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Simultaneous Column-and-Row Generation for Large-Scale Linear Programs with Column-Dependent-Rows

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ABSTRACT: In this paper, we develop a *simultaneous column-and-row generation* algorithm that could be applied to a general class of large-scale linear programming problems. These problems typically arise in the context of linear programming formulations with exponentially many variables. The defining property for these formulations is a set of linking constraints, which are either too many to be included in the formulation directly, or the full set of linking constraints can only be identified, if all variables are generated explicitly. Due to this dependence between columns and rows, we refer to this class of linear programs as problems with *column-dependent-rows*. To solve these problems, we need to be able to generate both columns and rows on-the-fly within an efficient solution approach. We emphasize that the generated rows are structural constraints and distinguish our work from the branch-and-cut-and-price framework. We first characterize the underlying assumptions for the proposed column-and-row generation algorithm. These assumptions are general enough and cover all problems with column-dependent-rows studied in the literature up until now to the best of our knowledge. We then introduce in detail a set of pricing subproblems, which are used within the proposed column-and-row generation algorithm. This is followed by a formal discussion on the optimality of the algorithm. To illustrate our approach, the paper is concluded by applying the proposed framework to the multi-stage cutting stock and the quadratic set covering problems.

Keywords: linear programming; column generation; column-and-row generation; row-and-column generation; pricing subproblem; multi-stage cutting stock; quadratic set covering; column-dependent-rows.

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1. Introduction. Column generation is a well-known method to solve large-scale linear programming (LP) problems, pioneered by Dantzig and Wolfe (1960) and Gilmore and Gomory (1961) among others. It is frequently employed to solve the LP relaxation of mixed integer programming problems with exponentially many variables. Some applications of column generation to integer programming problems include Desrochers et al. (1992), Desaulniers et al. (1997), Savelsbergh (1997), Barahona and Jensen (1998), Gamache et al. (1999), Vanderbeck (1999), and Akker et al. (1999). In such large-scale LPs the vast majority of the variables is zero at optimality, and thus the fundamental concept underlying column generation is to initialize the LP with a small set of columns, referred to as the restricted master problem, and then add new columns as required. This is accomplished iteratively by solving a pricing subproblem (PSP) following each optimization of the restricted master problem. In the PSP, the reduced cost of a column is minimized over the set of all columns, and upon solving PSP we either add a new column to the restricted master problem with a negative reduced cost (for minimization) or prove optimality. We refer to Desaulniers et al. (2005) and Lübbecke and Desrosiers (2005) for comprehensive surveys on column generation.

One of the pillars of the classical column generation framework is that the constraints in the master problem are all known explicitly. In this case, the number of rows in the restricted master problem is fixed, and complete dual information is supplied to the PSP from the restricted master problem, which allows us to compute the reduced cost of a column in the subproblem accurately. While this framework has been used successfully for solving a large number of problems over the years, it does not fit applications in which missing columns induce new linking constraints to be added to the restricted master problem. To motivate the discussion, consider a quadratic set covering model, where the binary variable y_k is set to 1, if column k is selected (see for example Saxena and Arora (1997); Bazaraa and Goode (1975)). We compute the total contribution from columns k and l as $c_k y_k + c_l y_l + c_{kl} y_k y_l$, where c_k and c_l are the individual contributions from columns k and l, respectively, and c_{kl} captures the cross-effect of having columns k and l simultaneously in the solution. A common linearization followed by relaxing the integrality constraints would lead to the large-scale LP below:

$$\begin{aligned} \text{minimize} \quad & \ldots + c_k y_k + c_l y_l + c_{kl} x_{kl} + \ldots \\ \text{subject to} \quad & \ldots \\ & y_k + y_l - x_{kl} \leq 1, \quad y_k - x_{kl} \geq 0, \quad y_l - x_{kl} \geq 0, \qquad (1) \\ & 0 \leq y_k, y_l, x_{kl} \leq 1, \\ & \ldots \end{aligned}$$

Note that this model contains three linking constraints for each pair of $y-$variables, and a large number of $y-$variables in an instance would prevent us from including all rows in the restricted master problem a priori. Thus, in this case both rows and columns need to be generated on-the-fly as required. Constraints of type (1) not present in the current restricted master problem may lead to two issues. First, primal feasibility may be violated with respect to the missing constraints. In order to address this issue, we should presumably add variable x_{kl} to the restricted master problem along with one of the variables y_k or y_l. Second, the reduced costs of the variables may be computed incorrectly in the PSP because no dual information associated with the missing constraints is passed from the restricted master problem to the PSP. For instance, assume that y_k is already a part of the restricted master problem, while y_l, x_{kl}, and the linking constraints (1) are absent from it. In this case, the PSP for y_l must anticipate the values of the dual variables associated with the missing constraints (1); otherwise, the reduced cost of y_l is calculated incorrectly. Thus, we conclude that in order to design a column generation algorithm for this particular linearization of the quadratic set covering problem, we need a subproblem definition that allows us to generate several variables and their associated linking constraints simultaneously by correctly estimating the dual values of the missing linking constraints. Later in the paper, we define precisely how we handle both of these issues formally and also provide illustrative examples. Note that this type of dependence between columns can be generalized if several columns interact simultaneously and would lead to a similar problem that grows both column- and row-wise.

The discussion in the preceding paragraph points to a major difficulty in column generation if the number of rows in the restricted master problem depends on the number of columns. We refer to such formulations as problems with *column-dependent-rows*, or briefly as CDR-problems. We emphasize that the solution of a CDR-problem is based on *simultaneous column-and-row generation*, and this is fundamentally different than a branch-and-cut-and-price algorithm that strengthens LP relaxations in a search tree by valid inequalities before applying column generation. We will elaborate on this important distinction further in Section 2.2 when we position our work with respect to the existing literature and refer to Desaulniers et al. (2011) and Desrosiers and Lübbecke (2011) for the use of cutting planes in a branch-and-price setting.

We have two main objectives and contributions in this paper. First, we develop a generic mathematical model for CDR-problems and argue that several important applications and formulations in the literature are captured by this model. Second, we propose a solution methodology for our generic model. The cornerstone of our approach is a subproblem definition that can simultaneously generate new columns as well as new structural constraints that are no longer redundant in the presence of these new columns. This is in marked contrast to traditional column generation, where all structural constraints are added to the restricted master problem at the outset. We also provide two detailed examples that illustrate our proposed modeling and solution methodology.

In the next section, we introduce our generic model, define the underlying assumptions, and motivate it by demonstrating that the multi-stage cutting stock (MSCS) and the quadratic set covering (QSC) problems may be formulated within this framework. After reviewing the related literature in Section 2.2, we develop our proposed generic column-and-row generation algorithm for CDR-problems in Section 3. This is followed by the applications of the proposed method to the MSCS and QSC problems in Section 4. An extension is discussed in Section 5. We conclude and point out potential research directions in Section 6.

2. CDR-Problems. In this section, we first specify the canonical form of the generic mathematical model representing the class of CDR-problems that we consider, and then, discuss the assumptions underlying our modeling and solution framework. We next briefly describe the MSCS and QSC problems and demonstrate that both of these problems satisfy our assumptions and may be formulated according to this generic model. These two problems are selected for their different characteristics that help us illustrate the different features and aspects of our proposed solution method. We mention other CDR-problems that fit into the proposed scheme while discussing the related literature in Section 2.2. The generic mathematical formulation of CDR-problems appears below, and we refer to it as the *master problem*, following the common terminology in column generation:

$$
\begin{array}{llllll}
(\mathbf{MP}) & \text{minimize} & \sum_{k\in K} c_k y_k + & \sum_{n\in N} d_n x_n, & & \\
& \text{subject to} & \sum_{k\in K} A_{jk} y_k & & \geq a_j, \quad j\in J, & \text{(MP-y)}\\
& & & \sum_{n\in N} B_{mn} x_n & \geq b_m, \quad m\in M, & \text{(MP-x)}\\
& & \sum_{k\in K} C_{ik} y_k + & \sum_{n\in N} D_{in} x_n & \geq r_i, \quad i\in I, & \text{(MP-yx)}\\
& & \multicolumn{3}{l}{y_k \geq 0, k\in K, \quad x_n \geq 0, n\in N.} &
\end{array}
$$

There may be exponentially many $y-$ and $x-$ variables in the formulation above, and we allow both types of variables to be generated in a column generation algorithm applied to solve the master problem. We assume that the set of constraints (MP-y) and (MP-x) are known explicitly and their cardinality is polynomially bounded in the size of the problem. On the other hand, a complete description of the set of linking constraints (MP-yx) may not be available. If this is the case, we may have to generate all $y-$ and $x-$ variables in the worst case to identify all linking constraints in a column generation algorithm. The discussion on a robust crew pairing problem studied by Muter et al. (2010b) in Section 2.2 provides an example for this case. Even if all linking constraints (MP-yx) are known explicitly a priori, there may be exponentially many of them. For instance, in the QSC example introduced in the previous section each pair of variables induces three linking constraints in the linearized formulation, and incorporating all $\mathcal{O}(\mid K\mid^2)$ linking constraints in the formulation directly is not a viable alternative for large $\mid K \mid$.

Based on the discussion in the preceding paragraph, the column-and-row generation algorithm for solving the master problem is initialized with subsets $\bar{K} \subset K$ and $\bar{N} \subset N$. The resulting model is

$$
\begin{aligned}
(\textbf{SRMP}) \qquad \text{minimize} \quad & \sum_{k\in\bar{K}} c_k y_k + \sum_{n\in\bar{N}} d_n x_n, \\
\text{subject to} \quad & \sum_{k\in\bar{K}} A_{jk} y_k \geq a_j, \quad j \in J, && \text{(SRMP-y)} \\
& \sum_{n\in\bar{N}} B_{mn} x_n \geq b_m, \quad m \in M, && \text{(SRMP-x)} \\
& \sum_{k\in\bar{K}} C_{ik} y_k + \sum_{n\in\bar{N}} D_{in} x_n \geq r_i, \quad i \in I(\bar{K},\bar{N}), && \text{(SRMP-yx)} \\
& y_k \geq 0, k \in \bar{K}, \quad x_n \geq 0, n \in \bar{N},
\end{aligned}
$$

where $I(\bar{K},\bar{N}) \subset I$ in (SRMP-yx) denotes the set of linking constraints formed by $\{y_k | k \in \bar{K}\}$, and $\{x_n | n \in \bar{N}\}$. During the column generation phase, new variables $\{y_k | k \in S_K\}$ and $\{x_n | n \in S_N\}$, where $S_K \subseteq (K \setminus \bar{K})$ and $S_N \subseteq (N \setminus \bar{N})$, are added to the restricted master problem iteratively as required as a result of solving different types of PSPs which we discuss in depth in Section 3. Moreover, these new variables may appear in new linking constraints currently absent from the restricted master problem, where the set of these new linking constraints is represented by $\Delta(S_K, S_N) = I(\bar{K} \cup S_K, \bar{N} \cup S_N) \setminus I(\bar{K}, \bar{N})$. Thus, the restricted master problem grows both vertically and horizontally during column generation, and due to this special structure we refer to the restricted master problem in our column-and-row generation algorithm as the *short restricted master problem* (SRMP).

Three main assumptions characterize the type of problems that fit into our generic model and that we can tackle by our proposed solution methodology. In the next section, we argue that all of these assumptions hold for our two illustrative CDR-problems; QSC and MSCS. Moreover, in Section 2.2 we consider other problems from the literature for which it is trivial to check that these assumptions also apply. The first assumption implies that the generation of the $x-$variables depends on the generation of the $y-$variables. Moreover, each $x-$variable is associated with only one set of linking constraints.

Assumption 2.1 *The generation of a new set of variables $\{y_k | k \in S_K\}$ prompts the generation of a new set of variables $\{x_n | n \in S_N(S_K)\}$. Furthermore, a variable $x_{n'}, n' \in S_N(S_K)$, does not appear in any* linking constraints *other than those indexed by $\Delta(S_K, S_N(S_K))$ and introduced to the SRMP along with $\{y_k | k \in S_K\}$ and $\{x_n | n \in S_N(S_K)\}$.*

Note that the dependence of $\bar{N}$ on $\bar{K}$ is designated by the index set $S_N(S_K)$. In the remainder of the paper, we will use the shorthand notation $\Delta(S_K)$ instead of $\Delta(S_K, S_N(S_K))$ whenever there is no ambiguity.

The next assumption requires the definition of a *minimal variable set*. A minimal variable set is a set of $y-$variables that triggers the generation of a set of $x-$variables and the associated linking constraints in the sense of Assumption 2.1. In the QSC formulation in Section 1, a minimal variable set given by $\{y_k, y_l\}$ consists of the variables y_k and y_l and generates a set of linking constraints of type (1) and the variable x_{kl}. We also note that in our subsequent discussion, we shall see that there may be several minimal variable sets associated with a set of linking constraints. Thus, we state the following assumption for the general case.

ASSUMPTION 2.2 *A linking constraint is redundant until* all *variables in at least one of the minimal variable sets associated with this linking constraint are added to the SRMP.*

This assumption implies that a feasible solution of SRMP does not violate any missing linking constraint before all variables in at least one of the associated minimal variable sets are added to the SRMP.

Assumptions 2.1 and 2.2 together define the goal of the fundamental subproblem in our proposed column-and-row generation approach. The objective of the *row-generating PSP* derived in Section 3 is to identify one or several minimal variable sets, where each minimal variable set $\{y_k | k \in S_K\}$ yields a set of variables $\{x_n | n \in S_N(S_K)\}$. These two sets of variables appear in a set of linking constraints indexed by $\Delta(S_K)$ currently not present in the SRMP, and we are also required to add these constraints to the SRMP to avoid violating the primal feasibility of the master problem (MP). Thus, for each new minimal variable set $\{y_k | k \in S_K\}$ to be introduced into the SRMP as an output of the row-generating PSP, the index sets defining SRMP are updated as $\bar{K} \leftarrow \bar{K} \cup S_K$, $\bar{N} \leftarrow \bar{N} \cup S_N(S_K)$, and a new set of constraints $\Delta(S_K)$ appear in the SRMP. Clearly, at least one of the currently generated $y-$variables must have a negative reduced cost.

The next assumption characterizes the signs of the coefficients in the linking constraints.

ASSUMPTION 2.3 *Suppose that we are given a minimal variable set $\{y_l | l \in S_K\}$ that generates a set of linking constraints $\Delta(S_K)$ and a set of associated $x-$variables $\{x_n | n \in S_N(S_K)\}$. When the set of linking constraints $\Delta(S_K)$ is* first *introduced into the SRMP during the column-and-row generation, then for each $k \in S_K$ there exists a constraint $i \in \Delta(S_K)$ of the form*

$$C_{ik} y_k + \sum_{n \in S_N(S_K)} D_{in} x_n \geq 0, \tag{2}$$

where $C_{ik} > 0$ and $D_{in} < 0$ for all $n \in S_N(S_K)$.

Assumption 2.3 ensures that a variable $x_n, n \in S_N(S_K)$, cannot assume a positive value until *all* variables in at least one of the minimal variable sets that generate $\Delta(S_K)$ are positive in the SRMP. In addition, we emphasize that although we use (2) throughout this paper, our analysis is also valid when a constraint of type (2) is given in a disaggregated form like

$$C_{ik} y_k + D_{in} x_n \geq 0, \quad n \in S_N(S_K).$$

Furthermore, linking constraints of type (2) may be specified as equalities in some CDR-problems. This case may also be handled with minor modifications to the analysis in Section 3. For ease of exposition, we omit the equality case and refer to Muter et al. (2010a) for details.

We further classify CDR-problems as *CDR-problems with interaction* and *CDR-problems with no interaction*. This distinction between two problem types plays an important role in our analysis.

Definition 2.1 *In a CDR-problem* with interaction*, the cardinality of any minimal variable set is larger than one. On the other hand, if each minimal variable set is a singleton, then the corresponding problem belongs to the class of CDR-problems* with no interaction.

Differentiating between CDR-problems with and with no interaction allows us to focus on the unique properties of these two types that affect the analysis of the row-generating PSP in Section 3. However, it is possible to combine the tools developed in this paper to tackle CDR-problems in which some minimal variable sets are singletons while others include more than one variable. This extension is discussed in Section 5.

2.1 Illustrative Examples. In the one-dimensional multi-stage cutting stock (MSCS) problem, operational restrictions impose that stock rolls are cut into finished rolls in more than one stage. (See Haessler (1971); Ferreira et al. (1990); Zak (2002a,b).) The objective is to minimize the number of stock rolls used for satisfying the demand for finished rolls, and appropriate cutting patterns need to be identified for each stage in the cutting process. We restrict our attention to the two-stage cutting stock problem similar to the study by Zak (2002b). In the first stage, a stock roll is cut into intermediate rolls, while finished rolls are produced from these intermediate rolls in the second stage. If we ignore the integrality restrictions, then the LP model for the MSCS problem is given by

$$\text{minimize} \quad \sum_{k\in K} y_k, \tag{3}$$

$$\text{subject to} \quad \sum_{n\in N} B_{mn}x_n \geq b_m, \quad m \in M, \tag{4}$$

$$\sum_{k\in K} C_{ik}y_k + \sum_{n\in N} D_{in}x_n \geq 0, \quad i \in I, \tag{5}$$

$$y_k \geq 0, k \in K, \quad x_n \geq 0, n \in N, \tag{6}$$

where the set of intermediate and finished rolls are denoted by I and M, respectively. The set of cutting patterns K for the first stage constitute the columns of C. Similarly, the columns of B are formed by the set of cutting patterns N for the second stage. The matrix D establishes the relationship between the cutting patterns in the first and the second stages. A single non-zero entry $D_{in} = -1$ in column n of D indicates that the cutting pattern n for the second stage is cut from the intermediate roll i. Constraints (4) ensure that the demand for finished rolls given by the vector b is satisfied, and constraints (5) impose that the consumption of the intermediate rolls does not exceed their production. The objective is to minimize the total number of stock rolls required. Clearly, this problem is a special case of the generic model (MP), where A, a, d, and r are zero, and c is a vector of all ones. In general, there may be exponentially many feasible cutting patterns in both stages, which prompts us to develop a column generation algorithm for solving this formulation. The challenging issue is that each generated cutting pattern for the first stage, which includes an intermediate roll currently absent from the restricted master problem, adds one more constraint to the model. Thus, the restricted master problem grows both horizontally and vertically and exhibits the structure of a CDR-problem. MSCS satisfies Assumption 2.1 because a cutting pattern for the second stage based on an intermediate roll i cannot be generated unless there exists at least one cutting pattern for the first stage that includes this intermediate roll i. Moreover, the associated linking constraint is redundant in this case as required by Assumption 2.2, and any cutting pattern for the first stage that contains a currently absent intermediate roll i constitutes a minimal variable set for the corresponding linking constraint. The last assumption does also hold because the linking constraint corresponding to a currently absent intermediate roll is of the form (2). We conclude that MSCS belongs to the class of CDR-problems with no interaction. Our proposed solution method will be applied to MSCS in Section 4.1.

In the QSC problem, the objective is to cover all items $j \in J$ by the sets $k \in K$ at minimum total cost. In addition to the sum of the individual costs of the sets, we also incorporate a cross-effect between each pair of sets $k, l \in K$ which results in a quadratic objective function. Bazaraa and Goode (1975) and Saxena and Arora (1997) study this problem. QSC is formulated as

$$\begin{aligned} \text{minimize} \quad & y^{\intercal} F y, \\ \text{subject to} \quad & Ay \geq 1, \\ & y \in \{0,1\}^{|K|}, \end{aligned}$$

where A is a binary $\mid J \mid \times \mid K \mid$ matrix of set memberships, and F is a symmetric positive semidefinite $\mid K \mid \times \mid K \mid$ cost matrix. To linearize the objective function, we add a binary variable x_{kl} for each pair of sets $k, l \in K$. A set of linking constraints mandates that $x_{kl} = 1$ if and only if $y_k = y_l = 1$. Relaxing the integrality restrictions leads to the following linear program:

$$\begin{aligned} \text{minimize} \quad & \sum_{k \in K} f_{kk} y_k + \sum_{(k,l) \in P, k<l} 2 f_{kl} x_{kl}, && && (7) \\ \text{subject to} \quad & \sum_{k \in K} A_{jk} y_k \geq 1, && j \in J, && (8) \\ & y_k + y_l - x_{kl} \leq 1, && (k,l) \in P, \ k < l, && (9) \\ & y_k - x_{kl} \geq 0, && (k,l) \in P, \ k < l, && (10) \\ & y_l - x_{kl} \geq 0, && (k,l) \in P, \ k < l, && (11) \\ & y_k \geq 0, && k \in K, && (12) \\ & x_{kl} \geq 0, && (k,l) \in P, \ k < l, && (13) \end{aligned}$$

where $P := K \times K$ is the set of all possible pairs, and $A_{jk} = 1$, if item j is covered by set k; and 0, otherwise. The first set of constraints is the coverage constraints and the remaining are the linking constraints. This problem is a special case of the generic model (MP) with both B and b equal to zero, and a is a vector of ones. The vector of cost coefficients c and d in (MP) are formed by the diagonal and off-diagonal entries of the cost matrix F, respectively. To solve this formulation by column generation, we select a subset of the columns from K and the associated linking constraints to form the initial SRMP. If a new variable, say y_k, enters SRMP, a set of linking constraints and $x-$variables for each pair (k, l) with $l \in \bar{K}$ are also added. We note that the variable x_{kl} and the set of linking constraints $y_k + y_l - x_{kl} \leq 1$, $y_k - x_{kl} \geq 0$, and $y_l - x_{kl} \geq 0$ are redundant until both of the variables y_k and y_l are part of the SRMP. Thus, the minimal variable set $\{y_k, y_l\}$ allows us to generate x_{kl} and the constraints that relate these three variables. We arrive at the conclusion that QSC is a CDR-problem with interaction that satisfies both Assumptions 2.1 and 2.2 stipulated previously. Moreover, the set of linking constraints induced by any minimal variable set $S_K = \{y_k, y_l\}$ conforms to the characterization in Assumption 2.3 because the constraints (10) and (11) are of the form (2). In Section 4.2, we show that our proposed solution method for CDR-problems can handle the formulation (7)-(13).

For some problems, the linking constraints (9)-(11) may be formed by a strict subset $\bar{P}$ of the set of all possible pairs P. If in addition an explicit complete description of $\bar{P}$ is not available a priori before invoking a column generation algorithm, then we refer to these problems as *QSC with restricted pairs* (see also the discussion in the paragraph immediately following the statement of problem (MP).) Typically, in QSC problems with restricted pairs the generation of the pairs that belong to $\bar{P}$ requires a call to an oracle. One example is studied by Muter et al. (2010b) discussed in the next section.

2.2 Related Literature. The literature on the CDR-problems is somewhat limited. In this section, we discuss the existing work in the literature and position our contributions. When it comes to the CDR-problems mentioned in this section, it is relatively easy to check that these problems satisfy our assumptions. Therefore, the proposed column-and-row generation algorithm indeed provides a generic approach to solve these problems.

To the best of our knowledge, the first column-and-row generation algorithm as we consider here was devised by Zak (2002b), who tries to solve a one-dimensional MSCS problem that was introduced in Section 2.1. In his column-and-row generation algorithm, Zak (2002b) defines three types of PSPs. The first PSP looks for a new first-stage cutting pattern, which only includes the intermediate rolls that are already present in the restricted master problem. In the second PSP, the objective is to identify new cutting patterns for the second stage based on the currently existing intermediate rolls. Both of these PSPs are classical knapsack problems. The final PSP considers the possibility of generating both new intermediate rolls and related cutting patterns simultaneously and results in a difficult nonlinear integer programming problem. This subproblem is solved heuristically under a restrictive assumption which dictates that only one new intermediate roll can be generated at each iteration. Thus, the solution method of Zak (2002b) may terminate prematurely at a suboptimal solution which is verified by applying our proposed solution method to an instance provided by Zak (2002a). This is discussed in Muter (2011). A two-stage batch scheduling problem that is structurally similar to MSCS is formulated by Wang and Tang (2010). The proposed column-and-row generation algorithm suffers from an analogous restrictive assumption.

Avella et al. (2006) study a time-constrained routing (TCR) problem motivated by an application that needs to schedule the visit of a tourist to a given geographical area as efficiently as possible in order to maximize her total satisfaction. The goal is to send the tourist on one tour during each day in the vacation period while ensuring that each attraction site is visited no more than once. This problem is formulated as a set packing problem with side constraints and solved heuristically by a column-and-row generation approach due to a potentially huge number of tours. The authors enumerate and store a large number of tours before invoking their column generation algorithm. The SRMP for solving the LP relaxation of the proposed formulation is initialized with a subset of the enumerated tours. A selected tour must be assigned to one of the days in the vacation period. Each generated tour during the column generation procedure introduces a set of variables and leads to a new linking constraint in the SRMP. The authors define an optimality condition for terminating their column generation algorithm based on the dual variables of the constraints in the current SRMP. Following each optimization of the SRMP, this condition is verified for each tour currently absent from the SRMP; i.e., no PSP is required. In Muter et al. (2010a), we demonstrate that this stopping condition fails to account for the dual variables of the missing linking constraints properly and may lead to a suboptimal LP solution at termination.

Avella et al. (2007) propose a branch-and-cut-and-price algorithm for the well-known P-median problem. In their formulation, a set of binary variables indicate the set of selected median nodes, and binary assignment variables designate the median node assigned to each node in the network. These two types of binary variables are linked by variable upper bound constraints. One of the main contributions of the authors is a column-and-row generation method for solving the LP relaxation of this formulation. The algorithm is invoked with a subset of the assignment variables and additional ones are generated as necessary. The generation of each assignment variable leads to a single new linking constraint added to the SRMP for primal feasibility, and the dual variable associated with this linking constraint is calculated correctly a priori due to the special structure of the formulation and incorporated directly into the reduced cost calculations. No PSP is required because all potential assignment variables are known explicitly. Similar to the formulation in the previous work of these authors on the TCR problem, we note that the P-median formulation investigated in Avella et al. (2007) is a special case of our generic formulation (MP) and can be handled by our proposed solution methodology.

Muter et al. (2010b) study a robust airline crew pairing problem for managing extra flights with the objective of hedging against a certain type of operational disruption by incorporating robustness into the pairings generated at the planning level. In particular, they address how a set of extra flights may be added into the flight schedule at the time of operation by modifying the pairings at hand and without delaying or canceling the existing flights in the schedule. Essentially, this is accomplished in two different ways. An extra flight may either be inserted into an existing pairing with ample connection time (a type-B solution) or the schedules of a pair of pairings are partially swapped to cover an extra flight while ensuring the feasibility of these two pairings before and after the swap (a type-A solution). In the latter case, there is a benefit of having a pair of pairings in the solution simultaneously. However, an additional complicating factor is that the set of type-A solutions and the associated linking constraints are not known explicitly. This is akin to the MSCS problem, where the set of intermediate rolls is not available a priori. Ultimately, the mathematical model proposed by Muter et al. (2010b) boils down to a QSC problem with restricted pairs and side constraints. The model is linearized by the same approach as that in (7)-(13) for the QSC problem. Muter et al. (2010b) devise a heuristic two-phase iterative column-and-row generation strategy to solve the LP relaxation of their master problem. In the first phase, the number of constraints in the SRMP is fixed and column generation is applied in a classical manner. Then, in the second phase additional type-A solutions are identified based on the pairings generated during the last call to the column generation with a fixed number of constraints, and the associated constraints are added to the SRMP before the next iteration of the algorithm resumes. We note that the problem in Muter et al. (2010b) is a CDR-problem and can be handled by the proposed methodology in this paper.

Finally, we refer to a recent work by Feillet et al. (2010). In this work, the optimality conditions for column-and-row generation are analyzed for two sample problems; the split delivery vehicle routing problem and the service network design problem for an urban rapid transit system. The authors claim that there is no simple rule to construct an optimal solution and thus, one has to define specifically how to proceed for every application case. Our work, however, does state a generic model and characterizes the type of problems that can be solved by column-and-row generation including those discussed by Feillet et al. (2010). Besides, we also propose an associated solution framework to design a column-and-row generation algorithm for CDR-problems.

Column-and-row generation (or row-and-column generation) is a term without a widely-agreed precise definition. Therefore, we conclude this section by distinguishing our work from others, who use the same term in a different context. For instance, both Frangioni and Gendron (2009), and Katayama et al. (2009) consider the multi-commodity capacitated network design problem and employ column-and-row generation algorithms. In both of these cases, the rows that are added to the formulation are valid inequalities that strengthen the LP relaxation in line with the general branch-and-cut-and-price paradigm (see Desaulniers et al. (2011); Desrosiers and Lübbecke (2011)). This is very different than our framework for CDR-problems, in which generated rows are structural constraints that are required for the validity of the formulation. Furthermore, as pointed out by Frangioni and Gendron (2009) the column- and row generation subproblems in the branch-and-cut-and-price context are either independent from each other or generated columns introduce new cuts with trivial separation problems. For a CDR-problem, the situation is completely different as we study thoroughly in the next section.

3. Proposed Solution Method. In this section, we develop a generic column-and-row generation algorithm that can handle all CDR-problems presented so far, including our prototype examples QSC and MSCS as well as those mentioned in Section 2.2. First, we discuss the rationale of the proposed algorithm at a higher level without going into the details of the specific PSPs, and then analyze each type of subproblem separately. We devote most of the discussion to the row-generating PSP. We conclude this section with a proof of optimality of the proposed algorithm before the mechanics of the algorithm is illustrated on the MSCS and QSC problems in Section 4.

For ease of exposition, we next state the dual of (MP):

$$
\begin{array}{llllll}
(\textbf{DMP}) & \text{maximize} & \sum_{j\in J} a_j u_j + \sum_{m\in M} b_m v_m + \sum_{i\in I} r_i w_i, & & & \\
& \text{subject to} & \sum_{j\in J} A_{jk} u_j + \sum_{i\in I} C_{ik} w_i \le c_k, & k\in K, & & \text{(DMP-y)}\\
& & \sum_{m\in M} B_{mn} v_m + \sum_{i\in I} D_{in} w_i \le d_n, & n\in N, & & \text{(DMP-x)}\\
& & u_j \ge 0, j\in J, \quad v_m \ge 0, m\in M, \quad w_i\ge 0, i\in I, & & &
\end{array}
$$

where u, v, and w denote the dual variables associated with the sets of constraints (MP-y), (MP-x), and (MP-yx), respectively.

As discussed in Section 1, the traditional column generation framework operates under the assumption that the number of constraints in the restricted master problem stays constant throughout the algorithm and all corresponding dual variables are known explicitly. This property is violated for CDR-problems, where generated columns introduce new constraints into the SRMP, and we need a new set of tools to solve these problems by column generation. In Section 1, we argued that the constraints missing in the SRMP may lead to a premature termination, if classical column generation is applied to the SRMP of a CDR-problem naively. To motivate our solution method and demonstrate our point formally, consider a set of variables $\{y_k | k \in S_K\}$, currently not present in the SRMP, and assume that adding these variables to the SRMP would also require adding a set of constraints $\Delta(S_K)$. Based on (DMP-y), the reduced cost $\bar{c}_k$ of $y_k, k\in S_K$, is then given by

$$
\bar{c}_k = c_k - \sum_{j\in J} A_{jk} u_j - \sum_{i\in I(\bar{K},\bar{N})} C_{ik} w_i - \sum_{i\in\Delta(S_K)} C_{ik} w_i, \tag{14}
$$

and ignoring the dual variables $\{w_i | i \in \Delta(S_K)\}$ could result in

$$\bar{c}_k < 0 \leq c_k - \sum_{j \in J} A_{jk} u_j - \sum_{i \in I(\bar{K}, \bar{N})} C_{ik} w_i. \tag{15}$$

In this case, we fail to detect that y_k prices out favorably. Avella et al. (2006) commit such an error as discussed in depth by Muter et al. (2010a).

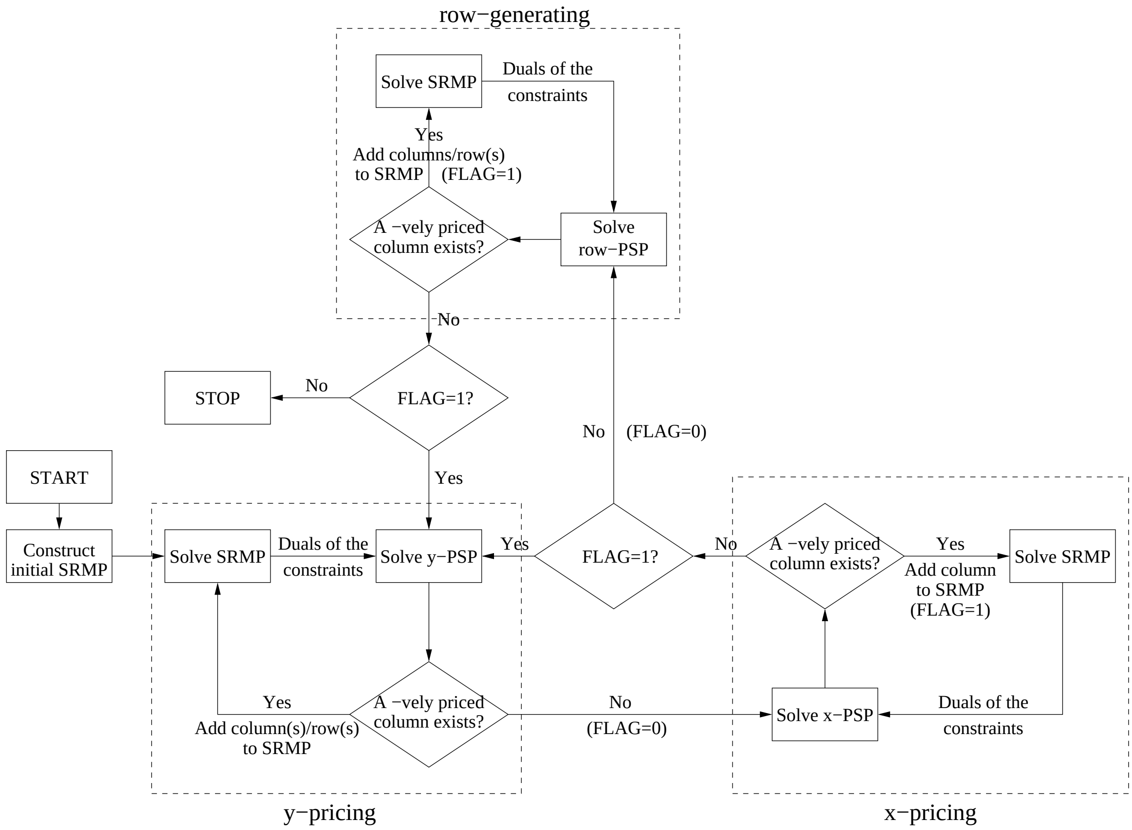

Figure 1: The flow of the proposed column-and-row-generation algorithm.

An overview of the proposed column-and-row generation algorithm is depicted in Figure 1. The $y-$ and $x-$PSPs search for new $y-$ and $x-$ variables, respectively, under the assumption that these variables price out favorably with respect to the *current* set of rows in the SRMP. The $y-$PSP may also incidentally induce new linking constraints as we explain in the next section. On the other hand, the row-generating PSP identifies at least one $y-$variable with a negative reduced cost, only if a set of new linking constraints and related $x-$variables are added to the SRMP. We note that not all CDR-problems give rise to all three PSPs as we discuss separately in the context of each PSP in the sequel. Theoretically, the order of invoking these subproblems does not matter; however, solving the row-generating PSP turns out to be computationally the most expensive in general. Therefore, we adopt the convention illustrated in Figure 1. The algorithm commences by calling the $y-$PSP repeatedly as long as new $y-$variables are generated, and then invokes the $x-$PSP in a similar manner. Finally, the row-generating PSP is called, if we can no longer generate $y-$ or $x-$ variables given the current set of constraints in the SRMP. Observe that we return to the $y-$PSP after solving a series of $x-$ or row-generating PSPs because the dual variables in the $y-$PSP are modified. The proposed column-and-row generation algorithm terminates, if solving the

$y-$, $x-$, and the row-generating PSPs *consecutively in a single pass* does not yield a negatively priced column (only when FLAG=0 in Figure 1). Next, we investigate each PSP in detail.

3.1 $y-$Pricing Subproblem. This subproblem checks the feasibility of the dual constraints (DMP-y) using the values of the known dual variables. The objective is to determine a variable y_k, $k \in (K \setminus \bar{K})$ with a negative reduced cost. The $y-$PSP is stated as

$$\zeta_y = \min_{k\in(K\setminus\bar{K})} \quad \{c_k - \textstyle\sum_{j\in J} A_{jk}u_j - \sum_{i\in I(\bar{K},\bar{N})} C_{ik}w_i\}, \tag{16}$$

where the dual variables $\{u_j|j \in J\}$ and $\{w_i|i \in I(\bar{K},\bar{N})\}$ are obtained from the optimal solution of the current SRMP. If ζ_y is nonnegative, we move to the next subproblem. Otherwise, there exists y_k with $\bar{c}_k < 0$, and SRMP grows by a single variable by setting $\bar{K} \leftarrow \bar{K} \cup \{k\}$. For example, a column-and-row generation algorithm for the problems MSCS and QSC with restricted pairs requires this PSP.

At this point we note that whenever a column y_k with a negative reduced cost is generated, one or several minimal variable sets may be coincidentally completed by the introduction of this new variable. Consequently, it may become necessary, particularly for CDR-problems with interaction, to add the associated sets of linking constraints as well as the $x-$variables to the SRMP before re-invoking the $y-$PSP. For MSCS, this subproblem generates a cutting pattern for the first stage composed of the existing intermediate rolls only. Hence, no new linking constraint can be added. However, consider the QSC problem with restricted pairs and a pair of columns y_k and y_l, where $(k,l) \in \bar{P}$. When the $y-$PSP generates y_k, the associated column y_l may already be present in the SRMP. This would then require augmenting the problem with new constraints of type (9)-(11). Ultimately, when the $y-$PSP is unable to produce any more new columns, it is guaranteed that all linking constraints, which are induced by the minimal variable sets that are currently in the SRMP, are already generated. Although the $y-$PSP may yield new sets of linking constraints, we stress that it differs fundamentally from the row-generating PSP. In the former case, new linking constraints are only a by-product of the newly generated columns. However, the latter problem is solved with the sole purpose of identifying new linking constraints that help us price out additional $y-$variables which otherwise possess nonnegative reduced costs.

3.2 $x-$Pricing Subproblem. This subproblem attempts to generate a new $x-$variable by identifying a violated constraint (DMP-x) and assumes that the number of constraints in the SRMP is fixed. Recall from our previous discussion that no new linking constraint may be induced in the SRMP without generating new $y-$variables in the proposed column-and-row generation algorithm; that is, $\Delta(\emptyset) = \emptyset$ for this PSP (see also Assumption 2.1). Thus, all dual variables that appear in this PSP are known explicitly. The $x-$PSP is then simply given by

$$\zeta_x = \min_{n\in N_{\bar{K}}} \quad \{d_n - \textstyle\sum_{m\in M} B_{mn}v_m - \sum_{i\in I(\bar{K},\bar{N})} D_{in}w_i\}, \tag{17}$$

where the dual variables $\{v_m|m \in M\}$ and $\{w_i|i \in I(\bar{K},\bar{N})\}$ are retrieved from the optimal solution of the current SRMP. In order to introduce a new variable x_n into the SRMP, we require that at least one associated minimal set of variables $\{y_k|k \in S_K\}$ is already present in the model; that is, $S_K \subseteq \bar{K}$. Consequently, the search for x_n with a negative reduced cost in this PSP is restricted to the set $N_{\bar{K}} \subseteq N$, where $N_{\bar{K}}$ is the index set of all $x-$variables that may be induced by the set of variables $\{y_k|k \in \bar{K}\}$ in the current SRMP. We update $\bar{N} \leftarrow \bar{N} \cup \{n\}$ if $\zeta_x < 0$, i.e., if the $x-$PSP determines a variable $x_n, n \in N_{\bar{K}}$ that prices out favorably. Otherwise, the column-and-row generation algorithm continues with the appropriate subproblem dictated by the flow of the algorithm in Figure 1. In the MSCS problem, the $x-$PSP identifies cutting patterns for the second stage that only consume intermediate rolls that are produced by the cutting patterns for the first stage in the current SRMP. This PSP is not needed in a column-and-row generation algorithm for QSC-type problems because the $x-$variables in the corresponding formulations are auxiliary and are only added to the SRMP along with a set of new linking constraints induced by a set of new $y-$variables.

3.3 Row-Generating Pricing Subproblem. Note that before invoking the row-generating PSP, we always ensure that no negatively priced variables exist with respect to the current set of constraints in the SRMP (see Figure 1). Therefore, the objective of this PSP is to identify new columns that price out favorably *only after* adding new linking constraints currently absent from the SRMP. The primary challenge here is to properly account for the values of the dual variables of the missing constraints, and thus be able to determine which linking constraints should be added to the SRMP together with a set of variables. Demonstrating that this task can be accomplished implicitly is a fundamental contribution of the proposed solution framework. Under the assumptions for CDR-problems stated in Section 2, we can correctly anticipate the *optimal* values of the dual variables of the missing constraints without actually introducing them into the SRMP first, and this thinking-ahead approach enables us to calculate all reduced costs correctly in our column-and-row generation algorithm for CDR-problems. Furthermore, recall that Assumption 2.3 stipulates that a variable x_n that appears in a new linking constraint cannot assume a positive value unless all $y-$variables in an associated minimal variable set are positive. Thus, while we generate $x-$ and $y-$variables simultaneously in this PSP along with a set of linking constraints, the ultimate goal is to generate at least one $y-$variable with a negative reduced cost. We formalize these concepts later in the discussion.

In the context of the row-generating PSP, we need to distinguish between CDR-problems with and with no interaction as specified in Definition 2.1. For CDR-problems with no interaction, a single variable $y_k, k \notin \bar{K}$, may induce one or several new linking constraints. For instance, in the MSCS problem a cutting pattern $y_k, k \notin \bar{K}$, for the first stage leads to one new linking constraint per intermediate roll that it includes and is currently missing in the SRMP. Thus, all linking constraints that are required in the SRMP to decrease the reduced cost of y_k below zero may be directly induced by adding y_k to the SRMP. However, in CDR-problems with interaction no single variable y_k induces a set of new linking constraints, and the row-generating PSP must be capable of identifying one or several minimal variable sets, each with a cardinality larger than one, to add to the SRMP so that y_k prices out favorably in the presence of these one or several new sets of linking constraints. To illustrate this point for QSC, assume that the reduced cost of $y_k, k \notin \bar{K}$, is positive if we only consider the minimal variable sets of the form $\{y_k, y_l\}, l \in \bar{K}$. However, the reduced cost of y_k may turn negative if it is generated along with $y_{l'}, l' \notin \bar{K}$. In this case, $\{y_k, y_{l'}\}$ is a separate minimal variable set that introduces an additional set of linking constraints of the form (9)-(11) into the SRMP.

Summarizing, the optimal solution of the row-generating PSP is a family $\mathcal{F}_k$ of index sets S_K^k, where each element $S_K^k \in \mathcal{F}_k$ is associated with a minimal variable set $\{y_l | l \in S_K^k\}$, and k in the superscript of the index set S_K^k denotes that $y_k \in \{y_l | l \in S_K^k\}$. Consequently, $\mathcal{F}_k$ is an element of the power set $\mathcal{P}_k$ of the set composed by the index sets of the minimal variable sets containing y_k. If the reduced cost $\bar{c}_k$ corresponding to the optimal family $\mathcal{F}_k$ is negative, then SRMP grows both horizontally and vertically with the addition of the variables $\{y_l|l \in \Sigma_k\}$, $\{x_n|n \in S_N(\Sigma_k)\}$, and the set of linking constraints $\Delta(\Sigma_k)$, where $\Sigma_k = \cup_{S_K^k \in \mathcal{F}_k} S_K^k$ denotes the index set of all $y-$variables introduced to the SRMP along with y_k. In the following discussion, SRMP$(\bar{K}, \bar{N}, I(\bar{K}, \bar{N}))$ refers to the current SRMP formed by $\{y_k|k \in \bar{K}\}$, $\{x_n|n \in \bar{N}\}$, and the set of linking constraints $I(\bar{K}, \bar{N})$ in addition to the structural constraints (SRMP-y)-(SRMP-x). Consequently, the outcome of the row-generating PSP is represented as SRMP$(\bar{K} \cup \Sigma_k, \bar{N} \cup S_N(\Sigma_k), I(\bar{K}, \bar{N}) \cup \Delta(\Sigma_k))$. In Table 1, we summarize our notation required for a detailed analysis of the row-generating PSP in the sequel.

A further distinction between CDR-problems with and with no interaction needs to be clarified before we delve into the mechanics of the row-generating PSP. The oracle that solves the row-generating PSP yields a family $\mathcal{F}_k$ –along with an associated index set Σ_k– so that $\bar{c}_k < 0$, if SRMP grows as specified above. For CDR-problems with no interaction, the optimal family of index sets reduces to a singleton, i.e., $\mathcal{F}_k = \{\{k\}\}$ and $\Sigma_k = \{k\}$. Furthermore, we must have $k \notin \bar{K}$; otherwise, $\bar{c}_k \geq 0$ would hold because $k \in \bar{K}$ implies $S_N(\{k\}) \subseteq \bar{N}$ and $\Delta(\{k\}) \subseteq I(\bar{K}, \bar{N})$, and the current SRMP would have been solved to optimality with all constraints relevant for y_k. On the other hand, for CDR-problems with interaction there may exist an $l \in \Sigma_k$ with $l \in \bar{K}$.

Table 1: Notation for the analysis of the row-generating PSP.

S_K	the index set of a minimal variable set $\{y_l|l \in S_K\}$.
S_K^k	index k denotes that y_k is a member of the minimal variable set $\{y_l|l \in S_K^k\}$.
$S_N(S_K)$	the index set of the $x-$variables induced by $\{y_l|l \in S_K\}$.
$\Delta(S_K)$	the index set of the linking constraints induced by $\{y_l|l \in S_K\}$.
$\mathcal{P}_k$	the power set of the set composed by the index sets of the minimal variable sets containing y_k.
$\mathcal{F}_k$	a family of the index sets of the minimal variable sets of the form S_K^k, i.e., $\mathcal{F}_k \in \mathcal{P}_k$.
Σ_k	$= \cup_{S_K^k \in \mathcal{F}_k} S_K^k$.
SRMP$(\bar{K}, \bar{N}, I(\bar{K}, \bar{N}))$	the current SRMP formed by $\{y_k|k \in \bar{K}\}$, $\{x_n|n \in \bar{N}\}$, and the set of linking constraints $I(\bar{K}, \bar{N})$ in addition to (SRMP-y)-(SRMP-x).

As explained previously, the minimal variable set $\{y_l|l \in S_K\}$ introduces $\Delta(S_K)$. In general, this relationship is not one-to-one; that is, y_k may appear in several sets of linking constraints, and the same set of linking constraints may be induced by several different minimal variable sets. To illustrate in the context of MSCS, if the intermediate rolls i, j and i, h appear in the first-stage cutting patterns k and l, respectively, then we have $\{(\{k\}, \{i\}), (\{k\}, \{j\}), (\{l\}, \{i\}), (\{l\}, \{h\})\}$, where a pair $(S_K, \Delta(S_K))$ specifies that the minimal variable set $\{y_l|l \in S_K\}$ introduces $\Delta(S_K)$. Therefore, $\{y_k\}$ and $\{y_l\}$ are the minimal variable sets for the sets of linking constraints $\{i, j\}$ and $\{i, h\}$, respectively, and the linking constraint i may be induced by both $\{y_k\}$ and $\{y_l\}$. In contrast, for the QSC problem, each set of linking constraints of the form (9)-(11) is introduced to the SRMP by a unique minimal variable set $\{y_k, y_l\}$, and we have $(\{k, l\}, \{i_1, i_2, i_3\})$, where i_1, i_2, i_3, are the indices of the associated linking constraints.

In general, adding new constraints and variables to an LP may destroy both the primal and the dual feasibility. In our case, Assumption 2.2 guarantees that the primal feasibility is preserved. Therefore, the goal of our analysis is to attach a correct set of values to each variable $w_i, i \in \Delta(\Sigma_k)$, and thus be able to calculate the reduced costs of y_k and $\{x_n | n \in S_N(\Sigma_k)\}$ to be inserted into the SRMP correctly. In particular, the ensuing analysis computes the optimal values of $\{w_i | i \in \Delta(\Sigma_k)\}$ without solving the SRMP explicitly under the presence of the currently missing associated set of linking constraints $\Delta(\Sigma_k)$. Moreover, it also guarantees that the optimal values of the dual variables $\{u_j | j \in J\}$, $\{v_m | m \in M\}$, and $\{w_i | i \in I(\bar{K}, \bar{N})\}$ retrieved from the optimal solution of the current SRMP would remain optimal with respect to the SRMP augmented with the set of linking constraints $\Delta(\Sigma_k)$ and $\{x_n | S_N(\Sigma_k)\}$. These properties, stated formally in Corollary 3.1a-b, are key to the correctness of the proposed column-and-row generation algorithm. Then, for any given y_k, an associated $\mathcal{F}_k$, and $S_K^k \in \mathcal{F}_k$, we have

$$\bar{c}_k = c_k - \sum_{j \in J} A_{jk} u_j - \sum_{i \in I(\bar{K}, \bar{N})} C_{ik} w_i - \sum_{i \in \Delta(\Sigma_k)} C_{ik} w_i, \tag{18}$$

$$\bar{d}_n = d_n - \sum_{m \in M} B_{mn} v_m - \sum_{i \in I(\bar{K}, \bar{N})} D_{in} w_i - \sum_{i \in \Delta(\Sigma_k)} D_{in} w_i \tag{19}$$

$$= d_n - \sum_{m \in M} B_{mn} v_m - \sum_{i \in \Delta(S_K^k)} D_{in} w_i, \tag{20}$$

where $\bar{c}_k$ and $\bar{d}_n$ are the reduced costs for y_k and $x_n, n \in S_N(S_K^k)$, respectively. The simplification of expression (19) to (20) follows from Assumption 2.1 which states that an $x-$variable appears in no more than one set of linking constraints. To reiterate, in (18)-(20) the values of the dual variables $\{u_j | j \in J\}$, $\{v_m | m \in M\}$, and $\{w_i | i \in I(\bar{K}, \bar{N})\}$ are retrieved from the optimal solution of the current SRMP, and $\{w_i | i \in \Delta(\Sigma_k)\}$ are unknown. Next, we introduce a series of conditions imposed on the reduced costs (20) as well as on the unknown dual variables $\{w_i | i \in \Delta(\Sigma_k)\}$ which ultimately leads to the formulation of the row-generating PSP. In this discussion, we also present how we can obtain a valid starting basis for the next optimization of the SRMP given that it is augmented by the variables $\{y_l | l \in \Sigma_k\}$, $\{x_n | n \in S_N(\Sigma_k)\}$, and the set of linking constraints $\Delta(\Sigma_k)$.

Suppose that for a given $\mathcal{F}_k$ and a set of associated dual variables $\{w_i | i \in \Delta(\Sigma_k)\}$ we have $\bar{c}_k \geq 0$ and $\bar{d}_{n'} < 0$ for some $n' \in S_N(S_K^k)$ with $S_K^k \in \mathcal{F}_k$. Hence, $\{y_l | l \in \Sigma_k\}$, $\{x_n | n \in S_N(\Sigma_k)\}$, and the set of linking constraints $\Delta(\Sigma_k)$ are added to the SRMP. This implies that $x_{n'}$ is eligible to enter the basis during the next iteration of solving the SRMP. However, this basis update would only result in a degenerate simplex iteration as the value of $x_{n'}$ is forced to zero in the basis by Assumption 2.3. That is, there exists a nonbasic variable y_l, $l \in S_K^k$, such that an associated constraint (2) is introduced into the SRMP. Note that the existence of such a nonbasic variable is guaranteed because $S_K^k \nsubseteq \bar{K}$. In order to avoid this type of degeneracy, we require that $\bar{d}_n \geq 0$ holds for all $n \in S_N(S_K^k)$ and for each $S_K^k \in \mathcal{F}_k$ while determining the values of $\{w_i | i \in \Delta(\Sigma_k)\}$. In other words, we impose the following set of constraints:

$$\sum_{m \in M} B_{mn} v_m + \sum_{i \in \Delta(S_K^k)} D_{in} w_i \leq d_n, \quad n \in S_N(S_K^k), S_K^k \in \mathcal{F}_k. \tag{21}$$

We underline that our proposed approach goes beyond the classical LP sensitivity analysis that would augment the basis with the surplus variables in the new linking constraints and then proceed to repair the infeasibility in the constraints (21). This is because setting $w_i = 0$, $i \in S_K^k$, $S_K^k \in \mathcal{F}_k$ may violate (21). Therefore, incorporating these constraints directly into the row-generating PSP may be regarded as a look-ahead feature. A further critical observation is that constraints (21) exhibit a block-diagonal structure. Given the optimal solution of the current SRMP, the first term on the left hand side of (21) is a constant for all n, and hence, we have

$$\sum_{i\in\Delta(S_K^k)} D_{in}w_i \le d_n - \sum_{m\in M} B_{mn}v_m, \quad n\in S_N(S_K^k), S_K^k\in\mathcal{F}_k, \tag{22}$$

which exposes the block-diagonal structure. The dual variables $\{w_i|i\in\Delta(S_K^k)\}$ do not factor into the reduced costs of any $x-$variables, except for $\{x_n|n\in S_N(S_K^k)\}$. Thus, the task of determining the values of $\{w_i|i\in\Delta(\Sigma_k)\}$ decomposes, and this property is also exploited in our analysis. We next show that enforcing the set of constraints (21) in the row-generating PSP does not change the *minimum value* of $\bar{c}_k$ and hence imposing (21) does not affect the correctness of the column-and-row generation procedure.

LEMMA 3.1 *For a given k, an associated $\mathcal{F}_k$, and $S_K^k\in\mathcal{F}_k$, imposing (21) on the set of unknown dual variables $\{w_i|i\in\Delta(S_K^k)\}$ while solving the row-generating PSP does not increase the minimum value of $\bar{c}_k$.*

PROOF. This result stems directly from Assumption 2.3 which states that there always exists a linking constraint $i'\in\Delta(S_K^k)$ of the form (2) such that $C_{i'k}>0$ and $D_{i'n}<0$ for all $n\in S_N(S_K^k)$. Coupling this with $w_i\ge 0, i\in\Delta(S_K^k)$ as required by the (DMP), we conclude that increasing $w_{i'}$ increases the reduced cost $\bar{d}_n$ given in (20) for all $\{x_n|n\in S_N(S_K^k)\}$ while reducing $\bar{c}_k$ in (18). Thus, (21) is always satisfied for the minimum value of $\bar{c}_k$. □

From the discussion so far it is evident that the row-generating PSP must provide us with a variable y_k and an associated family of index sets $\mathcal{F}_k$ so that the reduced cost $\bar{c}_k$ as defined in (18) is negative. Thus, for a given variable y_k we need to select a subset $\mathcal{F}_k\in\mathcal{P}_k$ so that $\bar{c}_k$ is minimized. During this optimization we must prescribe that the values determined for the unknown set of dual variables $\{w_i|i\in\Delta(S_K^k)\}$ satisfy the conditions set forth in (21) for each $S_K^k\in\mathcal{F}_k$. These arguments prompt us to pose the row-generating PSP as a two-stage optimization problem. In the first stage, we formulate and solve the problem of finding the minimum reduced cost for a given y_k as a subset selection problem. For any given $\mathcal{F}_k\in\mathcal{P}_k$, the problem of computing the optimal values of $\{w_i|i\in\Delta(\Sigma_k)\}$ decomposes into finding the optimal values of $\{w_i|i\in\Delta(S_K^k)\}$ for each $S_K^k\in\mathcal{F}_k$. In the second stage, we pick the $y-$variable with the most negative minimum reduced cost. We stop solving the row-generating PSP and proceed according to Figure 1 if the minimum reduced cost is nonnegative for all $y_k, k\in(K\setminus\bar{K})$.

The only missing piece in the approach described in the preceding paragraph is computing a valid reduced cost for y_k for a given $\mathcal{F}_k$ without changing the reduced costs of the variables in SRMP$(\bar{K},\bar{N},I(\bar{K},\bar{N}))$. This task is accomplished by showing that the optimal solution of the row-generating PSP corresponds to an *implicit construction of a basic optimal solution* to SRMP$(\bar{K},\bar{N}\cup S_N(\Sigma_k), I(\bar{K},\bar{N})\cup\Delta(\Sigma_k))$ that allows us to correctly price out $\{y_l|l\in\Sigma_k\}$. In particular, we prove that the optimal values of the dual variables $\{u_j|j\in J\}$, $\{v_m|m\in M\}$, and $\{w_i|i\in I(\bar{K},\bar{N})\}$ in SRMP$(\bar{K},\bar{N},I(\bar{K},\bar{N}))$ are identical to those in SRMP$(\bar{K},\bar{N}\cup S_N(\Sigma_k), I(\bar{K},\bar{N})\cup\Delta(\Sigma_k))$, and the values set for $\{w_i|i\in\Delta(\Sigma_k)\}$ in the row-generating PSP are optimal for SRMP$(\bar{K},\bar{N}\cup S_N(\Sigma_k), I(\bar{K},\bar{N})\cup\Delta(\Sigma_k))$ as stated in Corollary 3.1a-b. In addition, it turns out that we have $C_{il}w_i=0$ for a variable $y_l, l\in\bar{K}$ and for $i\in\Delta(\Sigma_k)$. In other words, a $y-$variable that currently exists in the SRMP does not appear in a new linking constraint with a positive dual variable, and this property guarantees that the reduced costs of $\{y_l|l\in\bar{K}\}$ are identical with respect to the optimal dual solutions of both SRMP$(\bar{K},\bar{N},I(\bar{K},\bar{N}))$ and SRMP$(\bar{K},\bar{N}\cup S_N(\Sigma_k), I(\bar{K},\bar{N})\cup\Delta(\Sigma_k))$ as stated in Corollary 3.1c.

To explain the construction of an optimal basis for SRMP$(\bar{K}, \bar{N} \cup S_N(\Sigma_k), I(\bar{K}, \bar{N}) \cup \Delta(\Sigma_k))$ based on the solution of the row-generating PSP, suppose that we are given a specific $\mathcal{F}_k$. We introduce $\{x_n | n \in S_N(\Sigma_k)\}$ and a set of new linking constraints $\Delta(\Sigma_k)$ into SRMP$(\bar{K}, \bar{N}, I(\bar{K}, \bar{N}))$ to obtain SRMP$(\bar{K}, \bar{N} \cup S_N(\Sigma_k), I(\bar{K}, \bar{N}) \cup \Delta(\Sigma_k))$. Warm starting the primal simplex method for this new SRMP would require us to augment the optimal basis of SRMP$(\bar{K}, \bar{N}, I(\bar{K}, \bar{N}))$ with $| \Delta(\Sigma_k) |$ new basic variables associated with the new set of linking constraints. To ensure complementary slackness, we determine the values of $\{w_i | i \in \Delta(\Sigma_k)\}$ such that the number of linearly independent active constraints among $w_i \geq 0, i \in \Delta(\Sigma_k)$ and (22) is at least $| \Delta(\Sigma_k) |$. This restriction is directly added to the definition of the row-generating PSP specified below. A tight constraint of the form (22) prescribes adding the corresponding $x-$variable to the basis, while $w_i = 0$ implies that the basis is extended by the corresponding primal surplus variable. In addition, in order to ensure that $C_{il} w_i = 0$ for $i \in \Delta(\Sigma_k)$ and for $\{y_l | l \in \bar{K}\}$ as discussed before, we only allow $w_i > 0$ if constraint $i \in \Delta(S_K^k)$ is of the form (2) as specified in Assumption 2.3 with $C_{ik} > 0$. Clearly, such a constraint does not include a variable $y_l, l \in \bar{K}$. The index set of constraints $\Delta(S_K^k)$ of the form (2) with $C_{ik} > 0$ is represented by $\Delta_+(S_K^k)$, and the complement of this set is denoted by $\Delta_0(S_K^k) = \Delta(S_K^k) \setminus \Delta_+(S_K^k)$. Thus, we always pick a surplus variable as basic for a constraint $i \in \Delta_0(S_K^k)$ for all $S_K^k \in \mathcal{F}_k$. For the other new linking constraints, we either designate an $x-$ or a surplus variable as basic. In Lemma 3.3, we first prove that the augmentation prescribed by the row-generating PSP is a valid basis for SRMP$(\bar{K}, \bar{N} \cup S_N(\Sigma_k), I(\bar{K}, \bar{N}) \cup \Delta(\Sigma_k))$, and then in Lemma 3.4, we prove that it is optimal. In particular, the values of the dual variables $\{w_i | i \in \Delta(\Sigma_k)\}$ set as described turn out to be optimal for SRMP$(\bar{K}, \bar{N} \cup S_N(\Sigma_k), I(\bar{K}, \bar{N}) \cup \Delta(\Sigma_k))$ as formalized in Corollary 3.1b. The row-generating PSP is then stated as:

$$\zeta_{yx} = \min_{k \in (K \setminus \bar{K})} \left\{ c_k - \sum_{j \in J} A_{jk} u_j - \sum_{i \in I(\bar{K}, \bar{N})} C_{ik} w_i - \max_{\mathcal{F}_k \in \mathcal{P}_k} \left(\sum_{S_K^k \in \mathcal{F}_k} \alpha_{S_K^k} \right) \right\}, \text{ where} \tag{23}$$

$$\alpha_{S_K^k} = \text{maximize} \quad \sum_{i \in \Delta(S_K^k)} C_{ik} w_i, \tag{24a}$$

$$\text{subject to} \quad \sum_{i \in \Delta(S_K^k)} D_{in} w_i \leq d_n - \sum_{m \in M} B_{mn} v_m, \quad n \in S_N(S_K^k), \tag{24b}$$

$$w_i = 0, \quad i \in \Delta_0(S_K^k), \tag{24c}$$

$$w_i \geq 0, \quad i \in \Delta_+(S_K^k), \tag{24d}$$

$$|\Delta(S_K^k)| \text{ many linearly independent tight constraints among (24b)-(24d).} \tag{24e}$$

The fundamental property of this formulation is that we solve (24) independently for each $S_K^k \in \mathcal{F}_k$ which allows us to calculate the minimum reduced cost of y_k efficiently. This decomposition relies on the block-diagonal structure previously discussed in the context of (22) and is exemplified in Section 4 when our generic methodology is applied to the MSCS and QSC problems. A potential source of difficulty is the constraint (24e) which mandates that the search for an optimal solution of (24) is restricted to the set of extreme points of the polyhedron described by (24b)-(24d). Without this restriction, the problem (24a)-(24d) is unbounded by a similar argument to that used in the proof of Lemma 3.1. Fortunately, in many cases (24) is amenable to simple solution approaches. This is illustrated on the MSCS and QSC problems in Section 4.

In summary, suppose that solving the row-generating PSP (23)-(24) results in $\bar{c}_k = \zeta_{yx} < 0$ and an associated family of index sets $\mathcal{F}_k$. Then, SRMP$(\bar{K}, \bar{N}, I(\bar{K}, \bar{N}))$ expands to SRMP$(\bar{K} \cup \Sigma_k, \bar{N} \cup S_N(\Sigma_k), I(\bar{K}, \bar{N}) \cup \Delta(\Sigma_k))$ before the primal simplex method is warm started based on the basis augmentation provided by the optimal solutions of (24) for $S_K^k \in \mathcal{F}_k$. This augmentation achieves two primary goals. First, the resulting basis is optimal for SRMP$(\bar{K}, \bar{N} \cup S_N(\Sigma_k), I(\bar{K}, \bar{N}) \cup \Delta(\Sigma_k))$, and the optimal objective function value ζ_{yx} of the row-generating PSP is the correct reduced cost of y_k under this augmentation. Second, we can invoke the primal simplex algorithm with this initial basis for SRMP$(\bar{K} \cup \Sigma_k, \bar{N} \cup S_N(\Sigma_k), I(\bar{K}, \bar{N}) \cup \Delta(\Sigma_k))$ so that y_k is the natural candidate to enter the basis. In the remainder of this section, we prove these properties of the proposed basis augmentation preceding a formal proof of the correctness of the proposed column-and-row generation approach for CDR-problems.

Let $\mathbf{B}$ and $\mathcal{B}$ be the optimal basis of SRMP$(\bar{K}, \bar{N}, I(\bar{K}, \bar{N}))$ and the associated basic sequence, respectively. Suppose that $\mathbf{B}$ is a $\beta \times \beta$ matrix, and $\delta :=\mid \Delta(\Sigma_k) \mid$ denotes the number of new constraints to be added to the SRMP. Recall that we always pick surplus variables as basic for the set of constraints $\Delta_0(S_K^k)$ for all $S_K^k \in \mathcal{F}_k$. However, for a constraint $i \in \Delta_+(S_K^k)$ we either select the corresponding surplus variable as basic if $w_i = 0$ or an $x-$variable that appears in this constraint, if its associated dual constraint (24b) is tight in the optimal solution of (24) for S_K^k. In other words, no more than $\mid \Delta_+(S_K^k) \mid$ of the variables $\{x_n | n \in S_N(S_K^k)\}$ are designated as basic by the optimal solution of (24). We denote the sets of new linking constraints associated with the new basic $x-$ and surplus variables as $\Delta_x(\Sigma_k)$ and $\Delta_s(\Sigma_k)$, respectively, where $\delta_x =\mid \Delta_x(\Sigma_k) \mid$, $\delta_s =\mid \Delta_s(\Sigma_k) \mid$, and $\Delta_x(\Sigma_k) \subseteq \cup_{S_K^k \in \mathcal{F}_k} \Delta_+(S_K^k)$. The resulting augmented matrix $\mathbf{B}_k$ is then obtained as:

$$\mathbf{B}_k = \left(\begin{array}{ccc|cc} \mathbf{A_1} & \mathbf{0} & \mathbf{E_1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B_1} & \mathbf{E_2} & \mathbf{B_2} & \mathbf{0} \\ \mathbf{C_1} & \mathbf{D_1} & \mathbf{E_3} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{D_2} & \mathbf{0} \\ \mathbf{C_2} & \mathbf{0} & \mathbf{0} & \mathbf{D_3} & -\mathbf{I} \end{array}\right) = \left(\begin{array}{c|cc} \mathbf{B} & \mathbf{F} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{D_2} & \mathbf{0} \\ \mathbf{G} & \mathbf{D_3} & -\mathbf{I} \end{array}\right), \tag{25}$$

where the coefficients of the new basic $x-$variables in the currently existing constraints in the SRMP are given by a $\beta \times \delta_x$ matrix $\mathbf{F} = \begin{pmatrix} \mathbf{0} \\ \mathbf{B_2} \\ \mathbf{0} \end{pmatrix}$. The $\delta_x \times \delta_x$ matrix $\mathbf{D_2}$ and the $\delta_s \times \delta_x$ matrix $\mathbf{D_3}$ specify the coefficients of these $x-$variables in the new linking constraints $\Delta_x(\Sigma_k)$ and $\Delta_s(\Sigma_k)$, respectively. The final column of $\mathbf{B}_k$ is associated with the new basic surplus variables, where $\mathbf{I}$ is a $\delta_s \times \delta_s$ identity matrix. The $\delta_x \times \beta$ matrix $\begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}$ in the fourth row of $\mathbf{B}_k$ and the $\delta_s \times \beta$ matrix $\mathbf{G} = \begin{pmatrix} \mathbf{C_2} & \mathbf{0} & \mathbf{0} \end{pmatrix}$ are constructed by the coefficients of the current basic variables in the new linking constraints $\Delta_x(\Sigma_k)$ and $\Delta_s(\Sigma_k)$, respectively. This partitioning is best explained in the context of the illustration in Figure 2 for the QSC problem, for which $\mathbf{B_1} = \mathbf{E_2} = \mathbf{B_2} = \mathbf{0}$ because the $x-$ variables appear only in the linking constraints. In this specific example, $\zeta_{yx} = \bar{c}_k < 0$ and $\mathcal{F}_k = \{\{k, l\}, \{k, m\}\}$. The variable y_l is already present in the current SRMP, and y_m is to be incorporated in the SRMP along with y_k. Along with these, we introduce x_{kl}, x_{km}, two sets of linking constraints of the form (9)-(11) associated with the pairs of variables y_k, y_l, and y_k, y_m, respectively, and a set of six surplus variables associated with the new linking constraints into the SRMP. The problem (24) designates x_{kl}, s_{l1}, and s_{l3} as basic for the constraints $y_k - x_{kl} - s_{l2} = 0$, $-y_k - y_l + x_{kl} - s_{l1} = -1$, and $y_l - x_{kl} - s_{l3} = 0$, respectively, where the first of these constraints belongs to the set $\Delta_+(\{k, l\})$ and the rest form the set $\Delta_0(\{k, l\})$, respectively. Note that s_{l2} may replace x_{kl} in the augmented basis depending on the optimal solution of (24) for $\{k, l\}$. The variables x_{km}, s_{m1}, and s_{m3} are selected as basic for the set of linking constraints $\Delta(\{k,m\})$ in a similar way. Thus, $\Delta_x(\{k,l,m\})$ consists of the new linking constraints $y_k - x_{kl} - s_{l2} = 0$ and $y_k - x_{km} - s_{m2} = 0$, while the rest of the new linking constraints belong to $\Delta_s(\{k,l,m\})$. Two crucial observations are due based on this discussion. First, no variable in the current SRMP is present in a constraint $i \in \Delta_+(S_K^k)$ for any $S_K^k \in \mathcal{F}_k$; that is, the submatrix in the first position in the fourth row of $\mathbf{B}_k$ is zero. Second, $\mathbf{D_2}$ is invertible as formalized by the next lemma. These two properties allow us to establish that $\mathbf{B}_k$ is a valid basis for $\text{SRMP}(\bar{K}, \bar{N} \cup S_N(\Sigma_k), I(\bar{K},\bar{N}) \cup \Delta(\Sigma_k))$ in Lemma 3.3. For other CDR-problems with interaction, we would need to define the sets $\Delta_+(S_K^k)$ and $\Delta_0(S_K^k)$ as appropriate for all $S_K^k \in \mathcal{F}_k$, and the structure of the submatrices $\mathbf{C_2}$, $\mathbf{D_2}$, and $\mathbf{D_3}$ would be different. Otherwise, the basis augmentation carries over in exactly the same way. The only extra provision for CDR-problems with no interaction is that $\mathbf{C_2} = \mathbf{0}$ because $\Sigma_k = \{k\}$ and $k \notin \bar{K}$.

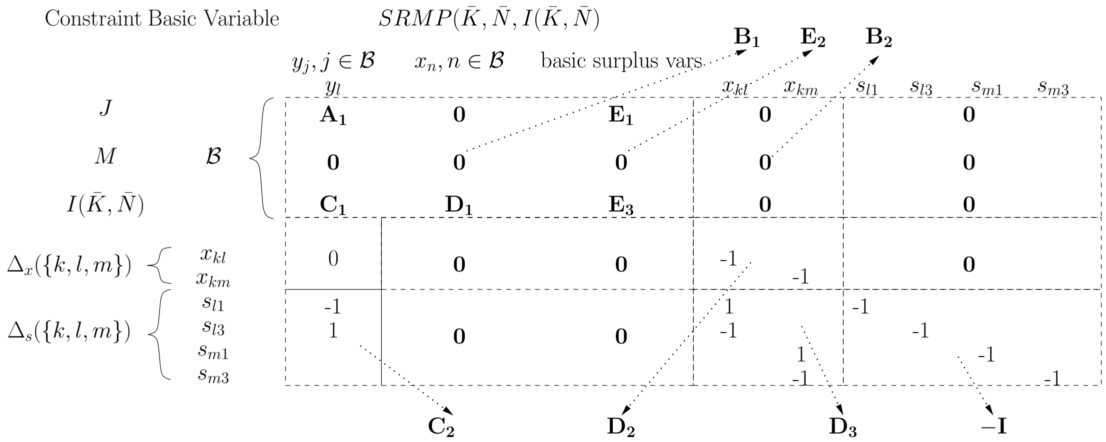

Figure 2: Basis augmentation for QSC, where $\mathcal{F}_k = \{\{k,l\},\{k,m\}\}$, and the new basic variables $\{x_{kl}, s_{l1}, s_{l3}\}$ and $\{x_{km}, s_{m1}, s_{m3}\}$ are associated with the new linking constraints $\Delta(\{k,l\})$ and $\Delta(\{k,m\})$, respectively.

Lemma 3.2 *The $\delta_x \times \delta_x$ matrix $\mathbf{D_2}$ is invertible.*

Proof. The matrix $\left(\begin{smallmatrix}\mathbf{D_2} & \mathbf{0}\\ \mathbf{D_3} & -\mathbf{I}\end{smallmatrix}\right)$ is constructed by solving (24) for each $S_K^k \in \mathcal{F}_k$ and exhibits a block-diagonal structure as discussed before. The columns in a given block are linearly independent as prescribed by (24e). Therefore, $\left(\begin{smallmatrix}\mathbf{D_2} & \mathbf{0}\\ \mathbf{D_3} & -\mathbf{I}\end{smallmatrix}\right)$ must be invertible, and by the uniqueness of the inverse we conclude that $\left(\begin{smallmatrix}\mathbf{D_2} & \mathbf{0}\\ \mathbf{D_3} & -\mathbf{I}\end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix}\mathbf{D_2}^{-1} & \mathbf{0}\\ \mathbf{D_3}\mathbf{D_2}^{-1} & -\mathbf{I}\end{smallmatrix}\right)$. Thus, $\mathbf{D_2}$ must be invertible. □

In Figure 2, one block in $\left(\begin{smallmatrix}\mathbf{D_2} & \mathbf{0}\\ \mathbf{D_3} & -\mathbf{I}\end{smallmatrix}\right)$ is formed by the coefficients of x_{kl}, s_{l1}, and s_{l3}, while x_{km}, s_{m1}, and s_{m3} construct the second block. The next lemma proves that $\mathbf{B}_k$ provides us with a basic solution for $\text{SRMP}(\bar{K}, \bar{N} \cup S_N(\Sigma_k), I(\bar{K},\bar{N}) \cup \Delta(\Sigma_k))$.

Lemma 3.3 *$\mathbf{B}_k$ is a $(\beta+\delta)$-dimensional basis for $\text{SRMP}(\bar{K}, \bar{N} \cup S_N(\Sigma_k), I(\bar{K},\bar{N}) \cup \Delta(\Sigma_k))$, and its inverse is obtained as:*

$$\begin{pmatrix}\mathbf{B} & \mathbf{F} & \mathbf{0}\\ \mathbf{0} & \mathbf{D_2} & \mathbf{0}\\ \mathbf{G} & \mathbf{D_3} & -\mathbf{I}\end{pmatrix}^{-1} = \begin{pmatrix}\mathbf{B}^{-1} & -\mathbf{B}^{-1}\mathbf{F}\mathbf{D_2}^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{D_2}^{-1} & \mathbf{0}\\ \mathbf{G}\mathbf{B}^{-1} & -\mathbf{G}\mathbf{B}^{-1}\mathbf{F}\mathbf{D_2}^{-1} + \mathbf{D_3}\mathbf{D_2}^{-1} & -\mathbf{I}\end{pmatrix}.$$

PROOF. The matrix $\mathbf{J} = \begin{pmatrix} \mathbf{B} & \mathbf{F} \\ \mathbf{0} & \mathbf{D_2} \end{pmatrix}$ is invertible because both $\mathbf{B}$ and $\mathbf{D_2}$ are invertible, and we compute $\mathbf{J}^{-1} = \begin{pmatrix} \mathbf{B}^{-1} & -\mathbf{B}^{-1}\mathbf{F}\mathbf{D_2}^{-1} \\ \mathbf{0} & \mathbf{D_2}^{-1} \end{pmatrix}$. Thus, $\mathbf{B}_k = \begin{pmatrix} \mathbf{J} & \mathbf{0} \\ \mathbf{K} & -\mathbf{I} \end{pmatrix}$, where $\mathbf{K} = \begin{pmatrix} \mathbf{G} & \mathbf{D_3} \end{pmatrix}$. Finally, we obtain

$$\mathbf{B}_k^{-1} = \begin{pmatrix} \mathbf{J} & \mathbf{0} \\ \mathbf{K} & -\mathbf{I} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{J}^{-1} & \mathbf{0} \\ \mathbf{K}\mathbf{J}^{-1} & -\mathbf{I} \end{pmatrix} = \begin{pmatrix} \mathbf{B}^{-1} & -\mathbf{B}^{-1}\mathbf{F}\mathbf{D_2}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{D_2}^{-1} & \mathbf{0} \\ \mathbf{G}\mathbf{B}^{-1} & -\mathbf{G}\mathbf{B}^{-1}\mathbf{F}\mathbf{D_2}^{-1} + \mathbf{D_3}\mathbf{D_2}^{-1} & -\mathbf{I} \end{pmatrix}$$

after plugging in $\mathbf{J}^{-1}$ and $\mathbf{K}\mathbf{J}^{-1}$ as appropriate. □

We next state one of our main results in this section and prove that $\mathbf{B}_k$ is in fact an optimal basis for $\text{SRMP}(\bar{K}, \bar{N} \cup S_N(\Sigma_k), I(\bar{K}, \bar{N}) \cup \Delta(\Sigma_k))$. We emphasize that this result does not require an optimal solution of (24) for $S_K^k \in \mathcal{F}_k$. It is sufficient to choose any extreme point feasible solution of (24b)-(24d) for each $S_K^k \in \mathcal{F}_k$ while constructing $\mathbf{B}_k$.

LEMMA 3.4 *$\mathbf{B}_k$ is an optimal basis for* $\text{SRMP}(\bar{K}, \bar{N} \cup S_N(\Sigma_k), I(\bar{K}, \bar{N}) \cup \Delta(\Sigma_k))$.

PROOF. It is sufficient to prove that $\mathbf{B}_k$ defines a pair of primal and dual basic feasible solutions since complementary slackness is always satisfied by a basic solution. In the following, $\mathbf{b}$ represents the right hand side coefficients of the constraints in $\text{SRMP}(\bar{K}, \bar{N}, I(\bar{K}, \bar{N}))$ while $\mathbf{c}_{\mathcal{B}}$ stands for the objective function coefficients of the variables in the optimal basic sequence $\mathcal{B}$ of $\text{SRMP}(\bar{K}, \bar{N}, I(\bar{K}, \bar{N}))$. Furthermore, the objective coefficients of the new basic $x-$variables are denoted by $\mathbf{c}_x$, and the right hand sides of the new linking constraints $\Delta_x(\Sigma_k)$ and $\Delta_s(\Sigma_k)$ are given by the vectors $\mathbf{r}_x$ and $\mathbf{r}_s$, respectively, where $\mathbf{r}_x = \mathbf{0}$ by (2) in Assumption 2.3. Thus, the vector $\mathbf{b}_k = \left(\begin{smallmatrix} \mathbf{b} \\ \mathbf{0} \\ \mathbf{r}_s \end{smallmatrix}\right)$ defines the right hand side of $\text{SRMP}(\bar{K}, \bar{N} \cup S_N(\Sigma_k), I(\bar{K}, \bar{N}) \cup \Delta(\Sigma_k))$.

For verifying the primal feasibility, we compute

$$\mathbf{B}_k^{-1}\mathbf{b}_k = \begin{pmatrix} \mathbf{B}^{-1} & -\mathbf{B}^{-1}\mathbf{F}\mathbf{D_2}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{D_2}^{-1} & \mathbf{0} \\ \mathbf{G}\mathbf{B}^{-1} & -\mathbf{G}\mathbf{B}^{-1}\mathbf{F}\mathbf{D_2}^{-1} + \mathbf{D_3}\mathbf{D_2}^{-1} & -\mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{b} \\ \mathbf{0} \\ \mathbf{r}_s \end{pmatrix} = \begin{pmatrix} \mathbf{B}^{-1}\mathbf{b} \\ \mathbf{0} \\ \mathbf{G}\mathbf{B}^{-1}\mathbf{b} - \mathbf{r}_s \end{pmatrix} \geq \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}. \tag{26}$$

The nonnegativity of $\mathbf{B}^{-1}\mathbf{b}$ follows from the optimality of $\mathbf{B}$ for $\text{SRMP}(\bar{K}, \bar{N}, I(\bar{K}, \bar{N}))$. By Assumption 2.2, the optimal solution of $\text{SRMP}(\bar{K}, \bar{N}, I(\bar{K}, \bar{N}))$ does not violate the linking constraints $\Delta_s(\Sigma_k)$ which are absent from $\text{SRMP}(\bar{K}, \bar{N}, I(\bar{K}, \bar{N}))$. Thus, we have $\mathbf{G}(\mathbf{B}^{-1}\mathbf{b}) \geq \mathbf{r}_s$, and $\mathbf{G}\mathbf{B}^{-1}\mathbf{b} - \mathbf{r}_s \geq \mathbf{0}$ as required.

In order to check the nonnegativity of the reduced costs in $\text{SRMP}(\bar{K}, \bar{N} \cup S_N(\Sigma_k), I(\bar{K}, \bar{N}) \cup \Delta(\Sigma_k))$, we first determine the values of the dual variables prescribed by $\mathbf{B}_k$. That is

$$\begin{pmatrix} \mathbf{c}_{\mathcal{B}} & \mathbf{c}_x & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{B}^{-1} & -\mathbf{B}^{-1}\mathbf{F}\mathbf{D_2}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{D_2}^{-1} & \mathbf{0} \\ \mathbf{G}\mathbf{B}^{-1} & -\mathbf{G}\mathbf{B}^{-1}\mathbf{F}\mathbf{D_2}^{-1} + \mathbf{D_3}\mathbf{D_2}^{-1} & -\mathbf{I} \end{pmatrix} = \begin{pmatrix} \mathbf{c}_{\mathcal{B}}\mathbf{B}^{-1} & (\mathbf{c}_x - \mathbf{c}_{\mathcal{B}}\mathbf{B}^{-1}\mathbf{F})\mathbf{D_2}^{-1} & \mathbf{0} \end{pmatrix}, \tag{27}$$

where the objective coefficients of the basic surplus variables are represented by $\mathbf{0}$. From (27), we conclude that the values of the dual variables $\{u_j | j \in J\}$, $\{v_m | m \in M\}$, and $\{w_i | i \in I(\bar{K}, \bar{N})\}$ are identical to those in the optimal solution of $\text{SRMP}(\bar{K}, \bar{N}, I(\bar{K}, \bar{N}))$. Moreover, we can show that the values of the dual variables $\{w_i | i \in \Delta(\Sigma_k)\}$ are precisely those assigned by the row-generating PSP. To this end, recall that we form an invertible $\delta \times \delta$ submatrix $\left(\begin{smallmatrix} \mathbf{D_2} & \mathbf{0} \\ \mathbf{D_3} & -\mathbf{I} \end{smallmatrix}\right)$ consisting of columns corresponding to $x-$ and surplus variables based on the solutions of (24) for each $S_K^k \in \mathcal{F}_k$. In addition, note that the objective function coefficient of $x_n, n \in S_N(S_K)$ in the primal LP corresponding to the dual LP (24a)-(24d) is given by $d_n - \sum_{m \in M} B_{mn} v_m$. Clearly, if x_n is selected as basic in $\text{SRMP}(\bar{K}, \bar{N} \cup S_N(\Sigma_k), I(\bar{K}, \bar{N}) \cup \Delta(\Sigma_k))$ this value is equal to the component of $\mathbf{c}_x - \mathbf{c}_{\mathcal{B}} \mathbf{B}^{-1} \mathbf{F}$ associated with x_n. Thus, the values assigned to $\{w_i | i \in \Delta(\Sigma_k)\}$ in the row-generating PSP are calculated as

$$\begin{pmatrix} \mathbf{c}_x - \mathbf{c}_{\mathcal{B}} \mathbf{B}^{-1} \mathbf{F} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{D_2} & \mathbf{0} \\ \mathbf{D_3} & -\mathbf{I} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{c}_x - \mathbf{c}_{\mathcal{B}} \mathbf{B}^{-1} \mathbf{F} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{D_2}^{-1} & \mathbf{0} \\ \mathbf{D_3} \mathbf{D_2}^{-1} & -\mathbf{I} \end{pmatrix} \tag{28}$$

$$= \begin{pmatrix} (\mathbf{c}_x - \mathbf{c}_{\mathcal{B}} \mathbf{B}^{-1} \mathbf{F}) \mathbf{D_2}^{-1} & \mathbf{0} \end{pmatrix}, \tag{29}$$

which are identical to those computed in (27) based on $\mathbf{B}_k$.

From the optimality of $\text{SRMP}(\bar{K}, \bar{N}, I(\bar{K}, \bar{N}))$, $c_l - \sum_{j \in J} A_{jl} u_j - \sum_{i \in I(\bar{K}, \bar{N})} C_{il} w_i \geq 0$ for a variable $y_l, l \in \bar{K}$ and $d_n - \sum_{m \in M} B_{mn} v_m - \sum_{i \in I(\bar{K}, \bar{N})} D_{in} w_i \geq 0$ for a variable $x_n, n \in \bar{N}$. Furthermore, no variable in $\text{SRMP}(\bar{K}, \bar{N}, I(\bar{K}, \bar{N}))$ is present in a linking constraint $i \in \Delta_x(\Sigma_k)$, and $w_i = 0$ for all $i \in \Delta_s(\Sigma_k)$. Thus, we conclude that $\bar{c}_l = c_l - \sum_{j \in J} A_{jl} u_j - \sum_{i \in I(\bar{K}, \bar{N})} C_{il} w_i - \sum_{i \in \Delta(\Sigma_k)} C_{il} w_i \geq 0$ for a variable $y_l, l \in \bar{K}$ and $\bar{d}_n = d_n - \sum_{m \in M} B_{mn} v_m - \sum_{i \in I(\bar{K}, \bar{N})} D_{in} w_i - \sum_{i \in \Delta(\Sigma_k)} D_{in} w_i \geq 0$ for a variable $x_n, n \in \bar{N}$ in $\text{SRMP}(\bar{K}, \bar{N} \cup S_N(\Sigma_k), I(\bar{K}, \bar{N}) \cup \Delta(\Sigma_k))$. For $x_n, n \in S_N(\Sigma_k)$, $\bar{d}_n = d_n - \sum_{m \in M} B_{mn} v_m - \sum_{i \in I(\bar{K}, \bar{N})} D_{in} w_i - \sum_{i \in \Delta(\Sigma_k)} D_{in} w_i = d_n - \sum_{m \in M} B_{mn} v_m - \sum_{i \in \Delta(S_K^k)} D_{in} w_i \geq 0$ by (24b) and because the values of the dual variables $\{w_i | i \in \Delta(\Sigma_k)\}$ employed in the row-generating PSP are optimal with respect to $\text{SRMP}(\bar{K}, \bar{N} \cup S_N(\Sigma_k), I(\bar{K}, \bar{N}) \cup \Delta(\Sigma_k))$. Thus, we arrive at the conclusion that the values of the dual variables calculated in (27) are dual feasible as required. □

The proof of Lemma 3.4 establishes formal arguments for some fundamental claims and propositions that we employed in the development of our column-and-row generation approach for CDR-problems. These are summarized in the following corollary.

Corollary 3.1

a. *The optimal values of the dual variables* $\{u_j | j \in J\}$, $\{v_m | m \in M\}$, *and* $\{w_i | i \in I(\bar{K}, \bar{N})\}$ *are identical for* $\text{SRMP}(\bar{K}, \bar{N}, I(\bar{K}, \bar{N}))$ *and* $\text{SRMP}(\bar{K}, \bar{N} \cup S_N(\Sigma_k), I(\bar{K}, \bar{N}) \cup \Delta(\Sigma_k))$.
b. *The values assigned to the dual variables* $\{w_i | i \in \Delta(\Sigma_k)\}$ *in the row-generating PSP are optimal for* $\text{SRMP}(\bar{K}, \bar{N} \cup S_N(\Sigma_k), I(\bar{K}, \bar{N}) \cup \Delta(\Sigma_k))$. *For* $i \in \Delta_s(\Sigma_k)$, *we have* $w_i = 0$ *at optimality.*
c. *The reduced costs of* $\{y_l | l \in \bar{K}\}$ *and* $\{x_n | n \in \bar{N}\}$ *are identical with respect to the optimal dual solutions of* $\text{SRMP}(\bar{K}, \bar{N}, I(\bar{K}, \bar{N}))$ *and* $\text{SRMP}(\bar{K}, \bar{N} \cup S_N(\Sigma_k), I(\bar{K}, \bar{N}) \cup \Delta(\Sigma_k))$.
d. *The reduced cost* $\bar{c}_k$ *computed in the row-generating PSP for any* $y_k, k \notin \bar{K}$ *and* $\mathcal{F}_k \in \mathcal{P}_k$ *is equal to the reduced cost of* y_k *with respect to the optimal solution of* $\text{SRMP}(\bar{K}, \bar{N} \cup S_N(\Sigma_k), I(\bar{K}, \bar{N}) \cup \Delta(\Sigma_k))$.
e. *A variable* $x_n, n \in S_N(\Sigma_k)$ *basic at the optimal solution of* $\text{SRMP}(\bar{K}, \bar{N} \cup S_N(\Sigma_k), I(\bar{K}, \bar{N}) \cup \Delta(\Sigma_k))$ *is equal to zero.*

We are now ready to prove that the column-and-row generation algorithm depicted in Figure 1 is an optimal algorithm for solving (MP) for CDR-problems characterized by Assumptions 2.1-2.3.

Theorem 3.1 *Given an optimal basis* $\mathbf{B}$ *for* $\text{SRMP}(\bar{K}, \bar{N}, I(\bar{K}, \bar{N}))$ *and a set of associated optimal values for the dual variables* $\{u_j | j \in J\}$, $\{v_m | m \in M\}$, *and* $\{w_i | i \in I(\bar{K}, \bar{N})\}$, *the proposed column-and-row generation algorithm terminates with an optimal solution for the master problem (MP) if* $\zeta_y \geq 0$, $\zeta_x \geq 0$, *and* $\zeta_{yx} \geq 0$ *in three consecutive calls to the* $y-$, $x-$, *and the row-generating PSPs, respectively.*

Proof. According to the flow of the proposed column-and-row generation algorithm in Figure 1, we invoke the row-generating PSP only after the $y-$ and $x-$PSPs fail to identify negatively priced $y-$ and $x-$ variables, respectively, given the optimal dual solution of SRMP$(\bar{K}, \bar{N}, I(\bar{K}, \bar{N}))$. If in addition the optimal objective value ζ_{yx} of the row-generating PSP is nonnegative given the optimal dual solution of SRMP$(\bar{K}, \bar{N}, I(\bar{K}, \bar{N}))$, then the algorithm terminates with an optimal solution to (MP) as we prove below. On the other hand, if the row-generating PSP terminates at least once successfully with a negatively priced new $y-$variable, then the optimal values of the dual variables are updated by re-optimizing SRMP augmented with new rows and columns. Therefore, in this case we cannot claim optimality following a subsequent unsuccessful optimization of the row-generating PSP with $\zeta_{yx} \geq 0$, and we must call the $y-$ and $x-$PSPs again (FLAG=1 at the termination of the row-generating PSP in Figure 1).

Now, assume that $\zeta_y \geq 0$, $\zeta_x \geq 0$, and $\zeta_{yx} \geq 0$ in three consecutive calls to the $y-$, $x-$, and the row-generating PSPs, respectively. In the sequel, we show that $\bar{c}_k \geq 0$ for $k \in K$ and $\bar{d}_n \geq 0$ for $n \in N$ even if we introduce the currently absent set of linking constraints $I \setminus I(\bar{K}, \bar{N})$ into the SRMP. Recall that the linking constraints feature a block-diagonal form, where a block is defined by a set of $x-$ and surplus variables that only appear in this block. For each block, we can choose any extreme point of a polyhedron similar to that defined by (24b)-(24d), designate new $x-$ and surplus variables as basic as prescribed, and then incorporate these linking constraints and the associated $x-$ and surplus variables in the SRMP. The resulting SRMP and the associated basis are denoted as SRMP$(\bar{K}, N, I)$ and $\mathbf{B}'$, respectively. Moreover, recall that Lemma 3.3 does not require that (24) is solved to optimality. Thus, we can develop a proof analogous to that of Lemma 3.4 and show that $\mathbf{B}'$ is an optimal basis for SRMP$(\bar{K}, N, I)$. Clearly, $\bar{d}_n \geq 0, n \in N$ and $\bar{c}_l \geq 0, l \in \bar{K}$ hold at the optimal solution of SRMP$(\bar{K}, N, I)$, and we have $x_n = 0$ for all $N \setminus \bar{N}$ by a straightforward extension of Corollary 3.1e. Finally, in order to complete the proof we need to argue that no variable $y_k, k \in K \setminus \bar{K}$ prices out favorably with respect to the optimal solution of SRMP$(\bar{K}, N, I)$. To this end, note that all missing linking constraints $\Delta(S_{\bar{K}}^k)$ induced by all minimal variable sets of the form $\{y_l | l \in S_{\bar{K}}^k\}$ are included in SRMP$(\bar{K}, N, I)$. The corresponding family of index sets is clearly an element of $\mathcal{P}_k$. Thus, we conclude that the reduced cost $\bar{c}_k$ of y_k is nonnegative because $0 \leq \zeta_{yx} \leq \bar{c}_k$, where ζ_{yx} denotes the minimum reduced cost of y_k computed over all possible members of $\mathcal{P}_k$ in the row-generating PSP in (23). □

4. Applications of The Proposed Method. In this section, the proposed solution method is applied to our illustrative problems, MSCS and QSC.

4.1 Multi-Stage Cutting Stock Problem. We develop the subproblems for a column-and-row generation algorithm that solves the LP relaxation of the one-dimensional MSCS problem given in (3)-(6) with exponentially many cutting patterns both in the first and second stages. We follow the steps of the generic framework for CDR-problems developed in Section 3.

As discussed in Section 2.1, MSCS is a CDR-problem with no interaction; that is, the cardinality of a minimal variable set is just one. A first-stage cutting pattern represented by y_k is generated by any feasible combination of existing and new intermediate rolls. For each new intermediate roll currently absent from the SRMP included in the pattern, a single new linking constraint is introduced into the SRMP. Thus, $\mathcal{F}_k = \{\{k\}\}$, $\Sigma_k = \{k\}$, and $\Delta(\Sigma_k)$ denotes the set of linking constraints that correspond to the new intermediate rolls in the pattern. All three types of PSPs introduced in Section 3 are required for the MSCS problem. In the sequel, we explain how each PSP is constructed. To this end, we first state the dual of the LP (3)-(6):

$$\text{maximize} \quad \sum_{m \in M} b_m v_m, \tag{30}$$

$$\text{subject to} \quad \sum_{i \in I} C_{ik} w_i \leq 1, \quad k \in K, \tag{31}$$

$$\sum_{m \in M} B_{mn} v_m + \sum_{i \in I} D_{in} w_i \leq 0, \quad n \in N, \tag{32}$$

$$v_m \geq 0, m \in M, \quad w_i \geq 0, i \in I, \tag{33}$$

where $\{v_m | m \in M\}$, and $\{w_i | i \in I\}$ are the dual variables corresponding to the primal constraints (4) and (5), respectively. Recall that a single non-zero entry $D_{in} = -1$ in column n of D indicates that the cutting pattern n for the second stage is cut from the intermediate roll i. This implies that for x_n associated with a cutting pattern obtained from the intermediate roll i, the inequality (32) reduces to $\sum_{m \in M} B_{mn} v_m \leq w_i$.

In the $y-$PSP given below, the objective is to identify a violated constraint (31) for a first-stage cutting pattern composed of the set of intermediate rolls $I(\bar{K}, \bar{N})$ present in the current SRMP:

$$\begin{aligned} \text{maximize} \quad & \sum_{i \in I(\bar{K}, \bar{N})} w_i C_i, \\ \text{subject to} \quad & \sum_{i \in I(\bar{K}, \bar{N})} \epsilon_i C_i \leq W, \\ & C_i \in \mathbb{Z}^+ \cup \{0\}, i \in I(\bar{K}, \bar{N}), \end{aligned} \tag{34}$$

where W is the stock roll width, ϵ_i is the width of the intermediate roll i, and C_i is the number of times intermediate roll i is cut from the stock roll. Clearly, the $y-$PSP is an integer knapsack problem that may be solved efficiently by well-known methods in the literature. If the optimal objective function value of (34) is larger than 1, then a new first-stage cutting pattern with a negative reduced cost is added to the SRMP. Representing this pattern by y_k, we have $\bar{K} \leftarrow \bar{K} \cup \{k\}$.

In the $x-$PSP, we search for a second-stage cutting pattern with a negative reduced cost that is cut from one of the existing intermediate rolls in the current SRMP. In other words, we determine whether one of the dual constraints $\sum_{m \in M} B_{mn} v_m \leq w_i, i \in I(\bar{K}, \bar{N})$, is violated:

$$\begin{aligned} \text{maximize} \quad & \sum_{m \in M} v_m B_m, \\ \text{subject to} \quad & \sum_{m \in M} \pi_m B_m \leq \epsilon_i, \\ & B_m \in \mathbb{Z}^+ \cup \{0\}, m \in M, \end{aligned} \tag{35}$$

where i is the index for the existing intermediate roll of width ϵ_i under consideration, π_m is the width of the finished roll $m \in M$, and B_m denotes the number of times finished roll m is cut from intermediate roll i. Similar to the previous case, the $x-$PSP is an integer knapsack problem. If the optimal objective function value of (35) is larger than w_i, then a new second-stage cutting pattern with a negative reduced cost is added to the SRMP. Representing this pattern by x_n, we have $\bar{N} \leftarrow \bar{N} \cup \{n\}$.

In the row-generating PSP, a first-stage cutting pattern may contain new intermediate rolls in addition to existing ones. Following the structure of the general formulation in (23)-(24), the row-generating PSP for MSCS is stated as:

$$\text{maximize} \quad \sum_{i \in I(\bar{K}, \bar{N})} w_i C_i + \sum_{i' \in I \setminus I(\bar{K}, \bar{N})} \alpha_{i'}, \tag{36a}$$

$$\text{subject to} \quad \sum_{i \in I(\bar{K},\bar{N})} \epsilon_i C_i + \sum_{i' \in I \setminus I(\bar{K},\bar{N})} \epsilon_{i'} C_{i'} \leq W, \tag{36b}$$

$$C_i \in \mathbb{Z}^+ \cup \{0\}, \qquad i \in I, \tag{36c}$$

where

$$\alpha_{i'} = \text{maximize} \quad w_{i'} C_{i'}, \tag{37a}$$

$$\text{subject to} \quad \sum_{m \in M} \pi_m B_m \leq \epsilon_{i'}, \tag{37b}$$

$$\min_{m \in M} \pi_m \leq \epsilon_{i'} \leq W, \tag{37c}$$

$$B_m \in \mathbb{Z}^+ \cup \{0\}, \qquad m \in M, \tag{37d}$$

$$\sum_{m \in M} v_m B_m \leq w_{i'}, \tag{37e}$$

$$w_{i'} \geq 0, \tag{37f}$$

$$\text{At least one of (37e) or (37f) is tight.} \tag{37g}$$

Due to their potential size, we cannot explicitly generate the sets of all feasible first- and second-stage cutting patterns. Therefore, the structural constraints (36b)-(36c) and (37b)-(37d) that define the feasible first- and second-stage cutting patterns, respectively, are incorporated in the row-generating PSP. The constraints (36b) and (37b) are the classical knapsack constraints for the first- and second-stage cutting patterns, respectively, and constraint (37c) imposes natural lower and upper bounds on the width of a new intermediate roll. This width cannot exceed W and can be no smaller than the width of the smallest finished roll. The constraint (37e) corresponds to (24b) and mandates that the unknown dual variable associated with the currently absent intermediate roll i' is larger than or equal to the sum of the dual variables for the finished rolls that are cut from this intermediate roll. Note that all new linking constraints in this problem are of type (2) as specified in Assumption 2.3. Therefore, $\Delta_+(\{k\}) = \Delta(\{k\})$ and $\Delta_0(\{k\}) = \emptyset$ which only requires $w_{i'} \geq 0$ as stated in (37f). Finally, constraint (37g) is the counterpart of (24e).

The row-generating PSP (36)-(37) itself is an integer program with exponentially many variables due to the size of $I \setminus I(\bar{K},\bar{N})$. The steps of a branch-and-price algorithm for solving the row-generating PSP are detailed in Muter (2011). Similar to the $y-$PSP, if the optimal objective function value of (36) is larger than 1, then a new first-stage cutting pattern with a negative reduced cost is identified. Representing this pattern by y_k, we have $\bar{K} \leftarrow \bar{K} \cup \{k\}$ and $\bar{N} \leftarrow S_N(\{k\})$, where each $x_n, n \in S_N(\{k\})$ is associated with a second-stage cutting pattern for a new intermediate roll. In addition, one new linking constraint per new intermediate roll is incorporated in the SRMP.

4.2 Quadratic Set Covering. In this section, we develop the subproblems for a column-and-row generation algorithm that solves the LP relaxation of the QSC problem which belongs to the class of CDR-problems with interaction. The cardinality of each minimal variable set for this problem is two, and three linking constraints of type (9)-(11) and one auxiliary $x-$variable are associated with a minimal variable set. As described in Section 2.1, the set of all $y-$variables is given explicitly, and the set of all possible pairs composed of the $y-$variables is denoted by P.

To solve the formulation (7)-(13) by column-and-row generation, we initialize the SRMP with a set of columns $\bar{K}$ that satisfy the covering constraints (8) in addition to the set of all linking constraints

(9)-(11) induced by $\{y_l | l \in \bar{K}\}$. A new variable y_k is always added to the SRMP along with three linking constraints of type (9)-(11) and a variable x_{kl} for each pair of variables y_k, y_l, where $l \in \bar{K}$. Thus, the column-and-row generation mechanism maintains that the SRMP is constituted by $\{y_l | l \in \bar{K}\}$, $\{x_n | n \in S_N(\bar{K})\}$, and the set of all linking constraints $\Delta(\bar{K})$ induced by $\bar{K}$ at all times during course of the algorithm. Moreover, the $x-$variables are auxiliary and only appear in the linking constraints. Therefore, for QSC we only need to solve the row-generating PSP developed below. We first state the dual of the LP (7)-(13):

$$
\begin{aligned}
\text{maximize} \quad & \sum_{j \in J} u_j + \sum_{(k,l) \in P, k<l} w_{kl}, & \\
\text{subject to} \quad & \sum_{j \in J} A_{jk} u_j + \sum_{(k,l) \in P, k<l} (w_{kl} + \varphi_{kl}) + \sum_{(l,k) \in P, l<k} (w_{lk} + \varphi'_{lk}) \leq f_{kk}, & k \in K, \\
& - w_{kl} - \varphi_{kl} - \varphi'_{kl} \leq 2 f_{kl}, & (k,l) \in P, k<l, \\
& \varphi_{kl}, \varphi'_{kl} \geq 0, w_{kl} \leq 0, & (k,l) \in P, k<l,
\end{aligned}
$$

where $\{u_j | j \in J\}$ are the dual variables corresponding to the covering constraints (8), and w_{kl}, φ_{kl}, φ'_{kl} for $(k,l) \in P, k < l$, are the dual variables associated with the linking constraints (9)-(11), respectively.

Following the structure of the general formulation in (23)-(24), the row-generating PSP for QSC is stated as:

$$
\zeta_{yx} = \min_{k \in (K \setminus \bar{K})} \quad \left\{ f_{kk} - \sum_{j \in J} A_{jk} u_j - \max_{\mathcal{F}_k \in \mathcal{P}_k} \left(\sum_{\{k,l\} \in \mathcal{F}_k, k<l} \alpha_{kl} + \sum_{\{l,k\} \in \mathcal{F}_k, l<k} \alpha_{lk} \right) \right\}, \text{ where} \tag{38}
$$

$$
\begin{aligned}
\alpha_{kl} = \text{maximize} \quad & w_{kl} + \varphi_{kl}, & \text{(39a)} \\
\text{subject to} \quad & - w_{kl} - \varphi_{kl} - \varphi'_{kl} \leq 2 f_{kl}, & \text{(39b)} \\
& w_{kl} = \varphi'_{kl} = 0, & \text{(39c)} \\
& \varphi_{kl} \geq 0, & \text{(39d)} \\
& \text{At least one of (39b) or (39d) is tight,} & \text{(39e)} \\
\alpha_{lk} = \text{maximize} \quad & w_{lk} + \varphi'_{lk}, & \text{(39f)} \\
\text{subject to} \quad & - w_{lk} - \varphi_{lk} - \varphi'_{lk} \leq 2 f_{lk}, & \text{(39g)} \\
& w_{lk} = \varphi_{lk} = 0, & \text{(39h)} \\
& \varphi'_{lk} \geq 0, & \text{(39i)} \\
& \text{At least one of (39g) or (39i) is tight,} & \text{(39j)}
\end{aligned}
$$

where $\mathcal{F}_k$ and the values of the dual variables $\{\varphi_{kl} | \{k,l\} \in \mathcal{F}_k, k < l\}$ and $\{\varphi'_{kl} | \{k,l\} \in \mathcal{F}_k, l < k\}$ are to be determined. From the discussion on Figure 2 in Section 3.3, recall that only the dual variable associated with a constraint $y_k - x_{kl} \geq 0$ may assume a positive value, and the rest of the constraints in $\Delta(\{k,l\})$ belong to the set $\Delta_0(\{k,l\})$. These restrictions for $\{k,l\} \in \mathcal{F}_k$ are imposed by the constraints (39c)-(39d) and (39h)-(39i) for $k < l$ and $l < k$, respectively. The dual feasibility of the $x-$variables in the new linking constraints is mandated by the constraints (39b) and (39g), and constraints (39e) and (39j) are the counterparts of (24e).

For $\{k,l\} \in \mathcal{F}_k, k < l$, the optimal solution of (39a)-(39e) is identified below, and the case for $l < k$ can be derived analogously.

(i) If $f_{kl} < 0$, the optimal value of $\varphi_{kl} = -2f_{kl}$ since φ_{kl} is a nonnegative variable that appears with a negative coefficient in (39b) and with a positive coefficient in the objective (39a). In this case, x_{kl} is the basic variable associated with the new linking constraint $y_k - x_{kl} \geq 0$. For the other two linking constraints in $\Delta(\{k,l\})$, the associated surplus variables are selected as basic.

(ii) If $f_{kl} \geq 0$, the optimal value of $\varphi_{kl} = 0$, and all new basic variables are surplus variables.

Thus, we conclude that $\alpha_{kl} = \max(-2f_{kl}, 0)$, where $2f_{kl}$ is the objective function coefficient of x_{kl} in (7)-(13). Based on this very simple structure of the optimal solutions of (39a)-(39e) and (39f)-(39j), we re-state the row-generating PSP for QSC:

$$\zeta_{yx} = \min_{k \in K \setminus \bar{K}} \left\{ f_{kk} - \sum_{j \in J} A_{jk} u_j - \max_{\mathcal{F}_k \in \mathcal{P}_k} \left\{ \sum_{\{k,l\} \in \mathcal{F}_k, k<l} \max(0, -2f_{kl}) + \sum_{\{l,k\} \in \mathcal{F}_k, l<k} \max(0, -2f_{lk}) \right\} \right\}. \tag{40}$$

If $\zeta_{yx} = \bar{c}_k < 0$ with an associated optimal family of index sets $\mathcal{F}_k$, then SRMP grows both horizontally and vertically with the addition of the variables $\{y_l | l \in \Sigma_k\}$, $\{x_n | n \in S_N(\Sigma_k)\}$, and the linking constraints $\Delta(\Sigma_k)$, where $\Sigma_k = \cup_{\{k,l\} \in \mathcal{F}_k} \{k, l\}$. Furthermore, as explained at the beginning of this section, all relevant linking constraints associated with the variables $\{y_l | l \in (\Sigma_k \setminus \{k\})\}$ are also incorporated into the SRMP. Note that some of the variables $\{y_l | l \in \Sigma_k\}$ are already a part of the SRMP. This is generally true for CDR-problems with interaction.

Example 4.1 *Consider the following symmetric positive semidefinite cost matrix* $\mathbf{F}$ *and the cover matrix* $\mathbf{A}$*:*

$$\mathbf{F} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Solving the QSC instance defined by $\mathbf{F}$ *and* $\mathbf{A}$ *by excluding the possibility of adding several* $y-$*variables simultaneously to the SRMP would result in the following PSP:*

$$\zeta_{yx} = \min_{k \in K \setminus \bar{K}} \left\{ f_{kk} - \sum_{j \in J} A_{jk} u_j - \sum_{l \in \bar{K}, k<l} \max(0, -2f_{kl}) - \sum_{l \in \bar{K}, l<k} \max(0, -2f_{lk}) \right\}. \tag{41}$$

Suppose that we form the initial SRMP with y_1 *only. The PSP defined in (41), which ignores the minimal variable sets of the form* $\{y_k, y_l\}, l \notin \bar{K}$ *for pricing out* $y_k, k \notin \bar{K}$*, cannot identify any new* $y-$*variable with a negative reduced cost, and the column generation terminates with* $y_1 = 1$ *and an objective function value of 2. Taking also into account the minimal variable sets* $\{y_k, y_l\}, l \notin \bar{K}$ *by solving the row-generating PSP (40) yields the optimal families of index sets* $\mathcal{F}_2 = \{\{1,2\}, \{2,3\}, \{2,4\}\}$*,* $\mathcal{F}_3 = \{\{1,3\}, \{2,3\}, \{3,4\}\}$*, and* $\mathcal{F}_4 = \{\{1,4\}, \{2,4\}, \{3,4\}\}$ *for* y_2, y_3*, and* y_4*, respectively. The associated reduced costs* $\bar{c}_2, \bar{c}_3, \bar{c}_4 < 0$*, and in all cases all remaining* $y-$*variables are incorporated into the SRMP. The resulting true optimal solution of (7)-(13) is given by* $y_1 = 0$*,* $y_2 = y_3 = y_4 = 1$ *with an objective value of 1.*

5. An Extension Keeping the discussion on the CDR-problems with and with no interaction separate helped us to highlight the differences in developing the row-generating PSP for these two types of problems in the past sections. These tools of analysis may however also be combined to tackle CDR-problems in which some minimal variable sets are of cardinality one, while others are composed of two or more $y-$variables. We aptly refer to such problems as *mixed* CDR-problems. To the best of our knowledge, there is no study in the literature that deals with a mixed CDR-problem. Therefore, to complement our discussion with such an extension, we briefly describe a fictitious mixed CDR-problem in the subsequent part of this section.

Consider a tactical network design and vehicle routing problem defined on a directed network, where the set of nodes is represented by K. A demand b_k is associated with each node of the network, and these demands have to be served daily by a set of routes originating and terminating at a depot, possibly located at two different nodes of the network. We pay a fixed cost of c_k for opening a depot at a node $k \in K$, and each vehicle route incurs a cost of d_n. Assume that split deliveries are allowed, where the number of units delivered to customer k by route n is given by B_{kn}. The objective is to minimize the total fixed and routing costs. No more than v_k vehicles may be dispatched from a depot at node k to return to the same location. Such routes are referred to as tours. Similarly, the number of vehicles which originate at node k and terminate at node l cannot exceed v_{kl}. The set of tours starting at node k is denoted by N_k, and N_{kl} represents the set of routes emanating from node k and finishing at node l. The set of all routes is given by $N = (\cup_{k \in K} N_k) \cup (\cup_{k,l \in K, k \neq l} N_{kl})$. The variable $y_k, k \in K$, takes the value one, if a depot is located at node k, and is zero otherwise. The binary variable $x_n, n \in N$, indicates whether tour/route n is selected in the solution. Then, the LP relaxation of this problem may be formulated as below:

$$
\begin{aligned}
\text{minimize} \quad & \sum_{k \in K} c_k y_k + \sum_{n \in N} d_n x_n, && (42)\\
\text{subject to} \quad & \sum_{n \in N} B_{kn} x_n \geq b_k, \quad k \in K, && (43)\\
& v_k y_k - \sum_{n \in N_k} x_n \geq 0, \quad k \in K, && (44)\\
& v_{kl} y_k - \sum_{n \in N_{kl}} x_n \geq 0, \quad k, l \in K,\ k \neq l, && (45)\\
& v_{kl} y_l - \sum_{n \in N_{kl}} x_n \geq 0, \quad k, l \in K,\ k \neq l, && (46)\\
& 0 \leq y_k \leq 1, k \in K, \quad 0 \leq x_n \leq 1, n \in N. && (47)
\end{aligned}
$$

The objective (42) minimizes the total cost of opening depots and serving the customers by a set of routes. Constraints (43) ensure that the customer demands are satisfied. The set of linking constraints (44) prescribe that a tour associated with $k \in K$ is not selected unless a depot is located at node k. Similarly, a route from node k to node l requires that a depot is present at both nodes k and l as described by the set of linking constraints (45)-(46).

This problem is a special case of the generic model (MP), where A, a, and r are zero, and $M = K$. In general, the number of routes in the problem may grow exponentially which motivates a column generation algorithm for solving the formulation (42)-(47). If, in addition, the number of nodes in the network is large, then this would merit a column-and-row generation approach because including all $\mathcal{O}(\mid K \mid^2)$ linking constraints in the formulation (42)-(47) directly would be computationally prohibitive. The column-and-row generation would be initialized with a small number of depots located at nodes $k \in \bar{K}$ and a set of associated routes so that the initial SRMP is feasible. For this problem, no $y-$PSP is required because the generation of a new $y-$variable is only meaningful if associated $x-$variables and linking constraints are introduced into the SRMP along with it. In the $x-$PSP, we either construct a tour associated with a depot at a node $k \in \bar{K}$ or a route from a depot at a node $k \in \bar{K}$ to a depot at a node $l \in \bar{K}$. In the row-generating PSP for y_k, $\{y_k\}$ is a minimal variable set of cardinality one for a single linking constraint (44). The remaining minimal variable sets are either of the form $\{y_k, y_l\}$ or $\{y_l, y_k\}$, depending on whether routes are generated from node k to node l or vice versa. That is, except for one, all minimal variable sets in the row-generating PSP for y_k are of cardinality two and induce two linking constraints (45)-(46). In both cases, $y-$variables generate the associated $x-$variables, and the linking constraints are redundant until all variables in the associated minimal variable set take positive values. Thus, both Assumptions 2.1 and 2.2 are satisfied. Note that there is a one-to-one correspondence between the minimal variable sets and the linking constraints. Moreover, the linking constraints are in the form mandated by (2) in Assumption 2.3. We conclude that the problem at hand is a mixed CDR-problem and is amenable to the column-and-row generation framework devised in this paper. We only need to pay attention to set up (24) in the row-generating PSP properly depending on the cardinality of the associated minimal variable set.

6. Conclusions and Future Research. In this paper, we presented and analyzed the first unified framework for large-scale linear programs with column-dependent-rows. We identified a set of properties that characterize CDR-problems and argued that this is a large class of problems. To the best of our knowledge, all CDR-problems studied so far in the literature belong to this class. Moreover, we devised a generic methodology for solving the LP relaxations of CDR-problems to optimality. The most important contribution of this methodology is the row-generating PSP that allows us to generate several variables and their associated linking constraints simultaneously by correctly estimating the dual values of the missing linking constraints.

The aim of this paper is not to concentrate on problem-specific algorithms but to show that a generic methodology can be developed to solve the LP relaxations of CDR-problems. To supplement the theory, we also apply the proposed generic approach to solve the LP relaxations of the MSCS and QSC problems. The application to MSCS leads to a row-generating PSP that itself is an integer program with exponentially many columns and is further analyzed in Muter (2011). Moreover, QSC is considered in the column generation context for the first time, and its unique structure helps us to generalize our approach to the class of problems we refer to as CDR-problems with interaction. We note that many routing and scheduling problems with interactions between individual routes or schedules may be cast as quadratic set covering problems with restricted pairs and side constraints, and devising computationally effective column-and-row generation algorithms for these problems is an important research question. An example appears in Muter et al. (2010b) and illustrates that this is not a simple endeavor by any means.

An important line of work we will pursue in the future is developing computationally effective implementations of our generic methodology for specific CDR-problems. Clearly, the ultimate goal is to embed the column-and-row generation algorithm into branch-and-price or branch-and-cut-and-price algorithms to solve large-scale integer programming problems with column-dependent-rows. Furthermore, we are currently working on alternate approaches based on Lagrangian relaxation and Benders decomposition for solving the LP relaxations of CDR-problems. This will enable us to look at CDR-problems from different angles and compare the computational performance of different algorithms.

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