CAUCHY PROBLEM FOR A HIGHER-ORDER BOUSSINESQ EQUATION

by

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to my family

&

to whom supported my study

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Abstract

In this thesis, we establish global well-posedness of the Cauchy problem for a particular higher-order Boussinesq equation.

At the microscopic level this sixth order Boussinesq equation was derived in [11] for the longitudinal vibrations of a dense lattice, in which a unit length of the lattice contains a large number of lattice points.

We take the initial data in the Sobolev space H^s with $s > \frac{1}{2}$. With smoothness assumptions on the nonlinear term, we establish local existence and uniqueness of the solution. Under further assumptions, we prove the global existence for $s \geq 1$. Finally, we show continuous dependence of the solution on the initial data.

YÜKSEK MERTEBEDEN BİR BOUSSINESQ DENKLEMİ İÇİN CAUCHY PROBLEMİ

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Özet

Bu tezde, yüksek mertebeden özel bir Boussinesq denklemi için yazılmış Cauchy probleminin global olarak iyi konulmuş olduğu gösterilmiştir.
Altıncı mertebeden olan bu Boussinesq denklemi mikroskobik düzeyde, bir yoğun latisin boyuna titreşimlerini tanımlamak için [11]'de türetilmiştir. Birim uzunluğunda çok sayıda latis noktası içeren latisler yoğun latis olarak adlandırılır.
Baslangıç verilerini $s > \frac{1}{2}$ için H^s Sobolev uzayında alarak, lineer olmayan terimin yeterince düzgün olduğu varsayımı altında çözümün lokal varlığı ve tekliği elde edilmiştir. Ek varsayımlar altında $s \geq 1$ için global varlık ispat edilmiştir. Son olarak, çözümün başlangıç verileri üzerine sürekli bağlılığı gösterilmiştir.

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Chapter 1

Introduction and Preliminary Concepts

In this thesis, we study a higher-order Boussinesq type equation and the related Cauchy problem. The higher-order Boussinesq equation models the longitudinal vibrations of a dense lattice. We prove the existence and uniqueness of the global solution for the Cauchy problem. For this purpose, we give some definitions and relations, then we introduce some typical problems and use the ideas in our problem. As far as we know, the results that we obtained are new.

The rest of Chapter 1 is devoted to the preliminaries. Mainly, we give a brief overview of nonlinear evolution equations, weak solutions, Fourier transform, and some special function spaces. We also give some useful theorems and inequalities. We refer to [6] for the notation and the definitions given. We also use some parts of [10], [16] to define weak derivatives and Fourier transform.

Before giving the higher-order Boussinesq equation we consider, we describe the physical model in Chapter 2. We derive the equation using quasi-continuum approximation for lattice dynamics equations. Moreover, we derive the conservation laws.

Cauchy problem for the generalized double dispersion equation was studied by Wang and Chen in [2]. In Chapter 3, we study their methods used to prove the global existence and uniqueness.

Finally, Chapter 4 contains our own results on the higher-order Boussinesq equation. Parallel to Chapter 3, we study the linearized problem and obtain the local existence and uniqueness by contraction mapping principle. Then, we prove the global existence and uniqueness theorem and show that the solution of the problem under consideration depends continuously on the initial data.

1.1 Nonlinear Evolution Equations

Partial differential equations with time t as one of the independent variables, namely nonlinear evolution equations arise not only from many fields of mathematics but also from other branches of science such as physics, mechanics and material science. For example, Navier-Stokes and Euler equations from fluid mechanics, nonlinear reaction-diffusion equations from heat transfer and biological sciences, nonlinear Klein-Gorden equations and nonlinear Schrödinger equations from quantum mechanics are special equations of this type. In particular, the Boussinesq-type equation that we study in this thesis describes non-linear vibrations arising in lattice dynamics.

The first question to ask is whether for a nonlinear evolution equation with given initial data there is a solution at least locally in time, and whether it is unique in the considered class. While the initial data are given at $t = 0$, we want to find the solution for later time $t > 0$. This problem has been solved for a wide class of nonlinear evolution equations by two powerful theorems in nonlinear analysis: the contraction mapping theorem and the Leray-Schauder fixed-point theorem [16]. Since the 1960's, much more attention has been paid to the question of global existence, in other words, whether a solution can be extended to all times. As we know from the theory of ordinary differential equations, there is a significant difference between linear and nonlinear equations. For a linear ordinary differential equation, we can often find a solution defined globally for all $t > 0$, while for a nonlinear equation it is not always possible. This is also true for nonlinear evolution equations which are clearly more complicated than ordinary differential equations.

The standard method for proving global existence and uniqueness is to combine a local existence and uniqueness result with estimates. A local existence result can be obtained as follows: First, the linearized problem is solved, then the original problem is transformed into a fixed point problem using this linearized problem. Finally, a suitable fixed point theorem will yield the result. Such results are local, because the fixed point theorems usually work for small time intervals. Obtaining estimates for global existence is usually a separate step in the proof since the special structure of nonlinear problem has to be exploited. A typical example of the scheme above is the existence and uniqueness theorem for ordinary differential equations [9].

We give some basic definitions and some useful inequalities below.

1.2 Classical and Weak Solutions

We say that a given problem for a partial differential equation is *well-posed* if

- the problem in fact has a solution;
- this solution is unique; and
- the solution depends continuously on the data given in the problem.

By solving a partial differential equation with initial/boundary conditions, we mean to show that the three conditions above are satisfied. In other words, for a solution, we want all derivatives involved in the equation to exist and satisfy the equation with initial/boundary conditions at each point of the domain. Such a solution is called a classical solution. Certain specific partial differential equations such as the wave equation can be solved in the classical sense; but if we wish to study conservation laws and recover the underlying physics, we must allow for solutions which are not continously differentiable or even not continous. As in the case of conservation laws, some equations can be described in weaker forms and may be satisfied by functions that are not sufficiently smooth. Moreover, a solution that starts smooth may eventually become singular as in the case of shock waves. To overcome this difficulty, we allow for generalized or weak solutions.

Example 1.2.1 Consider the Cauchy problem:

$$
\begin{aligned}
&u_{tt} - c^2 u_{xx} = 0 \quad x \in R, \quad t > 0,\\
&u(x,0) = \varphi(x) \quad x \in R,\\
&u_t(x,0) = \psi(x) \quad x \in R,
\end{aligned}
$$

where $\varphi(x)$ and $\psi(x)$ are arbitrary initial value functions. We know from D'Alembert's formula that if $\varphi(x) \in C^2(R)$ and $\psi(x) \in C^1(R)$ then the problem has a unique solution $u \in C^2(R \times R^+)$ given by the formula

$$u(x,t) = \frac{1}{2}(\varphi(x+ct) + \varphi(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds.$$

If the assumptions $\varphi(x) \in C^2(R)$ and $\psi(x) \in C^1(R)$ are not satisfied, then $u(x,t)$ given by this formula is not a classical solution but still describes a wave motion. For this reason, we define a weak (generalized) solution using approaches involving weak derivatives [6].

1.3 Weak Derivatives

Let $C_c^\infty(U)$ denote the space of infinitely differentiable functions $\phi : U \subset (R^n) \to R$, with compact support in U, an open subset of R^n. A function ϕ belonging to $C_c^\infty(U)$ is called a *test function*.

Consider a function $u \in C^1(U)$. Then if $\phi \in C_c^\infty$, we see from integration by parts formula that

$$\int_U u\phi_{x_i} dx = -\int_U u_{x_i}\phi dx \quad (i = 1, 2, ..., n), \tag{1.1}$$

where $dx = dx_1 dx_2 ... dx_n$. There are no boundary terms since ϕ has compact support in U and thus vanishes near the boundary ∂U. If k is a positive integer, $u \in C^k(U)$, and $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ is a multiindex of order $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n = k$, then

$$\int_U u D^\alpha \phi dx = (-1)^{|\alpha|} \int_U D^\alpha u \phi dx \tag{1.2}$$

where we use the multiindex notation

$$D^\alpha \phi = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} \phi.$$

Equation (1.2) is obtained by applying formula (1.1) $|\alpha|$ times. If u is not k times continously differentiable, then the expression "$D^\alpha u$" on the right hand side of (1.2) has no obvious meaning.

Definition 1.3.1 The linear space of all measurable functions $u : U \to C$, from an open subset U of R^n to the set of complex numbers C, for which

$$\|f\|_{L^p(U)} = (\int_U |f|^p \, dx)^{\frac{1}{p}} < \infty$$

where $1 \le p < \infty$, is defined as $L^p(U)$.

Definition 1.3.2 A function $u : U \to C$, from an open subset U of R^n to the set of complex numbers C, is essentially bounded on U if it is measurable and there exists a real number $M > 0$ such that $|u(x)| \le M$ for almost all $x \in U$. The infumum of all such numbers M is called *essential supremum* of u and is denoted by $\|u\|_\infty$. The set of all essentially bounded functions on U is denoted by $L^\infty(U)$.

Definition 1.3.3 Let U, V denote open subsets of R^n. We write

$$V \subset\subset U$$

and say V is *compactly contained* in U if $V \subset K \subset U$ for some compact set K.

Definition 1.3.4 Let $1 \leq p \leq \infty$. $L^p_{loc}(U)$ is the set of locally integrable functions,

$$L^p_{loc}(U) = \{u : U \to C \ \ | \ \ u \in L^p(V) \quad \text{for each} \quad V \subset\subset U\},$$

i.e. $u \in L^p$ if $u : U \to C$ satisfies $u \in L^p(V)$ for all $V \subset\subset U$.

Definition 1.3.5 Suppose $u, v \in L^1_{loc}(U)$ and α is a multiindex. If the equality

$$\int_U u D^\alpha \phi dx = (-1)^{|\alpha|} \int_U v \phi dx \tag{1.3}$$

is satisfied for all test functions $\phi \in C_c^\infty$, then v is called the *weak derivative of order* $|\alpha|$ *of the function* u in the domain U and is denoted by $D^\alpha u$, i.e. $v = D^\alpha u$. In other words, if we are given u and if there exists a function v which verifies (1.3) for all ϕ, we say that $D^\alpha u = v$ in the weak sense. If there does not exist such a function v, then u does not possess a weak α^{th}-partial derivative.

Fix $1 \leq p \leq \infty$ and let k be a nonnegative integer. We define now certain function spaces, whose members have weak derivatives of various orders lying in various L^p spaces.

Definition 1.3.6 The *Sobolev space* $W^{k,p}(U)$ consists of all integrable functions $u : U \to R$ such that for each multiindex α with $|\alpha| \leq k$, $D^\alpha u$ exists in the weak sense and belongs to $L^p(U)$.
Similarly we define the space $W^{k,p}_{loc}(U)$ using locally integrable functions instead of integrable ones.

We introduce a natural norm on the Sobolev space:

$$\|u\|_{W^{k,p}} = \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p}.$$

We have the following sequence of inclusions of Sobolev spaces:

$$L^p(U) = W^{0,p}(U) \supset W^{1,p}(U) \supset W^{2,p}(U) \supset ...$$

A sequence (u_k) converges to u in the Sobolev space $W^{k,p}$ if and only if $D^\alpha u_k \to D^\alpha u$ in $L^p(U)$ as $k \to \infty$ for all multiindices α such that $|\alpha| \leq k$.

Theorem 1.3.7 [6] For each $k = 1, 2, ...$ and $1 \leq p \leq \infty$, the Sobolev space $W^{k,p}(U)$ is a Banach space.

Remark: If $p = 2$, we write

$$H^k(U) = W^{k,2}(U) \quad (k = 0, 1, 2, ...).$$

The letter H is used since $H^k(U)$ is a Hilbert space. Note that $H^0(U) = L^2(U)$.

Remark: At the end of Example 1.2.1, we mentioned about some approaches for introducing weak solutions. One of them is to use weak derivatives so that the wave equation is satisfied in a form of integral identity. The other one is to define a weak (generalized) solution of the problem by approximating (φ, ψ) with smooth data $(\varphi_k, \psi_k) \in C^2(R) \times C^1(R)$ and passing to the L^2-limit in L^2 spaces of the corresponding solutions u_k.

Example 1.3.1 (Continuation of Example 1.2.1) Following the definition of weak derivative, a function $u \in W^{2,1}_{loc}(R^2)$ is said to be a weak solution of the wave equation $u_{tt} - c^2 u_{xx} = 0$ iff

$$\int_{R^2} u(\phi_{tt} - c^2 \phi_{xx}) dx dt = 0$$

for every test function $\phi \in C_0^\infty(R^2)$.

Another approach to weak solutions is as follows: A function $u \in W^{2,1}_{loc}(R^2)$ is said to be a weak solution of the wave equation $u_{tt} - c^2 u_{xx} = 0$ iff there exists a sequence of solutions $u_k(x,t) \in C^2(R^2)$ of the wave equation such that for every compact set $K \subset R^2$, $\|u_k - u\|_{W^{2,1}(K)} \to 0$ as $k \to \infty$.

In [14], these two approaches are shown to be equivalent.

1.4 Fourier Transform

The Fourier transform is of basic importance in various areas of analysis, especially in applications of partial differential equations and in the theory of probability.

The general idea for solving various problems (usually a partial or ordinary differential equation) using the Fourier method consists in the following three steps:

i. to convert the original problem to simpler one (ordinary differential equation or algebraic equation, respectively) using the Fourier transform;

ii. to solve the new equation;

iii. to obtain the solution of the original problem using the inverse Fourier transform.

Let $u \in L^1(R)$. Consider the function $(\xi, x) \mapsto e^{-ix\xi}u(x)$ from $R \times R$ into C. As for given $\xi \in R$ the function $x \mapsto e^{-ix\xi}u(x)$ is integrable on R since its absolute value equals $|u|$. Moreover, the function $\hat{u} : R \to C$ given by the integral

$$\hat{u}(\xi) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} e^{-ix\xi}u(x)dx$$

is well-defined.

Definition 1.4.1 The function $\hat{u}$ is called the *Fourier transform* of the function u and is also denoted by $F(u)$ or Fu.

Definition 1.4.2 For $v \in L^1(R)$, the function

$$\check{v}(x) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} e^{ix\xi}v(\xi)d\xi$$

is called the *inverse Fourier transform* of the function v.

We extend the definition of Fourier and inverse Fourier transform to functions $u \in L^2(R)$ by the following theorems([6], [10]).

Theorem 1.4.3 (Plancherel's Theorem) Assume $u \in L^1(R) \cap L^2(R)$. Then $\hat{u}, \check{u} \in L^2(R)$, and

$$\|\hat{u}\|_{L^2(R)} = \|\check{u}\|_{L^2(R)} = \|u\|_{L^2(R)} . \tag{1.4}$$

Theorem 1.4.4 Assume $u, v \in L^2(R)$. Then

i. $\int_R u\bar{v}dx = \int_R \hat{u}\bar{\hat{v}}d\xi$,

ii. $(D^\alpha u)\hat{}(\xi) = (i\xi)^\alpha \hat{u}(\xi)$ for each multiindex α such that $(D^\alpha u)\hat{} \in L^2(R)$,

iii. $\widehat{(u * v)} = (2\pi)^{\frac{1}{2}}\hat{u}\hat{v}$, where $u * v$ denotes the convolution of u and v (Convolution Theorem),

iv. $u = (\hat{u})\check{}$ (the Inversion Theorem)

where $\overline{z}$ denotes the complex conjugate of $z \in C$.

The Sobolev space $H^k(R)$ can be related to Fourier transform in the following sense where we write $\|.\|$ instead of $\|.\|_{L^2}$:

$$\|u\|^2_{H^k} = \|u\|^2_{W^{k,2}} = \sum_{|\alpha|\leq k} \|D^\alpha u\|^2 .$$

From Plancherel's theorem and Theorem 1.4.4 (ii),

$$\begin{aligned}
\sum_{|\alpha|\leq k} \|D^\alpha u\|^2 &= \sum_{|\alpha|\leq k} \left\|\widehat{D^\alpha u}\right\|^2 \\
&= \sum_{|\alpha|\leq k} \|(i\xi)^\alpha \widehat{u}\|^2 = \sum_{|\alpha|\leq k} \int_R |i\xi|^{2\alpha} |\widehat{u}(\xi)|^2 \, d\xi \\
&= \sum_{|\alpha|\leq k} \int_R \xi^{2\alpha} |\widehat{u}(\xi)|^2 \, d\xi = \int_R (\sum_{|\alpha|\leq k} \xi^{2\alpha}) |\widehat{u}(\xi)|^2 \, d\xi.
\end{aligned}$$

Let

$$\sum_{|\alpha|\leq k} \xi^{2\alpha} = 1 + \xi^2 + \xi^4 + ... + \xi^{2k} = P_k(\xi).$$

Since

$$\lim_{\xi\to\mp\infty} \frac{P_k(\xi)}{(1+\xi^2)^k} = 1$$

and

$$\frac{P_k(\xi)}{(1+\xi^2)^k} > 0,$$

there exists c_1,c_2 such that

$$c_1(1+\xi^2)^k \leq P_k(\xi) \leq c_2(1+\xi^2)^k$$

where $c_1 > 0$ and $c_2 = 1$. Therefore as a weight, $P_k(\xi)$ is equivalent to $(1+\xi^2)^k$. Using this equivalence, we obtain an equivalent definition of H^k as follows:

Proposition 1.4.1 The Sobolev space $H^k(R)$ can also be defined by

$$H^k(R) = \left\{ u \in L^2(R) \;\; | \;\; (1+\xi^2)^{\frac{k}{2}} \hat{u}(\xi) \in L^2(R) \right\}, \tag{1.5}$$

where $\xi \in R$, and $\hat{u}$ is the Fourier transform of u. The usual norm is equivalent to

$$\|u\|_{H^k(R)} = (\int_R (1+\xi^2)^k |\hat{u}(\xi)|^2 \, dt)^{\frac{1}{2}}. \tag{1.6}$$

We give a *scale of Sobolev spaces* $H^s(R)$ defined for all real numbers $s \geq 0$ instead of integers $k \geq 0$ by:

$$H^s(R) = \left\{ u \in L^2(R) \ \mid \ (1+\xi^2)^{\frac{s}{2}} \hat{u}(\xi) \in L^2(R) \right\}. \tag{1.7}$$

Thus, $u \in H^s(R)$ if and only if u is Lebesgue measurable and

$$\|u\|_{H^s(R)} = (\int_R (1+\xi^2)^s \left|\hat{u}(\xi)\right|^2 dt)^{\frac{1}{2}} < \infty.$$

For $s_1 < s_2$ we have a continuous imbedding

$$H^{s_2}(R) \subset H^{s_1}(R) \tag{1.8}$$

and $H^0(R) = L^2(R)$. We can prove this fact using (1.7). Note that $s_1 < s_2$ and $1+\xi^2 \geq 1$ together imply that $(1+\xi^2)^{s_1} \leq (1+\xi^2)^{s_2}$. Multiplying this inequality by $\left|\hat{u}(\xi)\right|^2$, then integrating over R, we obtain

$$\|u\|_{H^{s_1}} \leq \|u\|_{H^{s_2}}. \tag{1.9}$$

This means that the imbedding $H^{s_2}(R) \to H^{s_1}(R)$ is continuous. Functions from $H^s(R)$ are more differentiable as s increases, where s is the degree of regularity of functions. On the other hand, the Lebesgue spaces $L^p(R)$ do not satisfy this inclusion property because R is unbounded. Both $L^1(R) \setminus L^2(R)$ and $L^2(R) \setminus L^1(R)$ are nonempty sets.

1.5 Some Special Function Spaces

We define some suitable function spaces for nonlinear evolution equations so that we can obtain their global solutions using the methods mentioned in Section 1.1. They are all function spaces involving time t.

Let X be a Banach space, $1 \leq p < \infty$, $-\infty \leq a < b \leq \infty$. Then $L^p((a,b);X)$ denotes the space of L^p functions from (a,b) into X. In other words, a function $f \in L^p((a,b),X)$ if $f(t)$ belongs to X for each $t \in (a,b)$ and

$$\|f\|_{L^p((a,b);X)} = (\int_a^b (\|f(t)\|_X)^p dt)^{\frac{1}{p}} < \infty.$$

$L^p((a,b);X)$ is a Banach space with the norm given above.

For $p=\infty$, $L^\infty((a,b),X)$ is the space of measurable functions from (a,b) into X which are essentially bounded. It is a Banach space for the norm

$$\|f\|_{L^\infty((a,b);X)} = \text{ess sup}_{t\in(a,b)} \|f(t)\|_X .$$

Similarly, when $-\infty < a < b < \infty$ we can define the Banach spaces $C^k([a,b];X)$ with the norm

$$\|f\|_{C^k([a,b];X)} = \sum_{i=0}^{k} \max_{t\in[a,b]} \left\| \frac{d^i f}{dt^i}(t) \right\|_X .$$

They denote the functions which are k times differentiable, and which belong to X for each $t\in[a,b]$.

Example 1.5.1 When $X=L^2(R)$, $u\in L^2([0,T],L^2(R))$ is actually a function of two variables $u=u(x,t)$ satisfying

$$\|u\|_{L^2([0,T],L^2(R))} = \int_0^T \|u(t)\|_{L^2}^2\,dt = \int_0^T\int_{-\infty}^{\infty} |u(x,t)|^2\,dxdt = \|u\|_{L^2([0,T]\times R)} .$$

1.6 Useful Inequalities and Theorems

Lemma 1.6.1 (Minkowski's Inequality for Integrals) [4] If $1\le p\le\infty$, $u\in L^1(I,L^p(R))$ for a.e. t, where $I\subset[0,\infty)$, then

$$\left\|\int_I u(.,t)dt\right\|_{L^p} \le \int_I \|u(.,t)\|_{L^p}\,dt.$$

Remark: Obviously, Lemma 1.6.1 also holds for Sobolev spaces, i.e. $L^1(I,W^{k,p})$.

Lemma 1.6.2 (Integral Form of Gronwall's Inequality) [6]

i. Let $\phi(t)$ be the nonnegative, continuous function on $[0,T]$ which satisfies almost everywhere t the integral inequality

$$\phi(t)\le C_1\int_0^t \phi(s)ds + C_2$$

where C_1 and C_2 are nonnegative constants. Then,

$$\phi(t)\le C_2 e^{C_1 t}$$

for almost all $0\le t\le T$.

ii. In particular, if

$$\phi(t) \leq C_1 \int_0^t \phi(s)ds$$

for almost all $0 \leq t \leq T$, then $\phi(t) = 0$ almost everywhere.

Theorem 1.6.3 (Banach Fixed Point Theorem/Contraction Mapping Principle) Let f be a contraction mapping on a complete metric space. Then there exists a unique $z \in F$ such that $f(z) = z$. The typical case for this theorem is when $f : F \to F$ where F is a closed subset of a Banach space.

Theorem 1.6.4 (Sobolev Imbedding Theorem) [1] Let Ω be a domain in R^n, $j \geq 0$ and $m \geq 1$ be integers and $1 \leq p < \infty$.

i. If either $mp > n$ or $m = n$ and $p = 1$, then

$$W^{j+m,p}(\Omega) \to C_B^j(\Omega)$$

where $C_B^j(\Omega)$ is the space of functions having bounded, continuous derivatives up to order j on Ω.

Moreover,

$$W^{m,p}(\Omega) \to L^q(\Omega)$$

for $p \leq q \leq \infty$.

ii. If $mp = n$, then

$$W^{m,p}(\Omega) \to L^q(\Omega)$$

for $p \leq q < \infty$, where $q \geq p$ is an arbitrary number.

iii. If $mp < n$, then

$$W^{m,p}(\Omega) \to L^q(\Omega)$$

for $p \leq q \leq p^* = np/(n - mp)$.

Remark: In particular, for $n = 1$, $p = 2$ and $s > \frac{1}{2}$, we have $H^s(R) \subset L^\infty$ and moreover, there is some constant d, depending on s, so that $\|u\|_\infty \leq d \|u\|_{H^s}$ for $u \in H^s$.

Chapter 2

Physical Model

In this chapter, we show that the motion of dense lattices is modelled by the higher-order Boussinesq equation when higher order discrete effects are included. It is well-known that Boussinesq type equations like the Boussinesq equation, the improved Boussinesq equation and the modified improved Boussinesq equations occur in the continuum limit description of nonlinear lattices. The higher-order Boussinesq equation to be considered was first derived by Rosenau [11] using the quasicontinuum approximation for dense lattices. In Section 2.1, we rederive the higher-order Boussinesq equation using a similar approach. In Section 2.2, we present the conservation laws.

2.1 Derivation of the Higher Order Boussinesq Equation

In this section, as in [11], we derive a higher order Boussinesq type equation using the principles of lattice dynamics. We first consider the linear case related to vibrations of a harmonic lattice in Section 2.1.1. Then, in Section 2.1.2, we introduce the nonlinear case with vibrations of an anharmonic lattice. In Section 2.1.3, we give the quasicontinuum approximation of the discrete model. Finally, in section 2.1.4, we derive the higher-order Boussinesq equation.

2.1.1 Vibrations of a Harmonic Lattice

We now consider a one-dimensional chain of (equally spaced) particles as a simple model of crystal lattices. We assume that all the particles are identical and that the particles are interconnected by elastic springs. We denote the mass of the particles by m and the interparticle separation by h. The nearest neighbor interactions will be considered here. We label the particles by the number n and denote the displacement of a particle from the equilibrium position by Y_n. Note that Y_n is a function of the time t.

The kinetic energy of a particle is given by $\frac{1}{2}m(\dot{Y_n})^2$ where $\dot{Y_n} \equiv \frac{dY_n}{dt}$. The elastic energy stored in a spring is given by $\frac{\kappa}{2h^2}(Y_n - Y_{n-1})^2$ where κ is a constant related to the interparticle elastic constant (spring constant). The (total) kinetic energy of the discrete system of particles due to the displacements $Y_n(t)$ is

$$T = \sum_n \frac{1}{2}m(\dot{Y_n})^2.$$

The (total) potential energy, the deformation energy, of the discrete system is

$$V = \sum_n \frac{\kappa}{2}(\frac{Y_n - Y_{n-1}}{h})^2$$

Then, the Lagrangian of the discrete system of particles,

$$L = T - V,$$

is a function of the time t, and of the displacements Y_n and velocities $\dot{Y_n}$ of the particles in the system. The equations of motion for the system of particles can be derived using the principle of least action, which states that the motion of the discrete system of particles during the time interval $[t_1, t_2]$ minimizes the action functional

$$\int_{t_1}^{t_2} L dt$$

[7]. The Euler-Lagrange equations associated with the action are given by

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{Y_n}} - \frac{\partial L}{\partial Y_n} = 0. \tag{2.1}$$

A substitution of L into these equations provides the equations describing the dynamics of the discrete system

$$m\ddot{Y_n} = \frac{\kappa}{h^2}(Y_{n+1} - 2Y_n + Y_{n-1}). \tag{2.2}$$

These equations admit plane wave solutions of the type:

$$Y_n = A \exp[i(kx_n - \omega t)], \qquad x_n = nh \tag{2.3}$$

where k denotes a wave number and ω is the frequency. Substituting the plane wave solution, (2.3), into the equations of motion, (2.2), we obtain the (exact) dispersion relation of the harmonic lattice

$$\frac{\omega^2}{c^2} = \frac{4}{h^2}\sin^2(\frac{kh}{2}) \tag{2.4}$$

where $c = (\frac{\kappa}{m})^{\frac{1}{2}}$ is the speed of sound. The phase velocity is defined by $c_p = \frac{\omega}{k}$ and is the velocity of the propagation of the plane wave. We note that the phase velocity depends on wave number, that is, lattice waves are dispersive. The long wavelength limit (i.e. the limit of small wave numbers) implies that $kh << 1$. If the sine function in (2.4) is expanded in the limit, we obtain $\omega^2 = c^2k^2$ which implies that $c_p = c$ for long waves. That is, in the long wavelength limit, the wave propagation is nondispersive and the wave cannot identify the discreteness of the system in which it propagates.

2.1.2 Vibrations of an Anharmonic Lattice

In the preceeding analysis, the deformation energy involved only the terms of second degree in the displacements. This results in the linearized equations of motion. The approximation of the potential energy may not be appropriate for some specific cases, for instance the cases involving large displacements. Therefore, in this part we consider the potential energy for the one-dimensional chain with the nearest neighbor interactions in the form

$$V = \sum_n \kappa W(\frac{Y_n - Y_{n-1}}{h})$$

where W is an arbitrary function. Sometimes we use a decomposition of W into harmonic and anharmonic parts:

$$W(u) = \frac{1}{2}u^2 + G(u)$$

in which we recover the linear case $G \equiv 0$. Then we have

$$W'(u) = u + g(u)$$

where $g(u) = G'(u)$. We assume that the equilibrium potential energy of the discrete system is zero, i.e. $W(0) = 0$ which implies

$$G(0) = 0 \qquad \text{and} \qquad G(u) = \int_0^u g(p)dp.$$

The first-order derivative of the potential energy with respect to Y_n is the negative of the net force acting on the n^{th} particle. We assume that the net force at the equilibrium position is zero, i.e. $W'(0) = 0$ which implies $g(0) = 0$. Therefore, the first nonnegligible term in the Taylor series expansion of W about the equilibrium position is the quadratic term from which we obtain the harmonic approximation.
The Euler-Lagrange equations (2.1) obtained from the principle of least action are also valid for the nonlinear case. A substitution of the Lagrangian $L = T - V$ gives the equations of motion for the discrete system in the form

$$m\ddot{Y}_n = -\frac{\kappa}{h}[W'(\frac{Y_n - Y_{n-1}}{h}) - W'(\frac{Y_{n+1} - Y_n}{h})]$$

Introducing the notation

$$y_n = \frac{Y_n - Y_{n-1}}{h}$$

we can rewrite the equations of motion as

$$\ddot{y}_n = \frac{c^2}{h^2}[W'(y_{n+1}) - 2W'(y_n) + W'(y_{n-1})] \tag{2.5}$$

Note that the dispersion relation for the linearized form of these equations is given by (2.4).

2.1.3 Quasicontinuum Approximation of the Discrete Model

In this subsection, a quasicontinuum approximation of the dynamics of the lattice model is developed. Our aim is to incorporate correctly the leading effects of the discrete system in a continuum description. For this aim, we wish to find a smooth function $u(x,t)$ as an interpolation of the sampling $[x_n = nh, \quad y_n(t)]$ such that $u(x_n,t) = y_n(t)$. Note that the correspondence between the continuous variables resulting from the quasicontinuum approximation to be developed below and the discrete variables of the lattice dynamics will be in the form $n \Rightarrow x$, $Y_n(t) \Rightarrow U(x,t)$ and $y_n(t) \Rightarrow u(x,t) = U_x(x,t)$.
Consider a dense lattice, a lattice where a macroscopic unit length L of the lattice contains a large number of particles. This means that $h << 1$. Taylor series expansions for $(W'(y_{n\mp1}))$ near (x_n,t) give

$$W'(y_{n\mp1}) \equiv W'(u(x_n \mp h,t)) = W'(u)_{(x_n,t)} \mp [\frac{\partial}{\partial x}W'(u)]_{(x_n,t)}h + \frac{1}{2!}[\frac{\partial^2}{\partial x^2}W'(u)]_{(x_n,t)}h^2 + ... \tag{2.6}$$

If we substitute (2.6) into (2.5) and drop the index n, we obtain the following equation of motion for the second-order approximation

$$u_{tt} = c^2[W'(u)]_{xx} + O(h^2).$$

The corresponding linearized equation is

$$u_{tt} = c^2 u_{xx} + O(h^2).$$

For the plane wave solution $u(x,t) = A\ \exp[i(kx - \omega t)]$, the dispersion relation is obtained as $c_p = \frac{\omega}{k} = c$. This dispersion relation obtained from the quasicontinuum system is exactly the same as that obtained from the discrete system in the long wavelength. That is, the wave propagation in the present continuum system is nondispersive as in the long wavelength limit of the discrete system. The discrepancy between the dispersive wave propagation in the discrete system and the nondispersive wave propagation in the continuum system can be eliminated using a higher order approximation in the above derivation.

We now substitute (2.6) into (2.5) and drop the index n, thus we obtain

$$u_{tt} = c^2[W'(u)]_{xx} + \frac{c^2h^2}{12}[W'(u)]_{xxxx} + O(h^4)$$

for the fourth-order approximation. The corresponding linearized equation and its dispersion relation are in the form

$$u_{tt} = c^2 u_{xx} + \frac{c^2h^2}{12} u_{xxxx} + O(h^4) \tag{2.7}$$

and

$$\frac{\omega^2}{c^2k^2} = 1 - \frac{h^2}{12}k^2, \tag{2.8}$$

respectively. As it is expected, the present higher-order approximation provides a remedy for the above-mentioned discrepancy. In other words, the wave propagation in the continuum system is dispersive now.

We note that the dispersion relation (2.8) approximates the exact dispersion relation of the lattice up to $O(h^4)$. In other words, the two term Taylor series expansion of the exact dispersion relation gives (2.8). There exists still a discrepancy between the exact dispersion relation and (2.8). Recall that the exact dispersion relation is bounded for every k. However, the dispersion relation (2.8) becomes unbounded for large k. This discrepancy is eliminated by replacing the polynomial function in (2.8) by a rational function. We rewrite the dispersion relation (2.8) up to $O(h^4)$ in the form

$$\frac{\omega^2}{c^2k^2} = \frac{1}{1+\frac{h^2k^2}{12}}. \tag{2.9}$$

Note that the new dispersion relation (2.9) is bounded for every k and that it is equivalent to (2.8) for small k. The linear wave equation corresponding to the dispersion equation (2.9) is given by

$$u_{tt} = c^2 u_{xx} + \frac{h^2}{12} u_{xxtt}. \tag{2.10}$$

The Cauchy problem for this equation is well posed for all times while the Cauchy problem for (2.7) is ill posed.

The rest of this is devoted to the extension of the above results obtained for the linearized equations to the nonlinear case. We now assume that, with the use of the quasicontinuum approximation, the equations of motion for the discrete system, (2.5), can be written as

$$u_{tt} = c^2 L D^2 W'(u) \tag{2.11}$$

with

$$LD^2 = \frac{4}{h^2}\sinh^2\left(\frac{hD}{2}\right)$$
$$D \equiv \frac{\partial}{\partial x}, \qquad L = 1 + \frac{h^2}{12}D^2 + \frac{2h^4}{6!}D^4 + ...$$

As an application of this approach we first consider the second order approximation. That is, we restrict our attention to the invertible operators L_2 and L_2^{-1} as a second-order approximations of L and L^{-1}:

$$L_2 = 1 + \frac{h^2}{12}D^2, \qquad L_2^{-1} = 1 - \frac{h^2}{12}D^2.$$

An application of L_2^{-1} on (2.11) gives

$$L_2^{-1} u_{tt} = c^2 D^2 W'(u)$$

or explicitly

$$u_{tt} = c^2 [W'(u)]_{xx} + \frac{h^2}{12} u_{xxtt}. \tag{2.12}$$

The linearized form of this equation and the corresponding dispersion relation are given by (2.10) and (2.9), respectively. The above equation (2.12), which can be also written as

$$u_{tt} - c^2 u_{xx} - \frac{h^2}{12} u_{xxtt} = c^2 [g(u)]_{xx}, \tag{2.13}$$

is the so-called generalized "improved" Boussinesq equation ([2], [3]). The terms "improved" and "generalized" can be explained as follows. The equation

$$u_{tt} - c^2 u_{xx} - \frac{c^2 h^2}{12} u_{xxxx} = c^2 [g(u)]_{xx} \tag{2.14}$$

is the so-called generalized Boussinesq equation ([2], [3]). We have already shown that the linear dispersion relation for (2.13) leads to a nonphysical instability of linear waves. The term "improved" means that (2.13) does not admit such an instability of linear waves. Usually, the above evolution equations appear in the literature with the quadratic nonlinearities:

$$u_{tt} - c^2 u_{xx} - \frac{c^2 h^2}{12} u_{xxxx} = c^2 [u^2]_{xx} \tag{2.15}$$

and

$$u_{tt} - c^2 u_{xx} - \frac{h^2}{12} u_{xxtt} = c^2 [u^2]_{xx} \tag{2.16}$$

Equations (2.15) and (2.16) are called the Boussinesq equation (or the "bad" Boussinesq equation or the "ill-posed" Boussinesq equation) and the improved Boussinesq equation (or the "good" Boussinesq equation or the "well-posed" Boussinesq equation), respectively [15]. The term "generalized" means that the evolution equation is not restricted to the quadratic nonlinearity and that it involves an arbitrary function in the nonlinear term.

2.1.4 The Higher-Order Boussinesq Equation

In this subsection, we derive a nonlinear evolution equation such that its linear dispersion relation approximates the exact dispersion relation of the discrete system with a fourth order accuracy for every k. For this aim, we first consider the three-term Taylor series expansion of the exact dispersion relation (2.4) in the form

$$\frac{\omega^2}{c^2 k^2} = 1 - \frac{h^2}{12} k^2 + \frac{h^4}{360} k^4. \tag{2.17}$$

We approximate this polynomial function by the following rational function

$$\frac{\omega^2}{c^2k^2} = \frac{1}{1 + b_1k^2 + b_2k^4} \tag{2.18}$$

where b_1 and b_2 are constants. The coefficients b_1 and b_2 are determined by comparing the Taylor expansion of the rational function by the polynomial expansion (2.17). The taylor expansion of (2.18) up to $O(h^4)$ is given by

$$\frac{\omega^2}{c^2k^2} = 1 - b_1k^2 + (b_1^2 - b_2)k^4. \tag{2.19}$$

A comparison of (2.19) with (2.17) we obtain

$$b_1 = \frac{h^2}{12}, \qquad b_2 = \frac{h^4}{240}.$$

We now consider the invertible operators L_4 and L_4^{-1} as a fourth-order approximations of L and L^{-1}:

$$L_4 = 1 + \frac{h^2}{12}D^2 + \frac{h^4}{360}D^4, \qquad L_4^{-1} = 1 - \frac{h^2}{12}D^2 + \frac{h^4}{240}D^4.$$

An application of L_4^{-1} on (2.11) gives

$$L_4^{-1}u_{tt} = c^2D^2W'(u)$$

or explicitly

$$u_{tt} = c^2[W'(u)]_{xx} + \frac{h^2}{12}u_{xxtt} - \frac{h^4}{240}u_{xxxxtt}.$$

The linearized form of this equation is

$$u_{tt} = c^2u_{xx} + \frac{h^2}{12}u_{xxtt} - \frac{h^4}{240}u_{xxxxtt}.$$

The corresponding dispersion relation which is given by (2.18), is bounded for every k as in the exact dispersion relation of the discrete system. The above nonlinear evolution equation can be written as

$$u_{tt} - c^2u_{xx} - \frac{h^2}{12}u_{xxtt} + \frac{h^4}{240}u_{xxxxtt} = c^2[g(u)]_{xx}.$$

Henceforth this equation will be called the higher-order Boussinesq equation. If we introduce the following dimensionless variables

$$\xi = \frac{\sqrt{12}x}{h}, \qquad \tau = \frac{\sqrt{12}ct}{h}$$

the higher-order Boussinesq equation takes the following form

$$u_{\tau\tau} - u_{\xi\xi} - u_{\xi\xi\tau\tau} + \frac{144}{240}u_{\xi\xi\xi\xi\tau\tau} = [g(u)]_{\xi\xi}.$$

Replacing (ξ, τ) by (x, t), we rewrite the equation in the form

$$u_{tt} - u_{xx} - u_{xxtt} + \beta u_{xxxxtt} = g(u)_{xx}$$

where $\beta = \frac{144}{240} > 0$. Henceforth, the above form of the higher-order Boussinesq equation will be used in the rest of this study.

2.2 Conservation Laws

We will study the conservation laws for the higher order Boussinesq equation

$$u_{tt} - u_{xx} - u_{xxtt} + \beta u_{xxxxtt} = g(u)_{xx}. \tag{2.20}$$

We substitute $u = U_x$ into (2.20) and then, integrate on R with respect to x with the assumptions $U, U_x, U_t, U_{xx}, U_{tt}, U_{xt}... \to 0$ as $x \to \mp\infty$, to obtain

$$U_{tt} - U_{xx} - U_{xxtt} + \beta U_{xxxxtt} = [g(U_x]_x. \tag{2.21}$$

A Lagrangian density function for (2.21) is

$$L = \frac{1}{2}(U_t)^2 - W(U_x) + \frac{1}{2}(U_{xt})^2 + \frac{\beta}{2}(U_{xxt})^2$$

where $g(u) = -u + W'(u)$.

The equation is derived as the Euler-Lagrange equation of the functional (action)

$$S[U] = \int_{t_1}^{t_2}\int_{-\infty}^{\infty} L(U_t, U_x, U_{xt}, U_{xxt})dxdt. \tag{2.22}$$

Our aim is to state Noether's theorem for (2.22) and then to derive the conservation laws.

We first introduce the notation

$$\xi_1 = t, \quad \xi_2 = x, \quad v_1 = U, \quad v_2 = U_x, \quad v_3 = U_{xx}.$$

Then the above functional becomes

$$S[v_1, v_2, v_3] = \int\int L(\frac{\partial v_1}{\partial \xi_1}, \frac{\partial v_2}{\partial \xi_1}, \frac{\partial v_3}{\partial \xi_1}, \frac{\partial v_1}{\partial \xi_2})d\xi_1 d\xi_2$$

where

$$L = \frac{1}{2}(\frac{\partial v_1}{\partial \xi_1})^2 + \frac{1}{2}(\frac{\partial v_2}{\partial \xi_1})^2 + \frac{\beta}{2}(\frac{\partial v_3}{\partial \xi_1})^2 - W(\frac{\partial v_1}{\partial \xi_2}). \tag{2.23}$$

For simplicity, we introduce the vectors,

$$\xi = (\xi_1, \xi_2), \qquad v = (v_1, v_2, v_3)$$

and interprete ∇v as the tensor with components $\frac{\partial v_j}{\partial \xi_i}$, $(i = 1, 2; \quad j = 1, 2, 3)$. Using the vector notation, we can rewrite the functional in the form

$$S[v] = \int_\Omega L(\xi, v, \nabla v) d\xi \tag{2.24}$$

where $d\xi = d\xi_1 d\xi_2$ and Ω is the space-time region, i.e. $\Omega = [0, T] \times (-\infty, \infty)$. Now, consider a transformation

$$\xi_i^* = \Phi_i(\xi, v, \nabla v), \qquad (i = 1, 2) \tag{2.25}$$

$$v_j^* = \Psi_j(\xi, v, \nabla v), \qquad (j = 1, 2, 3) \tag{2.26}$$

Definition 2.2.1 [7] The functional (2.24) is said to be invariant under the transformation (2.25), (2.26) if $S[v^*] = S[v]$, i.e. if

$$\int_{\Omega^*} L(\xi^*, v^*, \nabla^* v^*) d\xi^* = \int_\Omega L(\xi, v, \nabla v) d\xi.$$

Theorem 2.2.2 (Noether) [7] Assume Ω is an arbitrary region. If the functional

$$S[v] = \int_\Omega L(\xi, v, \nabla v) d\xi$$

is invariant under the family of transformations

$$\xi_i^* = \Phi_i(\xi, v, \nabla v; \epsilon) \sim \xi_i + \epsilon \varphi_i(\xi, v, \nabla v) \qquad (i = 1, 2) \tag{2.27}$$

$$v_j^* = \Psi_j(\xi, v, \nabla v; \epsilon) \sim v_j + \epsilon \psi_j(\xi, v, \nabla v) \qquad (j = 1, 2, 3) \tag{2.28}$$

then

$$\sum_{i=1}^{2} \frac{\partial}{\partial \xi_i}[\sum_{j=1}^{3} \frac{\partial L}{\partial(\frac{\partial v_j}{\partial \xi_i})} \widetilde{\psi_j} + \varphi_i L] = 0 \tag{2.29}$$

on each extremal surface of $S[v]$, where

$$\widetilde{\psi_j} = \psi_j - \sum_{l=1}^{2} \frac{\partial v_j}{\partial \xi_l} \varphi_l. \tag{2.30}$$

Using equation (2.23), the identity (2.29) can be rewritten in the form

$$\frac{\partial}{\partial \xi_1}\left(\frac{\partial v_1}{\partial \xi_1}\widetilde{\psi_1} + \frac{\partial v_2}{\partial \xi_1}\widetilde{\psi_2} + \beta\frac{\partial v_3}{\partial \xi_1}\widetilde{\psi_3} + \varphi_1 L\right) + \frac{\partial}{\partial \xi_2}\left(-W'\left(\frac{\partial v_1}{\partial \xi_2}\right)\widetilde{\psi_1} + \varphi_2 L\right) = 0. \quad (2.31)$$

If we introduce the notation

$$\delta t = \epsilon\varphi_1, \quad \delta x = \epsilon\varphi_2, \quad \delta U = \epsilon\psi_1, \quad \delta U_x = \epsilon\psi_2, \quad \delta U_{xx} = \epsilon\psi_3$$

and write (2.30) in terms of the original variables, we obtain

$$\epsilon\widetilde{\psi_1} = -U_t\delta t - U_x\delta x + \delta U \quad (2.32)$$

$$\epsilon\widetilde{\psi_2} = -U_{xt}\delta t - U_{xx}\delta x + \delta U_x \quad (2.33)$$

$$\epsilon\widetilde{\psi_3} = -U_{xxt}\delta t - U_{xxx}\delta x + \delta U_{xx}. \quad (2.34)$$

Then the equation (2.31) becomes

$$P_t + Q_x = 0, \quad (2.35)$$

where

$$\begin{aligned} P = &-[(U_t)^2 + (U_{xt})^2 + \beta(U_{xxt})^2 - L]\delta t \\ &- (U_tU_x + U_{xt}U_{xx} + \beta U_{xxt}U_{xxx})\delta x \\ &+ U_t\delta U + U_{xt}\delta U_x + \beta U_{xxt}\delta U_{xx} \\ Q = &\ (U_t\delta t + U_x\delta x - \delta U)W'(U_x) + L\delta x. \end{aligned}$$

Recalling that U_t, U_x and the higher-order derivatives tend to zero as $x \to \mp\infty$, this implies $Q \to 0$ as $x \to \mp\infty$. Then, if we integrate equation (2.35) with respect to x from $-\infty$ to ∞, we obtain a conservation law

$$\frac{d}{dt}\int_{-\infty}^{\infty} P dx = 0 \quad (2.36)$$

or

$$\int_{-\infty}^{\infty} P dx = c \quad (2.37)$$

where c is a constant. Here P is called the density of the conserved quantity. If we introduce the notation

$$I(t) = \int_{-\infty}^{\infty} P dx$$

for the conserved quantity, the conservation law implies

$$I(t) = I(0).$$

Noether's theorem guarantees that if the action is invariant under a transformation, there exists a conservation law corresponding to the transformation. It also provides a general method of deriving conservation laws as discussed above for the higher-order Boussinesq equation. We now derive the three conservation laws of the higher-order Boussinesq equation which correspond to the conservation of mass, the conservation of energy and the conservation of momentum.

2.2.1 Conservation of Mass

The action functional of the higher-order Boussinesq equation is invariant under function translations, i.e. under the transformation $U^* = U + \epsilon$ where ϵ is arbitrary. Then we have

$$\delta t = \delta x = \delta U_x = \delta U_{xx} = 0, \qquad \delta U = \epsilon.$$

Equation (2.35) reduces to

$$(P_1)_t + (Q_1)_x = 0$$

with

$$P_1 = U_t, \qquad Q_1 = -W'(U_x).$$

Then the conserved quantity related to the mass is

$$I_1 = \int_{-\infty}^{\infty} P_1 dx.$$

2.2.2 Conservation of Energy

The action is invariant under time translations, i.e. under the transformation $t^* = t + \epsilon$. Then we have

$$\delta x = \delta U = \delta U_x = \delta U_{xx} = 0, \qquad \delta t = \epsilon.$$

Equation (2.35) becomes

$$(P_2)_t + (Q_2)_x = 0$$

with

$$P_2 = (U_t)^2 + (U_{xt})^2 + \beta(U_{xxt})^2 - L$$
$$= \frac{1}{2}[(U_t)^2 + (U_{xt})^2 + \beta(U_{xxt})^2] + W(U_x)$$
$$Q_2 = -U_t W^{'}(U_x)$$

where P_2 denotes the energy density. Equation (2.35) implies that the energy

$$I_2 = \int_{-\infty}^{\infty} P_2 dx$$

is conserved.

2.2.3 Conservation of Momentum

The action is invariant under space translations, i.e. under the transformation $x^* = x + \epsilon$. Then we have

$$\delta t = \delta U = \delta U_x = \delta U_{xx} = 0, \qquad \delta x = \epsilon.$$

We get from equation (2.35) that

$$(P_3)_t + (Q_3)_x = 0$$

with

$$P_3 = U_t U_x + U_{xt} U_{xx} + \beta U_{xxt} U_{xxx} \tag{2.38}$$
$$Q_3 = -U_x W^{'}(U_x) - L \tag{2.39}$$

where P_3 denotes the momentum density. Then the equation (2.36) corresponds to the conservation of the momentum defined by

$$I_3 = \int_{-\infty}^{\infty} P_3 dx.$$

Note that we can write I_3 explicitly in the form

$$I_3 = \int_{-\infty}^{\infty} U_x(U_t - U_{xxt} + \beta U_{xxxxt}) dx$$

by successive integration by parts. The conserved quantities I_1, I_2, I_3 derived above are equivalent to those given in equation (55) of [11].

Chapter 3

Cauchy Problem for the Generalized Double Dispersion Equation

In this chapter, we will study the Cauchy problem

$$u_{tt} - u_{xx} - u_{xxtt} + u_{xxxx} = g(u)_{xx}, \quad x \in R, \quad t > 0, \tag{3.1}$$

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x). \tag{3.2}$$

Here $\varphi(x)$ and $\psi(x)$ are the given initial functions and g is a given real-valued function, satisfying $g(0) = 0$.

This problem has been studied by Wang and Chen [2]. We will present here some of the results from [2]. This will also serve as a guide for the next chapter, where we study another Boussinesq type equation with a similar approach.

We introduced nonlinear evolution equations in Chapter 1, and mentioned some methods for proving global existence and uniqueness of their solutions. In this chapter, we use the standard method which we have explained. In Section 3.1, we solve the linear version of this problem, where $g(u(x,t))_{xx}$ is replaced by $(h(x,t))_{xx}$, and give estimates for the solution. We give the proof of existence and uniqueness of the local H^s- solution of the Cauchy problem (3.1), (3.2) by the *contraction mapping principle* in Section 3.2. Finally, in Section 3.3, we prove the existence and uniqueness of the global solution in a certain case.

3.1 Cauchy Problem for the Linearized Equation

The main theorem in this section is given below. Linear version of the Cauchy problem (3.1), (3.2) is described and the properties of the solution are proved.

Theorem 3.1.1 ([2], [12], [13]) Let $s \in R$. For any $T > 0$, suppose that $\varphi \in H^s$, $\psi \in H^{s-1}$ and $h \in L^1([0,T]; H^{s-1})$, then the Cauchy problem for the linear equation

$$u_{tt} - u_{xx} - u_{xxtt} + u_{xxxx} = (h(x,t))_{xx}, \quad x \in R, \quad t > 0, \tag{3.3}$$

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x) \tag{3.4}$$

has a unique solution $u \in C([0,T], H^s) \cap C^1([0,T], H^{s-1})$ and there is the estimation

$$\|u(t)\|_{H^s} + \|u_t(t)\|_{H^{s-1}} \leq 4(1+T)(\|\varphi\|_{H^s} + \|\psi\|_{H^{s-1}} + \int_0^t \|h(\tau)\|_{H^{s-1}} \, d\tau), \quad 0 \leq t \leq T. \tag{3.5}$$

Proof: Applying Fourier transform in (3.3) with respect to x, we get:

$$(1+\xi^2)\hat{u}_{tt} + \xi^2(1+\xi^2)\hat{u} = -\xi^2 \hat{h}(\xi,t).$$

Dividing by $(1+\xi^2)$, the problem becomes,

$$\hat{u}_{tt} + \xi^2 \hat{u} = -\frac{\xi^2}{1+\xi^2}\hat{h}(\xi,t) \tag{3.6}$$

$$\hat{u}(\xi,0) = \hat{\varphi}(\xi), \quad \hat{u}_t(\xi,0) = \hat{\psi}(\xi). \tag{3.7}$$

We observe that the partial differential equation turns into a non-homogenous ordinary differential equation with parameter ξ. The solution of this ordinary differential equation is of the form:

$$\hat{u}(\xi,t) = \hat{u}_{hom}(\xi,t) + \hat{u}_{part}(\xi,t)$$

where

$$\hat{u}_{hom}(\xi,t) = c_1(\xi)\cos(t\xi) + c_2(\xi)\sin(t\xi)$$

denotes the solution of the homogeneous equation.
For a particular solution $\hat{u}_{part}$ of (3.6), we use the variation of parameters method:

$$\hat{u}_{part}(\xi,t) = \hat{u}_1(\xi,t)\cos(t\xi) + \hat{u}_2(\xi,t)\sin(t\xi).$$

Taking derivative with respect to t,

$$(\hat{u}_{part})_t = -\xi\sin(t\xi)\hat{u}_1 + \xi\cos(t\xi)\hat{u}_2 + \cos(t\xi)(\hat{u}_1)_t + \sin(t\xi)(\hat{u}_2)_t.$$

In this method, we set

$$\cos(t\xi)(\hat{u}_1)_t + \sin(t\xi)(\hat{u}_2)_t = 0$$

and if we use this particular solution in (3.6), we see that

$$-\xi\sin(t\xi)(\hat{u}_1)_t+\xi\cos(t\xi)(\hat{u}_2)_t=-\frac{\xi^2}{1+\xi^2}\hat{h}(\xi,t).$$

Solving these equations, we find

$$(\hat{u}_1)_t \quad = \quad \sin(t\xi)\frac{\xi}{1+\xi^2}\hat{h}(\xi,t)$$

and

$$(\hat{u}_2)_t \quad = \quad -\cos(t\xi)\frac{\xi}{1+\xi^2}\hat{h}(\xi,t).$$

Integrating them over $[0,t]$,

$$\hat{u}_1 \quad = \quad \int_0^t \sin(\tau\xi)\frac{\xi}{1+\xi^2}\hat{h}(\xi,\tau)d\tau$$

and

$$\hat{u}_2 \quad = \quad -\int_0^t \cos(\tau\xi)\frac{\xi}{1+\xi^2}\hat{h}(\xi,\tau)d\tau.$$

Combining all of these,

$$\hat{u}_{part}(\xi,t)=-\int_0^t\frac{\xi\sin((t-\tau)\xi)}{1+\xi^2}\hat{h}(\xi,\tau)d\tau.$$

So,

$$\hat{u}(\xi,t)=c_1(\xi)\cos(t\xi)+c_2(\xi)\sin(t\xi)-\int_0^t\frac{\xi\sin((t-\tau)\xi)}{1+\xi^2}\hat{h}(\xi,\tau)d\tau.$$

Using the initial conditions (3.7), $c_1(\xi)=\hat{\varphi}(\xi)$, $c_2(\xi)=\frac{\hat{\psi}(\xi)}{\xi}$ and the solution is given by,

$$\hat{u}(\xi,t)=\hat{\varphi}(\xi)\cos(t\xi)+\frac{\hat{\psi}(\xi)}{\xi}\sin(t\xi)-\int_0^t\frac{\xi\sin((t-\tau)\xi)}{1+\xi^2}\hat{h}(\xi,\tau)d\tau. \tag{3.8}$$

We obtain the estimation for H^s norm of $u(x,t)$ by recalling the norm related to the Fourier transform:

$$\|w\|_{H^s}^2=\left\|(1+\xi^2)^{\frac{s}{2}}\hat{w}(\xi)\right\|^2=\int_R(1+\xi^2)^s\left|\hat{w}(\xi)\right|^2d\xi.$$

If we write $u=v_1+v_2+v_3$, where

$$\begin{aligned}
\hat{v}_1 &= \hat{\varphi}(\xi)\cos(t\xi)\\
\hat{v}_2 &= \frac{\hat{\psi}(\xi)}{\xi}\sin(t\xi)\\
\hat{v}_3 &= -\int_0^t\frac{\xi\sin((t-\tau)\xi)}{1+\xi^2}\hat{h}(\xi,\tau)d\tau,
\end{aligned}$$

then

$$\|u\|_{H^s} \le \|v_1\|_{H^s} + \|v_2\|_{H^s} + \|v_3\|_{H^s}\,. \tag{3.9}$$

Hence, we estimate the terms separately.

For the first term,

$$\|v_1\|_{H^s}^2 = \int_R (1+\xi^2)^s |\hat{\varphi}(\xi)|^2 \cos^2(t\xi)d\xi \le \int_R (1+\xi^2)^s |\hat{\varphi}(\xi)|^2 d\xi = \|\varphi\|_{H^s}^2\,.$$

For the second term, we observe that if we use the inequality $\left|\frac{\sin(t\xi)}{\xi}\right| \le \frac{1}{\xi}$, a singularity occurs at $\xi = 0$ whereas if we use $\left|\frac{\sin(t\xi)}{\xi}\right| \le \frac{|\xi t|}{|\xi|} = t$, we lose ξ and it affects the estimate we evaluate. Taking these differences into account, we divide the integral into two parts:

$$\begin{aligned}
\|v_2\|_{H^s}^2 &= \int_R (1+\xi^2)^s \frac{\sin^2(t\xi)}{\xi^2}\left|\hat{\psi}(\xi)\right|^2 d\xi \\
&= \int_{|\xi|<1} (1+\xi^2)^s \frac{\sin^2(t\xi)}{\xi^2}\left|\hat{\psi}(\xi)\right|^2 d\xi + \int_{|\xi|\ge 1} (1+\xi^2)^s \frac{\sin^2(t\xi)}{\xi^2}\left|\hat{\psi}(\xi)\right|^2 d\xi \\
&\le t^2 \int_{|\xi|<1} (1+\xi^2)^s \left|\hat{\psi}(\xi)\right|^2 d\xi + \int_{|\xi|\ge 1} (1+\xi^2)^s \frac{1}{\xi^2}\left|\hat{\psi}(\xi)\right|^2 d\xi.
\end{aligned}$$

For $|\xi| < 1$, $1+\xi^2 \le 2$, and for $|\xi| \ge 1$, we have $\frac{1}{\xi^2} \le 1$, so

$$\begin{aligned}
\|v_2\|_{H^s}^2 &\le 2t^2 \int_{|\xi|<1} (1+\xi^2)^{s-1} \left|\hat{\psi}(\xi)\right|^2 d\xi + 2\int_{|\xi|\ge 1} (1+\xi^2)^{s-1} \left|\hat{\psi}(\xi)\right|^2 d\xi \\
&\le 2(1+t^2) \int_R (1+\xi^2)^{s-1} \left|\hat{\psi}(\xi)\right|^2 d\xi = 2(1+t^2)\,\|\psi\|_{H^{s-1}}^2\,.
\end{aligned}$$

H^s norm of the third term in (3.8) can be evaluated similarly as follows:

$$\|v_3\|_{H^s} = \left\| - \int_0^t (1+\xi^2)^{\frac{s}{2}} \frac{\xi \sin((t-\tau)\xi)}{1+\xi^2} \hat{h}(\xi,\tau)d\tau \right\|.$$

From Minkowski's inequality for integrals [4],

$$\|v_3\|_{H^s} \le \int_0^t \left\| (1+\xi^2)^{\frac{s}{2}} \frac{\xi \sin((t-\tau)\xi)}{1+\xi^2} \hat{h}(\xi,\tau) \right\| d\tau = \int_0^t \|J(\tau)\|\, d\tau.$$

Now,

$$\begin{aligned}
\|J(\tau)\|^2 &= \int_{|\xi|<1} (1+\xi^2)^s \frac{\xi^2 \sin^2((t-\tau)\xi)}{(1+\xi^2)^2} \left|\hat{h}(\xi,\tau)\right|^2 d\xi \\
&+ \int_{|\xi|\ge 1} (1+\xi^2)^s \frac{\xi^2 \sin^2((t-\tau)\xi)}{(1+\xi^2)^2} \left|\hat{h}(\xi,\tau)\right|^2 d\xi.
\end{aligned}$$

Similar arguments mentioned before show that

$$\begin{aligned}\|J(\tau)\|^2 &\leq \int_{|\xi|<1}(t-\tau)^2(1+\xi^2)^s(\frac{\xi^2}{1+\xi^2})^2\left|\hat{h}(\xi,\tau)\right|^2 d\xi \\ &+ \int_{|\xi|\geq 1}(1+\xi^2)^s\frac{\xi^2}{(1+\xi^2)^2}\left|\hat{h}(\xi,\tau)\right|^2 d\xi.\end{aligned}$$

Since $\frac{\xi^2}{(1+\xi^2)^2} \leq 1$, and $0 \leq \tau \leq t$ implies $(t-\tau)^2 \leq t^2$

$$\|J(\tau)\|^2 \leq 2(t^2+1)\int_R (1+\xi^2)^{s-1}\left|\hat{h}(\xi,\tau)\right|^2 d\xi.$$

So,

$$\|\hat{v}_3\|_{H^s} \leq \sqrt{2}(1+t)\int_0^t \|h(\tau)\|_{H^{s-1}}\, d\tau.$$

From (3.9) and summing up the estimates, we get

$$\|u(t)\|_{H^s} \leq \|\varphi\|_{H^s} + \sqrt{2}(1+t)\|\psi\|_{H^{s-1}} + \sqrt{2}(1+t)\int_0^t \|h(\tau)\|_{H^{s-1}}\, d\tau. \tag{3.10}$$

We now obtain the estimate for u_t. Taking derivative with respect to t, we derive from (3.8)

$$\hat{u}_t(\xi,t) = -\xi\sin(t\xi)\hat{\varphi}(\xi) + \cos(t\xi)\hat{\psi}(\xi) - \int_0^t \cos((t-\tau)\xi)\frac{\xi^2}{1+\xi^2}\hat{h}(\xi,\tau)d\tau. \tag{3.11}$$

Now, we consider H^{s-1} norm of u_t. Similar to what we did above, since

$$\begin{aligned}\left\|-(1+\xi^2)^{\frac{s-1}{2}}\xi\sin(t\xi)\hat{\varphi}(\xi)\right\|^2 &= \int_R (1+\xi^2)^{s-1}\xi^2\sin^2(t\xi)\left|\hat{\varphi}(\xi)\right|^2 d\xi \\ &\leq \int_R (1+\xi^2)^s\left|\hat{\varphi}(\xi)\right|^2 d\xi = \|\varphi\|^2_{H^s},\end{aligned}$$

$$\int_R (1+\xi^2)^{s-1}\cos^2(t\xi)\left|\hat{\psi}(\xi)\right|^2 d\xi \leq \int_R (1+\xi^2)^{s-1}\left|\hat{\psi}(\xi)\right|^2 d\xi = \|\psi\|^2_{H^{s-1}},$$

$$\begin{aligned}(\int_R (1+\xi^2)^{s-1}\cos^2((t-\tau)\xi)(\frac{\xi^2}{1+\xi^2})^2\left|\hat{h}(\xi,\tau)\right|^2 d\xi)^{\frac{1}{2}} &\leq (\int_R (1+\xi^2)^{s-1}\left|\hat{h}(\xi,\tau)\right|^2 d\xi)^{\frac{1}{2}} \\ &= \|h\|_{H^{s-1}},\end{aligned}$$

we get

$$\|u_t(t)\|_{H^{s-1}} \leq \|\varphi\|_{H^s} + \|\psi\|_{H^{s-1}} + \int_0^t \|h(\tau)\|_{H^{s-1}}\, d\tau. \tag{3.12}$$

Therefore, estimation (3.5) holds. □

3.2 Local Existence for the Nonlinear Problem

In this section, our aim is to prove the local existence and uniqueness for the Cauchy problem (3.1), (3.2). By local existence, we mean the existence of the solution in small time interval. We first define a suitable function space.
For $s > \frac{1}{2}$, $\varphi \in H^s$, $\psi \in H^{s-1}$, and fixed $T > 0$, we consider the Banach space

$$X(T) = \left\{ u \in C([0,T], H^s) \cap C^1([0,T], H^{s-1}) \right\} \tag{3.13}$$

which is endowed with the norm

$$\|u\|_{X(T)} = \max_{t\in[0,T]} (\|u(t)\|_{H^s} + \|u_t(t)\|_{H^{s-1}}). \tag{3.14}$$

The Sobolev imbedding theorem [1] implies that in R, for $s > \frac{1}{2}$, $H^s \subset L^\infty$ with $\|.\|_\infty \leq d\,\|.\|_{H^s}$. Thus, for $u \in X(T)$, $u \in C([0,T], L^\infty)$ and $\|u(t)\|_\infty \leq d\,\|u(t)\|_{H^s}$.
For some constant $A > 0$ that we will later determine, we let

$$Y(T) = \{u \in X(T) \mid \|u\|_{X(T)} \leq A\}. \tag{3.15}$$

$Y(T)$ is a closed subset of $X(T)$.
For $\omega \in Y(T)$, we consider the problem

$$u_{tt} - u_{xx} - u_{xxtt} + u_{xxxx} = g(\omega)_{xx} \tag{3.16}$$

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x). \tag{3.17}$$

We observe that with $g(\omega(x,t)) = h(x,t)$, this problem reduces to the problem given by (3.3), (3.4). Thus, Theorem 3.1.1 can be applied. We let $\mathcal{S}(\omega) = u(x,t)$, where $u(x,t)$ is the unique solution of (3.16),(3.17). Here $\mathcal{S}$ denotes the map which carries ω into the unique solution of (3.16),(3.17). Our aim is to show that $\mathcal{S}$ has a unique fixed point in $Y(T)$ for appropriately chosen T and A.
For $\omega \in Y(T)$ (i.e. $\|\omega\|_{X(T)} \leq A$) and $u = \mathcal{S}(\omega)$, we have some problems that have to be solved:

1. What is the range of $Y(T)$ under the map $\mathcal{S}(\omega)$?
2. How can we obtain suitable estimates on $\|\mathcal{S}(\omega)\|_{X(T)}$?
3. Is $\mathcal{S}(\omega)$ a contraction mapping?

As in [2], we need the following two lemmas to control the non-linear term.

Lemma 3.2.1 [3] Assume that $f \in C^k(R)$, $f(0) = 0$, $u \in H^s \cap L^\infty$ and $k = [s]+1$, where $s \geq 0$. Then we have

$$\|f(u)\|_{H^s} \leq K_1(M)\|u\|_{H^s}$$

if $\|u\|_\infty \leq M$, where $K_1(M)$ is a constant dependent on M.

Lemma 3.2.2 [3] Assume that $f \in C^k(R)$, $u, v \in H^s \cap L^\infty$ and $k = [s]+1$, where $s \geq 0$. Then we have

$$\|f(u) - f(v)\|_{H^s} \leq K_2(M)\|u - v\|_{H^s}$$

if $\|u\|_\infty \leq M$, $\|v\|_\infty \leq M$, where $K_2(M)$ is a constant dependent on M.

Lemma 3.2.3 [2] Assume that $s > \frac{1}{2}$, $\varphi \in H^s$, $\psi \in H^{s-1}$ and $g \in C^{[s]+1}(R)$, then for T sufficiently small $\mathcal{S}$ is a contractive mapping from $Y(T)$ into itself .

Proof: Let $\omega \in Y(T)$ be given. Then, for $t \in [0, T]$

$$\|\omega(t)\|_\infty \leq d\,\|\omega(t)\|_{H^s} \leq d\,\|\omega\|_{X(T)} \leq dA.$$

By (1.9) and Lemma 3.2.1,

$$\|g(\omega(t))\|_{H^{s-1}} \leq \|g(\omega(t))\|_{H^s} \leq K_1(dA)\,\|\omega(t)\|_{H^s}$$

where $K_1(dA)$ is a constant dependent on d and A.
Letting $h(x,t) = g(\omega(x,t))$, it follows from Theorem 3.1.1 that the solution $u = \mathcal{S}(\omega)$ of problem (3.16), (3.17) belongs to $C([0,T], H^s) \cap C^1([0,T], H^{s-1})$ and

$$\|u(t)\|_{H^s} + \|u_t(t)\|_{H^{s-1}} \leq 4(1+T)(\|\varphi\|_{H^s} + \|\psi\|_{H^{s-1}} + \int_0^t \|g(\omega(\tau))\|_{H^{s-1}}\,d\tau).$$

So,

$$\begin{aligned}
\|\mathcal{S}(\omega)\|_{X(T)} &= \max_{t\in[0,T]}(\|u(t)\|_{H^s} + \|u_t(t)\|_{H^{s-1}}) \\
&\leq 4(1+T)(\|\varphi\|_{H^s} + \|\psi\|_{H^{s-1}}) + 4(1+T)T(\max_{t\in[0,T]}\|g(\omega(t))\|_{H^{s-1}}) \\
&\leq 4(1+T)(\|\varphi\|_{H^s} + \|\psi\|_{H^{s-1}}) + 4(1+T)TK_1(dA)\,\|\omega\|_{X(T)} \\
&\leq 4(1+T)(\|\varphi\|_{H^s} + \|\psi\|_{H^{s-1}}) + 4(1+T)TK_1(dA)A.
\end{aligned}$$

Call $\|\varphi\|_{H^s} + \|\psi\|_{H^{s-1}} = a$. In order to prove the lemma, we need to show that $\|\mathcal{S}(\omega)\|_{X(T)} \leq A$ so that $\mathcal{S}(Y(T)) \subset Y(T)$, i.e.

$$4(1+T)a + 4(1+T)TK_1(dA)A \leq A.$$

We observe that our constant A is related to a. This means that the setting of $Y(T)$ depends on $\|\varphi\|_{H^s} + \|\psi\|_{H^{s-1}}$. We try to determine A as a multiple of a, i.e. $A = ka$. We want

$$4(1+T)a + 4(1+T)TK_1(dka)ka \leq ka$$

or

$$4a + 4aT[1 + kK_1(dka) + kTK_1(dka)] \leq ka.$$

By choosing $k = 8$ and T small enough to have $T[1 + kK_1(dka) + kTK_1(dka)] \leq 1$, we get $\|\mathcal{S}(\omega)\|_{X(T)} \leq 8a = A$. Hence, for these values of A and T, $\mathcal{S}(Y(T)) \subset Y(T)$. Now, let $\omega, \tilde{\omega} \in Y(T)$ and $u = \mathcal{S}(\omega)$, $\tilde{u} = \mathcal{S}(\tilde{\omega})$. Set $V = u - \tilde{u}, W = \omega - \tilde{\omega}$. Then V satisfies

$$V_{tt} - V_{xx} - V_{xxtt} + V_{xxxx} = (g(\omega) - g(\tilde{\omega}))_{xx}, \tag{3.18}$$

$$V(x,0) = V_t(x,0) = 0. \tag{3.19}$$

Hence by Theorem 3.1.1 and Lemma 3.2.2,

$$\begin{aligned}
\|V(t)\|_{H^s} + \|V_t(t)\|_{H^{s-1}} &\leq 4(1+T)\int_0^t \|g(\omega(\tau)) - g(\bar{\omega}(\tau))\|_{H^{s-1}} \, d\tau \\
&\leq 4(1+T)\int_0^t \|g(\omega(\tau)) - g(\bar{\omega}(\tau))\|_{H^s} \, d\tau \\
&\leq 4(1+T)\int_0^t K_2(8da) \|\omega(\tau) - \bar{\omega}(\tau)\|_{H^s} \, d\tau \\
&= 4(1+T)\int_0^t K_2(8da) \|W(\tau)\|_{H^s} \, d\tau \\
&\leq 4(1+T)TK_2(8da) \max_{t\in[0,T]} \|W(t)\|_{H^s}.
\end{aligned}$$

Hence,

$$\|V\|_{X(T)} \leq 4(1+T)TK_2(8da) \|W\|_{X(T)}.$$

By choosing T small enough once more, so that $4(1+T)TK_2(8da) \leq \frac{1}{2}$, $\mathcal{S}$ becomes contractive. Thus the lemma is proved. □

As $\mathcal{S}$ is a contraction mapping from a closed subset $Y(T)$ of a Banach space $X(T)$ into $Y(T)$, Banach fixed point theorem states that there is a unique $\omega \in Y(T)$ such that $\mathcal{S}(\omega) = \omega$. So, we have proved the local existence result;

Theorem 3.2.4 [2] Assume that $s > \frac{1}{2}, \varphi \in H^s, \psi \in H^{s-1}$ and $g \in C^{[s]+1}(R)$, then there is some $T > 0$ such that the Cauchy problem (3.1),(3.2) has a unique solution $u \in C([0,T], H^s) \cap C^1([0,T], H^{s-1})$.

Remark: By (3.1),

$$u_{tt} = u_{xx} + \partial_x^2(1-\partial_x^2)^{-1}(g(u)).$$

If $u \in H^s$, then $u_{xx} \in H^{s-2}$. So, by the differential equation $u_{tt} \in H^{s-2}$. Thus, we observe that $u \in C^2([0,T], H^{s-2})$.

Summing up, the unique solution in Theorem 3.2.4 is in fact an element of $C([0,T], H^s) \cap C^1([0,T], H^{s-1}) \cap C^2([0,T], H^{s-2})$.

3.3 Global Existence for the Nonlinear Problem

In the previous section, we proved the existence and uniqueness of the local solution in some interval $[0,T]$. We now want to study a particular type of nonlinearity and show that in this case the solution exists for all $t \in [0,\infty)$. For this aim, we must first think about the extension of the solution to the maximal time interval. We will sketch below an outline of this standard extension process [9].

Let us consider the problem (3.1),(3.2). We proved that there is a number $T_1 > 0$ such that the solution exists uniquely in $[0,T_1]$. Now, we will look for a solution for $t \geq T_1$. We can write a shifted problem as follows:

$$u_{tt} - u_{xx} - u_{xxtt} + u_{xxxx} = g(u)_{xx}, \quad x \in R, \quad t > T_1$$
$$u(x,T_1) = \varphi_1(x), \quad u_t(x,T_1) = \psi_1(x)$$

where $\varphi_1 \in H^s$ and $\psi_1 \in H^{s-1}$ are obtained from the solution on $[0,T_1]$. Theorem 3.2.4 applied to the shifted problem gives a solution on the interval $[T_1,T_2]$ for some $T_2 > T_1$. Therefore, solution is extended to $[0,T_2]$. Continuing this way, solution can be extended to $[0,T_i]$ as long as $u(x,T_i) = \varphi_i \in H^s$ and $u_t(x,T_i) = \psi_i \in H^{s-1}$. This way we can extend the solution to some maximal interval $[0,T_{max})$. If $T_{max} < \infty$, the solution cannot be extended beyond T_{max}. The construction above shows that T_{max} can be characterized as follows:

Theorem 3.3.1 Assume that $s > \frac{1}{2}$, $\varphi \in H^s$, $\psi \in H^{s-1}$ and $g \in C^{[s]+1}(R)$, and the solution of (3.1), (3.2) is defined on the maximal interval $[0,T_{max})$. If $T_{max} < \infty$, we have

$$\lim \sup_{t \to T_{max}^-} [\|u(t)\|_{H^s} + \|u_t(t)\|_{H^{s-1}}] = \infty.$$

Remark: Theorem 3.3.1 says that if $T_{max} < \infty$, then the solution blows up at T_{max}. Conversely, if we know that the solution never blows up, then $T_{max} = \infty$.
Looking for the conditions that will yield global existence of solutions to the problem (3.1),(3.2), we first derive an energy identity for the equation:

$$u_{tt} - u_{xx} - u_{xxtt} + u_{xxxx} = g(u)_{xx}.$$

Let $\Lambda^{-\alpha} w = F^{-1}[|\xi|^{-\alpha} Fw]$, where F and F^{-1} denote Fourier transform and inverse Fourier transform in the x-variable respectively. It follows from Plancherel's theorem that $\Lambda^{-\alpha}$ is a self-adjoint operator [5] on $L^2(R)$. Note that Λ^2 is actually the positive operator $-\partial_x^2$. Then

$$\Lambda^{-2} u_{tt} + u + u_{tt} - u_{xx} = -g(u).$$

Multiplying both sides with u_t and integrating over R with respect to x, we get

$$(\Lambda^{-2} u_{tt} + u + u_{tt} - u_{xx} + g(u), u_t) = 0,$$

where $(.,.)$ denotes the inner product of L^2 space, i.e. $(f, g) = \int_R fg dx$.
Using the self-adjointness of Λ^{-1}, since $\Lambda^{-2} = \Lambda^{-1}\Lambda^{-1}$, this equation becomes

$$(\Lambda^{-1} u_{tt}, \Lambda^{-1} u_t) + (u_t, u) + (u_{tt}, u_t) - (u_{xx}, u_t) + (g(u), u_t) = 0.$$

We note that the left-hand side can be expressed as;

$$\frac{1}{2}\frac{d}{dt}(\left\|\Lambda^{-1} u_t\right\|^2 + \|u\|^2 + \|u_t\|^2 + \|u_x\|^2 + 2\int_R (\int_0^u g(p)dp)dx) = 0.$$

Hence we have proved:

Lemma 3.3.2 [2] Suppose that $g \in C(R)$, $G(u) = \int_0^u g(p)dp$, $\varphi \in H^1$, $\psi \in L^2$, $\Lambda^{-1}\psi \in L^2$ and $G(\varphi) \in L^1$. Then for the solution $u(x,t)$ of problem (3.1),(3.2), we have the energy identity

$$E(t) = \left\|\Lambda^{-1} u_t\right\|^2 + \|u\|^2 + \|u_t\|^2 + \|u_x\|^2 + 2\int_{-\infty}^{\infty} G(u)dx = E(0) \qquad (3.20)$$

for all $t > 0$ for which the solution exists.

Theorem 3.3.3 Assume that $s = 1$, $g \in C^2(R)$, $\varphi \in H^1$, $\psi \in L^2$, $\Lambda^{-1}\psi \in L^2$, $G(\varphi) \in L^1$ and $G(u) \geq 0$ for all $u \in R$, then the problem (3.1), (3.2) has a unique global solution $u \in C([0,\infty), H^1) \cap C^1([0,\infty), L^2)$.

Proof: If $G(u) \geq 0$, then from (3.20)

$$\left\|\Lambda^{-1}u_t\right\|^2 + \|u\|^2 + \|u_t\|^2 + \|u_x\|^2 \leq E(0).$$

In other words, both H^1-norm of u, i.e. $\|u\|^2 + \|u_x\|^2$, and L^2-norm of u_t are bounded by $E(0)$, and do not blow-up in finite time. More precisely,

$$\lim\, \sup_{t \to T}[\|u(t)\|_{H^1} + \|u_t(t)\|] \leq E(0) < \infty.$$

Therefore, by Theorem 3.3.1, the global solution $u(x,t) \in C([0,\infty), H^1) \cap C^1([0,\infty), L^2)$.

$\square$

Chapter 4

Cauchy Problem for the Higher-Order Boussinesq Equation

In this chapter, we deal with the higher order Boussinesq equation constructed in Chapter 2. Our aim is to prove the existence and uniqueness of the global solution and show that the Cauchy problem is well-posed. For this purpose, we follow the same procedure used in Chapter 3 and examine how it is applicable to our problem. Our problem is

$$u_{tt} - u_{xx} - u_{xxtt} + \beta u_{xxxxtt} = g(u)_{xx}, \quad x \in R, \quad t > 0, \tag{4.1}$$

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x). \tag{4.2}$$

Here $\varphi(x)$ and $\psi(x)$ are the given initial value functions and g is a given real-valued function of u, satisfying $g(0) = 0$; and β is a positive constant.

In Section 4.1, we solve the linear equation where $g(u)$ is replaced by $h(x,t)$. After giving the estimates of the solution at the end of Section 4.1, we prove the existence and uniqueness of the local solution in Section 4.2. In Section 4.3, we prove existence of the global solution in a certain case as in Theorem 3.3.3. In addition to these, in Section 4.4, we show that the solution depends continuosly on the given initial data, so the problem is well-posed.

4.1 Cauchy Problem for the Linearized Equation

We define the linear version of the equation (4.1) and give the estimates to its solution by the following theorem.

Theorem 4.1.1 Let $s \in R$, $T > 0$. Suppose that $\varphi \in H^s$, $\psi \in H^s$ and $h \in$

$L^1([0,T];H^{s-2})$, then the Cauchy problem for the linear equation

$$u_{tt} - u_{xx} - u_{xxtt} + \beta u_{xxxxtt} = (h(x,t))_{xx}, \quad x \in R, \quad t > 0, \tag{4.3}$$

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x) \tag{4.4}$$

has a unique solution $u \in C^1([0,T], H^s)$ and there is the estimation with some constant $m \geq 2$

$$\|u(t)\|_{H^s} + \|u_t(t)\|_{H^s} \leq m(1+T)(\|\varphi\|_{H^s} + \|\psi\|_{H^s} + \int_0^t \|h(\tau)\|_{H^{s-2}}\, d\tau), \quad 0 \leq t \leq T. \tag{4.5}$$

Proof: We take Fourier transform of (4.3) with respect to x.

$$(1 + \xi^2 + \beta\xi^4)\hat{u}_{tt} + \xi^2\hat{u} = -\xi^2\hat{h}. \tag{4.6}$$

Let $1 + \xi^2 + \beta\xi^4 = \lambda^2(\xi)$. Since

$$\lim_{\xi\to\infty} \frac{\lambda^2(\xi)}{(1+\xi^2)^2} = \beta > 0$$

and

$$\frac{\lambda^2(\xi)}{(1+\xi^2)^2} > 0,$$

there exists α_1,α_2 such that

$$\alpha_1^{-1}(1+\xi^2)^2 \leq \lambda^2(\xi) \leq \alpha_2(1+\xi^2)^2 \tag{4.7}$$

where $\alpha_1^{-1} > 0$, $\alpha_2 > 0$. Therefore, as a weight $\lambda^2(\xi)$ is equivalent to $(1+\xi^2)^2$. Dividing (4.6) by $\lambda^2(\xi)$,

$$\hat{u}_{tt} + \frac{\xi^2}{\lambda^2(\xi)}\hat{u} = -\frac{\xi^2}{\lambda^2(\xi)}\hat{h}(\xi,t) \tag{4.8}$$

$$\hat{u}(\xi,0) = \hat{\varphi}(\xi), \quad \hat{u}_t(\xi,0) = \hat{\psi}(\xi). \tag{4.9}$$

This is a non-homogenous initial value problem. We find the solution in two steps as in Section 3.1:

$$\hat{u}(\xi,t) = \hat{u}_{hom}(\xi,t) + \hat{u}_{part}(\xi,t)$$

$$\hat{u}_{hom}(\xi,t) = c_1(\xi)\cos(t\frac{\xi}{\lambda(\xi)}) + c_2(\xi)\sin((t\frac{\xi}{\lambda(\xi)})$$

$$\hat{u}_{part}(\xi,t) = -\int_0^t \sin((t-\tau)\frac{\xi}{\lambda(\xi)})\frac{\xi}{\lambda(\xi)}\hat{h}(\xi,\tau)d\tau.$$

Using initial conditions (4.9), $c_1 = \hat{\varphi}(\xi)$, $c_2 = \hat{\psi}(\xi)\frac{\lambda(\xi)}{\xi}$, and the solution is given by,

$$\hat{u}(\xi,t) = \hat{\varphi}(\xi)\cos(t\frac{\xi}{\lambda(\xi)}) + \hat{\psi}(\xi)\frac{\lambda(\xi)}{\xi}\sin(t\frac{\xi}{\lambda(\xi)}) - \int_0^t \sin((t-\tau)\frac{\xi}{\lambda(\xi)})\frac{\xi}{\lambda(\xi)}\hat{h}(\xi,\tau)d\tau. \tag{4.10}$$

Now, we find the suitable estimates for the H^s norm of $u(x,t)$ in a similar manner to the ones in the proof of Theorem 3.1.1.

First,

$$\int_R (1+\xi^2)^s |\hat{\varphi}(\xi)|^2 \cos^2(t\frac{\xi}{\lambda(\xi)})d\xi \le \int_R (1+\xi^2)^s |\hat{\varphi}(\xi)|^2 d\xi = \|\varphi\|^2_{H^s}.$$

Then,

$$\begin{aligned} \int_R (1+\xi^2)^s \left|\hat{\psi}(\xi)\right|^2 \frac{\lambda^2(\xi)}{\xi^2}\sin^2(t\frac{\xi}{\lambda(\xi)})d\xi &\le \int_R (1+\xi^2)^s \left|\hat{\psi}(\xi)\right|^2 \frac{\lambda^2(\xi)}{\xi^2} t^2 \frac{\xi^2}{\lambda^2(\xi)}d\xi \\ &\le t^2 \int_R (1+\xi^2)^s \left|\hat{\psi}(\xi)\right|^2 d\xi \\ &= t^2 \|\psi\|^2_{H^s}. \end{aligned} \tag{4.11}$$

Finally for the integrand of the third term,

$$\begin{aligned} &(\int_R (1+\xi^2)^s \sin^2((t-\tau)\frac{\xi}{\lambda(\xi)})(\frac{\xi}{\lambda(\xi)})^2 \left|\hat{h}(\xi,\tau)\right|^2 d\xi)^{\frac{1}{2}} \\ &\le (\int_R (1+\xi^2)^s((t-\tau)\frac{\xi}{\lambda(\xi)})^2(\frac{\xi}{\lambda(\xi)})^2 \left|\hat{h}(\xi,\tau)\right|^2)^{\frac{1}{2}} \\ &\le ((t-\tau)^2 \int_R (1+\xi^2)^s \frac{\xi^4}{\lambda^4(\xi)} \left|\hat{h}(\xi,\tau)\right|^2 d\xi)^{\frac{1}{2}} \\ &\le ((t-\tau)^2 \int_R (1+\xi^2)^s \left|\hat{h}(\xi,\tau)\right|^2 \frac{\alpha_1^2}{(1+\xi^2)^2} d\xi)^{\frac{1}{2}} \\ &\le (\alpha_1^2 t^2 \int_R (1+\xi^2)^{s-2} \left|\hat{h}(\xi,\tau)\right|^2 d\xi)^{\frac{1}{2}} = \alpha_1 t \|h(\tau)\|_{H^{s-2}}, \end{aligned}$$

where we used the inequality $\lambda^{-2}(\xi) \le \alpha_1(1+\xi^2)^{-2}$ from (4.7).

So, we obtain the following estimate for $\|u(x,t)\|_{H^s}$:

$$\|u(t)\|_{H^s} \le \|\varphi\|_{H^s} + t\|\psi\|_{H^s} + \alpha_1 t \int_0^t \|h(\tau)\|_{H^{s-2}} d\tau. \tag{4.12}$$

We now estimate for the H^s norm of $u_t(x,t)$. For this purpose, we first take the derivative of (4.10) with respect to t:

$$\hat{u}_t(\xi,t) = -\frac{\xi}{\lambda(\xi)}\sin(t\frac{\xi}{\lambda(\xi)})\hat{\varphi}(\xi) + \cos(t\frac{\xi}{\lambda(\xi)})\hat{\psi}(\xi) - \int_0^t \cos((t-\tau)\frac{\xi}{\lambda(\xi)})\frac{\xi^2}{\lambda^2(\xi)}\hat{h}(\xi,\tau)d\tau.$$

Then we continue with the usual method of estimating the terms separately:

$$\int_R (1+\xi^2)^s \frac{\xi^2}{\lambda^2(\xi)} \sin^2(t\frac{\xi}{\lambda(\xi)}) \left|\hat{\varphi}(\xi)\right|^2 d\xi \leq \int_R (1+\xi^2)^s \left|\hat{\varphi}(\xi)\right|^2 d\xi = \|\varphi\|_{H^s}^2, \quad (4.13)$$

$$\int_R (1+\xi^2)^s \cos^2(t\frac{\xi}{\lambda(\xi)}) \left|\hat{\psi}(\xi)\right|^2 d\xi \leq \int_R (1+\xi^2)^s \left|\hat{\psi}(\xi)\right|^2 d\xi = \|\psi\|_{H^s}^2, \quad (4.14)$$

$$(\int_R (1+\xi^2)^s \cos^2((t-\tau)\frac{\xi}{\lambda(\xi)}) \frac{\xi^4}{\lambda^4(\xi)} \left|\hat{h}(\xi,\tau)\right|^2 d\xi)^{\frac{1}{2}}$$

$$\leq (\alpha_1^2 \int_R (1+\xi^2)^{s-2} \left|\hat{h}(\xi,\tau)\right|^2 d\xi)^{\frac{1}{2}} = \alpha_1 \|h(\tau)\|_{H^{s-2}}. \quad (4.15)$$

These three inequalities imply

$$\|u_t(t)\|_{H^s} \leq \|\varphi\|_{H^s} + \|\psi\|_{H^s} + \alpha_1 \int_0^t \|h(\tau)\|_{H^{s-2}} \, d\tau. \quad (4.16)$$

We see u and u_t are all in H^s since their norms are bounded. This result verifies that there exists a unique solution in $C^1([0,T], H^s)$ for $0 \leq t \leq T$, and by (4.12) and (4.16), the estimation (4.5) holds for the constant $m = \max(\alpha_1, 2)$.

□

Remark: In deriving (4.11), the process differs from the one in Theorem 3.1.1. Here we only use the inequality $|\sin(w)| \leq |w|$ rather than $|\sin(w)| \leq 1$ because the latter would yield a worse estimate.

Moreover, when we consider H^s norm of $u_t(x,t)$, we observe that we could obtain a better estimate for the first term, but we will not need it for later.

4.2 Local Existence for the Nonlinear Problem

In this section, we use Theorem 4.1.1 and the contraction mapping principle to prove that the local solution to the Cauchy problem (4.1), (4.2) uniquely exists for data in H^s with $s > \frac{1}{2}$. We construct a complete metric space similar with the one in Section 3.2 and adapt Lemma 3.2.3 to our problem.

For $s > \frac{1}{2}$, $\varphi \in H^s$, $\psi \in H^s$, and fixed $T > 0$, consider the Banach space

$$X(T) = \{u \in C^1([0,T], H^s)\} \quad (4.17)$$

which is endowed with the norm

$$\|u\|_{X(T)} = \max_{t\in[0,T]} (\|u(t)\|_{H^s} + \|u_t(t)\|_{H^s}). \quad (4.18)$$

We define,

$$Y(T) = \{u \in X(T) \mid \|u\|_{X(T)} \leq A\} \tag{4.19}$$

for some constant $A > 0$. Y(T) is a closed subset of $X(T)$.

For $\omega \in Y(T)$, we consider the problem

$$u_{tt} - u_{xx} - u_{xxtt} + \beta u_{xxxxtt} = g(\omega)_{xx} \tag{4.20}$$

$$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x). \tag{4.21}$$

We see that for $g(\omega(x, t)) = h(x, t)$, this problem reduces to the linearized problem in Theorem 4.1.1. So, as in Section 3.2, we let $\mathcal{S}(\omega) = u(x, t)$; here $u(x, t)$ is the unique solution of (4.20), (4.21) and $\mathcal{S}$ denotes the map which carries ω into the unique solution of (4.20), (4.21). Our aim is again to show that for appropriately chosen T and A, $\mathcal{S}$ has a unique fixed point in $Y(T)$.

We prove the following lemma for given $\omega \in Y(T)$ and $u = \mathcal{S}(\omega)$.

Lemma 4.2.1 Assume that $s > \frac{1}{2}$, $\varphi \in H^s$, $\psi \in H^s$ and $g \in C^{[s]+1}(R)$, then for T sufficiently small, $\mathcal{S}$ is a contractive mapping from $Y(T)$ into itself.

Proof: Theorem 4.1.1 says that the solution $u = \mathcal{S}(\omega)$ of problem (4.20), (4.21) belongs to $C^1([0, T], H^s)$ and

$$\|u(t)\|_{H^s} + \|u_t(t)\|_{H^s} \leq m(1+T)(\|\varphi\|_{H^s} + \|\psi\|_{H^s} + \int_0^t \|g(\omega(\tau))\|_{H^{s-2}} \, d\tau), \quad 0 \leq t \leq T.$$

So,

$$\begin{aligned} \|\mathcal{S}(\omega)\|_{X(T)} &= \max_{t \in [0,T]} (\|u(t)\|_{H^s} + \|u_t(t)\|_{H^s}) \\ &\leq m(1+T)(\|\varphi\|_{H^s} + \|\psi\|_{H^s}) + m(1+T)T(\max_{t \in [0,T]} \|g(\omega(t))\|_{H^{s-2}}). \end{aligned}$$

Since $\|w(t)\|_\infty \leq d \|w(t)\|_{H^s} \leq A$, and Lemma 3.2.1 holds,

$$\|g(\omega(t))\|_{H^{s-2}} \leq \|g(\omega(t))\|_{H^s} \leq K_1(dA) \|\omega(t)\|_{H^s} \leq K_1(dA) \|\omega\|_{X(T)}$$

where $K_1(A)$ is a constant dependent on A. Then,

$$\begin{aligned} \|\mathcal{S}(\omega)\|_{X(T)} &\leq m(1+T)(\|\varphi\|_{H^s} + \|\psi\|_{H^s}) + m(1+T)TK_1(dA) \|\omega\|_{X(T)} \\ &\leq (1+T)(\|\varphi\|_{H^s} + \|\psi\|_{H^s}) + (1+T)TK_1(dA)A. \end{aligned}$$

Call $\|\varphi\|_{H^s} + \|\psi\|_{H^s} = a$. Since we want $\mathcal{S}(Y(T)) \subset Y(T)$, the inequality

$$\|\mathcal{S}(\omega)\|_{X(T)} \leq A$$

must hold. Thus we want

$$m(1+T)a + m(1+T)TK_1(dA)A \leq A. \tag{4.22}$$

As in Section 3.2, letting $A = ka$, $k > 0$, (4.22) becomes

$$m(1+T)a + m(1+T)TK_1(dka)ka \leq ka.$$

Equivalently,

$$ma + maT[1 + kK_1(dka) + kTK_1(dka)] \leq ka.$$

By choosing $k = 2m$ and T small enough to have $T[1+kK_1(dka)+kTK_1(dka)] \leq 1$, we get $\|\mathcal{S}(\omega)\|_{X(T)} \leq 2ma = A$. Hence, $\mathcal{S}(Y(T)) \subset Y(T)$.

Now, let $\omega, \bar{\omega} \in Y(T)$ and $u = \mathcal{S}(\omega)$, $\bar{u} = \mathcal{S}(\bar{\omega})$. Set $V = u - \bar{u}, W = \omega - \bar{\omega}$. Then U satisfies

$$V_{tt} - V_{xx} - V_{xxtt} + \beta V_{xxxxtt} = (g(\omega) - g(\bar{\omega}))_{xx}, \tag{4.23}$$

$$V(x,0) = V_t(x,0) = 0. \tag{4.24}$$

Hence by Theorem 4.1.1,

$$\begin{aligned} \|V(t)\|_{H^s} + \|V_t(t)\|_{H^s} &\leq m(1+T)\int_0^t \|g(\omega(\tau)) - g(\bar{\omega}(\tau))\|_{H^{s-2}}\, d\tau \\ &\leq m(1+T)TK_2(2mda) \max_{t\in[0,T]} \|W(t)\|_{H^s}. \end{aligned}$$

So,

$$\|V\|_{X(T)} \leq m(1+T)TK_2(2mda)\, \|W\|_{X(T)}.$$

If we further choose T small enough so that $m(1+T)TK_2(2mda) \leq \frac{1}{2}$, $\mathcal{S}$ becomes contractive. The lemma is proved since $\mathcal{S}$ is a contraction mapping from $Y(T)$ into itself for T sufficiently small.

$\square$

Hence, by Banach fixed point theorem, we obtain the following theorem which states local existence and uniqueness. We note that as in the Remark after Theorem 3.2.4, examining u_{tt}, we have the extra smoothness.

Theorem 4.2.2 Assume that $s > \frac{1}{2}, \varphi \in H^s, \psi \in H^s$ and $g \in C^{[s]+1}(R)$, then there is some $T > 0$ such that problem (4.1), (4.2) has a unique solution $u \in C^2([0,T], H^s)$.

4.3 Global Existence for the Nonlinear Problem

As we have seen before, looking for the global solution is equivalent to showing that there is no blow-up.

As a result of the extension process, we obviously have as in Theorem 3.3.1,

Theorem 4.3.1 Assume that $s > \frac{1}{2}$, $\varphi \in H^s$, $\psi \in H^s$ and $g \in C^{[s]+1}(R)$, and the solution of (4.1),(4.2) is defined on the maximal interval $[0, T_{max})$. If $T_{max} < \infty$, we have

$$\limsup_{t \to T_{max}^-} [\|u(t)\|_{H^s} + \|u_t(t)\|_{H^s}] = \infty,$$

When we intend to determine the conditions of the global existence of solutions to the problem (4.1),(4.2) for initial data $\varphi \in H^s$ and $\psi \in H^s$, we first derive an energy identity for the problem:

$$u_{tt} - u_{xx} - u_{xxtt} + \beta u_{xxxxtt} = g(u)_{xx}.$$

As in Section 3.3, we use the operator $\Lambda^{-\alpha} w = F^{-1}[|\xi|^{-\alpha} Fw]$. Then,

$$\Lambda^{-2} u_{tt} + u + u_{tt} - \beta u_{xxtt} = -g(u).$$

Multiplying both sides with u_t and integrating over R with respect to x, we get

$$(\Lambda^{-2} u_{tt} + u + u_{tt} - \beta u_{xxtt} + g(u), u_t) = 0$$

or

$$(\Lambda^{-1} u_{tt}, \Lambda^{-1} u_t) + (u_t, u) + (u_{tt}, u_t) + \beta(u_{xtt}, u_{xt}) + (g(u), u_t) = 0.$$

The left-hand side can be expressed as;

$$\frac{1}{2}\frac{d}{dt}(\left\|\Lambda^{-1} u_t\right\|^2 + \|u\|^2 + \|u_t\|^2 + \beta \|u_{xt}\|^2 + 2\int_R (\int_0^u g(p)dp)dx) = 0.$$

Thus, the following lemma has been proved:

Lemma 4.3.2 Suppose that $g \in C(R)$, $G(u) = \int_0^u g(p)dp$, $\varphi \in H^1$, $\psi \in H^1$, $\Lambda^{-1}\psi \in H^1$ and $G(\varphi) \in L^1$. Then for the solution $u(x,t)$ of problem (4.1),(4.2), we have the energy identity

$$E(t) = \left\|\Lambda^{-1} u_t\right\|^2 + \|u\|^2 + \|u_t\|^2 + \beta \|u_{xt}\|^2 + 2\int_{-\infty}^{\infty} G(u)dx = E(0) \qquad (4.25)$$

for all $t > 0$ for which the solution exists.

Remark: For $u = U_x$, the identity (4.25) and I_2, conserved energy formula shown in Chapter 2, are the same. This confirms that the energy identity we obtained in the lemma agrees with the physical structure mentioned in Chapter 2.

Theorem 4.3.3 Assume that $s \geq 1$, $g \in C^{s+1}(R)$, $\varphi \in H^s$, $\psi \in H^s$, $\Lambda^{-1}\psi \in H^s$, $G(\varphi) \in L^1$, and $G(u) \geq 0$ for all $u \in R$, then the problem (4.1), (4.2) has a unique global solution $u \in C^2([0, \infty), H^s)$.

Proof: We first prove the theorem for the case $s = 1$. If $G(u) \geq 0$, then from (4.25)

$$\left\|\Lambda^{-1}u_t\right\|^2 + \|u\|^2 + \|u_t\|^2 + \beta\|u_{xt}\|^2 \leq E(0) < \infty.$$

Hence, H^1 norm of u_t, i.e. $\|u_t\|^2 + \|u_{xt}\|^2$, is bounded, and does not blow-up in finite time. We need an estimate for $\|u(t)\|_{H^1}$; so we write $u(x,t)$ as an integral equation: Since $u(x,0) = \varphi(x)$,

$$u(x,t) = \varphi(x) + \int_0^t u_t(x,\tau)d\tau.$$

Then,

$$\|u(t)\|_{H^1} \leq \|\varphi\|_{H^1} + \int_0^t \|u_t(\tau)\|_{H^1} d\tau. \tag{4.26}$$

We had, $\|u_t\|_{H^1} \leq \gamma E(0)$, where $\gamma = \max\{1, \beta^{-1}\}$. Therefore, it can be easily seen from (4.26) that

$$\|u(t)\|_{H^1} \leq \|\varphi\|_{H^1} + \gamma t E(0).$$

Thus, for any finite $T > 0$,

$$\text{lim sup}_{t\to T^-}[\|u(t)\|_{H^1} + \|u_t(t)\|_{H^1}] \leq \|\varphi\|_{H^1} + \gamma(1+T)E(0) < \infty.$$

So, under the condition $G(u) \geq 0$, given $\varphi \in H^1$, and $\psi \in H^1$ imply $u \in C^2([0,\infty), H^1)$ by Theorem 4.3.1.

We now claim that if $\varphi, \psi \in H^s$, and $g \in C^{[s]+1}(R)$ for some $s > 1$, then $u \in C^2([0,\infty], H^s)$. By the above, we already know that $u \in C^2([0,\infty), H^1)$, so all we need to show is that $u(.,t)$, $u_t(.,t)$ (and $u_{tt}(.,t)$) are in H^s. We will apply an inductive process.

Suppose that $u(.,t)$, $u_t(.,t) \in H^r$ for some $r \leq s-2$. By Lemma 3.2.1,

$$\|g(u(t))\|_{H^r} \leq K_1(A)\|u(t)\|_{H^r}$$

with $A = \|u(t)\|_\infty \leq d\,\|u(t)\|_{H^1}$. Then the estimate of Theorem 4.1.1 with $h(x,t) = g(u(x,t))$ shows that for $0 \leq t \leq T$,

$$\begin{aligned}
\|u(t)\|_{H^{r+2}} + \|u_t(t)\|_{H^{r+2}} &\leq m(1+T)(\|\varphi\|_{H^{r+2}} + \|\psi\|_{H^{r+2}} + \int_0^t \|g(u(\tau))\|_{H^r}\,d\tau) \\
&\leq m(1+T)(\|\varphi\|_{H^s} + \|\psi\|_{H^s} + \int_0^t K_1(A)\,\|u(\tau)\|_{H^r}\,d\tau)
\end{aligned}$$

which shows that $\|u(t)\|_{H^{r+2}} + \|u_t(t)\|_{H^{r+2}} < \infty$ for all $t \in [0,T]$. Since T is arbitrary, this proves that $u \in C^2([0,\infty)\,, H^{r+2})$. The induction process allows us to obtain $u \in C^2([0,\infty)\,, H^s)$. □

4.4 Continuous Dependence on Initial Data

We now want to show that the solution of (4.1), (4.2) depends continuously on the initial data so that the problem is well-posed. For this purpose, we take two solutions u_1, u_2 of (4.1) with initial data (φ_1, ψ_1) and (φ_2, ψ_2) respectively defined on some interval $[0,T]$.

Let $v = u_1 - u_2$. Then v satisfies

$$\begin{aligned}
&v_{tt} - v_{xx} - v_{xxtt} + \beta v_{xxxxtt} = (g(u_1) - g(u_2))_{xx} \\
&v(x,0) = \varphi_1(x) - \varphi_2(x), \quad v_t(x,0) = \psi_1(x) - \psi_2(x).
\end{aligned}$$

By Theorem 4.1.1,

$$\begin{aligned}
\|u_1 - u_2\|_{H^s} &\leq m(1+T)(\|\varphi_1 - \varphi_2\|_{H^s} + \|\psi_1 - \psi_2\|_{H^s} + \int_0^t \|g(u_1) - g(u_2)\|_{H^{s-2}}\,d\tau) \\
&\leq m(1+T)(\|\varphi_1 - \varphi_2\|_{H^s} + \|\psi_1 - \psi_2\|_{H^s} + \int_0^t \|g(u_1) - g(u_2)\|_{H^s}\,d\tau).
\end{aligned}$$

By the Sobolev imbedding theorem, u_1 and u_2 are in L^∞. Letting $M = \max\{\|u_1\|_\infty\,, \|u_2\|_\infty\}$, from Lemma 3.2.2, we get

$$\|u_1 - u_2\|_{H^s} \leq m(1+T)(\|\varphi_1 - \varphi_2\|_{H^s} + \|\psi_1 - \psi_2\|_{H^s} + K_2(M)\int_0^t \|u_1 - u_2\|_{H^s}\,d\tau).$$

Integral form of Gronwall's inequality implies that

$$\|u_1 - u_2\|_{H^s} \leq m(1+T)(\|\varphi_1 - \varphi_2\|_{H^s} + \|\psi_1 - \psi_2\|_{H^s})e^{m(1+T)K_2(M)t},$$

for all $t \in [0,T]$.

Therefore, the solution depends continuously on the given initial data since it is bounded by a continuous function related with the difference of the initial data.

Bibliography

[1] R.A. Adams, J.F. Fournier, *Sobolev Spaces*, Academic Press, 2003

[2] G. Chen, S. Wang, *Cauchy problem of the generalized double dispersion equation*, Nonlinear Analysis, vol.64, 159-173, 2006

[3] G. Chen, S. Wang, *Small amptitude solutions of the generalized IMBq equation*, J.Math. Anal. Appl., vol.274, 846-866, 2002

[4] G. Chen, S. Wang, *The Cauchy problem for the generalized IMBq equation in* $W^{s,p}(R^n)$, J.Math. Anal. Appl., vol.266, 38-54, 2002

[5] B. Epstein, *Partial Differential Equations*, An Introduction, Mc-Graw-Hill, 1962

[6] L.C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics, vol.19, 1998

[7] I.M. Gelfand, S.V. Fomin, *Calculus of Variations*, Dover Publications, 2000

[8] C. Guowang, W. Shubin, *Existence and nonexistence of global solutions for the generalized IMBq equation*, Nonlinear Analysis, vol.36, 961-980, 1999

[9] W. Hurewicz, *Lectures on Ordinary Differential Equations*, The M.I.T. Press, 1970

[10] D. Mitrovic, D. Zubrinic, *Fundamentals of Applied Functional Analysis*, Distributions-Sobolev spaces-nonlinear elliptic equations, Addison Wesley Longman, 1998

[11] P. Rosenau, *Dynamics of Dense Discrete Systems*, Progress of Theoretical Physics, vol. 79, 1028-1042, 1988

[12] S. Selberg, *Multilinear Space-time estimates and applications to local existence theory for nonlinear wave equations*, Ph.D. Thesis, Princeton University, 1999

[13] C.D. Sogge, *Lectures on Nonlinear Wave Equations*, International Press Incorporated, Boston, 1995

[14] I.P. Stavroulakis, S.A. Tersian, *Partial Differential Equations*, An Introduction with Mathematica and MAPLE, World Scientific, 2004

[15] S.K. Turitsyn, *Blow-up in the Boussinesq equation*, Physical Review E, vol.47, R796, 1993

[16] S. Zheng, *Nonlinear Evolution Equations*, Chapman and Hall/CRC, 2004