

ON ISOMORPHISMS OF SPACES OF ANALYTIC FUNCTIONS OF  
SEVERAL COMPLEX VARIABLES

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Submitted to the Graduate School of Engineering and Natural Sciences  
in partial fulfillment of  
the requirements for the degree of  
Master of Science  
Sabancı University  
Spring 2007

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*To Mom*

*&*

*Dad*

## Acknowledgments

I would like to thank Prof. Dr. Vyacheslav Zakharyuta for his support and believing in my potential as a mathematician.

I would also like to thank my family for their support and influence over the years.

Many thanks to Ayça, Esen and Özcan for helping me with all sorts of details that had arisen during the writing of this thesis.

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Mathematics, Master of Science Thesis, 2007

Thesis Supervisor: Prof. Dr. Vyacheslav Zakharyuta

Keywords: Duality, Interpolation Estimates, Pluripotential Theory,  
Pluriregularity, Spaces of Analytic Functions, Stein Manifolds

## Abstract

In this thesis, we discuss results on isomorphisms of spaces of analytic functions of several complex variables in terms of pluripotential theoretic considerations. More specifically, we present the following result:

**Theorem 1** *Let  $\Omega$  be a Stein manifold of dimension  $n$ . Then,*

$$A(\Omega) \simeq A(U^n)$$

*if and only if  $\Omega$  is pluriregular and consists of at most finite number of connected components.*

The problem of isomorphic classification of spaces of analytic functions is also closely related to the problem of existence and construction of bases in such spaces. The essential tools we use in our approach are Hilbert methods and the interpolation properties of spaces of analytic functions which give us estimates of dual norms and help us to obtain extendable bases for pluriregular pairs.

# ÇOKLU KARMAŞIK DEĞİŞKENLİ ANALİTİK FONKSİYON UZAYLARINDA EŞBİÇİMLİLİK ÜZERİNE

Can Deha Karıksız

Matematik, Yüksek Lisans Tezi, 2007

Tez Danışmanları: Prof. Dr. Vyacheslav Zakharyuta

Anahtar Kelimeler: Stein Manifoldu, Analitik Fonksiyon Uzayları, Dual Uzaylar

## Özet

Bu tezde çoklu karmaşık değişkenli analitik fonksiyon uzaylarında eşbiçimliliğin karmaşık potansiyel teorisi yardımıyla karakterize edilmesi konusundaki gelişmelere değinilmekte ve  $n$  boyutlu bir Stein manifoldu üzerindeki analitik fonksiyon uzayının  $\mathbb{C}^n$ 'in birim diski üzerindeki analitik fonksiyon uzayına eşbiçimli olması için gerekli ve yeterli şartlara dair sonuca yer verilmektedir.

Analitik fonksiyon uzayları arasındaki eşbiçimlilik problemi bu uzaylarda tabanların varlığı ve bu tabanların oluşturulması problemiyle de yakın ilişki içerisindedir. Analitik fonksiyon uzaylarının interpolasyon özellikleri ve Hilbert teknikleri bu uzaylar için taban oluşturmada ve eşbiçimliliklerin belirlenmesinde önemli yardımlar sağlamaktadır.

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## CHAPTER 1

### SPACES OF ANALYTIC FUNCTIONS

#### 1.1. Stein Manifolds

A Hausdorff topological space  $\Omega$  is called a *manifold* (of dimension  $n$ ) if any point in  $\Omega$  has a neighbourhood which is homeomorphic to an open set in  $\mathbb{R}^n$ .

**Definition 1.1.1** *A manifold  $\Omega$  is called a complex analytic manifold if there is a given family  $F$  of homeomorphisms  $\kappa$  of open sets  $\Omega_\kappa \subset \Omega$  on open sets  $\tilde{\Omega}_\kappa \subset \mathbb{C}^n$  such that*

(i) *For  $\kappa, \kappa' \in F$ , the mapping*

$$\kappa' \kappa^{-1} : \kappa(\Omega_\kappa \cap \Omega_{\kappa'}) \rightarrow \kappa'(\Omega_\kappa \cap \Omega_{\kappa'})$$

*between open sets in  $\mathbb{C}^n$  is analytic,*

(ii)

$$\bigcup_{\kappa \in F} \Omega_\kappa = \Omega,$$

(iii) *For a homeomorphism  $\kappa_0$  of an open set  $\Omega_0 \subset \Omega$  on an open set in  $\mathbb{C}^n$  where the mapping*

$$\kappa \kappa_0^{-1} : \kappa_0(\Omega_0 \cap \Omega_\kappa) \rightarrow \kappa(\Omega_0 \cap \Omega_\kappa)$$

*and its inverse are analytic for every  $\kappa \in F$ , we have  $\kappa_0 \in F$ .*

If  $F$  satisfies (i) and (ii) only, then  $F$  can be extended in only one way to a family  $F'$  satisfying (i), (ii) and (iii), which is the set of all mappings satisfying (iii) relative to  $F$ . So, if we drop the last condition, then we can find different families defining the same complex analytic structure. Such a family is called a *complete set of complex analytic coordinate systems*. We also say that  $n$  complex valued functions  $(z_1, \dots, z_n)$  defined in a neighbourhood of a point  $w \in \Omega$  are a *local coordinate system at  $w$*  if they define a mapping of a neighbourhood of  $w$  into  $\mathbb{C}^n$  which is a complex analytic coordinate system.

We may define analyticity in several complex variables simply via Cauchy-Riemann conditions:

**Definition 1.1.2**  $u \in C^1(\Omega)$ , where  $\Omega$  is an open set in  $\mathbb{C}^n$ , is called analytic in  $\Omega$  if  $\bar{\partial}u = 0$  (i.e. it satisfies the Cauchy-Riemann equations).

Then, the concept of analyticity can be extended to functions on complex manifolds as follows:

**Definition 1.1.3** Let  $\Omega_1$  and  $\Omega_2$  be complex analytic manifolds. Then a mapping  $f : \Omega_1 \rightarrow \Omega_2$  is called analytic if  $\kappa_2 \circ f \circ \kappa_1^{-1}$  is analytic for all coordinate systems  $\kappa_1$  in  $\Omega_1$  and  $\kappa_2$  in  $\Omega_2$ .

Every open subset of a complex analytic manifold  $\Omega$  has a structure of a complex analytic manifold, so the concept of an analytic function on an open subset is also well defined. By the definition of a complex analytic manifold, analytic functions exist locally. Here is a class of complex manifolds where we can obtain globally defined analytic functions:

**Definition 1.1.4** An  $n$ -dimensional complex analytic manifold  $\Omega$  is called a Stein manifold if

(i)  $\Omega$  is countable at infinity, i.e. if there exists a countable number of compact subsets  $\{K_i : i \in \mathbb{N}\}$  such that every compact subset of  $\Omega$  is contained in some  $K_i$ .

(ii)  $\widehat{K} := \{z \in \Omega : |f(z)| \leq \sup_K |f| \quad \forall f \in A(\Omega)\}$  is a compact subset of  $\Omega$  for any compact subset  $K$  of  $\Omega$ .

(iii) for any different points  $z_1$  and  $z_2$  in  $\Omega$ , there exists  $f \in A(\Omega)$  such that  $f(z_1) \neq f(z_2)$ .

(iv) for any  $z \in \Omega$  there exist  $n$  functions  $f_1, \dots, f_n \in A(\Omega)$  which form a coordinate system at  $z$ .

Every domain of holomorphy in  $\mathbb{C}^n$  is a Stein manifold since an open set  $\Omega$  in  $\mathbb{C}^n$  is a domain of holomorphy if and only if  $K$  is relatively compact in  $\Omega$  implies  $\widehat{K}$  is relatively compact in  $\Omega$ . Also, any submanifold of a Stein manifold is a Stein manifold itself. For the definition of submanifold and the proof of this statement, one can look at [8].

## 1.2. Spaces of Analytic Functions

The space of all analytic functions on a complex manifold  $\Omega$ , with the topology of uniform convergence on compact subsets of  $\Omega$ , will be denoted by  $A(\Omega)$ . More precisely, the topology on  $A(\Omega)$  is the locally convex topology generated by the seminorms

$$|f|_K = \max_{z \in K} \{|f(z)|\} \quad (1.1)$$

where  $K$  is any compact set in  $\Omega$ . If  $\Omega$  is countable at infinity, e.g.  $\Omega$  is a Stein manifold, then the topology on  $A(\Omega)$  can be defined by some countable sequence of seminorms of the form (1.1). Also, completeness follows from the fact that for any sequence in  $A(\Omega)$  converging locally uniformly to a function  $f : \Omega \rightarrow \mathbb{C}$ , we have  $f \in A(\Omega)$ . Hence,  $A(\Omega)$  becomes a Fréchet space when  $\Omega$  is a Stein manifold.

For any arbitrary subset  $E$  of  $\Omega$ , we can also construct a locally convex space by using germs of analytic functions on  $E$  as follows:

Let  $\mathcal{N}(E)$  denote the collection of all open neighbourhoods of  $E$  in  $\Omega$ . We define an equivalence relation by stating that two functions  $f \in A(D_f)$  and  $g \in A(D_g)$ , where  $D_f, D_g \in \mathcal{N}(E)$ , are equivalent if there exists  $D \in \mathcal{N}(E)$  such that  $D \subset D_f \cap D_g$  and  $f(z) = g(z)$  for every  $z \in D$ . An equivalence class of this relation is called a *germ of analytic functions* (or, briefly a *germ*).

If  $E$  is a non-empty open set in  $\Omega$ , then if any two functions  $f, g \in A(E)$  are in the same germ, we have  $f \equiv g$  on  $E$ . Since  $E$  is a non-empty open set in  $\Omega$  and  $f - g \equiv 0$  on  $E$ , we should have  $f - g \equiv 0$  on  $\Omega$ . So,  $f \equiv g$  on any neighbourhood of  $E$ , which implies that any germ on  $E$  consists of a unique analytic function on  $E$ .

Let us denote by  $A(E)$ , all analytic germs on  $E$  equipped with the inductive limit topology

$$A(E) = \lim \operatorname{ind}_{D \in \mathcal{N}(E)} A(D), \quad (1.2)$$

i.e. the finest topology on  $A(E)$  for which the natural restriction mappings from  $A(D)$  to  $A(E)$ , where  $D \in \mathcal{N}(E)$ , are continuous. Then,  $A(E)$  is also a locally convex space.

If  $K$  is a compact set in  $\Omega$ , then we can represent  $A(K)$  as the countable inductive limit

$$A(K) = \lim \operatorname{ind}_{n \rightarrow \infty} A(D_n)$$

where  $D_n$  is any countable basis of  $\mathcal{N}(K)$ . Without loss of generality, we may choose  $D_n$  such that  $D_{n+1}$  is relatively compact in  $D_n$  for every  $n$  and no  $D_n$  contains a connected component disjoint from  $K$ . So, in this setting,  $x_n \rightarrow x$  in  $A(K)$  if there exists a neighbourhood  $D \in \mathcal{N}(K)$  such that  $x_n \in A(D)$  for every  $n$ ,  $x \in A(D)$  and  $(x_n)$  converges uniformly to  $x$  on any compact subset of  $D$ .

Let  $K$  be a compact set in  $\Omega$  and  $J : A(K) \rightarrow C(K)$  be the natural restriction homomorphism. We denote by  $AC(K)$ , the Banach space obtained from the completion of  $J(A(K))$  in  $C(K)$  with respect to the norm (1.1).

## CHAPTER 2

### ON DUALITY

#### 2.1. Analytic Functionals

Let  $\Omega$  be a Stein manifold. An element of the dual space  $A(\Omega)^*$  of  $A(\Omega)$ , i.e. the space of all linear continuous functionals on  $A(\Omega)$ , is called *an analytic functional*. If  $E$  is an arbitrary subset of a connected Stein manifold  $\Omega$ , then

$$j^* : A(E)^* \rightarrow A(\Omega)^* \tag{2.1}$$

which maps any  $x^* \in A(E)^*$  to its restriction on  $A(\Omega)^*$ , is linear and continuous.

If  $E$  is a Runge set in  $\Omega$ , i.e.  $A(\Omega)$  is dense in  $A(E)$ , then  $j^*$  becomes a dense imbedding. This result is obtained from the following proposition:

**Proposition 2.1.1** *Let  $X_0$  and  $X_1$  be separable locally convex spaces with a linear continuous dense imbedding  $j : X_1 \rightarrow X_0$  where  $X_1$  is reflexive. Then, the adjoint operator  $j^* : X_0^* \rightarrow X_1^*$ , where  $j^*(x^*) = x^* \circ j$  for every  $x^* \in X_0^*$ , is also a linear continuous dense imbedding.*

**Proof:** Linearity and continuity of  $j^*$  follows simply from the definition of  $j^*$ .

To show that  $j^*$  is injective, let  $j^*(x_1^*) = j^*(x_2^*)$  where  $x_1^*, x_2^* \in X_0^*$ . Then  $x_1^* \circ j = x_2^* \circ j$  and, since  $j$  is a dense imbedding, we have  $x_1^* = x_2^*$  on the dense

image set  $j(X_1) \subset X_0$ . Any continuous function is determined by its values on a dense subset, so  $x_1^* = x_2^*$  on  $X_0$ . Thus,  $j^*$  is injective.

To show that  $j^*$  is a dense mapping, we need to use the reflexivity of  $X_1$ . Since  $X_1$  is reflexive, there exists an isometric isomorphism

$$\begin{aligned} \lambda : X_1 &\rightarrow X_1^{**} \\ \lambda(x)(u) &= u(x), \quad u \in X_1^* \end{aligned}$$

and  $\lambda(X_1) = X_1^{**}$ . Now, fix  $v \in X_1^{**}$  such that  $v(y^*) = 0$  for every  $y^* \in j^*(X_0^*)$ . Since  $X_1$  is reflexive, there exists  $x \in X_1$  such that  $\lambda(x) = v$ . So, for all  $y^* \in j^*(X_0^*)$ ,  $v(y^*) = 0$  implies  $\lambda(x)(y^*) = 0$ , which implies  $y^*(x) = 0$ . Hence  $x^*(j(x)) = 0$  for every  $x^* \in X_0^*$ . Since  $X_0^*$  separates points of  $X_0$ , we have  $j(x) = 0$ , which implies that  $x = 0$  since  $j$  is injective. This implies that  $v = \lambda(x) = 0$ .

So, we have shown that for any  $v \in X_1^{**}$  such that  $v(y^*) = 0$  for all  $y^* \in L$ , we get  $v \equiv 0$ . Hence, by a corollary to Hahn-Banach theorem,  $L$  must be a dense subset of  $X_1^*$ . Thus,  $j^*$  is a dense imbedding.  $\square$

For a Runge set  $E \subset \Omega$ , let  $A'(E)$  denote the image of  $A(E)^*$  under  $j^*$  as in (2.1). Then, for Runge subsets  $E, F$  in  $\Omega$  where  $E \subset F$  and  $A(F)$  is dense in  $A(E)$ , we have the natural imbeddings  $A'(F) \subset A'(E) \subset A'(\Omega)$ .

## 2.2. GKS-Duality

The result of Grothendieck, Köthe and Silva, called GKS-duality, which we will state now, supplies us with the machinery in one-dimensional case so that for any set  $E \subset \overline{\mathbb{C}}$ , we can realize the dual space  $A(E)^*$  as the space of analytic functions  $A(E^*)$ , where  $E^* = \overline{\mathbb{C}} \setminus E$ , with the assumption that all germs of  $A(E)$  are equal to zero at the point  $\infty$  if  $\infty \in E$ .

**Theorem 2.2.1** *For any set  $E \subset \overline{\mathbb{C}}$  there exists an isomorphism*

$$\gamma : A(E)^* \rightarrow A(E^*)$$

such that

$$x^*(x) = \int_{\Gamma} x'(\zeta)x(\zeta)d\zeta, \quad x \in A(E),$$

where  $x' = \gamma(x^*)$  and  $\Gamma = \Gamma(x, x')$  is a rectifiable contour separating the singularities of the analytic germs  $x$  and  $x^*$ .

However, in several complex variables, there is no similar universal representation of  $A(E)^*$  as a space of analytic functions. Here is a simple special case that will be necessary for us:

**Proposition 2.2.2** *Let  $U_r$  be a polydisc in  $\mathbb{C}^n$  around zero with the polyradius  $r = (r_1, \dots, r_n)$  and define  $U_r^*$  as*

$$U_r^* := \{z = (z_\nu) \in \bar{\mathbb{C}}^n : |z_\nu| > r_\nu, \quad \nu = 1, \dots, n\}$$

*Then there exists a natural isomorphism  $J : A(U_r)^* \rightarrow A(U_r^*)$  such that for  $x' = J(x^*)$ , we have*

$$x^*(x) = \int_{\Gamma_\lambda} x(\zeta)x'(\zeta)d\zeta_1 \cdots d\zeta_n,$$

where

$$\Gamma_\lambda = \Gamma_\lambda(x^*) = \{\zeta = (\zeta_\nu) \in \bar{\mathbb{C}}^n : |\zeta_\nu| = \lambda r_\nu, \quad \nu = 1, \dots, n\}, \quad \lambda = \lambda(x^*) < 1$$

**Proof:** From the Cauchy integral formula, we know that any  $x \in A(U_r)$  can be represented as the integral

$$x(z) = \left(\frac{1}{2\pi i}\right)^n \int_{\Gamma} \frac{x(\zeta_1, \dots, \zeta_n)}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} d\zeta_1 \cdots d\zeta_n \quad (2.2)$$

where  $\Gamma_\lambda = \{\zeta = (\zeta_\nu) \in \mathbb{C}^n : |\zeta_\nu| = \lambda r_\nu, \nu = 1, \dots, n\}$  for  $0 < \lambda < 1$  and  $z = (z_1, \dots, z_n) \in A(U_r)$ . Let

$$u_\zeta(z) := \left(\frac{1}{2\pi i}\right)^n \frac{1}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)}$$

Then, for any  $\zeta \in \Gamma_\lambda$ , we have  $u_\zeta \in A(U_{\lambda r})$ , where

$U_{\lambda r} = \lambda U_r = \{z = (z_\nu) \in \bar{\mathbb{C}}^n : |z_\nu| < \lambda r_\nu, \quad \nu = 1, \dots, n\}$ . Also, if we denote by  $v_z(\zeta) := u_\zeta(z)$ , a function of  $\zeta$  where  $z \in U_{\lambda r}$  is fixed, then  $v_z(\zeta) \in A(U_{\lambda r}^*)$ .

Now, let  $x^* \in A(U_r)^*$ . Then there exist  $\gamma = \gamma(x^*)$  in  $(0, 1)$  and a positive constant  $C = C(x^*)$  so that  $|x^*(x)| \leq |x|_{U_{\gamma r}}$  for any  $x \in A(U_r)$ .



Let us take  $\lambda = \lambda(x^*)$  such that  $\gamma < \lambda < 1$ . Then

$$\begin{aligned} x(z) &= \left( \frac{1}{2\pi i} \right)^n \int_{\Gamma_\lambda} \frac{x(\zeta_1, \dots, \zeta_n)}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} d\zeta_1 \cdots d\zeta_n \\ &= \int_{\Gamma_\lambda} x(\zeta_1, \dots, \zeta_n) u_\zeta(z_1, \dots, z_n) d\zeta_1 \cdots d\zeta_n \end{aligned}$$

Now, if we take a sequence of partitions  $\{\Delta_j^{(m)}\}_{j=1}^m$  of  $\Gamma_\lambda$  with  $\zeta_j^{(m)} \in \Delta_j^{(m)}$  and  $\mu_j^{(m)}$  is the measure of  $\Delta_j^{(m)}$  for every  $j = 1, \dots, m$  such that  $\delta^{(m)} := \max_j \{\mu_j^{(m)}\} \rightarrow 0$  as  $m \rightarrow \infty$ , then, by the continuity of  $x(\zeta)u_\zeta(z)$ , for any fixed  $z$ , we have

$$x(z) = \lim_{m \rightarrow \infty} \sum_{j=1}^m x(\zeta_j^{(m)}) u_{\zeta_j^{(m)}}(z) \mu_j^{(m)}$$

Let  $\epsilon > 0$ , then by the additivity of integral we have

$$\int_{\Gamma_\lambda} x(\zeta) u_\zeta(z) d\zeta = \sum_{j=1}^m \int_{\Delta_j^{(m)}} x(\zeta) u_\zeta(z) d\zeta,$$

and

$$\begin{aligned} & \left| \int_{\Gamma_\lambda} x(\zeta) u_\zeta(z) d\zeta - \sum_{j=1}^m x(\zeta_j^{(m)}) u_{\zeta_j^{(m)}}(z) \mu_j^{(m)} \right| \\ &= \left| \sum_{j=1}^m \int_{\Delta_j^{(m)}} x(\zeta) u_\zeta(z) d\zeta - \sum_{j=1}^m x(\zeta_j^{(m)}) u_{\zeta_j^{(m)}}(z) \mu_j^{(m)} \right| \\ &\leq \sum_{j=1}^m \left| \int_{\Delta_j^{(m)}} \left( x(\zeta) u_\zeta(z) - x(\zeta_j^{(m)}) u_{\zeta_j^{(m)}}(z) \right) d\zeta \right| \end{aligned}$$

Since  $v_\zeta(z)$  is uniformly continuous on  $\overline{U}_{\gamma r}$ , for the given  $\epsilon$ , there exists  $\delta > 0$  such that  $|v_{\zeta^{(1)}}(z) - v_{\zeta^{(2)}}(z)| < \epsilon$  whenever  $|\zeta^{(1)} - \zeta^{(2)}| < \delta$ . So, if we choose a partition  $\Delta^{(m_0)}$  such that  $\delta^{(m_0)} < \delta$ , then we have

$$\sum_{j=1}^m \left| \int_{\Delta_j^{(m)}} \left( x(\zeta) u_\zeta(z) - x(\zeta_j^{(m)}) u_{\zeta_j^{(m)}}(z) \right) d\zeta \right| < \epsilon \sum_{j=1}^m \mu_j^{(m)}$$

whenever  $m \geq m_0$ . Hence, the partial sums converge to  $x(z)$  uniformly on  $A(\overline{U}_{\gamma r})$ .

Thus, if we apply  $x^*$  to both sides, we get

$$\begin{aligned} x^*(x) &= x^* \left( \lim_{m \rightarrow \infty} \sum_{j=1}^m x(\zeta_j^{(m)}) u_{\zeta_j^{(m)}}(z) \mu_j^{(m)} \right) \\ &= \lim_{m \rightarrow \infty} \sum_{j=1}^m x(\zeta_j^{(m)}) x^*(u_{\zeta_j^{(m)}}(z)) \mu_j^{(m)} \end{aligned}$$

By using the similar arguments as above, we can show that this limit tends to  $\int_{\Gamma_\lambda} x(\zeta)x'(\zeta)d\zeta$ , where  $x'(\zeta) = x^*(u_\zeta)$ . Thus,  $x^*(x) = \int_{\Gamma_\lambda} x(\zeta)x'(\zeta)d\zeta$ .

To show that  $x'$  is analytic on  $U_{\lambda^r}^*$ , let us look at the partial derivatives

$$\frac{\partial}{\partial \zeta_\nu} u_\zeta(z) = \lim_{h \rightarrow 0} \frac{u_{\zeta+he_\nu}(z) - u_\zeta(z)}{h}, \quad \zeta \in U_{\lambda^r}^* \quad (2.3)$$

where  $e_\nu$  are the unit vectors in the respective coordinates  $\zeta_\nu$ ,  $\nu = 1, \dots, n$ . Since we have uniform continuity on  $\bar{U}_{\gamma^r}$ ,

$$\begin{aligned} x^* \left( \lim_{h \rightarrow 0} \frac{u_{\zeta+he_\nu}(z) - u_\zeta(z)}{h} \right) &= \lim_{h \rightarrow 0} x^* \left( \frac{u_{\zeta+he_\nu} - u_\zeta}{h} \right) \\ &= \lim_{h \rightarrow 0} \frac{x^*(u_{\zeta+he_\nu}) - x^*(u_\zeta)}{h} \\ &= \frac{\partial}{\partial \zeta_\nu} (x^*(u_\zeta)) \\ &= \frac{\partial}{\partial \zeta_\nu} (x'(\zeta)) \end{aligned}$$

Hence,  $x'$  is analytic in each variable  $\zeta_\nu$ , and so  $x'$  is analytic on  $U_{\lambda^r}^*$ .  $\square$

### 2.3. Dual Form Of Cartan Theorem

Let  $M$  be a closed analytic submanifold of a Stein manifold  $\Omega$ . Then, according to Cartan theorem, the operator

$$R : A(\Omega) \rightarrow A(M) : Rx = x|_M, \quad x \in A(\Omega)$$

is a surjection. If we consider the adjoint operator  $R^* : A(M)^* \rightarrow A(\Omega)^*$  which maps any functional  $\phi \in A(M)^*$  to  $\psi = \phi \circ R \in A(\Omega)^*$ , we get a dual form of Cartan theorem:

**Proposition 2.3.1** *The adjoint operator  $R^* : A(M)^* \rightarrow A(\Omega)^*$  of the restriction operator  $R : A(\Omega) \rightarrow A(M)$  is an isomorphic imbedding.*

The proof can be found in [7].

## CHAPTER 3

### SOME TOPICS OF PLURIPOTENTIAL THEORY

#### 3.1. Maximal plurisubharmonic functions

Let  $PSH(D)$  denote the class of all plurisubharmonic functions in a domain  $D$ . For a detailed information on plurisubharmonic functions and pluripotential theory, one can refer to [9].

**Definition 3.1.1** *A function  $u \in PSH(D)$  is called maximal in  $D$  if for any relatively compact open subset  $G$  of  $D$  and any function  $v \in PSH(D)$ , upper semicontinuous on  $\bar{G}$  such that  $v \leq u$  on  $\partial G$ , we have  $v \leq u$  in  $G$ .*

Let us denote the class of all maximal plurisubharmonic functions on  $D$  by  $MPSH(D)$ . Due to Bremermann ([6]), we know that if  $u \in MPSH(D) \cap C^2(D)$ , then  $u$  is maximal if and only if

$$(dd^c u)^n = 0, \tag{3.1}$$

where  $d = \partial + \bar{\partial}$ ,  $d^c = i(\bar{\partial} - \partial)$ . The equation (3.1) is called the *homogeneous Monge-Ampère equation*.

Bedford and Taylor, in [3] and [4], had shown that the Monge-Ampère operator

$$u \rightarrow (dd^c u)^n$$

can be defined as a continuous operator from the space  $L_{loc}^\infty(D) \cap PSH(D)$  equipped with the topology induced by  $L_{loc}^\infty(D)$ , to the space  $M(D)$  of non-negative Borel measures on  $D$  equipped with the weak convergence topology, by continuously extending the corresponding differential operator acting on  $PSH(D) \cap C^2(D)$ . In light of these results, maximal plurisubharmonic functions can be characterized as follows:

**Proposition 3.1.1** *A function  $u \in PSH(D) \cap L^\infty(D)$  is maximal if and only if  $u$  satisfies the homogeneous Monge-Ampère equation,  $(dd^c u)^n = 0$ , in the generalized sense.*

### 3.2. Green Pluripotential

Let  $E$  be a subset of a Stein manifold  $\Omega$ . The following extremal function is called the *Green pluripotential* (or *MP-measure*):

$$\omega(z) = \omega(\Omega, E, z) = \limsup_{\zeta \rightarrow z} \omega^0(\Omega, E, \zeta), \quad z \in \Omega, \quad (3.2)$$

where

$$\begin{aligned} \omega^0(z) &= \omega^0(\Omega, E, z) = \sup\{u(z) : U \in P(E, \Omega)\}, \\ P(E, \Omega) &= \{u \in PSH(\Omega) : u|_E \leq 0, \quad u(z) < 1, \quad z \in \Omega\}. \end{aligned}$$

$\omega(z)$  is an analogue to the general solution of the Dirichlet problem for the class of harmonic functions in the domain  $\Omega = \Omega_1 \setminus \Omega_0 \subset \mathbb{R}^n$  with the boundary conditions  $u|_{\partial\Omega_1} \equiv 1, u|_{\partial\Omega_0} \equiv 0$ .

Here are some elementary properties of (3.2):

1.  $\omega \in PSH(\Omega)$ ,
2. if  $\Omega_1 \subset \Omega$ ,  $E_1 \subset E$ , then  $\omega(\Omega, E, z) \leq \omega(\Omega_1, E_1, z)$  for all  $z \in \Omega_1$ ,
3. if  $\Omega'$  is a connected component of  $\Omega$  and  $\Omega' \cap E = \emptyset$ , then  $\omega(\Omega, E, z) \equiv 1$  in  $\Omega'$ .

Bedford and Taylor had proven that  $\omega(z)$  satisfies the homogeneous Monge-Ampère equation on  $\Omega \setminus E$ , hence  $\omega(z)$  is maximal in  $\Omega \setminus E$ .

With the help of  $\omega(z)$ , we can state an analogue of the Two Constant Theorem for analytic functions of several complex variables:

**Proposition 3.2.2** *Let  $K$  be a compact subset of a Stein manifold  $\Omega$  and  $f$  be a bounded analytic function in  $\Omega$ . Then,*

$$|f(z)| \leq (|f|_{\Omega})^{\omega(z)} (|f|_K)^{1-\omega(z)}, \quad z \in \Omega. \quad (3.3)$$

**Proof:** Since  $f(z)$  is a bounded analytic function in  $\Omega$ ,  $\ln |f(z)|$  is plurisubharmonic in  $\Omega$ . Also, by the definition of the norms  $|f|_{\Omega}$  and  $|f|_K$ , we have  $\ln |f| \leq |f|_{\Omega}$  on  $\Omega$  and  $\ln |f| \leq |f|_K$  on  $K$ . Hence we can consider the function

$$\frac{\ln |f(z)| - \ln |f|_K}{\ln |f|_{\Omega} - \ln |f|_K}$$

which is plurisubharmonic in  $\Omega \setminus K$  and

$$\frac{\ln |f(z)| - \ln |f|_K}{\ln |f|_{\Omega} - \ln |f|_K} \leq \omega(z),$$

by the maximality of  $\omega(z)$ . Thus we have,

$$\ln |f(z)| \leq \omega(z) \ln |f|_{\Omega} + (1 - \omega(z)) \ln |f|_K$$

which implies

$$|f(z)| \leq |f|_{\Omega}^{\omega(z)} |f|_K^{(1-\omega(z))}.$$

□

### 3.3. Pluriregularity

**Definition 3.3.1** *A Stein manifold  $\Omega$  is called pluriregular (or strongly pseudoconvex) if there exists a plurisubharmonic function  $u \in PSH(\Omega)$  such that  $u(z) < 0$  for every  $z \in \Omega$  and  $u(z_j) \rightarrow 0$  for every sequence  $\{z_j\} \subset \Omega$  without limit points in  $\Omega$ .*

**Definition 3.3.2** A compact set  $K$  in a Stein manifold  $\Omega$  is called pluriregular on  $\Omega$  if  $\omega(D, K, z) = 0$  on  $K$  for some open neighbourhood  $D$  relatively compact in  $\Omega$ .  $K$  is called strongly pluriregular on  $\Omega$  if for any open neighbourhood  $D$  of  $K$ , relatively compact in  $\Omega$ , we have  $\omega(\tilde{D}, K, z) \equiv 0$  on  $K$ , where  $\tilde{D}$  is the envelope of holomorphy of  $D$ .

Pluriregularity of a compact set  $K$  in  $\Omega$  is equivalent to the pluriregularity of its envelope of holomorphy  $\widehat{K}_\Omega$ . Also,  $K$  is strongly pluriregular if and only if for every pseudoconvex covering domain  $\Omega_1$  over  $\Omega$  such that  $\Omega_1 \supset K$ , the envelope of holomorphy  $\widehat{K}_{\Omega_1}$  is pluriregular on  $\Omega_1$ . For the definition of envelope of holomorphy and some of its properties, one can look at [8], chapter 5.4.

**Definition 3.3.3** A pair  $(K, \Omega)$  is called pluriregular if  $K$  is a pluriregular holomorphically convex compact set on the pluriregular Stein manifold  $\Omega$  and every connected component of  $\Omega$  has a nonempty intersection with  $K$ .

Given a pluriregular pair  $(K, \Omega)$ , the Green pluripotential  $\omega(z)$  has the following useful properties:

**Theorem 3.3.1** If  $(K, \Omega)$  is a pluriregular pair, then  $\omega(z) = \omega(\Omega, K, z)$  is continuous in  $\Omega$  and satisfies

- (i)  $\omega(z) = 0, \quad z \in K,$
- (ii)  $0 < \omega(z) < 1, \quad z \in \Omega \setminus K,$
- (iii)  $\lim_{z \rightarrow \partial\Omega} \omega(z) = 1.$

The proof can be found in [16].

A set  $\Omega \in \mathbb{C}^n$  is called *circled* if for every  $a = (a_1, \dots, a_n) \in \Omega$ , the torus

$$\{z \in \mathbb{C}^n : z = (a_1 e^{i\theta_1}, \dots, a_n e^{i\theta_n}, \quad 0 \leq \theta_j \leq 2\pi\}$$

lies in  $\Omega$ . An open circled set in  $\mathbb{C}^n$  is called a *circular (Reinhardt) domain*. A circular domain  $\Omega$  is called *complete* if for every  $a \in \Omega$ , we have

$$\{z \in \mathbb{C}^n : |z_j| < |a_j|, \quad j = 1, \dots, n\} \subset \Omega.$$

Let  $D$  be a complete circular domain in  $\mathbb{C}^n$  and let us define the function

$$\psi_D(z) := \ln \inf\{t > 0 : \frac{z}{t} \in D\}, \quad z \in \mathbb{C}^n \quad (3.4)$$

$\psi_D(z)$  is upper semicontinuous in  $\mathbb{C}^n$  and it is known that it characterizes the pseudoconvexity of  $D$ . So, a complete circular domain  $D$  is pluriregular if and only if  $\psi_D(z)$  is plurisubharmonic in  $\mathbb{C}^n$  and continuous in  $\mathbb{C}^n \setminus \{0\}$ .

## CHAPTER 4

### INTERPOLATION PROPERTIES OF SPACES OF ANALYTIC FUNCTIONS

#### 4.1. Interpolation Estimates of the Norms of Analytic Functionals

In this section, we consider the two constant theorem in the case of analytic functionals. In the implicit form, it was considered in [16] as a result about Hilbert scales of analytic functions. Here, we give a proof based on the dual version of Cartan Theorem (theorem 2.3.1), without using Hilbert space techniques.

**Theorem 4.1.1** *Let  $(K, D)$  be a pluriregular pair on a Stein Manifold  $\Omega$ , where  $D$  is a strongly pluriregular open set on  $\Omega$ . Then, for any  $\epsilon > 0$  and  $\alpha \in (0, 1)$ , there exists a constant  $C = C(\alpha, \epsilon)$  such that for any  $x^* \in AC(K)^*$  the following estimate holds*

$$|x^*|_{D_\alpha}^* \leq C(|x^*|_K^*)^{1-\alpha+\epsilon}(|x^*|_D^*)^{\alpha-\epsilon},$$

where

$$D_\alpha = \{z \in D : \omega(D, K, z) < \alpha\}, \quad 0 < \alpha < 1 \quad (4.1)$$

Let us first consider a special case when  $D$  and  $K$  are analytic polyhedrons in  $\Omega$  defined by the same collection of analytic functions. We have a one-parameter family of polydiscs in  $\mathbb{C}^N$ :

$$V_\alpha = \{z = (z_\nu) \in \mathbb{C}^N : |z_\nu| < r_\nu(\alpha), \quad \nu = 1, \dots, N\}$$



where

$$r_\nu(\alpha) = R_\nu^\alpha r_\nu^{1-\alpha}, \quad r_\nu < R_\nu, \quad \nu = 1, \dots, N, \quad \alpha \in \mathbb{R}$$

For  $\alpha \in [0, 1]$ , let  $\Delta_\alpha := V_\alpha \cap \Omega$  and  $\tilde{\Delta}_\alpha := \bar{V}_\alpha \cap \Omega$  be two families of analytic polyhedrons in  $\Omega$  and let  $\Phi_\alpha$  and  $\tilde{\Phi}_\alpha$  be polyhedrons that are the connected components of  $\Delta_\alpha$  and  $\tilde{\Delta}_\alpha$  respectively, which have nonempty intersections with  $\tilde{\Phi}_0 = \tilde{\Delta}_0$ . Then, we have the following:

**Proposition 4.1.2** *For every  $\epsilon > 0$  and  $\alpha \in (0, 1)$ , there exists a constant  $C = C(\alpha, \epsilon)$  such that*

$$|x^*|_{\tilde{\Phi}_\alpha}^* \leq C \left( |x^*|_{\tilde{\Phi}_0}^* \right)^{1-\alpha+\epsilon} \left( |x^*|_{\tilde{\Phi}_1}^* \right)^{\alpha-\epsilon}, \quad x^* \in AC(\tilde{\Phi}_0)^* \quad (4.2)$$

**Proof:** We know that all polyhedrons  $\Phi_\alpha$  are closed submanifolds of polyhedrons  $V_\alpha$ , and so are  $\tilde{\Phi}_\alpha$ , since  $\tilde{\Phi}_\alpha = \bigcap_{\beta>\alpha} \Phi_\beta$ . So, for  $\alpha \in (0, 1)$ , we have

$$\tilde{\Phi}_0 \subset \Phi_\alpha \subset \tilde{\Phi}_\alpha \subset \Phi_1,$$

which implies that

$$A(\tilde{\Phi}_0) \supset A(\Phi_\alpha) \supset A(\tilde{\Phi}_\alpha) \supset A(\Phi_1).$$

Hence, taking the duals, we have the inclusions

$$A(\tilde{\Phi}_0)^* \subset A(\Phi_\alpha)^* \subset A(\tilde{\Phi}_\alpha)^* \subset A(\Phi_1)^*.$$

Thus, we get the natural imbeddings

$$A(\tilde{\Phi}_0)^* \rightarrow A(\Phi_\alpha)^* \rightarrow A(\tilde{\Phi}_\alpha)^* \rightarrow A(\Phi_1)^*.$$

Similarly, we obtain the natural imbeddings

$$A(\bar{V}_0)^* \rightarrow A(V_\alpha)^* \rightarrow A(\bar{V}_\alpha)^* \rightarrow A(V_1)^*.$$

Also, if we consider the polydiscs

$$V_\alpha^* = \{z = (z_\nu) \in \mathbb{C}^N : |z_\nu| > r_\nu(\alpha), \quad \nu = 1, \dots, N\},$$

then we have the inclusions  $\bar{V}_0^* \supset V_\alpha^* \supset \bar{V}_\alpha^* \supset V_1^*$ . Hence, we have the natural imbeddings

$$A(\bar{V}_0^*) \rightarrow A(V_\alpha^*) \rightarrow A(\bar{V}_\alpha^*) \rightarrow A(V_1^*).$$

Now, according to the dual form of Cartan theorem (theorem 2.3.1), for any  $\alpha \in [0, 1]$ , we get the isomorphic imbeddings  $T_\alpha : A(\Phi_\alpha)^* \rightarrow A(V_\alpha)^*$  and  $\tilde{T}_\alpha : A(\tilde{\Phi}_\alpha)^* \rightarrow A(\overline{V}_\alpha)^*$ . Also, by theorem 2.2.2, we have the isomorphisms  $S_\alpha : A(V_\alpha)^* \rightarrow A(V_\alpha^*)$  and  $\tilde{S}_\alpha : A(\overline{V}_\alpha)^* \rightarrow A(\overline{V}_\alpha^*)$ . Hence, we obtain the following diagram:

$$\begin{array}{ccccc}
A(\Phi_1)^* & \rightarrow & A(V_1)^* & \rightarrow & A(V_1^*) \\
\uparrow & & \uparrow & & \uparrow \\
A(\tilde{\Phi}_\alpha)^* & \rightarrow & A(\overline{V}_\alpha)^* & \rightarrow & A(\overline{V}_\alpha^*) \\
\uparrow & & \uparrow & & \uparrow \\
A(\Phi_\alpha)^* & \rightarrow & A(V_\alpha)^* & \rightarrow & A(V_\alpha^*) \\
\uparrow & & \uparrow & & \uparrow \\
A(\tilde{\Phi}_0)^* & \rightarrow & A(\overline{V}_0)^* & \rightarrow & A(\overline{V}_0^*)
\end{array}$$

So, if we apply the two constant theorem (theorem 3.2.2) for any given  $x \in A(V_\alpha^*)$ ,  $0 < \alpha < 1$ , then we have

$$|x|_{V_\alpha^*} \leq (|x|_{V_0^*})^{1-\alpha} (|x|_{V_1^*})^\alpha.$$

Furthermore, since we have the continuous imbeddings as shown in the diagram, given  $\epsilon > 0$  and  $0 < \alpha < 1$ , there exists a constant  $C = C(\alpha, \epsilon)$  such that

$$|x^*|_{\tilde{\Phi}_\alpha}^* \leq C(\alpha, \epsilon) |x|_{V_{\alpha-\epsilon}^*}.$$

So, combining these two inequalities, for any  $x^* \in A(\tilde{\Phi}_\alpha^*)$ , we get

$$|x^*|_{\tilde{\Phi}_\alpha}^* \leq C \left( |x^*|_{\tilde{\Phi}_0}^* \right)^{1-\alpha+\epsilon} \left( |x^*|_{\tilde{\Phi}_1}^* \right)^{\alpha-\epsilon}.$$

□

The main case can be obtained with the help of Lelong-Bremermann theorem about the uniform approximation of an arbitrary continuous plurisubharmonic function, which can be found in [5], by constructing the following plurisubharmonic function:

$$v(z) = \max\{\alpha_j \ln |f_j(z)| : j = 1, \dots, n\},$$

where  $\alpha_j > 0$  and  $f_j \in A(\Omega)$ , and considering the fact that level domains of  $v(z)$  are analytic polyhedrons that simultaneously approximate the corresponding level domains  $D_\alpha$ . Plurisubharmonicity of  $v(z)$  is shown in [16].

## 4.2. Hilbert Scales

Hilbert scales are a special class of analytic scales, so let us start by introducing the notion of analytic scale of spaces, as done in [10]. Let  $M$  be a normed space in which a family of linear operators  $T(z)$  acts satisfying the following conditions:

- (i)  $T(z)x$  is an entire function of the complex variable  $z$ .
- (ii)  $\|T(z)x\|_M$  is a bounded function on every straight line parallel to the imaginary axis.
- (iii)  $T(0)x = x$ .
- (iv)  $\sup_{\mu, \nu} \|T(\alpha + i\mu)T(\beta + i\nu)x\| \leq \sup_{\tau} \|T(\alpha + \beta + i\tau)x\|_M$ .
- (v)  $T(i\mu) \frac{T(z+\Delta z)x - T(z)x}{\Delta z} \rightarrow T(i\mu) (T(z)x)'$  uniformly in  $\mu$  as  $\Delta z \rightarrow 0$ .

The family  $E_\alpha$ ,  $-\infty < \alpha < \infty$ , of Banach spaces where  $E_\alpha$  are the completions of  $M$  with respect to the respective norms

$$\|x\|_\alpha = \sup_{-\infty < \tau < \infty} \|T(\alpha + i\tau)x\|_M$$

is called an *analytic scale of spaces*.

Now, let  $H_0$  be a complex Hilbert space and let  $j$  be an unbounded positive definite selfadjoint operator in  $H_0$  with a domain  $D_j$  such that

$$\|x\|_{H_0} \leq \|jx\|_{H_0} \quad x \in D_j.$$

Let us denote by  $E_\lambda$  the spectral resolution of the identity corresponding to  $j$ , and consider the set  $M$  of all elements representable in the form  $x = \int_1^N \lambda dE_\lambda x$  for some  $N < \infty$ . This set  $M$  is dense in  $H_0$  and on this set we can define the operators  $T(z) = j^z$  where

$$j^z x = \int_1^N \lambda^z dE_\lambda x.$$

This family of operators satisfies the conditions necessary to construct an analytic scale from them. So, on  $M$ , we introduce the norms

$$\|x\|_\alpha = \sup_{-\infty < \tau < \infty} \|j^{\alpha+i\tau} x\|_{H_0} = \|j^\alpha x\|_{H_0}.$$

The completions  $H_\alpha$  of  $M$  with respect to these norms are Hilbert Spaces and they form an analytic scale which is called a *Hilbert scale*.

Now let  $H^\alpha = (H_0)^{1-\alpha}(H_1)^\alpha$ ,  $\alpha \in \mathbb{R}$ , be a Hilbert scale generated by a pair  $H_1 \subset H_0$  of Hilbert spaces with continuous imbedding. If this imbedding is compact, i.e. every set bounded in the norm of  $H_1$  is relatively compact in  $H_0$ , then we have the following proposition:

**Proposition 4.2.1** *Let  $H_1 \hookrightarrow H_0$  be a linear dense compact imbedding, then there exists a common orthogonal basis  $\{e_k\}$  in every space  $H^\alpha$  such that  $\|e_k\|_{H_0} = 1$  and  $\mu_k := \|e_k\|_{H_1} \nearrow \infty$ .*

**Proof:** Given the pair of Hilbert spaces  $H_0$  and  $H_1$ , let us consider the restriction operator  $R : H_1 \rightarrow H_0$ , which is a linear dense compact imbedding. Then the adjoint operator  $R^* : H_0 \rightarrow H_1$  can be defined as

$$\langle Rx, y \rangle_{H_0} = \langle x, R^*y \rangle_{H_1} \quad x \in H_1, \quad y \in H_0.$$

Now, let  $S := R^*R$ , then  $S$  is a self adjoint operator and given  $x, y \in H_1$ , we have

$$\begin{aligned} \langle x, y \rangle_{H_0} &= \langle Rx, Ry \rangle_{H_0} && \text{by the definition of } R, \\ &= \langle x, R^*Ry \rangle_{H_1} \\ &= \langle x, Sy \rangle_{H_1} && \text{by the definition of } S, \\ &= \langle Sx, y \rangle_{H_1} && \text{since } S \text{ is selfadjoint.} \end{aligned}$$

From the above equality we get that  $\langle Sx, x \rangle_{H_1} = 0$  if and only if  $x = 0$ . Hence,  $S$  is strictly positively defined. Also,  $S$  is compact since it is the composition of a continuous and a compact operator. Therefore, there exists a complete orthonormal sequence of eigenvectors  $\{g_k\}$  with respective strictly positive eigenvalues  $\lambda_k$ , where  $\lambda_k \rightarrow 0$ . Without loss of generality we can take  $\lambda_k \searrow 0$ . Then, for any  $m, n$  we have

$$\langle g_m, g_n \rangle_{H_0} = \langle Sg_m, g_n \rangle_{H_1} = \langle \lambda_m g_m, g_n \rangle_{H_1} = \lambda_m \langle g_m, g_n \rangle_{H_1} = \lambda_m \delta_{m,n}$$

Hence, we have  $\|g_k\|_{H_0} = \sqrt{\lambda_k}$  and  $\|g_k\|_{H_1} = 1$  where  $\{g_k\}$  is a common orthogonal basis in  $H_0$  and  $H_1$ . Setting  $e_k = \frac{1}{\lambda_k} g_k$ , we get the common orthogonal basis with the properties  $\|e_k\|_{H_0} = 1$  and  $\mu_k := \|e_k\|_{H_1} \nearrow \infty$ , where  $\mu_k = \frac{1}{\sqrt{\lambda_k}}$ .

Using this basis, the Hilbert scale is determined by the norms:

$$\|x\|_{H_\alpha} = \left( \sum_{k=1}^{\infty} |\xi_k|^2 \mu_k^{2\alpha} \right)^{\frac{1}{2}}, \quad x = \sum_{k=1}^{\infty} \xi_k e_k.$$

□

**Theorem 4.2.2** *Let  $D$  be a strongly pluriregular open set on the Stein manifold  $\Omega$  and a compact set  $K \subset D$  be pluriregular on  $D$ . Given a pair of Hilbert spaces  $H_0, H_1$  with the continuous imbeddings*

$$A(K) \hookrightarrow H_0 \hookrightarrow AC(K),$$

$$A(\bar{D}) \hookrightarrow H_1 \hookrightarrow A(D),$$

*the following continuous imbeddings hold:*

$$A(K_\alpha) \subset H^\alpha \subset A(D_\alpha), \quad 0 < \alpha < 1.$$

**Sketch of proof:** Using the common orthogonal basis from the previous proposition,  $\{e_k\} \subset H_1 \subset A(D)$ , and we obtain the estimate

$$|e_k|_{D_\alpha} \leq C(\alpha, \epsilon) \mu_k^{\alpha+\epsilon}$$

by using the projective limit topology properties of the spaces considered and applying the two constant theorem for analytic functions. Also, applying the interpolational estimates as in theorem 4.1.1, we get

$$|e_k^*|_{D_\alpha}^* \leq C(\alpha, \epsilon) \mu_k^{-\alpha+\epsilon}.$$

Furthermore, we use the fact that the imbedding  $H_1 \subset H_0$  is nuclear and so for any  $\delta > 0$  we get

$$\sum_{k=1}^{\infty} \mu_k^{-\delta} < \infty$$

Thus we can obtain the norm estimates which provides us with the continuous imbeddings  $A(K_\alpha) \subset H^\alpha \subset A(D_\alpha)$ .

The following generalization of the previous theorem can be stated by means of Vogt's idea, which will be presented in the next section. This statement was also proved in [16] under the assumption of existence of a basis in  $A(D)$ :

**Theorem 4.2.3** *Let  $(K, \Omega)$  be a pluriregular pair. Then there exists a Hilbert space  $H_1 \subset A(\Omega)$  such that for any Hilbert space  $H_0$  satisfying*

$$A(K) \hookrightarrow H_0 \hookrightarrow AC(K),$$

*the following continuous imbeddings hold:*

$$A(K_\alpha) \subset H^\alpha \subset A(D_\alpha), \quad 0 < \alpha < 1.$$

The proof can be found in [17].

The conditions on  $H_0$  can also be weakened by considering the  $H$ -space  $AL^2(E, \mu)$ , which is the completion of the set  $A(E)$  in the space  $L^2(E, \mu)$  with respect to the norm

$$\|x\| = \left( \int_E |x(z)|^2 d\mu \right)^{1/2}$$

where  $\mu$  is some Borel measure on an unrestricted Borel set  $E \subset \Omega$ . Let us state this fact formally in the next theorem:

**Theorem 4.2.4** *Let  $E$  be a relatively compact Borel set on a pluriregular Stein manifold  $\Omega$  not pluripolar in any connected component of  $\Omega$ , and  $\mu$  be a Borel measure supported on  $E$  satisfying*

$$\int_A (dd^c \omega(\Omega, K, z))^n \leq \left( \ln \frac{1}{\mu A} \right)^{-n(1+\delta)}$$

*for any Borel set  $A \subset E$  with a sufficiently small measure  $\mu A \leq \epsilon(\delta)$ . Then there exists a Hilbert space  $H_1 \subset A(\Omega)$  such that for any Hilbert space  $H_0$  satisfying the continuous imbeddings*

$$A(\overline{E}) \subset H_0 \subset AL^2(E, \mu)$$

*the following continuous imbeddings hold:*

$$A(E_\alpha) \subset H^\alpha \subset A(\Omega_\alpha), \quad 0 < \alpha < 1,$$

where

$$\Omega_\alpha = \{z \in \Omega : \omega(\Omega, E, z) < \alpha\},$$

$$E_\alpha = \{z \in \Omega : \omega^0(\Omega, \overline{E}, z) \leq \alpha\}.$$

### 4.3. Dragilev Classes

Let  $X$  be an  $F$ -space with topology defined by the system of norms

$$\{\|x\|_p, p \in \mathbb{N}\}.$$

Then, in the strong dual space  $X^*$ , we have the system of dual unbounded norms

$$\|x^*\|_p^* := \sup\{|x^*(x)| : x \in U_p\}, \quad x^* \in X^*, \quad p \in \mathbb{N},$$

where

$$U_p := \{x \in X : \|x\|_p \leq 1\}.$$

Also, let us introduce another notation

$$U_p^0 := \{x^* \in X^* : \|x^*\|_p^* \leq 1\}, \quad p \in \mathbb{N}.$$

**Definition 4.3.1** *A Fréchet space  $X$  belongs*

(i) *to the class  $\mathcal{D}_1$  if*

$$\exists p \forall q \exists r \exists C \mid \|x\|_q^2 \leq C\|x\|_p\|x\|_r, \quad x \in X,$$

(ii) *to the class  $\mathcal{D}_2$  if*

$$\forall p \exists q \forall r \exists C \mid (\|x^*\|_q^*)^2 \leq C\|x^*\|_p^*\|x^*\|_r^*, \quad x^* \in X^*.$$

The inequalities in conditions (i) and (ii) are equivalent to the following relations respectively:

$$\begin{aligned} U_q^0 &\subset tU_p^0 + \frac{C}{t}U_r, \quad t > 0, \\ U_q &\subset tU_r + \frac{C}{t}U_p, \quad t > 0. \end{aligned}$$

With the help of these relations, the next theorem was proved by Vogt in [13], which was also used in the generalization of existence of a Hilbert space  $H_1$  as in the previous section.

**Theorem 4.3.1** *Let  $X$  be a Fréchet-Schwarz space. Then,  $X \in \mathcal{D}_2$  if and only if there exists a bounded closed absolutely convex set  $B \subset X$  such that*

$$\forall p \forall \mu : 0 < \mu < 1 \exists q \exists c \mid U_q \subset t^\mu B + \frac{C}{t^{1-\mu}}U_p, \quad t > 0. \quad (4.3)$$

The statement was also proved in [16] under the assumption of existence of an unconditional basis in a Hilbert space (not necessarily Schwarz).

The inclusion in (4.3) can also be written in an equivalent form with the same quantifiers as

$$\|x^*\|_q^* \leq C (\|x^*\|)^{1-\mu} (\|x^*\|_p^*)^{1-\mu}, \quad x^* \in X^*, \quad (4.4)$$

where

$$\|x^*\|^* := \sup\{|x^*(x)| : x \in B\}, \quad x^* \in X^*.$$

**Definition 4.3.2** *A Banach space  $E$  continuously imbedded in  $X$  is called a Vogt space, denoted by  $E \in \mathcal{V}(X)$ , if (4.4) holds for the norm  $\|x^*\|^*$  defined above.*

Now, we can relate the Dragilev classes to the spaces of analytic functions in terms of pluripotential theoretic considerations.

**Theorem 4.3.2** *Let  $\Omega$  be a Stein manifold. Then,  $A(\Omega) \in \mathcal{D}_2$  if and only if  $\Omega$  is pluriregular.*

**Sketch of proof:** For sufficiency, given  $p$ ,  $\|x\|_p$  is defined on a compact set, say  $K$ , and  $(K, \Omega)$  becomes a pluriregular pair. So, given a strongly pluriregular open set  $D$  in  $\Omega$  and  $\mu$  such that  $0 < \mu < 1$ , by theorem 4.1.1, we have the interpolational estimate where for every  $\epsilon > 0$  there exists  $C$  such that

$$\|x^*\|_{D_\mu}^* \leq C (\|x^*\|_K^*)^{1-\mu+\epsilon} (\|x^*\|_D^*)^{\mu-\epsilon}.$$

So, we can apply theorem 4.3.1 to obtain that  $A(\Omega) \in \mathcal{D}_2$ .

In order to prove necessity, without loss of generality, we may assume that  $\Omega$  is connected. We can also choose a suitable compact set  $K \subset \Omega$ , a Hilbert space  $H_0$  such that  $A(K) \hookrightarrow H_0 \hookrightarrow AC(K)$ , and a space  $H_1 \in \mathcal{V}(A(\Omega))$  such that there is a common orthogonal basis  $\{e_k\}$  for  $H_0$  and  $H_1$  satisfying  $\|e_k\|_{H_0} = 1$  and  $\mu_k = \|e_k\|_{H_1} \nearrow \infty$ . Then the function

$$u(z) = \overline{\lim}_{\zeta \rightarrow z} \overline{\lim}_{k \rightarrow \infty} \left( \frac{\ln |e_k(\zeta)|}{\ln \mu_k} - 1 \right)$$

is a negative plurisubharmonic function such that  $\lim_{z \rightarrow \partial\Omega} u(z) = 0$ . This fact was proved in [16]. Hence,  $\Omega$  is pluriregular.

Similarly, one can state the following theorem:

**Theorem 4.3.3** *Let  $K$  be a compact set on a Stein manifold  $\Omega$ . Then  $A(K)^* \in \mathcal{D}_2$  if and only if  $K$  is strongly pluriregular on  $\Omega$ .*



## CHAPTER 5

### ISOMORPHISMS OF SPACES OF ANALYTIC FUNCTIONS

#### 5.1. Extendable Basis For a Pluriregular Pair

**Theorem 5.1.1** *Let  $(K, D)$  be a pluriregular pair. Then, there exists a common basis  $\{x_i(z)\}$  in the spaces  $A(D)$ ,  $A(K)$ ,  $A(D_\alpha)$ ,  $A(K_\alpha)$ ,  $0 < \alpha < 1$ , satisfying the asymptotic estimate*

$$\limsup_{\zeta \rightarrow z} \limsup_{i \rightarrow \infty} \frac{\ln |x_i(z)|}{a_i} = \omega(D, K, z), \quad z \in D \setminus K \quad (5.1)$$

where

$$K_\alpha = \{z \in D : \omega(D, K, z) \leq \alpha\}, \quad D_\alpha = \{z \in D : \omega(D, K, z) < \alpha\}$$

and  $\{a_i\}$  is some nondecreasing sequence of positive numbers such that

$$a_i \asymp i^{\frac{1}{\dim D}}, \quad i \rightarrow \infty.$$

Let us give a sketch of the proof. We can take a common orthogonal basis  $\{x_i(z)\}$  for some pair of Hilbert spaces  $H_0, H_1$  with the continuous imbeddings

$$H_1 \hookrightarrow A(D) \hookrightarrow A(K) \hookrightarrow H_0.$$

According to a method proposed in [14], we can choose  $H_0$  and  $H_1$  so that  $H_1 \in \mathcal{V}(A(D))$  and  $H_0^* \in \mathcal{V}(A(K)^*)$ . By the proposition 2.1.1, the dual space  $H_0^*$  is naturally imbedded in  $A(K)^*$ . The existence of such spaces follows from the theorems in section . The system  $\{x_i(z)\}$  is normed and ordered by theorem 4.2.1, so if we take  $a_i = \ln \mu_i(H_0, H_1)$  then the conclusion follows from the inequalities

$$|x_i|_{D_\alpha} \leq C(\alpha, \epsilon) \mu_i^{\alpha+\epsilon}, \quad |x_i^*|_{D_\alpha^*} \leq C(\alpha, \epsilon) \mu_i^{-\alpha+\epsilon}, \quad \sum_{k=1}^{\infty} \mu_k^{-\delta} < \infty.$$

How to obtain the asymptotic equality is explained in [16].

There exists simpler sufficient conditions for  $H_1 \in \mathcal{V}(A(D))$  and  $H_0^* \in \mathcal{V}(A(K)^*)$ . For example, it is sufficient for the space  $H_0$  to satisfy the continuous imbeddings

$$A(K) \hookrightarrow H_0 \hookrightarrow AL^2(K, \mu)$$

where  $\mu$  is a Borel measure satisfying the conditions of Theorem 4.2.4. Also, if a manifold  $D$  is strongly pluriregular on the Stein manifold  $\Omega$ , then the continuity of the imbeddings

$$A(\overline{D}) \hookrightarrow H_1 \hookrightarrow A(D)$$

imply  $H_1 \in \mathcal{V}(A(D))$ . However, in the case of arbitrary pluriregular Stein manifolds, we have only the fact of existence of Vogt spaces.

As an application of the previous theorem, we can state the following proposition, which was considered in [14]:

**Proposition 5.1.2** *Let  $D_0, D$  be pluriregular circular (Reinhardt) domains where  $\overline{D}_0 \subset D$ . Then, there exists a common basis in  $A(D)$  and  $A(\overline{D}_0)$  consisting of homogeneous polynomials.*

## 5.2. Isomorphisms of Pluriregular Domains and Compact Sets

By means of extendable bases considered in the previous section, we obtain the following result:

**Theorem 5.2.1** *Let  $\Omega$  be a Stein manifold of dimension  $n$ . Then,*

$$A(\Omega) \simeq A(U^n)$$

*if and only if  $\Omega$  is pluriregular and consists of at most finite number of connected components.*

**Sketch of Proof:** Let  $\Omega$  be a Stein manifold of dimension  $n$  satisfying  $A(\Omega) \simeq A(U^n)$ . Then  $A(\Omega) \in \mathcal{D}^2$  since it is isomorphic to  $A(U^n)$ . So, the necessity follows from the fact that  $A(\Omega) \in \mathcal{D}^2$  if and only if  $\Omega$  is pluriregular.

Conversely, let  $\Omega$  be pluriregular, consisting of at most finite number of connected components and let  $K$  be a pluriregular compact set having a non-empty intersection with every connected component of  $\Omega$  with the property  $K = \hat{K}_\Omega$ . Given a common basis  $\{x_i(z)\}$ , which exists due to theorem 5.1.1, we have the isomorphism  $T : A(\Omega) \rightarrow A(U^n)$  defined by

$$x_i(z) \rightarrow (\exp a_i) e_i(z), \quad i \in \mathbb{N},$$

where  $\{e_i\}$  is a common orthogonal basis such that

$$\|e_i\|_{H_0} = 1, \quad \mu_i := \|e_i\|_{H_1} \nearrow \infty.$$

It should be pointed out here that the result of Vogt which provides us a suitable Hilbert space  $H_1$  was used.

Under some additional assumptions on  $\Omega$ , such as strong pluriregularity or existence of a basis in  $A(\Omega)$ , the sufficiency was also proved in [16].

Similarly, we can obtain the following result:

**Theorem 5.2.2** *Let  $K$  be a compact set on a Stein manifold  $\Omega$ . Then,*

$$A(K) \simeq A(U^n)$$

*if and only if  $K$  has a Runge neighbourhood in  $\Omega$  and  $K$  is strongly pluriregular.*

Necessity follows from the fact that  $A(K)^* \in \mathcal{D}^2$  if and only if  $K$  is strongly pluriregular on  $\Omega$ . For sufficiency, the isomorphism can be obtained by using the basis from Theorem 5.1.1 for the pluriregular pair  $(K, \tilde{D})$ , where  $D$  is some Runge neighbourhood of  $K$  such that its envelope of holomorphy  $\tilde{D}$  is pluriregular.

Here is a corollary of Theorem 5.2.1:

**Proposition 5.2.3** *Let  $\Omega$  be a complete pseudoconvex circular domain in  $\mathbb{C}^n$  and*

$$\psi_{\Omega}(z) := \ln \inf \left\{ t > 0 : \frac{z}{t} \in \Omega \right\}, \quad z \in \mathbb{C}^n.$$

*Then,  $A(\Omega) \simeq A(U^n)$  if and only if  $\psi_{\Omega}(z)$  is continuous in  $\mathbb{C}^n$ .*

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