SABANCI UNIVERSITY
Orhanlı-Tuzla, 34956 Istanbul, Turkey
Phone: +90 (216) 483-9500
Fax: +90 (216) 483-9550
http://www.sabanciuniv.edu

http://algopt.sabanciuniv.edu/projects
November 19, 2010

A Note on "A LP-based Heuristic for a Time-Constrained Routing Problem"

İbrahim Muter, Ş. İlker Birbil, Kerem Bülbül, and Güvenç Şahin
Sabancı University, Manufacturing Systems and Industrial Engineering, Orhanlı-Tuzla, 34956 Istanbul, Turkey
imuter@su.sabanciuniv.edu, sibirbil@sabanciuniv.edu, bulbul@sabanciuniv.edu, guvencs@sabanciuniv.edu

ABSTRACT: Avella et al. (2006) [Avella, P., D'Auria, B., Salerno, S. (2006). A LP-based heuristic for a time-constrained routing problem. European Journal of Operational Research 173:120-124] investigate a time-constrained routing (TCR) problem. The core of the proposed solution approach is a large-scale linear program (LP) that grows both row- and column-wise when new variables are introduced. Thus, a column-and-row generation algorithm is proposed to solve this LP optimally, and an optimality condition is presented to terminate the column-and-row generation algorithm. We demonstrate that this optimality condition is incorrect and may lead to a suboptimal solution at termination. We identify the source of this error and discuss how the generic column-and-row generation algorithm proposed by Muter et al. (2010) may be applied to this TCR problem in order to solve the proposed large-scale LP correctly.

Keywords: Time-constrained routing; large-scale optimization; linear programming; column generation; column-and-row generation; column-dependent-rows.

1. Introduction. Avella et al. (2006) study a time-constrained routing problem motivated by an application that schedules the visit of a tourist to a given geographical area. The problem is to send the tourist on one tour during each day in the vacation period by maximizing her total satisfaction while ensuring that each attraction is visited no more than once. The authors refer to this problem as the Intelligent Tourist Problem (ITP).

ITP is formulated as a set packing problem with side constraints. The set of sites that may be visited by a tourist in a vacation period T is denoted by V, and J represents the set of daily tours that originate from and terminate at the same location. Each tour is a sequence of sites to be visited on the same day, provided that it satisfies the time-windows restrictions of the tourist and the other feasibility criteria. The total satisfaction of the tourist from participating in tour j is given by r_j, and the binary variable y_j is set to one, if tour j is incorporated into the itinerary of the tourist. If tour j is performed on day t, the binary variable x_{jt} takes the value one. The overall mathematical model for ITP then becomes

$$\begin{aligned}
\text{maximize} \quad & \sum_{j \in J} r_j y_j, && && (1)\\
\text{subject to} \quad & \sum_{j \in D(i)} y_j \leq 1, && i \in V, && (2)\\
& y_j - \sum_{t \in S(j)} x_{jt} = 0, && j \in J, && (3)\\
& \sum_{j \in F(t)} x_{jt} = 1, && t \in T, && (4)
\end{aligned}$$

$$y_j \in \{0,1\}, \qquad j \in J, \tag{5}$$

$$x_{jt} \in \{0,1\}, \qquad j \in J, T \in T, \tag{6}$$

where $D(i) \subset J$ denotes the subset of the tours containing site i, $S(j) \subset T$ represents the set of days on which tour j can be performed, and $F(t) \subset J$ denotes the subset of the tours allowed on day t. By constraints (2), at most one tour in the selected itinerary includes site i. Constraints (3) impose that a tour to be included in the itinerary is assigned to one of the days in T on which the tour is allowed. We also require that exactly one tour is selected on each day of the vacation period as prescribed by constraints (4). Finally, the objective function (1) maximizes the total satisfaction of the tourist over the vacation period T.

Avella et al. (2006) solve ITP heuristically in two steps. In the first step, the LP relaxation of (1)-(6) is solved by a column-and-row generation approach due to a potentially huge number of tours. To this end, a large number of possible tours is enumerated and added to J. A subset $\bar{J} \subset J$ of these tours is selected to form the restricted master problem (RMP) for the column-and-row generation procedure. At each iteration, a set of new tours $j \in L \subseteq (J \setminus \bar{J})$ is introduced to the RMP. For each $j \in L$, this implies adding $x_{jt}, t \in S(j)$, and the associated linking constraint $y_j - \sum_{t \in S(j)} x_{jt} = 0$ to the RMP. The column-and-row generation procedure terminates when the condition stated below in Theorem 1.1 is satisfied. Observe that the authors evaluate this condition for each tour in $j \in (J \setminus \bar{J})$ following each optimization of the RMP, where J is known explicitly. Furthermore, in the computational experiments $| L |= 1$; that is, the tour that violates the condition in Theorem 1.1 to the largest extent is added to the RMP. In the second step of the proposed solution approach, (1)-(6) is solved by a commercial solver over the set of tours currently present in the RMP to obtain an integer feasible solution for ITP.

Now, let $\pi_i, i \in V$, $\gamma_j, j \in J$, and $\lambda_t, t \in T$, denote the dual variables associated with the constraints (2)-(4), respectively. The following theorem, given in Avella et al. (2006) without a proof, defines the stopping condition for the column-and-row generation algorithm applied to ITP by the authors.

THEOREM 1.1 *The solution of the current RMP is optimal for the LP relaxation of* (1)-(6) *if* $\bar{r}_j = r_j - \sum_{i:j \in D(i)} \pi_i - \sum_{t:j \in F(t)} \lambda_t \leq 0$, *for each* $j \in J$.

The statement of the theorem above corrects two typos in the original Theorem 3.1 given by Avella et al. (2006), where the termination condition appears as $\bar{r}_j = r_j - \sum_{i:j \in D(i)} \pi_i - \sum_{t:j \in F(t)} \lambda_t \geq 0$, for each $t \in T$ and $j \in D(t)$. Next, we demonstrate that the stopping condition in Theorem 1.1 is incorrect and may lead to a suboptimal solution when the column-and-row generation algorithm proposed by Avella et al. (2006) terminates. Then, we discuss how the generic column-and-row generation algorithm proposed by Muter et al. (2010) may be applied here to solve the proposed large-scale LP correctly.

The following is the dual of the linear programming relaxation of (1)-(6):

$$\text{minimize} \quad \sum_{i \in V} \pi_i + \sum_{t \in T} \lambda_t, \tag{7}$$

$$\text{subject to} \quad \sum_{i:j \in D(i)} \pi_i + \gamma_j \geq r_j, \qquad j \in J, \tag{8}$$

$$-\gamma_j + \lambda_t \geq 0, \qquad j \in J, t \in S(j), \tag{9}$$

$$\pi_i \geq 0, \qquad i \in V. \tag{10}$$

Given the optimal dual variables associated with the current RMP, the resulting pricing subproblem to be solved becomes

$$\zeta_{yx} = \max_{j \in J \setminus \bar{J}} \quad \bar{c}_j, \tag{11}$$

where $\bar{c}_j = r_j - \sum_{i:j \in D(i)} \pi_i - \gamma_j$ denotes the reduced cost of tour j. If the optimal objective function value of this subproblem is positive with $\bar{c}_j > 0$, the variables y_j, $x_{jt}, t \in S(j)$, and the primal constraint $y_j - \sum_{t \in S(j)} x_{jt} = 0$ should be added to the RMP. Otherwise, the optimal solution of the current RMP is declared as optimal for the LP relaxation of (1)-(6), and the algorithm terminates. The challenge here is that the value of the dual variable $\gamma_j, j \in J \setminus \bar{J}$ is unknown because the corresponding constraint is currently absent from the RMP. Hence, in order to design an optimal column-and-row generation algorithm for ITP, we must devise a method that anticipates the correct value of $\gamma_j, j \in J \setminus \bar{J}$ to be incorporated into the pricing subproblem. According to Theorem 1.1, $\sum_{t:j \in F(t)} \lambda_t = \sum_{t \in S(j)} \lambda_t$ is considered as an appropriate estimate for $\gamma_j, j \in J \setminus \bar{J}$. In the sequel, we first show that this is incorrect, and then illustrate how to determine the correct value of $\gamma_j, j \in J \setminus \bar{J}$ based on the generic column-and-row generation framework given by Muter et al. (2010).

Consider an iteration of the column-and-row-generation algorithm, where the current RMP is solved to optimality and the associated optimal dual solution is represented by $\pi_i, i \in V$, $\gamma_j, j \in \bar{J}$, and $\lambda_t, t \in T$. Suppose that $y_{j'}, j' \in J \setminus \bar{J}$ is to be priced out, where $\bar{r}_{j'} = r_{j'} - \sum_{i:j' \in D(i)} \pi_i - \sum_{t:j' \in F(t)} \lambda_t = r_{j'} - \sum_{i:j' \in D(i)} \pi_i - \sum_{t \in S(j')} \lambda_t \leq 0$. Observe for the currently unknown dual variable $\gamma_{j'}$ that the dual constraints (9) imply

$$\gamma_{j'} \leq \min_{t \in S(j')} \lambda_t, \tag{12}$$

and if $\mid S(j') \mid > 1$ and $\max_{t \in S(j')} \lambda_t > 0$, then we may have

$$\gamma_{j'} \leq \min_{t \in S(j')} \lambda_t < \sum_{t \in S(j')} \lambda_t. \tag{13}$$

Clearly, this may lead to

$$\bar{r}_{j'} = r_{j'} - \sum_{i:j' \in D(i)} \pi_i - \sum_{t \in S(j')} \lambda_t \leq 0 < r_{j'} - \sum_{i:j' \in D(i)} \pi_i - \gamma_{j'} = \bar{c}_{j'}. \tag{14}$$

Thus, we conclude that while $\bar{r}_j \leq 0$ for all $j \in J$ as required by Theorem 1.1 due to Avella et al. (2006), there may exist a tour j' with $\bar{c}_{j'} > 0$. In other words, $\sum_{t \in S(j')} \lambda_t$ is not an appropriate estimate of the missing dual variable $\gamma_{j'}$, and the column-and-row generation algorithm of Avella et al. (2006) may terminate prematurely, while there exists a tour j' with a positive reduced cost. On the other hand, if $\lambda_t \leq 0, t \in S(j')$, then

$$\sum_{t \in S(j')} \lambda_t < \gamma_{j'} \leq \min_{t \in S(j')} \lambda_t \tag{15}$$

may hold, and this may result in

$$\bar{c}_{j'} = r_{j'} - \sum_{i:j' \in D(i)} \pi_i - \gamma_{j'} \leq 0 < r_{j'} - \sum_{i:j' \in D(i)} \pi_i - \sum_{t \in S(j')} \lambda_t = \bar{r}_{j'}. \tag{16}$$

That is, a tour j' with a nonpositive reduced cost may violate the condition in Theorem 1.1 and it is unnecessarily added to the RMP.

Using the column-and-row generation algorithm given in Muter et al. (2010), the pricing subproblem becomes a two-stage problem:

$$\zeta_{yx} = \max_{j \in J \setminus \bar{J}} \quad r_j - \sum_{i:j \in D(i)} \pi_i - \alpha_j, \tag{17}$$

where

$$\alpha_j = \text{minimize} \quad \gamma_j, \tag{18}$$

$$\text{subject to} \quad -\gamma_j + \lambda_t \geq 0, \qquad t \in S(j), \tag{19}$$

$$\text{at least one constraint in (19) is tight.} \tag{20}$$

To calculate the correct reduced cost of y_j, we solve the problem (18)-(20) separately for each $j \in J \setminus \bar{J}$. Constraints (19) in the problem (18)-(20) impose that the dual constraints (9) corresponding to the variables $x_{jt}, t \in S(j)$, induced by any $j \in J \setminus \bar{J}$ are not violated. Consequently, if the optimal objective of (17) is $\zeta_{yx} = \bar{c}_j > 0$, then the RMP is augmented with y_j and $x_{jt}, t \in S(j)$, along with one new linking constraint of type (3), and y_j is the natural candidate to enter the basis at the next iteration of the column-and-row generation algorithm. In order to warm-start the primal simplex algorithm to re-optimize the RMP, the optimal basis from the previous iteration must be augmented with a new variable. This new basic variable x_{jt} is provided by the optimal solution of (18)-(20) for y_j and corresponds to a tight constraint in (19). Thus, the complementary slackness condition for the new basic variable is satisfied.

We observe that the problem (18)-(20) is unbounded without the restriction (20) for all $j \in J \setminus \bar{J}$. Therefore, for each $j \in J \setminus \bar{J}$, the solution of (18)-(20) is trivial and $\alpha_j = \gamma_j = \min_{t \in S(j)} \lambda_t$. Consequently, the correct termination condition for the column-row-generation algorithm that solves the LP relaxation of (1)-(6) optimally is formalized in the theorem below.

THEOREM 1.2 *Let $\pi_i, i \in V$, $\gamma_j, j \in \bar{J}$, and $\lambda_t, t \in T$ be the optimal dual solution corresponding to the optimal basis* $\mathbf{B}$ *of the current RMP. The primal solution associated with* $\mathbf{B}$ *is optimal for the LP relaxation of* (1)-(6) *if $r_j - \sum_{i:j \in D(i)} \pi_i - \min_{t \in S(j)} \lambda_t \leq 0$ for every $j \in J \setminus \bar{J}$.*

The proof of this theorem follows from the analysis of the column-and-row generation algorithm developed in Muter et al. (2010).

To support these observations, we generated a random instance of ITP. The column-and-row generation algorithm is first applied to the LP relaxation of this instance using the stopping condition of Avella et al. (2006) as stated in Theorem 1.1. We observe that this stopping condition leads to a premature termination with a suboptimal solution. Then, we employ the stopping condition of Theorem 1.2 and determine the true optimal solution. These results confirm also empirically that the stopping condition given by Avella et al. (2006) is not correct[1].

References

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[1]This counterexample is provided at http://people.sabanciuniv.edu/sibirbil/ejorsupp/.