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Twisted partial difference sets, twisted LP-packings and bent partitions

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Abstract

Recently, the first constructions of bent partitions of elementary abelian groups that do not induce partial difference sets or Latin square type partial difference set packings (LP-packings) have been presented (Anbar et al.; Wang et al., 2025). Motivated by observations on the differential properties of examples of these bent partitions, the notions of twisted partial difference sets and twisted LP-packings in abelian groups $\mathcal{G}$ are introduced in this article. Basic properties of twisted partial difference sets and twisted LP-packings, as well as properties of the corresponding character values, are investigated. It is shown that the sets arising from all recently introduced bent partitions are twisted partial difference sets, and that these bent partitions induce twisted LP-packings. As a consequence, all nontrivial bent partitions of elementary abelian groups known so far are shown to correspond either to LP-packings or to twisted LP-packings.

Keywords Bent functions · Vectorial bent functions · Bent partitions · (Twisted) partial difference sets · (Twisted) LP-packings

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1 Introduction

Let $\mathcal{G}$ be an abelian group of order v, written additively, and let $\mathcal{G}^* = \mathcal{G} \setminus \{0\}$. Let D be a subset of $\mathcal{G}$ of size κ. The set D is called a $(v, \kappa, \lambda, \mu)$ partial difference set (PDS) in $\mathcal{G}$

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if every nonzero element of D can be written as a difference $d_1 - d_2$, with $d_1, d_2 \in D$, in exactly λ ways, and every element of $\mathcal{G}^* \backslash D$ can be written as such a difference in exactly μ ways.

A PDS D is called *regular* if $-D = D$ and $0 \notin D$. This assumption is not restrictive, since by [12, Proposition 1.2] every partial difference set D with $\lambda \neq \mu$ (that is, D is not a difference set) satisfies $-D = D$, and, moreover, the set $D \backslash \{0\}$ is again a PDS.

Throughout this article, attention is restricted to PDSs that are not difference sets. Moreover, it is assumed that D satisfies $D \neq \emptyset$, $D \neq \{0\}$, $D \neq \mathcal{G}$ and $D \neq \mathcal{G} \setminus \{0\}$. These conditions imply that $2 \leq \kappa \leq v - 2$. Note that if D is a partial difference set in $\mathcal{G}$, then its complement $\mathcal{G} \backslash D$ is also a partial difference set.

It is easily verified that the parameters of a PDS satisfy

$$\kappa^2 = (\kappa - \mu) + \kappa(\lambda - \mu) + \mu v.$$

If the parameters of a regular PDS D are of the form $(v, \kappa, \lambda, \mu) = \left(n^2,\ s(n-1),\ n + s^2 - 3s,\ s^2 - s\right)$ for some positive integers n and s, then D is said to be of (n, s) Latin square type.

Recently, in [11], the concept of a *Latin square partial difference set packing (LP-packing)* was introduced. Let $t > 1$ and $c > 0$ be integers, let $\mathcal{G}$ be an abelian group of order t^2c^2, and let U be a subgroup of $\mathcal{G}$ of order tc. A (c, t) *LP-packing* in $\mathcal{G}$ relative to U is a collection $\{P_1, \ldots, P_t\}$ of t pairwise disjoint regular (tc, c) Latin square type partial difference sets in $\mathcal{G}$ such that $\bigcup_{i=1}^{t} P_i = \mathcal{G} \setminus U$.

Partial difference sets, and more recently LP-packings, have been investigated in connection with bent and vectorial bent functions; see [1, 2, 7, 9, 14, 17, 18]. Let $\mathbb{V}_n^{(p)}$ denote an n-dimensional vector space over the prime field $\mathbb{F}_p$. Recall that a function $f : \mathbb{V}_n^{(p)} \to \mathbb{F}_p$ is called a *bent function* if its *derivative* $D_a f(x) = f(x + a) - f(x)$ is balanced for all nonzero $a \in \mathbb{V}_n^{(p)}$, i.e., for every $c \in \mathbb{F}_p$, the inverse image set $(D_a f)^{-1}(c) = \{x \in V_n^{(p)} : D_a f(x) = c\}$ has the same cardinality.

Alternatively, the function f is bent if the *Walsh transform* of f,

$$\mathcal{W}_f(b) = \sum_{x \in \mathbb{V}_n^{(p)}} \zeta_p^{f(x) - \langle b, x \rangle_n}, \quad \zeta_p = e^{2\pi\iota/p}, \tag{1}$$

where $\langle u, v \rangle_n$ denotes a fixed (non-degenerate) inner product on $\mathbb{V}_n^{(p)}$ and ι is a complex primitive 4-th root of unity, has absolute value $p^{n/2}$ for every $b \in \mathbb{V}_n^{(p)}$.

Observe that the Walsh transform of a Boolean bent function f satisfies $\mathcal{W}_f(b) = 2^{n/2}(-1)^{f^*(b)}$, where f^* is a Boolean function, called the *dual* of f, which is always bent as well. Since the Walsh transform $\mathcal{W}_f$ is integer-valued when $p = 2$, Boolean bent functions exist only for even values of n. In contrast, p-ary bent functions $f : \mathbb{V}_n^{(p)} \to \mathbb{F}_p$ exist for both even and odd values of n, and their Walsh transform satisfies

$$\mathcal{W}_f(b) = \begin{cases} \pm\zeta_p^{f^*(b)}\, p^{n/2}, & \text{if } n \text{ is even, or if } n \text{ is odd and } p \equiv 1 \mod 4, \\ \pm\iota\, \zeta_p^{f^*(b)}\, p^{n/2}, & \text{if } n \text{ is odd and } p \equiv 3 \mod 4, \end{cases}$$

where $f^* : \mathbb{V}_n^{(p)} \to \mathbb{F}_p$ is again called the *dual* of f.

We distinguish between *weakly regular* bent functions, namely those for which $\mathcal{W}_f(b) = \epsilon\, \zeta_p^{f^*(b)}\, p^{n/2}$, where $\epsilon \in \{\pm 1, \pm\iota\}$ is a constant independent of b, and *non-weakly regular* bent functions, for which the value $\epsilon = \epsilon_b$ depends on b. In particular, a bent function f is said to be *regular* if it is weakly regular with $\epsilon = 1$. Consequently, all Boolean bent functions are regular. The dual of a weakly regular bent function is again bent, whereas this property does not necessarily hold for non-weakly regular bent functions; see [10, 13].

A *vectorial function* $F : \mathbb{V}_n^{(p)} \to \mathbb{V}_k^{(p)}$ is called *bent* (or *vectorial bent*) if its derivative $D_a F(x) = F(x+a) - F(x)$ is balanced for every nonzero $a \in \mathbb{V}_n^{(p)}$. Equivalently, F is bent if every *component function* $F_\alpha(x) = \langle \alpha, F(x)\rangle_k$, $\alpha \in \mathbb{V}_k^{(p)} \setminus \{0\}$, is a bent function, i.e., if

$$\mathcal{W}_{F_\alpha}(b) = \sum_{x \in \mathbb{V}_n^{(p)}} \zeta_p^{\langle \alpha, F(x)\rangle_k - \langle b, x\rangle_n}, \quad \zeta_p = e^{2\pi\iota/p},$$

has absolute value $p^{n/2}$ for all nonzero $\alpha \in \mathbb{V}_k^{(p)}$ and all $b \in \mathbb{V}_n^{(p)}$.

2 Preliminary results

In this section, we collect some preliminary results on partial difference sets, LP-packings, and *bent partitions*, a concept that has recently been introduced in [8].

2.1 Results on partial difference sets

A useful tool for investigating PDSs is Corollary 3.3 in [12], where PDSs are characterized in terms of character sums. For a subset D of $\mathcal{G}$ and a character χ of $\mathcal{G}$, we define $\chi(D) = \sum_{d \in D} \chi(d)$.

Lemma 1 *Let $\mathcal{G}$ be an abelian group of order v, and let $D \subseteq \mathcal{G}$ be a subset satisfying $-D = D$. Then D is a $(v, \kappa, \lambda, \mu)$ partial difference set in $\mathcal{G}$ if and only if, for every nontrivial character χ of $\mathcal{G}$, the value $\chi(D)$ assumes exactly two distinct values. More precisely,*

$$\chi(D) = (\beta \pm \sqrt{\Delta})/2,$$

where $\beta = \lambda - \mu$ and $\Delta = \beta^2 + 4\gamma$, with $\gamma = \kappa - \mu$ if $0 \notin D$, and $\gamma = \kappa - \lambda$ if $0 \in D$. If $D \neq \emptyset$ and $D \neq \mathcal{G} \setminus \{0\}$, then the parameters satisfy

$$0 \le \lambda \le \kappa - 1 \quad \text{and} \quad 0 \le \mu \le \kappa. \tag{2}$$

Moreover, if $0 \in D$ and D has parameters $(v, \kappa, \lambda, \mu, \beta, \Delta)$, then the set $D \backslash \{0\}$ is a partial difference set with parameters

$$(v, \widetilde{\kappa}, \widetilde{\lambda}, \widetilde{\mu}, \widetilde{\beta}, \widetilde{\Delta}) = \big(v,\ \kappa - 1,\ \lambda - 2,\ \mu,\ \beta - 2,\ \Delta\big). \tag{3}$$

Remark 1 Since we assume that D is a partial difference set that is not a difference set, we have $\beta \neq 0$. Moreover, under the assumptions $D \neq \emptyset$, $D \neq \{0\}$, $D \neq \mathcal{G}$ and $D \neq \mathcal{G} \setminus \{0\}$, it follows that $\gamma \geq 0$; see Eq. (2). Consequently, we always have $\Delta = \beta^2 + 4\gamma > 0$.

In particular, by Lemma 1, for an (n, s) Latin square type partial difference set D we have $\chi(D) \in \{-s, n - s\}$; see also [11, Lemma 2.5].

Let $\mathcal{G}$ be an abelian group of order v, and let D be a $(v, \kappa, \lambda, \mu)$ PDS in $\mathcal{G}$. We denote by $\widehat{\mathcal{G}}$ the character group of $\mathcal{G}$, and define a subset D^+ of $\widehat{\mathcal{G}}$ by

$$D^+ = \big\{\chi \in \widehat{\mathcal{G}} \ :\ \chi(D) = (\beta + \sqrt{\Delta})/2\,\big\}. \tag{4}$$

Note that $D^+ = (D \backslash \{0\})^+$; hence it is sufficient to consider D^+ for a regular PDS D.

Lemma 2 [12, Theorem 3.4] *Let $\mathcal{G}$ be an abelian group of order v, and let D be a regular $(v, \kappa, \lambda, \mu)$ PDS in $\mathcal{G}$. Then the set D^+ defined in (4) is a regular PDS in $\widehat{\mathcal{G}}$. In particular, if D is a regular (n, s) Latin square type PDS in $\mathcal{G}$, then D^+ is a regular (n, s) Latin square type PDS in $\widehat{\mathcal{G}}$.*

The PDS D^+ in Lemma 2 is called the *dual* of the PDS D. Since $\mathcal{G}$ and $\widehat{\mathcal{G}}$ are isomorphic, the set D^+ may be identified with a subset of $\mathcal{G}$, and its dual may be considered accordingly.

Lemma 3 [12, Theorem 3.7] *If D is a regular PDS in $\mathcal{G}$, then $(D^+)^+ = D$.*

2.2 Partial difference sets and bent partitions

Partial difference sets appear as preimage sets of certain classes of bent and vectorial bent functions. They form the basic constituents of bent partitions, which in turn generate a large number of bent and vectorial bent functions. We refer to [6, 8, 17] and the survey paper [3] for further details.

Definition 1 [8, Definition 1] Let $n = 2m$ be even. A partition $\Omega = \{A_1, \ldots, A_K\}$ of $\mathbb{V}_n^{(p)}$ into K sets $A_1, \ldots, A_K$ is called a *bent partition* of $\mathbb{V}_n^{(p)}$ of *depth* K if every function $f : \mathbb{V}_n^{(p)} \rightarrow \mathbb{F}_p$ satisfying the following property is bent:

> Every $c \in \mathbb{F}_p$ has exactly K/p of the sets $A_1, \ldots, A_K$ in its preimage set $f^{-1}(c) = \{x \in \mathbb{V}_n^{(p)} : f(x) = c\}$.

Bent partitions of $\mathbb{V}_n^{(p)}$ of depth $K > 3$ exist only for even dimensions n; see [2, Remark 3]. Hence, throughout this article, bent and vectorial bent functions are always considered as functions from $\mathbb{V}_n^{(p)}$ to $\mathbb{F}_{p^k}$, $k \geq 1$, for an even integer n. Consequently, bent functions f that are weakly regular but not regular have *Walsh coefficients* of the form $\mathcal{W}_f(b) = -\zeta_p^{f^*(b)} p^{n/2}$.

As shown in [8], two types of bent partitions $\Omega = \{A_1, A_2, \ldots, A_K\}$ can be distinguished:

Type I. $|A_2| = |A_3| = \cdots = |A_K|$, and $|A_1| = |A_2| + p^{n/2}$,
Type II. $|A_2| = |A_3| = \cdots = |A_K|$, and $|A_1| = |A_2| - p^{n/2}$.

All bent partitions of depth $K > 4$ known so far are of Type I and give rise to regular bent functions. Moreover, their depth satisfies $K = p^k$, i.e., K is a power of p; see [15, Theorem 3]. In addition, the larger set A_1 contains an $(n/2)$-dimensional subspace U. Separating U from A_1 then yields a *normal bent partition*, which is defined as follows.

Definition 2 [8, Definition 2] A partition of $\mathbb{V}_n^{(p)}$ into an $(n/2)$-dimensional subspace U and sets $A_1, A_2, \ldots, A_K$ is called a *normal bent partition* of *depth* K if every function $f : \mathbb{V}_n^{(p)} \rightarrow \mathbb{F}_p$ satisfying the following properties is bent:

(I) For every $c \in \mathbb{F}_p$, exactly K/p of the sets A_j are contained in the preimage set $f^{-1}(c)$, and
(II) f is constant on the subspace U.

A key result was obtained in [17], where certain classes of bent partitions were related to some classes of vectorial bent functions:

Let $F : \mathbb{V}_n^{(p)} \rightarrow \mathbb{V}_k^{(p)}$ be a vectorial bent function. We say that F is *vectorial dual-bent* if the duals of its component functions, $(F_\alpha)^* = (\langle \alpha, F(x) \rangle_k)^*$, $\alpha \in \mathbb{V}_k^{(p)} \setminus \{0\}$, are the component functions of another vectorial bent function F^*, which is called a *vectorial dual* of F. Observe that in this case we have

$$\langle \alpha, F \rangle_k^* = (F_\alpha)^* = F_{\sigma(\alpha)}^* = \langle \sigma(\alpha), F^* \rangle_k,$$

for some permutation σ of $\mathbb{V}_k^{(p)}$ satisfying $\sigma(0) = 0$.

A vectorial dual-bent function $F : \mathbb{V}_n^{(p)} \to \mathbb{V}_k^{(p)}$, with $n = 2m$, is said to satisfy *Condition A* if all component functions of F are regular, or all are weakly regular but not regular, and if for every pair of nonzero elements $\alpha, \beta \in \mathbb{V}_k^{(p)}$ with $\alpha + \beta \neq 0$ we have

$$(F_\alpha)^* + (F_\beta)^* = (F_{\alpha+\beta})^*,$$

i.e., the sum of the duals equals the dual of the sum. We remark that if $p \neq 3$, then the case where all component functions are weakly regular but not regular cannot occur; see [16, Proposition 1]. Moreover, even for $p = 3$ and $k > 1$, no such example is currently known. In the following, we therefore mainly consider the case where all component functions of F are regular.

The relationships between bent partitions and vectorial dual-bent functions on the one hand, and partial difference sets on the other hand, can be summarized as follows. We refer to [1, 16, 17] and the survey paper [3] for further details.

Proposition 1 *Let $F : \mathbb{V}_n^{(p)} \to \mathbb{V}_k^{(p)}$ be a vectorial dual-bent function such that all component functions are regular and $(F_\alpha)^* + (F_\beta)^* = (F_{\alpha+\beta})^*$ for all nonzero $\alpha, \beta \in \mathbb{V}_k^{(p)}$ with $\alpha + \beta \neq 0$. Then the following statements hold.*

(i) *The preimage set partition*

$$\{F^{-1}(c) : c \in \mathbb{V}_k^{(p)}\}, \quad \textit{where } F^{-1}(c) = \{x \in \mathbb{V}_n^{(p)} : F(x) = c\},$$

is a bent partition for which all generated bent functions are regular. Moreover, for every $c \in \mathbb{V}_k^{(p)}$ we have $\mathbb{F}_p^* F^{-1}(c) = F^{-1}(c)$, i.e., each preimage set is closed under multiplication by nonzero elements of the prime field.

(ii) All sets $F^{-1}(c)\setminus\{0\}$ are Latin square type PDSs.

Proposition 2 *Let $F : \mathbb{V}_n^{(p)} \to \mathbb{V}_k^{(p)}$ be a vectorial dual-bent function such that all component functions are regular and $(F_\alpha)^* + (F_\beta)^* = (F_{\alpha+\beta})^*$ for all nonzero $\alpha, \beta \in \mathbb{V}_k^{(p)}$ with $\alpha + \beta \neq 0$. Suppose that F is constant c_0 on an $(n/2)$-dimensional subspace U. Then the following statements hold.*

(i) *The collection $\Omega = \{U,\ F^{-1}(c_0)\setminus U,\ F^{-1}(c) : c \in \mathbb{V}_k^{(p)},\ c \neq c_0\}$ is a normal bent partition.*

(ii) *The sets $F^{-1}(c_0)\setminus U$ and $F^{-1}(c)$, for $c \neq c_0$, are (n, s) Latin square type PDSs, where $n = p^{n/2}$ and $s = p^{n/2-k}$.*

(iii) *The collection $\{F^{-1}(c_0)\setminus U,\ F^{-1}(c) : c \in \mathbb{V}_k^{(p)},\ c \neq c_0\}$ is a $(p^{n/2-k},\ p^k)$ LP-packing in $\mathbb{V}_n^{(p)}$ relative to U.*

Example 1 Let k be a divisor of m, and let e be an integer satisfying

- $e \equiv p^l \mod (p^k - 1)$ for some integer l,
- $\gcd(p^m - 1, e) = 1$.

Define the function $F : \mathbb{F}_{p^m} \times \mathbb{F}_{p^m} \to \mathbb{F}_{p^k}$ by

$$F(x, y) = \mathrm{Tr}_k^m\big(yx^{-e}\big).$$

Then F is a vectorial dual-bent function satisfying Condition A and is identically zero on the m-dimensional subspace $U = \{(0, y) : y \in \mathbb{F}_{p^m}\}$. The partition

$$\Omega = \{U,\ F^{-1}(0)\setminus U,\ F^{-1}(c) : c \in \mathbb{F}_{p^k}^*\}$$

is a normal bent partition. Moreover, the partition Ω can be viewed as a generalization of the Desarguesian spread, since Ω reduces to the Desarguesian spread when $k = m$. For this reason, Ω is called a *generalized Desarguesian spread*, and the function F is referred to as a *vectorial generalized* PS_{ap} *function*; see [5] for further details.

All sets in Ω are Latin square type partial difference sets. Furthermore, the collection

$$\{F^{-1}(0)\backslash U,\ F^{-1}(c) : c \in \mathbb{F}_{p^k}^*\}$$

forms a $(p^{m-k},\ p^k)$ LP-packing in $\mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$ relative to U. For further details, we refer the reader to [5, 7, 17] and the survey [3].

In the meantime, many further examples—including secondary constructions—of bent partitions, and equivalently of vectorial dual-bent functions satisfying Condition A, have been obtained; see, for example, [1, 4, 6].

2.3 Non-PDS bent partitions

In [4], the first construction of bent partitions whose sets are not partial difference sets and which do not correspond to LP-packings was obtained by modifying the vectorial generalized PS$_{ap}$ function in Example 1.

Let k be a divisor of m, let a, b be integers, and let $M : \mathbb{F}_{p^m} \to \mathbb{F}_{p^k}$ be a function such that

- $a \equiv b \equiv p^l \mod (p^k - 1)$ for some integer l,
- $\gcd(p^m - 1, a) = \gcd(p^m - 1, b) = 1$,
- $M(cx) = cM(x)$ for all $c \in \mathbb{F}_{p^k}$.

Define the function

$$F(x, y) = \mathrm{Tr}_k^m\big(yx^{-a}\big) + M(x^{-b}). \tag{5}$$

It is shown in [4, Sect. 3.2] that the preimage set partition of F forms a bent partition

$$\Omega = \{F^{-1}(c) : c \in \mathbb{F}_{p^k}\}.$$

Moreover, by separating the subspace $U = \{(0, y) : y \in \mathbb{F}_{p^m}\}$ from $F^{-1}(0)$, one obtains a normal bent partition of $\mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$, namely,

$$\widetilde{\Omega} = \{U,\ F^{-1}(0)\backslash U,\ F^{-1}(c) : c \in \mathbb{F}_{p^k}^*\},$$

which is called in [4] a *modified generalized Desarguesian spread.*

All bent functions obtained from this bent partition are regular. However, if M is not the zero function, then the property $\mathbb{F}_p^* F^{-1}(c) = F^{-1}(c)$ for all $c \in \mathbb{F}_{p^k}$ does not hold. It is then pointed out that, if M is not the zero function, none of the sets in this bent partition is a partial difference set. In particular, this bent partition does not correspond to an LP-packing.

In [15], bent partitions of a similar form are further investigated under the name $\mathcal{WBP}$ bent partitions, which refers to bent partitions for which all obtained bent functions are regular, or all are weakly regular but not regular. We remark that so far only examples in which all bent functions are regular have been found, which implies that these bent partitions are of Type I. Furthermore, all of these bent partitions can be transformed into normal bent partitions.

The constructions in [15] are based on the following lemma.

Lemma 4 *Let $n = 2m$, and let k be an integer with $k \geq 2$ when $p = 2$. Let $F : \mathbb{V}_n^{(p)} \to \mathbb{V}_k^{(p)}$ be a vectorial bent function whose component functions are all regular* (*or all weakly regular* *but not regular). Suppose that there exist functions* $G : \mathbb{V}_n^{(p)} \to \mathbb{V}_k^{(p)}$ *and* $h : \mathbb{V}_n^{(p)} \to \mathbb{F}_p$ *such that, for all nonzero* $\alpha \in \mathbb{V}_k^{(p)}$,

$$(F_\alpha)^*(x) = G_\alpha(x) + h(x). \tag{6}$$

Then the preimage set partition of F, $\{F^{-1}(c) : c \in \mathbb{V}_k^{(p)}\}$, *is a bent partition of* $\mathbb{V}_n^{(p)}$ *belonging to the class* $\mathcal{WBP}$.

Note that the function F in Lemma 4 is vectorial dual-bent if and only if h is the zero function. Otherwise, the preimage sets of F, i.e., the sets of the corresponding bent partition, are not PDSs.

However, the following observations indicate that the sets arising from at least some bent partitions in the $\mathcal{WBP}$ class are close to being partial difference sets.

Observation 1 *(I) In [4, Example 3.3], a function* F *of the form* (5) *is given. For the parameters* $m = 6, k = 2, a = 59$, *and* $b = 2$, *it is observed that the sets* $D_0 = F^{-1}(0)\setminus U$ *and* $D_c = F^{-1}(c)$, *for* $c \in \mathbb{F}_{p^k}^*$, *have the following property. For each* $c \in \mathbb{F}_{p^k}$, *there exists a set* A_c *and integers* λ, μ *such that every nonzero element of* A_c *can be written as a difference of elements of* D_c *in exactly* λ *ways, and every nonzero element not contained in* A_c *can be written as such a difference in exactly* μ *ways. The set* A_c *is a PDS.*

(II) From the proof of Theorem 4 in [15], it follows that the character sums over a set of a bent partition in the $\mathcal{WBP}$ *class assume only two* absolute values. *More precisely, let* $F : \mathbb{V}_n^{(p)} \to \mathbb{V}_k^{(p)}$ *be a vectorial bent function with regular components satisfying* (6). *Then, for a nontrivial character* $\chi_a(z) = \zeta_p^{\langle a,z\rangle_n}$ *of* $\mathbb{V}_n^{(p)}$, *we have*

$$\chi_a\big(F^{-1}(c)\big) = \begin{cases} \zeta_p^{h(-a)}\big(p^{n/2} - p^{n/2-k}\big), & \text{if } c = G(-a), \\ -\,\zeta_p^{h(-a)}\, p^{n/2-k}, & \text{otherwise.} \end{cases}$$

3 Twisted partial difference sets

Motivated by Observation 1 (I), we introduce the notion of a *twisted partial difference set* as follows.

Definition 3 Let $\mathcal{G}$ be an abelian group of order v, and let $D \subseteq \mathcal{G}$ be a subset of size κ. The set D is called a $(v, \kappa, \lambda, \mu)$ twisted partial difference set in $\mathcal{G}$ if there exists a $(v, \kappa, \lambda, \mu)$ partial difference set $A \subseteq \mathcal{G}$ such that every nonzero element of A can be expressed as a difference $d_1 - d_2$, with $d_1, d_2 \in D$, in exactly λ ways, and every nonzero element of $\mathcal{G}\setminus A$ can be expressed as such a difference in exactly μ ways.

If A is an (n, s) Latin square type PDS, then D is called an (n, s) Latin square type twisted PDS. In Definition 3, we may assume that $D \neq A$ and $D \neq \mathcal{G}\setminus A$; otherwise, D itself is a PDS.

A trivial example of a twisted PDS is given by $D = A + a$, where A is a partial difference set in $\mathcal{G}$ and $a \in \mathcal{G}$. As pointed out in Observation 1 (I), the inverse images of a function F of the form (5) yield twisted PDSs. We refer to [4, Example 3.3] for more details. In fact, we will show in Sect. 5 that not only the bent partition given in [4], but also the ones constructed in [15], yield twisted PDSs.

Remark 2 (i) In contrast to PDSs (which are not difference sets), twisted PDSs do not, in general, satisfy the condition $-D = D$. Counterexamples arise, for instance, from translates $A + a$ of a PDS A, or from the sets occurring in modified generalized Desarguesian spreads; see Corollary 2.

(ii) In contrast to PDSs, for which the sets $A \cup \{0\}$ and $A \backslash \{0\}$ are again PDSs, a twisted PDS D does not, in general, share this property. More precisely, if $0 \notin D$ (respectively, $0 \in D$), then the set $D \cup \{0\}$ (respectively, $D \backslash \{0\}$) is in general not a twisted PDS:
For $D \cup \{0\}$ we see the additional differences $d - 0 = d$, $0 - d = -d$. Therefore

- $u \in A, u \notin D \cup -D$ appears λ times as a difference,
- $u \in A, u \in D \cup -D$ appears $\lambda + 1$ times as a difference,
- $u \in \mathcal{G}^* \setminus A, u \notin D \cup -D$ appears μ times as a difference,
- $u \in \mathcal{G}^* \setminus A, u \in D \cup -D$ appears $\mu + 1$ times as a difference

(with a slight adaptation for characteristic 2 or more generally for elements with $-d = d$). If not $\lambda = \mu$ or $D \cup -D = D = -D = A$ $(\mathcal{G}^* \setminus A)$, then $D \cup \{0\}$ is not a twisted PDS. An analogous argument applies to $D \backslash \{0\}$ in the case where $0 \in D$.

Before establishing a connection between the character values of a twisted PDS and those of the corresponding PDS, we introduce some notation from the group ring $\mathbb{Z}[\mathcal{G}]$.

Let $\mathcal{G}$ be a finite abelian group. As is common when working with group rings over finite abelian groups, we denote the identity element of $\mathcal{G}$ by $1_{\mathcal{G}}$ throughout this section. The group ring $\mathbb{Z}[\mathcal{G}]$ consists of all formal sums $\sum_{g \in \mathcal{G}} a_g g$ with coefficients $a_g \in \mathbb{Z}$. For such an element, we define $\left(\sum a_g g\right)^{(-1)} = \sum a_g g^{-1}$. As usual, a subset $A \subseteq \mathcal{G}$ is identified with the group ring element $\sum_{g \in A} g$.

A character of $\mathcal{G}$ is a group homomorphism $\chi : \mathcal{G} \to \mathbb{C}^*$, which extends linearly to a homomorphism from $\mathbb{Z}[\mathcal{G}]$ to $\mathbb{C}$.

Theorem 1 *Let $\mathcal{G}$ be a finite abelian group of order v with identity element $1_{\mathcal{G}}$, and let $D \subseteq \mathcal{G}$ be a subset of size κ. Then D is a $(v, \kappa, \lambda, \mu)$ twisted partial difference set in $\mathcal{G}$ if and only if there exists a $(v, \kappa, \lambda, \mu)$ partial difference set $A \subseteq \mathcal{G}$ such that*

$$\chi(D) = \zeta_\chi \, \chi(A) \tag{7}$$

for every character χ of $\mathcal{G}$, where $\zeta_\chi \in \mathbb{C}$ satisfies $|\zeta_\chi| = 1$.

Proof Suppose first that D is a $(v, \kappa, \lambda, \mu)$ twisted partial difference set in $\mathcal{G}$, and let A be the associated $(v, \kappa, \lambda, \mu)$ partial difference set. For the trivial character χ_0, we have

$$\chi_0(D) = |D| = \kappa = |A| = \chi_0(A),$$

and hence $\zeta_{\chi_0} = 1$.

We first consider the case where $1_{\mathcal{G}} \notin A$. Since A is a $(v, \kappa, \lambda, \mu)$ partial difference set, in group-ring notation, we have

$$AA^{(-1)} = \kappa 1_{\mathcal{G}} + \lambda A + \mu\big(\mathcal{G} - 1_{\mathcal{G}} - A\big).$$

Applying a nontrivial character χ yields

$$\begin{aligned} \chi\big(AA^{(-1)}\big) = |\chi(A)|^2 &= \kappa + \lambda\chi(A) + \mu\chi\big(\mathcal{G} - 1_{\mathcal{G}} - A\big) \\ &= (\kappa - \mu) + (\lambda - \mu)\chi(A), \end{aligned} \tag{8}$$

where we used $\chi(\mathcal{G}) = 0$ for a nontrivial character. By the definition of a twisted partial difference set, we also have

$$DD^{(-1)} = \kappa 1_{\mathcal{G}} + \lambda A + \mu\big(\mathcal{G} - 1_{\mathcal{G}} - A\big).$$

Consequently,

$$\chi\big(DD^{(-1)}\big) = |\chi(D)|^2 = (\kappa - \mu) + (\lambda - \mu)\chi(A) = |\chi(A)|^2,$$

where the final equality follows from (8).

Similarly, if $1_{\mathcal{G}} \in A$, then an analogous calculation gives

$$|\chi(D)|^2 = (\kappa - \lambda) + (\lambda - \mu)\chi(A) = |\chi(A)|^2.$$

Therefore, in both cases we obtain $|\chi(D)| = |\chi(A)|$.

If $\chi(A) = 0$, then necessarily $\chi(D) = 0$, and without loss of generality we define $\zeta_\chi = 1$. Otherwise, setting $\zeta_\chi = \chi(D)/\chi(A)$ yields $|\zeta_\chi| = 1$ and

$$\chi(D) = \zeta_\chi\, \chi(A).$$

Conversely, suppose that A is a $(v, \kappa, \lambda, \mu)$ partial difference set in $\mathcal{G}$ and that D is a κ-subset of $\mathcal{G}$ such that

$$\chi(D) = \zeta_\chi\, \chi(A)$$

for every character χ of $\mathcal{G}$, where $|\zeta_\chi| = 1$. Then, for every character χ,

$$\chi\big(DD^{(-1)}\big) = |\chi(D)|^2 = |\chi(A)|^2 = \chi\big(AA^{(-1)}\big).$$

By [11, Proposition 2.2], it follows that

$$DD^{(-1)} = AA^{(-1)} \quad \text{in} \quad \mathbb{Z}[\mathcal{G}].$$

Since A is a $(v, \kappa, \lambda, \mu)$ partial difference set, this equality shows that D is a $(v, \kappa, \lambda, \mu)$ twisted partial difference set in $\mathcal{G}$, with associated PDS A. □

Note that if D is a subgroup of $\mathcal{G}$, then for any nontrivial character χ of $\mathcal{G}$, we have $\chi(D) = 0$ or $\chi(D) = |D|$.

As already indicated in Remark 2, twisted partial difference sets and partial difference sets exhibit somewhat different behavior. Using Theorem 1, we can at least show that the complement of a twisted partial difference set is again a twisted partial difference set.

Corollary 1 *Let D be a twisted PDS in an abelian group $\mathcal{G}$. Then the complement $\overline{D} = \mathcal{G}\backslash D$ is also a twisted PDS. If A is the PDS associated with D such that $\chi(D) = \zeta_\chi \chi(A)$, then $\chi(\overline{D}) = \zeta_\chi \chi(\overline{A})$ for every nontrivial character χ of $\mathcal{G}$, where $\overline{A} = \mathcal{G}\backslash A$.*

Proof For every nontrivial character χ of $\mathcal{G}$, we have

$$\chi(\overline{D}) = -\,\chi(D) = -\,\zeta_\chi \chi(A) = \zeta_\chi \chi(\overline{A}).$$

Since $\overline{A}$ is a PDS whenever A is, the result follows from Theorem 1. □

Remark 3 Let D be a twisted PDS, and let A' denote the set of elements that can be written in exactly λ ways as a difference of two elements of D. Then both A' and $A' \cup \{1_{\mathcal{G}}\}$ are PDSs, and exactly one of them has the same parameters as D; this set is therefore the PDS A associated with D.

Clearly, the identity element $1_{\mathcal{G}}$ belongs to exactly one of the sets D and $\overline{D}$, and to exactly one of the sets A and $\overline{A}$. If $1_{\mathcal{G}} \in D$, it does not necessarily follow that $1_{\mathcal{G}} \in A$.

We now employ Theorem 1 to confirm that the differential properties stated in Observation 1 (I) apply to the sets arising from any modified generalized Desarguesian spread $\widetilde{\Omega}$. We denote the sets of the normal bent partition $\widetilde{\Omega}$, obtained from the function $F(x, y) = \mathrm{Tr}_k^m(yx^{-a}) + M(x^{-b})$ in (5), by $U = \{(0, y) : y \in \mathbb{F}_{p^m}\}$, $\widetilde{\mathcal{A}}(0) = F^{-1}(0) \setminus U$ and $\widetilde{\mathcal{A}}(\gamma) = F^{-1}(\gamma)$, $\gamma \in \mathbb{F}_{p^k}^*$.

Corollary 2 *For every $\gamma \in \mathbb{F}_{p^k}$, the set $\widetilde{\mathcal{A}}(\gamma)$ in the normal bent partition $\widetilde{\Omega}$ defined above is a twisted PDS in $\mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$ of $(p^m,\ p^{m-k})$ Latin square type.*

Proof For $(u, v) \in \mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$, we denote by $\chi_{(u,v)}$ the character defined by $\chi_{(u,v)}(x, y) = \zeta_p^{\mathrm{Tr}_1^m(ux+vy)}$. By Proposition 6 in [4], for $(u, v) \in \mathbb{F}_{p^m} \times \mathbb{F}_{p^m} \setminus \{(0, 0)\}$, we have $\chi_{(u,v)}(\widetilde{\mathcal{A}}(\gamma)) = -p^{m-k}$ for all $\gamma \in \mathbb{F}_{p^k}$ if $v = 0$. If $v \neq 0$, then

$$\chi_{(u,v)}\big(\widetilde{\mathcal{A}}(\gamma)\big) = \begin{cases} \zeta_p^{-\mathrm{Tr}_1^k\left(M(v^{db})\right)}\big(p^m - p^{m-k}\big), & \text{if } \gamma = -\mathrm{Tr}_k^m\big((uv^{-d})^{p^l}\big), \\ -\zeta_p^{-\mathrm{Tr}_1^k\left(M(v^{db})\right)}\ p^{m-k}, & \text{otherwise}, \end{cases}$$

where d is an integer satisfying $da \equiv 1 \mod (p^m - 1)$.

On the other hand, the generalized Desarguesian spread Ω in Example 1, obtained from the function $G(x, y) = \mathrm{Tr}_k^m(yx^{-a})$, gives rise to an LP-packing. That is, $\mathcal{A}(0) = G^{-1}(0) \setminus U$ and $\mathcal{A}(\gamma) = G^{-1}(\gamma)$, $\gamma \in \mathbb{F}_{p^k}^*$, are all $(p^m,\ p^{m-k})$ Latin square type partial difference sets, where $U = \{(0, y) : y \in \mathbb{F}_{p^m}\}$. Moreover, we have (see [5, 8])

$$\chi_{(u,v)}\big(\mathcal{A}(\gamma)\big) = \begin{cases} p^m - p^{m-k}, & \text{if } v \neq 0 \text{ and } \gamma = -\mathrm{Tr}_k^m\big((uv^{-d})^{p^l}\big), \\ -p^{m-k}, & \text{otherwise}. \end{cases}$$

By Theorem 1, it follows that for every $\gamma \in \mathbb{F}_{p^k}$ the set $\widetilde{\mathcal{A}}(\gamma)$ is a $(p^m,\ p^{m-k})$ Latin square type twisted PDS in $\mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$, for which the associated partial difference set is $\mathcal{A}(\gamma)$. □

Remark 4 The sets arising from modified generalized Desarguesian spreads provide the first non-trivial examples of twisted PDSs. Indeed, in [4, Example 3.3] it is pointed out that these sets are, in general, not obtained as translates of any PDS. Moreover, the normal bent partition $\widetilde{\Omega}$ and the associated generalized Desarguesian spread Ω are not equivalent, i.e., $\widetilde{\Omega}$ cannot be obtained from Ω by a coordinate transformation.

The proof of Corollary 2 relies on the fact that a PDS $\mathcal{A}(\gamma)$ is already known such that $\chi_{(u,v)}\big(\widetilde{\mathcal{A}}(\gamma)\big) = \zeta_{\chi_{(u,v)}}\, \chi_{(u,v)}\big(\mathcal{A}(\gamma)\big)$. This follows from the observation that $\widetilde{\Omega}$ is obtained by modifying the generalized Desarguesian spread Ω in Example 1, whose sets are all PDSs. The argument then makes use of the analysis of the relevant character sum values given in [4, 5, 8].

By Lemma 1 on the character sum values of partial difference sets, a subset A can be classified as a PDS solely by inspecting its character values. In the next section, we investigate properties of character sums over twisted PDSs.

4 Character sum properties of twisted PDSs

In this section, we propose a generalization of Lemma 1 to twisted PDSs. As might be expected, the situation is more involved than for conventional PDSs.

We require the following proposition concerning the character values of twisted partial difference sets. This proposition also reveals properties of the mapping $\chi \mapsto \zeta_\chi$ from $\widehat{\mathcal{G}}$ to $\mathbb{C}$, defined in (7), in terms of character sums. To this end, we first introduce some preparatory results.

Let $\mathcal{G}$ be a finite abelian group. In contrast to the notation used in Sect. 3, we adopt additive notation and denote the identity element of $\mathcal{G}$ by $0_{\mathcal{G}}$. When the group $\mathcal{G}$ is clear from the context, we simply write 0.

Recall that $\mathcal{G}$ is canonically isomorphic to its character group $\widehat{\mathcal{G}}$. Under this identification, the mapping $\chi \mapsto \zeta_\chi$ defined in (7) induces a function $\Psi : \mathcal{G} \to \mathcal{U}$, where $\mathcal{U} = \{u \in \mathbb{C} : |u| = 1\}$. In particular, under this identification, the identity element $0_{\mathcal{G}}$ corresponds to the trivial character of $\mathcal{G}$, and hence $\Psi(0_{\mathcal{G}}) = 1$. Consequently, the properties of the mapping $\chi \mapsto \zeta_\chi$ in (7) can be described in terms of the properties of the function $\Psi : \mathcal{G} \to \mathbb{C}$.

For some abelian groups $\mathcal{G}_1, \dots, \mathcal{G}_n$ let $\mathcal{G}$ be the abelian group $\mathcal{G} = \mathcal{G}_1 \times \cdots \times \mathcal{G}_n$. We recall that then every character χ of $\mathcal{G}$ can be written uniquely in the form

$$\chi = \chi_1 \cdots \chi_n$$

for some $\chi_i \in \widehat{\mathcal{G}_i}$. Equivalently, for $x = (x_1, \dots, x_n) \in \mathcal{G}$, we have

$$\chi(x) = \chi_1(x_1) \cdots \chi_n(x_n).$$

By the fundamental theorem of finite abelian groups, we identify $\mathcal{G}$ with $\mathbb{Z}_{p_1^{r_1}} \times \cdots \times \mathbb{Z}_{p_n^{r_n}}$ for some primes $p_1, \dots, p_n$ and positive integers $r_1, \dots, r_n$. Under this identification, for any $i = (i_1, \dots, i_n)$ and $j = (j_1, \dots, j_n)$ in $\mathcal{G}$, we have

$$\chi_i(j) = \zeta_1^{i_1 j_1} \cdots \zeta_n^{i_n j_n} = \chi_j(i),$$

where ζ_k denotes a primitive $p_k^{r_k}$-th root of unity for each $k = 1, \dots, n$.

Proposition 3 *For some primes $p_1, \dots, p_n$ and positive integers $r_1, \dots, r_n$, let $\mathcal{G}$ be the abelian group*

$$\mathcal{G} = \mathbb{Z}_{p_1^{r_1}} \times \cdots \times \mathbb{Z}_{p_n^{r_n}}.$$

Let $D \subseteq \mathcal{G}$ be a subset of size κ, for which each character χ_i corresponding to a nonzero $i \in \mathcal{G}$ satisfies

$$\chi_i(D) = \frac{\beta \pm \sqrt{\Delta}}{2} \Psi(i), \tag{9}$$

for some function $\Psi : \mathcal{G} \to \mathcal{U}$. Define

$$C(D) = \left\{ i \in \mathcal{G} : |\chi_i(D)| = \tfrac{\beta + \sqrt{\Delta}}{2} \right\} \cup \{(0, \dots, 0)\}. \tag{10}$$

Then for any nonzero $j \in \mathcal{G}$,

$$\chi_j(C(D)) = \frac{1}{\sqrt{\Delta}} \left(\frac{\beta + \sqrt{\Delta}}{2} - \kappa + \sum_{d \in D} \sum_{i \in \mathcal{G}} \overline{\Psi(i)}\, \chi_i(d + j) \right), \tag{11}$$

and

$$|C(D)| = \frac{1}{\sqrt{\Delta}} \left(\frac{\beta + \sqrt{\Delta}}{2} - \kappa - \frac{\beta - \sqrt{\Delta}}{2} |\mathcal{G}| + \sum_{d \in D} \sum_{i \in \mathcal{G}} \overline{\Psi(i)}\, \chi_i(d) \right), \tag{12}$$

where $\overline{\Psi(i)}$ denotes the complex conjugate of $\Psi(i)$.

Proof By our assumption in Eq. 9 and Remark 1, for every nontrivial character χ_i of $\mathcal{G}$,

$$\frac{2\chi_i(D)\overline{\Psi(i)}-\beta}{\sqrt{\Delta}}=\pm 1,$$

and this expression equals 1 if and only if $i\in C(D)\backslash\{0\}$. Hence, the function $f:\mathcal{G}\backslash\{0\}\to\{0,1\}$ defined by

$$f(i)=\frac{1}{2}\left(\frac{2\chi_i(D)\overline{\Psi(i)}-\beta}{\sqrt{\Delta}}+1\right)=\frac{2\chi_i(D)\overline{\Psi(i)}-\beta+\sqrt{\Delta}}{2\sqrt{\Delta}}$$

is the characteristic function of $C(D)\backslash\{0\}$. Consequently, for $j\neq 0$, using the facts that $\chi_j(i)=\chi_i(j)$, $\chi_0(D+j)=|D+j|=|D|=\kappa$, $\overline{\Psi(0)}=1$, and $\sum_{i\in\mathcal{G}\backslash\{0\}}\chi_i(j)=-1$, we obtain the following equalities.

$$\begin{aligned}\chi_j\big(C(D)\setminus\{0\}\big)&=\sum_{i\in C(D)\backslash\{0\}}\chi_j(i)=\sum_{i\in\mathcal{G}\backslash\{0\}}\chi_j(i)\,f(i)\\&=\sum_{i\in\mathcal{G}\backslash\{0\}}\chi_j(i)\left(\frac{2\chi_i(D)\overline{\Psi(i)}-\beta+\sqrt{\Delta}}{2\sqrt{\Delta}}\right)\\&=\frac{1}{\sqrt{\Delta}}\sum_{i\in\mathcal{G}\backslash\{0\}}\chi_j(i)\chi_i(D)\,\overline{\Psi(i)}\;-\;\frac{\beta-\sqrt{\Delta}}{2\sqrt{\Delta}}\sum_{i\in\mathcal{G}\backslash\{0\}}\chi_j(i)\\&=\frac{1}{\sqrt{\Delta}}\sum_{i\in\mathcal{G}\backslash\{0\}}\chi_i(D+j)\,\overline{\Psi(i)}\;-\;\frac{\beta-\sqrt{\Delta}}{2\sqrt{\Delta}}\sum_{i\in\mathcal{G}\backslash\{0\}}\chi_i(j)\\&=\frac{1}{\sqrt{\Delta}}\sum_{i\in\mathcal{G}}\chi_i(D+j)\,\overline{\Psi(i)}\;-\;\frac{\kappa}{\sqrt{\Delta}}\;+\;\frac{\beta-\sqrt{\Delta}}{2\sqrt{\Delta}}.\end{aligned}\tag{13}$$

Hence, for $j\neq 0$, using the identity $\chi_j\big(C(D)\big)=\chi_j\big(C(D)\backslash\{0\}\big)+1$, we obtain

$$\chi_j\big(C(D)\big)=\frac{1}{\sqrt{\Delta}}\sum_{i\in\mathcal{G}}\chi_i(D+j)\,\overline{\Psi(i)}\;-\;\frac{\kappa}{\sqrt{\Delta}}\;+\;\frac{\beta+\sqrt{\Delta}}{2\sqrt{\Delta}},$$

which yields Eq. (11).

Using the fact that $\sum_{i\in\mathcal{G}}\chi_i(d)=0$ for any nonzero $d\in\mathcal{G}$, we obtain

$$\sum_{i\in\mathcal{G}}\chi_i\big(C(D)\setminus\{0\}\big)=\sum_{d\in C(D)\backslash\{0\}}\sum_{i\in\mathcal{G}}\chi_i(d)=0,$$

that is,

$$\chi_0\big(C(D)\backslash\{0\}\big)=-\sum_{j\in\mathcal{G}\backslash\{0\}}\chi_j\big(C(D)\backslash\{0\}\big).$$

Then, by Eq. (13), we have

$$\begin{aligned}\chi_0\big(C(D)\setminus\{0\}\big)=&-\frac{1}{\sqrt{\Delta}}\sum_{j\in\mathcal{G}\backslash\{0\}}\sum_{i\in\mathcal{G}}\chi_i(D+j)\,\overline{\Psi(i)}\\&+(|\mathcal{G}|-1)\left(\frac{\kappa}{\sqrt{\Delta}}-\frac{\beta-\sqrt{\Delta}}{2\sqrt{\Delta}}\right).\end{aligned}\tag{14}$$

Note that

$$\sum_{j\in\mathcal{G}\setminus\{0\}}\sum_{i\in\mathcal{G}}\chi_i(D+j)\,\overline{\Psi(i)}=\sum_{j\in\mathcal{G}}\sum_{i\in\mathcal{G}}\chi_i(D+j)\,\overline{\Psi(i)}-\sum_{i\in\mathcal{G}}\chi_i(D)\,\overline{\Psi(i)}. \tag{15}$$

We observe that

$$\sum_{j\in\mathcal{G}}\chi_i(d+j)=\sum_{j\in\mathcal{G}}\chi_i(j)=\chi_i(\mathcal{G}),$$

and that $\chi_i(\mathcal{G})=0$ if $i\neq 0$, and $|\mathcal{G}|$ otherwise. We thus obtain

$$\begin{aligned}\sum_{j\in\mathcal{G}}\sum_{i\in\mathcal{G}}\chi_i(D+j)\,\overline{\Psi(i)}&=\sum_{j\in\mathcal{G}}\sum_{i\in\mathcal{G}}\sum_{d\in D}\chi_i(d+j)\,\overline{\Psi(i)}\\&=\sum_{i\in\mathcal{G}}\overline{\Psi(i)}\left(\sum_{d\in D}\sum_{j\in\mathcal{G}}\chi_i(d+j)\right)\\&=|D|\sum_{i\in\mathcal{G}}\overline{\Psi(i)}\,\chi_i(\mathcal{G})=|D|\,|\mathcal{G}|=\kappa\,|\mathcal{G}|.\end{aligned}$$

Hence, by Eq. (15), we have

$$\sum_{j\in\mathcal{G}\setminus\{0\}}\sum_{i\in\mathcal{G}}\chi_i(D+j)\,\overline{\Psi(i)}=\kappa\,|\mathcal{G}|-\sum_{i\in\mathcal{G}}\chi_i(D)\,\overline{\Psi(i)}.$$

Then, by (14), we obtain

$$\chi_0\big(C(D)\setminus\{0\}\big)=\frac{1}{\sqrt{\Delta}}\left(\sum_{i\in\mathcal{G}}\chi_i(D)\,\overline{\Psi(i)}-\kappa-(|\mathcal{G}|-1)\frac{\beta-\sqrt{\Delta}}{2}\right).$$

Combining this with the identities $\chi_0\big(C(D)\big)=\chi_0\big(C(D)\setminus\{0\}\big)+1$ and $\chi_0\big(C(D)\big)=|C(D)|$, we obtain the desired conclusion in Eq. (12). □

Recall that for a finite abelian group $\mathcal{G}$ and a function $\Psi:\mathcal{G}\to\mathbb{C}$, the (discrete) Fourier transform of Ψ is the function $\mathcal{F}_\Psi:\widehat{\mathcal{G}}\to\mathbb{C}$ defined by

$$\mathcal{F}_\Psi(\chi)=\sum_{i\in\mathcal{G}}\Psi(i)\,\overline{\chi(i)}.$$

In the remainder of this article, we consider the complex conjugate of the Fourier transform associated with Ψ. This allows us to characterize twisted PDSs in the elementary abelian group $\mathbb{V}_n^{(p)}$ in terms of the Walsh transform of p-ary functions on $\mathbb{V}_n^{(p)}$ defined in (1); see Corollary 4.

As a consequence of Eq. (11) in Proposition 3, we obtain the following condition, expressed in terms of the Fourier transform of Ψ.

Corollary 3 *Let $\mathcal{G}$, D, Ψ and $C(D)$ be as in Proposition 3. Then $C(D)$ is a partial difference set in $\mathcal{G}$ if and only if*

$$\sum_{d\in D}\overline{\mathcal{F}_\Psi(\chi_{d+j})}$$

assumes exactly two distinct real values as j ranges over the nonzero elements of $\mathcal{G}$.

We are now ready to present the main theorem of this section, which can be viewed as a generalization of Lemma 1 (the characterization of PDSs via character sums) to the setting of twisted PDSs.

Theorem 2 *Let D be a subset of size κ of a finite abelian group $\mathcal{G}$ of order v, with $2 \leq \kappa \leq v-2$. Suppose that for each character χ_i corresponding to a nonzero $i \in \mathcal{G}$ we have*

$$\chi_i(D) = \frac{\beta \pm \sqrt{\Delta}}{2} \Psi(i)$$

for some function $\Psi : \mathcal{G} \to \mathcal{U}$. Then D is a twisted partial difference set if and only if for every $j \in \mathcal{G}$ the sum

$$\sum_{d \in D} \overline{\mathcal{F}_{\Psi}(\chi_{d+j})}$$

takes a value in $\{0,\ |\mathcal{G}|\}$.

Proof We first will observe that if $C(D)$ is a partial difference set, then we also have $C(D) = -C(D)$ (hence $C(D) \setminus \{0\}$ is a regular partial difference set). Recall that $i \in C(D) \setminus \{0\}$ if and only if $|\chi_i(D)| = (\beta + \sqrt{\Delta})/2$. Since $\chi_{-i}(D) = \overline{\chi_i(D)}$, it follows that

$$|\chi_{-i}(D)| = |\chi_i(D)|,$$

and hence $-i \in C(D)$ whenever $i \in C(D)$.

Now suppose that D is a $(v, \kappa, \lambda, \mu)$ twisted partial difference set, and let A be the corresponding $(v, \kappa, \lambda, \mu)$ partial difference set. Since $|\chi_i(A)| = |\chi_i(D)|$ for all $i \in \mathcal{G}$, it follows that

$$A^{+} = C(D) \setminus \{0\}.$$

In particular, $C(D) \setminus \{0\}$, and hence $C(D)$, is a partial difference set.

Since A is a partial difference set, it is in particular a twisted partial difference set, in which case the function $\Psi(i)$ is identically equal to 1. Moreover, the identity $A^{+} = C(D) \setminus \{0\}$ implies that $C(A) = C(D)$. Hence, by Proposition 3 [see Eq. (11)], for every $j \in \mathcal{G} \setminus \{0\}$ we obtain

$$\begin{aligned} \chi_j(C(A)) &= \frac{1}{\sqrt{\Delta}} \left(\frac{\beta + \sqrt{\Delta}}{2} - \kappa + \sum_{a \in A} \sum_{i \in \mathcal{G}} \chi_i(a+j) \right) \\ &= \frac{1}{\sqrt{\Delta}} \left(\frac{\beta + \sqrt{\Delta}}{2} - \kappa + \sum_{d \in D} \sum_{i \in \mathcal{G}} \chi_i(d+j)\, \overline{\Psi(i)} \right) = \chi_j(C(D)). \end{aligned}$$

Consequently, for all nonzero $j \in \mathcal{G}$,

$$\sum_{d \in D} \sum_{i \in \mathcal{G}} \overline{\Psi(i)}\, \chi_i(d+j) = \sum_{a \in A} \sum_{i \in \mathcal{G}} \chi_i(a+j). \tag{16}$$

For $j = 0$, Eq. (12) similarly yields

$$\sum_{d \in D} \sum_{i \in \mathcal{G}} \overline{\Psi(i)}\, \chi_i(d) = \sum_{a \in A} \sum_{i \in \mathcal{G}} \chi_i(a). \tag{17}$$

Since $A \setminus \{0\}$ is a partial difference set and $A = -A$, we have

$$\sum_{a \in A} \sum_{i \in \mathcal{G}} \chi_i(a+j) = \begin{cases} |\mathcal{G}|, & \text{if } j \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Together with (16) and (17), this proves assertion.

Conversely, suppose that for every $j \in \mathcal{G}$,

$$S_j := \sum_{d \in D} \sum_{i \in \mathcal{G}} \overline{\Psi(i)}\, \chi_i(d+j) \in \{0, |\mathcal{G}|\}. \tag{18}$$

Since

$$\sum_{b \in \mathcal{G}} \chi_i(b) = \delta_0(i)\, |\mathcal{G}| \quad \text{and} \quad \Psi(0) = 1,$$

we obtain

$$\begin{aligned}\sum_{j \in \mathcal{G}} S_j &= \sum_{j \in \mathcal{G}} \sum_{d \in D} \sum_{i \in \mathcal{G}} \overline{\Psi(i)}\, \chi_i(d+j) = \sum_{d \in D} \sum_{i \in \mathcal{G}} \overline{\Psi(i)} \sum_{j \in \mathcal{G}} \chi_i(d+j)\\ &= |\mathcal{G}| \sum_{d \in D} \overline{\Psi(0)} = |\mathcal{G}|\, |D| = v\kappa.\end{aligned}$$

Hence $S_j = |\mathcal{G}|$ for exactly κ of the S_j, and since $2 \le \kappa \le v-2$, both values in (18) must occur at least twice as j ranges over $\mathcal{G}$.

By Corollary 3, the set $C(D)$ is a partial difference set. Let $A = C(D)^+$ denote the partial difference set with parameters $(v, \kappa', \lambda', \mu')$. By Lemma 3 and the identity

$$C(D)^+ = (C(D)\backslash\{0\})^+,$$

we obtain $C(A) = C(D)$. Write

$$\chi_j(A) = \frac{\beta' \pm \sqrt{\Delta'}}{2},$$

where β' and Δ' are determined by the parameters of A. Since $C(A) = C(D)$, it follows that $\chi_j(C(A)) = \chi_j(C(D))$ for all $j \in \mathcal{G}$. Since both values 0 and $|\mathcal{G}|$ occur for nonzero $j \in \mathcal{G}$, Eq. (11) yields

$$\frac{1}{\sqrt{\Delta'}}\left(\frac{\beta' + \sqrt{\Delta'}}{2} - \kappa' + |\mathcal{G}|\right) = \frac{1}{\sqrt{\Delta}}\left(\frac{\beta + \sqrt{\Delta}}{2} - \kappa + |\mathcal{G}|\right),$$

$$\frac{1}{\sqrt{\Delta'}}\left(\frac{\beta' + \sqrt{\Delta'}}{2} - \kappa'\right) = \frac{1}{\sqrt{\Delta}}\left(\frac{\beta + \sqrt{\Delta}}{2} - \kappa\right).$$

These equalities imply

$$\Delta = \Delta' \quad \text{and} \quad \frac{\beta - \beta'}{2} = \kappa - \kappa'. \tag{19}$$

Moreover, applying Eq. (12) for $j = 0$ (that is, when $|C(A)| = |C(D)|$) and using $\Delta = \Delta'$, we obtain

$$(\kappa - \kappa')v = \frac{\beta - \beta'}{2} v = \sum_{d \in D} \sum_{i \in \mathcal{G}} \overline{\Psi(i)}\, \chi_i(d) - \sum_{a \in A} \sum_{i \in \mathcal{G}} \chi_i(a).$$

Hence, in view of the assumption that $S_j \in \{0, v\}$ and Eq. (19), we conclude that

$$\kappa - \kappa' \in \{-1, 0, 1\}.$$

We now distinguish the following cases.

Case (I): $\kappa - \kappa' = -1$. This occurs if and only if

$$\sum_{d \in D} \sum_{i \in \mathcal{G}} \overline{\Psi(i)}\, \chi_i(d) = 0 \quad \text{and} \quad \sum_{a \in A} \sum_{i \in \mathcal{G}} \chi_i(a) = v.$$

Note that the second equality is equivalent to $0 \in A$. Let $\widetilde{A} = A \setminus \{0\}$. Since $(A \setminus \{0\})^+ = A^+$, we obtain $C(\widetilde{A}) = C(D)$. By (3),

$$\widetilde{\kappa} = \kappa' - 1 = \kappa, \quad \widetilde{\Delta} = \Delta,$$

and therefore $\widetilde{\beta} = \beta$. Thus, by Theorem 1, $\widetilde{A}$ is the partial difference set corresponding to D.

Case (II): $\kappa - \kappa' = 0$. Then (19) implies $\beta' = \beta$, and hence A itself is the partial difference set corresponding to D.

Case (III): $\kappa - \kappa' = 1$. This occurs if and only if

$$\sum_{d \in D} \sum_{i \in \mathcal{G}} \overline{\Psi(i)}\, \chi_i(d) = v \quad \text{and} \quad \sum_{a \in A} \sum_{i \in \mathcal{G}} \chi_i(a) = 0.$$

In this case, $0 \notin A$ and we define $\widetilde{A} = A \cup \{0\}$. Again, $C(\widetilde{A}) = C(D)$ and

$$\widetilde{\kappa} = \kappa' + 1 = \kappa, \quad \widetilde{\Delta} = \Delta,$$

which implies $\widetilde{\beta} = \beta$. Thus, $\widetilde{A}$ is the partial difference set corresponding to D. □

Remark 5 As noted in the proof of Theorem 2, if D is a partial difference set, then $\Psi(i) = 1$ for all $i \in \mathcal{G}$. In this case, we have

$$\sum_{d \in D} \sum_{i \in \mathcal{G}} \overline{\Psi(i)}\, \chi_i(d + j) = \sum_{d \in D} \sum_{i \in \mathcal{G}} \chi_i(d + j) = \begin{cases} 0, & -j \notin D, \\ |\mathcal{G}|, & -j \in D. \end{cases}$$

Hence, the second condition in Theorem 2 is always satisfied, and Theorem 2 reduces to Lemma 1 on the characterization of partial difference sets via character sums.

Let $\mathcal{G} = \mathbb{V}_n^{(p)} = \mathbb{F}_p^n$. For any subset $D \subseteq \mathcal{G}$ and any character χ_i of $\mathcal{G}$, we have $\chi_i(D) \in \mathbb{Q}(\zeta_p)$. In fact, the mapping Ψ defined in (9) is a function from $\mathbb{F}_p^n$ into the group

$$\{\zeta_p^{\ell} : \ell = 0, 1, \dots, p - 1\} \subseteq \mathbb{Q}(\zeta_p).$$

Therefore, Ψ can be written in the form

$$\Psi(i) = \zeta_p^{h(i)},$$

for some p-ary function $h : \mathbb{F}_p^n \to \mathbb{F}_p$.

Corollary 4 *Let $D \subseteq \mathbb{F}_p^n$ be a subset of size κ. Assume that for each nonzero $i \in \mathbb{F}_p^n$, the corresponding character χ_i satisfies*

$$\chi_i(D) = \frac{\beta \pm \sqrt{\Delta}}{2}\, \zeta_p^{h(i)}, \tag{20}$$

where $h : \mathbb{F}_p^n \to \mathbb{F}_p$ is a p-ary function. Then D is a twisted partial difference set if and only if for every $j \in \mathbb{F}_p^n$,

$$\sum_{d \in D} \mathcal{W}_{-h}(j - d) \in \{0,\ p^n\}. \tag{21}$$

Proof By Theorem 2, D is a twisted partial difference set if and only if for every $j \in \mathbb{F}_p^n$,

$$\sum_{d \in D} \sum_{i \in \mathbb{F}_p^n} \zeta_p^{-h(i)}\, \chi_i(d + j) \in \{0, p^n\}.$$

In our setting,

$$\sum_{d\in D}\sum_{i\in\mathbb{F}_p^n}\zeta_p^{-h(i)}\,\chi_i(d+j)=\sum_{d\in D}\sum_{i\in\mathbb{F}_p^n}\zeta_p^{-h(i)+\langle i,\,d+j\rangle}=\sum_{d\in D}\mathcal{W}_{-h}(-j-d),$$

where $\langle\,,\,\rangle$ denotes the standard dot product on $\mathbb{F}_p^n$. This yields the desired condition as j varies over $\mathbb{F}_p^n$. □

Example 2 In Corollary 2, for each $\gamma\in\mathbb{F}_{p^k}$, the twisted partial difference set $\widetilde{\mathcal{A}}(\gamma)$ in $\mathbb{F}_{p^m}\times\mathbb{F}_{p^m}$ obtained from $F(x,y)=\mathrm{Tr}_k^m(yx^{-a})+M(x^{-b})$ satisfies

$$\chi_{(u,v)}\big(\widetilde{\mathcal{A}}(\gamma)\big)=\zeta_p^{-\mathrm{Tr}_1^k(M(v^{db}))}\,\chi_{(u,v)}\big(A(\gamma)\big),$$

for all $(u,v)\in\mathbb{F}_{p^m}\times\mathbb{F}_{p^m}\setminus\{(0,0)\}$, where $A(\gamma)$ is defined by $\mathcal{A}(\gamma)=G^{-1}(\gamma)$ for $\gamma\in\mathbb{F}_{p^k}^*$, and $\mathcal{A}(0)=G^{-1}(0)\setminus U$, with $G(x,y)=\mathrm{Tr}_k^m(yx^{-a})$ and $U=\{(0,y):y\in\mathbb{F}_{p^m}\}$. Recall that $a\equiv b\equiv p^l \mod (p^k-1)$ and $ad\equiv 1 \mod (p^m-1)$. In this case, we have $h(i)=h(i_1,i_2)=-\mathrm{Tr}_1^k\big(M(i_2^{db})\big)$. By Proposition 5 in [4], for a fixed element $\beta\in\mathbb{F}_{p^m}$ with $\mathrm{Tr}_k^m(\beta)=1$ and $\mathcal{Z}=\{\alpha\in\mathbb{F}_{p^m}:\mathrm{Tr}_k^m(\alpha)=0\}$ we have

$$\widetilde{\mathcal{A}}(\gamma)=\{(x,\ (\beta(\gamma-M(x^{-b}))+\alpha)x^a):x\in\mathbb{F}_{p^m}^*,\ \alpha\in\mathcal{Z}\},$$

see [4, Proposition 5]. Hence, we have the following equalities.

$$\begin{aligned}
\mathcal{S}(j_1,j_2)&:=\sum_{(x,y)\in\widetilde{\mathcal{A}}(\gamma)}\mathcal{W}_{-h}(j_1-x,\,j_2-y)\\
&=\sum_{(x,y)\in\widetilde{\mathcal{A}}(\gamma)}\ \sum_{(i_1,i_2)\in\mathbb{F}_{p^m}\times\mathbb{F}_{p^m}}\zeta_p^{\mathrm{Tr}_1^k(M(i_2^{db}))-\mathrm{Tr}_1^m(i_1(j_1-x)+i_2(j_2-y))}\\
&=\sum_{(x,y)\in\widetilde{\mathcal{A}}(\gamma)}\ \sum_{i_2\in\mathbb{F}_{p^m}}\zeta_p^{\mathrm{Tr}_1^k(M(i_2^{db}))-\mathrm{Tr}_1^m(i_2(j_2-y))}\sum_{i_1\in\mathbb{F}_{p^m}}\zeta_p^{-\mathrm{Tr}_1^m(i_1(j_1-x))}.
\end{aligned}$$

If $j_1=0$, then $\mathcal{S}(j_1,j_2)=0$ as for any $(x,y)\in\widetilde{\mathcal{A}}(\gamma)$, we have $x\neq 0$. Suppose that $j_1\neq 0$. Then the inner sum is 0 unless $x=j_1$. In this case, we have $y=(\beta(\gamma-M(j_1^{-b}))+\alpha)j_1^a$. Hence

$$\begin{aligned}
\mathcal{S}(j_1,j_2)&=p^m\sum_{\alpha\in\mathcal{Z}}\sum_{i_2\in\mathbb{F}_{p^m}}\zeta_p^{\mathrm{Tr}_1^k(M(i_2^{db}))-\mathrm{Tr}_1^m\left(i_2\left(j_2-(\beta(\gamma-M(j_1^{-b}))+\alpha)j_1^a\right)\right)}\\
&=p^m\sum_{i_2\in\mathbb{F}_{p^m}}\zeta_p^{\mathrm{Tr}_1^k(M(i_2^{db}))-\mathrm{Tr}_1^m\left(i_2\left(j_2-\beta(\gamma-M(j_1^{-b}))j_1^a\right)\right)}\sum_{\alpha\in\mathcal{Z}}\zeta_p^{\mathrm{Tr}_1^m(\alpha\,i_2\ j_1^a)}\\
&=p^{2m-k}\sum_{i_2\in\mathbb{F}_{p^m},\,i_2\,j_1^a\in\mathcal{Z}^\perp}\zeta_p^{\mathrm{Tr}_1^k(M(i_2^{db}))-\mathrm{Tr}_1^m\left(i_2\left(j_2-\beta(\gamma-M(j_1^{-b}))j_1^a\right)\right)},
\end{aligned}$$

where $\mathcal{Z}^\perp$ is the orthogonal complement of $\mathcal{Z}$. As $\mathcal{Z}^\perp=\mathbb{F}_{p^k}$ (see the proof of Proposition 5 in [6]), we set $z=i_2\,j_1^a$, i.e., $i_2=zj_1^{-a}$. This implies that $i_2^{db}=z^{db}\,j_1^{-abd}=zj_1^{-b}$ since

$bd \equiv 1 \mod (p^k-1)$ and $ad \equiv 1 \mod (p^m-1)$. Then by the fact that $M(cx) = cM(x)$ for all $c \in \mathbb{F}_{p^k}$ and $\mathrm{Tr}_k^m(\beta) = 1$, we have

$$\begin{aligned}
\mathcal{S}(j_1, j_2) &= p^{2m-k} \sum_{z \in \mathbb{F}_{p^k}} \zeta_p^{\mathrm{Tr}_1^k\left(M(zj_1^{-b})\right) - \mathrm{Tr}_1^m\left(z\left(j_1^{-a} j_2 - \beta(\gamma - M(j_1^{-b}))\right)\right)} \\
&= p^{2m-k} \sum_{z \in \mathbb{F}_{p^k}} \zeta_p^{\mathrm{Tr}_1^k\left(zM(j_1^{-b})\right) - \mathrm{Tr}_1^m\left(z\left(j_1^{-a} j_2 - \beta(\gamma - M(j_1^{-b}))\right)\right)} \\
&= p^{2m-k} \sum_{z \in \mathbb{F}_{p^k}} \zeta_p^{\mathrm{Tr}_1^k\left(zM(j_1^{-b})\right) - \mathrm{Tr}_1^k\left(z\left(\mathrm{Tr}_k^m(j_1^{-a} j_2) - (\gamma - M(j_1^{-b}))\right)\right)} \\
&= p^{2m-k} \sum_{z \in \mathbb{F}_{p^k}} \zeta_p^{\mathrm{Tr}_1^k\left(z\left(\gamma - \mathrm{Tr}_k^m(j_1^{-a} j_2)\right)\right)} \\
&= \begin{cases} p^{2m}, & \text{if } j_1 \neq 0 \text{ and } \mathrm{Tr}_k^m(j_1^{-a} j_2) = \gamma, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

Hence, with Corollary 4, we have an alternative argument for $\widetilde{\mathcal{A}}(\gamma)$ being a twisted PDS in $\mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$.

5 Bent partitions and twisted partial difference sets

In this section, we analyse the differential properties of the sets arising from (normal) bent partitions in the $\mathcal{WBP}$ class. As shown in Corollary 2, all sets of a modified generalized Desarguesian spread are twisted partial difference sets. Amongst others, we will show that this also applies to the recent constructions of such bent partitions given in [15], which are obtained from the preimage sets of vectorial bent functions $F : \mathbb{V}_n^{(p)} \to \mathbb{V}_k^{(p)}$ with regular components satisfying (6), i.e., there exist functions $G : \mathbb{V}_n^{(p)} \to \mathbb{V}_k^{(p)}$ and $h : \mathbb{V}_n^{(p)} \to \mathbb{F}_p$ such that, for every nonzero $\alpha \in \mathbb{V}_k^{(p)}$,

$$(F_\alpha)^*(x) = G_\alpha(x) + h(x).$$

As all components of F in the constructions in [15] are regular, the resulting bent partitions $\Omega = \{F^{-1}(j) : j \in \mathbb{V}_k^{(p)}\}$ are of Type I. More precisely, there exists a unique element $j \in \mathbb{V}_k^{(p)}$, which we may assume to be $j = 0$ without loss of generality, such that

$$|F^{-1}(0)| = p^{n-k} - p^{n/2-k} + p^{n/2} \quad \text{and} \quad |F^{-1}(j)| = p^{n-k} - p^{n/2-k}$$

for all $j \in \mathbb{V}_k^{(p)} \setminus \{0\}$; see [3, Remark 3].

In contrast to vectorial dual-bent functions, which give rise to bent partitions consisting of PDSs as in Proposition 1, the functions F in (6) do not satisfy $\mathbb{F}_p^*(F^{-1}(c)) = F^{-1}(c)$. That is, the sets $F^{-1}(c)$ are not closed under multiplication by nonzero elements of the prime field. However, as shown in the following proposition, the vectorial function G in (6) does satisfy this property.

Proposition 4 *Let $n = 2m$, and let k be a positive integer, with $k \geq 2$ when $p = 2$. Let $F : \mathbb{V}_n^{(p)} \to \mathbb{V}_k^{(p)}$ be a vectorial bent function with regular (or weakly regular but not regular) components such that $F(0) = 0$. Suppose that F satisfies (6) for some functions $G : \mathbb{V}_n^{(p)} \to \mathbb{V}_k^{(p)}$ and $h : \mathbb{V}_n^{(p)} \to \mathbb{F}_p$. Then $G(ab) = G(a)$ for all $a \in \mathbb{V}_n^{(p)}$ and $b \in \mathbb{F}_p^*$. In particular, $\mathbb{F}_p^*(G^{-1}(c)) = G^{-1}(c)$ for every $c \in \mathbb{V}_k^{(p)}$.*

Proof For $a \in \mathbb{V}_n^{(p)} \setminus \{0\}$ and $c \in \mathbb{V}_k^{(p)}$, define

$$S_{a,c} = \sum_{b \in \mathbb{F}_p^*} \chi_{ab}(F^{-1}(c)).$$

Let σ_d be the automorphism of the cyclotomic field $\mathbb{Q}(\zeta_p)$ defined by $\sigma_d(\zeta_p) = \zeta_p^d$ for $d \in \mathbb{F}_p^*$. Since $S_{a,c}$ is fixed by every automorphism σ_d with $d \in \mathbb{F}_p^*$, it follows that $S_{a,c} \in \mathbb{Z}$; see the proof of Theorem 3 in [15]. By Eq. (30) in the proof of Theorem 4 in [15], the quantity $S_{a,c}$ can then be expressed as

$$S_{a,c} = \sum_{b \in \mathbb{F}_p^*} \chi_{ab}(F^{-1}(c)) = \sum_{b \in \mathbb{F}_p^*} \zeta_p^{h(-ab)}\, p^{m-k}\big(p^k\, \delta_0\big(G(-ab) - c\big) - 1\big). \tag{22}$$

We divide the proof into two cases.

Case 1. Let $a \in \mathbb{V}_n^{(p)} \setminus \{0\}$ be such that $h(-a) \neq 0$. Set

$$C_b = p^{m-k}\big(p^k\, \delta_0\big(G(-ab) - c\big) - 1\big), \quad b \in \mathbb{F}_p^*.$$

Then, by Eq. (22), we have

$$S_{a,c} = \zeta_p^{h(-a)} C_1 + \zeta_p^{h(-2a)} C_2 + \cdots + \zeta_p^{h(-(p-1)a)} C_{p-1}.$$

We now show that every power ζ_p^j, $j \in \{1, 2, \ldots, p-1\}$, appears among the terms of $S_{a,c}$. Recall that $\sigma_d(S_{a,c}) = S_{a,c}$ for all $d \in \mathbb{F}_p^*$. Since $h(-a) \neq 0$, for each $j \in \{1, 2, \ldots, p-1\}$ there exists $d \in \mathbb{F}_p^*$ such that $h(-a)d \equiv j \mod p$. Therefore, the term ζ_p^j appears in $\sigma_d(S_{a,c}) = S_{a,c}$. This implies that

$$\big\{\zeta_p^{h(-a)}, \zeta_p^{h(-2a)}, \ldots, \zeta_p^{h(-(p-1)a)}\big\} = \big\{\zeta_p, \zeta_p^2, \ldots, \zeta_p^{p-1}\big\}.$$

Since $\{\zeta_p, \zeta_p^2, \ldots, \zeta_p^{p-1}\}$ is linearly independent over $\mathbb{Q}$, and since $S_{a,c} \in \mathbb{Z}$, it follows that

$$C_1 = C_2 = \cdots = C_{p-1} = -S_{a,c}.$$

Consequently, if $G(-a) = c$, then $G(-ab) = c$ for all $b \in \mathbb{F}_p^*$.

Case 2. Suppose that $h(-a) = 0$ for $a \in \mathbb{V}_n^{(p)} \setminus \{0\}$. We claim that $h(-ba) = 0$ for all $b \in \mathbb{F}_p^*$. Indeed, if there exists $b \in \mathbb{F}_p^*$ such that $h(-ba) \neq 0$, then setting $\tilde{a} = ba$ would place us in Case 1. This would imply that $h(-b\tilde{a}) \neq 0$ for all $b \in \mathbb{F}_p^*$, and in particular that $h(-a) \neq 0$, a contradiction. Thus $h(-ba) = 0$ for all $b \in \mathbb{F}_p^*$.

In this case, Eq. (22) reduces to

$$S_{a,c} = \sum_{b \in \mathbb{F}_p^*} p^{m-k}\big(p^k\, \delta_0\big(G(-ab) - c\big) - 1\big). \tag{23}$$

As shown in the proof of Theorem 3 in [15], the quantity $S_{a,c}$ in (23) assumes exactly two possible values as c ranges over $\mathbb{V}_k^{(p)}$, namely,

$$S_{a,c} = (p-1)\bigl(p^m - p^{m-k}\bigr) \quad \text{or} \quad S_{a,c} = -(p-1)p^{m-k}.$$

Observe that, by Eq. (23), the value $S_{a,c} = (p-1)(p^m - p^{m-k})$ occurs if and only if $G(-ab) = c$ whenever $G(-a) = c$.

Combining Cases 1 and 2 we have $G(ab) = G(a)$ for all $a \in \mathbb{V}_n^{(p)}$ and $b \in \mathbb{F}_p^*$. In particular, $\mathbb{F}_p^*(G^{-1}(c)) = G^{-1}(c)$, $c \in \mathbb{V}_k^{(p)}$. □

By Proposition 4, the function G in (6) shares several properties with vectorial dual-bent functions satisfying Condition A. Note that, for the vectorial *modified generalized* PS_{ap} *function* F in (5), which induces the modified generalized Desarguesian spread, the corresponding function G is the generalized PS_{ap} function $G(x, y) = \mathrm{Tr}_k^m(yx^{-a})$. This function is, in fact, a vectorial dual-bent function satisfying Condition A.

In the next theorem, we reveal the differential properties of $\mathcal{WBP}$ bent partitions obtained from (6) in Lemma 4, in the case that G is a vectorial dual-bent function satisfying Condition A.

Theorem 3 *Let $n = 2m$, and let k be a positive integer, with $k \geq 2$ when $p = 2$. Let $F : \mathbb{V}_n^{(p)} \to \mathbb{V}_k^{(p)}$ be a vectorial bent function with regular components satisfying $F(0) = 0$. Suppose that for every nonzero $\alpha \in \mathbb{V}_k^{(p)}$,*

$$(F_\alpha)^*(x) = G_\alpha(x) + h(x),$$

for some functions $G : \mathbb{V}_n^{(p)} \to \mathbb{V}_k^{(p)}$ and $h : \mathbb{V}_n^{(p)} \to \mathbb{F}_p$. If G is a vectorial dual-bent function satisfying Condition A, then all preimage sets

$$F^{-1}(c) = \{x \in \mathbb{V}_n^{(p)} : F(x) = c\}$$

are twisted partial difference sets.

Proof By Eq. (30) in the proof of Theorem 4 in [15], for each character χ_a of $\mathbb{V}_n^{(p)}$ we have

$$\chi_a(F^{-1}(c)) = p^{2m-k}\,\delta_0(a) + \zeta_p^{h(-a)}\,p^{m-k}\bigl(p^k\,\delta_0\bigl(G(-a) - c\bigr) - 1\bigr). \tag{24}$$

Since G is a vectorial dual-bent function satisfying Condition A, it is the vectorial dual of some vectorial bent function H, which also satisfies Condition A. By the proof of Theorem 1 in [17], we have

$$\chi_a(H^{-1}(c)) = p^{2m-k}\,\delta_0(a) + p^{m-k}\bigl(p^k\,\delta_0\bigl(G(-a) - c\bigr) - 1\bigr) \tag{25}$$

for each character χ_a of $\mathbb{V}_n^{(p)}$. By Lemma 1, $H^{-1}(c)$ is a partial difference set for all $c \in \mathbb{V}_k^{(p)}$.

We note that $h(0) = 0$. Then, by Eqs. (24) and (25), for each $c \in \mathbb{V}_k^{(p)}$ the character values $\chi_a(F^{-1}(c))$ satisfy

$$\chi_a(F^{-1}(c)) = \zeta_p^{h(-a)}\,\chi_a(H^{-1}(c)),$$

i.e., the values $\chi_a(F^{-1}(c))$ satisfy Eq. (7). Hence, by Theorem 1, the set $F^{-1}(c)$ is a twisted partial difference set for every $c \in \mathbb{V}_k^{(p)}$.

□

Observe that Theorem 3 confirms once more that twisted PDSs are obtained from every modified generalized Desarguesian spread.

In [15], three further constructions of bent partitions are proposed which, in general, do not induce PDSs or LP-packings. In the first construction, conditions are given under which vectorial Maiorana–McFarland bent functions induce bent partitions in the $\mathcal{WBP}$ class. The second and third constructions are secondary constructions.

Construction 1: [15, Proposition 3]. Let k be a divisor of m with $k < m$ and $k > 1$ if $p = 2$, π be a permutation of $\mathbb{F}_{p^m}$ satisfying

$$\pi(\alpha x) = \Theta(\alpha)\pi(x) \quad \text{for all}\, \alpha \in \mathbb{F}_{p^k} \tag{26}$$

for some permutation Θ of $\mathbb{F}_{p^k}$ with the property that

$$\Theta^{-1}(\alpha^{-1}) + \Theta^{-1}(\beta^{-1}) = \Theta^{-1}((\alpha+\beta)^{-1}) \quad \text{for any}\, \alpha \neq -\beta \in \mathbb{F}^{*}_{p^k}. \tag{27}$$

For a function $M : \mathbb{F}_{p^m} \to \mathbb{F}_{p^k}$ satisfying

$$M(\alpha x) = \Theta(\alpha)M(x), \quad \alpha \in \mathbb{F}_{p^k}, \tag{28}$$

let $F : \mathbb{F}_{p^m} \times \mathbb{F}_{p^m} \to \mathbb{F}_{p^k}$ be the vectorial Maiorana–McFarland bent function

$$F(x, y) = \mathrm{Tr}^m_k(x\pi(y)) + M(y). \tag{29}$$

Then the preimage set partition of F is a bent partition in the $\mathcal{WBP}$ class.

Corollary 5 *Let F be a vectorial Maiorana–McFarland bent function given as in* (29)*. Then the preimage sets $F^{-1}(c)$, for $c \in \mathbb{F}_{p^k}$, are twisted partial difference sets.*

Proof First, we show that F satisfies (6), i.e., there exist functions $G : \mathbb{F}_{p^m} \times \mathbb{F}_{p^m} \to \mathbb{F}_{p^k}$ and $h : \mathbb{F}_{p^m} \times \mathbb{F}_{p^m} \to \mathbb{F}_p$ such that, for every $\alpha \in \mathbb{F}^{*}_{p^k}$, we have

$$(F_\alpha)^*(x, y) = G_\alpha(x, y) + h(x, y). \tag{30}$$

These functions are not given explicitly in [15]. Observing that G is vectorial dual-bent and satisfies Condition A, we then conclude the proof by Theorem 3.

For $\alpha \in \mathbb{F}^{*}_{p^k}$, the component function $F_\alpha(x, y) = \mathrm{Tr}^m_1(\alpha x\pi(y)) + \mathrm{Tr}^k_1(\alpha M(y))$ has the dual

$$(F_\alpha)^*(x, y) = \mathrm{Tr}^m_1\big(-y\,\pi^{-1}(x/\alpha)\big) + \mathrm{Tr}^k_1\big(\alpha\, M(\pi^{-1}(x/\alpha))\big).$$

Observing that, by (26), we have $\pi^{-1}(\alpha x) = \Theta^{-1}(\alpha)\,\pi^{-1}(x)$ for all $\alpha \in \mathbb{F}_{p^k}$, and applying (28), we obtain

$$(F_\alpha)^*(x, y) = \mathrm{Tr}^m_1\big(-\Theta^{-1}(\alpha^{-1})\,\pi^{-1}(x)\,y\big) + \mathrm{Tr}^k_1\big(M(\pi^{-1}(x))\big).$$

Note that the function $h(x, y) := \mathrm{Tr}^k_1\big(M(\pi^{-1}(x))\big)$ is independent of α. Let $H(x, y) = \mathrm{Tr}^m_k\big(-\pi^{-1}(x)\,y\big)$, which is, by [4, Corollary 12], a vectorial dual-bent function satisfying Condition A. Then

$$\mathrm{Tr}^m_1\big(-\Theta^{-1}(\alpha^{-1})\,\pi^{-1}(x)\,y\big) = H_{\Theta^{-1}(\alpha^{-1})}(x, y).$$

Observe that, by (27), the mapping $\alpha \mapsto \Theta^{-1}(\alpha^{-1})$ is linear. Hence, we may replace H by an equivalent vectorial function G such that $H_{\Theta^{-1}(\alpha^{-1})} = G_\alpha$. With this choice, Eq. (30) is satisfied. □

Remark 6 Clearly, the modified generalized PS_{ap} functions belong to the class of functions in Construction 1, with $\pi(x) = x^{-e}$, and Θ is the identity. In the representation in Example 1 the variables x, y are exchanged.

Construction 2 (generalized version): This construction is an application of the secondary vectorial bent function construction in [17, Theorem 4]. The construction in [17, Theorem 4] was then generalized in [1, Theorem 4]. We here consider the generalized version given in [1], and therefore we first state a generalization of [15, Proposition 4].

Let n and m be even integers, and let k be an integer satisfying $k \le m/2 \le n/2$. Let $s < k$ be a divisor of k, and let $e : \mathbb{V}_m^{(p)} \to \mathbb{F}_{p^k}$ be a vectorial dual-bent function satisfying Condition A with sign ϵ_e, i.e., $\epsilon_e = 1$ if all components of e are regular, and $\epsilon_e = -1$ if all components of e are weakly regular but not regular.

Let $\eta, \beta \in \mathbb{F}_{p^k}$ be linearly independent over $\mathbb{F}_{p^s}$. Let $F^{(\gamma)} : \mathbb{V}_n^{(p)} \to \mathbb{F}_{p^s}$, $\gamma \in \mathbb{F}_{p^s}$, be a family of vectorial bent functions, and define the function $H : \mathbb{V}_n^{(p)} \times \mathbb{V}_m^{(p)} \to \mathbb{F}_{p^s}$ by

$$H(x, y) = F^{\left(\mathrm{Tr}_s^k(\eta e(y))\right)}(x) + \mathrm{Tr}_s^k\big(\beta e(y)\big). \tag{31}$$

Then H is a vectorial bent function; see [1, Theorem 4], [4, Proposition 17].

Proposition 5 *Let H be defined as in (31) for a vectorial dual-bent function $e : \mathbb{V}_m^{(p)} \to \mathbb{F}_{p^k}$ satisfying Condition A, and let $F^{(\gamma)} : \mathbb{V}_n^{(p)} \to \mathbb{F}_{p^s}$, $\gamma \in \mathbb{F}_{p^s}$, be vectorial bent functions such that, for each γ, all component functions are regular (or all are weakly regular but not regular). Suppose that for each $\gamma \in \mathbb{F}_{p^s}$ there exist functions*

$$G^{(\gamma)} : \mathbb{V}_n^{(p)} \to \mathbb{F}_{p^s}, \quad h^{(\gamma)} : \mathbb{V}_n^{(p)} \to \mathbb{F}_p,$$

such that (6) holds, i.e., for every nonzero $\alpha \in \mathbb{F}_{p^s}$,

$$\big(F_\alpha^{(\gamma)}\big)^*(x) = G_\alpha^{(\gamma)}(x) + h^{(\gamma)}(x). \tag{32}$$

Then the preimage set partition $\{H^{-1}(c) : c \in \mathbb{F}_{p^s}\}$ of H is a bent partition of $\mathbb{V}_n^{(p)} \times \mathbb{V}_m^{(p)}$ belonging to the class $\mathcal{WBP}$.

Proof By the proof of Theorem 4 in [1], for every nonzero $\alpha \in \mathbb{F}_{p^s}$, we have

$$\begin{aligned}
\mathcal{W}_{H_\alpha}(a, b) &= \epsilon_e\, p^{\frac{m}{2}}\, \zeta_p^{\mathrm{Tr}_1^s\left(\alpha\, \mathrm{Tr}_s^k(\beta e^*(b))\right)}\, \mathcal{W}_{F_\alpha^{(\mathrm{Tr}_s^k(\eta e^*(b)))}}(a) \\
&= \epsilon\, \epsilon_e\, p^{\frac{m+n}{2}}\, \zeta_p^{\mathrm{Tr}_1^s\left(\alpha\, \mathrm{Tr}_s^k(\beta e^*(b))\right)}\, \zeta_p^{\left(F_\alpha^{(\mathrm{Tr}_s^k(\eta e^*(b)))}\right)^*(a)} \\
&= \epsilon\, \epsilon_e\, p^{\frac{m+n}{2}}\, \zeta_p^{\mathrm{Tr}_1^s\left(\alpha\, \mathrm{Tr}_s^k(\beta e^*(b))\right)}\, \zeta_p^{G_\alpha^{(\mathrm{Tr}_s^k(\eta e^*(b)))}(a) + h^{(\mathrm{Tr}_s^k(\eta e^*(b)))}(a)}.
\end{aligned}$$

Here $\epsilon = 1$ if all components of the functions $F^{(\gamma)}$ are regular, and $\epsilon = -1$ if all components of the functions $F^{(\gamma)}$ are weakly regular but not regular. Hence, we have

$$(H_\alpha)^*(x, y) = G_\alpha^{(\mathrm{Tr}_s^k(\eta e^*(y)))}(x) + \mathrm{Tr}_1^s\big(\alpha\, \mathrm{Tr}_s^k(\beta e^*(y))\big) + h^{(\mathrm{Tr}_s^k(\eta e^*(y)))}(x).$$

Define the functions

$$\widetilde{G}(x, y) = G^{\left(\mathrm{Tr}_s^k(\eta e^*(y))\right)}(x) + \mathrm{Tr}_s^k\big(\beta e^*(y)\big),$$

$$\widetilde{h}(x, y) = h^{\left(\mathrm{Tr}_s^k(\eta e^*(y))\right)}(x). \tag{33}$$

Then, for every nonzero $\alpha \in \mathbb{F}_{p^s}$,

$$(H_\alpha)^*(x, y) = \widetilde{G}_\alpha(x, y) + \widetilde{h}(x, y).$$

Consequently, by Lemma 4, the preimage set partition of H is a bent partition belonging to the class $\mathcal{WBP}$. □

Corollary 6 *Let H be the vectorial bent function defined in (31). Suppose that, for all $\gamma \in \mathbb{F}_{p^k}$, the function $G^{(\gamma)}$ in (32) is a vectorial dual-bent function satisfying Condition A. Then the preimage sets $H^{-1}(c)$, $c \in \mathbb{F}_{p^s}$, are twisted partial difference sets.*

Proof If for all $\gamma \in \mathbb{F}_{p^s}$ the function $G^{(\gamma)}$ is a vectorial dual-bent function satisfying Condition A, then by [1, Theorem 4], the function $\widetilde{G}(x, y)$ is a vectorial dual-bent function satisfying Condition A as well. The assertion follows then with Theorem 3. □

Remark 7 The only candidates for functions $F^{(\gamma)}$ satisfying (32), which we need for the construction of $\widetilde{G}$ in (33), are modified generalized PS_{ap} functions (5), and more generally functions in Construction 1. Note that for all of those functions the corresponding function G given as in (6) is vectorial dual-bent satisfying Condition A.

Construction 3: [15, Theorem 6] Let n_1, n_2 be even integers and $k_1 \le n_1/2, k_2 \le n_2/2, r$ be integers satisfying $r \le k_1, k_2$ and $k_1, k_2 \ge 2$ if $p = 2$. Let $R^{(1)} : \mathbb{V}_{n_1}^{(p)} \to \mathbb{V}_{k_1}^{(p)}$ and $R^{(2)} : \mathbb{V}_{n_2}^{(p)} \to \mathbb{V}_{k_2}^{(p)}$ be vectorial bent functions for which all components are regular (or all are weakly regular but not regular). Suppose that there exist functions $G^{(i)} : \mathbb{V}_{n_i}^{(p)} \to \mathbb{V}_{k_i}^{(p)}$ and $h^{(i)} : \mathbb{V}_{n_i}^{(p)} \to \mathbb{F}_p$, $i = 1, 2$, such that

$$\begin{aligned}(R_\alpha^{(1)})^*(x) &= G_\alpha^{(1)}(x) + h^{(1)}(x) \quad \text{for all}\, \alpha \in \mathbb{V}_{k_1}^{(p)*},\\ (R_\alpha^{(2)})^*(x) &= G_\alpha^{(2)}(x) + h^{(2)}(x) \quad \text{for all}\, \alpha \in \mathbb{V}_{k_2}^{(p)*}.\end{aligned} \tag{34}$$

For a function $K : \mathbb{V}_{k_1}^{(p)} \times \mathbb{V}_{k_2}^{(p)} \to \mathbb{V}_r^{(p)}$ define for any $\bar{x} \in \mathbb{V}_{n_1}^{(p)}$, $S^{(\bar{x})}(y) = K(\bar{x}, y)$, and for any $\bar{y} \in \mathbb{V}_{n_2}^{(p)}$ define $T^{(\bar{y})}(x) = K(x, \bar{y})$. If for all $\bar{x} \in \mathbb{V}_{n_1}^{(p)}$ the function $S^{(\bar{x})}$ is balanced and for all $y \in \mathbb{V}_{n_2}^{(p)}$ the function $T^{(\bar{y})}$ is balanced, then the preimage set partition of $F : \mathbb{V}_{n_1}^{(p)} \times \mathbb{V}_{n_2}^{(p)} \to \mathbb{V}_r^{(p)}$

$$F(x, y) = K(R^{(1)}(x), R^{(2)}(y)) \tag{35}$$

is a bent partition in the $\mathcal{WBP}$ class. Moreover, F is vectorial dual-bent if and only if $h^{(1)}$ and $h^{(2)}$ are constant functions, and $h^{(1)}(x) = -h^{(2)}(y)$, $x \in \mathbb{V}_{n_1}^{(p)}$, $y \in \mathbb{V}_{n_2}^{(p)}$.

Remark 8 When $K(x, y) = x + y$, then $K(R^{(1)}(x), R^{(2)}(y)) = R^{(1)}(x) + R^{(2)}(y)$ is the direct sum of vectorial bent functions.

For all so far known classes of functions $R^{(1)}, R^{(2)}$ in (34), the functions $G^{(1)}, G^{(2)}$ are vectorial dual-bent satisfying Condition A. Hence the following corollary covers all available constructions.

Corollary 7 *Suppose that $G^{(1)}$ and $G^{(2)}$ in (34) are vectorial-dual bent functions satisfying Condition A, and let F be a vectorial bent function given in (35). Then the preimage sets $F^{-1}(c)$, $c \in \mathbb{F}_{p^k}$, of F are twisted partial difference sets.*

Proof By the proof of Theorem 6 in [15], for $\alpha \in \mathbb{V}_r^{(p)^*}$ we have

$$\begin{aligned}&(F_\alpha)^*(x,y) = G_\alpha(x,y) + h(x,y), \quad \text{where} \\ &G(x,y) = K(G^{(1)}(x), G^{(2)}(y)), \quad h(x,y) = h^{(1)}(x) + h^{(2)}(y).\end{aligned} \tag{36}$$

Let $\widetilde{R}^{(1)}$ and $\widetilde{R}^{(2)}$ be the vectorial duals of $G^{(1)}$ and $G^{(2)}$. Then $(\widetilde{R}^{(1)}_\alpha)^*(x) = G^{(1)}_\alpha(x)$ for all $\alpha \in \mathbb{V}_{k_1}^{(p)^*}$ and $(\widetilde{R}^{(2)}_\alpha)^*(x) = G^{(2)}_\alpha(x)$ for all $\alpha \in \mathbb{V}_{k_2}^{(p)^*}$, and (34) is satisfied with $h^{(1)} = 0$, $h^{(2)} = 0$. Hence we can apply Construction 3 and obtain $\widetilde{F}$ satisfying

$$(\widetilde{F}_\alpha)^*(x,y) = G_\alpha(x,y), \quad \text{where } G(x,y) = K(G^{(1)}(x), G^{(2)}(y)),$$

which implies that $\widetilde{F}$, G are vectorial dual-bent functions satisfying Condition A. With Theorem 3 we conclude the proof.

□

6 Bent partitions and twisted LP-packings

Let $\{P_1, \ldots, P_t\}$ be a (c,t) LP-packing in an abelian group $\mathcal{G}$ of order t^2c^2, relative to a subgroup U of order tc. Then the partition $\Omega = \{U, P_1, \ldots, P_t\}$ is a normal bent partition (generalized to arbitrary abelian groups), see [2, Proposition 2], [11]. Moreover, the union of any number of the sets from Ω is again a Latin square type partial difference set, see e.g. [11, Lemma 3.9]. In this section, we extend the concept of an LP-packing to twisted partial difference sets. Recall that, by Theorem 1, the character sums over a twisted partial difference set attain only two distinct absolute values.

Definition 4 For integers $t > 1$ and $c > 0$, let $\mathcal{G}$ be an abelian group of order t^2c^2, and let U be a subgroup of $\mathcal{G}$ of order tc. Let $\{Q_1, \ldots, Q_t\}$ be a collection of t pairwise disjoint (tc, c) Latin square type twisted partial difference sets in $\mathcal{G}$. We call $\{Q_1, \ldots, Q_t\}$ a (c,t) twisted LP-packing in $\mathcal{G}$ relative to U if $\bigcup_{i=1}^t Q_i = \mathcal{G} \setminus U$ and if there exists a function $\Psi : \mathcal{G}^* \to \mathcal{U}$, (using that $\widehat{\mathcal{G}}$ and $\mathcal{G}$ are isomorphic) such that $\chi(Q_j) \in \{\Psi(\chi)(t-1)c, -\Psi(\chi)c\}$.

Note that Ψ in Definition 4 is independent from j.

Remark 9 Let $\{P_1, \ldots, P_t\}$ be a (c,t) LP-packing in an abelian group $\mathcal{G}$ of order t^2c^2, relative to a subgroup U. Then, for each character χ of $\mathcal{G}$, we have $\chi(P_j) \in \{(t-1)c, -c\}$. Hence, a conventional LP-packing satisfies the character value condition in Definition 4 with $\Psi(\chi) = 1$. Therefore, twisted LP-packings may be viewed as a natural generalization of LP-packings.

6.1 General properties

In this subsection, we collect some properties of twisted LP-packings, which closely relate them to conventional LP-packings.

Proposition 6 *Let $\mathcal{G}$ be an abelian group of order t^2c^2 and U be a subgroup of order tc. Let $Q_1, \ldots, Q_t$ be the twisted PDSs of a (c,t) twisted LP-packing relative to the subgroup U.*

(i) *The PDSs $A_1, \ldots, A_t$ corresponding to the twisted PDSs Q_j, $j = 1, \ldots, t$, form a (c,t) LP-packing in $\mathcal{G}$ relative to U.*

(ii) *The union of any number of sets from* $\Omega = \{U, Q_1, \dots, Q_t\}$ *is a twisted PDS of Latin square type.*

(iii) *If* $\mathcal{G} = \mathbb{V}_n^{(p)}$, *n even, is an elementary abelian group, then* $\Omega = \{U, Q_1, \dots, Q_t\}$ *is a bent partition of* $\mathbb{V}_n^{(p)}$.

Proof For any nontrivial character χ of $\mathcal{G}$ we have $\chi(U) = 0$ or $\chi(U) = tc$, and per definition $\chi(Q_j) = -\Psi(\chi)c$ or $\chi(Q_j) = \Psi(\chi)(t-1)c$, $1 \le j \le t$. Let $\chi(U) = 0$. Then with $\chi(\mathcal{G}) = 0$ we observe that $\chi(Q_j) = \Psi(\chi)(t-1)c$ for exactly one index j. If $\chi(U) = ct$ then we must have $\chi(Q_j) = -c$ for all $j = 1, \dots, t$.

(i) By the definition of twisted LP-packings and Theorem 1, for every character χ of $\mathcal{G}$ we have (in group-ring notation)

$$\Phi = \chi(\sum_{i=1}^{t} Q_i) = \Psi(\chi)\chi\Big(\sum_{i=1}^{t} A_i\Big).$$

If $\chi(U) = 0$, then $\Phi = 0$, and hence $\chi\Big(\sum_{i=1}^t Q_i\Big) = \chi\Big(\sum_{i=1}^t A_i\Big) = 0$. Note that if $\chi(U) = tc$, then $\Psi(\chi) = 1$ and hence $\chi\Big(\sum_{i=1}^t Q_i\Big) = \chi\Big(\sum_{i=1}^t A_i\Big) = -ct$. In any case, $\chi\Big(\sum_{i=1}^t Q_i\Big) = \chi\Big(\sum_{i=1}^t A_i\Big)$ for all $\chi \in \widehat{\mathcal{G}}$, hence by [11, Proposition 2.2], $\sum_{i=1}^t A_i = \sum_{i=1}^t Q_i = \mathcal{G} - U$. Consequently, the (tc, c) Latin square type PDSs A_i are pairwise disjoint, $\bigcup_{i=1}^t A_i = \mathcal{G} \setminus U$, and therefore $\{A_1, \dots, A_t\}$ is an LP-packing relative to U.

(ii) Let Q be a union of r of the sets Q_j. If $\chi(U) = 0$, then $\chi(Q) = \Psi(\chi)(-rc)$ or $\chi(Q) = \Psi(\chi)(tc - rc)$, depending if the unique Q_j for which $\chi(Q_j) = \Psi(\chi)(t-1)c$ is in the union Q or not. If $\chi(U) = ct$, then we have $\chi(Q) = -rc$. Hence the character sums $\chi(Q)$ take on two absolute values when χ varies. Similarly, when U is in the union, for characters χ with $\chi(U) = ct$ we have $\chi(Q) = tc - rc$, and again the character sums $\chi(Q)$ take on two absolute values. It remains to show that for the union A of the corresponding PDSs (which is also a PDS), we have $\Psi(\chi)\chi(A) = \chi(Q)$ for all nontrivial characters χ. This follows from $\Psi(\chi)\chi(A_i) = \chi(Q_i)$, $i = 1, \dots, t$, $\Psi(\chi) = 1$ if $\chi(U) = ct$, and (i).

(iii) follows with the proof of Theorem 4 in [15].

□

6.2 Skew LP-packings from $\mathcal{WBP}$ bent partitions

All bent partitions arising from vectorial dual-bent functions $F : \mathbb{V}_n^{(p)} \to \mathbb{V}_k^{(p)}$ satisfying Condition A, whose components are all regular, consist of Latin square type partial difference sets. We remark again that, so far, no examples are known with weakly regular but not regular components, except for preimage set partitions arising from certain ternary bent functions. All known examples of vectorial dual-bent functions F are constant on an $(n/2)$-dimensional subspace U. Consequently, they give rise to normal bent partitions of $\mathbb{V}_n^{(p)}$, which correspond to LP-packings of $\mathbb{V}_n^{(p)}$ relative to U.

In this subsection, we point out that all bent partitions constructed in [4, 17] that do not correspond to partial difference sets or LP-packings nevertheless induce twisted LP-packings.

Let $F : \mathbb{V}_n^{(p)} \to \mathbb{V}_k^{(p)}$ be a vectorial bent function satisfying (6), i.e., there exist functions $G : \mathbb{V}_n^{(p)} \to \mathbb{V}_k^{(p)}$ and $h : \mathbb{V}_n^{(p)} \to \mathbb{F}_p$ such that, for every nonzero $\alpha \in \mathbb{V}_k^{(p)}$, we have $(F_\alpha)^*(x) = G_\alpha(x) + h(x)$. In all known examples, all components of F are regular. Consequently, the associated bent partition $\Omega = \{F^{-1}(c) : c \in \mathbb{V}_k^{(p)}\}$ is of Type I. Moreover, in these examples the function F is constant (without loss of generality equal to 0) on an $(n/2)$-dimensional subspace U, and the collection

$$\{U,\ F^{-1}(0)\backslash U,\ F^{-1}(c) : c \in \mathbb{V}_k^{(p)}\backslash\{0\}\}$$

forms a normal bent partition.

We will use the following well-known lemma. For the convenience of the reader, we include the argument.

Lemma 5 *Let $n = 2m$, and let $F : \mathbb{V}_n^{(p)} \to \mathbb{V}_k^{(p)}$ be a vectorial bent function whose component functions are all regular. Then F vanishes on an m-dimensional subspace U of $\mathbb{V}_n^{(p)}$ if and only if for every nonzero $\alpha \in \mathbb{V}_k^{(p)}$ the dual $(F_\alpha)^*$ of F_α vanishes on $U^\perp$.*

Proof Clearly, a vectorial bent function vanishes on a subspace if and only if all its component functions vanish on that subspace. Hence, it suffices to prove the statement for p-ary bent functions.

Let f be a regular p-ary bent function, and let U be an m-dimensional subspace of $\mathbb{V}_n^{(p)}$, where $n = 2m$. Then

$$\sum_{u\in U^\perp} \mathcal{W}_f(u) = \sum_{x\in\mathbb{V}_n^{(p)}} \zeta_p^{f(x)} \sum_{u\in U^\perp} \zeta_p^{-\langle x,u\rangle} = p^m \sum_{x\in U} \zeta_p^{f(x)}.$$

Since f is regular, we have $\mathcal{W}_f(u) = \zeta_p^{f^*(u)}\, p^{n/2}$ for all $u \in \mathbb{V}_n^{(p)}$, and consequently,

$$\sum_{u\in U^\perp} \mathcal{W}_f(u) = p^m \sum_{u\in U^\perp} \zeta_p^{f^*(u)} = p^m \sum_{x\in U} \zeta_p^{f(x)}.$$

It follows that $f(u) = 0$ for all $u \in U$ if and only if $f^*(u) = 0$ for all $u \in U^\perp$.

□

Theorem 4 *Let $n = 2m$, and let k be a positive integer, with $k \geq 2$ when $p = 2$. Let $F : \mathbb{V}_n^{(p)} \to \mathbb{V}_k^{(p)}$ be a vectorial bent function whose components are all regular, and suppose that for every nonzero $\alpha \in \mathbb{V}_k^{(p)}$,*

$$(F_\alpha)^*(x) = G_\alpha(x) + h(x),$$

for some functions $G : \mathbb{V}_n^{(p)} \to \mathbb{V}_k^{(p)}$ and $h : \mathbb{V}_n^{(p)} \to \mathbb{F}_p$. Assume that there exists an m-dimensional subspace U of $\mathbb{V}_n^{(p)}$ such that

$$U^\perp \subseteq G^{-1}(0) \quad \textit{and} \quad h(a) = 0 \quad \textit{for all } a \in U^\perp.$$

Then the collection

$$\{F^{-1}(0)\backslash U,\ F^{-1}(c) : c \in \mathbb{V}_k^{(p)}\backslash\{0\}\}$$

forms a (p^{m-k}, p^k) twisted LP-packing in $\mathbb{V}_n^{(p)}$ relative to U if and only if G is a vectorial dual-bent function satisfying Condition A.

Proof First note that with Lemma 5, we have $U \subseteq F^{-1}(0)$. Consequently, $\{U,\ F^{-1}(0) \setminus U,\ F^{-1}(c) : c \in \mathbb{V}_k^{(p)} \setminus \{0\}\}$ is a normal bent partition.

Suppose that G is a vectorial dual-bent function satisfying Condition A. Then by Theorem 3 $F^{-1}(c)$ is a twisted PDS for every $c \in \mathbb{V}_k^{(p)}$. By the proof of Theorem 3, the PDS corresponding to $F^{-1}(c)$ is $H^{-1}(c)$, where H is a vectorial dual of G. More precisely, with Eqs. (24) and (25), for $H^{-1}(c)$, $c \in \mathbb{V}_k^{(p)}$, we have

$$\chi_a(F^{-1}(c)) = \zeta_p^{h(-a)}\, \chi_a(H^{-1}(c)) = \zeta_p^{h(-a)}\, p^{m-k}\big(p^k\, \delta_0\big(G(-a)-c\big) - 1\big),$$

for every nontrivial character χ_a of $\mathbb{V}_n^{(p)}$. Since for $c \neq 0$, $H^{-1}(c)$ is a (p^m, p^{m-k}) Latin square type PDS, $F^{-1}(c)$ is a (p^m, p^{m-k}) Latin square type twisted PDS. Note that by Lemma 5, H vanishes on U, hence H induces an LP-packing relative to U, i.e., $H^{-1}(0) \setminus U$ is a (p^m, p^{m-k}) Latin square type PDS as well. It remains to show that $F^{-1}(0) \setminus U$ is a twisted PDS with $H^{-1}(0) \setminus U$ as corresponding PDS. We therefore investigate the character sums $\chi_a(F^{-1}(0) \setminus U)$ for every nontrivial character χ_a.

First suppose that $a \notin U^{\perp}$ and hence $\chi_a(U) = 0$. Then

$$\chi_a\big(F^{-1}(0)\setminus U\big) = \chi_a(F^{-1}(0)) = \zeta_p^{h(-a)}\, \chi_a(H^{-1}(0)) = \zeta_p^{h(-a)}\, \chi_a\big(H^{-1}(0)\setminus U\big).$$

If on the other hand $a \in U^{\perp}$, then with the assumption that $h(a) = 0$ we obtain

$$\chi_a\big(F^{-1}(0)\setminus U\big) = \chi_a(F^{-1}(0)) - \chi_a(U) = \chi_a(H^{-1}(0)) - \chi_a(U) = \chi_a\big(H^{-1}(0)\setminus U\big).$$

Hence $F^{-1}(0) \setminus U$ is in fact a twisted PDS with $H^{-1}(0) \setminus U$ as corresponding PDS, and the collection $\{F^{-1}(0) \setminus U,\ F^{-1}(c) : c \in \mathbb{V}_k^{(p)} \setminus \{0\}\}$ forms a $(p^{m-k},\ p^k)$ twisted LP-packing in $\mathbb{V}_n^{(p)}$ relative to U.

Conversely, suppose that the collection

$$\{F^{-1}(0)\setminus U,\ F^{-1}(c) : c \in \mathbb{V}_k^{(p)}\setminus\{0\}\}$$

is a $(p^{m-k},\ p^k)$ twisted LP-packing in $\mathbb{V}_n^{(p)}$ relative to U. We aim to show that

$$\{G^{-1}(0)\setminus U^{\perp},\ G^{-1}(c) : c \in \mathbb{V}_k^{(p)}\setminus\{0\}\}$$

is a $(p^{m-k},\ p^k)$ LP-packing in $\mathbb{V}_n^{(p)}$ relative to $U^{\perp}$. By [1, Proposition 3], G is then a vectorial dual-bent function satisfying Condition A.

Let A_c be the PDS corresponding to $F^{-1}(c)$, $c \in \mathbb{V}_k^{(p)}\setminus\{0\}$, and let A_0 be the PDS corresponding to $F^{-1}(0)\setminus U$, i.e.,

$$\chi_a(F^{-1}(c)) = \zeta_{\chi_a}\, \chi_a(A_c), \quad \text{and} \quad \chi_a(F^{-1}(0) \setminus U) = \zeta_{\chi_a}\, \chi_a(A_0), \tag{37}$$

for all $a \in \mathbb{V}_n^{(p)}$. Since the set $\{F^{-1}(0) \setminus U,\ F^{-1}(c) : c \in \mathbb{V}_k^{(p)} \setminus \{0\}\}$ forms a $(p^{m-k},\ p^k)$ twisted LP-packing relative to U, by Proposition 6 the set $\{A_c : c \in \mathbb{V}_k^{(p)}\}$ forms a $(p^{m-k},\ p^k)$ LP-packing relative to U.

Note that by Eq. (37), using the notation in (10), we have

$$C(F^{-1}(c))\setminus\{0\} = \{a \in \mathbb{V}_n^{(p)}\setminus\{0\} : |\chi_a(A_c)| = p^{m-k}(p^k - 1)\} = A_c^{+},$$

i.e., $C(F^{-1}(c))\setminus\{0\}$ is precisely the dual PDS of A_c. Moreover, by Eq. (24), we have

$$C(F^{-1}(c))\setminus\{0\} = A_c^{+} = \{a \in \mathbb{V}_n^{(p)}\setminus\{0\} : G(-a) = c\} = -\,G^{-1}(c).$$

Since $-A_c^+ = A_c^+$, it follows that $G^{-1}(c) = A_c^+$. In other words, $G^{-1}(c)$ is a $(p^{m-k},\ p^k)$ Latin square type PDS for each $c \in \mathbb{V}_k^{(p)} \setminus \{0\}$ by Lemma 2.

We now consider the set $F^{-1}(0)\setminus U$. Under the assumptions that $h(a) = 0$ for all $a \in U^\perp$ and that $U^\perp \subseteq G^{-1}(0)$, we obtain, for each nonzero $a \in \mathbb{V}_n^{(p)}$,

$$\chi_a\big(F^{-1}(0) \setminus U\big) = \chi_a(F^{-1}(0)) - \chi_a(U) = \begin{cases} -\,p^{m-k}, & \text{if } a \in U^\perp, \\ \chi_a(F^{-1}(0)), & \text{otherwise.} \end{cases} \tag{38}$$

Combining Eqs. (24) and (38), we conclude that

$$\chi_a\big(F^{-1}(0) \setminus U\big) = \begin{cases} p^m - p^{m-k}, & \text{if } a \in G^{-1}(0) \setminus U^\perp, \\ -\,p^{m-k}, & \text{otherwise.} \end{cases} \tag{39}$$

Equation (39) yields $A_0^+ = G^{-1}(0)\setminus U^\perp$. This completes the proof by Lemma 2.

□

As observed in the previous section, for all available constructions of vectorial bent functions F that satisfy (6), the function G is a vectorial dual-bent function satisfying Condition A. In the sequel, we point out that F is also constant on an $(n/2)$-dimensional subspace U; hence, the preimage set partition of F gives rise to a normal bent partition. Note that, without loss of generality, we may assume that $U \subseteq F^{-1}(0)$.

For the vectorial dual-bent ingredient functions G used in the secondary constructions, we may also assume that the $(n/2)$-dimensional subspace is contained in $G^{-1}(0)$ (which will be $U^\perp$, and hence h vanishes on $U^\perp$).

Construction 1: Observe that for the permutation Θ in (27) we have $\Theta(0) = 0$. Otherwise, $\Theta^{-1}(\alpha^{-1}) = 0$ for some $\alpha \neq 0$, and therefore $\Theta^{-1}(\beta^{-1}) = \Theta^{-1}(\alpha^{-1} + \beta^{-1})$ for some $\beta \in \mathbb{F}_{p^k}$, which contradicts the fact that Θ is a permutation. This implies that $\pi(0) = 0$ and $M(0) = 0$. Consequently, the function F in Construction 1 vanishes on $U = \{(x, 0) : x \in \mathbb{F}_{p^m}\}$, and $h(x, y) = \mathrm{Tr}_1^k\big(M(\pi^{-1}(x))\big) = 0$ on $U^\perp = \{(0, y) : y \in \mathbb{F}_{p^m}\}$, which also implies that G vanishes on $U^\perp$.

Construction 2: We may suppose that the vectorial dual-bent function e satisfying Condition A vanishes on the $(m/2)$-dimensional subspace U, and hence that its dual e^* vanishes on $U^\perp$. Moreover, assume that the vectorial dual-bent function $G^{(0)}$, satisfying Condition A, and the function $h^{(0)}$ both vanish on the $(n/2)$-dimensional subspace W. Then the functions $\widetilde{G}$ and $\widetilde{h}$ in (33) vanish on $W \times U^\perp$. Consequently, the function H in (31) vanishes on $V \times U$, where $V = W^\perp$.

Construction 3: For $i = 1, 2$, we may assume that $G^{(i)}$ in (34) is a vectorial dual-bent function satisfying Condition A that vanishes on the $(n_i/2)$-dimensional subspace $U_i^\perp$, and that $h^{(i)}(a) = 0$ for all $a \in U_i^\perp$. Suppose that $K(0, 0) = c$, and without loss of generality let $c = 0$. Then, for $x \in U_1^\perp$ and $y \in U_2^\perp$, the vectorial dual-bent function G and the function h defined in (36) satisfy that $G(x, y) = K\big(G^{(1)}(x),\ G^{(2)}(y)\big) = K(0, 0) = 0$ and $h(x, y) = 0$. Hence, the function F in (36) vanishes on the subspace $U_1 \times U_2$.

Together with the above observations and Theorem 4, we get the following corollary.

Corollary 8 *The functions in Constructions* 1, 2, *and* 3 *yield twisted LP-packings.*

7 Perspectives

Motivated by observations on the differential properties of several recently introduced examples of bent partitions, we introduced the notions of twisted partial difference sets and twisted LP-packings. We investigated the properties of twisted partial difference sets and twisted LP-packings not only for elementary abelian groups, but also for arbitrary abelian groups. Moreover, we showed that all bent partitions of elementary abelian groups known to date with depth $K > 3$ correspond to twisted LP-packings.

Our results may potentially initiate further research in several directions. Since the main objective of this article is to investigate the differential properties of the sets in bent partitions, we begin by posing several questions concerning bent partitions and twisted PDSs in elementary abelian groups.

In general, it would be of interest to find additional constructions and new classes of bent partitions, LP-packings, and twisted LP-packings. In particular, the following question may be addressed:

Are there bent partitions in the class $\mathcal{WBP}$ whose sets are not twisted partial difference sets, and therefore they do not correspond to twisted LP-packings?

By Theorem 3, the preimage sets of a vectorial bent function F satisfying (6) with a vectorial dual-bent function G satisfying Condition A, are twisted PDSs. Hence, the question above leads to the following questions concerning vectorial bent functions.

- Do there exist vectorial bent functions $F : \mathbb{V}_n^{(p)} \to \mathbb{V}_k^{(p)}$ satisfying (6) for some functions $G : \mathbb{V}_n^{(p)} \to \mathbb{V}_k^{(p)}$ and $h : \mathbb{V}_n^{(p)} \to \mathbb{F}_p$, such that G is not a vectorial dual-bent function satisfying Condition A (and, moreover, no such representation exists with a vectorial dual-bent function G)?
- Do there exist bent partitions in the class $\mathcal{WBP}$ that are not preimage set partitions of vectorial bent functions F satisfying (6)? In other words, does the converse of Lemma 4 hold (or fail to hold)? See [15, Remark 2].

We have just introduced the concepts of twisted PDSs and twisted LP-packings. Hence, many questions naturally remain open and unaddressed.

PDSs correspond to strongly regular graphs, see [12, Proposition 1.1]. PDSs also appear in twisted partial difference sets D as corresponding structures A, as defined in Definition 3. The dual PDS of A coincides with the dual of the twisted PDS D. However, a direct graph-theoretic interpretation of twisted PDSs does not seem to be obvious. We remark that, in general, $-D \neq D$. Similarly, LP-packings correspond to (amorphic) association schemes, equivalently to edge decompositions of complete graphs into strongly regular graphs of Latin square type. Hence, a natural question is whether the binary relations arising from twisted LP-packings induce a similar structure.

Twisted PDSs may also be investigated more generally for arbitrary abelian groups. Some interesting questions and research problems include the following:

Can one construct twisted partial difference sets with different parameters in abelian groups, beyond the elementary abelian case, and not arising from shifts of partial difference sets?

In [11], it is shown that LP-packings exist in many abelian groups. Can one obtain analogous results for twisted LP-packings?

In Theorem 2, twisted PDSs D of a finite abelian group $\mathcal{G}$ are characterized by the character sum condition

$$\chi_i(D) = \frac{\beta \pm \sqrt{\Delta}}{2}\,\Psi(i), \tag{40}$$

for some function $\Psi : \mathcal{G} \to \mathcal{U}$ satisfying

$$\left| \sum_{d \in D} \overline{\mathcal{F}_{\Psi}(\chi_{d+j})} \right| \in \{0, \ |\mathcal{G}|\}. \tag{41}$$

Do there exist subsets D of an abelian group $\mathcal{G}$ that satisfy Condition (40) but not Condition (41)? If not, then Condition (41) on the Fourier transform would be redundant.

Elementary abelian version. Do there exist subsets of $\mathbb{V}_n^{(p)}$ that satisfy the character value condition (20), but for which the corresponding function h does not satisfy the Walsh transform condition (21)?

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