Global existence and blow-up for a class of nonlocal nonlinear Cauchy problems arising in elasticity

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Abstract
We study the initial-value problem for a general class of nonlinear nonlocal wave equations arising in one-dimensional nonlocal elasticity. The model involves a convolution integral operator with a general kernel function whose Fourier transform is nonnegative. We show that some well-known examples of nonlinear wave equations, such as Boussinesq-type equations, follow from the present model for suitable choices of the kernel function. We establish global existence of solutions of the model assuming enough smoothness on the initial data together with some positivity conditions on the nonlinear term. Furthermore, conditions for finite time blow-up are provided.

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1. Introduction

In this paper we focus on global existence and blow-up of solutions to the nonlocal nonlinear Cauchy problem

$$u_{tt} = (\beta * (u + g(u)))_{xx}, \qquad x \in \mathbb{R}, \quad t > 0, \tag{1.1}$$

$$u(x, 0) = \varphi(x), \qquad u_t(x, 0) = \psi(x), \tag{1.2}$$

where $u = u(x, t)$, g is a nonlinear function of u, the subscripts denote partial derivatives, the symbol $*$ denotes convolution in the spatial domain, β is an integrable function whose Fourier transform satisfies a set of conditions specified in section 2.

The predictions of classical (local) elasticity theory become inaccurate when the characteristic length of an elasticity problem is comparable to the atomic length scale. To remedy this situation, a nonlocal theory of elasticity was introduced (see [1–3] and the references cited therein) and the main feature of the new theory is the fact that its predictions were more down to earth than those of the classical theory. For other generalizations of elasticity we refer the reader to [4–7]. Equation (1.1) models one-dimensional motion in an infinite medium with nonlinear and nonlocal elastic properties within the context of the nonlocal theory [8]. The well-posedness of the Cauchy problem (1.1)–(1.2) depends crucially on the presence of a suitable kernel. However, the choice of an appropriate kernel function remains an open problem in the nonlocal theory of elasticity and nowadays, attention is focused on the selection of the kernel function [9]. Then the question that naturally arises is which of the possible forms of the kernel functions are relevant for the global well-posedness of the corresponding initial-value problem. In [8], for a special form of the kernel function which reduces (1.1) to a higher-order Boussinesq equation, it has been proved that the corresponding initial-value problem is globally well posed. However, the effect of more general kernel functions on global existence and blow-up must be addressed. In this study, as a partial answer to this question, we consider (1.1) with a general class of kernel functions (see (2.6) below) and provide both global existence and blow-up results for the solutions of the initial-value problem (1.1)–(1.2). The kernel functions most frequently used in the literature are particular cases of this general class of kernel functions.

Section 2 presents a derivation of (1.1) from the one-dimensional theory of nonlinear nonlocal elasticity. This section also presents both a discussion about the assumptions we make about β and a brief overview of the kernel functions used in the literature. In section 3 we present a local existence theory for solutions of the Cauchy problem (1.1)–(1.2) for given initial data in suitable Sobolev spaces. In section 4 we prove global existence of solutions of (1.1)–(1.2) assuming some positivity conditions on the nonlinear function g together with enough smoothness on the initial data. Finally, in section 5 we discuss finite time blow-up of solutions.

In what follows $H^s = H^s(\mathbb{R})$ will denote the L^2 Sobolev space on $\mathbb{R}$. For the H^s norm we use the Fourier transform representation $\|u\|_s^2 = \int_{\mathbb{R}}(1+\xi^2)^s|\hat{u}(\xi)|^2\,\mathrm{d}\xi$. We use $\|u\|_\infty$, $\|u\|$ and $\langle u, v\rangle$ to denote the L^∞ and L^2 norms and the inner product in L^2, respectively. D_x stands for the classical partial derivative with respect to x.

2. The nonlocal nonlinear Cauchy problem

2.1. Physical motivation

We now discuss the physical motivation behind (1.1). In the classical (or local) theory of elasticity, the stress at a spatial point depends uniquely on the strain at the same point. However, in the nonlocal theory of elasticity the stress at a point depends on the strain field at every point in the body and consequently it is written as a functional of the strain field (see [1] and the references cited therein). We now show that in one space dimension, the dimensionless form of the equation governing the resulting dynamics is given by (1.1).

Consider a one-dimensional, homogeneous, nonlinearly and non-locally elastic infinite medium. Let a scalar-valued function $u(X, t)$ be the displacement of a reference point X at time t. In the absence of body forces the equation of motion for the displacement is

$$\rho_0 u_{tt} = (S(u_X))_X, \tag{2.1}$$

where ρ_0 is the mass density of the medium, $S = S(u_X)$ is the (nonlocal) stress [1, 8]. In contrast with classical elasticity, we employ a nonlocal model of constitutive equation, which gives the stress S as a general nonlinear nonlocal function of the strain $\epsilon = u_X$. The constitutive equation for nonlinear nonlocal elastic response considered here has the following form

$$S(X, t) = \int_{\mathbb{R}} \alpha(|X - Y|)\sigma(Y, t)\,\mathrm{d}Y, \qquad \sigma(X, t) = W'(\epsilon(X, t)), \tag{2.2}$$

where σ is the classical (local) stress, W is the local strain-energy density function, Y denotes a generic point of the medium, α is a kernel function to be specified below and the symbol $'$ denotes differentiation [8]. The kernel α serves as a weight on the relative contribution of the local stress $\sigma(Y,t)$ at a point Y in a neighbourhood of X to the nonlocal stress $S(X,t)$. So, when the kernel becomes the Dirac delta measure, the classical constitutive relation of a hyperelastic material is recovered. We emphasize the distinction between our proposed constitutive relation (2.2), which involves a nonlinear dependence of the stress on both local and nonlocal effects, and the standard constitutive relations, which include a linear term describing nonlocal effects.

If a stress-free undistorted state is considered as the reference configuration, the strain-energy function must satisfy $W(0) = W'(0) = 0$. Without loss of generality, for convenience, we decompose the derivative of the strain-energy density function into its linear and nonlinear parts:

$$W'(\epsilon) = \gamma[\epsilon + g(\epsilon)], \qquad g(0) = 0, \tag{2.3}$$

where γ is a constant with the dimension of stress, or equivalently

$$W(\epsilon) = \gamma\left[\tfrac{1}{2}\epsilon^2 + G(\epsilon)\right]$$

with

$$G(\epsilon) = \int_0^{\epsilon} g(s)\,\mathrm{d}s. \tag{2.4}$$

Differentiating both sides of (2.1) with respect to X we obtain the equation of motion for the strain:

$$\rho_0\epsilon_{tt} = \gamma\left\{\int_{\mathbb{R}} \alpha(|X-Y|)[\epsilon + g(\epsilon)]\,\mathrm{d}Y\right\}_{XX}. \tag{2.5}$$

Now we define the dimensionless independent variables

$$x = \frac{X}{l}, \qquad \tau = \frac{t}{l}\sqrt{\frac{\gamma}{\rho_0}}$$

where l is a characteristic length and from now on, and for simplicity, we use u for ϵ and t for τ. Thus, (2.5) takes the form given in (1.1), with

$$\beta * v = \int_{\mathbb{R}} \beta(x-y)v(y)\,\mathrm{d}y$$

and $\beta(x) = l\alpha(|x|)$.

2.2. *The kernel function*

An important open question in the nonlocal theory of elasticity is how to choose the kernel function which represents the details of the atomic scale effects. The triangular kernel, the exponential kernel and the Gaussian kernel (see, for instance, equations (3.3), (3.4) and (3.5) of [10], respectively) are examples of only the most commonly used kernel functions. In general it is assumed that the kernel function α is a monotonically decreasing nonnegative function of the distance $|x-y|$ between x and y. We refer the reader to [9] for an example of a non-monotone, sign-changing kernel function. In this study we consider a general class of kernel functions, which covers both types of the kernels reported in the literature. Throughout the rest of this work we assume that the kernel β is an integrable function whose Fourier transform satisfies

$$0 \leqslant \hat{\beta}(\xi) \leqslant C(1+\xi^2)^{-r/2} \qquad \text{for all } \xi \tag{2.6}$$

for a suitable constant $C > 0$. Here the exponent r can be any real number (not necessarily an integer). The order of β is $-r$, where r is the largest such exponent.

The kernels used in the literature all satisfy the positivity condition $\hat{\beta}(\xi) \geqslant 0$. This is a natural consequence of the wave character of the equation of motion, which means the nondissipative nature of the dynamics. On the other hand, in the literature the following two conditions are imposed on the kernel: $\beta(0) \geqslant \beta(x)$ and $\beta(-x) = \beta(x)$. In fact, these conditions are implied by the positivity of $\hat{\beta}(\xi)$ through Bochner's theorem [11]. In this study we only consider kernels with $r \geqslant 2$. When $r < 2$, the model is linearly unstable with unbounded growth rate at short wavelengths and thus this case seems to be of a different nature.

In the next subsection we present several examples of kernel functions, showing how the general class of kernels defined by (2.6) covers the most common used kernels in the literature. Before this, for convenience we rewrite (1.1) in a slightly different form. Since differentiation and convolution commute, $(\beta * h(u))_{xx} = K * h(u)$, where $K = \beta_{xx}$ in the distribution sense. Obviously $\widehat{K}(\xi) = -\xi^2\widehat{\beta}(\xi)$. Then (1.1)–(1.2) becomes

$$u_{tt} = K * f(u), \qquad x \in \mathbb{R}, \quad t > 0, \tag{2.7}$$

$$u(x,0) = \varphi(x), \qquad u_t(x,0) = \psi(x), \tag{2.8}$$

where to shorten the notation we write $f(u) = u + g(u)$.

2.3. *Examples for the kernel*

The following list of kernels contains the most commonly used kernels.

(i) *The Dirac measure*: $\beta = \delta$. In this case $r = 0$, and we recover the classical theory of one-dimensional elasticity. The equation of motion, (1.1), is just a nonlinear wave equation

$$u_{tt} - u_{xx} = g(u)_{xx}.$$

(ii) *The triangular kernel* [10]:

$$\beta(x) = \begin{cases} 1 - |x|, & |x| \leqslant 1 \\ 0, & |x| \geqslant 1. \end{cases}$$

Since $\widehat{\beta}(\xi) = (4/\xi^2)\sin^2(\frac{\xi}{2})$, we have $r = 2$. Abusing notation slightly, note that

$$K(x) = \delta(x-1) - 2\delta(x) + \delta(x+1).$$

So

$$(\beta * v)_{xx} = v(x-1) - 2v(x) + v(x+1)$$

and the equation of motion, (1.1), becomes a differential–difference equation.

(iii) *The exponential kernel* [10]: $\beta(x) = \frac{1}{2}\mathrm{e}^{-|x|}$. Since $\widehat{\beta}(\xi) = (1+\xi^2)^{-1}$, we have $r = 2$. Note that β is Green's function for the operator $1 - D_x^2$ so that

$$(\beta * v)_{xx} = (1 - D_x^2)^{-1} v_{xx} = \beta * v - v.$$

The equation of motion, (1.1), becomes the generalized improved Boussinesq equation,

$$u_{tt} - u_{xx} - u_{xxtt} = (g(u))_{xx}.$$

The global existence of the Cauchy problem for this equation has been studied by many authors (see [12–14]).

(iv) *The double-exponential kernel* [15]: $\beta(x) = 1/(2(c_1^2 - c_2^2))(c_1 e^{-|x|/c_1} - c_2 e^{-|x|/c_2})$, where c_1 and c_2 are real and positive constants. Since $\widehat{\beta}(\xi) = (1 + \gamma_1\xi^2 + \gamma_2\xi^4)^{-1}$, where $\gamma_1 = c_1^2 + c_2^2$ and $\gamma_2 = c_1^2 c_2^2$, we have $r = 4$. As above, β is Green's function for the operator $1 - \gamma_1 D_x^2 + \gamma_2 D_x^4$ and

$$(\beta * v)_{xx} = (1 - \gamma_1 D_x^2 + \gamma_2 D_x^4)^{-1} v_{xx}.$$

The equation of motion, (1.1), becomes the higher-order Boussinesq equation,

$$u_{tt} - u_{xx} - \gamma_1 u_{xxtt} + \gamma_2 u_{xxxxtt} = (g(u))_{xx}.$$

The global existence of the Cauchy problem of the higher-order Boussinesq equation has been proved in [8, 16, 17].

(v) *The Gaussian kernel* [10]: $\beta(x) = (1/\sqrt{2\pi})e^{-x^2/2}$. Note that $\widehat{\beta}(\xi) = e^{-\xi^2/2}$.

(v) *A sign-changing kernel* [9]: $\beta(x) = (1/\sqrt{2\pi})(1 - x^2)e^{-x^2/2}$. Then $\widehat{\beta}(\xi) = \xi^2 e^{-\xi^2/2}$.

In these last two examples the kernel function β hence its Fourier transform $\widehat{\beta}$ is rapidly decreasing, and we can take any r in (2.6). The equation of motion, (1.1), is an integro-differential equation.

In order to capture the equations corresponding to the classical (local) theory of elasticity one expects that the kernel functions must behave like the Dirac measure as the characteristic length tends to zero. Notice that the characteristic length l does not appear in the above dimensionless forms of the kernel functions. However, if we rewrite the kernel functions in terms of the original quantities X and α, then we indeed get the Dirac measure as l goes to zero.

The rest of this study addresses some issues related to global existence and blow-up of solutions for the Cauchy problem (1.1)–(1.2).

3. Local well posedness

Below we will give several versions of local well posedness depending on the properties of the kernel β. We first need two lemmas concerning the behaviour of the nonlinear term [13, 18].

Lemma 3.1. *Let $s \geqslant 0$, $f \in C^{[s]+1}(\mathbb{R})$ with $f(0) = 0$. Then for any $u \in H^s \cap L^\infty$, we have $f(u) \in H^s \cap L^\infty$. Moreover there is some constant $A(M)$ depending on M such that for all $u \in H^s \cap L^\infty$ with $\|u\|_\infty \leqslant M$*

$$\|f(u)\|_s \leqslant A(M)\|u\|_s.$$

Lemma 3.2. *Let $s \geqslant 0$, $f \in C^{[s]+1}(\mathbb{R})$. Then for any $M > 0$ there is some constant $B(M)$ such that for all $u, v \in H^s \cap L^\infty$ with $\|u\|_\infty \leqslant M$, $\|v\|_\infty \leqslant M$ and $\|u\|_s \leqslant M$, $\|v\|_s \leqslant M$ we have*

$$\|f(u) - f(v)\|_s \leqslant B(M)\|u - v\|_s \qquad \text{and} \qquad \|f(u) - f(v)\|_\infty \leqslant B(M)\|u - v\|_\infty.$$

For $s > \frac{1}{2}$, by the Sobolev embedding theorem, $H^s \subset L^\infty$. Then the bounds on L^∞ norms in lemma 3.2 become redundant and we get the following.

Corollary 3.3. *Let $s > \frac{1}{2}$, $f \in C^{[s]+1}(\mathbb{R})$. Then for any $M > 0$ there is some constant $B(M)$ such that for all $u, v \in H^s$ with $\|u\|_s \leqslant M$, $\|v\|_s \leqslant M$ we have*

$$\|f(u) - f(v)\|_s \leqslant B(M)\|u - v\|_s.$$

We will assume throughout this paper that $f \in C^{[s]+1}(\mathbb{R})$ with $f(0) = 0$ for some $s \geqslant 0$; note that the function g appearing in (1.1) will also satisfy the same assumptions.

Theorem 3.4. *Let $s > 1/2$ and $r \geqslant 2$. Then there is some $T > 0$ such that the Cauchy problem (2.7)–(2.8) (equivalently (1.1)–(1.2)) is well posed with solution in $C^2([0,T], H^s)$ for initial data $\varphi, \psi \in H^s$.*

Proof. Converting the problem into an H^s valued system of ordinary differential equations

$$u_t = v \qquad u(0) = \varphi$$
$$v_t = K * f(u) \qquad v(0) = \psi,$$

we can use the standard well posedness result [19] for ordinary differential equations once we check whether the right-hand side is Lipschitz on H^s. As

$$|\widehat{K}(\xi)| = |-\xi^2\widehat{\beta}(\xi)| \leqslant C\xi^2(1+\xi^2)^{-r/2} \leqslant C,$$

then

$$\begin{aligned}\|K * v\|_s &= \|(1+\xi^2)^{s/2}\widehat{K}(\xi)\widehat{v}(\xi)\| \\ &\leqslant C\|(1+\xi^2)^{s/2}\widehat{v}(\xi)\| = C\|v\|_s\,. \end{aligned} \tag{3.1}$$

Thus $K * (\quad)$ is a bounded linear map on H^s. Then from corollary 3.3, $K * f(u)$ is locally Lipschitz on H^s. ∎

Now we remove the restriction $s > \frac{1}{2}$. Note that the main obstacle is the following: to control the nonlinear term we need an L^∞ estimate. Even if we start with L^∞ data, $K * (\quad)$ may not stay in L^∞. We overcome this obstacle in two different cases. One case is theorem 3.5, where we assume that the order of β is sufficiently large. This will ensure that $K * f(u)$ is sufficiently smooth, hence in L^∞. The other case is in theorem 3.6.

Theorem 3.5. *Let $s \geqslant 0$ and $r > \frac{5}{2}$. Then there is some $T > 0$ such that the Cauchy problem (2.7)–(2.8) (equivalently (1.1)–(1.2)) is well posed with solution in $C^2([0,T], H^s \cap L^\infty)$ for initial data $\varphi, \psi \in H^s \cap L^\infty$.*

Proof. As in the proof of theorem 3.4 we convert the problem into an ODE system on $H^s \cap L^\infty$ where the space is endowed with the norm $\|v\|_{s,\infty} = \|v\|_s + \|v\|_\infty$. Then all we need is to show that $K * f(u)$ is Lipschitz on $H^s \cap L^\infty$. Since

$$|\widehat{K}(\xi)| = |-\xi^2\widehat{\beta}(\xi)| \leqslant C\xi^2(1+\xi^2)^{-r/2} \leqslant C(1+\xi^2)^{-(r-2)/2},$$

we have

$$\begin{aligned}\|K * v\|_{s+r-2} &= \|(1+\xi^2)^{(s+r-2)/2}\widehat{K}(\xi)\widehat{v}(\xi)\| \\ &\leqslant C\|(1+\xi^2)^{s/2}\widehat{v}(\xi)\| = C\|v\|_s. \end{aligned}$$

Then $K * (\quad)$ is a bounded linear map from H^s into H^{s+r-2}. Since $s \geqslant 0$ and $r > \frac{5}{2}$ we have $s + r - 2 > \frac{5}{2} - 2 = \frac{1}{2}$. Again the Sobolev embedding theorem implies that $K * (\quad)$ is a bounded linear map from $H^s \cap L^\infty$ into $H^s \cap L^\infty$. Lemma 3.2 implies the Lipschitz condition on $H^s \cap L^\infty$. ∎

For the second case, we observe that it suffices to have $K * (\quad)$ map from $H^s \cap L^\infty$ into itself. This holds for $K \in L^1$ as well as for the triangular and the exponential kernel of section 2.3 where K equals an L^1 function plus some Dirac measures. Theorem 3.6 generalizes this to finite measures.

Theorem 3.6. *Let $s \geqslant 0$ and let K be a finite measure on $\mathbb{R}$. Then there is some $T > 0$ such that the Cauchy problem (2.7)–(2.8) (equivalently (1.1)–(1.2)) is well posed with solution in $C^2([0,T], H^s \cap L^\infty)$ for initial data $\varphi, \psi \in H^s \cap L^\infty$.*

Proof. When K is a finite measure

$$(K * v)(x) = \int_{\mathbb{R}} v(x-y)\,\mathrm{d}K(y),$$

so that for $v \in L^\infty$, $\|K * v\|_\infty \leqslant \|v\|_\infty \int \mathrm{d}|K| = \|K\|\|v\|_\infty$, where $\|K\|$ is the total variation of K. Also,

$$\widehat{K}(\xi) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}x\xi}\,\mathrm{d}K(x)$$

so that $|\widehat{K}(\xi)| \leqslant \|K\|$. Then for $v \in H^s$,

$$\|K * v\|_s = \|(1+\xi^2)^{s/2}\widehat{K}(\xi)\widehat{v}(\xi)\| \leqslant \|K\|\|(1+\xi^2)^{s/2}\widehat{v}(\xi)\| = \|K\|\|v\|_s.$$

Thus $K * (\quad)$ is a bounded linear map on $H^s \cap L^\infty$ and the Lipschitz condition follows from lemma 3.2. ∎

When $\beta \in L^1$ and K is a finite measure, we have $(1+\xi^2)\widehat{\beta}(\xi) \leqslant \|\beta\|_{L^1} + \|K\|$, so that $r = 2$.

Remark 3.7. In theorems 3.4, 3.5 and 3.6 we have not used the assumption $\widehat{\beta}(\xi) \geqslant 0$; so in fact these results hold for more general kernels.

Remark 3.8. Going back to the kernels given in section 2.3, we see that $r > \frac{5}{2}$ in (iv), (v) and (vi) so theorem 3.5 applies; whereas for (ii) and (iii), $r = 2$. In (ii) K is a finite measure so theorem 3.6 applies. In (iii) computation shows that $K = \beta - \delta$ so that $\mathrm{d}K = \mathrm{e}^{-|x|}\,\mathrm{d}x - \mathrm{d}\delta$ and theorem 3.6 applies again.

By the Sobolev embedding theorem $H^s \cap L^\infty = H^s$ for $s > \frac{1}{2}$, and the norm $\|\quad\|_{s,\infty}$ is equivalent to the H^s norm. The solution in theorems 3.4, 3.5 and 3.6 can be extended to a maximal interval $[0, T_{\max})$ where finite $T_{\max}$ is characterized by the blow-up condition

$$\limsup_{t \to T_{\max}^-} (\|u(t)\|_{s,\infty} + \|u_t(t)\|_{s,\infty}) = \infty.$$

Clearly $T_{\max} = \infty$, i.e. there is a global solution iff

$$\text{for any } T < \infty, \text{ we have } \limsup_{t \to T^-} (\|u(t)\|_{s,\infty} + \|u_t(t)\|_{s,\infty}) < \infty.$$

Lemma 3.9. *Suppose the conditions of theorems 3.4, 3.5 or 3.6 hold and u is the solution of the Cauchy problem (2.7)–(2.8) (equivalently (1.1)–(1.2)). Then there is a global solution if and only if*

$$\textit{for any } T < \infty, \textit{ we have } \limsup_{t \to T^-} \|u(t)\|_\infty < \infty.$$

Proof. Clearly if $\limsup_{t \to T^-} (\|u(t)\|_{s,\infty} + \|u_t(t)\|_{s,\infty}) < \infty$ then $\limsup_{t \to T^-} \|u(t)\|_\infty < \infty$. Conversely, assume the solution exists for $t \in [0, T)$ and $\|u(t)\|_\infty \leqslant M$ for all $0 \leqslant t \leqslant T$. Integrating the equation twice and calculating the resulting double integral as an iterated integral, we have

$$u(x,t) = \varphi(x) + t\psi(x) + \int_0^t (t-\tau)(K * f(u))(x,\tau)\,\mathrm{d}\tau \tag{3.2}$$

$$u_t(x,t) = \psi(x) + \int_0^t (K * f(u))(x,\tau)\,\mathrm{d}\tau. \tag{3.3}$$

Hence for all $t \in [0, T)$

$$\|u(t)\|_s \leqslant \|\varphi\|_s + T\|\psi\|_s + T\int_0^t \|(K * f(u))(\tau)\|_s\,\mathrm{d}\tau,$$

$$\|u_t(t)\|_s \leqslant \|\psi\|_s + \int_0^t \|(K * f(u))(\tau)\|_s\,\mathrm{d}\tau.$$

But $\|(K * f(u))(\tau)\|_s \leqslant C\|(f(u))(\tau)\|_s \leqslant CA(M)\|u(\tau)\|_s$ where the first inequality follows from (3.1) and the second from lemma 3.1. Then

$$\|u(t)\|_s + \|u_t(t)\|_s \leqslant \|\varphi\|_s + (T+1)\|\psi\|_s + (T+1)CA(M)\int_0^t \|u(\tau)\|_s\,\mathrm{d}\tau,$$

and Gronwall's lemma implies

$$\|u(t)\|_s + \|u_t(t)\|_s \leqslant (\|\varphi\|_s + (T+1)\|\psi\|_s)\mathrm{e}^{(T+1)CA(M)T} < \infty$$

for all $t \in [0, T)$. ■

4. Conservation of energy and global existence

In the rest of the study we will assume that

$$0 < \widehat{\beta}(\xi) \leqslant C(1+\xi^2)^{-r/2}.$$

We define the operator P as $Pv = \mathcal{F}^{-1}(|\xi|^{-1}(\widehat{\beta}(\xi))^{-1/2}\widehat{v}(\xi))$ with the inverse Fourier transform $\mathcal{F}^{-1}$. Then $P^{-2}v = -K * v$.

Lemma 4.1. *Suppose the conditions of theorems 3.4, 3.5 or 3.6 hold and the solution of the Cauchy problem (1.1)–(1.2) exists in* $C^2([0, T), H^s \cap L^\infty)$ *for some* $s \geqslant 0$. *If* $P\psi \in L^2$, *then* $Pu_t \in C^1([0, T), L^2)$. *If moreover* $P\varphi \in L^2$, *then* $Pu \in C^1([0, T), L^2)$.

Proof. From (3.3) we get

$$Pu_t(x, t) = P\psi(x) - \int_0^t (P^{-1}f(u))(x, \tau)\,\mathrm{d}\tau.$$

But for fixed t, we have $f(u) \in H^s$. Also $P^{-1}v = \mathcal{F}^{-1}(|\xi|(\widehat{\beta}(\xi))^{1/2}\widehat{v}(\xi))$ thus $P^{-1}(f(u)) \in H^{s+\frac{r}{2}-1} \subset L^2$. The second statement follows similarly from (3.2). ■

Lemma 4.2. *Suppose the conditions of theorems 3.4, 3.5 or 3.6 hold and u satisfies (1.1)–(1.2) on some interval* $[0, T)$. *If* $P\psi \in L^2$ *and the function* $G(\varphi)$ *defined by (2.4) belongs to* L^1, *then for any* $t \in [0, T)$ *the energy*

$$E(t) = \|Pu_t(t)\|^2 + \|u(t)\|^2 + 2\int_{\mathbb{R}} G(u)\,\mathrm{d}x$$

is constant in $[0, T)$.

Proof. By lemma 4.1 $Pu_t(t) \in L^2$. The equation of motion, (1.1), can be rewritten as $P^2u_{tt} + u + g(u) = 0$. Multiplying by $2u_t$, integrating in x and using Parseval's identity we get that $\mathrm{d}E/\mathrm{d}t = 0$. ■

Theorem 4.3. *Let* $s \geqslant 0$ *and* $r > 3$. *Let* $\varphi, \psi \in H^s \cap L^\infty$, $P\psi \in L^2$ *and* $G(\varphi) \in L^1$. *If there is some* $k > 0$ *so that* $G(r) \geqslant -kr^2$ *for all* $r \in \mathbb{R}$, *then the Cauchy problem (1.1)–(1.2) has a global solution in* $C^2([0, \infty), H^s \cap L^\infty)$.

Proof. Since $r > 3$, by theorem 3.5 we have local existence in $C^2([0,T), L^2)$ for some $T > 0$. The hypothesis implies that $E(0) < \infty$. Assume that u exists on $[0, T)$. Since $G(u) \geqslant -ku^2$, we get for all $t \in [0, T)$

$$\|Pu_t(t)\|^2 + \|u(t)\|^2 \leqslant E(0) + 2k\|u(t)\|^2. \tag{4.1}$$

Since $\widehat{\beta}(\xi) \leqslant C(1+\xi^2)^{-r/2}$; we have

$$\begin{aligned}
\|Pu_t(t)\|^2 &= \int_{\mathbb{R}} \xi^{-2}(\widehat{\beta}(\xi))^{-1}(\widehat{u}_t(\xi,t))^2 \,\mathrm{d}\xi \\
&\geqslant C^{-1}\int_{\mathbb{R}} \xi^{-2}(1+\xi^2)^{r/2}(\widehat{u}_t(\xi,t))^2 \,\mathrm{d}\xi \\
&\geqslant C^{-1}\int_{\mathbb{R}} (1+\xi^2)^{(r-2)/2}(\widehat{u}_t(\xi,t))^2 \,\mathrm{d}\xi \\
&\geqslant C^{-1}\|u_t(t)\|^2_{\frac{r}{2}-1}.
\end{aligned} \tag{4.2}$$

By the triangle inequality, for any Banach space valued differentiable function v we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\| \leqslant \left\|\frac{\mathrm{d}}{\mathrm{d}t}v(t)\right\|.$$

Then putting together (4.1)-(4.2)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{\frac{r}{2}-1} &= 2\|u(t)\|_{\frac{r}{2}-1}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{\frac{r}{2}-1} \\
&\leqslant 2\|u_t(t)\|_{\frac{r}{2}-1}\|u(t)\|_{\frac{r}{2}-1} \\
&\leqslant \|u_t(t)\|^2_{\frac{r}{2}-1} + \|u(t)\|^2_{\frac{r}{2}-1} \\
&\leqslant C\|Pu_t(t)\|^2 + \|u(t)\|^2_{\frac{r}{2}-1} \\
&\leqslant C(E(0) + 2k\|u(t)\|^2) + \|u(t)\|^2_{\frac{r}{2}-1} \\
&\leqslant CE(0) + (2Ck+1)\|u(t)\|^2_{\frac{r}{2}-1}.
\end{aligned}$$

Gronwall's lemma implies that $\|u(t)\|_{\frac{r}{2}-1}$ stays bounded in $[0, T)$. But since $(r/2) - 1 > \frac{1}{2}$, we conclude that $\|u(t)\|_\infty$ also stays bounded in $[0, T)$. By lemma 3.9 this implies a global solution. ■

Remark 4.4. If $g'(r)$ is bounded from below, $g'(r) \geqslant -2k$, then integrating we get $G(r) \geqslant -kr^2$. Hence the condition is satisfied when g is a polynomial of odd order with positive leading term.

When $r = 2$, we can prove global existence in a certain case; typically when the kernel is of the form $\beta(x) = \mu(|x|)$ with some smooth μ that is decreasing sufficiently fast. Then $K = \beta_{xx} = \mu''(|x|) - 2\mu'(0)\delta$ will satisfy the condition in the following theorem.

Theorem 4.5. *Let $s \geqslant 0$ and let the kernel β be such that $\beta_{xx} * v = K * v = h * v - \lambda v$ for some $\lambda > 0$ and for some $h \in L^1 \cap L^\infty$. Let $\varphi, \psi \in H^s \cap L^\infty$, $P\psi \in L^2$ and $G(\varphi) \in L^1$. If there is some $C > 0$ and $q > 1$ so that $|g(r)|^q \leqslant CG(r)$ for all $r \in \mathbb{R}$; then the Cauchy problem (1.1)–(1.2) has a global solution in $C^2([0,\infty), H^s \cap L^\infty)$.*

Proof. As K is a finite measure by theorem 3.6 we have a local solution. We follow the idea in [13]. Suppose the solution u exists for $t \in [0, T)$. For fixed $x \in \mathbb{R}$ let

$$e(t) = \tfrac{1}{2}(u_t(x,t))^2 + \lambda\left(\tfrac{1}{2}(u(x,t))^2 + G(u(x,t))\right).$$

Then

$$\begin{aligned} e'(t) &= (u_{tt} + \lambda(u + g(u)))u_t \\ &= ((\beta * (u + g(u)))_{xx} + \lambda(u + g(u)))u_t \\ &= (h * u)u_t + (h * g(u))u_t \\ &\leqslant u_t^2 + \tfrac{1}{2}(\|h * u\|_\infty^2 + \|h * g(u)\|_\infty^2). \end{aligned}$$

Since $h \in L^1 \cap L^\infty$, we have $h \in L^P$ for all $p \geqslant 1$. By Young's inequality

$$e'(t) \leqslant u_t^2 + \tfrac{1}{2}(\|h\|^2\|u\|^2 + \|h\|_{L^p}^2\|g(u)\|_{L^q}^2),$$

where $(1/p) + (1/q) = 1$. Now the last two terms may be estimated as $\|u\|^2 \leqslant E(0)$ and

$$\|g(u)\|_{L^q}^2 = \left(\int_{\mathbb{R}} |g(u)|^q \, \mathrm{d}x\right)^{\frac{2}{q}} \leqslant \left(C \int_{\mathbb{R}} G(u) \, \mathrm{d}x\right)^{\frac{2}{q}} \leqslant (CE(0))^{\frac{2}{q}}$$

so that

$$e'(t) \leqslant D + 2e(t)$$

for some constant D depending on $\|h\|_{L^p}$, $\|h\|$ and $E(0)$. This inequality holds for all $x \in \mathbb{R}$, $t \in [0, T)$. Gronwall's lemma then implies that $e(t)$ and thus $u(x, t)$ stays bounded. ∎

The condition $|g(r)|^q \leqslant CG(r)$ is satisfied in particular when $g(r) = cr^{2n+1}$ with $c > 0$ and positive integer n.

We want to conclude with some comments on the condition $P\psi \in L^2$. The estimate (4.2) shows that when $P\psi \in L^2$, we will have $\psi \in H^{\frac{r}{2}-1}$. The converse is not necessarily true since, without any further assumptions on the kernel, the factor $(\widehat{\beta}(\xi))^{-1/2}$ may be quite large. In examples (v) and (vi) of section 2.3, $P\psi \in L^2$ implies a very strong smoothness condition on ψ.

5. Blow-up in finite time

We will use the following lemma to prove blow-up in finite time.

Lemma 5.1 ([20]). *Suppose $H(t)$, $t \geqslant 0$ is a positive, twice differentiable function satisfying $H''H - (1 + \nu)(H')^2 \geqslant 0$, where $\nu > 0$. If $H(0) > 0$ and $H'(0) > 0$, then $H(t) \to \infty$ as $t \to t_1$ for some $t_1 \leqslant H(0)/\nu H'(0)$.*

We first rewrite the energy identity as

$$E(t) = \|Pu_t(t)\|^2 + 2\int_{\mathbb{R}} F(u) \, \mathrm{d}x = E(0),$$

where $F(r) = \int_0^r f(\rho) \, \mathrm{d}\rho$ with $f(u) = u + g(u)$ as before.

Theorem 5.2. *Suppose that the conditions of theorems 3.4, 3.5 or 3.6 hold, $P\varphi, P\psi \in L^2$ and $G(\varphi) \in L^1$. If there is some $\nu > 0$ such that*

$$pf(p) \leqslant 2(1 + 2\nu)F(p) \qquad \text{for all } p \in \mathbb{R},$$

and

$$E(0) = \|P\psi\|^2 + 2\int_{\mathbb{R}} F(\varphi) \, \mathrm{d}x < 0,$$

then the solution u of the Cauchy problem (2.7)–(2.8) blows up in finite time.

Proof. Assume that there is a global solution. Then $Pu(t)$, $Pu_t(t) \in L^2$ for all $t > 0$. Let $H(t) = \|Pu(t)\|^2 + b(t+t_0)^2$ for some positive b and t_0 that will be determined later. We have

$$\begin{aligned} H'(t) &= 2\langle Pu, Pu_t\rangle + 2b(t+t_0), \\ H''(t) &= 2\|Pu_t\|^2 + 2\langle Pu, Pu_{tt}\rangle + 2b. \end{aligned}$$

Note that

$$\begin{aligned} 2\langle Pu, Pu_{tt}\rangle = 2\langle u, P^2u_{tt}\rangle = -2\langle u, f(u)\rangle &= -2\int_{\mathbb{R}} uf(u)\,\mathrm{d}x \\ &\geqslant -4(1+2\nu)\int_{\mathbb{R}} F(u)\,\mathrm{d}x \\ &= 2(1+2\nu)(\|Pu_t\|^2 - E(0)), \end{aligned}$$

so that

$$H''(t) \geqslant 4(1+\nu)\|Pu_t\|^2 - 2(1+2\nu)E(0) + 2b.$$

On the other hand, we have

$$\begin{aligned} (H'(t))^2 &= 4[\langle Pu, Pu_t\rangle + b(t+t_0)]^2 \\ &\leqslant 4[\|Pu\|\|Pu_t\| + b(t+t_0)]^2 \\ &= 4[\|Pu\|^2\|Pu_t\|^2 + 2\|Pu\|\|Pu_t\|b(t+t_0) + b^2(t+t_0)^2] \\ &\leqslant 4[\|Pu\|^2\|Pu_t\|^2 + b\|Pu\|^2 + b\|Pu_t\|^2(t+t_0)^2 + b^2(t+t_0)^2]. \end{aligned}$$

Thus

$$\begin{aligned} &H''(t)H(t) - (1+\nu)(H'(t))^2 \\ &\quad\geqslant [4(1+\nu)\|Pu_t\|^2 - 2(1+2\nu)E(0) + 2b][\|Pu\|^2 + b(t+t_0)^2] \\ &\qquad -4(1+\nu)[\|Pu\|^2\|Pu_t\|^2 + b\|Pu\|^2 + b\|Pu_t\|^2(t+t_0)^2 + b^2(t+t_0)^2] \\ &\quad= [-2(1+2\nu)E(0) + 2b - 4b(1+\nu)][\|Pu\|^2 + b(t+t_0)^2] \\ &\quad= -2(1+2\nu)(b+E(0))H(t)\,. \end{aligned}$$

Now if we choose $b \leqslant -E(0)$, this gives

$$H''(t)H(t) - (1+\nu)(H'(t))^2 \geqslant 0\,.$$

Moreover

$$H'(0) = 2\langle P\varphi, P\psi\rangle + 2bt_0 \geqslant 0$$

for sufficiently large t_0. According to lemma 5.1, this implies that $H(t)$, and thus $\|Pu(t)\|^2$ blows up in finite time contradicting the assumption that the global solution exists. ■

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