J}^T \left(\mathbf{J A}^{-1} \mathbf{J}^T\right)^{-1}$ it is easy to verify relations $\mathbf{J A}^{-1} \mathbf{\Gamma}_P^T = \mathbf{0}^{p\times(n-p)}$ and $\mathbf{\Gamma}_P \mathbf{A}^{-1} \mathbf{J}^T = \mathbf{0}^{(n-p)\times p}$. Then, the force distribution matrix in (20) is block diagonal and its inverse will be also block diagonal and the dynamics (20) can be written as

$$\begin{aligned} &\mathbf{\Lambda}_x \ddot{\mathbf{x}} + \mathbf{\Lambda}_x \mathbf{J A}^{-1}(\mathbf{b}+\mathbf{g}) - \mathbf{\Lambda}_x \dot{\mathbf{J}} \dot{\mathbf{q}} = \mathbf{f}_x \\ &\mathbf{\Lambda}_P \ddot{\mathbf{x}}_P + \mathbf{\Lambda}_P \mathbf{\Gamma}_P \mathbf{A}^{-1}(\mathbf{b}+\mathbf{g}) - \mathbf{\Lambda}_P \dot{\mathbf{\Gamma}}_P \dot{\mathbf{q}} = \mathbf{f}_P \\ &\mathbf{\Lambda}_x = \left(\mathbf{J A}^{-1} \mathbf{J}^T\right)^{-1}; \ \mathbf{\Lambda}_P = \left(\mathbf{\Gamma}_P \mathbf{A}^{-1} \mathbf{\Gamma}_P^T\right)^{-1} \end{aligned} \tag{21}$$

The task and the posture become dynamically decoupled and forces $\mathbf{f}_x$ and $\mathbf{f}_P$ can be selected separately. Note the similarities of the transformations with constrained systems

Let task and posture tracking errors be

$$\begin{aligned} &\mathbf{e}_x\left(\mathbf{x}, \mathbf{x}^{ref}\right) = \mathbf{x}(\mathbf{q}) - \mathbf{x}^{ref}, && \mathbf{e}_x \in \Re^{p\times 1} \\ &\mathbf{e}_P\left(\mathbf{x}_P, \mathbf{x}_P^{ref}\right) = \mathbf{x}_P(\mathbf{q}) - \mathbf{x}_P^{ref}, && \mathbf{e}_P \in \Re^{(n-p)\times 1} \end{aligned} \tag{22}$$

Here $\mathbf{x}^{ref}$ and $\mathbf{x}_P^{ref}$ stand or the task and posture references. Then dynamics of the task and posture errors is

$$\begin{aligned} \ddot{\mathbf{e}}_x &= \mathbf{\Lambda}_x^{-1}\left(\mathbf{f}_x - \left(\mathbf{\Lambda}_x \mathbf{J A}^{-1}(\mathbf{b}+\mathbf{g}) - \mathbf{\Lambda}_x \dot{\mathbf{J}} \dot{\mathbf{q}}\right)\right) - \ddot{\mathbf{x}}^{ref} \\ \ddot{\mathbf{e}}_P &= \mathbf{\Lambda}_P^{-1}\left(\mathbf{f}_P - \left(\mathbf{\Lambda}_P \mathbf{\Gamma}_P \mathbf{A}^{-1}(\mathbf{b}+\mathbf{g}) - \mathbf{\Lambda}_P \dot{\mathbf{\Gamma}}_P \dot{\mathbf{q}}\right)\right) - \ddot{\mathbf{x}}_P^{ref} \end{aligned} \tag{23}$$

Selecting manifolds

$$\begin{aligned}\boldsymbol{\sigma}_x &= \mathbf{C}_x\mathbf{e}_x + \dot{\mathbf{e}}_x = \mathbf{0}\\ \boldsymbol{\sigma}_P &= \mathbf{C}_P\mathbf{e}_P + \dot{\mathbf{e}}_P = \mathbf{0}\end{aligned} \tag{24}$$

The accelerations inducing no change in the rate of change of the rate of change $\dot{\boldsymbol{\sigma}}_x = \mathbf{0}$ and $\dot{\boldsymbol{\sigma}}_P = \mathbf{0}$ (so-called equivalent acceleration) can be simply obtained as

$$\begin{aligned}\ddot{\mathbf{x}}_x^{eq} &= \ddot{\mathbf{x}}^{ref} - \mathbf{C}_x\dot{\mathbf{e}}_x\\ \ddot{\mathbf{x}}_P^{eq} &= \ddot{\mathbf{x}}_P^{ref} - \mathbf{C}_P\dot{\mathbf{e}}_P\end{aligned} \tag{25}$$

The force inducing acceleration (25) in task and posture control can be expressed as equivalent controls in task and posture become

$$\begin{aligned}\mathbf{f}_x^{eq} &= \boldsymbol{\Lambda}_x\left(\mathbf{J}\mathbf{A}^{-1}(\mathbf{b}+\mathbf{g}) - \dot{\mathbf{J}}\dot{\mathbf{q}}\right) + \boldsymbol{\Lambda}_x\ddot{\mathbf{x}}_x^{eq}\\ \mathbf{f}_P^{eq} &= \boldsymbol{\Lambda}_P\left(\boldsymbol{\Gamma}_P\mathbf{A}^{-1}(\mathbf{b}+\mathbf{g}) - \dot{\boldsymbol{\Gamma}}_P\dot{\mathbf{q}}\right) + \boldsymbol{\Lambda}_P\ddot{\mathbf{x}}_P^{eq}\end{aligned} \tag{26}$$

Let, for task and posture, Lyapunov function candidate be selected in the form $V_i = \boldsymbol{\sigma}_i^T\boldsymbol{\sigma}_i/2, i = x, P$. Stability conditions require derivative of the Lyapunov function candidate to be negative definite. Let $\dot{V}_i = \boldsymbol{\sigma}_i^T\dot{\boldsymbol{\sigma}}_i = -\rho_i\boldsymbol{\sigma}_i^T\boldsymbol{\Psi}_i(\boldsymbol{\sigma}_i) < 0$ where function $\boldsymbol{\Psi}_i(\boldsymbol{\sigma}_i)$ satisfies component-vise $sign(\Psi_{ii}(\boldsymbol{\sigma}_i)) = sign(\sigma_{ii})$ and let $\rho_i > 0$. Then task and posture control input are

$$\begin{aligned}\mathbf{f}_x &= \mathbf{f}_x^{eq} - \rho_x\boldsymbol{\Lambda}_x\boldsymbol{\Psi}_x(\boldsymbol{\sigma}_x)\\ \mathbf{f}_P &= \mathbf{f}_P^{eq} - \rho_P\boldsymbol{\Lambda}_P\boldsymbol{\Psi}_P(\boldsymbol{\sigma}_P)\end{aligned} \tag{27}$$

These forces can be expressed in more compact way by introducing the task and posture disturbance forces $\mathbf{f}_x^{dis}$ and $\mathbf{f}_P^{dis}$ to obtain expressions similar to those obtained in the acceleration control framework [6]

$$\begin{aligned}\mathbf{f}_x^{eq} &= \mathbf{f}_x^{dis} + \boldsymbol{\Lambda}_x\left(\ddot{\mathbf{x}}_x^{eq} - \rho_x\boldsymbol{\Psi}_x(\boldsymbol{\sigma}_x)\right)\\ \mathbf{f}_P^{eq} &= \mathbf{f}_P^{dis} + \boldsymbol{\Lambda}_P\left(\ddot{\mathbf{x}}_P^{eq} - \rho_P\boldsymbol{\Psi}_P(\boldsymbol{\sigma}_P)\right)\\ \mathbf{f}_x^{dis} &= \boldsymbol{\Lambda}_x\left(\mathbf{J}\mathbf{A}^{-1}(\mathbf{b}+\mathbf{g}) - \dot{\mathbf{J}}\dot{\mathbf{q}}\right)\\ \mathbf{f}_P^{dis} &= \boldsymbol{\Lambda}_P\left(\boldsymbol{\Gamma}_P\mathbf{A}^{-1}(\mathbf{b}+\mathbf{g}) - \dot{\boldsymbol{\Gamma}}_P\dot{\mathbf{q}}\right)\end{aligned} \tag{28}$$

The equivalent forces and the disturbance forces can be estimated component-vise. Control (27) enforce stability of the equilibrium $\boldsymbol{\sigma}_x = \mathbf{0}$ and $\boldsymbol{\sigma}_P = \mathbf{0}$. Strictly speaking sliding mode will be enforced in the intersection of manifolds $\boldsymbol{\sigma}_x = \mathbf{0}$ and $\boldsymbol{\sigma}_P = \mathbf{0}$ if $\boldsymbol{\Psi}(\boldsymbol{\sigma})$ is selected such that intersection is reached in finite time. In the discrete-time implementation $\boldsymbol{\Psi}(\boldsymbol{\sigma})$ can be selected continuous (in the discrete-time sense) [3]. The equations of motion in sliding mode are given by (24). If asymptotic convergence, instead of sliding mode is selected equations of motion are then given by $\boldsymbol{\sigma}_i^T\dot{\boldsymbol{\sigma}}_i + \rho_i\boldsymbol{\sigma}_i^T\boldsymbol{\Psi}_i(\boldsymbol{\sigma}_i) = \mathbf{0}$.

The configuration space force can be expressed as

$$\boldsymbol{\tau} = \mathbf{J}^T\mathbf{f}_x + \boldsymbol{\Gamma}_P^T\mathbf{f}_P \tag{29}$$

Inserting (25), (28) and (29) into (8) results in the operational space desired acceleration

$$\ddot{\mathbf{q}}^{des} = \mathbf{J}^{\#}\left(\ddot{\mathbf{x}}^{des} - \dot{\mathbf{J}}\dot{\mathbf{q}}\right) + \boldsymbol{\Gamma}_P^{\#}\left(\ddot{\mathbf{x}}_P^{des} - \dot{\boldsymbol{\Gamma}}_P\dot{\mathbf{q}}\right) \tag{30}$$

Here $\mathbf{J}^{\#} = \mathbf{A}^{-1}\mathbf{J}^T\boldsymbol{\Lambda}_x$ stand for the task Jacobian pseudoinverse and $\boldsymbol{\Gamma}_P^{\#} = \mathbf{A}^{-1}\boldsymbol{\Gamma}_P^{\mathrm{T}}\boldsymbol{\Lambda}_P$ stands for the posture Jacobian $(\mathbf{J}_P\boldsymbol{\Gamma})$ pseudoinverse. The result show full correspondence with constrained system control. The solution offers a way of combining the task and the posture control or instead of posture, control of another task.

The application of the SMC leads to the same structure of the control in operational space and in the constrained systems. Such similarity is very important for the motion control of systems that may be constrained and at the same time need to implement certain task. The fact that dynamics of constrained systems and tasks are described by equations that have the same form and that projection of the velocities and the forces are consistent in both cases opens possibility of combining the constraints and tasks into an augmented description and treating them within the same framework.

4.3. *Constraints in Operational Space*

Assume configuration space dynamical model for $n-$dof multi-body system as in (8), task defined by $\mathbf{x}(\mathbf{q}) \in \Re^{m\times 1}$ and operational space Jacobian matrix $\mathbf{J}_x \in \Re^{m\times n}, m < n$. Let select input force such that end-effecter is constrained to smooth surface defined by

$$\phi(\mathbf{x}(\mathbf{q})) = \phi^{ref}(t), \phi \in \Re^{p\times 1}, p < m < n \tag{31}$$

The $\phi^{ref}(t)$ stands for the time dependent sufficiently smooth reference. Relation between operational space acceleration and acceleration in constrained direction can be expressed by twice differentiating (31)

$$\ddot{\phi}(\mathbf{x}(\mathbf{q})) = \mathbf{J}_\phi\ddot{\mathbf{x}} + \dot{\mathbf{J}}_\phi\dot{\mathbf{x}}; \quad \mathbf{J}_\phi = \left(\frac{\partial\phi}{\partial\mathbf{x}}\right) \in \Re^{p\times m} \tag{32}$$

Here $\mathbf{J}_\phi \in \Re^{p\times m}$ stands for constraint Jacobian in operational space. Projection of operational space dynamics (21) in constrained direction can be written as

$$\begin{aligned}&\boldsymbol{\Lambda}_\phi\ddot{\phi} + \boldsymbol{\Lambda}_\phi\left(\mathbf{J}_\phi\boldsymbol{\Lambda}_x^{-1}\boldsymbol{\mu}_x - \dot{\mathbf{J}}_\phi\dot{\mathbf{x}}\right) + \boldsymbol{\Lambda}_\phi\mathbf{J}_\phi\boldsymbol{\Lambda}_x^{-1}\mathbf{v}_x = \mathbf{f}_\phi\\ &\boldsymbol{\Lambda}_\phi = \left(\mathbf{J}_\phi\left(\mathbf{J}_x\mathbf{A}^{-1}\mathbf{J}_x^T\right)^{-1}\mathbf{J}_\phi^T\right)^{-1}\\ &\mathbf{f}_{x\phi} = \mathbf{J}_\phi^T\mathbf{f}_\phi\end{aligned} \tag{33}$$

Structure of the matrix $\boldsymbol{\Lambda}_\phi = \left(\mathbf{J}_\phi\boldsymbol{\Lambda}_x^{-1}\mathbf{J}_\phi^T\right)^{-1}$ illustrates result of the two consecutive transformations from configuration space - the projection into operational space and then projection into constrained direction.

Enforcing sliding mode in $\boldsymbol{\sigma}_\phi = \mathbf{C}_\phi\mathbf{e}_\phi + \mathbf{e}_\phi = 0$, $\mathbf{C}_\phi > 0$ with control error $\mathbf{e}_\phi = \phi - \phi^{ref}$ will guaranty convergence to equilibrium (31). The control force $\mathbf{f}_\phi$ guarantying sliding mode motion in manifold $\boldsymbol{\sigma}_\phi = 0$ is selected as

$$\begin{aligned}\mathbf{f}_\phi &= \mathbf{f}_\phi^{dis} + \boldsymbol{\Lambda}_\phi\ddot{\phi}^{des}\\ \mathbf{f}_\phi^{dis} &= \boldsymbol{\Lambda}_\phi\left(\mathbf{J}_\phi\boldsymbol{\Lambda}_x^{-1}\boldsymbol{\mu}_x - \dot{\mathbf{J}}_\phi\dot{\mathbf{x}}\right) + \boldsymbol{\Lambda}_\phi\mathbf{J}_\phi\boldsymbol{\Lambda}_x^{-1}\mathbf{v}_x\\ \ddot{\phi}^{des} &= \ddot{\phi}^{ref} - \mathbf{C}_\phi\dot{\mathbf{e}}_\phi - \rho_\phi\boldsymbol{\Psi}_\phi(\boldsymbol{\sigma}_\phi)\end{aligned} \tag{34}$$

Here $\ddot{\phi}^{des}$ stands for the desired acceleration in the constrained direction, $\ddot{\phi}^{ref}$ stands for the reference acceleration.

As shown in analysis of constrained systems dynamics decoupling of the remaining $(m-p)$ degrees of freedom requires projection in orthogonal complement subspace. This can be obtained by projecting operational space velocity into unconstrained direction $\dot{\mathbf{z}} = \mathbf{J}_{z\phi}\dot{\mathbf{x}} = \mathbf{J}_z \mathbf{\Gamma}_{x\phi}\dot{\mathbf{x}}$ with $\mathbf{J}_z \in \Re^{(m-p)\times m}$ as a full row rank matrix and $\mathbf{\Gamma}_{x\phi} \in \Re^{m\times m}$ as null space projection matrix $\mathbf{\Gamma}_{x\phi} = \mathbf{I} - \mathbf{J}_\phi^{\#}\mathbf{J}_\phi$ associated with constraint Jacobian in the operational space, and $\mathbf{J}_\phi^{\#} = \mathbf{\Lambda}_x^{-1}\mathbf{J}_\phi^T\left(\mathbf{J}_\phi\mathbf{\Lambda}_x^{-1}\mathbf{J}_\phi^T\right)^{-1}$ stands for dynamically consistent pseudo inverse. The dynamics $\ddot{\mathbf{z}} = \mathbf{J}_{z\phi}\ddot{\mathbf{x}} + \dot{\mathbf{J}}_{z\phi}\dot{\mathbf{x}}$ can be expressed as

$$\begin{aligned}
&\mathbf{\Lambda}_{z\phi}\ddot{\mathbf{z}} + \mathbf{\Lambda}_{z\phi}\left(\mathbf{J}_{z\phi}\mathbf{\Lambda}_x^{-1}\boldsymbol{\mu}_x - \dot{\mathbf{J}}_{z\phi}\dot{\mathbf{x}} + \mathbf{J}_{z\phi}\mathbf{\Lambda}_x^{-1}\boldsymbol{\nu}\right) = \mathbf{f}_{z\phi} \\
&\mathbf{\Lambda}_{z\phi} = \left(\mathbf{J}_{z\phi}\left(\mathbf{J}_x\mathbf{A}^{-1}\mathbf{J}_x^T\right)^{-1}\mathbf{J}_{z\phi}^T\right)^{-1} \\
&\mathbf{f}_{xz} = \mathbf{J}_{z\phi}^T\mathbf{f}_{z\phi}
\end{aligned} \tag{35}$$

Tracking of reference $\mathbf{z}^{ref}$ can be enforced by establishing sliding mode motion ins manifold $\boldsymbol{\sigma}_z = \mathbf{C}_z\mathbf{e}_z + \mathbf{e}_z = \mathbf{0}$, $\mathbf{C}_z > 0$, $\mathbf{e}_z = \mathbf{z} - \mathbf{z}^{ref}$. Then desired acceleration and corresponding force become

$$\begin{aligned}
&\mathbf{f}_{z\phi} = \mathbf{f}_z^{dis} + \mathbf{\Lambda}_{z\phi}\ddot{\mathbf{z}}^{des} \\
&\ddot{\mathbf{z}}^{des} = \ddot{\mathbf{z}}^{ref} - \mathbf{C}_\phi\dot{\mathbf{e}}_z - \rho_z\mathbf{\Psi}_z(\boldsymbol{\sigma}_z) \\
&\mathbf{f}_z^{dis} = \mathbf{\Lambda}_{z\phi}\left(\mathbf{J}_{z\phi}\mathbf{\Lambda}_x^{-1}\boldsymbol{\mu}_x - \dot{\mathbf{J}}_{z\phi}\dot{\mathbf{x}} + \mathbf{J}_{z\phi}\mathbf{\Lambda}_x^{-1}\boldsymbol{\nu}\right)
\end{aligned} \tag{36}$$

Operational space force can be expressed as

$$\mathbf{f}_x = \mathbf{J}_\phi^T\mathbf{f}_\phi + \mathbf{J}_{z\phi}^T\mathbf{f}_{z\phi} \tag{37}$$

This result is equivalent to concurrent task and posture control in operational space. If desired acceleration $\ddot{\phi}^{des}$ is selected as control in constrained direction force tracking loop, then (37) realizes concurrent force control and motion control in operational space. The enforcement of constraints in operational space establishes algebraic relation between certain number of operational space coordinates and thus limiting the set of motions that can be realized by the system.

If (31) describes hard constraints with $\phi^{ref}(t) \equiv \mathbf{0}$ operational space interaction force $\mathbf{f}_{\text{int}} = \mathbf{J}_\phi^T\lambda$ should be added to dynamics. Here λ stands for the Lagrange multiplier. These multipliers can be determined using the same idea as applied in section 4.1

4.4. *Hierarchy of Tasks in Operational Space*

So far we have been analyzing dynamics and control issues for multi-body systems subject to constraints and the single task and the posture. The solution has been found in selecting primary goal (enforcement of the constraints) and then solving secondary goal by using projection into Null space. Let us now address issues in analysis and control of multi-body systems with constraints and multiple tasks. A $n-$dof multi-body system is required to maintain functional constraints while fulfilling selected tasks;

- Constraint is defined by function $\phi(\mathbf{q}) = 0 \in \Re^{m_c\times 1}$ with constraint Jacobian $\mathbf{\Phi} \in \Re^{m_c\times n}$;
- One of the tasks is defined by $\mathbf{x}(\mathbf{q}) \in \Re^{m_x\times 1}$ with task Jacobian $\mathbf{J}_x \in \Re^{m_x\times n}$;
- Second task is defined by $\mathbf{y}(\mathbf{q}) \in \Re^{m_y\times 1}$ with task Jacobian $\mathbf{J}_y \in \Re^{m_y\times n}$
- The priority of task $\mathbf{x}(\mathbf{q})$ is higher than priority of task $\mathbf{y}(\mathbf{q})$;
- All matrices $\mathbf{\Phi} \in \Re^{m_c\times n}$, $\mathbf{J}_x \in \Re^{m_x\times n}$ and $\mathbf{J}_y \in \Re^{m_y\times n}$ are assumed to have full row rank, thus constraints and tasks are linearly independent,
- Without loss of generality let allocation of available configuration space degrees of freedom is such that constraints and tasks can be implemented concurrently and no free dof-s are left, thus $m_c + m_x + m_y = n$.

With these operational requirements application of the so far discussed approach we can write constraint-task velocity mapping in the following form

$$\begin{bmatrix}\dot{\phi}\\ \dot{\mathbf{x}}\\ \dot{\mathbf{y}}\end{bmatrix} = \begin{bmatrix}\mathbf{\Phi}\\ \mathbf{J}_1\\ \mathbf{J}_2\end{bmatrix}\dot{\mathbf{q}}, \quad \boldsymbol{\eta} = \begin{bmatrix}\phi\\ \mathbf{x}\\ \mathbf{y}\end{bmatrix}; \quad \mathbf{J} = \begin{bmatrix}\mathbf{\Phi}\\ \mathbf{J}_1\\ \mathbf{J}_2\end{bmatrix} \tag{38}$$

Matrices $\mathbf{J}_1 \in \Re^{m_x\times n}$ and $\mathbf{J}_2 \in \Re^{m_y\times n}$ are assumed to have full row rank and should be determined as function of constraint and task Jacobian matrices in such a way that dynamics of constraints and tasks are decoupled. By assumptions constraint-tasks Jacobian $\mathbf{J} \in \Re^{n\times n}$ has full rank $\det(\mathbf{J}) \neq 0$. Formally constraint and task attributed velocities and accelerations can be expressed as

$$\dot{\boldsymbol{\eta}} = \mathbf{J}\dot{\mathbf{q}}; \quad \ddot{\boldsymbol{\eta}} = \mathbf{J}\ddot{\mathbf{q}} + \dot{\mathbf{J}}\dot{\mathbf{q}}, \tag{39}$$

Configuration space force is

$$\begin{aligned}
&\boldsymbol{\tau} = \mathbf{\Phi}^T\mathbf{f}_\Phi + \mathbf{J}_1^T\mathbf{f}_x + \mathbf{J}_2^T\mathbf{f}_y = \mathbf{J}^T\mathbf{f} \\
&\mathbf{J}^T = \left[\mathbf{\Phi}^T \vdots \mathbf{J}_1^T \vdots \mathbf{J}_2^T\right]; \ \mathbf{f}^T = \left[\mathbf{f}_\Phi \quad \mathbf{f}_x \quad \mathbf{f}_y\right]
\end{aligned} \tag{40}$$

Here $\mathbf{f}_\Phi \in \Re^{m_c\times 1}, \mathbf{f}_x \in \Re^{m_x\times 1}, \mathbf{f}_y \in \Re^{m_{yc}\times 1}$ are control forces associated with constraint and tasks.

The configuration space dynamics is then described by

$$\begin{aligned}
&\mathbf{A}\ddot{\mathbf{q}} + \mathbf{b}(\mathbf{q},\dot{\mathbf{q}}) + \mathbf{g}(\mathbf{q}) + \mathbf{J}^T\mathbf{f}_c = \boldsymbol{\tau} \\
&\mathbf{J}^T\mathbf{f}_c = \mathbf{\Phi}^T\mathbf{f}_{\Phi c} + \mathbf{J}_1^T\mathbf{f}_{xc} + \mathbf{J}_2^T\mathbf{f}_{yc}
\end{aligned} \tag{41}$$

By inserting (40) and (41) into (39) formally, the constraint-operational space dynamics can be expressed as

$$\begin{aligned}
&\mathbf{\Lambda}\ddot{\boldsymbol{\eta}} + \mathbf{\Lambda}\left(\mathbf{J}\mathbf{A}^{-1}(\mathbf{b}+\mathbf{g}) - \dot{\mathbf{J}}\dot{\mathbf{q}}\right) = \mathbf{f} - \mathbf{f}_c \\
&\mathbf{\Lambda} = \left(\mathbf{J}\mathbf{A}^{-1}\mathbf{J}^T\right)^{-1}
\end{aligned} \tag{42}$$

The dynamical coupling are present in all three terms the inertia matrix $\mathbf{\Lambda}$ and coupling forces $\boldsymbol{\mu}(\mathbf{q},\dot{\mathbf{q}}), \boldsymbol{\nu}(\mathbf{q})$ while constraint and tasks associated forces are decoupled. Here we would like to establish first dynamical decoupling at least for the acceleration induced forces and then apply sliding mode control. In order to find decoupling conditions let us first analyze constraint-operational space control distribution matrix

$$\mathbf{\Lambda}^{-1} = \begin{bmatrix} \mathbf{\Phi A}^{-1}\mathbf{\Phi}^T & \mathbf{\Phi A}^{-1}\mathbf{J}_1^T & \mathbf{\Phi A}^{-1}\mathbf{J}_2^T \\ \mathbf{J}_1\mathbf{A}^{-1}\mathbf{\Phi}^T & \mathbf{J}_1\mathbf{A}^{-1}\mathbf{J}_1^T & \mathbf{J}_1\mathbf{A}^{-1}\mathbf{J}_2^T \\ \mathbf{J}_2\mathbf{A}^{-1}\mathbf{\Phi}^T & \mathbf{J}_2\mathbf{A}^{-1}\mathbf{J}_1^T & \mathbf{J}_2\mathbf{A}^{-1}\mathbf{J}_2^T \end{bmatrix} \tag{43}$$

Matrix $\mathbf{\Lambda}^{-1}$ shows the dynamical coupling of the acceleration terms. In order to have dynamically decoupled acceleration terms the extra-diagonal elements $\mathbf{\Lambda}_{ij}\,(i \neq j)$ of control distribution matrix must be zero. That gives a set of matrix equations to be solved. If the following requirements are met then (43) will be reduced into block diagonal form

$$\begin{aligned} \mathbf{\Phi A}^{-1}\mathbf{J}_1^T &= \mathbf{0}^{m_c \times m_x} \\ \mathbf{\Phi A}^{-1}\mathbf{J}_2^T &= \mathbf{0}^{m_c \times m_y} \\ \mathbf{J}_1\mathbf{A}^{-1}\mathbf{J}_2^T &= \mathbf{0}^{m_x \times m_y} \end{aligned} \tag{44}$$

These conditions will ensure the constraints and tasks dynamical decoupling. Recalling the structure of weighted pseudoinverse $\mathbf{J}^{\#} = \mathbf{A}^{-1}\mathbf{J}^T\left(\mathbf{J}\mathbf{A}^{-1}\mathbf{J}^T\right)^{-1}$ and its orthogonal complement $\mathbf{\Gamma} = \mathbf{I} - \mathbf{J}^{\#}\mathbf{J}$ we can verify that conditions (44) are met by selecting either $\mathbf{J}_1$ and $\mathbf{J}_2$ proportional to orthogonal complement of the constraint Jacobian $\mathbf{\Phi}$. Under assumption that task $\mathbf{x}(\mathbf{q})$ has priority then matrix $\mathbf{J}_1$ associate to this task should be selected as $\mathbf{J}_1 = \mathbf{J}_x\mathbf{\Gamma}_\Phi \in \Re^{m_x \times n}$. Then $\mathbf{\Phi A}^{-1}\mathbf{J}_1^T = \mathbf{0}^{m_c \times m_x}$ is true.

The equations $\mathbf{J}_2\mathbf{A}^{-1}\mathbf{\Phi}^T = \mathbf{0}$ and $\mathbf{J}_2\mathbf{A}^{-1}\mathbf{J}_1^T = \mathbf{0}$ lead to the selection of matrix $\mathbf{J}_2$ as

$$\begin{aligned} \mathbf{J}_2\mathbf{A}^{-1}\mathbf{\Phi}^T &= \mathbf{0}^{m_y \times mc} \Rightarrow \mathbf{J}_2 = \mathbf{J}_y\mathbf{\Gamma}_\Phi \in \Re^{m_y \times n} \\ \mathbf{J}_2\mathbf{A}^{-1}\mathbf{J}_1^T &= \mathbf{0}^{m_y \times m_x} \Rightarrow \mathbf{J}_2 = \mathbf{J}_y\mathbf{\Gamma}_{J1} \in \Re^{m_y \times n} \end{aligned} \tag{45}$$

Structure of matrix $\mathbf{J}_2$ can be derived by combining these two solutions. This leads to

$$\mathbf{J}_2 = \mathbf{J}_y\left(\mathbf{I} - \mathbf{\Phi}^{\#}\mathbf{\Phi} - \mathbf{J}_1^{\#}\mathbf{J}_1\right) \tag{46}$$

More general solution for task hierarchy can be found in [7]. The transformation from configuration space into the constraint and operational spaces can be determined by premultiplying the configuration space equations of motion by $\mathbf{\Phi A}^{-1}$, $\mathbf{J}_1\mathbf{A}^{-1}$ and $\mathbf{J}_2\mathbf{A}^{-1}$ respectively and recalling $\boldsymbol{\tau} = \mathbf{\Phi}^T\mathbf{f}_\Phi + \mathbf{J}_1^T\mathbf{f}_x + \mathbf{J}_2^T\mathbf{f}_y$ to obtain

$$\begin{aligned} \mathbf{\Lambda}_\Phi\ddot{\phi} + \boldsymbol{\mu}_\Phi(\mathbf{q},\dot{\mathbf{q}}) + \mathbf{v}_\Phi(\mathbf{q}) &= \mathbf{f}_\Phi - \mathbf{f}_{\Phi c} \\ \mathbf{\Lambda}_x\ddot{\mathbf{x}} + \boldsymbol{\mu}_x(\mathbf{q},\dot{\mathbf{q}}) + \mathbf{v}_x(\mathbf{q}) &= \mathbf{f}_x - \mathbf{f}_{xc} \\ \mathbf{\Lambda}_y\ddot{\mathbf{y}} + \boldsymbol{\mu}_y(\mathbf{q},\dot{\mathbf{q}}) + \mathbf{v}_y(\mathbf{q}) &= \mathbf{f}_y - \mathbf{f}_{yc} \end{aligned} \tag{47}$$

Here $\mathbf{\Lambda}_\Phi = \left(\mathbf{\Phi A}^{-1}\mathbf{\Phi}^T\right)^{-1}$ stands for the inertia matrix in the constraint direction, $\mathbf{\Lambda}_x = \left(\mathbf{J}_1\mathbf{A}^{-1}\mathbf{J}_1^T\right)^{-1}$ and $\mathbf{\Lambda}_y = \left(\mathbf{J}_2\mathbf{A}^{-1}\mathbf{J}_2^T\right)^{-1}$ stand for the inertia matrices in the operational spaces; the $\boldsymbol{\mu}_i(\mathbf{q},\dot{\mathbf{q}}) + \mathbf{v}_i(\mathbf{q}), i = \Phi, x, y$ stand for the projections of the configuration space disturbance $\mathbf{b}(\mathbf{q},\dot{\mathbf{q}}) + \mathbf{g}(\mathbf{q})$ and the velocity induced forces. The $\mathbf{f}_\Phi, \mathbf{f}_x, \mathbf{f}_y$ stand for the control forces in the corresponding operational spaces.

Having dynamics of the system decomposed as in (47) the application of the SMC method leads to the selection of the constraint control as in (33) and the task control as in (27) thus having

$$\begin{aligned} \mathbf{f}_\phi &= \mathbf{f}_\phi^{eq} - \mathbf{\Lambda}_\phi\rho_\phi\mathbf{\Psi}_\phi\left(\boldsymbol{\sigma}_\phi\right) \\ \mathbf{f}_x &= \mathbf{f}_x^{eq} - \rho_x\mathbf{\Lambda}_x\mathbf{\Psi}_x\left(\boldsymbol{\sigma}_x\right) \\ \mathbf{f}_y &= \mathbf{f}_y^{eq} - \rho_y\mathbf{\Lambda}_y\mathbf{\Psi}_y\left(\boldsymbol{\sigma}_y\right) \end{aligned} \tag{48}$$

With sliding mode manifolds

$$\begin{aligned} \boldsymbol{\sigma}_\phi &= \mathbf{C}_\phi\mathbf{e}_\phi + \mathbf{e}_\phi; \;\; \mathbf{e}_\phi = \phi - \phi^{ref} \\ \boldsymbol{\sigma}_x &= \mathbf{C}_x\mathbf{e}_x + \dot{\mathbf{e}}_x; \;\; \mathbf{e}_x = \mathbf{x} - \mathbf{x}^{ref} \\ \boldsymbol{\sigma}_y &= \mathbf{C}_y\mathbf{e}_y + \dot{\mathbf{e}}_y; \;\; \mathbf{e}_y = \mathbf{y} - \mathbf{y}^{ref} \end{aligned} \tag{49}$$

The result shows that the subsequent tasks in the hierarchy are executed in the orthogonal complement space of the preceding task. The closed loop dynamics in sliding mode reduces to $\boldsymbol{\sigma}_\phi = \mathbf{0}, \boldsymbol{\sigma}_x = \mathbf{0}, \boldsymbol{\sigma}_y = \mathbf{0}$ [8].

5. *Conclusions*

The SMC control of the multi-body mechanical systems is discussed to some details. It has been shown that sliding mode methods can be directly applied for control in configuration and operational space. Moreover it has been shown that equations of motion of the constrained systems and task redundant systems can be found using equivalent control method. The dynamical decoupling of the constraints and tasks are obtained using general inverse which have the same structure as the sliding mode projection matrix.

6. *Acknowledgement*

The work on this research is in part supported by TUBITAK project 108M520.

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