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ORIGINAL PAPER

Hypercyclic shifts on lattice graphs

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Abstract
Recently K.-G. Grosse-Erdmann and D. Papathanasiou described hypercyclic shifts in weighted spaces on directed trees. In this note we discuss several simple examples of graphs which are not trees, e.g., the lattice graphs, and study hypercyclicity of the corresponding backward shifts.

Keywords Weighted shift operator · Directed graph · Hypercyclic operator · Mixing operator

Mathematics Subject Classification 47A15 . 47B37 . 05C63 . 46A45

1 Introduction

Let $G = (V, E)$ be a connected directed graph consisting of a countable set V of vertices where $E \subset V \times V$ is its set of edges. For $v, u \in V$ we write $v \to u$ if $(v, u) \in E$. Given a vertex $v \in V$ we denote by $Chi(v)$ the set of its "children":

$$Chi(v) = \{u \in V : \ v \to u\}.$$

More generally, for $n \geq 1$, we put $Chi^n(v) = Chi(Chi(...(Chi(v))))$ (n times). Analogously, we denote by $Par(v)$ the set of those u for which $v \in Chi(u)$ ("parents" of v),

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and $Par^n(v) = Par(Par(...(Par(v))))$ n times. In what follows we will assume that each vertex in the graph G has a finite degree, that is, the sets $Chi(v)$ and $Par(v)$ are finite for every $v \in V$.

For any graph G and function $f : V \to \mathbb{C}$ we define the (unweighted) backward shift

$$(Bf)(v) = \sum_{u \in Chi(v)} f(u), \qquad v \in V.$$

Recall that a continuous linear operator T on a separable Banach space X is said to be *hypercyclic* if there exists $x \in X$ such that the set $\{T^n x : n \in \mathbb{N}_0\}$ is dense in X (here $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$). The operator T is said to be *weakly mixing* if $T \oplus T$ is hypercyclic on $X \oplus X$. Finally, T is said to be *mixing* if for any nonempty open sets U and V there exists N such that $T^n(U) \cap V \neq \emptyset$ for $n \geq N$. It is well known that mixing implies weak mixing and weak mixing implies hypercyclicity. For the theory of hypercyclic operators see [2, 4].

Weighted shifts are among the most well-known examples of hypercyclic operators. Hypercyclic shifts on weighted spaces $\ell^p(\mathbb{N}, \mu)$ and $\ell^p(\mathbb{Z}, \mu)$ of one-sided and two-sided sequences were described by Salas [11]. Note that $\mathbb{N}$ and $\mathbb{Z}$ can be considered as simplest examples of a rooted and unrooted tree respectively.

We will consider the hypercyclicity properties of the backward shift on the standard spaces $\ell^p(V, \mu)$ and $c_0(V, \mu)$ on V. Let $\mu = (\mu_v)_{v \in V}$ be a family of non-zero (real or complex, but not necessarily positive) numbers, called a weight. For $1 \leq p < \infty$ put

$$\ell^p(V, \mu) = \left\{ f : V \to \mathbb{C} : \ \|f\|^p_{\ell^p(V,\mu)} = \sum_{v \in V} |f(v)\mu_v|^p < \infty \right\}.$$

The space $c_0(V, \mu)$ is defined as

$$c_0(V, \mu) = \{f : V \to \mathbb{C} : \ \forall \varepsilon > 0 \ \exists F \subset V, \text{finite, such that } |f(v)\mu_v| < \varepsilon, \ v \in V \backslash F\};$$

it is equipped with the sup-norm $\|f\|_{c_0(V,\mu)} = \sup_{v \in V} |f(v)\mu_v|$.

For the case of directed trees (both rooted or not) without leaves, a solution of the hypercyclicity problem was obtained by Grosse-Erdmann and the third author [5]. Let us formulate their result for a rooted tree in the case $1 < p < \infty$.

Theorem [5, Theorem 4.3]. *Let $G = (E, V)$ be a rooted directed tree, $1 < p < \infty$, $1/p + 1/q = 1$. The operator B on $\ell^p(V, \mu)$ is hypercyclic if and only if it is weakly mixing and if and only if there is an increasing sequence (n_k) of positive integers such that, for each $v \in V$,*

$$\sum_{u \in Chi^{n_k}(v)} |\mu_u|^{-q} \to \infty, \qquad k \to \infty.$$

Previously hypercyclicity of shifts on directed trees was studied by Martínez-Avendaño [8], while in a recent preprint [6] chaotic weighted shifts on trees are characterized. For further results about shifts on trees see [1, 7].

We notice that the graph structure on the set of vertices is only used at the definition of the operator and not the underlying space. Concerning the weights, one may consider a weighted ℓ^p space indexed by the set of vertices, and an unweighted shift acting on it. A closely related approach is to consider an unweighted ℓ^p space and a weighted graph (where the weights can be ascribed on the vertices, or more generally on the edges) which gives rise to a weighted shift operator. As can be seen from [5] the two viewpoints are equivalent when the underlying graph is a tree. In this paper we mainly consider unweighted shifts on weighted spaces.

The aim of this note is to study hypercyclicity of the backward shift for some concrete simple examples of graphs which are not trees. Let us mention the following general question: *to describe those directed graphs for which there exist a measure μ on V such that B is hypercyclic on* $\ell^p(V, \mu)$. One of the trivial obstacles is (as noted already in [5]) the existence of vertices $v \in V$ such that $Chi(v) = \emptyset$. For another simple example of a graph which does not carry a hypercyclic weighted shift see Sect. 7. It seems that existence of cycles also makes it more difficult (but not impossible) to have a hypercyclic weighted shift.

2 Model examples

One can expect that the next natural class of graphs for which the hypercyclic shifts can be described is the class of Cartesian products of trees. Recall that given two graphs $G = (V(G), E(G))$, $H = (V(H), E(H))$ (directed or not), the vertex set of $G \times H$ is the Cartesian product $V(G) \times V(H)$ and two vertices (u, u') and (v, v') are adjacent in $G \times H$ if and only if either $u = v$ and $(u', v') \in E(H)$, or $u' = v'$ and $(u, v) \in E(G)$.

Even in the class of Cartesian products of trees hypercyclicity of shifts seems to be a difficult problem. In this note we consider the simplest cases of directed lattice graphs, i.e., Cartesian products $[1, 2, \ldots, m] \times \mathbb{N}$, $[1, 2, \ldots, m] \times \mathbb{Z}$ and $\mathbb{N} \times \mathbb{N}$, where $[1, 2, \ldots, m]$, $\mathbb{N}$, $\mathbb{Z}$ are considered as directed graphs with natural orientations of the edges. Namely, for $m \in \mathbb{N}$, consider the graph $G_m = (V_m, E_m)$ such that $V_m = (v_{i,j})_{1 \le i \le m, j \ge 1} \cong [1, 2, \ldots, m] \times \mathbb{N}$ and

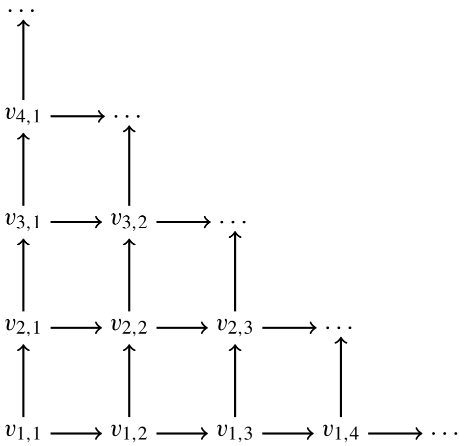
$$(v_{i,j}, u) \in E_m \iff u = v_{i,j+1} \quad \text{or} \quad u = v_{i+1,j},\ 1 \le i \le m-1. \tag{2.1}$$

Below we show the picture of the graph G_m with $m = 3$:

$$\begin{array}{ccccccc}
v_{3,1} & \longrightarrow & v_{3,2} & \longrightarrow & v_{3,3} & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
v_{2,1} & \longrightarrow & v_{2,2} & \longrightarrow & v_{2,3} & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
v_{1,1} & \longrightarrow & v_{1,2} & \longrightarrow & v_{1,3} & \longrightarrow & \cdots
\end{array}$$

Analogously, we define the graph $\tilde{G}_m = (\tilde{E}_m, \tilde{V}_m)$ with $\tilde{V}_m = (v_{i,j})_{1 \le i \le m, j \in \mathbb{Z}} \cong [1, 2, \ldots, m] \times \mathbb{Z}$, whose vertices are also given by (2.1) but with $j \in \mathbb{Z}$.

Similarly, we consider the lattice graphs G_∞ and $\tilde{G}_\infty$ such that $V_\infty \cong \mathbb{N} \times \mathbb{N}$, $\tilde{V}_\infty \cong \mathbb{Z} \times \mathbb{N}$ and all edges are of the form $(v_{i,j}, v_{i,j+1})$ or $(v_{i,j}, v_{i+1,j})$. Below we give the picture of the graph G_∞:



It is obvious that B is bounded on $\ell^p(V, \mu)$, $1 \le p < \infty$, or on $c_0(V, \mu)$ for each of the above lattice graphs if and only if there exists $C > 0$ such that for all admissible (i, j)

$$|\mu_{v_{i,j}}| \le C \min(|\mu_{v_{i+1,j}}|, |\mu_{v_{i,j+1}}|). \tag{2.2}$$

We start with a hypercyclicity/weak mixing criterion for the graph G_m.

Theorem 2.1 *Let B be the backward shift on G_m, $m \in \mathbb{N}$, and let X be any of the spaces $\ell^p(V_m, \mu)$, $1 \le p < \infty$, or $c_0(V_m, \mu)$. Assume that B is bounded on X. Then the following are equivalent*:

(i) *B is hypercyclic on X*;

(ii) *B is weakly mixing on X*;

(iii) *there exists an increasing sequence (n_k) of positive integers such that for any $1 \le i \le m$, $j \ge 1$,*

$$n_k^{m-i} |\mu_{v_{i,j+n_k}}| \to 0, \qquad k \to \infty. \tag{2.3}$$

(iv) *there exists an increasing sequence (n_k) of positive integers such that for any $1 \le i \le m$,*

$$n_k^{m-i} |\mu_{v_{i,n_k}}| \to 0, \qquad k \to \infty. \tag{2.4}$$

We see that there is a substantial difference with the case of a tree, where it is sufficient that μ tends to 0 along a subset of children of any vertex. A novel feature is that the weights should tend to zero with various speed depending on the layer (by i-th layer, $1 \le i \le m$, we mean the set of vertices $\{v_{i,j} : j \in \mathbb{N}\}$).

A similar result is true for the case $\tilde{G}_m = [1, 2, \dots, m] \times \mathbb{Z}$. As in the classical case of $\mathbb{Z}$ the weights should tend to zero along some symmetric subsequences.

Theorem 2.2 *Let B be the backward shift on $\tilde{G}_m$, $m \in \mathbb{N}$, and let X be any of the spaces $\ell^p(\tilde{V}_m, \mu)$, $1 \le p < \infty$, or $c_0(\tilde{V}_m, \mu)$. Assume that B is bounded on X. Then the following are equivalent*:

(i) *B is hypercyclic on X*;

(ii) *B is weakly mixing on X*;

(iii) *there exists an increasing sequence (n_k) of positive integers such that for any $1 \le i \le m$, $j \in \mathbb{Z}$,*

$$n_k^{m-i} (|\mu_{v_{i,j+n_k}}| + |\mu_{v_{i,j-n_k}}|) \to 0, \qquad k \to \infty. \tag{2.5}$$

We see that when the number of rows in the graph G_m or $\tilde{G}_m$ grows the conditions on the weight become more and more restrictive. It is therefore a bit suprising that in the case of doubly infinite lattice $\mathbb{N} \times \mathbb{N}$ the backward shift is weakly mixing under much milder conditions on the weight. The following result gives a sufficient condition of hypercyclicity/weak mixing which is sharp in a rough exponential scale. It is an interesting open problem to find a necessary and sufficient condition for hypercyclicity of the shift on $\mathbb{N} \times \mathbb{N}$.

Theorem 2.3 *Let B be the backward shift on $G_\infty = \mathbb{N} \times \mathbb{N}$ and let X be any of the spaces $\ell^p(V_\infty, \mu)$, $1 \le p < \infty$, or $c_0(V_\infty, \mu)$. Assume that B is bounded on X.*

1. *If*

$$\limsup_{i+j\to\infty} |\mu_{v_{i,j}}|^{1/(i+j)} < 2,$$

then B is mixing (in particular, weakly mixing and hypercyclic) on X.

2. *Assume that there is $c > 0$ such that*

$$|\mu_{v_{i,j}}| \ge c2^{i+j}. \tag{2.6}$$

Then B on $G_\infty = \mathbb{N} \times \mathbb{N}$ is not hypercyclic.

We leave the case of lattices $\mathbb{N} \times \mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z}$ for a future research. It is clear that in this case the class of weights for which B is weak mixing or hypercyclic must be large as in the case of G_∞.

Our last result applies to the case when the weght μ on the lattice depends only on one of the coordinates. In this case, making use of recent results of Menet and the third author [9] we show, by reducing the problem to a theorem of Salas, that the backward shift is always mixing whenever it is bounded. We would like to thank Q. Menet for sharing his insights on identifying the backward shift on G_∞ with a generalized shift in the sense of [9].

Theorem 2.4 *Assume that the weight μ on the graph G_∞ or $\tilde{G}_\infty$ depends on one coordinate only, that is, $\mu_{v_{i,j}} = \mu_i$ for any $j \in \mathbb{N}$ or, respectively, $j \in \mathbb{Z}$. Let X be any of the spaces $\ell^p(V, \mu)$, $1 \le p < \infty$, or $c_0(V, \mu)$, where $V = V_\infty$ or $V = \tilde{V}_\infty$. If the backward shift B is bounded on X, then it is mixing.*

3 Proof of Theorem 2.1

As in [5] we use the classical Hypercyclicity Criterion (see, e.g., [4, Theorem 3.12]).

Theorem (Hypercyclicity Criterion). *Let X be a separable Banach space and let T be a bounded operator on X. Assume that there exist dense subsets X_0, Y_0 of X, an increasing sequence $\{n_k\}$ of positive integers, and maps $R_{n_k} : Y_0 \to X$ such that, for any $x \in X_0$ and $y \in Y_0$,*

(i) $T^{n_k} x \to 0$,
(ii) $R_{n_k} y \to 0$,
(iii) $T^{n_k} R_{n_k} y \to y$,

as $k \to \infty$. Then T is weakly mixing and, in particular, hypercyclic. If furthermore, T satisfies the Hypercyclicity Criterion for the full sequence $\{n\}$, then T is mixing.

We denote by e_v the function such that $e_v(v) = 1$ and $e_v(u) = 0$, $u \ne v$. Note that $Be_v = \sum_{u \in Par(v)} e_u$.

Proof of Theorem 2.1 The implication (ii) $\Longrightarrow$ (i) is trivial.

(iii) $\Longrightarrow$ (ii) Note that for $n \ge m$ we have $Chi^n(v_{i,j}) = \{v_{i,j+n}, v_{i+1,j+n-1}, \dots, v_{m,j+n+i-m}\}$, while $Par^n(v_{i,j+n}) = \{v_{i,j}, v_{i-1,j+1}, \dots, v_{1,i+j-1}\}$. Also,

$$\begin{aligned} B(e_{v_{i,j}}) &= e_{v_{i-1,j}} + e_{v_{i,j-1}}, \\ B^2(e_{v_{i,j}}) &= e_{v_{i-2,j}} + 2e_{v_{i-1,j-1}} + e_{v_{i,j-2}}, \\ &\dots \\ B^n(e_{v_{i,j}}) &= \sum_{l=0}^{n} \binom{n}{l} e_{v_{i-l,j-(n-l)}}, \end{aligned} \tag{3.1}$$

for those values of l for which $e_{v_{i-l,j-(n-l)}}$ makes sense. We agree to understand $e_{v_{i-l,j-(n-l)}}$ as a zero function if $i \le l$ or $j \le n-l$. Thus,

$$\begin{aligned} B^n(e_{v_{1,j+n}}) &= e_{v_{1,j}}, \\ B^n(e_{v_{2,j+n}}) &= ne_{v_{1,j+1}} + e_{v_{2,j}}, \\ &\dots \\ B^n(e_{v_{i,j+n}}) &= \sum_{l=0}^{i-1} \binom{n}{l} e_{v_{i-l,j+l}}. \end{aligned}$$

Note that the set of finite functions is dense in X and that $B^n f = 0$ for any finite f and sufficiently large n. We need to define the operators R_{n_k} so that $R_{n_k} f \to 0$ and $B^{n_k} R_{n_k} f \to f$ on finite sequences. Put

$$R_{n_k}(e_{v_{1,j}}) := e_{v_{1,j+n_k}}, \qquad R_{n_k}(e_{v_{2,j}}) := e_{v_{2,j+n_k}} - n_k e_{v_{1,j+1+n_k}},$$

and, assuming that R_{n_k} is already defined on the layers with numbers $1, \dots, i-1$, we put

$$R_{n_k}(e_{v_{i,j}}) = e_{v_{i,j+n_k}} - \sum_{l=1}^{i-1} \binom{n_k}{l} R_{n_k}(e_{v_{i-l,j+l}}). \tag{3.2}$$

Then, for any v, we have $B^{n_k} R_{n_k}(e_v) = e_v$. Note that

$$R_{n_k}(e_{v_{i,j}}) = \sum_{s=1}^{i} \alpha_{i,s} e_{v_{s,i+j-s+n_k}} \tag{3.3}$$

for some coefficients $\alpha_{i,s}$ independent on j (but, of course, depending on n_k). It is easy to show by induction that

$$|\alpha_{i,s}| \le C n_k^{i-s}, \tag{3.4}$$

where the constant C may depend on m only. Indeed, it follows from (3.2) and (3.3) that

$$\begin{aligned} R_{n_k}(e_{v_{i,j}}) &= e_{v_{i,j+n_k}} - \sum_{l=1}^{i-1} \binom{n_k}{l} \sum_{s=1}^{i-l} \alpha_{i-l,s} e_{v_{s,i+j-s+n_k}} \\ &= e_{v_{i,j+n_k}} - \sum_{s=1}^{i-1} \left(\sum_{l=1}^{i-s} \binom{n_k}{l} \alpha_{i-l,s} \right) e_{v_{s,i+j-s+n_k}}. \end{aligned}$$

Assume that we know that $|\alpha_{i-l,s}| \le Cn_k^{i-l-s}$, $1 \le l \le i-s$. Since $\binom{n_k}{l} \le n_k^l$ and $i \le m$, it follows that

$$|\alpha_{i,s}| = \left|\sum_{l=1}^{i-s} \binom{n_k}{l} \alpha_{i-l,s}\right| \le Cmn_k^{i-s}, \qquad 1 \le s \le i-1.$$

In particular, we have $\alpha_{i,s} = O(n_k^{m-s})$ for all $1 \le i \le m$. By the hypothesis (2.3) $n_k^{m-s}\mu_{v_{s,i+j-s+n_k}} \to 0$ for any i, j, s and we conclude that $R_{n_k} f \to 0$ for any finite f. Thus, B is weakly mixing.

(i) $\Longrightarrow$ (iii) We follow here the idea of the proof from [5]. If B is hypercyclic, then for any $\varepsilon, \delta > 0$ and $K \in \mathbb{N}$ we can choose an arbitrarily large n and $f \in X$ such that $\|f\|_X < \varepsilon$ and

$$\left\| B^n f - \sum_{j=1}^{K} e_{v_{m,j}} \right\|_X < \delta \min_{v \in F} |\mu_v|,$$

where $F = \{(v_{i,j}), 1 \le i \le m, 1 \le j \le K\}$ (take as f an appropriate multiple of some hypercyclic vector for B). Then we have, for $1 \le j \le K$,

$$\left|(B^n f)(v_{m,j}) - 1\right| < \delta, \qquad \left|(B^n f)(v_{i,j})\right| < \delta, \qquad 1 \le i \le m-1.$$

Note that for $n > m$

$$(B^n f)(v_{i,j}) = \sum_{l=0}^{m-i} \binom{n}{l} f(v_{i+l,j+n-l}).$$

In particular, $(B^n f)(v_{m,j}) = f(v_{m,j+n})$, whence $|f(v_{m,j+n}) - 1| < \delta$, $1 \le j \le K$. Next

$$\left|(B^n f)(v_{m-1,j})\right| = \left|f(v_{m-1,j+n}) + nf(v_{m,j-1+n})\right| < \delta,$$

whence

$$|f(v_{m-1,j+n}) + n| < \delta(1+n), \qquad 2 \le j \le K.$$

Continuing the estimates we obtain by induction that for any $l < m$ and for sufficiently large n we have

$$\left| f(v_{m-l,j+n}) + \frac{(-1)^{l+1}}{l!} n^l \right| < C\delta n^l, \qquad l+1 \le j \le K,$$

where the constant $C > 0$ depends on m only. Indeed,

$$\left|(B^n f)(v_{m-l,j})\right| = \left| f(v_{m-l,j+n}) + \sum_{k=1}^{l} \binom{n}{k} f(v_{m-l+k,j+n-k}) \right| < \delta.$$

Suppose that we have already shown that the quantities

$$\rho_{l,k} = f(v_{m-l+k,j+n-k}) + \frac{(-1)^{l-k+1}}{(l-k)!} n^{l-k}$$

satisfy $|\rho_{l,k}| \le C\delta n^{l-k}$, $1 \le k \le l$. Then

$$\left| f(v_{m-l,j+n}) + \sum_{k=1}^{l} \binom{n}{k} \frac{(-1)^{l-k}}{(l-k)!} n^{l-k} + \sum_{k=1}^{l} \binom{n}{k} \rho_{l,k} \right| < \delta.$$

Note that $\binom{n}{k} = \frac{n^k}{k!} + O(n^{k-1})$ when $n \to \infty$. It follows that, for sufficiently large n,

$$\left| f(v_{m-l,j+n}) + (-1)^l n^l \sum_{k=1}^{l} \frac{(-1)^k}{k!(l-k)!} \right| < C\delta n^l.$$

It remains to notice that

$$\sum_{k=1}^{l} \frac{(-1)^k}{k!(l-k)!} = -\frac{1}{l!}.$$

Thus, if we choose $\delta > 0$ to be sufficiently small, we get

$$|f(v_{i,j+n})| \geq Cn^{m-i}, \qquad 1 \leq i \leq m, \; m \leq j \leq K,$$

where $C > 0$ is (another) constant depending only on m. Replacing n by $n+m-1$, we get $|f(v_{i,j+n})| \geq Cn^{m-i}$ for $1 \leq i \leq m$, $1 \leq j \leq K-m+1$, Recall that $\|f\|_X < \varepsilon$. Since $|\mu_v f(v)| \leq \|f\|_X$, we conclude that $n^{m-i}|\mu_{v_{i,j+n}}| \leq C^{-1}\varepsilon$, $1 \leq i \leq m$, $1 \leq j \leq K-m+1$.

Finally, since $\varepsilon > 0$ and $K \in \mathbb{N}$ were arbitrary, we repeat this procedure for sequences $\varepsilon_k \to 0$ and $K_k \to \infty$ and find a sequence $n_k \to \infty$ satisfying (2.3).

(iv) $\Longrightarrow$ (iii) It remains to show that a formally weaker condition (2.4) implies (2.3). From the boundedness of B (see (2.2)) it follows that for each $l \geq 0$ we have $(n_k - l)^{m-i}|\mu_{v_{i,n_k-l}}| \to 0$. It is now not difficult to show (see [4, Lemma 4.2]) that there exists an increasing sequence m_k such that $(m_k + j)^{m-i}|\mu_{v_{i,m_k+j}}| \to 0$ for any $j \geq 0$.

4 Proof of Theorem 2.2

The proof for the case of $[1, 2, \ldots, m] \times \mathbb{Z}$ is similar to the proof of Theorem 2.1.

(iii) $\Longrightarrow$ (ii) Define operators R_{n_k} by the same formula (3.2). Then $R_{n_k} f \to 0$ and $B^{n_k} R_{n_k} f \to f$ on finite sequences. In the proof of Theorem 2.1 (formulas (3.1)) we have seen that

$$B^{n_k}(e_{v_{i,j}}) = \sum_{l=0}^{i-1} \binom{n_k}{l} e_{v_{i-l,j+l-n_k}},$$

and so (2.5) implies also that $B^{n_k} f \to 0$ on finite sequences. It remains to apply the Hypercyclicity Criterion.

(i) $\Longrightarrow$ (iii) To prove necessity of (2.5) consider a finite set $F = \{(v_{i,j}), 1 \leq i \leq m, -K \leq j \leq K\}$. If B is hypercyclic, then it is topologically transitive and so for any $\varepsilon \in (0, 1/2)$ there exist an arbitrarily large n and $f \in X$ such that

$$\left\| f - \sum_{j=-K}^{K} e_{v_{m,j}} \right\|_X < \varepsilon \min(1, \min_{v \in F} |\mu_v|),$$

and

$$\left\| B^n f - \sum_{j=-K}^{K} e_{v_{m,j}} \right\|_X < \varepsilon \min_{v \in F} |\mu_v|. \tag{4.1}$$

We can also choose n large enough so that $v_{i,j+n}, v_{i,j-n} \notin F$ for any $v_{i,j} \in F$. Repeating the arguments from the proof of Theorem 2.1 we show that if ε is sufficiently small, then

$|f(v_{i,j+n})| \geq Cn^{m-i}$ for $1 \leq i \leq m$, $-K \leq j \leq K$, where $C > 0$ is some constant depending on m only. Since $\left\| f - \sum_{j=-K}^{K} e_{v_{m,j}} \right\|_X < \varepsilon$ and $v_{i,j+n} \notin F$ for any $v_{i,j} \in F$ we conclude that $|\mu_{v_{i,j+n}} f(v_{i,j+n})| < \varepsilon$ for $v_{i,j} \in F$. Thus,

$$n^{m-i}|\mu_{v_{i,j+n}}| \leq C^{-1}\varepsilon, \qquad 1 \leq i \leq m, \; -K \leq j \leq K.$$

It remains to prove a similar estimate for $\mu_{v_{i,j-n}}$. Since, for $-K \leq j \leq K$,

$$|f(v_{m,j}) - 1| \cdot |\mu_{v_{m,j}}| \leq \left\| f - \sum_{l=-K}^{K} e_{v_{m,l}} \right\|_X < \varepsilon \min(1, \min_{v \in F} |\mu_v|),$$

we conclude that for $-K \leq j \leq K$ we have $|f(v_{m,j})| \geq 1/2$ whereas $|f(v_{i,j})| < \varepsilon$, $1 \leq i \leq m-1$. Now we write $f = \sum_{i=1}^{m} \sum_{j \in \mathbb{Z}} f(v_{i,j}) e_{v_{i,j}}$. It holds that

$$B^n f = \sum_{i=1}^{m} \sum_{j \in \mathbb{Z}} f(v_{i,j}) \sum_{l=0}^{i-1} \binom{n}{l} e_{v_{i-l,j+l-n}} = \sum_{k=1}^{m} \sum_{j \in \mathbb{Z}} \left[\sum_{i=k}^{m} \binom{n}{i-k} f(v_{i,j-i+k}) \right] e_{v_{k,j-n}}.$$

Indeed, setting $k = i - l$, we have

$$\sum_{l=0}^{i-1} \binom{n}{l} e_{v_{i-l,j+l-n}} = \sum_{k=1}^{i} \binom{n}{i-k} e_{v_{k,i+j-k-n}}.$$

A change in the summation order gives

$$\sum_{i=1}^{m} \sum_{k=1}^{i} f(v_{i,j}) \binom{n}{i-k} e_{v_{k,i+j-k-n}} = \sum_{k=1}^{m} \sum_{i=k}^{m} f(v_{i,j}) \binom{n}{i-k} e_{v_{k,i+j-k-n}},$$

which in turn implies that

$$\sum_{i=1}^{m} \sum_{j \in \mathbb{Z}} f(v_{i,j}) \sum_{l=0}^{i-1} \binom{n}{l} e_{v_{i-l,j+l-n}} = \sum_{k=1}^{m} \sum_{i=k}^{m} \sum_{j \in \mathbb{Z}} f(v_{i,j}) \binom{n}{i-k} e_{v_{k,i+j-k-n}}.$$

It remains to rename $i + j - k$ by j on the right hand side. For sufficiently large n one has

$$\begin{aligned} \left| \sum_{i=k}^{m} \binom{n}{i-k} f(v_{i,j-i+k}) \right| &\geq \binom{n}{m-k} |f(v_{m,j-m+k})| - \sum_{i=k}^{m-1} \binom{n}{i-k} |f(v_{i,j-i+k})| \\ &\geq C_1 n^{m-k} - \varepsilon n^{m-k-1} m \geq C_2 n^{m-k} \end{aligned}$$

for all $1 \leq k \leq m$, $-K + m - 1 \leq j \leq K$. Since $v_{k,j-n} \notin F$, $-K \leq j \leq K$, we conclude as above that $n^{m-k}|\mu_{v_{k,j-n}}| \leq C\varepsilon$, $1 \leq k \leq m$, $-K + m - 1 \leq j \leq K$.

Finally, since $\varepsilon > 0$ and $K \in \mathbb{N}$ were arbitrary, we can find a sequence $n_k \to \infty$ satisfying (2.5). □

5 Proof of Theorem 2.3

We give two slightly different proofs of Statement 1 in Theorem 2.3. In the first of them we use the following simple sufficient condition of mixing known as the Godefroy–Shapiro Criterion (see [3], [2, Corollary 1.10] or [4, Theorem 3.1]), while the second proof is based on the Hypercyclicity Criterion.

Theorem (Godefroy–Shapiro Criterion). *If, for a bounded linear operator T on a separable Banach space X, both $\cup_{|\lambda|<1}\mathrm{Ker}\,(T-\lambda I)$ and $\cup_{|\lambda|>1}\mathrm{Ker}\,(T-\lambda I)$ span a dense subspace in X, then T is mixing.*

Proof of Theorem 2.3 Throughout the proof we will consider the shift on the lattice $\mathbb{N}_0\times\mathbb{N}_0$ in place of $\mathbb{N}\times\mathbb{N}$. This will slightly simplify the formulas.

First proof of Statement 1. For $r\ge 1$, $s\in\mathbb{C}$, consider the function $f_{r,s}$ defined on $\mathbb{N}_0\times\mathbb{N}_0$ by

$$f_{r,s}(v_{i,j}) = r^{i+2j}s^{i+j},$$

which can be conveniently represented as a matrix with $f_{r,s}(v_{i,j})$ situated at the vertex $v_{i,j}$ (recall that in our notation i is the number of a row and j is the number of a column):

$$f_{r,s} = \begin{pmatrix} \dots & \dots & \dots & \dots & \dots & \dots \\ r^4s^4 & \dots & \dots & \dots & \dots & \dots \\ r^3s^3 & r^5s^4 & \dots & \dots & \dots & \dots \\ r^2s^2 & r^4s^3 & r^6s^4 & \dots & \dots & \dots \\ rs & r^3s^2 & r^5s^3 & r^7s^4 & \dots & \dots \\ 1 & r^2s & r^4s^2 & r^6s^3 & r^8s^4 & \dots \end{pmatrix}.$$

Since $\limsup_{i+j\to\infty}|\mu_{v_{i,j}}|^{1/(i+j)} = q < 2$ (note that if B is bounded then $q>0$), it is clear that $f_{r,s}\in X$ whenever $qr^2|s|<1$. On the other hand

$$Bf_{r,s} = s(r^2+r)f_{r,s}.$$

Thus, the set of eigenvalues contains the disc $\left\{|\lambda| < \frac{r^2+r}{qr^2} = \frac{1}{q}\left(1+\frac{1}{r}\right)\right\}$ for any $r\ge 1$. Since $q<2$, its radius is greater than 1 when r is sufficiently close to 1. More precisely, choose $\delta>0$ so small that $\frac{1}{qr^2}-\delta>\frac{1}{2}$ and $\delta(r^2+r)<1$ for any $r\in(1,1+\delta)$. Put

$$U_1 = \left\{(r,s):\ r\in(1,1+\delta),\ \frac{1}{qr^2}-\delta<|s|<\frac{1}{qr^2}\right\},$$
$$U_2 = \{(r,s):\ r\in(1,1+\delta),\ |s|<\delta\}.$$

Then $|s|(r^2+r)>1$ when $(r,s)\in U_1$ and $|s|(r^2+r)<1$ when $(r,s)\in U_2$. To apply the Godefroy–Shapiro criterion, we show that the families $\{f_{r,s}:\ (r,s)\in U_1\}$ and $\{f_{r,s}:\ (r,s)\in U_2\}$ are complete in X.

Indeed, let $g=\{g(v_{i,j})\}=\{g_{i,j}\}\in X^*$ be a sequence in the annihilator of $\{f_{r,s}:\ (r,s)\in U_1\}$. Then we have, for any $r\in(1,1+\delta)$ and $\frac{1}{qr^2}-\delta<|s|<\frac{1}{qr^2}$,

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} f(v_{k,n-k})\bar{g}_{k,n-k} = \sum_{n=0}^{\infty} s^n r^n \sum_{k=0}^{n}\bar{g}_{k,n-k}r^{n-k} = 0.$$

Considering this series as a power series with respect to the variable s we conclude that for any $n\in\mathbb{N}_0$

$$\sum_{k=0}^{n}\bar{g}_{k,n-k}r^{n-k} = 0, \qquad r\in(1,1+\delta),$$

whence, obviously, $g_{k,n-k}=0$ for any $n\ge 0$, $0\le k\le n$. Completeness of $\{f_{r,s}:\ (r,s)\in U_2\}$ is analogous.

Second proof of Statement 1. For $k \geq 0$ we let

$$X_k = \operatorname{span}\{e_{v_{i+j}} : i + j = k\}$$

be the $(k+1)$-dimensional space of sequences with non-zero entries on the diagonal $\{(i, j) : i + j = k\}$. The subspace

$$Y = \operatorname{span}\left(\bigcup_{k=0}^{\infty} X_k\right)$$

is dense in the space. Let $\{a_i\}_{i=0}^{\infty}$ be a sequence of pairwise distinct, non-zero real scalars. For $k \geq 0$ we set

$$f_i^k = \sum_{j=0}^{k} a_i^j e_{v_{k-j,j}}, \quad 0 \leq i \leq k.$$

Notice that the set $\{f_i^k\}_{i=0}^k$ forms a base for X_k since the determinant of the coefficients is a non-zero Vandermonde determinant. For $n \geq 0$, define R_n on $\{f_i^k\}_{i=0}^k$ by

$$R_n f_i^k = \frac{1}{(1+a_i)^n} \sum_{j=0}^{k+n} a_i^j e_{v_{k+n-j,j}}.$$

Note that as in (3.1), we get

$$B^n e_{v_{i,j}} = \sum_{l=0}^{n} \binom{n}{l} e_{v_{i-l,j-(n-l)}},$$

where, as before, we consider $e_{v_{i-l,j-(n-l)}}$ as the zero function, whenever $i < l$ or $j < n-l$. Therefore, by a direct computation,

$$\begin{aligned} B^n R_n f_i^k &= \frac{1}{(1+a_i)^n} \sum_{l=0}^{n} \binom{n}{l} \sum_{j=n-l}^{n+k-l} a_i^j e_{v_{k+n-j-l,j-(n-l)}} \\ &= \frac{1}{(1+a_i)^n} \sum_{l=0}^{n} \binom{n}{l} \sum_{s=0}^{k} a_i^{n-l+s} e_{v_{k-s,s}} = f_i^k. \end{aligned}$$

Also, $B^n f_i^k = 0$, for each $n > k$. Extend R_n linearly to get first an operator from X_k to X_{k+n} and then to get an operator on Y. It remains to show that $R_n f_i^k \to 0$.

Let $q \in (1, 2)$ be such that $\limsup_{i+j\to\infty} |\mu_{v_{i,j}}|^{1/(i+j)} < q$ and choose $a_i \in (q-1, 1)$, $i \geq 0$. If $X = \ell^p(V_\infty, \mu)$ we get that

$$\|R_n f_i^k\| = \frac{1}{(1+a_i)^n} \left(\sum_{j=0}^{k+1} a_i^{jp} |\mu_{v_{k+n-j,j}}|^p\right)^{1/p} \leq \frac{C q^{k+n}}{(1+a_i)^n (1-a_i^p)^{1/p}} \to 0.$$

The Hypercyclicity Criterion now applies for the full sequence $\{n\}$, and ensures that B is mixing. The case $X = c_0(V_\infty, \mu)$ is analogous.

Proof of Statement 2. Assume that B is hypercyclic. Then for any $\varepsilon \in (0, 1/2)$ and $N \in \mathbb{N}$ there exists $f \in X$ and $n > N$ such that $\|f\|_X < \varepsilon$ and

$$\|B^n f - e_{v_{0,0}}\| < \varepsilon |\mu_{v_{0,0}}|.$$

Since $(B^n f)(v_{0,0}) = \sum_{k=0}^{n} \binom{n}{l} f(v_{k,n-k})$, we have

$$\left| \sum_{k=0}^{n} \binom{n}{k} f(v_{k,n-k}) \right| > 1/2.$$

Let k_0 be such that $|f(v_{k_0,n-k_0})| = \max_{0\le k\le n} |f(v_{k,n-k})|$. Using the fact that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$, we conclude that $|f(v_{k_0,n-k_0})| \ge 2^{-n-1}$. It follows from (2.6) that $\|f\|_X \ge c/2$, a contradiction with $\|f\|_X < \varepsilon$ if ε is sufficiently small. □

We conclude this section with a necessary condition for hypercyclicity of the backward shift on graphs, which also proves Statement 2 of Theorem 2.3. For simplicity we formulate it for $1 < p < \infty$. The cases of $\ell^1(V_\infty, \mu)$ and $c_0(V_\infty, \mu)$ require an obvious modification.

Proposition 5.1 *Let B be bounded on $X = \ell^p(V_\infty, \mu)$, $1 < p < \infty$, and let $1/p + 1/p' = 1$. If B is hypercyclic on X, then there exists an increasing sequence (n_k) of positive integers such that for any $i, j \in \mathbb{N}$,*

$$\sum_{l=0}^{n_k} \frac{\binom{n_k}{l}^{p'}}{|\mu_{v_{i+l,j+n_k-l}}|^{p'}} \to \infty, \qquad k \to \infty.$$

Proof of Theorem 2.1 The proof is analogous to [5, Lemma 4.2]. If B is hypercyclic, then for any $\varepsilon \in (0, 1/2)$ and $K, N \in \mathbb{N}$ there exist $f \in X$ and $n > N$ such that

$$\|f\|_X < \varepsilon, \qquad \left\| B^n f - \sum_{1\le i,j\le K} e_{v_{i,j}} \right\|_X < \varepsilon \min_{1\le i,j\le K} |\mu_{v_{i,j}}|.$$

Since $(B^n f)(v_{i,j}) = \sum_{l=0}^{n} \binom{n}{l} f(v_{i+l,j+n-l})$, we have for $i, j \le K$

$$1/2 \le \left| \sum_{l=0}^{n} \binom{n}{l} f(v_{i+l,j+n-l}) \right| \le \|f\|_X \left(\sum_{l=0}^{n} \frac{\binom{n}{l}^{p'}}{|\mu_{v_{i+l,j+n-l}}|^{p'}} \right)^{1/p'}$$
$$\le \varepsilon \left(\sum_{l=0}^{n} \frac{\binom{n}{l}^{p'}}{|\mu_{v_{i+l,j+n-l}}|^{p'}} \right)^{1/p'}.$$

Repeating the procedure for $K_k \to \infty$ and $\varepsilon_k \to 0$ we find a sequence n_k as required. □

In the case of trees considered in [5] this natural condition turns out be also sufficient. We do not know whether it is the case for the lattice $\mathbb{N} \times \mathbb{N}$.

6 Proof of Theorem 2.4

We start with the following simple but useful observation.

Proposition 6.1 *Let $G = (V, E)$ be a subgraph of a directed graph $\tilde{G} = (\tilde{V}, \tilde{E})$ such that if $v \in V$ and $(v, u) \in \tilde{E}$, then $u \in V$ and $(v, u) \in E$, i.e., G includes all edges which start in V. Let μ be a weight on $\tilde{G}$ such that the backward shift $\tilde{B}$ is bounded on $\ell^p(\tilde{V}, \mu)$, $1 \le p < \infty$, or $c_0(\tilde{V}, \mu)$. Then the backward shift B on $\ell^p(V, \mu)$ or $c_0(V, \mu)$ inherits all dynamical properties (mixing, weak mixing, hypercyclicity) of $\tilde{B}$ on $\ell^p(\tilde{V}, \mu)$ or, respectively, $c_0(\tilde{V}, \mu)$.*

Proof of Theorem 2.1 Consider the restriction operator $R : \ell^p(\tilde{V}, \mu) \to \ell^p(V, \mu)$, $1 \leq p < \infty$, or $R : c_0(\tilde{V}, \mu) \to c_0(V, \mu)$, $Rf = f|_V$. It is clear that R is a surjective operator of norm one, and, by the properties of G,

$$R \circ \tilde{B} = B \circ R.$$

This means that B is quasi-conjugate to $\tilde{B}$ and so inherits all dynamical properties of $\tilde{B}$. □

In the following proof we will use recent results from [9] about generalized shifts. Given an operator T on the Banach space X, the generalized shift B_T is defined on $X^{\mathbb{N}}$ or $X^{\mathbb{Z}}$ as

$$B_T(x_k)_k = (Tx_{k+1})_k.$$

Thus, if $T = I$, then B_T is a usual shift.

Consider the following spaces:

$$\ell^p(X, \mathbb{Z}) = \left\{ (x_k)_k \in X^{\mathbb{Z}} : \sum_k \|x_k\|_X^p < \infty \right\},$$

$$c_0(X, \mathbb{Z}) = \left\{ (x_k)_k \in X^{\mathbb{Z}} : \lim_{|k|\to\infty} \|x_k\|_X = 0 \right\}.$$

It is shown in [9, Corollary 2.8] that B_T on $\ell^p(X, \mathbb{Z})$, $1 \leq p < \infty$, or on $c_0(X, \mathbb{Z})$ is hypercyclic if and only if it is weakly mixing and if and only if T is weakly mixing on X, while B_T is mixing if and only if T is mixing [9, Corollary 3.2].

Proof of Theorem 2.4 It follows from Proposition 6.1 that it is sufficient to prove the statement for the graph $\tilde{G}_\infty = (\tilde{V}_\infty, \tilde{E}_\infty)$ since its subgraph G_∞ satisfies the conditions of the proposition. Note that the backward shift B on G_∞ when considered as an operator on either $\ell^p(V_\infty, \mu)$, $1 \leq p < \infty$, or $c_0(V_\infty, \mu)$ is bounded if and only if the backward shift $\tilde{B}$ on $\tilde{G}_\infty$, when considered as an operator on any of $\ell^p(\tilde{V}_\infty, \mu)$, $1 \leq p < \infty$, or $c_0(\tilde{V}_\infty, \mu)$ is bounded, which happens precisely when $\sup_{i\in\mathbb{N}} \frac{|\mu_i|}{|\mu_{i+1}|} < \infty$.

Now set for $k \in \mathbb{Z}$,

$$D_k = \{v_{i,j} \in \tilde{V}_\infty : i + j = k\}$$

and notice that, due to the fact that μ depends only on the first coordinate, $\mu_{v_{i,j}} = \mu_i$, we can identify the spaces $\ell^p(D_k, \mu)$, $k \in \mathbb{Z}$, with $\ell^p(\mathbb{N}, \mu)$, $1 \leq p < \infty$, and similarly $c_0(D_k, \mu)$, $k \in \mathbb{Z}$, with $c_0(\mathbb{N}, \mu)$ via the identification $v_{i,k-i} \leftrightarrow i$. This allows us to further identify for $1 \leq p < \infty$

$$\ell^p(\tilde{V}_\infty) \cong \ell^p(\ell^p(\mathbb{N}, \mu), \mathbb{Z}) = \left\{ (f_k)_{k\in\mathbb{Z}} \in \ell^p(\mathbb{N}, \mu)^{\mathbb{Z}} : \sum_{k=-\infty}^{\infty} \|f_k\|_{\ell^p(\mathbb{N},\mu)}^p < \infty \right\}$$

and

$$c_0(\tilde{V}_\infty) \cong c_0(c_0(\mathbb{N}, \mu), \mathbb{Z}) = \left\{ (f_k)_{k\in\mathbb{Z}} \in c_0(\mathbb{N}, \mu)^{\mathbb{Z}} : \lim_{|k|\to\infty} \|f_k\|_{c_0(\mathbb{N},\mu)} \to 0 \right\}$$

by $(f(v_{i,j}))_{i\in\mathbb{N}, j\in\mathbb{Z}} \mapsto (f_k)_{k\in\mathbb{Z}} = ((f(v_{i,k-i}))_{i\in\mathbb{N}})_{k\in\mathbb{Z}}$.

Let B_0 be the unweighted unilateral backward shift on either $\ell^p(\mathbb{N}, \mu)$, $1 \leq p < \infty$, or $c_0(\mathbb{N}, \mu)$. Under the above identifications, we notice that the backward shift $\tilde{B}$ can be viewed as the generalized shift B_{I+B_0} on $\ell^p(\ell^p(\mathbb{N}, \mu), \mathbb{Z})$ or $c_0(c_0(\mathbb{N}, \mu), \mathbb{Z})$, defined by

$$B_{I+B_0}(f_k)_{k\in\mathbb{Z}} = ((I + B_0)(f_{k+1}))_{k\in\mathbb{Z}}.$$

Indeed,

$$(\tilde{B}f)(v_{i,k-i}) = f(v_{i,k+1-i}) + f(v_{i+1,k-i}) = f(v_{i,k+1-i}) + f(v_{i+1,k+1-(i+1)}).$$

Therefore, if we write $f_k = (f(v_{i,k-i}))_{i\in\mathbb{N}}$, then $\tilde{B}(f_k)_{k\in Z} = (f_{k+1} + B_0 f_{k+1})_{k\in\mathbb{Z}}$. By [9], the unilateral generalized shift B_{I+B_0} is mixing if and only its symbol $I + B_0$ is mixing. Since by a theorem of Salas [11] (see also [4, Theorem 8.2]), $I + B_0$ is always mixing on $\ell^p(\mathbb{N}, \mu)$, $1 \le p < \infty$, or on $c_0(\mathbb{N}, \mu)$ provided it is bounded, we conclude that $\tilde{B}$ is mixing. □

Remark 6.2 If we drop the assumption that μ depends on one coordinate only and if $\tilde{\mu}$ is a weight on $\tilde{V}_\infty$ that extends μ and makes $\tilde{B}$ bounded, then we can still repeat the above argument and see $\tilde{B}$ as the bilateral generalized shift B_{I+B_0} (which is bounded) but in this case the symbol $I + B_0$ on the n-th coordinate is an operator from $\ell^p(\mathbb{N}, \tilde{\mu}|_{D_{n+1}})$ to $\ell^p(\mathbb{N}, \tilde{\mu}|_{D_n})$. In this case Salas' result cannot be used to conclude that $I + B_0$ is mixing.

Also, if we consider the shift on the lattice $\mathbb{Z} \times \mathbb{Z}$ in the case when μ depends on one coordinate only, we can identify it with B_{I+B_0}, where B_0 is the usual bilateral shift on $\ell^p(\mathbb{Z}, \mu)$. However, in the bilateral case it is not known whether $I + B_0$ is mixing or, at least, hypercyclic.

7 Examples

We start with an obvious example when a graph cannot carry a hypercyclic backward shift even if we have a freedom in the choice of a measure.

Example 7.1 Let $V = \{v_{1,1}, v_{1,2}\} \cup \{v_{i,i-1}, v_{i,i}, v_{i,i+1},\ i \ge 2\}$. Assume that $v_{i,i-1} \to v_{i,i}$, $v_{i-1,i} \to v_{i,i}$, $v_{i,i} \to v_{i+1,i}$, $v_{i,i} \to v_{i,i+1}$, and there are no other edges in G. Then it is clear that $(B^n f)(v_{i,i-1}) = (B^n f)(v_{i-1,i}) = (B^{n-1} f)(v_{i,i})$ when $i \ge 2$, and so B is not hypercyclic.

More generally, if G is a graph with a vertex v satisfying that $|Par(v)| > 1$ and $|Chi(Par(v))| = 1$ (i.e., $Chi(Par(v)) = \{v\}$), then the same argument applies and B is not hypercyclic. Here we denote by $|E|$ the number of elements in the set E.

Example 7.2 Let $V = \{v_j\}_{j\ge1} \cong \mathbb{N}$ and $E = \{(v_i, v_{i+1}),\ i \ge 1\} \cup \{(v_2, v_1)\}$. Thus, V is the tree $\mathbb{N}$ with one added edge making a cycle. We consider the "Rolewicz operator" αB (considered for the first time in [10]), where $\alpha \in \mathbb{C}$, $|\alpha| > 1$, on the unweighted space ℓ^p. By a trivial computation, for a sequence $f = (f_j)_{j\ge1}$, we have

$$B^{2n-1} f = \left(\sum_{j=1}^{n} f_{2j}, \sum_{j=1}^{n+1} f_{2j-1}, f_{2n+2}, f_{2n+3}, \dots \right),$$

$$B^{2n} f = \left(\sum_{j=1}^{n+1} f_{2j-1}, \sum_{j=1}^{n+1} f_{2j}, f_{2n+3}, f_{2n+4}, \dots \right).$$

We show that αB is hypercyclic (and even mixing) in ℓ^p for $1 < p < \infty$. Denote by Y the set of all finite sequences (x_k) such that $\sum_{k\text{ – odd}} x_k = \sum_{k\text{ – even}} x_k = 0$. Let us show that Y is dense in ℓ^p for $1 < p < \infty$. For any finite vector $x \in \ell^p$ put $a = \sum_{k\text{ – odd}} x_k$ and

$b = \sum_{k\,-\,\text{even}} x_k$ and let

$$\tilde{x}_n = x - \frac{a}{n}\sum_{k=1}^{n} e_{v_{2k-1}} - \frac{b}{n}\sum_{k=1}^{n} e_{v_{2k}}.$$

Then $\tilde{x}_n \in Y$ and, obviously, $\tilde{x}_n \to x$ in ℓ^p when $p > 1$.

Now let U be a non-empty subset of ℓ^p and let W be a neighborhood of zero. If $x \in U \cap Y$, then $(aB)^n x = 0$ eventually, which means that the return set $N(U, W) = \{n : (aB)^n(U) \cap W \neq \emptyset\}$ is cofinite. If we set

$$y_n = \frac{1}{a^n}\sum_{k=1}^{\infty} x_k e_{v_{k+n}},$$

then $y_n \to 0$ and $(aB)^n y_n = x$ for each $n \in \mathbb{N}$ which shows that also $N(W, U)$ is cofinite. By [4, Proposition 2.37], we conclude that aB is mixing.

Remark 7.3 Note that any bounded operator T acting on a Banach space X with a Schauder base can be interpreted as a *weighted* backward shift on some Banach space of sequences indexed by the vertices of some graph. Indeed, if $\{e_n\}_{n=1}^{\infty}$ is a Schauder basis and

$$Te_n = \sum_{i=1}^{\infty} \alpha_{i,n} e_i$$

for some coefficients $\alpha_{i,n}$, then we define the graph G to have vertices e_n and we assume that $e_i \to e_n$ if and only if $\alpha_{i,n} \neq 0$. We define the weight on the edge (e_i, e_n) as $\alpha_{i,n}$. Thus, for $x = \sum_n c_n e_n$ we have $Tx = \sum_i \left(\sum_n \alpha_{i,n} c_n\right) e_i$, and so T is a weighted backward shift on the coefficient space of the basis $\{e_n\}$ equipped with its standard norm (see, e.g., [12, Chapter 1]).

Example 7.4 In the next example we again consider an unweighted backward shift. Let $V = \{v_j\}_{j\geq 1} \cong \mathbb{N}$ and assume that $v_j \to v_{j+1}$ and $v_j \to v_{j+2}$, $j \geq 1$, and all edges are of this form. Then we have $(Bf)(v_j) = f(v_{j+1}) + f(v_{j+2})$ and so B essentially coincides with $B_0(I + B_0)$ where B_0 is the usual backward shift on $\mathbb{N}$.

Assume that $\mu = \{\mu_n\}$ is a weight such that the backward shift B_0 is bounded on X where $X = \ell^p(\mathbb{N}, \mu)$, $1 \leq p < \infty$, or $X = c_0(\mathbb{N}, \mu)$. Let $q = \limsup_{n\to\infty} |\mu_n|^{1/n}$. Then for any $s \in \mathbb{C}$ with $|s| < q^{-1}$ we have $f_s = \{s^n\}_{n\geq 1} \in X$ and $B_0(I + B_0)f_s = s(1+s)f_s$. If $q < \frac{1+\sqrt{5}}{2}$, then $q^{-1}(1 + q^{-1}) > 1$, and we have an open set of eigenvalues λ of $B_0(I + B_0)$ with $|\lambda| < 1$ and an open set of eigenvalues λ of $B_0(I + B_0)$ with $|\lambda| > 1$. Corresponding families of eigenvectors are complete and so $B_0(I + B_0)$ is mixing by the Godefroy–Shapiro criterion.

As in Theorem 2.3, one easily shows that $q_0 = \frac{1+\sqrt{5}}{2}$ is the critical value. Assume that there is $C > 0$ such that $|\mu_n| \geq Cq_0^n$. We show that in this case $B_0(I + B_0)$ is not hypercyclic. Indeed, if $B_0(I + B_0)$ is hypercyclic, then for any $\varepsilon \in (0, 1/2)$ and $N \in \mathbb{N}$ there exists $f = (f_k)_{k\in\mathbb{N}} \in X$ and $n > N$ such that $\|f\|_X < \varepsilon$ and $\|(B_0(I + B_0))^n f - e_1\|_X < \varepsilon|\mu_1|$ where $e_1 = (1, 0, 0, \ldots)$. On one hand, we have $|f_k| \leq C^{-1}\varepsilon q_0^{-k}$. On the other hand,

$$\left|\sum_{k=0}^{n}\binom{n}{k} f_{k+n} - 1\right| < \varepsilon,$$

a contradiction, when ε is sufficiently small, since $\sum_{k=0}^{n} \binom{n}{k} q_0^{-k-n} = q_0^{-n}(q_0^{-1} + 1)^n = 1$. This is in contrast with the case of the operator $I + B_0$ which is weakly mixing on $\ell^p(\mathbb{N}, \mu)$, $1 \le p < \infty$, or $c_0(\mathbb{N}, \mu)$ whenever B_0 is bounded [4, Theorem 8.2].

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