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RESEARCH ARTICLE

Families of twists of tuples of hyperelliptic curves

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Abstract

Let $f \in \mathbb{Q}[x]$ be a square-free polynomial of degree at least 3, m_i, $i = 1, 2, 3$, odd positive integers, and a_i, $i = 1, 2, 3$, non-zero rational numbers. We show the existence of a rational function $D \in \mathbb{Q}(v_1, v_2, v_3, v_4)$ such that the Jacobian of the quadratic twist of $y^2 = f(x)$ and the Jacobian of the m_i-twist, respectively, $2m_i$-twist, of $y^2 = x^{m_i} + a_i^2$, $i = 1, 2, 3$, by D are all of positive Mordell–Weil ranks. As an application, we present families of hyperelliptic curves with large Mordell–Weil rank.

1. Introduction

Given an abelian variety A defined over a number field K, the Mordell–Weil theorem asserts that the abelian group of rational points $A(K)$ is finitely generated. This implies that $A(K)$ is isomorphic to $\mathbb{T} \times \mathbb{Z}^r$, where $\mathbb{T}$ is the torsion subgroup of $A(K)$ and r is a non-negative integer, called the Mordell–Weil rank of $A(K)$. In this article, we are interested in the following question. Fix an integer $n \geq 2$. Given (hyper)elliptic curves C_i defined over $\mathbb{Q}$ of genus g_i, and tuples of positive integers r_i, $i = 1, \ldots, n$, is there a twist of C_i such that the Mordell–Weil rank of the Jacobian of this twist is at least r_i, $i = 1, \ldots, n$?

The problem was initially discussed in [4] where it was proved that for a pair of elliptic curves E_1 and E_2 over $\mathbb{Q}$ whose j-invariants are not simultaneously 0 or 1728, there exist infinitely many square-free rational numbers d such that the quadratic twists of E_1 and E_2 by d are both of positive Mordell–Weil rank. In [1], it was proved that there exist families of pairs of elliptic curves E_1 and E_2 and infinitely many square-free rationals d such that the quadratic twists of E_1 and E_2 by d are both of Mordell–Weil rank at least 2. The positivity of the Mordell–Weil rank of families of twists of triples and quadruples of elliptic curves with j-invariants 0 or 1728 was established in [8, 9]. This motivates the investigation of the positivity of Mordell–Weil rank of families of twists of higher-dimensional abelian varieties.

Given non-zero rational numbers a, b, c, it was proved that there exists a polynomial $d(t) \in \mathbb{Q}[t]$ such that the Jacobians of the curves given by $y^2 = x^n + ad(t)$, $y^2 = x^n + bd(t)$, $y^2 = x^n + cd(t)$, and $n \geq 3$, both have positive Mordell–Weil rank over $\mathbb{Q}(t)$, see [5]. The latter result was extended to four such hyperelliptic curves when n is odd. If $f \in \mathbb{Q}[x]$ is a square-free polynomial of degree at least 3, it was proved in [6] that there exists a function $D \in \mathbb{Q}(u, v, w)$ such that the Jacobians of the curves $Dy^2 = f(x)$, $y^2 = Dx^m + b$, and C all have positive Mordell–Weil ranks over $\mathbb{Q}(u, v, w)$ where C is either $y^2 = Dx^m + c$ or $y^2 = x^m + cD$, and $m \geq 3$ is odd. The question was investigated for other families of quadruples of hyperelliptic curves in [3].

In this work, we extend the latter results to include several families of quadruples of twisted hyperelliptic curves. Namely, we consider the following sets of hyperelliptic curves:

$$Dy^2 = f(x), \quad y^2 = Dx^{m_1} + a, \quad y^2 = Dx^{m_2} + b, \quad y^2 = Dx^{m_3} + c,$$

or

$$Dy^2 = f(x), \quad y^2 = x^{m_1} + aD, \quad y^2 = x^{m_2} + bD, \quad y^2 = x^{m_3} + cD,$$

or

$$y^2 = Dx^{m_1} + a, \quad y^2 = Dx^{m_2} + b, \quad y^2 = Dx^{m_3} + c, \quad y^2 = Dx^{m_4} + d,$$

where $f(x) \in \mathbb{Q}[x]$ is a square-free polynomial of degree at least 3, and $m_i \geq 3$ is an odd integer, $i = 1, 2, 3, 4$. We prove that if a, b, and c are non-zero rational squares, then D can be chosen to be a rational function in $\mathbb{Q}(u, v_1, v_2, v_3)$ such that the Mordell–Weil ranks of all four hyperelliptic curves over $\mathbb{Q}(u, v_1, v_2, v_3)$ are positive. The proof depends on associating a system of Diophantine equations to these quadruples of hyperelliptic curves. Finding such a rational D corresponds to finding a rational solution to this system of equations.

As a byproduct, fixing an odd integer $m \geq 3$, we show that there exists a rational $D \in \mathbb{Q}(u, v_1, v_2, v_3)$ such that the Mordell–Weil rank of the Jacobian of $Dy^2 = f(x)$ is positive and the Mordell–Weil rank of the Jacobian of C is at least 3, where C is either $y^2 = Dx^m + a^2$ if $\deg f$ is at least 3 or $y^2 = x^m + a^2 D$ if $f(x)$ is of degree 3 or 4.

2. A quadratic twist and three higher-degree twists

Let $f(x) \in \mathbb{Q}[x]$ be a square-free polynomial of degree at least 3. Fix odd integers $m_1, m_2, m_3 \geq 3$. We are investigating the existence of non-square rational numbers D such that the Jacobians of the (hyperelliptic) curves

$$C : Dy^2 = f(x), \quad C_1 : y^2 = Dx^{m_1} + a, \quad C_2 : y^2 = Dx^{m_2} + b, \quad C_3 : y^2 = Dx^{m_3} + c$$

are of positive Mordell–Weil rank over $\mathbb{Q}$.

We set $M = \operatorname{lcm}(m_1, m_2, m_3)$ and $M_i = M/m_i$, $i = 1, 2, 3$. The rational number D can be found by solving the following system of equations:

$$\frac{y_1^2 - a}{x_1^{m_1}} = \frac{y_2^2 - b}{x_2^{m_2}} = \frac{y_3^2 - c}{x_3^{m_3}} = \frac{f(x_4)}{y_4^2}. \tag{2.1}$$

More precisely, we are looking for solutions x_i, y_i, $i = 1, 2, 3, 4$, for the system (2.1) where

$$x_1 x_2 x_3 \left(y_1^2 - a\right)\left(y_2^2 - b\right)\left(y_3^2 - c\right) y_4 f(x_4) \neq 0.$$

In what follows, we suggest how one can obtain a family of parametric solutions to (2.1). We set

$$x_1 = \frac{1}{v_1^2 T^{M_1}}, \quad x_2 = \frac{1}{v_2^2 T^{M_2}}, \quad x_3 = \frac{1}{v_3^2 T^{M_3}}, \quad x_4 = u, \quad y_1 = p, \quad y_2 = q, \quad y_3 = r, \quad y_4 = \frac{1}{T^{(M-1)/2}}$$

where u, v_1, v_2, v_3 are rational parameters. We observe that

$$T = \frac{f(u)}{v_3^{2m_3}(r^2 - c)},$$
$$v_1^{2m_1} T^M (p^2 - a) = v_2^{2m_2} T^M (q^2 - b) = v_3^{2m_3} T^M (r^2 - c).$$

In other words, the system (2.1) can be simplified to

$$T = \frac{f(u)}{v_3^{2m_3}(r^2 - c)}, \quad v_1^{2m_1}(p^2 - a) = v_2^{2m_2}(q^2 - b) = v_3^{2m_3}(r^2 - c).$$

Geometrically, the second equation defines an intersection $\mathcal{C}_{a,b,c}$ of two quadratic surfaces in $\mathbb{P}^3$ over $\mathbb{Q}(v_1, v_2, v_3)$. Since $[p : q : r : s] = [\pm v_2^{m_2} v_3^{m_3} : \pm v_1^{m_1} v_3^{m_3} : \pm v_1^{m_1} v_2^{m_2} : 0]$ are rational points on $\mathcal{C}_{a,b,c}$, it follows that $\mathcal{C}_{a,b,c}$ is an elliptic curve over $\mathbb{Q}(v_1, v_2, v_3)$. One may check that the latter four projective rational points form a subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

We now assume, further, that a, b, and c are non-zero rational squares. After a transformation, the curve $\mathcal{C}_{a,b,c}$ is defined by

$$x^2 - T_1^2 = y^2 - T_2^2 = z^2 - T_3^2,$$

where $T_1 = av_1^{m_1}$, $T_2 = bv_2^{m_2}$, and $T_3 = cv_3^{m_3}$.

Proposition 2.1. *Let C_{T_1,T_2,T_3} be the elliptic curve defined by*

$$x^2 - T_1^2 = y^2 - T_2^2 = z^2 - T_3^2$$

over $\mathbb{Q}(T_1, T_2, T_3)$. The curve C_{T_1,T_2,T_3} has positive Mordell–Weil rank over $\mathbb{Q}(T_1, T_2, T_3)$. In particular, except for a thin set of triples $(t_1, t_2, t_3) \in \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$, the curve C_{t_1,t_2,t_3} has positive Mordell–Weil rank over $\mathbb{Q}$.

Proof. The point $[T_1 : T_2 : T_3 : 1] \in C_{T_1,T_2,T_3}(\mathbb{Q}(T_1, T_2, T_3))$. In addition, it specializes to a point of infinite order on $C_{1,2,3}$ when $[1 : 1 : 1 : 0]$ is regarded as the identity element, Magma [2]. For the convenience of the reader, we include the code as it will be used later.

```
K<T1,T2,T3>:=FunctionField(Rationals(),3);
P<x,y,z,w>:=ProjectiveSpace(K,3);
C:=Curve(P, [(x^2-T1^2*w^2)-(y^2-T2^2*w^2),(y^2-T2^2*w^2)-(z^2-T3^2*w^2)]);
P:=C![1,1,1,0];
Q:=C![T1,T2,T3];
E,phi:=EllipticCurve(C,P);
Q1:=phi(Q);
```

Now the statement follows from Silverman's specialization theorem [7, Section 20, Theorem 20.1]. □

One remarks that when the point $[w_1 : w_2 : w_3 : 1]$ is a point of infinite order in $\mathcal{C}_{a,b,c}$, then any of its multiples can be used to generate a solution to the system (2.1).

Doubling the point $P := [a : b : c : 1] \in \mathcal{C}_{a^2,b^2,c^2}(\mathbb{Q}(v_1, v_2, v_3))$ yields that $2P = [w_1 : w_2 : w_3 : 1]$ is given by

$$\left[\frac{a^2b^2v_1^{2m_1}v_2^{2m_2} + a^2c^2v_1^{2m_1}v_3^{2m_3} - b^2c^2v_2^{2m_2}v_3^{2m_3}}{2abcv_1^{2m_1}v_2^{m_2}v_3^{m_3}} : \frac{a^2b^2v_1^{2m_1}v_2^{2m_2} - a^2c^2v_1^{2m_1}v_3^{2m_3} + b^2c^2v_2^{2m_2}v_3^{2m_3}}{2abcv_1^{m_1}v_2^{2m_2}v_3^{m_3}} \right.$$
$$\left. : \frac{-a^2b^2v_1^{2m_1}v_2^{2m_2} + a^2c^2v_1^{2m_1}v_3^{2m_3} + b^2c^2v_2^{2m_2}v_3^{2m_3}}{2abcv_1^{m_1}v_2^{m_2}v_3^{2m_3}} : 1\right].$$

We now set

$$T = \frac{f(u)}{v_3^{2m_3}\left(w_3^2 - c^2\right)}, \quad D = \left(w_1^2 - a^2\right)v_1^{2m_1}T^M.$$

We consider the following twists of hyperelliptic curves:

$$C : Dy^2 = f(x), \quad C_1 : y^2 = Dx^{m_1} + a^2, \quad C_2 : y^2 = Dx^{m_2} + b^2, \quad C_3 : y^2 = Dx^{m_3} + c^2.$$

Now, the points $P_i := (x_i, y_i) = \left(\frac{1}{v_i^2 T^{M_i}}, w_i\right)$ are $\mathbb{Q}(v_1, v_2, v_3)$-rational points in C_i, $i = 1, 2, 3$, whereas $P := (x_4, y_4) = \left(u, \frac{1}{T^{(M-1)/2}}\right)$ is a $\mathbb{Q}(u, v_1, v_2, v_3)$-rational point in C.

In [8, 9], explicit constructions of families of triples and quadruples of elliptic curves with j-invariants 0 or 1728 were given such that there is a rational function D for which the twists of the curves in each tuple by D are of positive Mordell–Weil rank. This was extended to pairs and triples of Jacobians of hyperelliptic curves of the same genus in [5, 6]. Following the argument above, we obtain the following theorem which can be considered as an extension of the aforementioned results to quadruples of Jacobians of hyperelliptic curves that are not necessarily of the same genus.

Theorem 2.2. *Let $f \in \mathbb{Q}[x]$ be a square-free polynomial of degree at least 3. Let $m_1, m_2, m_3 \geq 3$ be odd integers. Consider the hyperelliptic curves*

$$E : y^2 = f(x), \quad E_1 : y^2 = x^{m_1} + a^2, \quad E_2 : y^2 = x^{m_2} + b^2, \quad E_3 : y^2 = x^{m_3} + c^2,$$

where a, b, c are non-zero rational numbers. Then there exists a rational function $D_{2,m_1,m_2,m_3} \in \mathbb{Q}(u, v_1, v_2, v_3)$ such that the Jacobian of the quadratic twist of the curve E by D_{2,m_1,m_2,m_3} and the Jacobians of the m_i-twists of the curves E_i, $i = 1, 2, 3$, by D_{2,m_1,m_2,m_3} have positive Mordell–Weil rank over $\mathbb{Q}(u, v_1, v_2, v_3)$.

Proof. Let C and C_i, $i = 1, 2, 3$, be the twists of E and E_i, $i = 1, 2, 3$, described above. Let P_i, $i = 1, 2, 3$, and P be the rational points above. We set J and J_i to be the Jacobians of C and C_i, $i = 1, 2, 3$, respectively. We consider the rational divisors $D = (P) - (\infty)$ on C and $D_i = (P_i) - (\infty)$, $i = 1, 2, 3$, on C_i. Since C is a non-constant quadratic twist of a constant curve E, it follows that the divisor D defines a rational point of infinite order in $J(\mathbb{Q}(u, v_1, v_2, v_3))$.

Due to [5, Proposition 2.1], the divisor D_i corresponds to a rational point of infinite order in $J_i(\mathbb{Q}(u, v_1, v_2, v_3))$ as P_i, $i = 1, 2, 3$, has non-constant coordinates with a non-zero y-coordinate. $\square$

Example 2.3. *The curves $E : Dy^2 = x^5 + x + 1$ and $E_i : y^2 = Dx^{4i+1} + i^2$, $i = 1, 2, 3$, are of positive Mordell–Weil rank over $\mathbb{Q}$, where*

$$\begin{aligned}
D := D(u, v_1, v_2, v_3) &= \frac{\left(u^5 + u + 1\right)^{585}\left(w_1^2 - 1\right)v_1^{10}}{v_3^{15210}(w_3^2 - 9)^{585}} \quad \text{where} \\
w_1 &= \frac{3v_3^{13}}{4v_2^9} + v_2^9\left(\frac{1}{3v_3^{13}} - \frac{3v_3^{13}}{v_1^{10}}\right), \\
w_2 &= \frac{3v_3^{13}}{v_1^5} + v_1^5\left(\frac{1}{3v_3^{13}} - \frac{3v_3^{13}}{4v_2^{18}}\right), \\
w_3 &= \frac{3v_2^9}{v_1^5} + v_1^5\left(\frac{3}{4v_2^9} - \frac{v_2^9}{3v_3^{26}}\right), \qquad u, v_1, v_2, v_3 \in \mathbb{Q} \setminus \{0\}.
\end{aligned}$$

Due to Silverman's specialization theorem of abelian varieties, one knows that for all but a thin set of values of u, v_1, v_2, v_3 in $\mathbb{Q}$, the points $\left(u, \frac{v_3^{7592}(w_3^2-9)^{292}}{(u^5+u+1)^{292}}\right) - (\infty)$ on the Jacobian of E and the point $(P_i) - (\infty)$, $i = 1, 2, 3$, on the Jacobian of E_i, where

$$P_i = \left(\frac{v_3^{26M_i}(w_3^2 - 9)^{M_i}}{v_i^2(u^5 + u + 1)^{M_i}}, w_i\right), \ M_1 = 117, \ M_2 = 65, \ M_3 = 45,$$

are points of infinite order.

In Corollaries 3.2 and 4.2 of [6], the authors established the existence of pairs of hyperelliptic curves C_1 and C_2 together with a rational function D such that the Mordell–Weil rank of the Jacobian of a twist of C_1 by D is positive, whereas the Mordell–Weil rank of the Jacobian of a twist of C_2 by D is at least 2. In the following theorem, we present such examples of pairs of hyperelliptic curves where the Mordell–Weil rank of the Jacobian of a twist of C_2 by D is at least 3.

Theorem 2.4. *Let $f \in \mathbb{Q}[x]$ be a square-free polynomial of degree at least 3. Let $m \geq 3$ be an odd integer. Consider the hyperelliptic curves*

$$E : y^2 = f(x), \quad E' : y^2 = x^m + a^2, \quad a \in \mathbb{Q} \setminus \{0\}.$$

Then there exists a rational function $D_{2,m} \in \mathbb{Q}(u, v_1, v_2, v_3)$ such that the Jacobian of the quadratic twist of the curve E by $D_{2,m}$ is of positive Mordell–Weil rank and the Mordell–Weil rank of the Jacobian of the m-twist of the curve E' by $D_{2,m}$ over $\mathbb{Q}(u, v_1, v_2, v_3)$ is at least 3.

Proof. In Theorem 2.2, we set $a = b = c$ and $m := m_1 = m_2 = m_3$. Choosing $(p, q, r) = (w_1, w_2, w_3)$ as defined above, we obtain the three rational points $P_i = \left(\frac{1}{v_i^2}, w_i\right)$, $i = 1, 2, 3$, in $E'(\mathbb{Q})$ where E' is defined by $y^2 = Dx^m + a$.

We consider the automorphisms ϕ_i, $1 \le i \le 3$, of $K = \mathbb{Q}(u, v_1, v_2, v_3)$ defined by

$$\phi_i(u) = u, \quad \phi_i(v_i) = -v_i, \quad \phi_i(v_j) = v_j \text{ for } i \neq j.$$

Since $\phi_i(T) = T$ and $\phi_i(D) = D$, the automorphism ϕ_i induces a map Φ_i on E' and its Jacobian J. In particular,

$$\Phi_i(P_i) = P_i,\ i = 1, 2, 3, \quad \Phi_i(P_j) = \left(\frac{1}{v_j^2}, -w_j\right),\ i \neq j.$$

We define the K-rational divisors $D_i := (P_i) - (\infty)$, $i = 1, 2, 3$, on E'. In accordance with the proof of Theorem 2.2, the divisors D_i define rational points of infinite order in J. In addition, one has

$$\Phi_i(D_i) = D_i,\ i = 1, 2, 3, \qquad \Phi_i(D_j) = \left(\left(\frac{1}{v_j^2}, -w_j\right)\right) - (\infty) \sim -D_j,\ i \neq j.$$

We now assume that there are integers α, β, γ such that $\alpha D_1 + \beta D_2 + \gamma D_3 \sim 0$. Applying the automorphism Φ_1, one gets $\alpha D_1 - \beta D_2 - \gamma D_3 \sim 0$. Adding the two linear equivalence relations, one obtains $2\alpha D_1 \sim 0$, hence $\alpha = 0$. Similarly, one applies the automorphism Φ_2 and obtains $\beta = 0$. It follows that D_1, D_2, and D_3 are linearly independent. ☐

3. A quadratic twist and three twists of even degrees

Let $f(x) \in \mathbb{Q}[x]$ be a square-free polynomial of degree at least 3. Fix odd integers $m_1, m_2, m_3 \ge 3$. We are investigating the existence of non-square rational numbers D such that the Jacobians of the (hyperelliptic) curves

$$H : Dy^2 = f(x), \quad H_1 : y^2 = x^{m_1} + aD, \quad H_2 : y^2 = x^{m_2} + bD, \quad H_3 : y^2 = x^{m_3} + cD$$

are of positive Mordell–Weil rank over $\mathbb{Q}$.

Setting $M = \operatorname{lcm}(m_1, m_2, m_3)$ and $M_i = M/m_i$, $i = 1, 2, 3$. We find the rational number D by solving the following system of equations

$$\frac{y_1^2 - x_1^{m_1}}{a} = \frac{y_2^2 - x_2^{m_2}}{b} = \frac{y_3^2 - x_3^{m_3}}{c} = \frac{f(x_4)}{y_4^2}, \tag{3.1}$$

where the solutions x_i, y_i, $i = 1, 2, 3, 4$, satisfy $x_i y_i f(x_4)(y_1^2 - x_1^{m_1})(y_2^2 - x_2^{m_2})(y_3^2 - x_3^{m_3}) \neq 0$. To produce a family of parametric solutions to the system (3.1), we set

$$\begin{aligned} x_1 &= v_1^2 T^{M_1},\ x_2 = v_2^2 T^{M_2},\ x_3 = v_3^2 T^{M_3},\ x_4 = u, \\ y_1 &= pT^{(M-1)/2},\ y_2 = qT^{(M-1)/2},\ y_3 = rT^{(M-1)/2},\ y_4 = \frac{1}{T^{(M-1)/2}}, \end{aligned}$$

where u, v_1, v_2, v_3 are rational parameters. After simplification, one obtains

$$T = \frac{p^2 - af(u)}{v_1^{2m_1}} = \frac{q^2 - bf(u)}{v_2^{2m_2}} = \frac{r^2 - cf(u)}{v_3^{2m_3}}.$$

It follows that solving the system (3.1) is equivalent to finding rational points on the curve

$$v_2^{2m_2} v_3^{2m_3}(p^2 - af(u)) = v_1^{2m_1} v_3^{2m_3}(q^2 - bf(u)) = v_1^{2m_1} v_2^{2m_2}(r^2 - cf(u))$$

which is an intersection $\mathcal{H}_{a,b,c}$ of two quadratic surfaces in $\mathbb{P}^3$ over $\mathbb{Q}(v_1, v_2, v_3)$. In addition, the curve $\mathcal{H}_{a,b,c}$ possesses the four projective rational points $[p : q : r : s] = [\pm v_1^{m_1} : \pm v_2^{m_2} : \pm v_3^{m_3} : 0]$. Thus, $\mathcal{H}_{a,b,c}$ is an elliptic curve.

Theorem 3.1. *Let E be an elliptic curve defined by $y^2 = f(x)$ where $f \in \mathbb{Q}[x]$ is a polynomial of degree* 3 *or* 4*. Assume, moreover, that the Mordell–Weil rank of E is positive. Let $m_1, m_2, m_3 \geq 3$ be odd integers. Consider the hyperelliptic curves*

$$E_1 : y^2 = x^{m_1} + a^2, \qquad E_2 : y^2 = x^{m_2} + b^2, \qquad E_3 : y^2 = x^{m_3} + c^2,$$

where a, b, c are non-zero rational numbers. Then there exists a rational function $D_{2,2m_1,2m_2,2m_3} \in \mathbb{Q}(v_1, v_2, v_3)$ such that the quadratic twist of the curve E and the Jacobians of the $2m_i$-twists of the curves E_i, $i = 1, 2, 3$, by $D_{2,2m_1,2m_2,2m_3}$ have positive Mordell–Weil rank over $\mathbb{Q}(v_1, v_2, v_3)$.

Proof. Following the discussion above, one needs to find a rational point on the elliptic curve

$$\mathcal{H}_{a^2,b^2,c^2} : v_2^{2m_2} v_3^{2m_3}(p^2 - a^2 f(u)) = v_1^{2m_1} v_3^{2m_3}(q^2 - b^2 f(u)) = v_1^{2m_1} v_2^{2m_2}(r^2 - c^2 f(u)).$$

Since E is of positive Mordell–Weil rank, we choose u such that $(u, y_u) \in E(\mathbb{Q})$ is of infinite order. Now, the elliptic curve $\mathcal{H}_{a^2,b^2,c^2}$ is birational to

$$x^2 - T_1^2 = y^2 - T_2^2 = z^2 - T_3^2$$

where $T_1 = a v_2^{m_2} v_3^{m_3} y_u$, $T_2 = b v_1^{m_1} v_3^{m_3} y_u$, and $T_3 = c v_1^{m_1} v_2^{m_2} y_u$. According to Proposition 2.1, the latter curve is of positive Mordell–Weil rank over $\mathbb{Q}(T_1, T_2, T_3)$. Now, the proof is similar to the proof of Theorem 2.2. ☐

We remark that doubling the point of infinite order $[a y_u : b y_u : c y_u : 1] \in \mathcal{H}_{a^2,b^2,c^2}(\mathbb{Q}(v_1, v_2, v_3))$ yields the point $[w_1 : w_2 : w_3 : 1]$ given by

$$\left[\frac{a^2b^2v_3^{2m_3}y_u + a^2c^2v_2^{2m_2}y_u - b^2c^2v_1^{2m_1}y_u}{2abcv_2^{m_2}v_3^{m_3}} : \frac{a^2b^2v_3^{2m_3}y_u - a^2c^2v_2^{2m_2}y_u + b^2c^2v_1^{2m_1}y_u}{2abcv_1^{m_1}v_3^{m_3}}\right.$$
$$\left. : \frac{-a^2b^2v_3^{2m_3}y_u + a^2c^2v_2^{2m_2}y_u + b^2c^2v_1^{2m_1}y_u}{2abcv_1^{m_1}v_2^{m_2}} : 1\right] \in \mathcal{H}_{a^2,b^2,c^2}(\mathbb{Q}(v_1, v_2, v_3)).$$

One obtains the following result.

Theorem 3.2. *Let E be an elliptic curve defined by $y^2 = f(x)$ where $f \in \mathbb{Q}[x]$ is a polynomial of degree* 3 *or* 4*. Assume, moreover, that the Mordell–Weil rank of E is positive. Let $m \geq 3$ be an odd integer. Consider the hyperelliptic curves $E' : y^2 = x^m + a^2$, where a is a non-zero rational number. Then there exists a rational function $D_{2,2m} \in \mathbb{Q}(v_1, v_2, v_3)$ such that the quadratic twist of the curve E by $D_{2,2m}$ is of positive Mordell–Weil rank and the Mordell–Weil rank of the Jacobian of the $2m$-twist of the curve E' by $D_{2,2m}$ over $\mathbb{Q}(v_1, v_2, v_3)$ is at least* 3*.*

Proof. We set $m := m_1 = m_2 = m_3$ and $a = b = c$ in Theorem 3.1. The three $\mathbb{Q}(v_1, v_2, v_3)$-rational points $P_i := \left(v_i^2, w_i T^{(m-1)/2}\right)$, $i = 1, 2, 3$, on the curve $y^2 = x^m + aD$ are of infinite order.

We consider the automorphisms ϕ_i, $i = 1, 2, 3$, of $\mathbb{Q}(v_1, v_2, v_3)$ defined by

$$\phi_i(v_i) = -v_i, \quad \phi_i(v_j) = v_j \text{ for } i \neq j.$$

Now the proof follows as in the proof of Theorem 2.4. ☐

4. One even-degree twist and three odd-degree twists

Given the curves

$$y^2 = x^{m_1} + a, \quad y^2 = x^{m_2} + b, \quad y^2 = x^{m_3} + c, \quad y^2 = x^{m_4} + d,$$

where a, b, c, d are non-zero rationals and m_i, $i = 1, 2, 3, 4$, are odd integers, we investigate the existence of a rational function D such that the Jacobians of the following twists are of positive

Mordell–Weil rank:

$$y^2 = Dx^{m_1} + a, \quad y^2 = Dx^{m_2} + b, \quad y^2 = Dx^{m_3} + c, \quad y^2 = x^{m_4} + dD.$$

The rational D can be obtained by solving the system:

$$\frac{y_1^2 - a}{x_1^{m_1}} = \frac{y_2^2 - b}{x_2^{m_2}} = \frac{y_3^2 - c}{x_3^{m_3}} = \frac{y_4^2 - x_4^{m_4}}{d}$$

where $x_i y_i (y_1^2 - a)(y_2^2 - b)(y_3^2 - c)(y_4^2 - x_4^{m_4}) \neq 0$. Let $M = \operatorname{lcm}(m_1, m_2, m_3, m_4)$ with $M_i = M/m_i$, $i = 1, 2, 3, 4$. We set

$$x_1 = \frac{1}{v_1^2 T^{M_1}}, \quad x_2 = \frac{1}{v_2^2 T^{M_2}}, \quad x_3 = \frac{1}{v_3^2 T^{M_3}}, \quad x_4 = v_4^2 T^{m_4},$$
$$y_1 = p, \quad y_2 = q, \quad y_3 = r, \quad y_4 = uT^{(M-1)/2}$$

where u, v_1, v_2, v_3, v_4 are rational parameters. One now obtains that

$$(p^2 - a)v_1^{2m_1} T = (q^2 - b)v_2^{2m_2} T = (r^2 - c)v_3^{2m_3} T = \frac{u^2 - v_4^{2m_4} T}{d}.$$

In other words, one needs to solve the system:

$$T = \frac{u^2}{v_4^{2m_4} + d(r^2 - c)v_3^{2m_3}}, \quad v_1^{2m_1}(p^2 - a) = v_2^{2m_2}(q^2 - b) = v_3^{2m_3}(r^2 - c).$$

We now obtain the following result.

Theorem 4.1. *Let $m_1, m_2, m_3, m_4 \geq 3$ be odd integers. Consider the hyperelliptic curves*

$$E_1 : y^2 = x^{m_1} + a^2, \quad E_2 : y^2 = x^{m_2} + b^2, \quad E_3 : y^2 = x^{m_3} + c^2, \quad E_4 : y^2 = x^{m_4} + d$$

where a, b, c, d are non-zero rational numbers. Then there exists a rational function $D_{m_1,m_2,m_3,2m_4} \in \mathbb{Q}(u, v_1, v_2, v_3)$ such that the Jacobian of the m_i-twists of the curves E_i, $i = 1, 2, 3$ by $D_{m_1,m_2,m_3,2m_4}$ and the Jacobian of the $2m_4$-twist of E_4 by $D_{m_1,m_2,m_3,2m_4}$ have positive Mordell–Weil rank over $\mathbb{Q}(u, v_1, v_2, v_3)$.

Proof. Proposition 2.1 implies that the elliptic curve $v_1^{2m_1}(p^2 - a^2) = v_2^{2m_2}(q^2 - b^2) = v_3^{2m_3}(r^2 - c^2)$ has positive Mordell–Weil rank over $\mathbb{Q}(u, v_1, v_2, v_3)$. The result now follows as in the proofs of Theorems 2.2 and 3.1. □

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