On Algorithmic Solutions to Simple Allocation Problems

Özgür Kıbrıs†
Sabancı University
August 23, 2008

Abstract

We interpret solution rules to a class of simple allocation problems as data on the choices of a policy-maker. We study the properties of rational rules. We show that every rational rule falls into a class of algorithmic rules that we describe. The Equal Gains rule is a member of this class and it uniquely satisfies rationality, continuity, and equal treatment of equals. Its dual, the Equal Losses rule, uniquely satisfies continuity, equal treatment of equals, and two properties that constitute the dual of rationality: translation down and translation up.

JEL Classification numbers: D11, D81

Keywords: Rational, contraction independence, continuity, equal treatment of equals, translation, duality.

*Part of this paper was written while I was visiting the University of Rochester. I would like to thank this institution for its hospitality. I would also like to thank Walter Bossert, Tarık Kara, Yves Sprumont, İpek Gürsel Tapkı, William Thomson, Rakesh Vohra, and seminar participants at the University of Montreal and the University of Rochester for comments and suggestions. Finally, I gratefully acknowledge the research support of the Turkish Academy of Sciences via a TUBA-GEBIP fellowship.

†Faculty of Arts and Social Sciences, Sabancı University, 34956, Istanbul, Turkey. E-mail: ozgur@sabanciuniv.edu Tel: +90-216-483-9267 Fax: +90-216-483-9250
1 Introduction

Revealed preference theory studies conditions under which by observing the choice behavior of an agent, one can discover the underlying preferences that govern it. Choice rules for which this is possible are called rational. Most of the earlier work on rationality analyzes consumers’ demand choices from budget sets (e.g. see Samuelson, 1938, 1948). The underlying premise that choices reveal information about preferences, however, is applicable to a wide range of choice situations. For example, applications of the theory to bargaining games (Nash, 1950) characterize bargaining rules which can be “rationalized” as maximizing the underlying preferences of an impartial arbitrator (Peters and Wakker, 1991; Bossert, 1994; Ok and Zhou, 1999; Sánchez, 2000).

In this paper, we study the implications of rationality on a class of solutions to simple allocation problems. A simple allocation problem for a society $N$ is an $|N| + 1$ dimensional nonnegative real vector $(c_1, \ldots, c_{|N|}, E) \in \mathbb{R}^N_+$ satisfying $\sum_N c_i \geq E$ where $E$, the endowment has to be allocated among agents in $N$ who are characterized by $c$, the characteristic vector. Simple allocation problems have a wide range of applications. We discuss them in detail in Subsection 1.1.

Our results are as follows. In Section 3, we describe a class of algorithmic rules. Such a rule chooses an allocation for each problem $(c, E)$ by following an algorithm which gradually reduces each agent $i$’s characteristic value from $E$ to $c_i$. At each Step $k \in \{0, \ldots, n\}$, the algorithm takes into account only the characteristics of $k$ agents (that is, for each $|S| = k$, defining $c^S$ as $c^S = (c_S, E_{N\setminus S})$, the algorithm calculates $x^S = F(c^S, E)$). If there is a feasible allocation $x^S$ (i.e. satisfying $x^S \leq c$), the algorithm chooses it. Otherwise, it moves to the next step.

Proposition 2 states that every rational rule is algorithmic. The converse, however, is not true, even for continuous rules. Proposition 4 states that every rational rule which is

---

1Thus, as discussed in Subsection 1.1, we interpret an allocation rule on simple allocation problems as representing the choices of a decision-maker (e.g. a public-policy maker, a tax codifier or a bankruptcy judge).

2With an abuse of notation, we use $E_{N\setminus S}$ to denote an $|N \setminus S|$ dimensional vector whose every coordinate is $E$. 
continuous with respect to the characteristic vector is an *algorithmic* rule that additionally satisfies the following properties: first, for problems \((c, E)\) with \(\sum_{i=1}^{N} c_i > E\), the algorithm stops at a Step \(k < n\); second, if the algorithm stops at Step \(k\) with coalition \(S\), then each member of \(S\) receives his characteristic value.

Section 4 contains the main result of the paper: Theorem 1 uses the above results to show that the *Equal Gains rule* uniquely satisfies *rationality*, *continuity*, and *equal treatment of equals* (a fairness property which requires that agents with identical characteristics should receive identical shares).

The literature contains other characterizations of the *Equal Gains rule*. Dagan (1996) shows that this rule uniquely satisfies *equal treatment of equals*, *truncation invariance*, and *composition up*.

Schummer and Thomson (1997) show that the allocation chosen by the *Equal Gains rule* minimizes (i) the difference between the largest and the smallest share and (ii) the variance of the shares. In a related result, Bosmans and Lauwers (2006) show that the allocation chosen by the *Equal Gains rule* Lorenz dominates every other allocation. Herrero and Villar (2002) and Yeh (2004) show that the *Equal Gains rule* uniquely satisfies *conditional full compensation* and *composition down*. Finally, Yeh (2006) shows that the *Equal Gains rule* uniquely satisfies *conditional full compensation* and *own-claim monotonicity*.

Our characterization is logically independent from these previous results. Furthermore, the main principles employed in these characterizations (such as “composition”, *full compensation*, or *Lorenz domination*) are quite different than our main axiom: *rationality*. Also, with the exception of Schummer and Thomson (1997) and Bosmans and Louwers (2006), the above characterizations use properties that relate the rule’s behavior at different social endowment levels. This is not the case for Theorem 1.

---

3 *Composition up* requires that dividing the social endowment in two, first allocating one part, revising the characteristic vector accordingly, and then allocating the rest produces the same final allocation as allocating all the social endowment at once.

4 *Conditional full compensation* roughly requires agents with sufficiently small characteristic values to receive their characteristic values. *Composition down* deals with the following scenario: after the social endowment is allocated, we discover that the actual social endowment is smaller; then, it requires that using the original characteristic vector or the initially chosen allocation should produce the same final outcome.
Our final results are in Section 5 where we first introduce two properties: translation down and translation up. Both are concerned with the implications of translating a problem by simultaneously changing, at the same amount, the characteristic value of an agent and the endowment. For such translations, these properties require that the initial allocation be translated the same way. We show, in Lemma 5, that a rule satisfies translation down and translation up if and only if its dual rule is rational. In Theorem 2, we then use this lemma and Theorem 1 to show that the Equal Losses rule uniquely satisfies translation down, translation up, continuity, and equal treatment of equals.

In a companion paper (Kibris, 2008), we carry out a revealed preference analysis on simple allocation problems and study rational, transitive-rational, and representable rules. There, we show that an allocation rule is rational if and only if it satisfies a standard property called contraction independence (also called independence of irrelevant alternatives in the context of bargaining by Nash (1950) and Property α in the context of consumer choice by Sen (1971)). In this paper, we make extensive use of this equivalence.

In the next subsection, we discuss the various applications of our analysis. In Section 2, we present our model and further discuss rational rules. In the following sections, we present our results as summarized above.

1.1 Examples and Applications

A simple allocation problem for a society $N$ is an $|N| + 1$ dimensional nonnegative real vector $(c_1, \ldots, c_{|N|}, E)$ which, with the exception of the last application below, is interpreted as follows. A social endowment $E$ of a perfectly divisible commodity is to be allocated among members of $N$. Each agent $i \in N$ is characterized by an amount $c_i$ of the commodity. Next, we discuss the alternative interpretations of $c$ and $E$ at various applications.

1. **Taxation:** A public authority is to collect an amount $E$ of tax from a society $N$. Each agent $i$ has income $c_i$. This is a central and very old problem in public finance. For example, see Edgeworth (1898) and the following literature. Young (1987) proposes a class of “parametric solutions” to this problem.

2. **Bankruptcy:** A bankruptcy judge is to allocate the remaining assets $E$ of a bankrupt
firm among its creditors, $N$. Each agent $i$ has credited $c_i$ to the bankrupt firm and now, claims this amount. For example, see O’Neill (1982) and the following literature. For a detailed review of the extensive literature on taxation and bankruptcy problems, see Thomson (2003 and 2007).

3. **Permit Allocation**: The Environmental Protection Agency is to allocate an amount $E$ of pollution permits among firms in $N$ (such as $CO_2$ emission permits allocated among energy producers). Each firm $i$, depending on its location, is imposed by the local authority an emission constraint $c_i$ on its pollution level. For more on this application, see Kibris (2003) and the literature cited therein.

4. **Single-peaked or Saturated Preferences**: A social planner is to allocate $E$ units of a perfectly divisible commodity among members of $N$. Each agent $i$ is known to have preferences with peak (saturation point) $c_i$. The rest of the preference information is disregarded as typical in several well-known solutions to this problem, such as the Uniform rule or the Proportional rule. For example, see Sprumont (1991) and the following literature.

5. **Demand Rationing**: A supplier is to allocate its production $E$ among demanders in $N$. Each demander $i$ demands $c_i$ units of the commodity. The supply-chain management literature contains detailed analysis of this problem. For example, see Cachon and Lariviere (1999) and the literature cited therein.

6. **Bargaining with Quasilinear Preferences and Claims**: An arbitrator is to allocate $E$ units of a numeriare good among agents who have quasilinear preferences with respect to it. Each agent holds a claim $c_i$ on what he should receive. For examples of bargaining problems with claims, see Chun and Thomson (1992) and the following literature. For bargaining problems with quasilinear preferences, see Moulin (1985) and the following literature.

7. **Surplus Sharing**: A social planner is to allocate the return $E$ of a project among its investors in $N$. Each investor $i$ has invested $s_i$. The project is profitable, that is,

---

5We would like to thank Rakesh Vohra for bringing this application to our attention.
The surplus sharing problem can now be analyzed as a simple allocation problem. For more on surplus-sharing problems, see Moulin (1985 and 1987) and the following literature.

8. **Consumer Choice under fixed prices and rationing:** A consumer has to allocate his *income* $E$ among a set $N$ of commodities. The prices of the commodities are fixed and thus, do not change from one problem to another. (With appropriate choice of consumption units, normalize the price vector so that all commodities have the same price.) As typical in the fixed-price literature, the consumer also faces “rationing constraints” on how much he can consume of each commodity. Let $c_i$ be the agent’s *consumption constraint* on commodity $i$. See Benassy (1993) or Kibrıs and Küçüksenel (2008) for more on rationing rules.

## 2 Model

Let $N = \{1, ..., n\}$ be the set of agents. For $i \in N$, let $e_i$ be the $i^{th}$ unit vector in $\mathbb{R}_{+}^N$. Let $e = \sum_N e_i$. We use the vector inequalities $\leq, \leq, <$. For $c \in \mathbb{R}_{+}^N$, $\alpha \in \mathbb{R}_{+}$, and $S \subseteq N$, with an abuse of notation, we write $(c_S, \alpha_{N \setminus S})$ to denote the vector which coincides with $c$ on $S$ and which chooses $\alpha$ for every coordinate in $N \setminus S$.

A **simple allocation problem** for $N$ is a pair $(c, E) \in \mathbb{R}_{+}^N \times \mathbb{R}_{+}$ such that $\sum_N c_i \geq E$ (please see Figures 1 and 2). We call $E$ the endowment and $c$ the characteristic vector. As discussed in Subsection 1.1, depending on the application, $E$ can be an asset or a liability and $c$ can be a vector or incomes, claims, demands, preference peaks, or consumption constraints. Let $\mathcal{C}$ be the set of all simple allocation problems for $N$. Given a simple allocation problem $(c, E) \in \mathcal{C}$, let $X(c, E) = \{x \in \mathbb{R}_{+}^N \mid x \leq c \text{ and } \sum x_i \leq E\}$ be the choice set of $(c, E)$.

An allocation rule $F: \mathcal{C} \to \mathbb{R}_{+}^N$ assigns each simple allocation problem $(c, E)$ to an allocation $F(c, E) \in X(c, E)$ such that $\sum_N F_i(c, E) = E$. Note that each rule $F$ satisfies
\( F(c, E) \leq c. \) Depending on the application, this might be interpreted as satisfying the consumption constraints or as an efficiency requirement (as in the case of single-peaked preferences) or that no agent be taxed more than his income. Also, \( \sum_{i \in N} F_i(c, E) = E \) can be interpreted as an efficiency property or, as in taxation, a feasibility requirement or as in consumer choice, the Walras law.

The following are some well-known examples of rules. The **Proportional rule** allocates the endowment proportional to the characteristic values: for each \( i \in N, \text{PRO}_i(c, E) = \frac{c_i}{\sum_{j} c_j} E. \) In the taxation literature, this rule is called a *Linear Tax*. The **Equal Gains rule** allocates the endowment equally, subject to no agent receiving more than his characteristic value: for each \( i \in N, \text{EG}_i(c, E) = \min \{c_i, \lambda\} \) where \( \lambda \in \mathbb{R}_+ \) satisfies \( \sum_{i \in N} \min \{c_i, \lambda\} = E. \) In the single-peaked allocation literature, this rule is called the *Uniform rule*, in the bankruptcy literature it is called the *Constrained Equal Awards rule*, and in the taxation literature, it is called the *Leveling Tax*. The **Equal Losses rule** equalizes the losses agents incur, subject to no agent receiving a negative share: for each \( i \in N, \text{EL}_i(c, E) = \max \{0, c_i - \lambda\} \) where \( \lambda \in \mathbb{R}_+ \) satisfies \( \sum_{i \in N} \max \{0, c_i - \lambda\} = E. \) In the single-peaked allocation literature, this rule is called the *Equal Distance rule*, in the bankruptcy literature it is called
Figure 2: A three-agent simple allocation problem.

the Constrained Equal Losses rule, and in the taxation literature, it is called the Head Tax. The Talmud rule (Aumann and Maschler, 1985) assigns equal gains until each agent receives half his characteristic value and then uses the equal losses idea: \( TAL(c, E) = EG\left(\frac{c}{2}, \min\left\{ E, \frac{1}{2} \sum_{i} c_i \right\}\right) + EL\left(\frac{c}{2}, \max\{0, E - \sum_{i} c_i\}\right) \).

A rule \( F \) is rational if there is a binary relation \( B \subseteq \mathbb{R}_+^N \times \mathbb{R}_+^N \) such that for each \( (c, E) \in C \), \( F(c, E) = \{ x \in X(c, E) \mid \text{for each } y \in X(c, E), xBy \} \). Kibris (2008) shows that rationality is equivalent to the following property. A rule \( F \) satisfies contraction independence if a chosen alternative from a set is still chosen from subsets (contractions) that contain it: for each pair \( (c, E), (c', E) \in C \), \( F(c, E) \in X(c', E) \subseteq X(c, E) \) implies \( F(c', E) = F(c, E) \). In the literature, this property is also referred to as independence of irrelevant alternatives (Nash, 1950) or Sen’s property \( \alpha \) (Sen, 1971).

**Theorem A. (Kibris, 2008)** A rule \( F \) is rational if and only if it is contraction independent.

The following lemma is from Kibris (2008). For completeness, we include the simple proof.
Lemma 1 A rule $F$ is rational if and only if for each $(c, E), (c', E) \in C$ it satisfies the following properties

Property (i). if for each $i \in N$, $\min \{c_i, E\} = \min \{c_i', E\}$, then $F(c, E) = F(c', E)$,

Property (ii). if $F(c, E) \leq c' \leq c$, then $F(c', E) = F(c, E)$.

Proof. ($\Rightarrow$) Assume that $F$ is rational. Then, by Theorem A, it satisfies contraction independence. Let $(c, E), (c', E) \in C$. First, assume that for each $i \in N$, $\min \{c_i, E\} = \min \{c_i', E\}$. Let $x \in \mathbb{R}_+^N$ satisfy $\sum_N x_i \leq E$. Then, $x \leq c$ if and only if $x \leq c'$. Thus $X(c, E) = X(c', E)$. This implies $F(c, E) = F(c', E)$. Next, assume that $F(c, E) \leq c' \leq c$. Then, $F(c, E) \in X(c', E)$, which by contraction independence, implies $F(c, E) = F(c', E)$.

($\Leftarrow$) Assume that (i) and (ii) are satisfied. Let $(c, E), (c', E) \in C$ be such that $F(c, E) \in X(c', E) \subseteq X(c, E)$. Then for each $i \in N$, either $c_i' \leq c_i$ or $\min \{c_i', E\} = \min \{c_i, E\}$. Let $S = \{i \in N \mid c_i' \leq c_i\}$. Let $c'' = (c'_S, c_{N\setminus S})$. Then $F(c, E) \leq c'' \leq c$ and by (ii), $F(c', E) = F(c'', E)$. Now, for each $i \in N$, $\min \{c_i', E\} = \min \{c''_i, E\}$. Thus, by (i), $F(c'', E) = F(c', E)$. Altogether, we obtain $F(c, E) = F(c', E)$. Thus, $F$ satisfies contraction independence. Then, by Theorem A, $F$ is rational.

Property (i) of Lemma 1 is called truncation-invariance for rules on bankruptcy and taxation problems (Thomson, 2003 and 2007). Property (ii) says that a decrease in characteristic values does not change the initially chosen allocation as long as it remains feasible.

In what follows, we will make extensive use of the equivalence stated in Lemma 1.

3 Rationality vs Algorithmic Rules

Using Lemma 1, it is straightforward to check that the Equal Gains rule is rational while the Proportional rule, Equal Losses rule, and the Talmudic rule are not. One important difference of the Equal Gains rule from the others is that it can be alternatively defined as choosing the outcome of the following algorithm: let $(c, E) \in C$,

Step 1. Propose equal division of the endowment among all agents, that is, let $x^1 =$
If no agent receives more than his characteristic value, that is, if \( x^1 \leq c \), stop and let \( EG(c, E) = x^1 \). Otherwise, let \( S_1 = \{ i \in N \mid x^1_i \geq c_i \} \) and move to Step 2.

**Step k.** (for \( k = 2, ..., |N| \)) Propose to each \( i \in S_{k-1} \) his characteristic value. (These are agents who, in Step \( k - 1 \), were proposed a share at least as large as their characteristic values). Propose equal division of the remaining endowment among the remaining agents, that is, let \( x^k = EG(c_{S_{k-1}}, E_{N \setminus S_{k-1}}, E) \). If no agent receives more than his characteristic value, that is, if \( x^k \leq c \), stop and let \( EG(c, E) = x^k \). Otherwise, let \( S_k = \{ i \in N \mid x^k_i \geq c_i \} \) and move to Step \( k + 1 \).

In what follows, we present a property that generalizes this idea.

**Definition 1** A rule \( F \) is **algorithmic** if for each \((c, E) \in C\), \( F(c, E) \) is the outcome of the following algorithm:

**Step 0.** If \( F(E_N, E) \leq c \), let \( F(c, E) = F(E_N, E) \). Otherwise, move to Step 1.

**Step 1.** If there is \( S \subseteq N \) such that \( |S| = 1 \) and \( F(c_S, E_{N \setminus S}, E) \leq c \), let \( F(c, E) = F(c_S, E_{N \setminus S}, E) \). Otherwise, move to Step 2.

For \( k = 2, ..., |N| \):

**Step k.** If there is \( S \subseteq N \) such that \( |S| = k \) and \( F(c_S, E_{N \setminus S}, E) \leq c \), let \( F(c, E) = F(c_S, E_{N \setminus S}, E) \). Otherwise, move to Step \( k + 1 \).

If \( F \) is not an **algorithmic** rule, two things can happen: either (i) the algorithm finds a feasible allocation at some step \( k \), but another allocation is chosen for the original problem, or (ii) the algorithm finds multiple feasible allocations at some step \( k \) and it is not clear which one to choose.

The **Equal Gains rule** is not the only **algorithmic rule**. Any rational rule satisfies the property.

**Proposition 2** If a rule \( F \) is rational then it is an **algorithmic rule**.

**Proof.** Let \( F \) be rational. We first show that for each pair \( S, T \subseteq N \) and each \((c, E) \in C\) such that \( F_T(c_S, E_{N \setminus S}, E) \leq c_T \) and \( F_S(c_T, E_{N \setminus T}, E) \leq c_S \), we have \( F(c_S, E_{N \setminus S}, E) = \)

\( \text{Note that } E_N \text{ denotes a claims vector whose every coordinate is } E. \)
To see this, assume \( F_T(c_S, E_{N\setminus S}, E) \leq c_T \) and \( F_S(c_T, E_{N\setminus T}, E) \leq c_S \). Then
\[
F(c_S, E_{N\setminus S}, E) \leq (c_{S\cup T}, E_{(S\cup T)^c}) \leq (c_S, E_{N\setminus S}).
\]
By Lemma 1, \( F(c_{S\cup T}, E_{(S\cup T)^c}, E) = F(c_S, E_{N\setminus S}, E) \). Similarly,
\[
F(c_T, E_{N\setminus T}, E) \leq (c_{S\cup T}, E_{(S\cup T)^c}) \leq (c_T, E_{N\setminus T})
\]
implies \( F(c_{S\cup T}, E_{(S\cup T)^c}, E) = F(c_T, E_{N\setminus T}, E) \). Thus, \( F(c_S, E_{N\setminus S}, E) = F(c_T, E_{N\setminus T}, E) \).

We now show that \( F \) is an algorithmic rule. Let \((c, E) \in C\). First assume that \( F(E_N, E) \leq c \). Then by Lemma 1, \( F(c, E) = F(E_N, E) \). Thus, the algorithm yields the allocation at Step 0. Now suppose \( F(E_N, E) \not\leq c \).

Since \( F(c_N, E) \leq c \), there is some \( k \in \{1, \ldots, n\} \) such that \((i)\) for each \( T \subseteq N \) such that \(|T| < k\), we have \( F(c_T, E_{N\setminus T}, E) \not\leq c \) and \((ii)\) for some \( S \subseteq N \) such that \(|S| = k\), we have \( F(c_S, E_{N\setminus S}, E) \leq c \). We will next show that the algorithm then stops at Step \( k \) and yields \( F(c_S, E_{N\setminus S}, E) \).

Because of \((i)\), the algorithm does not stop at any Step \( l < k \). Note that for each \( S' \subseteq N \) such that \(|S'| = k\), if \( F(c_{S'}, E_{N\setminus S'}, E) \leq c \), then by the first paragraph, \( F(c_{S'}, E_{N\setminus S'}, E) = F(c_S, E_{N\setminus S}, E) \). Applying Lemma 1 to this unique allocation, \( F(c, E) = F(c_S, E_{N\setminus S}, E) \), the desired conclusion. Thus, \( F \) is algorithmic.

Surprisingly, not every algorithmic rule is rational. The following example demonstrates this point.

**Example 1** (An algorithmic rule that is not rational) Let \( N = \{1, 2\} \) and let \( f : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) be defined as \( f(x) = \max\{0, 2x - 10\} \).

\[
F(c, E) = \begin{cases} 
(0, 10) & \text{if } E = 10, c_2 \geq 10 \\
(10 - f(c_2), f(c_2)) & \text{if } E = 10, c_2 < 10, c_1 \geq 10 - f(c_2) \\
(c_1, 10 - c_1) & \text{if } E = 10, c_2 < 10, c_1 < 10 - f(c_2) \\
EG(c, E) & \text{if } E \neq 10.
\end{cases}
\]

The rule \( F \) in the example is algorithmic. The reason is, first, \( F(E_N, E) = (0, 10) \) is chosen in every problem for which it is feasible. All other problems have \( c_2 < 10 \) and for
them, \( F(E_1, c_2, E) = (10 - f(c_2), f(c_2)) \) is chosen whenever it is feasible. All remaining problems have \( c_1 < 10 - f(c_2) \) and \( c_2 < 10 \) and the definition of an algorithmic rule does not determine how they should be solved. The rule \( F \) is not rational since \( F(10, 9, 10) = (2, 8) \) and \( F(3, 8, 10) \neq (2, 8) \) violates Property (ii) of Lemma 1. Finally note that \( F \) is also continuous.

We next introduce a subclass of algorithmic rules.

**Definition 2** A rule \( F \) is an **algorithmic rule with boundary condition** if (i) \( F \) is an algorithmic rule, (ii) for each \((c, E) \in C\), if \( F(c, E) \) is first obtained at Step \( k \) with the coalition \( S \subseteq N \) such that \( |S| = k \), then \( F_S(c, E) = c_S \), (iii) for each \((c, E) \in C\) such that \( \sum N c_i > E \), there is \( k < n \) such that \( F(c, E) \) is first obtained at a Step \( k \) of the algorithm.

It turns out that any **contraction independent** and \( c \)-**continuous** rule is of this form. To prove this result, we use the following lemma.

**Lemma 3** Assume that \( F \) is rational and \( c \)-continuous. Then, for each \((c, E) \in C, i \in N, \) and \( \delta \in \mathbb{R}_+ \), \( F_i(c + \delta e_i, E) > c_i \) implies \( F_i(c, E) = c_i \).

**Proof.** Let \((c, E) \in C, i \in N, \) and \( \delta \in \mathbb{R}_+ \) satisfy \( F_i(c + \delta e_i, E) > c_i \). Suppose \( F_i(c, E) < c_i \). Then by **claims-continuity**, there is \( \varepsilon < \delta \) such that \( F_i(c + \varepsilon e_i, E) = c_i \). But then, \( F(c + \varepsilon e_i, E) \leq c \leq c + \varepsilon e_i \), by Lemma 1, implies \( F(c, E) = F(c + \varepsilon e_i, E) \), a contradiction. ■

We next present the proposition.

**Proposition 4** If a rule \( F \) is rational and \( c \)-continuous, then it is an **algorithmic rule with boundary condition**.

**Proof.** Let \( F \) satisfy the given properties. Then, by **Proposition 2**, \( F \) is an algorithmic rule.

Next, let \((c, E) \in C \) and assume that \( F(c, E) \) is first obtained at Step \( k \) with the coalition \( S \subseteq N \) such that \( |S| = k \). Then, \( F(c, E) = F(c_S, E_{N \setminus S}, E) \). If \( S = \emptyset \), \( F_S(c, E) = c_S \) is trivially satisfied. So assume \( S \neq \emptyset \) and let \( i \in S \). Since \( F(c, E) \) is not obtained at Step \( k - 1 \),
$F(c_{S\setminus i}, E_{N\setminus (S\setminus i)}, E) \not\leq c$. Since $F(c_S, E_{N\setminus S}, E) \leq c$, this implies $F_i(c_{S\setminus i}, E_{N\setminus (S\setminus i)}, E) > c_i$.

Then, by Lemma 3, $F_i(c_S, E_{N\setminus S}, E) = c_i$. Since this conclusion holds for all $i \in S$, we have $F_S(c, E) = c_S$.

Finally, let $(c, E) \in C$ be such that $\sum N c_i > E$. We show that $F(c, E)$ is first obtained at a Step $k < n$. For contradiction, suppose $F(c, E)$ is first obtained at Step $n$. Then, by the previous paragraph, $F(c, E) = c$. But then, $\sum N F_i(c, E) = \sum N c_i = E < \sum N c_i$, a contradiction. ■

For two-agent problems, any algorithmic rule with boundary condition is c-continuous.

The following example demonstrates that for larger societies, this relationship does not hold anymore.

**Example 2** (An algorithmic rule with boundary condition which is not c-continuous) Let $N = \{1, 2, 3\}$, let $\omega = (3, 1, 1)$, and let $EG^\omega$ be the weighted Equal Gains rule with weight vector $\omega$, defined as follows: for each $(c, E) \in C$, $EG^\omega_1(c, E) = \min \{c_1, 3\rho\}$ and for $i \in \{2, 3\}$, $EG^\omega_i(c, E) = \min \{c_i, \rho\}$ where $\rho \in \mathbb{R}_+$ satisfies $\sum N EG^\omega_i(c, E) = E$. Finally, let $F$ be defined as

$$F(c, E) = \begin{cases} EG(c, E) & \text{if } c_3 > 0, \\ EG^\omega(c, E) & \text{if } c_3 = 0. \end{cases}$$

### 4 A Characterization of the Equal Gains Rule

The following is an important fairness notion. A rule $F$ satisfies **equal treatment of equals** if two agents with identical characteristics are always awarded equal shares: for each $(c, E) \in C$ and $i, j \in N$, $c_i = c_j$ implies $F_i(c, E) = F_j(c, E)$. A large class of rules, including the four central ones introduced in Section 2, satisfy this property (e.g. see Young, 1987). Among them, however, the Equal Gains rule is the only rational rule.

**Theorem 1** A rule $F$ satisfies rationality, c-continuity, and equal treatment of equals if and only if it is the Equal Gains rule.
Proof. It is straightforward to show that $EG$ satisfies the given properties. Conversely, let $F$ be a rule that satisfies them. We next show $F = EG$. First note that, since $F$ is rational, it satisfies the two properties of Lemma 1.

Let $(c, E) \in \mathcal{C}$. If $\sum_N c_i = E$, $F(c, E) = EG(c, E) = c$. Alternatively, assume $\sum_N c_i > E$. Note that, by Proposition 4, $F$ and $EG$ are both algorithmic rules with boundary condition.

Step 1: For each Step $k \in \{0, ..., n-1\}$ of the algorithm and for each $S \subseteq N$ such that $|S| = k$, we have $F(c, S_{N \setminus S}, E) = EG(c, S_{N \setminus S}, E)$. To prove this, we use induction. Initially, let $k = 0$. Thus, $S = \emptyset$ and by equal treatment of equals, $F(c, S_{N \setminus S}, E) = EG(c, S_{N \setminus S}, E)$. Now let $k \in \{1, ..., n-1\}$ and assume that the statement holds for each $l < k$. Let $S \subseteq N$ be such that $|S| = k$.

Case 1: There is $l < k$ and $T \subseteq S$ such that $|T| = l$ and $F(c, S_{N \setminus S}, E) = F(c, T, S_{N \setminus T}, E)$. Then, by our assumption, $F(c, S_{N \setminus T}, E) = EG(c, S_{N \setminus T}, E)$. Thus, $EG(c, S_{N \setminus T}, E) \leq (c, S_{N \setminus S})$. By Lemma 1 applied to $EG$, $EG(c, S_{N \setminus T}, E) = EG(c, S_{N \setminus S}, E)$. Combining the equalities, we then have $F(c, S_{N \setminus S}, E) = EG(c, S_{N \setminus S}, E)$.

Case 2: For each $l < k$ and $T \subseteq S$ with $|T| = l$, $F(c, S_{N \setminus S}, E) \neq F(c, T, S_{N \setminus T}, E)$. Thus, $F(c, S_{N \setminus S}, E)$ is first obtained at Step $k$. Since, $F$ is algorithmic with boundary condition, then $F_S(c, S_{N \setminus S}, E) = c$. Since $F$ satisfies equal treatment of equals, for each $i \in N \setminus S$, $F_i(c, S_{N \setminus S}, E) = \frac{E - \sum_{j \in S} c_j}{n - |S|}$. By the induction hypothesis, for each $l < k$ and $T \subseteq N$ with $|T| = l$, $F(c, T, S_{N \setminus T}, E) = EG(c, T, S_{N \setminus T}, E)$. By assumption of Case 2, for each $l < k$ and $T \subseteq S$ with $|T| = l$, $F(c, S_{N \setminus S}, E) \neq F(c, T, S_{N \setminus T}, E)$, and thus, by Lemma 1 applied to $F$, $F(c, S_{N \setminus S}, E) \neq (c, S_{N \setminus S})$. Thus, for each $l < k$ and $T \subseteq S$ with $|T| = l$, $EG(c, T, S_{N \setminus T}, E) \neq (c, S_{N \setminus S}, E)$. Therefore, $EG(c, S_{N \setminus S}, E)$ is first obtained at Step $k$. Since $EG$ is algorithmic with boundary condition, then $EG_S(c, S_{N \setminus S}, E) = c$. Since $EG$ satisfies equal treatment of equals, for each $i \in N \setminus S$, $EG_i(c, S_{N \setminus S}, E) = \frac{E - \sum_{j \in S} c_j}{n - |S|}$. Combining this with the observations on $F$, we have $F(c, S_{N \setminus S}, E) = EG(c, S_{N \setminus S}, E)$.

Step 2: $F(c, E) = EG(c, E)$. First, since $F$ is algorithmic with boundary condition, there is $k < n$ and $S \subseteq N$ with $|S| = k$ such that $F(c, E) = F(c, S_{N \setminus S})$. Then, by Step 1, $F(c, S_{N \setminus S}, E) = EG(c, S_{N \setminus S}, E)$. Therefore, $EG(c, S_{N \setminus S}, E) \leq c$. Then, by Lemma 1 applied to $EG$, we have $EG(c, E) = EG(c, S_{N \setminus S}, E)$. Combining the equalities, $F(c, E) = EG(c, E)$. □
The above characterization is tight. Without rationality, Proportional rule becomes admissible. Without equal treatment of equals, all algorithmic rules with boundary condition, such as the weighted versions of the Equal Gains rule, become admissible. Finally, the following example presents a rule that violates c-continuity but satisfies the other properties.

**Example 3** *(A rule that satisfies rationality and equal treatment of equals but not c-continuity)* Let \( N = \{1, 2\} \). Let \( F \) be defined as

\[
F(c, E) = \begin{cases} 
\left( \frac{E}{2}, \frac{E}{2} \right) & \text{if } c_1 \geq \frac{E}{2} \text{ and } c_2 \geq \frac{E}{2}, \\
(E, 0) & \text{if } c_1 \geq E \text{ and } c_2 < \frac{E}{2}, \\
(c_1, E - c_1) & \text{if } E - c_2 < c_1 < E \text{ and } c_2 < \frac{E}{2}, \\
(0, E) & \text{if } c_1 < \frac{E}{2} \text{ and } c_2 \geq E, \\
(E - c_2, c_2) & \text{if } c_1 < \frac{E}{2} \text{ and } E - c_1 < c_2 < E.
\end{cases}
\]

**5 Translation Properties and Duality**

We start this section by introducing two properties that require that the solution to a problem be covariant under certain translations of the problem. Similar properties have been analyzed in bargaining theory (*e.g.* see Thomson, 1981).

A rule \( F \) satisfies **translation down** *(Figure 3, left)* if for each \((c, E) \in C\), each \( i \in N \), and each \( \delta \in (0, F_i(c, E)] \), we have \( F(c - \delta e_i, E - \delta) = F(c, E) - \delta e_i \). It satisfies **translation up** *(Figure 3, right)* if for each \((c, E) \in C\), each \( i \in N \) such that \( c_i \geq \sum_N c_j - E \), and each \( \delta \in (0, \infty) \), we have \( F(c + \delta e_i, E + \delta) = F(c, E) + \delta e_i \). Both properties are concerned with the implications of translating a problem by simultaneously changing, at the same amount, the characteristic value of an agent and the endowment. For such translations, these properties require that the initial allocation be translated the same way.

It turns out that these two properties are very closely related to rationality. The **dual of a rule** \( F, F^d \), allocates what is available in the same way as \( F \) allocates what is missing, that is, for each \((c, E) \in C\), \( F^d(c, E) = c - F(c, \sum_N c_i - E) \). Aumann and Maschler (1985) quote several passages from the Talmud where the notion of duality is implicitly discussed and self-duality of a rule (that is, the rule coinciding with its dual) is promoted.
The duality of rules can also be used to define a notion of duality for properties. A property $\Pi$ is the dual of another property $\Pi^d$ if whenever a rule $F$ satisfies $\Pi$, its dual rule $F^d$ satisfies $\Pi^d$. Some properties, such as $c$-continuity or equal treatment of equals, are self-dual. That is, a rule $F$ satisfies $c$-continuity (or equal treatment of equals) if and only if its dual $F^d$ satisfies the same property.

The following result shows that translation down and translation up are, together, the dual of rationality (or equivalently, contraction independence).

**Lemma 5** A rule $F$ satisfies rationality if and only if its dual $F^d$ satisfies translation up and translation down.

**Proof.** First assume that $F$ satisfies rationality. Then it satisfies the two properties of Lemma 1. Let $(c, E) \in \mathcal{C}$, $i \in N$, and $\delta \in (0, \infty)$.

**Claim 1.** $F^d$ satisfies translation down. To see this, assume $\delta \leq F_i^d (c, E)$. Let $E' = \sum_N c_i - E$. Then, $\delta \leq c_i - F_i (c, E)$ implies $F (c, E') \leq c - \delta e_i \leq c$. Then, by Property (ii) of Lemma 1, $F (c - \delta e_i, E - \delta) = F (c, E')$. This implies $F^d (c - \delta e_i, E - \delta) = F^d (c, E) - \delta e_i$.  

Figure 3: Translation down (left) and translation up (right).
Claim 2. \( F^d \) satisfies translation up. To see this, assume \( c_i \geq \sum_N c_j - E \). Let \( \overline{E} = \sum_N c_i - E \). Then, \( \min \{c_i, \overline{E}\} = \overline{E} = \min \{c_i + \delta, \overline{E}\} \). Then, by Property (i) of Lemma 1, \( F(c + \delta e_i, \overline{E}) = F(c, \overline{E}) \). This implies \( F^d(c + \delta e_i, E + \delta) = F^d(c, E) + \delta e_i \).

Next, assume that \( F^d \) satisfies translation down and translation up. Let \((c, E), (c', E) \in \mathcal{C}\).

Claim 3. \( F \) satisfies Property (ii) of Lemma 1. Assume \( F(c, E) \leq c' \leq c \). Let \( A = \{j \in N \mid c'_j < c_j\} \). If \( A = \emptyset \), \( F(c', E) = F(c, E) \) trivially holds. Alternatively, let \( i \in A \). Let \( \delta = c_i - c'_i \) and \( \overline{E} = \sum_N c_i - E \). Then \( F_i(c, E) \leq c_i - \delta \) implies \( \delta \in (0, F_i^d(c, \overline{E})] \). Thus, by translation down, \( F^d(c - \delta e_i, \overline{E} - \delta) = F^d(c, \overline{E}) - \delta e_i \). This implies \( F(c - \delta e_i, E) = F(c, E) \). Applying the same argument to each \( i \in A \), we obtain \( F(c', E) = F(c, E) \).

Claim 4. \( F \) satisfies Property (i) of Lemma 1. Assume for each \( j \in N \), \( \min \{c_j, E\} = \min \{c'_j, E\} \). Let \( A = \{j \in N \mid c'_j < c_j\} \), \( B = \{j \in N \mid c'_j > c_j\} \), and \( c'' = (c_A, c_{N \setminus A}) \). By Claim 3, \( F(c'', E) = F(c, E) \). We will next show \( F(c', E) = F(c'', E) \). If \( B = \emptyset \), this trivially holds. Alternatively, let \( i \in B \). Let \( \delta = c'_i - c''_i \) and \( \overline{E} = \sum_N c''_j - E \). Note that \( c''_i = c_i \). Then \( \min \{c''_i, E\} = \min \{c_i, E\} \) implies \( E \leq c''_i \). Thus, \( c'_i \geq \sum_N c''_j - \overline{E} \). Then, by translation up, \( F^d(c'' + \delta e_i, \overline{E} + \delta) = F^d(c'', \overline{E}) + \delta e_i \). This implies \( F(c'' + \delta e_i, E) = F(c'', E) \). Applying the same argument to each \( i \in B \), we obtain \( F(c', E) = F(c'', E) \). Together, \( F(c', E) = F(c, E) \).

Claims 3 and 4, by Lemma 1, imply that \( F \) is rational.

The following result is a corollary of Theorem 1 and Lemma 5. It also uses the fact that the Equal Gains and Equal Losses rules are dual rules (e.g. see Thomson, 2003).

**Theorem 2** A rule \( F \) satisfies translation up, translation down, c-continuity, and equal treatment of equals if and only if it is the Equal Losses rule.

**Proof.** It is straightforward to show that \( EL \) satisfies the given properties. Conversely, let \( F \) be a rule that satisfies them. We next show \( F = EL \). By Lemma 5, \( F^d \) satisfies rationality. Since c-continuity and equal treatment of equals are self-dual properties, \( F^d \) also satisfies them. Thus, by Theorem 1, \( F^d = EG \). Since \( EG \) and \( EL \) are dual rules, then, \( F = EL \).
References


[26] Thomson, W., 2007, How to Divide When There Isn’t Enough: From the Talmud to Game Theory, book manuscript.

