Incentives under Collusion in a Two-Agent Hidden-Action Model of a Financial Enterprize

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Abstract

This study analyzes collusion in an enterprize in which concerns about hedging cannot be ignored. In our two-agent single-task hidden-action model, where all the parties involved have exponential utility functions and the principal owning normally distributed observable and verifiable returns is restricted to offer linear contracts, agents may exploit all feasible collusion opportunities via enforceable side contracts. Hence in general, an optimal incentive compatible and individually rational contract is not necessarily immune to collusion.

We demonstrate that collusion may be ignored when making the agents work with the highest effort profile is profitable for the principal and either of the following holds: (1) mean of the return is only affected by the first agent's effort level, whereas variance of that is only affected by the second agent's, (2) mean is increasing and variance is decreasing separately in effort levels of both of them. On the other hand, for situations in which any of these assumptions are violated, numerical examples, showing that collusion may make the principal strictly worse off, are provided.

For the justification of linear contracts as was done in the model of Holmstrom and Milgrom (1987) we consider a variant of its generalization given by Sung (1995), into which collusion possibilities are incorporated. In that continuous-time repeated agency problem including collusion, we prove the optimality of linear contracts.

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1 Introduction

A standard principal-agent problem[1] has individual rationality and incentive compatibility constraints, which ensure that (1) none of the agents get a lower expected utility than provided by their outside employment opportunities and (2) none of the agents has a higher expected utility by deviating alone and choosing an effort level other than the one desired by the principal. However, when agents can observe each other's effort level, tacit collusion which might arise due to repeated interaction may be analyzed in a one-shot moral hazard model in which the agents are allowed to write binding side contracts.[2] In such a setting these standard constraints are not necessarily sufficient to eliminate collusion opportunities, hence, collusion cannot be ignored.

We study a two-agent single-task hidden-action framework with collusion where all parties involved dislike risk. Thus, including the non-wealth-constrained principal, all parties involved are risk averse and in particular have exponential, i.e. constant absolute risk aversion (CARA), utility functions, a common assumption made in contract and finance theory in order to eliminate income effects.[3] The distribution of returns from the project depends on and is normal for any given effort profile agents choose. Although we work with the general model where each agent separately may control the mean and the variance, an interesting case happens when the effort choice of the first agent, *the product manager*, has more impact increasing the mean, and that of the second, *the finance manager*, more impact decreasing the variance of the return process. In order to restrict the principal to consider linear state-contingent compensation schemes, our work employs the significant results and techniques of Sung (1995) which generalizes Holmstrom and Milgrom (1987) to the case where not only the mean but also variance of output can be controlled independently of each other.

In this model the interaction between incentive compatibility and collusion proofness is analyzed. We establish that collusion may be ignored when arranging the agents work with the highest effort profile is optimal for the principal and either of the following holds: (1) mean of the return is affected only by the effort level of the first agent, whereas variance only by that of the second, (2)

[1]Early theoretical work in a single-agent moral hazard model includes Wilson (1969), Spence and Zeckhauser (1971), Ross (1973), Mirlees (1974), Mirlees (1976), Harris and Raviv (1979), Shavell (1979), Holmstrom (1979), Grossman and Hart (1983), Rogerson (1985), Holmstrom and Milgrom (1987), Holmstrom and Milgrom (1991), Holmstrom and Milgrom (1994), Schattler and Sung (1993), Sung (1995), Schattler and Sung (1997), and Hellwig and Schmidt (2002) among others. Models with many agents, on the other hand, have been studied by Baiman and Demski (1980), Lazaer and Rosen (1981), Holmstrom (1982), Demski and Sappington (1984), Mookherjee (1984), Demski, Sappington, and Spiller (1988), Ma, Moore, and Turnbull (1988), Itoh (1991) and many others.

[2]In the absence of repeated interaction, we think of a situation where a principal employs two agents each of whom can observe the other's effort level, and engage in enforceable side contracts. A particular franchising example would be a manufacturing firm delegating its sales to a dealer network (agent 1) while using a financial auditing company (agent 2) to manage its accounts.

[3]Indeed, we think of the principal and agents to be hedgers as modelled in the finance literature.

mean is increasing and variance is decreasing separately in effort levels of both of them. On the other hand, for situations in which these assumptions are violated, by providing numerical examples we demonstrate that collusion could make the principal strictly worse off.

The significance of collusion in hidden-action models was first observed by Demski and Sappington (1984), Demski, Sappington, and Spiller (1988), and Mookherjee (1984). In their hidden-action framework they modelled the interaction between the agents with a non-cooperative game and identified multiple Bayesian Nash equilibria that are not payoff equivalent for the principal. Ma, Moore, and Turnbull (1988) eliminated the undesirable equilibria from the point of view of the principal, by using a mechanism where an agent acts as a designated "police" under the promise of extra utility, never getting realized in equilibrium. Because that the success of their mechanism critically depends on single agent deviation due to the use of non-cooperative game theoretic tools, Barlo (2003) proposed to analyze the interaction among the agents through cooperative game theoretic means. When agents can observe and verify each others effort choices and have the opportunity to write binding side contracts, he shows that collusion proofness can be replaced by the following: principal's offer, an effort profile and a state-contingent compensation scheme, must be in the core of the bargaining game it induces among the agents.[4] For more we refer the reader to third chapter of Barlo (2003).

The use of linear contracts obtained from normally distributed returns and exponential utility functions has been a key ingredient in applied contract theory literature also incorporating multiple tasks into the analysis. We cite Holmstrom and Milgrom (1991), Lafontaine (1992), Holmstrom and Milgrom (1994), and Slade (1996) to that regard. The pioneer work in establishing theoretical justifications for such models (and also resolving the non-existence result due to Mirlees (1974)) is Holmstrom and Milgrom (1987), which was generalized by Sung (1995). In those studies they present repeated agency settings in which the lack of income effects due to exponential utility functions were employed to show the optimality of linear contracts. An immediate implication of this kind of linearity results is that instead of considering a complicated repeated setting, it is as if the situation were to be that the agent were to choose the mean of a normal distribution only once, and the principal were restricted to employ linear sharing rules.

In order to reach a similar conclusion, we generalize the repeated agency model provided by Sung (1995) to incorporate potential collusion between two agents. Proving the optimality of linear contracts, thus, enables us to conclude that the linearity restriction might be defended in the same fashion as in Holmstrom and Milgrom (1987).

In section 2 we present and discuss the ingredients of our model and section 3 demonstrates our main result characterizing the cases in which collusion may or may not be ignored. Finally, in section

[4]The principal only knows that agents are rational, but is not aware of the particular relative bargaining power each agent possesses. Thus, the principal's only knowledge about the bargaining process is that it has to produce a Pareto optimal and participatory result.

4 we present a continuous-time repeated agency model with collusion for which the optimality of linear contracts is proved.

2 Model

Ours is a two-agent single-task hidden-action model with state-contingent, observable and verifiable returns. The principal possessing an asset has to employ two agents, and is not wealth-constrained. If operated, this asset delivers state-contingent observable and verifiable returns drawn from a normal distribution whose mean and variance is determined by employees' effort choices. The principal cannot observe or verify their effort levels, hence, contracts he may offer cannot depend on agents' effort profile. However, each agent observes the other's effort choice and by writing binding state-contingent side contracts they can exploit all collusion opportunities.

We assume that E, the set of effort levels, is finite. Asset's state-contingent returns $x \in \Re$ are distributed with $F(x \mid e)$, such that for all $e = (e_1, e_2) \in E^2$, $F(x \mid e)$ is the normal cumulative distribution given by the mean $\mu(e)$ and the variance $\sigma^2(e)$. We define the *highest effort profile*, as an effort profile $e \in E^2$ where $\mu(e) \geqslant \mu(e')$ and $\sigma^2(e) \leqslant \sigma^2(e')$ for all $e' \neq e$.

The principal and agents are all risk-averse, and have exponential utility functions with the following coefficients of risk aversion: R for the principal, and r_i for agent $i = 1, 2$. Agent i has a utility function $u_i(y, e_i) = -\exp\{-r_i(y - c_i(e_i))\}$, where costs of effort are in terms of return. Additionally, each agent has an outside employment opportunity resulting in reserve certainty equivalent of W_i, $i = 1, 2$. Given a wage contract $(S_i(x))_{i=1,2,x\in\Re}$ which makes both agents accept and exert the effort level $e \in E^2$, expected utilities for the principal and agent i, are:

$$\begin{aligned} Eu_p(S_1, S_2 \mid e) &= \int -\exp\{-R(x - \bar{S}_x)\} \mathrm{d}F(x \mid e) \\ Eu_i(S_i \mid e_i, e_{-i}) &= \int -\exp\{-r_i(S_i(x) - c_i(e_i))\} \mathrm{d}F(x \mid e_i, e_{-i}), \end{aligned}$$

where $\bar{S}_x = S_1(x) + S_2(x)$.

We restrict attention to linear contracts of the form

$$S_i(x) = \gamma_i x + \rho_i, \tag{1}$$

where $\gamma_i, \rho_i \in \Re$, $i = 1, 2$. Thus, we can use standard techniques of determining the certainty equivalent from an exponential utility and a normal distribution, and obtain the certainty equivalent of agent $i = 1, 2$ when $e \in E^2$ is exerted as follows:

$$CE_i(\gamma_i, \rho_i \mid e) = \gamma_i \mu(e) + \rho_i - \frac{r_i}{2}\gamma_i^2 \sigma^2(e) - c_i(e_i).$$

Similarly, the certainty equivalent of the principal is:

$$CE_p(\gamma, \rho \mid e) = \left(1 - \sum_{i=1}^{2} \gamma_i\right) \mu(e) - \frac{R}{2}\left(1 - \sum_{i=1}^{2} \gamma_i\right)^2 \sigma^2(e) - \sum_{i=1}^{2} \rho_i.$$

In this setting, the individual rationality constraint for agent $i = 1, 2$, who has a given reserve certainty equivalent of W_i is:

$$\gamma_i \mu(e) + \rho_i - \frac{r_i \gamma_i^2}{2} \sigma_x^2(e) - c_i(e_i) \geqslant W_i. \tag{IR_i}$$

Similarly, the incentive compatibility constraint for agent $i = 1, 2$ is:

$$\gamma_i \left(\mu(e) - \mu(e_i', e_{-i})\right) - \frac{r_i \gamma_i^2}{2} \left(\sigma_x^2(e) - \sigma^2(e_i', e_{-i})\right) \geqslant c_i(e_i) - c_i(e_i'), \quad \forall e_i'. \tag{IC_i}$$

So far the model described is basically the two-agent single-task version of the one given by Holmstrom and Milgrom (1991).

2.1 Collusion

Collusion proofness constraint as introduced by Laffont and Martimort (1997) in a hidden-information model, has been formulated for two-agent hidden-action models by Barlo (2003). The requirement is that the principal is restricted to offer contracts that none of the agents can strictly benefit upon by deviating jointly to a feasible side contract and effort profile. Barlo (2003) deals with this constraint by formulating the interaction among agents with a bargaining game where the principal's offer, a compensation scheme and effort profile, is given. Agents jointly try to extract all the surplus due to more efficient risk sharing and coordination in effort choices, by considering feasible joint deviations[5] with the use of binding side contracts. Moreover, the bargaining power of agents should not be observable by the principal. Then, Barlo (2003) proves the existence of an optimal collusion proof, incentive compatible and individually rational contract whenever existence of an optimal incentive compatible and individually rational solution is given. He also shows that for interior solutions[6], collusion proofness constraint can be replaced by another set of constraints which make (1) the principal take care of all the joint deviations in effort profiles, and (2) offer contracts under which the two agents' marginal rate of substitution between two states is equal to one another.

Hence the first set of collusion constraints we will impose (for agent i) is as follows:

$$\gamma_i \left(\mu(e_1, e_2) - \mu(e_1', e_2')\right) - \frac{r_i \gamma_i^2}{2} \left(\sigma^2(e_1, e_2) - \sigma^2(e_1', e_2')\right) \geqslant c_i(e_i) - c_i(e_i'), \quad \forall (e_1', e_2') \in E^2. \tag{CC_i}$$

Moreover, in order to establishing the requirement that two agents' marginal rate of substitution between two states must be equal to one another it enough to have

$$\frac{\gamma_1}{\gamma_2} = \frac{r_2}{r_1}. \tag{CC'}$$

due to the specific form of the exponential utility functions (lack of income effects) and linear wages as given in equation 1.

[5]These deviations are required to satisfy a participation constraint, i.e. cannot make any of the agents strictly worse off than the expected utility level they would get from the principal's offer.

[6]Conditions on the utility functions that guarantee the optimal contract be interior are the same as given in assumption A1 of Grossman and Hart (1983).

2.2 The Principal's Problem

Under the requirement of linear compensation schemes as defined in equation 1, we are now ready to present the principal's problem $\mathcal{P}1$ including collusion proofness constraints:[7]

$$\max_{(\gamma_i,\rho_i,e_i)_{i=1,2}} \left\{ \left(1 - \frac{r_1 + r_2}{r_2}\gamma_1\right) \mu(e) - \frac{R}{2}\left(1 - \frac{r_1 + r_2}{r_2}\gamma_1\right)^2 \sigma^2(e) - \sum_{i=1}^{2} \rho_i \right\}, \qquad (\mathcal{P}1)$$

subject to

$$(IR_1) \quad : \quad \gamma_1\mu(e) + \rho_1 - \frac{r_1\gamma_1^2}{2}\sigma^2(e) - c_1(e_1) \geqslant W_1, \qquad (2)$$

$$(IR_2) \quad : \quad \frac{r_1}{r_2}\gamma_1\mu(e) + \rho_2 - \frac{r_1^2\gamma_1^2}{2r_2}\sigma^2(e) - c_2(e_2) \geqslant W_2, \qquad (3)$$

$$(CC_1) \quad : \quad \gamma_1\left(\mu(e_1, e_2) - \mu(e_1', e_2')\right) - \frac{r_1\gamma_1^2}{2}\left(\sigma^2(e_1, e_2) - \sigma^2(e_1', e_2')\right) \geqslant c_1(e_1) - c_1(e_1'), \qquad \forall (e_1', e_2') \in E^2, \qquad (4)$$

$$(CC_2) \quad : \quad \frac{r_1\gamma_1}{r_2}\left(\mu(e_1, e_2) - \mu(e_1', e_2')\right) - \frac{r_1^2\gamma_1^2}{2r_2}\left(\sigma^2(e_1, e_2) - \sigma^2(e_1', e_2')\right) \geqslant c_2(e_2) - c_2(e_2'), \qquad \forall (e_1', e_2') \in E^2. \qquad (5)$$

2.3 Linear Compensation Schemes

The use of linear compensation schemes in hidden action models with exponential utilities and normally distributed returns provides obvious technical convenience. However, it needs to be pointed out that unless the linearity restriction is justified by the features of the context in which the agency relation occurs, it is quite strong.

As far as empirical evidence is concerned Lafontaine (1992) reports that "franchise contracts generally involve the payment, from the franchisee to the franchisor, of a lump-sum franchise fee as well as a proportion of sales in royalties, with the latter usually constant over all sales levels." Furthermore, another observation is given in Slade (1996) where the author notes that only linear contracts are used by the oil companies engaged in franchising in retail-gasoline markets in the city of Vancouver.

On the other hand, the pioneer work displaying the optimality of linear contracts in a repeated agency setting with exponential utilities and normally distributed returns is Holmstrom and Milgrom (1987). In this study, they consider a principal-agent pair involved in a repeated agency relation to determine the drift rate of a Brownian motion governing the returns of an asset belonging to the principal who is able to observe only the time-path of total returns (at any instant).[8]

[7] Please notice that in $\mathcal{P}1$ we substituted γ_2 using equation CC'.

[8] Note that unlike it is in the discrete model, in the continuous-time setting the assumption that the principal be able to observe only the time-path of total returns accrued until that instant (for any instant), does not imply that

Using the specific nature of the exponential utility function due to which parties involved are immune to income effects, and time-state independent cost functions, Holmstrom and Milgrom (1987) first establishes that the agent's optimal responses become time-state independent. Based on that result[9], using the assumption that the principal can observe only the time path of total profits they prove that among all possible sharing mechanisms, an optimal one is stationary and linear in profits. Therefore, it is as if the agent were to choose the mean of a normal distribution only once and the principal were restricted to employ linear sharing rules.

Whether or not this important result is due to the specific method of approximating the continuous-time model via a discrete-time version has been analyzed by Hellwig and Schmidt (2002) in which they "explicitly derive the continuous-time model as a limit of discrete-time models with ever shorter periods and show that the optimal incentive scheme in the continuous model, which is *linear in accounts*, can be approximated by a sequence of optimal incentive schemes in the discrete models".

Under the same set of assumptions Sung (1995) considers a continuous-time version of these models where the agent can also control the diffusion rate of the Brownian motion governing the returns. Again, time-state independent technology and exponential utility functions imply that at every instant the agent's best responses are time-state independent even when he can control the diffusion rate. The resulting problem becomes similar to that in Holmstrom and Milgrom (1987) with an additional time-state independent constraint (thus, features a similar stationary decision-making environment), and he proves that the linearity in total profits result holds.

In section 4 in order to render a similar theoretical justification for our model, under the same set of assumptions we consider a two-agent version of the model of Sung (1995), in which the two agents can also exploit all the collusion opportunities at every instant. Borrowing the same techniques and adding an extra assumption (to be discussed in the next paragraph) about the behavior of agents during the collusion phase, we prove that a stationary and linear in total profits sharing rule is optimal. Thus, it is as if the agents, who can exploit all collusion opportunities, were to choose the mean and the variance of a normal distribution only once and the principal were restricted to employ linear sharing rules.

The extra assumption we have to impose on the two-agent version of the model of Sung (1995) also gives birth to a requirement needed to associate the one-shot problem $\mathcal{P}1$ with the continuous-time model. This assumption is concerned with the collusion aspect and is on the relative bargaining

he observes the instantaneous incremental return at any instant. We refer the reader to the introduction of Hellwig and Schmidt (2002)[p.2226] for a detailed discussion of this point.

[9]This result immediately implies what Hellwig and Schmidt (2002) calls *linearity in accounts*: "If the principal observes the outcome paths, then the intertemporal incentive problem has a stationary solution with an incentive scheme that is a linear function of the frequencies with which the different "one-shot" outcomes are observed." However, to obtain linearity in total profits, which is quite different from the linearity in accounts, they note that the assumption that the principal observes only the time-path of total profits, is critical.

power of each agent when they try to decide which side contract to implement. Even though the principal is still not informed of the relative bargaining power of each agent against the other, we assume that the ratio of agents relative bargaining powers must equal to the inverse ratio of their coefficients of risk aversion. In other words, this restriction makes an agent's relative bargaining power against the other be equal to the coefficient of risk aversion of his opponent.[10] We need this assumption because then, in the continuous-time model we could reduce the agents' instantaneous collective bargaining problem to a simple form which is suited to use techniques given in Sung (1995). Therefore, under this particular requirement it is as if there is only one agent that the principal faces.

Another restriction to associate the one-shot problem, $\mathcal{P}1$ with the continuous-time repeated setting described and solved in section 4 is about the cost functions. Indeed, in $\mathcal{P}1$ the cost functions are not fully tailored for the continuous-time model. To that regard we need to have agents' cost functions depend only on the mean and variance of output.[11]

Finally it needs to be said that finite effort levels are used in $\mathcal{P}1$ only to keep the analysis and numerical programming simple.

3 When Does Collusion Matter?

The following Theorem will provide a full characterization of the situations when it is sufficient to restrict attention to optimal incentive compatible and individually rational contracts, i.e. when collusion may be ignored.

Theorem 1 *Suppose there exists a highest effort profile and that its implementation is optimal for the principal with incentive compatibility and individual rationality constraints, and either of the following holds:*

A. $\mu(e_1, e_2) = \mu(e_1)$ *is increasing in* e_1*, and* $\sigma^2(e_1, e_2) = \sigma^2(e_2)$ *is decreasing in* e_2*; or*

B. $\mu(e_1, e_2)$ *is increasing, and* $\sigma^2(e_1, e_2)$ *decreasing separately in both* e_1 *and* e_2*.*

Then incentive compatibility implies collusion proofness, hence, an optimal incentive compatible contract is collusion proof. Moreover, for any other case not covered by the hypothesis, collusion may make the principal strictly worse off.

[10] Indeed, these kinds of restrictions are generally needed in many aggregation exercises in economics. Furthermore, we argue that such a requirement is not that restrictive. Because in the symmetric case, i.e. two agents have the same coefficient of risk aversion, this assumption implies that they have the same relative bargaining power. And if they have different coefficients of risk aversion, the one with the lower value would have the higher coefficient as his relative bargaining power, which we believe has intuitive appeal.

[11] Recall that in $\mathcal{P}1$ agents' costs are functions of their own effort levels, and not of the mean and variance. Thus, the cost function that would be suitable in our case should have the following form: $c_i(e_i, e_{-i}) = c_i(\mu(e_i, e_{-i}), \sigma^2(e_i, e_{-i}))$.

As was mentioned in the introduction, an interesting feature of our model happens when the effort choice of the first agent (the product manager) has more impact on the mean and that of the second (the finance manager) more on the variance. This Theorem demonstrates that collusion may be ignored, whenever either the first agent only affects the mean, whereas the second agent only affects the variance of the return process; or they both increase the mean and decrease the variance separately as their effort levels increase, provided that it is optimal for the principal to induce the highest effort level with incentive compatibility and individual rationality constraints.

Proof of Theorem 1. Let (e_1, e_2) be the highest effort profile in E^2.

Case A. With these assumptions collusion proofness constraints turns out to be

$$\begin{aligned}
(CC_1) \quad &: \quad \gamma_1 \left(\mu(e_1) - \mu(e_1')\right) - \frac{r_1\gamma_1^2}{2} \left(\sigma^2(e_2) - \sigma^2(e_2')\right) \geqslant \\
&\qquad\qquad c_1(e_1) - c_1(e_1'), \qquad \forall (e_1', e_2') \in E^2. \\
(CC_2) \quad &: \quad \frac{r_1\gamma_1}{r_2} \left(\mu(e_1) - \mu(e_1')\right) - \frac{r_1^2\gamma_1^2}{2r_2} \left(\sigma^2(e_2) - \sigma^2(e_2')\right) \geqslant \\
&\qquad\qquad c_2(e_2) - c_2(e_2'), \qquad \forall (e_1', e_2') \in E^2.
\end{aligned}$$

On the other hand, incentive compatibility constraints are:

$$\begin{aligned}
(IC_1) \quad &: \quad \gamma_1 \left(\mu(e_1) - \mu(e_1')\right) \geqslant c_1(e_1) - c_1(e_1'), \quad \forall e_1' \in E. \\
(IC_2) \quad &: \quad \frac{-r_1^2\gamma_1^2}{2r_2} \left(\sigma^2(e_2) - \sigma^2(e_2')\right) \geqslant c_2(e_2) - c_2(e_2'), \quad \forall e_2' \in E.
\end{aligned}$$

Clearly, the left hand side of (IC_i) is less than the left hand side of (CC_i) for $i = 1, 2$, which implies that any solution satisfying (IC_i) also satisfies (CC_i) for $i = 1, 2$. ■

Case B. Similarly, with these assumptions,

$$\begin{aligned}
(CC_1) \quad &: \quad \gamma_1 \left(\mu(e_1, e_2) - \mu(e_1', e_2')\right) - \frac{r_1\gamma_1^2}{2} \left(\sigma^2(e_1, e_2) - \sigma^2(e_1', e_2')\right) \geqslant \\
&\qquad\qquad c_1(e_1) - c_1(e_1'), \qquad \forall (e_1', e_2') \in E^2. \\
(CC_2) \quad &: \quad \frac{r_1\gamma_1}{r_2} \left(\mu(e_1, e_2) - \mu(e_1', e_2')\right) - \frac{r_1^2\gamma_1^2}{2r_2} \left(\sigma^2(e_1, e_2) - \sigma^2(e_1', e_2')\right) \geqslant \\
&\qquad\qquad c_2(e_2) - c_2(e_2'), \qquad \forall (e_1', e_2') \in E^2.
\end{aligned}$$

And,

$$\begin{aligned}
(IC_1) \quad &: \quad \gamma_1 \left(\mu(e_1, e_2) - \mu(e_1', e_2)\right) - \frac{r_1\gamma_1^2}{2} \left(\sigma^2(e_1, e_2) - \sigma^2(e_1', e_2)\right) \geqslant \\
&\qquad\qquad c_1(e_1) - c_1(e_1'), \qquad \forall e_1' \in E. \\
(IC_2) \quad &: \quad \frac{r_1\gamma_1}{r_2} \left(\mu(e_1, e_2) - \mu(e_1, e_2')\right) - \frac{r_1^2\gamma_1^2}{2r_2} \left(\sigma^2(e_1, e_2) - \sigma^2(e_1, e_2')\right) \geqslant \\
&\qquad\qquad c_2(e_2) - c_2(e_2'), \qquad \forall e_2' \in E.
\end{aligned}$$

Again, left hand side of (IC_i) is less than the left hand side of (CC_i) for $i = 1, 2$, which implies that any solution satisfying (IC_i) also satisfies (CC_i) for $i = 1, 2$. ■

This and the following two examples conclude the proof of the Theorem. ■

The rest of this section will display that whenever in cases not covered by the hypothesis of Theorem 1, collusion might make the principal strictly worse off.

The following example will drop the assumption that making the agents work with the highest effort vector is optimal for the principal. It will be shown that collusion binds, in terms of not only risk sharing contracts, but also joint effort deviations.

In this simple nevertheless interesting example, the mean of the return depends positively only on the effort level of agent 1, so that $\mu(e) = \mu(e_1)$; and the variance negatively only on the effort level of agent 2, so that $\sigma^2(e) = \sigma^2(e_2)$. Assume there are two levels of effort for both agents, that is, $e_i \in \{0, 1\}$, $i = 1, 2$. Reserve certainty equivalence figures for the agents are $W_1 = 1$ and $W_2 = 1.5$, respectively. The costs are given as $c_1(1) = 1$, $c_1(0) = 0$ for agent 1 and $c_2(1) = 0.5$, $c_2(0) = 0$ for agent 2. The coefficient of absolute risk aversion for the principal is $R = 2$ and those for the agents are $r_1 = r_2 = 10$.

The mean and the variance of returns depending on the effort vector is:

(e_1, e_2)	$\mu(e_1)$	$\sigma^2(e_2)$
$(1, 1)$	10	1
$(1, 0)$	10	2
$(0, 1)$	5	1
$(0, 0)$	5	2

When the principal wants to induce the highest effort level, given by $e = (1, 1)$, his certainty equivalent will be 4.86. Note that particularly at the highest effort profile in this problem, incentive compatibility implies collusion proofness. However, when agents work with effort profile $(1, 0)$, the principal's optimal level of certainty equivalent with incentive compatibility and individual rationality is 4.98. Hence, the assumption of Theorem 1 that the highest effort level be optimal for the principal to implement, does not hold. Consequently, if collusion proofness constraint is added to the problem at effort profile $(1, 0)$, the principal's return decreases to 4.57. Thus, without the collusion possibility, the most profitable effort profile for the principal is $(1, 0)$. But, if he considers the collusion possibility among agents, he would want them to work with effort vector $(1, 1)$. Hence, the highest expected return generating effort profile and compensation scheme the principal can implement changes with by the addition of collusion proofness constraint.

In another example where each agent's effort level exhibits the opposite effect of that of the other, we identify a situation in which collusion matters. We display that the equivalence of incentive compatibility and collusion proofness constraints is broken down by the agent whose control variable is adversely affected by that of the other. In particular, if increasing agent 2's effort decreases the mean and increasing agent 1's effort decreases the variance then the equivalence of incentive compatibility and collusion proofness given in Theorem 1 breaks down since any contract satisfying (IC_1) may not satisfy (CC_1). Similarly, for the case when agent 1's effort positively affects the variance while agent 2's effect is negative, but both of their controls increase the mean, then the equivalence again does not hold because any contract satisfying (IC_2) may not satisfy (CC_2).

Particularly, suppose that the mean and variance of the return depend on effort levels of both agents. However, agents effort choices affect the variables in opposite directions, i.e we assume that agent 2's control decreases the mean of the return, whereas agent 1's increases the variance of it, hence, assumption **B** of Theorem 1 does not hold. More specifically, the reserve certainty equivalence figures for the agents are $W_1 = 0.5$ and $W_2 = 1.5$, and the effort costs $c_1(1) = 0.75$, $c_1(0) = 0$ for agent 1 and $c_2(1) = 0.01$, $c_2(0) = 0$ for agent 2. The coefficient of absolute risk aversion for the principal is $R = 2$ and that for the agents is $r_1 = r_2 = 10$, as before. The mean and the variance of the return is summarized in the following matrix

(e_1, e_2)	$\mu(e_1, e_2)$	$\sigma^2(e_1, e_2)$
$(1, 1)$	30	1
$(1, 0)$	31	2
$(0, 1)$	26.5	1/2
$(0, 0)$	28.3	4/3

In this example where there is no highest effort profile, making the agents work with the effort profile $(1, 1)$ is the most profitable for the principal both with and without collusion. The principal's optimal certainty equivalent figure is 26.31 with incentive compatibility and individual rationality constraints. On the other hand, the same figure drops to 26.02 with the addition of collusion proofness constraint. Note that the reason for this observation is that with collusion agents can share each other's risk. Therefore, the principal, in order to obtain the effort profile $(1, 1)$, he has to perform the costly effort of making the agents face more risk when collusion is considered. Consequently, this intuition is confirmed in this example as well: the sharing rule coefficients are $\gamma_1 = \gamma_2 = 0.2641$ with incentive compatibility and individual rationality constraint, and $\gamma_1 = \gamma_2 = 0.3326$ with addition of collusion proofness constraints.

4 A Continuous-Time Model with Collusion

We consider a variant of the model given by Sung (1995) in which both of the two agents employed can control not only the mean but also variance of output, and instantaneously exploit all collusion opportunities. In this setting we prove that among the optimal sharing rules there exists a stationary and linear one.

4.1 Model

The principal and agents will be interacting over the time interval $t \in [0, 1]$ to determine the drift and diffusion rates of a stochastic process governed by a Brownian motion given by

$$\mathrm{d}X_t = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}B_t,$$

where we think of the intermediate outcome X_t as the total returns up to period $t \in [0, 1]$.

Note that μ_t and σ_t^2 are the instantaneous mean and variance of the return at time t. We let agents' instantaneous time-state independent cost functions be given by $c_i(\mu_t, \sigma_t)$, where $c_i : U \times \mathbb{S} \to \Re$ and is assumed to be twice continuously differentiable. We assume c_i and $c_{i\mu}$ (derivative with respect to mean) are bounded. And the total costs incurred by the agents are given by $\int_0^1 c_i(\mu_t, \sigma_t)\mathrm{d}t$, $i = 1, 2$. Agents are compensated according to the salary functions S_1 and S_2 that depend on the principal's observation of the outcome at time 1.

Before stating the problem explicitly, we have to give the technical side borrowed from Schattler and Sung (1993) and Sung (1995): It is assumed that the probability space is given by $(\Omega, \mathcal{F}, P)$. Ω be the space $C = C([0, 1])$ of all continuous functions on the interval $[0, 1]$ with values in $\Re^n$. The space Ω is like a complete description of the uncertainty associated with the underlying asset. Let W_t be the coordinate process on Ω, $W_t(w) = w(t)$ for $w \in \Omega$. w represents one possible history of the assets over the time period and $w(t)$ describes the correct state of the asset at time t if w occurs. Let $\mathcal{F}_t^0$ be the filtration generated by W until time t and P be the Wiener measure on $(\Omega, \mathcal{F}_t^0)$. $\mathcal{F}_t$ is defined to be augmentation of $\mathcal{F}_t^0$ by all null sets of $\mathcal{F}_1^0$. The σ-algebra $\mathcal{F}_t$ contains the entire information based on the complete history of all possible realizations up to time t and is augmented by all null sets. Wiener measure gives an underlying probability measure. The control laws μ and σ are $\mathcal{F}_t$-predictable mappings $\mu : [0, 1] \times \Omega \to U$ and $\sigma : [0, 1] \times \Omega \to \mathbb{S}$, where U is a bounded open subset of $\Re$ and $\mathbb{S}$ is a compact subset of $\Re_{++}$. These control laws, μ and σ, determine the instantaneous values of μ_s and σ_s at each date $s \in [0, t]$ as functions of the history of the process X up to time t, for every $t \in [0, 1]$. Moreover, we adopt the notation of calling the controls determined at time t, by μ_t and σ_t, i.e. $\mu_t = \mu(t, X)$ and $\sigma_t = \sigma(t, X)$. Furthermore, we assume σ satisfies a uniform Lipschitz condition in Z, $\bar{Z} \in C[0, 1]$ and there exists a constant K such that

$$|\sigma(t, Z) - \sigma(t, \bar{Z})| \leq K \sup_{0 \leq s \leq t} \|Z(s) - \bar{Z}(s)\|.$$

We denote $\Phi^k(U)$ the class of functionals $\phi : [0, 1] \times C \times U \to \Re^n$ which have the following properties:

i. For $(t, X) \in [0, 1] \times C$ fixed, the functional $\phi(t, X, .)$ is k-times continuously differentiable on some open neighborhood of U.

ii. For $u \in U$ fixed, the process $\phi(., ., u)$ is $\mathcal{F}_t$-predictable.

iii. There exists a constant K such that

$$\|\phi(t,X,u)\| \le K(1+\max_{0\le s\le t}\|X_s\|) \text{ for all } (t,X,u)\in[0,1]\times C\times U.$$

Define $\Phi_b^k(U)$ as the class of functionals ϕ which satisfy

iii'. ϕ is bounded; there exists a constant K such that $\|\phi(t,X,u)\| \le K$ for all $(t,X,u)\in[0,1]\times C\times U$.

4.2 The Principal's Problem

Now we are ready to state the principal's problem.

Definition 1 *Principal chooses salary functions $\hat{S}_1$ and $\hat{S}_2$, and control laws $(\hat{\mu},\hat{\sigma})$, where the salary functions depend on X_1 (which is the final outcome) to maximize $E[-\exp\{-R(X_1-\hat{S}_1(X_1)-\hat{S}_2(X_1))\}]$ subject to*

(i) *(Feasibility) For all $t\in[0,1]$*

$$\mathrm{d}X_t = \hat{\mu}_t \mathrm{d}t + \hat{\sigma}_t \mathrm{d}B_t,$$

(ii) *(Collusion) Salary functions $(\hat{S}_1,\hat{S}_2)$, and control laws $(\hat{\mu},\hat{\sigma})$ must be such that $((\hat{S}_1,\hat{S}_2),(\hat{\mu},\hat{\sigma}))$ maximizes*

$$E\left[-\left(\exp\left\{-r_1(S_1-\int_0^1 c_1(\mu_t,\sigma_t)\mathrm{d}t)\right\}\right)^{\theta}\left(\exp\left\{-r_2(S_2-\int_0^1 c_2(\mu_t,\sigma_t)\mathrm{d}t)\right\}\right)^{(1-\theta)}\right] \tag{6}$$

subject to

$$\mathrm{d}X_t = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}B_t, \tag{7}$$

$$S_1+S_2 \le \hat{S}_1+\hat{S}_2, \tag{8}$$

$$-\exp\left\{-r_i(S_i-\int_0^1 c_i(\mu_t,\sigma_t)\mathrm{d}t)\right\} \ge -\exp\left\{-r_i(\hat{S}_i-\int_0^1 c_i(\hat{\mu}_t,\hat{\sigma}_t)\mathrm{d}t)\right\}, \tag{9}$$

for all $\theta\in[0,1]$.

(iii) *(Individual rationality) For all $i=1,2$,*

$$E\left[-\exp\left\{-r_i(\hat{S}_i-\int_0^1 c_i(\hat{\mu}_t,\hat{\sigma}_t)\mathrm{d}t)\right\}\right] \ge -\exp\{r_iW_i\}$$

The feasibility and individual rationality requirement in the above definition is standard. On the other hand, the collusion requirement needs more explanation.

The collusion constraint tells us that principal's choices must be such that no matter what relative bargaining powers of agents are, the *bargaining power weighted expected utility figure*[12], equation 6, must be maximized particularly at the values the principal desires, even when agents may use different and feasible control laws (equation 7) and different resource feasible side contracting schemes (equation 8) which are required to be participatory[13] (equation 9). For more details on this aspect we refer the reader to our section on collusion, i.e. section 2.1.

4.3 Optimality of Linear Contracts

First we need to borrow the following assumption needed from Schattler and Sung (1993):

Assumption 1 (A1.) *The diffusion rate $\sigma : [0,1] \times C \rightarrow \Re^{n\times n}$ is locally bounded, $\mathcal{F}_t$-predictable, and non-singular matrix-valued functional which satisfies a uniform Lipschitz condition in $Z \in C$, i.e., there exists a constant K such that*

$$|\sigma_{ij}(t,Z) - \sigma_{ij}(t,\bar{Z})| \leq K \sup_{0\leq s\leq t} \|Z(s) - \bar{Z}(s)\|.$$

Assumption 2 (A2.) $\sigma^{-1} f \in \Phi(U)$, $a \in \Phi_b(U)$ *(and a takes values in $\Re$) and* $b \in \Phi_b(U)$.

Assumption 3 (A3.) $F : C \rightarrow \Re$ *is $\mathcal{F}_1$-measurable and bounded below.(It is allowed that $F(X)$ depends on the entire history in the agents' problem. In the principal's problem $F(X)$ is usually made to be a function of the terminal X_1 alone.)*

Assumption 4 (A4.) *Suppose that for every $(t, X, p, p_0) \in [0,1] \times C \times \Re^n \times \Re_-$, there exists an $u^* \in U$ such that*

$$H(t, X, p, p_0, u^*) = \max_{u\in U} H(t, X, p, p_0, u)$$

where for $(t, X, p, p_0, u) \in [0,1] \times C \times \Re^n \times \Re_- \times U$, H as the Hamiltonian is defined to be

$$H(t, X, p, p_0, u) := p^T (f(t, X, u) + \sigma(t, X) b(t, X, u)) + p_0 (a(t, X, u) + \frac{1}{2}\|b(t, X, u)\|^2).$$

Note that this is needed when U is not compact.

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[12] Note that due to the specific nature of exponential utility functions, the utility aggregation for colluding agents we will use is as follows:

$$E\left[-|U_1|^{\theta}|U_2|^{(1-\theta)}\right].$$

[13] None of the agents should get a lower expected utility figure than the level promised by the principal's offer.

For detailed discussions of these assumptions, we refer reader Schattler and Sung (1993).

The next assumption is needed to extend the work of Schattler and Sung (1993) to our setting:

Assumption 5 (A5.) *The exogenously given relative bargaining power of each agent (that the principal is not aware of) is given by* $\theta = \frac{r_2}{r_1+r_2}$ *for agent 1, and* $1-\theta = \frac{r_1}{r_1+r_2}$ *for agent 2.*

Please refer to the final paragraphs of section 2.3 for a discussion of assumption (A5).

The following Theorem, the main result of section 4, will establish that among the optimal collusion proof salary functions one pair has to be stationary and linear in profits:

Theorem 2 *Suppose (A1) - (A5) hold. Then an optimal collusion proof stationary contract profile is as follows:*

$$\begin{aligned} S_1(\mu^*, \sigma^*) &= \frac{r_2}{r_1+r_2} S(\mu^*, \sigma^*) \\ S_2(\mu^*, \sigma^*) &= \frac{r_1}{r_1+r_2} S(\mu^*, \sigma^*), \end{aligned}$$

where

$$\begin{aligned} S(\mu^*, \sigma^*) &= W_{10} + W_{20} + c_1(\mu^*, \sigma^*) + c_2(\mu^*, \sigma^*) \\ &\quad + (c_{1\mu}(\mu^*, \sigma^*) + c_{2\mu}(\mu^*, \sigma^*))\,(X_1 - X_0) + (c_{1\mu}(\mu^*, \sigma^*) + c_{2\mu}(\mu^*, \sigma^*))\,\mu^* \\ &\quad + \frac{r_1 r_2}{2(r_1+r_2)} \left(c_{1\mu}(\mu^*, \sigma^*) + c_{2\mu}(\mu^*, \sigma^*)\right)^2 \sigma^{*2}. \end{aligned}$$

The rest of this section is devoted to the proof of Theorem 2.

Our proof entails modifications of selected theorems from Schattler and Sung (1993) and Sung (1995) and has three steps: Step 1 turns the principal's problem into one that is more suitable for these techniques. Then in step 2 we will use Theorem 3.1 and modify the Representation Theorem of Schattler and Sung (1993)[14], Theorem 4.1, to come up with a representation rule for the salary function of our two-agent hidden action model. Finally, in step 3 first this Representation Theorem will be modified for our situation in order to make the two agents control the variance as well. Using this construction we will prove the linearity result in the same fashion as was done in Sung (1995).

In order to follow techniques discussed above we have to work with the following problem which is more suited for this analysis then the one given in definition 1.

Definition 2 *Principal chooses salary functions* S_1 *and* S_2 *that depends on* X_1 *(which is the final outcome) to maximize* $E[-\exp\{-R(X_1 - S_1(X_1) - S_2(X_1))\}]$ *subject to*

[14]In this paper the agent is not allowed to control the diffusion rate of the Brownian motion governing the return process. Moreover, the next two generalities are that the control variable u is assumed to be in $U \subset \Re^n$ for modeling multi-task problems, and the salary function depends not only on X_1 but also the entire process X.

(i) *(Feasibility) For all $t \in [0,1]$*

$$\mathrm{d}X_t = \hat{\mu}_t \mathrm{d}t + \hat{\sigma}_t \mathrm{d}B_t,$$

(ii) *(Collusion) Given S_1 and S_2, $(\hat{\mu}, \hat{\sigma})$ is a pair of control laws to maximize*

$$E\left[-\left(\exp\left\{-r_1(S_1 - \int_0^1 c_1(\mu_t, \sigma_t)\mathrm{d}t)\right\}\right)^{\theta}\left(\exp\left\{-r_2(S_2 - \int_0^1 c_2(\mu_t, \sigma_t)\mathrm{d}t)\right\}\right)^{(1-\theta)}\right] \quad (10)$$

subject to

$$\mathrm{d}X_t = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}B_t,$$

for all $\theta \in [0,1]$.

(iii) *(Individual rationality) For all $i = 1, 2$,*

$$E\left[-\exp\left\{-r_i(S_i - \int_0^1 c_i(\hat{\mu}_t, \hat{\sigma}_t)\mathrm{d}t)\right\}\right] \geq -\exp\{r_i W_i\}.$$

It should be pointed out that the primary difference between these problems (i.e. problem given in definition 1 versus that given in definition 2) is about the collusion items involved. For convenience, we will refer to the problem given in definition 1 as the first problem, and to the problem given in definition 2 as the second.

We note that the set of constraints of the first problem is a subset of that of the second problem: If $(\hat{S}_1, \hat{S}_2; \hat{\mu}, \hat{\sigma})$ solves the first nested maximization problem (i.e. is in the constraint set of the first problem), then clearly $\hat{\mu}, \hat{\sigma})$ solves the second nested maximization problem at given salary functions $(\hat{S}_1, \hat{S}_2)$ (thus, $(\hat{S}_1, \hat{S}_2; \hat{\mu}, \hat{\sigma})$ is in the constraint set of the second problem).

Therefore, from this point on until the very end of the proof of Theorem 2, we will work with the second problem in which the principal is more relaxed. Indeed, the final steps to establish the proof of Theorem 2 will be to show that the optimal solution to the second problem is in the constraint set of the first problem.

Note that from the point of view of the principal (who does not know the relative bargaining power of the agents) the agents solve the following maximization problem:

$$E_u\left[-\left|-\exp\left\{-r_1(S_1(X) - \int_0^1 c_1(t, X, u)\mathrm{d}t)\right\}\right|^{\theta}\left|-exp\left\{-r_2(S_2(X) - \int_0^1 c_2(t, X, u)\mathrm{d}t)\right\}\right|^{(1-\theta)}\right]$$

for all θ over all admissible controls u, subject to

$$dX_t = f(t, X, u)\mathrm{d}t + \sigma(t, X)dB_t.$$

We assume that principal tries to maximize his expected utility after the salaries are paid. So his maximization problem is

$$E[-\exp\{-R(X_1 - S_1(X) - S_2(X))\}]$$

for all $\theta \in [0,1]$ over all salary functions $S_1 = S_1(X)$ and $S_2 = S_2(X)$ which guarantee the agents' reservation utility subject to

$$\mathrm{d}X_t = f(t, X, u)\mathrm{d}t + \sigma(t, X)\mathrm{d}B_t,$$

where u is the optimal control implemented by S_1 and S_2.

In theorem 3.1 of Schattler and Sung (1993) they give the mathematical model for a class of maximization problems which include both the agents' and principal's problems as given above. This class of maximization problems is: maximize

$$E\left[-\exp\left\{-F(Y) + \int_0^1 a(t, Y, u)\mathrm{d}t + \int_0^1 b^T(t, Y, u)\mathrm{d}B_t\right\}\right]$$

over all admissible controls u, subject to

$$\mathrm{d}Y_t = f(t, Y, u)\mathrm{d}t + \sigma(t, Y)\mathrm{d}B_t.$$

Recall that the possible actions of the agents are represented by the class $\mathcal{U}$ of all $\mathcal{F}_t$-predictable processes u with values in some control set U, $u : [0,1] \times C \to U$, $(t, X) \to u(t, X)$. It is assumed that $U(\subset \Re^m)$ can be written as a countable union of compact subsets. (U may be a compact set or be any open set.)

Condition (A2) with conditions on functionals Φ enables a change of measure. The control problem can be stated as the following with this change of measure: [15]

$$J(u) = E_u\left[-\exp\left\{-F(Y) + \int_0^1 a(t, Y, u)\mathrm{d}t + \int_0^1 b^T(t, Y, u)\mathrm{d}B_t^u\right\}\right].$$

The following Theorem due to Schattler and Sung (1993) gives the main results of Martingale approach to stochastic control problems adjusted for exponential utility functions. Note that as they have done the symbol ∇ denotes the row vector.

Theorem 3 (Theorem 3.1 of Schattler and Sung (1993)) *Under assumptions (A1)-(A4), the stochastic control problem to maximize*

$$J(u) = E_u\left[-\exp\left\{-F(Y) + \int_0^1 a(t, Y, u)\mathrm{d}t + \int_0^1 b^T(t, Y, u)\mathrm{d}B_t^u\right\}\right]$$

[15] For the details of this nontrivial yet standard step, the reader is referred to Schattler and Sung (1993).

over all $u \in \mathcal{U}$ has a solution. Furthermore, there exist $\mathcal{F}_t$-adapted processes $\mathcal{V}_t$ and $\nabla\mathcal{V}_t$ such that $u^* \in \mathcal{U}$, $\int_0^1 |\nabla\mathcal{V}_t\sigma|^2 \mathrm{d}t < \infty$ a.e, such that $u^* \in \mathcal{U}$ is optimal if and only if

$$u^* \in \arg\max_u \left\{ \nabla\mathcal{V}_t \times (f(t,X,u) + \sigma(t,X)b(t,X,u)) + \mathcal{V}_t \times \left(a(t,X,u) + \frac{1}{2}\|b(t,X,u)\|^2 \right) \right\}$$

for almost all $(t,w) \in [0,1] \times \Omega$. Furthermore, the value process $\mathcal{V}_t$ has an Itô differential of the form

$$\begin{aligned} \mathcal{V}_t &= \mathcal{V}_0 - \int_0^t \nabla\mathcal{V}_t \left(f(t,X,u^*) + \sigma(t,X)b(t,X,u^*) \right) + \mathcal{V}_t \left(a(t,X,u^*) + \frac{1}{2}\|b(t,X,u^*)\|^2 \right) \mathrm{d}t \\ &\quad + \int_0^t \nabla\mathcal{V}_t \mathrm{d}X_t. \end{aligned}$$

In order to apply Theorem 3 to our principal-agent problem we have to convert our control problems into the following format:

$$J(u) = E_u \left[-\exp\left\{ -F(Y) + \int_0^1 a(t,Y,u)\mathrm{d}t + \int_0^1 b^T(t,Y,u)\mathrm{d}B_t^u \right\} \right]$$

over all $u \in \mathcal{U}$ (recall that here u stands for the control variable and not the utility).

We first consider the problem of maximizing $J(u)$ defined by

$$E_u \left[- \left| -\exp\left\{ -r_1 \left(S_1(X) - \int_0^1 c_1(t,X,u)\mathrm{d}t \right) \right\} \right|^{\theta} \left| -\exp\left\{ -r_2 \left(S_2(X) - \int_0^1 c_2(t,X,u)\mathrm{d}t \right) \right\} \right|^{(1-\theta)} \right]$$

over all $u \in \mathcal{U}$. Note that this problem captures the bargaining between the two agents the first having a weight of $\theta \in [0,1]$, and the second $1-\theta$. As usual we will be considering sharing rules $S_i : C \to \Re$ of the following structure:

$$S_i(X) = S_{i1}(X) + \int_0^1 \alpha_i(t,X)\mathrm{d}t + \int_0^1 \beta_i^T(t,X)\mathrm{d}X_t.$$

As in Schattler and Sung (1993) we make the following assumptions: $S_i : C \to \Re$, $i = 1,2$, is and $\mathcal{F}_1$-measurable which is bounded below and α_i, β_i, $i = 1,2$ are all $\mathcal{F}_t$ predictable, α_i is bounded and β_i obeys $\beta_i^T f \in \Phi_b(U)$, and $\beta_i\sigma$ is bounded. We let Σ be the class of all $\mathcal{F}_1$-measurable random variables which can be written as described above.

The following Theorem and its corollary, in which we deal with the agents' problem, is our version of Theorem 4.1 from Schattler and Sung (1993).

Theorem 4 *Let $\sigma^{-1}f \in \Phi(U)$; $c_i \in \Phi_b(U)$, $i = 1,2$; and suppose $S_1, S_2 \in \Sigma$. If condition (A4) holds, then the problem to maximize is $J(u)$ defined by*

$$E_u \left[- \left| -\exp\left\{ -r_1 \left(S_1(X) - \int_0^1 c_1(t,X,u)\mathrm{d}t \right) \right\} \right|^{\theta} \left| -exp\left\{ -r_2 \left(S_2(X) - \int_0^1 c_2(t,X,u)\mathrm{d}t \right) \right\} \right|^{(1-\theta)} \right]$$

over all $u \in \mathcal{U}$ has a solution for all $\theta \in [0,1]$. There exist $\mathcal{F}_t$-adapted processes $\mathcal{V}_t^\theta$ and $\hat{\nabla}\mathcal{V}_t^\theta$ such that $u^* \in \mathcal{U}$ is an optimal control if and only if u^* maximizes

$$\hat{\nabla}\mathcal{V}_t^\theta f(t,X,u) + \mathcal{V}_t^\theta(r_1\theta c_1(t,X,u) + r_2(1-\theta)c_2(t,X,u))$$

over U for almost every $t \in [0,1]$, $w \in \Omega$. Furthermore, if u^* is an optimal control, then

$$\begin{aligned} S_1(X) + S_2(X) &= \frac{r_1+r_2}{r_2}\theta W_{10} + \frac{r_1+r_2}{r_1}(1-\theta)W_{20} + \frac{r_1+r_2}{r_2}\theta\int_0^1 c_1(t,X,u^*)\mathrm{d}t \\ &+\frac{r_1+r_2}{r_1}(1-\theta)\int_0^1 c_2(t,X,u^*)\mathrm{d}t - \frac{r_1+r_2}{r_1r_2}\int_0^1 \frac{\hat{\nabla}\mathcal{V}_t^\theta\sigma(t,X)}{\mathcal{V}_t^\theta}\mathrm{d}B_t^{u^*} \\ &+\frac{1}{2}\frac{r_1+r_2}{r_1r_2}\int_0^1 \frac{\| \hat{\nabla}\mathcal{V}_t^\theta\sigma(t,X) \|^2}{\mathcal{V}_t^{\theta 2}}\mathrm{d}t, \end{aligned}$$

where the constants W_{10} and W_{20} are the agents' certainty equivalent at time 0.

Proof. The following proof is a revised version of the proof of Theorem 4.1 from Schattler and Sung (1993).

$$J(u) = E_u[-\exp\{-r_1\theta(S_1(X) - \int_0^1 c_1(t,X,u)\mathrm{d}t) - r_2(1-\theta)(S_2(X) - \int_0^1 c_2(t,X,u)\mathrm{d}t)\}].$$

Since $S_1, S_2 \in \Sigma$,

$$\begin{aligned} J(u) &= E_u[-\exp\{-r_1\theta(S_{11}(X) + \int_0^1 \alpha_1(t,X)\mathrm{d}t + \int_0^1 \beta_1^T(t,X)\mathrm{d}X_t - \int_0^1 c_1(t,X,u)\mathrm{d}t) \\ &-r_2(1-\theta)(S_{21}(X) + \int_0^1 \alpha_2(t,X)\mathrm{d}t + \int_0^1 \beta_2^T(t,X)\mathrm{d}X_t - \int_0^1 c_2(t,X,u)\mathrm{d}t)\}]. \end{aligned}$$

Since $\mathrm{d}X_t = f(t,X,u)\mathrm{d}t + \sigma(t,X)\mathrm{d}B_t^u$, the utility becomes

$$\begin{aligned} J(u) &= E_u[-\exp\{-r_1\theta S_{11}(X) - r_2(1-\theta)S_{21}(X) \\ &+\int_0^1 [r_1\theta(c_1(t,X,u) - \alpha_1(t,X) - \beta_1^T(t,X)f(t,X,u)) \\ &\quad + r_2(1-\theta)(c_2(t,X,u) - \alpha_2(t,X) - \beta_2^T(t,X)f(t,X,u))]\mathrm{d}t \\ &-\int_0^1 \left(r_1\theta\beta_1^T(t,X) + r_2(1-\theta)\beta_2^T(t,X)\right)\sigma(t,X)\mathrm{d}B_t^u\}]. \end{aligned}$$

By Theorem 3 the stochastic control problem has a solution and there exist $\mathcal{F}_t$-adapted processes $\mathcal{V}_t^\theta$ and $\nabla\mathcal{V}_t^\theta$ such that $u^* \in \mathcal{U}$ is an optimal control if and only if u^* maximizes

$$\begin{aligned} \nabla\mathcal{V}_t^\theta &\times \{f(t,X,u) + \sigma(t,X)\sigma^T(t,X)(r_1\theta\beta_1^T(t,X) + r_2(1-\theta)\beta_2^T(t,X))\} \\ &+\mathcal{V}_t^\theta \times \{r_1\theta(c_1(t,X,u) - \alpha_1(t,X) - \beta_1^T(t,X)f(t,X,u)) \\ &\quad + r_2(1-\theta)(c_2(t,X,u) - \alpha_2(t,X) - \beta_2^T(t,X)f(t,X,u)) \\ &\quad + \frac{1}{2}\|\sigma^T(t,X)(r_1\theta\beta_1(t,X) + r_2(1-\theta)\beta_2(t,X))\|^2\} \end{aligned}$$

over the control set. If we define a new $\mathcal{F}_t$-adapted process $\hat{\nabla}\mathcal{V}_t^\theta$ by

$$\hat{\nabla}\mathcal{V}_t^\theta := \nabla\mathcal{V}_t^\theta - \mathcal{V}_t^\theta(r_1\theta\beta_1^T(t,X) + r_2(1-\theta)\beta_2^T(t,X)),$$

then the control dependent part of the function H is of the form

$$\hat{\nabla}\mathcal{V}_t^\theta.f(t,X,u) + \mathcal{V}_t^\theta.(r_1\theta c_1(t,X,u) + r_2(1-\theta)c_2(t,X,u)),$$

so u^* maximizes the previous expressing over U for almost every $t \in [0,1]$, $w \in \Omega$.

To derive the representation for salary function, we define the agents' certainty equivalent wealth processes as follows:

$$\begin{aligned} W_{1t}^\theta &= -\frac{r_2^2}{r_1(r_1+r_2)^2}\frac{1}{\theta^2}\ln(-\mathcal{V}_t^\theta), \\ W_{2t}^\theta &= -\frac{r_1^2}{r_2(r_1+r_2)^2}\frac{1}{(1-\theta)^2}\ln(-\mathcal{V}_t^\theta). \end{aligned}$$

So,

$$W_t^\theta = \left(\frac{(r_1+r_2)^2\theta^2}{r_2^2}W_{1t} + \frac{(r_1+r_2)^2(1-\theta)^2}{r_1^2}W_{2t}\right) = -K\ln(-\mathcal{V}_t^\theta),$$

where $K \equiv \left(\frac{1}{r_1} + \frac{1}{r_2}\right)$, then By Itô's rule

$$\mathrm{d}W_t^\theta = -(\frac{1}{r_1} + \frac{1}{r_2})\frac{1}{\mathcal{V}_t^\theta}\mathrm{d}\mathcal{V}_t^\theta + \frac{1}{2}\frac{\|\nabla\mathcal{V}_t^\theta\sigma(t,X)\|^2}{(\mathcal{V}_t^\theta)^2}(\frac{1}{r_1} + \frac{1}{r_2})\mathrm{d}t.$$

By Theorem 3 $\mathcal{V}_t^\theta$ has an Itô differential of the form

$$\begin{aligned} \mathcal{V}_t^\theta &= \mathcal{V}_o^\theta + \int_0^t \{\nabla\mathcal{V}_t^\theta.\left(f(t,X,u) + \sigma(t,X)\sigma^T(t,X)(r_1\theta\beta_1^T(t,X) + r_2(1-\theta)\beta_2^T(t,X))\right) \\ &\quad + \mathcal{V}_t^\theta.(r_1\theta(c_1(t,X,u) - \alpha_1(t,X) - \beta_1^T(t,X)f(t,X,u)) \\ &\qquad + r_2(1-\theta)(c_2(t,X,u) - \alpha_2(t,X) - \beta_2^T(t,X)f(t,X,u)) \\ &\qquad + \frac{1}{2}\|\sigma^T(t,X)(r_1\theta\beta_1(t,X) + r_2(1-\theta)\beta_2(t,X))\|^2)\}\mathrm{d}t + \int_0^t \nabla\mathcal{V}_t^\theta\mathrm{d}X_t. \end{aligned}$$

So,

$$\begin{aligned} \mathrm{d}W_t^\theta &= -(\frac{1}{r_1} + \frac{1}{r_2})\beta_1^T\mathrm{d}X_t \\ &\quad +(\frac{1}{r_1} + \frac{1}{r_2})\frac{1}{\mathcal{V}_t^\theta}\{\nabla\mathcal{V}_t^\theta\left(f(t,X,u) - \sigma(t,X)\sigma^T(t,X)(r_1\theta\beta_1(t,X) + r_2(1-\theta)\beta_2(t,X))\right) \\ &\qquad + \mathcal{V}_t^\theta(r_1\theta(c_1(t,X,u) - \alpha_1(t,X) - \beta_1^T(t,X)f(t,X,u)) + r_2(1-\theta)(c_2(t,X,u) \\ &\qquad - \alpha_2(t,X) - \beta_2^T(t,X)f(t,X,u)) + \frac{1}{2}\|\sigma^T(t,X)(r_1\theta\beta_1(t,X) \\ &\qquad + r_2(1-\theta)\beta_2(t,X))\|^2)\}\mathrm{d}t \\ &\quad +\frac{1}{2}\frac{\|\nabla\mathcal{V}_t^\theta\sigma(t,X)\|^2}{(\mathcal{V}_t^\theta)^2}(\frac{1}{r_1} + \frac{1}{r_2})\mathrm{d}t. \end{aligned}$$

Since $K = \left(\frac{1}{r_1} + \frac{1}{r_2}\right)$, then

$$\begin{aligned} dW_t^\theta &= \{Kr_1\theta c_1(t,X,u) + Kr_2(1-\theta)c_2(t,X,u)\}dt \\ &\quad -\{Kr_1\theta\alpha_1(t,X) + Kr_2(1-\theta)\alpha_2(t,X)\}dt + \{Kr_1\theta\beta_1^T + Kr_2(1-\theta)\beta_2^T\}dX_t \\ &\quad -K\frac{\hat{\nabla}\mathcal{V}_t^\theta}{\mathcal{V}_t^\theta}\sigma(t,X)dB_t^{u^*} + \frac{K}{2}\frac{\|\hat{\nabla}\mathcal{V}_t^\theta\sigma(t,X)\|^2}{(\mathcal{V}_t^\theta)^2}dt. \end{aligned}$$

When we integrate between 0 and 1, the result follows. ■

Corollary 1 *Suppose $m = n$ and assume that the matrix $D_u f(t,X,u)$ is invertible for all $(t,X,u) \in [0,1] \times \Omega \times U$. If an optimal control u^* for the agents' problem (given in definition 2) takes values in the interior of U, then the following representation of the sharing function $S_1 + S_2$ in terms of the optimal control u^* is: $S_1(X) + S_2(X)$ equals*

$$\begin{aligned} &\frac{r_1+r_2}{r_2}\theta W_{10} + \frac{r_1+r_2}{r_1}(1-\theta)W_{20} \\ &+\frac{r_1+r_2}{r_2}\theta\int_0^1 c_1(t,X,u^*)dt + \frac{r_1+r_2}{r_1}(1-\theta)\int_0^1 c_2(t,X,u^*)dt \\ &-\frac{r_1+r_2}{r_1r_2}\int_0^1 (r_1\theta\nabla_u c_1(t,X,u^*) + r_2(1-\theta)\nabla_u c_2(t,X,u^*))D_u^{-1}f(t,X,\bar{u})\sigma(t,X)dB_t^{u^*} \\ &+\frac{1}{2}\frac{r_1+r_2}{r_1r_2}\int_0^1 \|(r_1\theta\nabla_u c_1(t,X,u^*) + r_2(1-\theta)\nabla_u c_2(t,X,u^*))D_u^{-1}f(t,X,\bar{u})\sigma(t,X)\|^2dt. \end{aligned}$$

Proof. Since u^* maximizes $\hat{\nabla}\mathcal{V}_t^\theta f(t,X,u) + \mathcal{V}_t^\theta(r_1\theta c_1(t,X,u) + r_2(1-\theta)c_2(t,X,u))$,

$$\hat{\nabla}\mathcal{V}_t^\theta D_u f(t,X,u) + \mathcal{V}_t^\theta(r_1\theta\nabla_u c_1(t,X,u) + r_2(1-\theta)\nabla_u c_2(t,X,u)) = 0.$$

Then,

$$\frac{\hat{\nabla}\mathcal{V}_t^\theta}{\mathcal{V}_t^\theta} = -(r_1\theta\nabla_u c_1(t,X,u) + r_2(1-\theta)\nabla_u c_2(t,X,u))D_u^{-1}f(t,X,u).$$

The result follows by plugging this to the representation of the sharing function $S_1 + S_2$. ■

It is imperative to deal with controls which the principal can enforce in the problem given in definition 2. If there are controls $\bar{u}$ for which the representation theorem holds, but which are not optimal in the agents' problem (given in definition 2) for the associated sharing function $\bar{S}_1 + \bar{S}_2$, then the principal cannot make the agents follow $\bar{u}$ by assigning $\bar{S}_1 + \bar{S}_2$.

Definition 3 *A control u is "implementable" if the sharing function $\bar{S}_1 + \bar{S}_2 = \bar{S}_1(u) + \bar{S}_2(u)$ in the representation theorem with control u is assigned to the agents as salary function and u comes as an optimal control for the agents' problem (given in definition 2) with $\bar{S}_1(u) + \bar{S}_2(u)$.*

Now we are set to go into step 2. Note that in step 1, the class of admissible salary rules was narrowed down to the above given form. The next step will be to adopt this Representation Theorem to the problem given in definition 2, where agents choose $(\mu, \sigma) = (\{\mu_t, \sigma_t\}_t)$ as their control laws, and then characterize the implementable ones.

First we have another corollary to Theorem 4:

Corollary 2 *Suppose that given an arbitrary admissible salary function S_1, S_2, the agents optimal control law (to be obtained from the problem given in definition 2) is $(\hat{\mu}, \hat{\sigma})$, where $\hat{\mu}_t, t \in [0,1]$ almost everywhere lies in the interior of U, and that the resulting agents' expected utility satisfies the participation constraints. Then the salary function can be represented as the following:*

$$\begin{aligned} S_1(\hat{\mu}, \hat{\sigma}) + S_2(\hat{\mu}, \hat{\sigma}) &= \frac{r_1 + r_2}{r_2}\theta W_{10} + \frac{r_1 + r_2}{r_1}(1-\theta)W_{20} \\ &+ \int_0^1 \left[\frac{r_1 + r_2}{r_2}\theta c_1(\hat{\mu}_t, \hat{\sigma}_t) + \frac{r_1 + r_2}{r_1}(1-\theta)c_2(\hat{\mu}_t, \hat{\sigma}_t)\right] \mathrm{d}t \\ &+ \int_0^1 \left[\frac{r_1 + r_2}{r_2}\theta c_{1\mu}(\hat{\mu}_t, \hat{\sigma}_t) + \frac{r_1 + r_2}{r_1}(1-\theta)c_{2\mu}(\hat{\mu}_t, \hat{\sigma}_t)\right] \mathrm{d}X_t \\ &- \int_0^1 \left[\frac{r_1 + r_2}{r_2}\theta c_{1\mu}(\hat{\mu}_t, \hat{\sigma}_t) + \frac{r_1 + r_2}{r_1}(1-\theta)c_{2\mu}(\hat{\mu}_t, \hat{\sigma}_t)\right] \hat{\mu}_t \mathrm{d}t \\ &+ \frac{1}{2}\frac{r_1 + r_2}{r_1 r_2}\int_0^1 \left(r_1\theta c_{1\mu}(\hat{\mu}_t, \hat{\sigma}_t) + r_2(1-\theta)c_{2\mu}(\hat{\mu}_t, \hat{\sigma}_t)\right)^2 \hat{\sigma}_t^2 \mathrm{d}t. \end{aligned} \tag{11}$$

Proof. From Theorem 4, we see that for any admissible diffusion-rate processes σ_t, if $\hat{\mu}_t$ is optimal, then the salary function can be written as $S_1(\hat{\mu}, \sigma) + S_2(\hat{\mu}, \sigma)$. Suppose that $(\hat{\mu}_t, \hat{\sigma}_t)$ is an optimal control pair for the agents' problem (given in definition 2). This means that given any $\hat{\sigma}_t$, the optimal drift rate is $\hat{\mu}_t$. Hence, the salary function can be represented as $S_1(\hat{\mu}, \hat{\sigma}) + S_2(\hat{\mu}, \hat{\sigma})$. ■

This representation is only a necessary condition for an admissible salary function S_1, S_2 to be an optimal salary function. But the representation does not consider the agents' first order condition with respect to σ. Moreover, the representation is $\mathcal{F}_1$-measurable, i.e, depends on the entire history. However, in the problem given in definition 2, principal can only implement X_1-measurable salary rule. Because of these two reasons, the representation gives a too large class of salary functions. Thus, we have to further narrow it down. This leads to the following Theorem, which is a modified version of proposition A1 of Sung (1995).

Theorem 5 *If for almost all $t \in [0,1]$, $(\bar{\mu}_t, \bar{\sigma}_t) \in U \times \mathbb{S}$, where U is a bounded open subset of $\Re$ and $\mathbb{S}$ is a compact subset of $\Re_{++}$, maximizes $\Phi^a(\mu_t, \sigma_t; \bar{\mu}_t, \bar{\sigma}_t)$ defined by*

$$\begin{aligned} \Phi^a(\mu_t, \sigma_t; \bar{\mu}_t, \bar{\sigma}_t) &= \frac{r_1 + r_2}{r_1 r_2}\left(r_1\theta c_{1\mu}(\bar{\mu}_t, \bar{\sigma}_t) + r_2(1-\theta)c_{2\mu}(\bar{\mu}_t, \bar{\sigma}_t)\right)\mu_t \\ &- \frac{r_1 + r_2}{r_1 r_2}\left(r_1\theta c_1(\mu_t, \sigma_t) + r_2(1-\theta)c_2(\mu_t, \sigma_t)\right) \\ &- \frac{1}{2}\frac{r_1 + r_2}{r_1 r_2}\left(r_1\theta c_{1\mu}(\bar{\mu}_t, \bar{\sigma}_t) + r_2(1-\theta)c_{2\mu}(\bar{\mu}_t, \bar{\sigma}_t)\right)^2 \sigma_t^2, \end{aligned}$$

then $(\bar{\mu}, \bar{\sigma})$ *is implementable. Furthermore, converse holds if assumption 5 is satisfied (i.e.* $\theta = \frac{r_2}{r_1+r_2}$ *and* $(1-\theta) = \frac{r_1}{r_1+r_2}$*).*

Proof. The agents' problem given a salary function as defined in equation 11 is to choose $\mathcal{F}_t$-predictable control laws, $\{\mu_t, \sigma_t\}_t$, in order to maximize

$$E\left[-\left|-\exp\left\{-r_1(\bar{S}_1(\bar{\mu}_t,\bar{\sigma}_t) - \int_0^1 c_1(\mu_t,\sigma_t)\mathrm{d}t)\right\}\right|^{\theta}\left|-exp\left\{-r_2(\bar{S}_2(\bar{\mu}_t,\bar{\sigma}_t) - \int_0^1 c_2(\mu_t,\sigma_t)\mathrm{d}t)\right\}\right|^{(1-\theta)}\right].$$

Recall that $K \equiv \frac{r_1+r_2}{r_1 r_2}$, and additionally doing the required substitutions on the above term renders

$$\begin{aligned}
&E\{-\exp\{-\frac{1}{K}[\frac{(r_1+r_2)^2}{r_2^2}\theta^2 W_{10} + \frac{(r_1+r_2)^2}{r_1^2}(1-\theta)^2 W_{20} \qquad (12)\\
&\frac{(r_1+r_2)^2}{r_2^2}\theta^2\int_0^1 c_1(\bar{\mu}_t,\bar{\sigma}_t)\mathrm{d}t + \frac{(r_1+r_2)^2}{r_1^2}(1-\theta)^2\int_0^1 c_2(\bar{\mu}_t,\bar{\sigma}_t)\mathrm{d}t\\
&+\frac{(r_1+r_2)}{r_1r_2}\int_0^1 \left(r_1\theta c_{1\mu}(\bar{\mu}_t,\bar{\sigma}_t) + r_2(1-\theta)c_{2\mu}(\bar{\mu}_t,\bar{\sigma}_t)\right)\mu_t\mathrm{d}t\\
&-\frac{(r_1+r_2)}{r_1r_2}\int_0^1 \left(r_1\theta c_{1\mu}(\bar{\mu}_t,\bar{\sigma}_t) + r_2(1-\theta)c_{2\mu}(\bar{\mu}_t,\bar{\sigma}_t)\right)\bar{\mu}_t\mathrm{d}t\\
&+\frac{1}{2}\frac{(r_1+r_2)}{r_1r_2}\int_0^1 \left(r_1\theta c_{1\mu}(\bar{\mu}_t,\bar{\sigma}_t) + r_2(1-\theta)c_{2\mu}(\bar{\mu}_t,\bar{\sigma}_t)\right)^2\bar{\sigma}_t^2\mathrm{d}t\\
&-\frac{(r_1+r_2)}{r_1r_2}\int_0^1 \left(r_1\theta c_1(\mu_t,\sigma_t) + r_2(1-\theta)c_2(\mu_t,\sigma_t)\right)\mathrm{d}t\\
&+\frac{(r_1+r_2)}{r_1r_2}\int_0^1 \left(r_1\theta c_{1\mu}(\bar{\mu}_t,\bar{\sigma}_t) + r_2(1-\theta)c_{2\mu}(\bar{\mu}_t,\bar{\sigma}_t)\right)\sigma_t\mathrm{d}B_t]\}\}.
\end{aligned}$$

Consequently by Lemma A1 of Sung (1995) the agents' dynamic programming equation takes the following form: [16]

$$\begin{aligned}
0 \equiv\; &\frac{\partial V}{\partial t}(t,X) + \max_{\mu_t,\sigma_t}\{\frac{\partial V}{\partial X_t}(t,X)[\mu_t - \theta^2\left(r_1\theta c_{1\mu}(\bar{\mu}_t,\bar{\sigma}_t) + r_2(1-\theta)c_{2\mu}(\bar{\mu}_t,\bar{\sigma}_t)\right)] \qquad (13)\\
&+\frac{\sigma_t^2}{2}\frac{\partial^2 V}{\partial X_t^2}(t,X) + \frac{1}{K}(\frac{r_1+r_2}{r_1r_2}\left(r_1\theta c_1(\mu_t,\sigma_t) + r_2(1-\theta)c_2(\mu_t,\sigma_t)\right)\\
&+\frac{r_1+r_2}{r_1r_2}\left(r_1\theta c_{1\mu}(\bar{\mu}_t,\bar{\sigma}_t) + r_2(1-\theta)c_{2\mu}(\bar{\mu}_t,\bar{\sigma}_t)\right)\bar{\mu}_t\\
&-\frac{r_1+r_2}{r_1r_2}\left(r_1\theta c_{1\mu}(\bar{\mu}_t,\bar{\sigma}_t) + r_2(1-\theta)c_{2\mu}(\bar{\mu}_t,\bar{\sigma}_t)\right)\mu_t\\
&-\frac{(r_1+r_2)^2}{r_2^2}\theta^2 c_1(\bar{\mu}_t,\bar{\sigma}_t) - \frac{(r_1+r_2)^2}{r_1^2}(1-\theta)^2 c_2(\bar{\mu}_t,\bar{\sigma}_t)\\
&-\frac{1}{2}\frac{r_1+r_2}{r_1r_2}\left(r_1\theta c_{1\mu}(\bar{\mu}_t,\bar{\sigma}_t) + r_2(1-\theta)c_{2\mu}(\bar{\mu}_t,\bar{\sigma}_t)\right)\bar{\sigma}_t^2\\
&+\frac{1}{2}\frac{r_1+r_2}{r_1r_2}\left(r_1\theta c_{1\mu}(\bar{\mu}_t,\bar{\sigma}_t) + r_2(1-\theta)c_{2\mu}(\bar{\mu}_t,\bar{\sigma}_t)\right)\sigma_t^2)V(t,X_t)\}.
\end{aligned}$$

[16] Note that $V(t,X_t)$ defined by $V(t,X_t) \equiv \sup_{\{\mu_t,\sigma_t\}_t} J(\{\mu_t,\sigma_t\}_t; t, X)$ depends on the current state X_t rather than the whole past given by $X^t \equiv \{X_{t'} : t' \in [0,t]\}$.

with the the terminal condition

$$V(1,X) \quad = \quad -\exp\left\{-\frac{1}{K}\left[\frac{(r_1+r_2)^2}{r_2^2}\theta^2 W_{10} + \frac{(r_1+r_2)^2}{r_1^2}(1-\theta)^2 W_{20}\right]\right\}.$$

If $\{\bar{\mu}_t, \bar{\sigma}_t\}_t$ maximizes Φ^a for all most all $t \in [0,1]$ and defining $V(t,X) = -\exp\{-\frac{1}{K}[\frac{(r_1+r_2)^2}{r_2^2}\theta^2 W_{10} + \frac{(r_1+r_2)^2}{r_1^2}(1-\theta)^2 W_{20}]\}$, then it satisfies the dynamic programming equation with $\{\bar{\mu}_t, \bar{\sigma}_t\}_t$. As was done in Sung (1995), then by the verification theorem of the dynamic equation in Schattler and Sung (1993), $\{\bar{\mu}_t, \bar{\sigma}_t\}_t$ is the optimal control pair where $V(t,X)$ as given above is the value function.

Now for the converse, we need the additional assumption that $\theta = \frac{r_2}{r_1+r_2}$. Suppose that $\{\bar{\mu}_s, \bar{\sigma}_s\}$ is implementable for all $s \in [t,1]$. Then by substituting for θ into equation 12 we get that the agents' optimal remaining expected utility at time t is

$$\begin{aligned}
&-\exp\{-\frac{1}{K}[W_{10}+W_{20}]\} \times \\
&E[\exp\{-\frac{1}{2}\int_t^1 \left(-\frac{1}{K}(c_{1\mu}(\bar{\mu}_s,\bar{\sigma}_s)+c_{2\mu}(\bar{\mu}_s,\bar{\sigma}_s))\right)^2 \bar{\sigma}_s^2 \mathrm{d}s \\
&\qquad\qquad + \int_t^1 \left(-\frac{1}{K}(c_{1\mu}(\bar{\mu}_s,\bar{\sigma}_s)+c_{2\mu}(\bar{\mu}_s,\bar{\sigma}_s))\right)\bar{\sigma}_s\}\mathrm{d}B_s].
\end{aligned}$$

The second term is the expectation of a Girsanov density, which is 1. Hence, if $\{\bar{\mu}_s, \bar{\sigma}_s\}_s$ is implementable and $\theta = \frac{r_2}{r_1+r_2}$, the agents' optimal remaining expected utility at time t is $V(t,X) = -\exp\{-\frac{1}{K}[W_{10}+W_{20}]\}$. Since $-\exp\{-\frac{1}{K}[W_{10}+W_{20}]\}$ meets all assumption in Lemma A1 of Sung (1995), the agents' optimal remaining expected utility satisfies equation 13, which implies $\{\bar{\mu}_s, \bar{\sigma}_s\}_s \in \operatorname{argmax}_{(\mu_s,\sigma_s)} \Phi^a(\mu_s,\sigma_s;\bar{\mu}_s,\bar{\sigma}_s)$ for almost all $s \in [t,1]$. As the choice of t is arbitrary, this holds for $s \in [0,1]$ ■

Let us now define a set of controls that can induce the agents to choose the controls implied by the salary function given in the form of the representation as done in Sung (1995):

$$\mathbf{Z} \equiv \left\{\{(\bar{\mu}_t,\bar{\sigma}_t)\} | t \in [0,1], \{\bar{\mu}_t,\bar{\sigma}_t\}_t \in \operatorname{argmax}_{(\mu_t,\sigma_t)} \Phi^a(\mu_t,\sigma_t;\bar{\mu}_t,\bar{\sigma}_t)\right\}.$$

But, note that not every implementable control is in that set.(In the case of $\theta = \frac{r_2}{r_1+r_2}$, then every implementable control should be in $\mathbf{Z}$.) $\mathbf{Z}$ is still a large class of controls from the point of view of the principal since the set contains controls that are not necessarily X_1-measurable. We have to take into account that:

$$\mathbf{Z}' = \mathbf{Z} \cap \{S_1(\{\bar{\mu}_t,\bar{\sigma}_t\}) + S_2(\{\bar{\mu}_t,\bar{\sigma}_t\}) \text{ as in equation 11 and is } X_1\text{-measurable}\}$$

Set $\mathbf{Z}'$ gives the complete description of the implementable contracts (when $\theta = \frac{r_2}{r_1+r_2}$). Now, we are ready to deal with the principal's problem given in definition 2.

$$\begin{aligned}
E[-\exp\{-R(X_1 - W_{10} - W_{20} - \int_0^1 (c_1(\hat{\mu}_t,\hat{\sigma}_t)+c_2(\hat{\mu}_t,\hat{\sigma}_t))\mathrm{d}t - \int_0^1 (c_{1\mu}(\hat{\mu}_t,\hat{\sigma}_t)+c_{2\mu}(\hat{\mu}_t,\hat{\sigma}_t))\hat{\sigma}_t)\mathrm{d}B_t \\
-\frac{1}{2K}\int_0^1 (c_{1\mu}(\hat{\mu}_t,\hat{\sigma}_t)+c_{2\mu}(\hat{\mu}_t,\hat{\sigma}_t))^2\hat{\sigma}_t^2)\mathrm{d}t\}].
\end{aligned}$$

As was done in Sung (1995) it is enough to consider stationary solutions to this problem by letting principal choose $(\{\hat{\mu}_t, \hat{\sigma}\}_t) \in \mathbf{Z}$, such that $(\hat{\mu}_t, \hat{\sigma}_t) = (\hat{\mu}, \hat{\sigma})$ for almost all the $t \in [0,1]$. Indeed, following the same steps we identify the optimal salary functions with stationary controls and observe that they are linear. Moreover, employing the same techniques it can be shown that these optimal salary functions, identified by using equation 11, are also a solution to the principal's problem (given in definition 2) where the controls are chosen from $\mathbf{Z}'$.

$$\begin{aligned} S(\mu^*, \sigma^*) &= S_1(\mu^*, \sigma^*) + S_2(\mu^*, \sigma^*) \\ &= W_{10} + W_{20} + c_1(\mu^*, \sigma^*) + c_2(\mu^*, \sigma^*) \\ &\quad + \left(c_{1\mu}(\mu^*, \sigma^*) + c_{2\mu}(\mu^*, \sigma^*)\right)(X_1 - X_0) + \left(c_{1\mu}(\mu^*, \sigma^*) + c_{2\mu}(\mu^*, \sigma^*)\right)\mu^* \\ &\quad + \frac{r_1 r_2}{2(r_1 + r_2)} \left(c_{1\mu}(\mu^*, \sigma^*) + c_{2\mu}(\mu^*, \sigma^*)\right)^2 \sigma^{*2}. \end{aligned}$$

Observing $S_1(\mu^*, \sigma^*) = \frac{r_2}{r_1 + r_2} S(\mu^*, \sigma^*)$ and $S_2(\mu^*, \sigma^*) = \frac{r_1}{r_1 + r_2} S(\mu^*, \sigma^*)$ leads us to the final step where we have to show that these salary functions and control laws are a member of the constraint set of the problem given in definition 1.

We have to point out that for the solution to the problem given in definition 2 (that we will refer to as the second problem), clearly feasibility (item *i*) and individual rationality (item *iii*) requirements of the problem given in definition 1 (that we call the first problem) are satisfied. So the remaining part is to show that the collusion requirement of the first problem (item *ii* which we may refer to as the first nested maximization problem) is also satisfied. In other words we have to establish that $(S_1^*, S_2^*; \mu^*, \sigma^*)$ solves the first nested maximization problem where (μ^*, σ^*) derived as the optimal control laws to the second nested maximization problem at values (S_1^*, S_2^*).

Here we present the intuitive argument to establish this relation which easily can be verified by checking the first order conditions of the first nested maximization problem. Because that there are no income effects with exponential utility functions, and (μ^*, σ^*) solves the second nested maximization problem at salary functions (S_1^*, S_2^*) derived above, agents do not possess any joint deviation opportunities in terms of control laws when salary functions (S_1^*, S_2^*) is offered. Second, since the ratio of the salary functions $\frac{S_1^*}{S_2^*}$ is equal to $\frac{r_2}{r_1}$ in every contingencies, the agents cannot share risk more efficiently by deviating to resource feasible and participatory side contracting scheme. Thus, $(S_1^*, S_2^*; \mu^*, \sigma^*)$ solves the first nested maximization problem, therefore, $(S_1^*, S_2^*; \mu^*, \sigma^*)$ is in the constraint set of the first problem. Since the constraint set of the first problem is a subset of the constraint set of the second[17], and $(S_1^*, S_2^*; \mu^*, \sigma^*)$ solves the second problem, $(S_1^*, S_2^*; \mu^*, \sigma^*)$ also is a solution to the first problem. This finishes the proof of Theorem 2.

[17]Please refer to the discussion in the second paragraph right after definition 2.

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