A PLURIPOTENTIAL THEORY FOR BANACH LATTICE VALUED FUNCTIONS

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ABSTRACT

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In this thesis, we establish a foundation of a pluripotential theory for functions attaining values in a Banach lattice. For this purpose, we generalize notions such as semi-continuity, subharmonicity, plurisubharmonicity and maximality for vector valued functions, study their properties and the cones of plurisubharmonic functions. Moreover, we investigate operator valued Jensen measures for a given cone of vector valued functions, and upper, lower envelopes of a given function with respect to the given cone. With these at our disposal, we prove Edwards' Theorem in Banach lattice settings. As an result our findings, we provide a Perron method of solution for a Dirichlet Problem for vector valued harmonic/maximal plurisubharmonic functions with continuous boundary data.

ÖZET

BANACH LATİS DEĞERLİ FONKSİYONLAR İÇİN BİR ÇOKLU-POTANSİYEL TEORİSİ

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Anahtar Kelimeler: Vektör Değerli Fonksiyonlar, Jensen Ölçüleri, Edwards Teoremi, Çoklu Altharmonik Fonksiyonlar, Dirichlet Problemi

Bu tez çalışmasında, bir Banach latisinde değer alan fonksiyonlar için bir çoklu-potansiyel teorisinin temellerini kuracağız. Bu hedefle, vektör değerli fonksiyonlar için yarı-süreklilik, altharmoniklik, çoklu altharmoniklik ve maksimalite gibi kavramlarını genelleştireceğiz ve böyle fonksiyonların özelliklerini ile bu tür fonksiyonlardan ibaret konileri çalışacağız. Ayrıca, vektör değerli foksiyonların bir konisinin operatör değerli Jensen ölçülerini ve verili bir fonksiyon için üst ve alt zarf fonksiyonlarını inceleyeceğiz. Elde ettiklerimizle, Banach latisi koşullarında Edwards Teoremi'ni kanıtlayacağız. Neticesinde, sürekli sınır veri fonksiyonlu harmonik/ maksimal çoklu altharmonik vektör değerli fonksiyonlar için Dirichlet problemlerinin Perron çözüm yöntemini vereceğiz.

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"Always look on the bright side of life!"
Song by Monty Python

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1. INTRODUCTION

The primary purpose of this thesis is to establish a pluripotental theory for vector valued functions and present a Perron method for Dirichlet problems for vector valued maximal plurisubharmonic functions. Thus, the first task is to choose a suitable order and a topological structure, that are compatible with each other, on a given vector space. Throughout our search for such spaces, Dedekind complete Banach lattices and in particular ones with order continuous norm seemed as most fitting for our purposes.

With the well-known usual order on the class $S(\mathcal{H})$ of self-adjoint linear operators on a complex Hilbert space $\mathcal{H}$, defined by positive definiteness, $S(\mathcal{H})$ turns out to be a partially ordered vector space yet not a vector lattice. Sherman (1951) proved that a subalgebra $\mathcal{A}$ of a C^*-algebra of self-adjoint operators on a complex Hilbert space forms a lattice if and only if $\mathcal{A}$ is commutative, and as a special case, a subalgebra $\mathcal{A}$ of $S(\mathcal{H})$ is a lattice if and only if $\mathcal{A}$ is commutative. For that reason, we are in the need of commutativity condition on a subalgebra of $S(\mathcal{H})$ in order to have a lattice structure. Another order, so called the spectral order on $S(\mathcal{H})$ (indeed, on the class of self-adjoint operators on a Hilbert space) was defined by Olson (1971). It was proved in Olson (1971) that $S(\mathcal{H})$ becomes a conditionally complete lattice under the spectral order but it is not a vector lattice. It was proved again in Olson (1971) that if $\mathcal{Y}$ is a commutative subalgebra of $S(\mathcal{H})$, the usual order and the spectral order on $\mathcal{Y}$ are equivalent to each other, and furthermore $\mathcal{Y}$ with the usual order becomes a conditionally complete vector lattice. Therefore, neither of these partial orders will be very useful in noncommutative settings. In this thesis, we introduce the notion of Γ-order on Banach spaces with unconditional Schauder basis and noncommutative matrix spaces that is given via a specific map Γ and our notion of Γ-order overcomes this difficulty. We prove that this partial order makes the given space a conditionally complete vector lattice and give several examples of Γ-order.

In scalar case, Jensen measures attracted the attention of quite a number of mathematicians to explain different phenomena of pluripotential theory. Edwards' theorem provides a duality between a positive cone of upper semi-continuous functions on a compact space and the set of Jensen measures for this cone. This theorem found many applications in functional analysis, uniform algebras, pluripotential theory and optimization. Additionally, Perron solution of Dirichlet problem heavily relies on the investigation of upper envelopes. One of the purposes of this thesis is to prove an analogue of Edwards' Theorem for Banach lattice valued functions. Another novelty is the introduction of the notion of Jensen measures, which are operator valued measures in our settings. In order to achieve such result, we need a class $\mathcal{F}$ of vector-valued functions that is a cone of functions and also if $\{u_\alpha\}$ is a family of functions in $\mathcal{F}$, we want the supremum $\sup_\alpha u_\alpha$ to be a well-defined function.

Coifman & Semmes (1993) considered subharmonic norm-valued functions N_z on a finite dimensional complex Banach space V. In particular, they paid attention to the following the matrix-valued Dirichlet problem

$$\sum_{i=1}^{d} \bar{\partial}_i\Big(P^{-1}\partial_i P\Big) = 0, P = \omega \text{ on } \partial\Omega, \tag{1.1}$$

where ω is a positive definite $n \times n$ matrix-valued function from the class $C(\partial\Omega)$.

Lempert (2017) studied the form $R^P = \bar{\partial}(P^{-1}\partial P)$ on some open subset Ω of $\mathbb{C}$ where $P : \Omega \to EndV$ is a C^∞ map attaining values in positive invertible operators. Herein, V is a finite or an infinite dimensional separable Hilbert space, $EndV$ is the space of continuous linear maps on V to itself. Lempert (2017) provides further information about the solution of the mentioned Dirichlet problem.

On the other hand, Enflo & Smithies (2001) studied compact operator-valued functions and showed that harmonic compact operator-valued functions are characterized by having harmonic diagonal matrix coefficients in any choice of basis.

Arendt (2016) generalized the notion of harmonicity for functions with values in a real Banach space $\mathcal{E}$, characterized harmonicity for norm bounded functions and proved that the Dirichlet problem of finding a harmonic function $u : \Omega \to \mathcal{E}$ continuous on the closure of Ω which satisfies $u|_{\partial\Omega} = \varphi$, where $\varphi : \partial\Omega \to \mathcal{E}$ is continuous, has unique solution whenever the bounded open set Ω is Dirichlet regular. Arendt, Bernhard & Kreuter (2020) further obtained a necessary and sufficient condition of harmonicity for functions u with values in $\mathcal{E}$.

Kreuter (2020) demonstrated that a generalized Dirichlet problem on a bounded open set $\Omega \subset \mathbb{R}^n$ with a continuous boundary data function φ with values in a real Banach space $\mathcal{E}$ has a unique solution. Kreuter also studied the same generalized Dirichlet problem in Banach lattice settings with the aim of providing a Perron method of solution via envelopes of continuous sub/super solutions.

One of the main results of this thesis is the following theorem which is a generalization of Edwards' Theorem to the case of Banach lattice valued functions. In this context, we first introduce semi-continuity, and then we prove several approximation results for vector-valued functions in Chapter 3. At the beginning of Chapter 4, we define operator-valued Jensen measures and envelope functions of a vector valued function. For a given order continuous function $\varphi : K \to \mathcal{E}$ and a given nonempty set $\mathcal{F}$ of functions $u : K \to \mathcal{E}$ containing constant functions, we define the upper and lower envelopes of φ at a given point $z \in K$ as

$$S^{\mathcal{F}}\varphi(z) = \curlyvee\{u(z) : u \in \mathcal{T}[\varphi]\},\ \ I^{\mathcal{F}}\varphi(z) = \curlywedge\left\{\int_K \varphi d\mu : \mu \in \mathcal{J}_z^{\mathcal{F}}\right\}$$

respectively, where $\mathcal{T}[\varphi] = \{u \in \mathcal{F} : u \preceq \varphi \text{ on } K\}$ and $\mathcal{J}_z^{\mathcal{F}}$ is the class of Jensen measures for $\mathcal{F}$ with barycenter z.

Theorem 1.1 (Edwards' Theorem in Banach Lattices)**.** *Let (K,ρ) be a compact metric space, $(\mathcal{E}, \preceq, \|\cdot\|)$ a Banach lattice with order continuous norm. Let $\mathcal{F} \subset o$-$USC(K;\mathcal{E})$ be a cone possessing the constant function property of functions and $\widetilde{\mathcal{F}} \subset USC(K;\mathcal{E})$ the cone of functions obtained by taking the closure of $\mathcal{F}$ with respect to the topology of uniform convergence on K. For any given function $\varphi \in o$-$C(K;\mathcal{E})$,*

$$S^{\mathcal{F}}\varphi(z) = I^{\mathcal{F}}\varphi(z) = S^{\widetilde{\mathcal{F}}}\varphi(z) = I^{\widetilde{\mathcal{F}}}\varphi(z)$$

holds for any $z \in K$.

Chapter 5 starts with a short list of results due to Arendt (2016), Arendt et al. (2020) and Kreuter (2020) on harmonic functions and the Perron method for vector valued functions. In the second section of Chapter 5, we take an approach different than the one in Kreuter (2020) by defining order subharmonic functions and subharmonic functions as the uniform limits of sequences of order subharmonic functions on compact subsets of the domain of definition. With this deviation between our definitions, we present a Perron method for Dirichlet problems.

In Chapter 6, we lay foundations of a pluripotential theory for vector valued functions. We first define order plurisubharmonic, plurisubharmonic and maximal plurisubharmonic vector valued functions. In the theorems below, let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. For an open subset Ω of $\mathbb{K}^n, n \geq 1$, we denote by $\mathcal{F}$ the cone o-$SH(\Omega;\mathcal{E}) \cap o$-

$USC(\overline{\Omega};\mathcal{E})$ if $\mathbb{K}=\mathbb{R}$. If $\mathbb{K}=\mathbb{C}$, $\mathcal{F}$ denotes the cone $o\text{-}PSH(\Omega;\mathcal{E})\cap o\text{-}USC(\overline{\Omega};\mathcal{E})$. We denote by $\widetilde{\mathcal{F}}$ the cones obtained by taking the closure of $\mathcal{F}$ with respect to the uniform topology on $\overline{\Omega}$ when $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$, respectively.

Similar to the scalar case, we observe that vector valued subharmonic and plurisubharmonic functions have much properties in common. Thus, we will provide the proofs of these two theorems only in the case $\mathbb{K}=\mathbb{C}$ with the remark that the proofs of the theorems in the case $\mathbb{K}=\mathbb{R}$ are almost identical.

Theorem 1.2. *Let Ω be a bounded domain in $\mathbb{K}^n$, $(\mathcal{E},\preceq,\|\cdot\|)$ a Banach lattice with order continuous norm. For any given $\varphi\in C_{oba}(\overline{\Omega};\mathcal{E})$, the function $S^{\mathcal{F}}\varphi$ belongs to $\mathcal{T}[\varphi]=\{u\in\mathcal{F}:u\preceq\varphi \text{ on } \overline{\Omega}\}$. For any $\varphi\in o\text{-}C(\overline{\Omega};\mathcal{E})$, $S^{\mathcal{F}}\varphi=I^{\mathcal{F}}\varphi$ on $\overline{\Omega}$.*

The theorem below presents a method for generating a subharmonic or plurisubharmonic function of the cone via given boundary data function. For given $\varphi\in C_{oba}(S(\Omega);\mathcal{E})$, we consider the collection

$$\check{\mathcal{U}}[\varphi]=\{u\in\mathcal{F}:u\preceq\varphi \text{ on } S(\Omega)\},$$

the Perron envelope of φ is defined as

$$\check{\mathbf{P}}\varphi(z)=\curlyvee\{u(z):u\in\check{\mathcal{U}}[\varphi]\},z\in\Omega,$$

and its upper semi-continuous regularization of $\check{\mathbf{P}}\varphi$ on $\overline{\Omega}$ is defined as

$$(\check{\mathbf{P}}\varphi)^*(w)=\limsup_{\Omega\ni z\to w}\check{\mathbf{P}}\varphi(z),w\in\overline{\Omega}.$$

Herein, $S(\Omega)$ is the Shilov boundary of $\Omega\subset\mathbb{C}^n$.

Theorem 1.3 (Perron Envelope)**.** *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$ and $(\mathcal{E},\preceq,\|\cdot\|)$ a Banach lattice with order continuous norm. For every $\varphi\in C_{oba}(S(\Omega);\mathcal{E})$, $(\check{\mathbf{P}}\varphi)^*$ is well-defined, belongs to $\check{\mathcal{U}}[\varphi]$ and $(\check{\mathbf{P}}\varphi)^*=\check{\mathbf{P}}\varphi$ on Ω.*

While it is significant on its own, Theorem 1.3 has further importance in our theory such as providing Perron method of solution for Dirichlet problems for subharmonic /plurisubharmonic functions. Let $H_b(\Omega;\mathcal{E})$ be the space of norm bounded, harmonic functions if $\mathbb{K}=\mathbb{R}$ and $MPSH(\Omega;\mathcal{E})$ the class of maximal plurisubharmonic functions if $\mathbb{K}=\mathbb{C}$. Our result on the Perron solution particularly states that the solution of the given Dirichlet problem order converges to the given boundary data, hence our result differs from the results of Arendt (2016) and Kreuter (2020). Herein, $\partial_r\Omega$ is the set of all (Dirichlet) regular boundary points of $\Omega\subset\mathbb{R}^n$.

Theorem 1.4 (Perron Solution for Dirichlet Problem).

Let $(\mathcal{E},\preceq,\|\cdot\|)$ *be a Banach lattice with order continuous norm and* Ω *a bounded domain in* $\mathbb{K}^n$*.*

(i) Let $\Omega\subset\mathbb{R}^n$ *be a bounded domain. If* $\varphi\in C_{oba}(\partial\Omega;\mathcal{E})$*, the function*

$$\mathbf{H}\varphi(z)=\curlyvee\{u(z):u\in\mathcal{U}[\varphi]\},z\in\Omega,$$

is the unique solution of

$$\begin{cases} u\in H_b(\Omega;\mathcal{E}), \\ o\text{-}\lim_{\Omega\ni z\to\zeta}u(z)=\varphi(\zeta),\zeta\in\partial_r\Omega. \end{cases} \tag{1.2}$$

If $\varphi\in C(\partial\Omega;\mathcal{E})\setminus C_{oba}(\partial\Omega;\mathcal{E})$*, there exists a monotone increasing sequence uniformly converging* $\{\varphi_n\}\subset C_{oba}(\partial\Omega;\mathcal{E})$ *and* $\curlyvee_n\mathbf{H}\varphi_n$ *is the unique solution of (1.2).*
Moreover, the solution of the problem (1.2) coincides with the function

$$\begin{cases} \hat{\mathbf{H}}\varphi,\ \text{if } \varphi\in C_{oba}(\partial\Omega;\mathcal{E}), \\ \curlyvee_n\hat{\mathbf{H}}\tilde{\varphi}_n,\ \text{if } \varphi\in C(\partial\Omega;\mathcal{E})\setminus C_{oba}(\partial\Omega;\mathcal{E}), \end{cases}$$

where

$$\hat{\mathbf{H}}\varphi(z)=\curlyvee\{u(z):u\in o\text{-}SH(\Omega;\hat{\mathcal{E}})\cap o\text{-}USC(\overline{\Omega};\hat{\mathcal{E}}),u\preceq\varphi \text{ on } \partial\Omega\},z\in\Omega,$$

for $\varphi\in C_{oba}(\partial\Omega;\mathcal{E})$*, and* $\{\tilde{\varphi}_n\}\subset C_{oba}(\partial\Omega;\hat{\mathcal{E}})$ *is a monotone increasing sequence uniformly converging* φ *if* $\varphi\in C(\partial\Omega;\mathcal{E})\setminus C_{oba}(\partial\Omega;\mathcal{E})$ *and* $\hat{\mathcal{E}}$ *is the Banach sublattice of* $\mathcal{E}$ *generated by the range of* φ*.*

(ii) Let Ω *be a bounded hyperconvex domain in* $\mathbb{C}^n$*. For any* $\varphi\in C_{oba}(S(\Omega);\mathcal{E})$*,* $\check{\mathbf{P}}\varphi$ *is a solution of the problem*

$$\begin{cases} u\in MPSH(\Omega;\mathcal{E}), \\ u^*=\varphi \text{ on } S(\Omega). \end{cases} \tag{1.3}$$

2. PRELIMINARIES

In this chapter, we lay the groundwork necessary for a pluripotential theory of vector valued functions . For the definitions, results we mention in regards to vector lattices and functionals, we refer to Akilov & Kantorovich (1982), Aliprantis & Burkinshaw (2006), Fremlin (2002), Meyer-Nieberg (1991), Schaefer (1975) and Wnuk (1999).On Schauder bases for Banach spaces, our main references are James (1982) and Singer (1970). For the fundamentals of vector measures and integration theory, we refer the reader to the monographs Dinculeanu (1967) and Dinculeanu (2000). In this chapter, we also introduce a partial order relation called Γ-order on a vector space and provide examples for this order on Banach spaces with unconditional Schauder Bases and on non-commutative matrix spaces.

2.1 Topology, Vector Lattices and Γ-Order

Definition 2.1. *A real vector space $\mathcal{E}$ is said to be an ordered vector space if it is equipped with a partial order relation $'' \preceq ''$ possessing the following properties:*

(i) If $x \preceq y$, then $x+z \preceq y+z$, for every $x,y,z \in \mathcal{E}$,
(ii) If $x \preceq y$, then $\alpha x \preceq \alpha y$, for every $x,y \in \mathcal{E}$ and every $\alpha \in \mathbb{R}^+ = [0,+\infty)$.

A vector lattice $(\mathcal{E},\preceq)$ is an ordered vector space such that for any given vectors $x,y \in \mathcal{E}$, both the supremum and the infimum of the set $\{x,y\}$ exist in $\mathcal{E}$. If a set $A \subset \mathcal{E}$ has an infimum/ supremum in $\mathcal{E}$, we use the notations $\curlywedge A = \curlywedge\{a : a \in A\}$ and $\curlyvee A = \curlyvee\{a : a \in A\}$ for the infimum and the supremum of A, respectively. We may interchangeably use the notation $x \curlyvee y$ and $x \curlywedge y$ for $\curlywedge\{x,y\}$ and $\curlyvee\{x,y\}$, respectively.

For a given vector $x \in \mathcal{E}$ and a nonempty subset A of a vector lattice $\mathcal{E}$ such that $\curlyvee A$ exists in $\mathcal{E}$, one can show that the supremum of the set $x+A = \{x+a : a \in A\}$

exists in $\mathcal{E}$ *and* $\curlyvee(x+A) = x + \curlyvee A$*. If the infimum of the set* $-A = \{-a : a \in A\}$ *exists in* $\mathcal{E}$*,* $\curlywedge(-A) = -\curlyvee A$ *(cf. (Aliprantis & Burkinshaw, 2006, p.3,4)). The set of all positive elements in* $\mathcal{E}$*, i.e. those* $x \in \mathcal{E}$ *so that* $0 \preceq x$*, is denoted by* $\mathcal{E}^+$*.*

Lemma 2.1 (The Infinite Distributive Law)**.** *(Aliprantis & Burkinshaw, 2006, Theorem 1.8) If* A *is a nonempty subset of a vector lattice* $(\mathcal{E},\preceq)$ *with its supremum* $\curlyvee A$ *exists in* $\mathcal{E}$*, then for any given vector* x*, the supremum of the set* $x \curlywedge A = \{x \curlywedge a : a \in A\}$ *exists in* $\mathcal{E}$ *and* $\curlyvee(x \curlywedge A) = x \curlywedge (\curlyvee A)$*. Similarly, if* $\curlywedge A$ *exists, then* $\curlywedge(x \curlyvee A)$ *exists where* $x \curlyvee A = \{x \curlyvee a : a \in A\}$*, and* $\curlywedge(x \curlyvee A) = x \curlyvee (\curlywedge A)$*.*

Definition 2.2. *A vector lattice* $(\mathcal{E},\preceq)$ *is said to be conditionally complete if one (and as we have a vector space structure, both) of the conditions that "every nonempty subset with an upper bound in* $\mathcal{E}$ *has a supremum in the vector lattice", and "every nonempty subset with a lower bound in* $\mathcal{E}$ *has an infimum in the vector lattice" holds.*

Definition 2.3. *A nonempty subset* $\mathcal{W}$ *of a vector space is called a cone if the following conditions hold:*

(i) $x+y$ *is in* $\mathcal{W}$ *whenever* $x,y \in \mathcal{W}$ *,*
(ii) αx *is in* $\mathcal{W}$ *whenever* $x \in \mathcal{W}$ *and* $\alpha \in \mathbb{R}^+$*.*

It is elementary to show that $\mathcal{E}^+$ is a cone in a given vector lattice $(\mathcal{E},\preceq)$.

Definition 2.4. *Let* $(\mathcal{E},\preceq)$ *be a vector lattice. A net* $\{x_i\}_{i\in\mathcal{I}}$ *in* $\mathcal{E}$ *is said to be decreasing if for every* $i_1, i_2 \in \mathcal{I}$ *with* $i_1 \leq i_2$*, the relation* $x_{i_2} \leq x_{i_1}$ *holds and we write* $x_i\downarrow$*. The notation* $x_i \downarrow x$ *means that* $x_i\downarrow$*, the infimum of the set* $\{x_i : i \in \mathcal{I}\}$ *exists in* $\mathcal{E}$ *and* $x = \curlywedge\{x_i : i \in \mathcal{I}\}$*. We similarly define the relations* $x_i\uparrow$ *and* $x_i \uparrow x$*.*

Lemma 2.2. *(Aliprantis & Burkinshaw, 2006, p.14) A vector lattice* $(\mathcal{E},\preceq)$ *is conditionally complete if and only if for every net* $\{x_i\}_{i\in\mathcal{I}}$ *with* $0 \preceq x_i \uparrow\preceq x$ *for some* $x \in \mathcal{E}$*,* $\curlyvee\{x_i : i \in \mathcal{I}\}$ *exists in* $\mathcal{E}$*.*

Definition 2.5. *Let* $(\mathcal{E},\preceq)$ *be a vector lattice .*

(i) For $x \in \mathcal{E}$*, we define* $x^+ = \curlyvee\{x,0\}$ $x^- = \curlyvee\{-x,0\}$ *and* $|x| = \curlyvee\{x,-x\}$ *that are called respectively the positive part, the negative part and the modulus (or absolute value) of* x*. By their definitions,* x^+, x^- *and* $|x|$ *are positive elements of* $\mathcal{E}$*;*

(ii) A subset A *of* $\mathcal{E}$ *is said to be solid whenever* $|x| \preceq |y|$ *and* $y \in A$ *imply* $x \in A$*. A solid vector subspace of* $\mathcal{E}$ *is called an ideal of* $\mathcal{E}$*;*

(iii) A subset A *of* $\mathcal{E}$ *is said to be downwards directed if for every* $x,y \in A$*, there is a* $z \in A$ *so that* $z \preceq x,y$*. A subset* A *of* $\mathcal{E}$ *is said to be upwards directed if for every* $x,y \in A$*, there is a* $z \in A$ *so that* $x,y \preceq z$*;*

(iv) Let $(\mathcal{E},\preceq)$ *be a normed space with the norm* $\|\cdot\|$*. If* $\|x\| \leq \|y\|$ *whenever* $0 \preceq |x| \preceq |y|, x,y \in \mathcal{E}$*, then* $(\mathcal{E},\preceq,\|\cdot\|)$ *is called a normed Riesz space and the*

norm $\|\cdot\|$ is called a lattice norm;

(v) If a normed Riesz space $(\mathcal{E}, \preceq, \|\cdot\|)$ is complete with respect to the norm $\|\cdot\|$, then it is called a Banach lattice.

(vi) Let $(\mathcal{E}, \preceq, \|\cdot\|)$ be a normed Riesz space. The norm $\|\cdot\|$ is said to be order continuous if for every downwards directed set A in $\mathcal{E}$ with $\curlywedge A = 0$, we have $\inf_{x \in A} \|x\| = 0$.

(vii) For a vector lattice $(\mathcal{E}, \preceq)$ and a non-empty subset $\mathcal{A}$ of $\mathcal{E}$, denote

$$\mathcal{A}^{\curlyvee} = \{x \in \mathcal{E} : x = \curlyvee_{i=1}^{n} x_i, x_1, x_2, \dots, x_n \in \mathcal{A}\},$$
$$\mathcal{A}^{\curlywedge} = \{x \in \mathcal{E} : x = \curlywedge_{i=1}^{n} x_i, x_1, x_2, \dots, x_n \in \mathcal{A}\}.$$

If $(\mathcal{E}, \preceq, \|\cdot\|)$ is a normed Riesz space, we denote by $\overline{\mathcal{A}^{\curlyvee\curlywedge}}$ the norm closure of $\mathcal{A}^{\curlyvee\curlywedge}$.

Let $(\mathcal{E}, \preceq, \|\cdot\|)$ be a normed Riesz space. If $A \subset \mathcal{E}$ is an order bounded set , that is $|x| \preceq y$ for some $y \in \mathcal{E}^+$ and every $x \in A$, then $\|x\| \leq \|y\|$ for every $x \in A$. In other words, an order bounded set in a normed Riesz space is norm bounded.

Definition 2.6. *Let $(\mathcal{E}, \|\cdot\|)$ be a normed space, (M, ρ) be a metric space. For a given function $u : M \to \mathcal{E}$, we will denote by*

$$\|u\|_{\mathcal{E},M} = \sup_{z \in M} \|u(z)\|.$$

Lemma 2.3. *(Aliprantis & Burkinshaw, 2006, p. 204-205, Exercises 8-9)*

(i) If $\mathcal{A}$ is a vector subspace of a vector lattice $(\mathcal{E}, \preceq)$, then $\mathcal{A}^{\curlyvee\curlywedge} = A^{\curlywedge\curlyvee}$ is the vector sublattice generated by $\mathcal{A}$, which means that $\mathcal{A}^{\curlyvee\curlywedge}$ is the smallest vector sublattice containing $\mathcal{A}$, and $\mathcal{A}^{\curlyvee\curlywedge} = \mathcal{A}^{\curlyvee} - \mathcal{A}^{\curlyvee} = \mathcal{A}^{\curlywedge} - \mathcal{A}^{\curlywedge}$.

(ii) If $\mathcal{A}$ is a separable vector subspace of a Banach lattice $(\mathcal{E}, \preceq, \|\cdot\|)$, then $\overline{\mathcal{A}^{\curlyvee\curlywedge}}$ is a separable Banach sublattice of $\mathcal{E}$. In other words, every separable vector subspace of a Banach lattice is contained in a separable Banach sublattice.

Lemma 2.4. *(Aliprantis & Burkinshaw, 2006, Theorem 1.9) For arbitrary x, y and z in a vector lattice $(\mathcal{E}, \preceq)$, we have the following inequalities;*

$$\Big||x| - |y|\Big| \preceq |x+y| \preceq |x| + |y| \text{ (The Triangle Inequality)},$$
$$|x \curlyvee z - y \curlyvee z| \preceq |x-y| \text{ and } |x \curlywedge z - y \curlywedge z| \preceq |x-y| \text{ (Birkhoff's Inequalities)}.$$

Lemma 2.5. *(Schaefer, 1975, Ch. 2, Proposition 1.4) Let $(\mathcal{E}, \preceq)$ be a vector lattice. For every $x, y, z, w \in \mathcal{E}$,*

$$|x \curlyvee y - z \curlyvee w| \preceq |x-z| + |y-w|$$
$$|x \curlywedge y - z \curlywedge w| \preceq |x-z| + |y-w|.$$

Definition 2.7. *Let* $(\mathcal{E},\preceq),(\hat{\mathcal{E}},\lesssim)$ *be two vector lattices. Let us denote by* $\curlyvee,\curlywedge$ *the supremum and infimum operations in both of* $\mathcal{E}$ *and* $\hat{\mathcal{E}}$. *A linear mapping* $T:\mathcal{E}\to\hat{\mathcal{E}}$ *is said to be a lattice homomorphism if it preserves lattice operations, that is* $T(x\curlyvee y)=T(x)\curlyvee T(y)$ *(and hence* $T(x\curlywedge y)=T(x)\curlywedge T(y)$*) for all* $x,y\in\mathcal{E}$. *If* T *is an isomorphism, then it is called a lattice isomorphism. In addition, whenever* $(\mathcal{E},\preceq,\|\cdot\|)$ *and* $(\hat{\mathcal{E}},\lesssim,\|\cdot\|_{\hat{\mathcal{E}}})$ *are normed Riesz spaces, and* $\|T(x)\|_{\hat{\mathcal{E}}}=\|x\|$ *for every* $x\in\mathcal{E}$ *(norm preserving),* T *is called a lattice isometry.*

If $(\mathcal{E},\|\cdot\|)$ is a normed space, we denote by $\mathcal{E}'$ the space of all continuous linear functionals on $\mathcal{E}$.

Definition 2.8. *Let* $(\mathcal{E},\preceq)$ *and* $(\hat{\mathcal{E}},\lesssim)$ *be two vector lattices. A linear map* $S:\mathcal{E}\to\hat{\mathcal{E}}$ *is said to be positive if* $S(\mathcal{E}^+)\subset\hat{\mathcal{E}}^+$.

Lemma 2.6. *(Aliprantis & Burkinshaw, 2006, Theorem 4.3) Let* $(\mathcal{E},\preceq,\|\cdot\|)$ *be a Banach lattice and* $(\hat{\mathcal{E}},\lesssim,\|\cdot\|_{\hat{\mathcal{E}}})$ *a normed Riezs space. Then, any positive linear map* $S:\mathcal{E}\to\hat{\mathcal{E}}$ *is continuous.*

Lemma 2.7 (Nakano)**.** *(Aliprantis & Burkinshaw, 2006, Theorem 4.9) For a Banach lattice* $(\mathcal{E},\preceq,\|\cdot\|)$*, the following are equivalent;*

(i) The norm $\|\cdot\|$ *on* $\mathcal{E}$ *is order continuous.*
(ii) For given $0\preceq x_n\uparrow\preceq x$ *in* $\mathcal{E}$*, the sequence* $\{x_n\}$ *is Cauchy in the norm* $\|\cdot\|$.
(iii) $\mathcal{E}$ *is an ideal of* $\mathcal{E}''$ *(in the sense that the canonical map* $J:\mathcal{E}\to\mathcal{E}''$ *maps* $\mathcal{E}$ *onto an ideal of* $\mathcal{E}''$*).*

As a consequence of Lemma 2.7, any reflexive Banach lattice has order continuous norm.

Lemma 2.8. *(Aliprantis & Burkinshaw, 2006, Corollary 4.10) A Banach lattice* $(\mathcal{E},\preceq,\|\cdot\|)$ *with order continuous norm is conditionally complete.*

Definition 2.9. *A net* $\{x_i\}_{i\in\mathcal{I}}$ *in a vector lattice* $(\mathcal{E},\preceq)$ *is said to be order convergent to a vector* $x\in\mathcal{E}$ *if there exists a net* $\{y_i\}_{i\in\mathcal{I}}$ *(with the same index set* $\mathcal{I}$*) such that* $y_i\downarrow 0$ *and* $|x_i-x|\preceq y_i$ *for all indices* $i\in\mathcal{I}$. *In such case, we write* $x_i\xrightarrow{o}x$ *or* $o\text{-}\lim_i x_i=x$.

Lemma 2.9. *(Meyer-Nieberg, 1991, Propositon 1.1.10) Let* $(\mathcal{E},\preceq)$ *be a conditionally complete vector lattice. An order bounded net* $\{x_i\}_{i\in\mathcal{I}}$ *in* $\mathcal{E}$ *is order convergent to* $x\in\mathcal{E}$ *if and only if*

$$x=\curlyvee_{i\in\mathcal{I}}\{\curlywedge\{x_j:j\geq i\}\}=\curlywedge_{i\in\mathcal{I}}\{\curlyvee\{x_j:j\geq i\}\}.$$

Lemma 2.10 (Amemiya)**.** *(Akilov & Kantorovich, 1982, Theorem 10.3.2, p. 290)*

For a normed Riesz space $(\mathcal{E},\preceq,\|\cdot\|)$*, the following statements are equivalent:*

(i) $\mathcal{E}$ *is complete with respect to the norm* $\|\cdot\|$*;*

(ii) *For every Cauchy sequence* $\{x_n\}$ *in* $(\mathcal{E},\preceq,\|\cdot\|)$ *satisfying* $0\preceq x_n\uparrow$, $x=\curlyvee\{x_n : n\in\mathbb{N}\}$ *exists in* $\mathcal{E}$*;*

(iii) *For every Cauchy sequence* $\{x_n\}$ *in* $(\mathcal{E},\preceq,\|\cdot\|)$ *satisfying* $0\preceq x_n\uparrow$, x_n *converges in norm to some* $x\in\mathcal{E}$.

Definition 2.10. *Let* $\mathcal{E}$ *and* $\mathcal{Y}$ *be two real vector spaces with the following properties:*

(i) $\mathcal{Y}$ *equipped with a partial order relation* $\leq$ *is a conditionally complete vector lattice. Let us denote by* $\vee$ *and* $\wedge$ *the supremum and infimum operations on* $\mathcal{Y}$.

(ii) *There exists a bijective, real linear, continuous map* $\Gamma:\mathcal{E}\to\mathcal{Y}$.

Let us consider the relation $\preceq$ *on* $\mathcal{E}$ *given by*

$$x\preceq y \Longleftrightarrow \Gamma(x)\leq\Gamma(y),\ x,y\in\mathcal{E}.$$

We call the relation $\preceq$ *on* $\mathcal{E}$ *the* Γ*-order.*

$\mathcal{E}$ inherits a similar structure as $\mathcal{Y}$ and we prove this first.

Theorem 2.1. $(\mathcal{E},\preceq)$ *is a conditionally complete vector lattice.*

Proof. Clearly, $\preceq$ is reflexive and transitive. It is symmetric since Γ is injective. Hence, $\preceq$ is a partial order on $\mathcal{E}$. For a, b, $c\in\mathcal{E}$ and a scalar $\lambda\geq 0$ with $a\preceq b$, we have $\Gamma(a+c)=\Gamma(a)+\Gamma(c)\leq\Gamma(b)+\Gamma(c)=\Gamma(b+c)$, that is, $a+c\preceq b+c$. Likewise, $\Gamma(\lambda a)=\lambda\Gamma(a)\leq\lambda\Gamma(b)=\Gamma(\lambda b)$, that is, $\lambda a\preceq\lambda b$. Let $\{x_\alpha\}$ be a nonempty family in $\mathcal{E}$ and $b\in\mathcal{E}$ so that $x_\alpha\preceq b$ for each α. Set $A=\vee_\alpha\Gamma(x_\alpha)$ and $a=\Gamma^{-1}(A)$. Then $A=\Gamma(a)\leq\Gamma(b)$, that is, $a\preceq b$. By definition, a is an upper bound for $\{x_\alpha\}$. Hence, $a=\curlyvee\{x_\alpha\}\in\mathcal{E}$. Similarly, one can show that the infimum of a nonempty family order bounded below in $\mathcal{E}$ exists in $\mathcal{E}$. □

Definition 2.11. *Let* $\mathcal{E}$ *be a real vector space and* $\mathcal{Y}$ *an ordered vector space with a partial order relation* $\leq$. *A function* $p:\mathcal{E}\to\mathcal{Y}$ *is said to be sublinear if*

(i) $p(x_1+x_2)\leq p(x_1)+p(x_2)$ *for all* $x_1,x_2\in\mathcal{E}$,

(ii) $p(\alpha x)=\alpha p(x)$ *for any* $x\in\mathcal{E}$ *and any* $\alpha\in\mathbb{R}^+$.

A function $p:\mathcal{E}\to\mathcal{Y}$ *is called superlinear if* $-p$ *is sublinear.*

We remind that $\mathbb{R}$ with its usual order $\leq$ is a conditionally complete vector lattice.

Proposition 2.1 (Hahn-Banach Theorem)**.** *(Aliprantis & Burkinshaw, 2006, Theorem 1.25) Let* $\mathcal{E}$ *be a real vector space,* $(\tilde{\mathcal{E}},\preceq)$ *a conditionally complete vector lattice and* $p:\mathcal{E}\to\tilde{\mathcal{E}}$ *a sublinear function. If* $\mathcal{Z}$ *is a vector subspace of* $\mathcal{E}$ *and* $S:\mathcal{Z}\to\tilde{\mathcal{E}}$ *is a linear map such that* $S(x)\preceq p(x)$ *for all* $x\in\mathcal{Z}$, *then there exists a linear map* $\tilde{S}:\mathcal{E}\to\tilde{\mathcal{E}}$ *such that:*

(i) $\tilde{S} = S$ *on* $\mathcal{Z}$.

(ii) $\tilde{S}(x) \preceq p(x)$ *for every* $x \in \mathcal{E}$.

2.2 Linear Functionals

Let $\mathcal{E}$ be a vector space and $\mathcal{W}$ a set of linear functionals on $\mathcal{E}$. We say that $\mathcal{W}$ separates points of $\mathcal{E}$ if for given nonzero vector $x \in \mathcal{E}$, there exists a $f \in \mathcal{W}$ such that $f(x) \neq 0$. Equivalently, $\mathcal{W}$ separates points of $\mathcal{E}$ if for given distinct $x, y \in \mathcal{E}$, there exists $f \in \mathcal{W}$ so that $f(x) \neq f(y)$.

Definition 2.12. *Let* $(\mathcal{E}, \preceq)$ *be a vector lattice.*

(i) A linear functional $f : \mathcal{E} \to \mathbb{R}$ *is said to be positive if* $f(x) \geq 0$ *holds for every* $x \in \mathcal{E}^+$. *If for every* $x \in \mathcal{E}^+, x \neq 0$, *we have* $f(x) > 0$, *then we say that* f *is strictly positive. In case* $(\mathcal{E}, \preceq, \|\cdot\|)$ *is a normed Riesz space, we denote the set of all positive continuous linear functionals on* $\mathcal{E}$ *by* $\mathcal{E}'_+$.

(ii) A linear functional $f : \mathcal{E} \to \mathbb{R}$ *is said to be regular (or order bounded) if* f *maps order bounded sets in* $\mathcal{E}$ *to bounded sets in* $\mathbb{R}$. *We denote by* $\mathcal{E}^\sim$ *the vector space of all regular linear functionals on* $\mathcal{E}$, *and* $\mathcal{E}^\sim$ *is called the order dual of* $\mathcal{E}$.

(iii) A linear functional $f : \mathcal{E} \to \mathbb{R}$ *is said to be order continuous if* $x_i \xrightarrow{o} 0$ *in* $\mathcal{E}$ *implies* $f(x_i) \to 0$ *in* $\mathbb{R}$. *We denote by* $\mathcal{E}_n^\sim$ *the vector space of all order continuous functionals on* $\mathcal{E}$, *and* $\mathcal{E}_n^\sim$ *is called the order continuous dual of* $\mathcal{E}$.

Lemma 2.11. *Let* $(\mathcal{E}, \preceq)$ *be a vector lattice. Then,*

(i) (Aliprantis & Burkinshaw, 2006, p.58) $\mathcal{E}^\sim$ *is the vector space generated by the positive linear functionals on* $\mathcal{E}$. *That is, a functional* $f \in \mathcal{E}^\sim$ *can be written as the difference of two positive linear functionals on* $\mathcal{E}$. $\mathcal{E}^\sim$ *is a conditionally complete vector lattice with the relation*

$$f \geq g,\ f, g \in \mathcal{E}^\sim \iff f(x) \geq g(x) \text{ for every } x \in \mathcal{E}^+. \tag{2.1}$$

We define the positive and negative parts f^+, f^- *as well as the modulus* $|f|$ *of a functional* $f \in \mathcal{E}^\sim$ *by*

$$f^+(\cdot) = \sup\{f(\cdot), 0\}, \quad f^-(\cdot) = \sup\{-f(\cdot), 0\},$$
$$|f|(\cdot) = \sup\{f^+(\cdot), f^-(\cdot)\}.$$

(ii) (Aliprantis & Burkinshaw, 2006, Lemma 1.54) $\mathcal{E}_n^\sim \subset \mathcal{E}^\sim$.

We note that $(\mathbb{R}, \leq, |\cdot|)$ is a conditionally complete Banach lattice. Then, we have the following corollary of (Aliprantis & Burkinshaw, 2006, Theorem 1.56).

Lemma 2.12. *Let $f \in \mathcal{E}^{\sim}$. The following are equivalent;*

(i) *f is order continuous;*
(ii) *If $x_i \downarrow 0$ holds in $\mathcal{E}$, then $f(x_i) \to 0$ holds in $\mathbb{R}$.*
(iii) *If $x_i \downarrow 0$ holds in $\mathcal{E}$, then $\inf_i |f(x_i)| = 0$ in $\mathbb{R}$.*
(iv) *f^+ and f^- are order continuous;*
(v) *$|f|$ is order continuous.*

Hence, a functional $x' \in \mathcal{E}'_+$ on a normed Riesz space $(\mathcal{E}, \preceq, \|\cdot\|)$ is order continuous if for every upwards directed set $A \subset \mathcal{E}$ with supremum $\curlyvee A$ in $\mathcal{E}$, we have $x'(\curlyvee A) = \sup_{x\in A} x'(x)$. Moreover, $\mathcal{E}'$ is an ideal of $\mathcal{E}^{\sim}$ and thus $\mathcal{E}'$ equipped with the order relation (2.1) is a conditionally complete Banach lattice (cf. (Aliprantis & Burkinshaw, 2006, Theorem 3.49, Theorem 4.1)). The following is a corollary to (Schaefer, 1975, Theorem 5.14, p.94);

Lemma 2.13. *Every separable conditionally complete Banach lattice has order continuous norm.*

Lemma 2.14. *(Fremlin, 2002, Proposition 356D) A normed Riesz space $(\mathcal{E}, \preceq, \|\cdot\|)$ has order continuous norm if and only if $\mathcal{E}' \subset \mathcal{E}^{\sim}_n$. If $(\mathcal{E}, \preceq, \|\cdot\|)$ is a Banach lattice as well, then $\mathcal{E}' = \mathcal{E}^{\sim} = \mathcal{E}^{\sim}_n$.*

As a consequence, we have the following useful result:

Corollary 2.1. *If $(\mathcal{E}, \preceq, \|\cdot\|)$ is a Banach lattice with order continuous norm, then every $x' \in \mathcal{E}'_+$ is order continuous.*

Lemma 2.15. *Let $(\mathcal{E}, \preceq,)$ be a vector lattice.*

(i) *(Aliprantis & Burkinshaw, 2006, Theorem 1.66) A vector $x \in \mathcal{E}$ belongs to the positive cone $\mathcal{E}^+$ if and only if $f(x) \geq 0$ for all positive functionals $f \in \mathcal{E}^{\sim}$.*
(ii) *(Aliprantis & Burkinshaw, 2006, Corollary 4.5) If $(\mathcal{E}, \preceq, \|\cdot\|)$ is a Banach lattice, then $\mathcal{E}'$ of $\mathcal{E}$ coincides with $\mathcal{E}^{\sim}$. Then, a vector $x \in \mathcal{E}^+$ if and only if $x'(x) \geq 0$ for every $x' \in \mathcal{E}'_+$.*

Remark 2.1. *If $\{a_i\}$ is a net in a Banach lattice $(\mathcal{E}, \preceq, \|\cdot\|)$ with order continuous norm so that $\lim_i \|a_i - a\| \to 0$ and $x \preceq a_i \preceq y$ for some $x, y \in \mathcal{E}$ implies that $x \preceq a \preceq y$. Then for any $x' \in \mathcal{E}'_+$, we have $\lim_i x'(a_i) = x'(a)$ and so $\limsup_i x'(a_i) = \liminf_i x'(a_i) = x'(a)$. As x' is arbitrary and $\{a_i\}$ is order bounded, $o\text{-}\lim_i a_i = a$.*

2.3 Vector Measures and Integration

Let $(\mathcal{E}, \|\cdot\|)$ be a Banach space, (M,ρ) a metric space and Σ the family of all Borel subsets of M. We denote by $B(\mathcal{E})$ the space of all continuous linear operators $T:\mathcal{E}\to\mathcal{E}$ and by $\|\cdot\|_{B(\mathcal{E})}$ the operator norm on $B(\mathcal{E})$.

Definition 2.13. *Let $\mu:\Sigma\to B(\mathcal{E})$ be a set function satisfying $\mu(\varnothing)=0$.*

(i) *$\mu:\Sigma\to B(\mathcal{E})$ is said to be finitely additive if for every finite $\{A_j : j=1,2,\dots,N\}$ of pairwise disjoint sets from Σ so that $A=\cup_j^N A_j$,*

$$\mu(A)=\sum_{j=1}^{N}\mu(A_j)$$

holds.

(ii) *$\mu:\Sigma\to B(\mathcal{E})$ is said to be countably additive if for every sequence $\{A_j\}_{j\in\mathbb{N}}$ of pairwise disjoint sets from Σ so that $A=\cup_j A_j$,*

$$\mu(A)=\sum_{j=1}^{\infty}\mu(A_j)$$

holds. A countably additive set function $\mu:\Sigma\to B(\mathcal{E})$ is called a (operator valued) measure.

(iii) *For a given subset A of M and a set function $\mu:\Sigma\to B(\mathcal{E})$, we define*

$$\tilde{\mu}(A)=\sup_{\mathcal{I}}\sum_{i\in\mathcal{I}}\|\mu(A_i)x_i\|$$

where the supremum is taken over all finite families $\{A_i\}_{i\in\mathcal{I}}$ of pairwise disjoint sets from the collection Σ so that $A_i\subset A$ and all finite family $\{x_i\}\subset\mathcal{E}$ of norm $\|x_i\|\le 1$ for each i. The set function $\tilde{\mu}(A)$ is called the semi-variation of μ on A. We say that μ has finite semi-variation if $\tilde{\mu}(A)$ is finite for every $A\in\Sigma$. We denote by $\widetilde{\mathcal{M}}(M;B(\mathcal{E}))$ the real vector space of all $B(\mathcal{E})$-valued finitely additive set functions of finite semi-variation defined on Σ.

(iv) *Similarly, we define*

$$|\mu|(A)=\sup_{\mathcal{I}}\sum_{i\in\mathcal{I}}\|\mu(A_i)\|_{B(\mathcal{E})}$$

where the supremum is taken over all finite families $\{A_i\}_{i\in\mathcal{I}}$ of pairwise disjoint sets from the collection Σ so that $\cup_i A_i=A$. The set function $|\mu|(A)$ is called the variation of μ on A, and if μ is countably additive, its variation $|\mu|$ is a positive (scalar valued) Borel measure on M. We say that μ has finite variation *if $|\mu|(M)$ is finite. We denote by $\mathcal{M}(M;B(\mathcal{E}))$ the real vector space of all $B(\mathcal{E})$-valued measures of finite variation defined on Σ.*

Definition 2.14. *Let $\mu \in \widetilde{\mathcal{M}}(M;B(\mathcal{E}))$. A set $A \in \Sigma$ is said to be μ-negligible if $\mu(B) = 0$ for every set $B \in \Sigma$ so that $B \subset A$. Moreover, a set $B \subset M$ is said to be μ-negligible if there exists a μ-negligible set $A \in \Sigma$ such that $B \subset A$. A function $f : M \to \mathcal{E}$ is said to be μ-measurable if there exists a sequence of simple functions $\{s_n\}$ converging to f pointwise on M except on a μ-negligible set.*

As a consequence of the definition, a function $f : M \to \mathcal{E}$ is μ-measurable function if there exists a μ-neglibigle set $A \subset M$ so that $f|_{M\setminus A}$ is a Borel function and $f(M \setminus A)$ is separable (cf. (Dinculeanu, 2000, Theorem 1.1.8, p. 5) and (Dinculeanu, 2000, p. 77,78)).

Definition 2.15. *A function of the form*

$$s(z) = \sum_{i\in\mathcal{I}} \chi_{A_i}(z)x_i,\ z \in M,$$

is called a simple function where the index set $\mathcal{I}$ is finite, $A_i \in \Sigma$ are pairwise disjoint subsets of M, χ_{A_i} is the characteristic function of A_i and $x_i \in \mathcal{E}$ for every $i \in \mathcal{I}$. We will call the integral of s with respect to a $\mu \in \widetilde{\mathcal{M}}(M;B(\mathcal{E}))$ over a set $A \in \Sigma$, denoted by $\int_A s d\mu$, given by

$$\int_A s d\mu = \sum_{i\in\mathcal{I}} \mu(A_i \cap A)x_i.$$

Definition 2.16. *A function $f : M \to \mathcal{E}$ is said to be totally measurable if there exists a sequence $\{s_n\}$ of simple functions such that s_n uniformly converges to f on M. We denote by $\mathcal{TM}(M;\mathcal{E})$ the real vector space of totally measurable functions $f : M \to \mathcal{E}$.*

$\mathcal{TM}(M;\mathcal{E})$ endowed with the supremum norm $\|\cdot\|_{\mathcal{E},M}$ is a Banach space. If (K,ρ) is a compact metric space, then $C(K;\mathcal{E})$ is a subspace of $\mathcal{TM}(K;\mathcal{E})$.

Suppose that $(\mathcal{E}, \preceq, \|\cdot\|)$ is a Banach lattice. Consider $\mathcal{TM}(K;\mathcal{E})$ with the order relation

$$f \lesssim g \Leftrightarrow f(z) \preceq g(z), \text{ for every } z \in K, \quad f,g \in \mathcal{TM}(K;\mathcal{E}).$$

For given $f,g \in \mathcal{TM}(K;\mathcal{E})$, there exist sequences $\{s_n\}$ and $\{t_n\}$ of simple functions such that $\|s_n - f\|_{\mathcal{E},K} \to 0$ and $\|t_n - g\|_{\mathcal{E},K} \to 0$ as $n \to \infty$. For fixed $n \in \mathbb{N}$, $s_n = \sum_i \chi_{A_i^{(n)}} x_i^{(n)}$ and $t_n = \sum_i \chi_{B_i^{(n)}} y_i^{(n)}$ where $A_i^{(n)}, B_i^{(n)} \in \Sigma$, $A_i^{(n)} \cap A_j^{(n)} = \varnothing$, $B_i^{(n)} \cap B_j^{(n)} = \varnothing$ for $i \neq j$ and $x_i^{(n)}, y_i^{(n)} \in \mathcal{E}$. The functions $s_n \curlyvee t_n$ and $s_n \curlywedge t_n$ are simple functions, and by Lemma 2.5, $\|s_n \curlyvee t_n - f \curlyvee g\|_{\mathcal{E},K} \to 0$ and $\|s_n \curlywedge t_n - f \curlywedge g\|_{\mathcal{E},K} \to 0$ as $n \to \infty$. Hence, $\mathcal{TM}(K;\mathcal{E})$ equipped with the order relation $\lesssim$ and the supremum norm $\|\cdot\|_{\mathcal{E},K}$ is a Banach lattice.

Definition 2.17. *For a function $f \in \mathcal{TM}(M;B(\mathcal{E}))$, we define its integral over a set $A \in \Sigma$ with respect to a given $\mu \in \widetilde{\mathcal{M}}(M;B(\mathcal{E}))$ as the limit*

$$\int_A f d\mu = \lim_{n\to\infty} \int_A s_n d\mu,$$

where $\{s_n\}$ is a sequence of simple functions converging uniformly f on M.

For $A \in \Sigma$, a function $f \in \mathcal{TM}(M;\mathcal{E})$ vanishing outside A and $\mu \in \widetilde{\mathcal{M}}(M;B(\mathcal{E}))$,

$$\left\| \int_A f d\mu \right\| \leq \|f\|_{\mathcal{E},M} \tilde{\mu}(A) \leq \|f\|_{\mathcal{E},M} \tilde{\mu}(M).$$

Lemma 2.16. *(Dinculeanu, 1967, Theorem 1.9.1, p.145) Let $S : \mathcal{TM}(M;\mathcal{E}) \to \mathcal{E}$ be a continuous linear operator. Then, there exists a unique $\mu \in \widetilde{\mathcal{M}}(M;B(\mathcal{E}))$ so that*

$$S(f) = \int_M f d\mu$$

and the operator norm $\|S\| = \tilde{\mu}(M)$.

Definition 2.18. *A function $f : M \to \mathcal{E}$ is said to be μ-integrable or integrable with respect to $\mu \in \mathcal{M}(M;B(\mathcal{E}))$ over $A \in \Sigma$ if there exists a sequence $\{s_n\}$ of simple functions so that*

$$\lim_{n\to\infty} \int_A \|s_n - f\| d|\mu| = 0.$$

The integral of f with respect to μ over $A \in \Sigma$, $\int_A f d\mu \in \mathcal{E}$ is then defined by the relation

$$\int_A f d\mu = \lim_{n\to\infty} \int_A s_n d\mu.$$

For a given $\mu \in \mathcal{M}(M;B(\mathcal{E}))$, we denote by $L_{\mathcal{E}}(\mu)$ the space of μ-integrable functions $u : M \to \mathcal{E}$.

As long as we deal with a measure $\mu \in \mathcal{M}(M;B(\mathcal{E}))$, one can show that a μ-measurable function f is μ-integrable over $A \in \Sigma$ if and only if $\|f(\cdot)\|$ is $|\mu|$-integrable over A. In that case, the inequality

$$\left\| \int_A f d\mu \right\| \leq \int_A \|f(\cdot)\| d|\mu|$$

holds. Moreover, with respect to a given $\mu \in \mathcal{M}(M;B(\mathcal{E}))$, the notions of integral in Definitions 2.17 and 2.18 coincide for totally measurable functions.

If λ is a scalar valued Borel measure on M, a function $f : M \to \mathcal{E}$ is said to be Bochner integrable with respect to λ if f is λ-measurable and $\|f(\cdot)\|$ is λ-integrable.

If a function $f: M \to \mathcal{E}$ is Bochner integrable with respect to λ, then

$$x'\left(\int_M f d\lambda\right) = \int_M x' \circ f d\lambda$$

holds for every $x' \in \mathcal{E}'$.

Definition 2.19. *If $(\mathcal{E}, \preceq, \|\cdot\|)$ is a Banach lattice, we say that a measure $\mu \in \mathcal{M}(M; B(\mathcal{E}))$ is positive if for any μ-integrable function φ with values in $\mathcal{E}^+$, the integral $\int_A \varphi d\mu$ belongs to $\mathcal{E}^+$ for every $A \in \Sigma$.*

Definition 2.20. *A continuous linear operator $S: \mathcal{TM}(M; \mathcal{E}) \to \mathcal{E}$ is said to be dominated if for every $A \in \Sigma$,*

$$|||S_A||| = \sup \sum_i \|S(\chi_{A_i} x_i)\| < \infty,$$

where the supremum is taken over all $f = \sum_i \chi_{A_i} x_i \in \mathcal{TM}(M; \mathcal{E})$ such that f vanishes outside A, all families $\{A_i\} \subset \Sigma$ of pairwise disjoint sets and $\|f\|_{\mathcal{E},M} \leq 1$.

One also computes $|||S_A|||$ by the formula

$$|||S_A||| = \sup \sum_i \|S(f_i)\|,$$

where the supremum is taken over all finite families $\{f_i\} \in \mathcal{TM}(M; \mathcal{E})$ such that f_i vanishes outside A, $\|f_i\|_{\mathcal{E},M} \leq 1$ and $\|f_i(\cdot)\| \|f_j(\cdot)\| = 0$ on M for $i \neq j$ (cf. (Dinculeanu, 1967, Proposition 2.9.5, p. 144)). In particular, if $S: \mathcal{TM}(K; \mathcal{E}) \to \mathcal{E}$ is dominated, where (K, ρ) is a compact metric space, then $\hat{S}$, the restriction of S to $C(K; \mathcal{E})$, is dominated, that is, for every $A \in \Sigma$,

$$\left|\left|\left|\hat{S}_A\right|\right|\right| = \sup \sum_i \|S(f_i)\| < \infty,$$

where the supremum is taken over all finite families $\{f_i\} \in C(K; \mathcal{E})$ such that f_i vanishes outside A, $\|f_i\|_{\mathcal{E},K} \leq 1$ and $\|f_i(\cdot)\| \|f_j(\cdot)\| = 0$ on K for $i \neq j$.

Lemma 2.17. *(Dinculeanu, 1967, Theorem 3.19.2, p. 380) Let (K, ρ) be a compact metric space and $(\mathcal{E}, \|\cdot\|)$ a Banach space. If $S: C(K; \mathcal{E}) \to \mathcal{E}$ is a dominated linear operator, there exists a unique measure $\mu \in \mathcal{M}(K; B(\mathcal{E}))$ such that*

$$S(\Phi) = \int_K \Phi d\mu$$

for every $\Phi \in C(K; \mathcal{E})$ and the operator norm $\|S\| = |\mu|(K)$.

Proposition 2.2. *Let (K, ρ) be a compact metric space and $(\mathcal{E}, \preceq, \|\cdot\|)$ a Banach lattice with order continuous norm. For any given positive linear operator*

$S:C(K;\mathcal{E})\to\mathcal{E}$, there exists a unique positive measure $\mu\in\mathcal{M}(K;B(\mathcal{E}))$ such that

$$S(\Phi)=\int_K \Phi d\mu$$

for every $\Phi\in C(K;\mathcal{E})$ and the operator norm $\|S\|=|\mu|(K)$.

Proof. By Proposition 2.1, for any given positive linear operator $S:C(K;\mathcal{E})\to\mathcal{E}$, there exists a positive linear operator $\tilde{S}:\mathcal{TM}(K;\mathcal{E})\to\mathcal{E}$ such that $\tilde{S}=S$ on $C(K;\mathcal{E})$. Lemma 2.16 shows that there exists a unique $\mu\in\widetilde{\mathcal{M}}(K;B(\mathcal{E}))$ so that

$$\tilde{S}(f)=\int_K f d\mu$$

for every $f\in\mathcal{TM}(K;\mathcal{E})$. Let $A\in\Sigma$ and let $s=\sum_i \chi_{A_i}x_i$ be a simple function vanishing outside A so that $\|x_i\|\leq 1$. Note that $|s|=\sum_i\chi_{A_i}|x_i|$ is a simple function as well. As $\tilde{S}$ is a positive operator, $\tilde{S}(s)\preceq S(|s|)$ and $\tilde{S}(-s)\preceq S(|s|)$. Then, without loss of generality, we assume that s is a simple function with values in $\mathcal{E}^+$. It follows that

$$\|\tilde{S}(\chi_{A_i}x_i)\|\leq\tilde{\mu}(A_i)\|x_i\|\leq\tilde{\mu}(A_i),$$
$$\|\tilde{S}(s)\|=\left\|\int_K s d\mu\right\|=\left\|\sum_i\mu(A_i)x_i\right\|\leq\sum_i\|\mu(A_i)x_i\|\leq\tilde{\mu}(K).$$

Since $\tilde{S}$ is a continuous operator,

$$\left\|\left|\tilde{S}_A\right|\right\|\leq\tilde{\mu}(K).$$

In conclusion, $\left\|\left|\tilde{S}_A\right|\right\|\leq\tilde{\mu}(K)<\infty$ for every $A\in\Sigma$, and hence, $\tilde{S}$ is a dominated operator. The rest follows from Lemma 2.17. □

We now present a generalization of Monotone Convergence Theorem in Banach lattice settings.

Proposition 2.3 (Monotone Convergence Theorem)**.** *Let $(\mathcal{E},\preceq,\|\cdot\|)$ be a Banach lattice with order continuous norm. Let $\mu\in\mathcal{M}(M;B(\mathcal{E}))$ be a positive measure on a metric space (M,ρ). Let $\{u_k\}$, $u_k:M\to\mathcal{E}$, be a monotone decreasing sequence of μ-integrable functions on M. Suppose that there exists a $|\mu|$-integrable $\Phi:M\to\mathbb{R}$ so that $\|u_k(\cdot)\|\leq\Phi(\cdot)$ $|\mu|$-almost everywhere on M for each k. If the pointwise limit $u=\lim_{k\to\infty}u_k$ exists, then it is a μ-integrable function on M satisfying*

$$\int_M u d\mu=\lim_{k\to\infty}\int_M u_k d\mu.$$

Proof. By Lemma 2.7 and Lemma 2.10, the pointwise norm limit of u_k equals $\curlywedge_k u_k$.

Since

$$\Big| \|u_k(\cdot)\| - \|u(\cdot)\| \Big| \le \|u_k(\cdot) - u(\cdot)\|, k \in \mathbb{N},$$

$\|u_k(\cdot)\| \to \|u(\cdot)\|$ pointwise on M and as $\|u_k(\cdot)\| \le \Phi(\cdot)$ $|\mu|$-almost everywhere on M, the function u is μ-integrable on M (cf. (Dinculeanu, 1967, Theorem 9, p.136)). Then, $u - u_k$ is μ-integrable and

$$0 \le \left\| \int_M (u - u_k) d\mu \right\| \le \int_M \|u - u_k\| d|\mu| \to 0,$$

as $k \to \infty$. □

2.4 Examples of Banach lattices

Example 2.1. *A Banach lattice $(\mathcal{E}, \preceq, \|\cdot\|)$ is said to be an abstract L_p-space (AL_p space, in short) for some $p \in [1,\infty)$, if the norm $\|\cdot\|$ is p-additive, that is $\|x+y\|^p = \|x\|^p + \|y\|^p$ holds for all $x, y \in \mathcal{E}^+$ with $x \curlywedge y = 0$.*

We note that every AL_p space has order continuous norm (cf. (Aliprantis & Burkinshaw, 2006, p. 194)). For a measure space (Ω, Σ, μ), $L_p(\mu), p \in [1,\infty)$, the vector space of all μ-measurable functions $f : \Omega \to \mathbb{R}$ satisfying $\int_\Omega |f|^p d\mu < \infty$ are examples for AL_p-spaces. We further note that a Banach lattice is an AL_p-space for some $p \in [1,\infty)$ if and only if it is lattice isometric to some $L_p(\mu)$ space (cf. (Aliprantis & Burkinshaw, 2006, Theorem 4.27)).

Example 2.2. *(Meyer-Nieberg, 1991, p. 9) The space $C_b(0,1) = \{u \in C(0,1) : \sup_{x\in(0,1)} |u(x)| < \infty\}$ is a Banach lattice with the usual pointwise order relation of functions and the sup norm $\|\cdot\|_\infty$. The sequence $\{t^n\} \subset C_b(0,1)$ decreases to the constant function 0, i.e. $t^n \downarrow 0$, yet $\|t^n\|_\infty = 1$ for each n. Therefore, $\|\cdot\|_\infty$ is not order continuous.*

Example 2.3. *(Meyer-Nieberg, 1991, p. 86) The sequence space $\ell^\infty = \{x = (x_m)_{m\in\mathbb{N}} : \|x\|_\infty = \sup_m |x_m| < \infty\}$ with the term by term comparison relation and the norm $\|\cdot\|_\infty$ is a Banach lattice. Consider the sequence $(x^{(n)}), x^{(n)} = (x^{(n)}_m)_{m\in\mathbb{N}}$,*

$$x^{(n)}_m = \begin{cases} 0, & \text{if } m \le n, \\ 1, & \text{if } m > n. \end{cases}$$

One immediately sees that $x^{(n)} \downarrow 0$, yet $\|x^{(n)}\|_\infty = 1$. Hence, the norm $\|\cdot\|_\infty$ is not *order continuous.*

2.4.1 Banach spaces with Unconditional Schauder Basis

Definition 2.21. *Let $(\mathcal{E}, \|\cdot\|)$ be an infinite dimensional (real or complex) Banach space. A sequence $\{e_j\}_{j\in\mathbb{N}}$ in $\mathcal{E}$ is called a Schauder basis of $\mathcal{E}$ if for every $x \in \mathcal{E}$, there exists a unique sequence of scalars $\{x_j\}$ such that $x = \sum_j x_j e_j$. In addition, if $\{e_{\pi(j)} : j \in \mathbb{N}\}$ is a Schauder basis of $\mathcal{E}$ for every permutation $\pi : \mathbb{N} \to \mathbb{N}$, then $\{e_j\}$ is called an unconditional Schauder basis of $\mathcal{E}$.*

Example 2.4. *The sequence spaces $\ell^p, p \in [1,\infty)$, c_0 are Banach spaces with unconditional Schauder bases $\{e_j\}$ where e_j are canonical basis elements. Any separable infinite dimensional Hilbert space has a complete orthonormal system which is an unconditional Schauder basis.*

The following is a well known result on Banach spaces with Schauder bases:

Lemma 2.18. *(James, 1982, Theorem 3.1, Theorem 7.2) Let $(\mathcal{E}, \|\cdot\|)$ be a Banach space with a Schauder basis $\{e_j\}$ and let $\|\cdot\|_a : \mathcal{E} \to [0,\infty)$ be defined as*

$$\|x\|_a = \sup_{n\geq 1} \left\| \sum_{j=1}^{n} x_j e_j \right\|,$$

where $x = \sum_j x_j e_j \in \mathcal{E}$. Then, $\|\cdot\|_a$ is a norm on $\mathcal{E}$ which is equivalent to the norm $\|\cdot\|$. In addition, if $\{e_j\}$ is an unconditional Schauder basis, $\|\cdot\|_b$ given by

$$\|x\|_b = \sup_{\|(z_j)\|_\infty \leq 1} \left\| \sum_{j=1}^{\infty} z_j x_j e_j \right\|,$$

where (z_j) is a bounded sequence in $\mathbb{R}$ and $x = \sum_j x_j e_j \in \mathcal{E}$, defines a norm on $\mathcal{E}$ that is equivalent to the norm $\|\cdot\|$.

Corollary 2.2 (James (1982))**.** *If $\{e_j\}$ is an unconditional Schauder basis for a Banach space $(\mathcal{E}, \|\cdot\|)$, the norm $\|\cdot\|_b$ has the property that*

$$\left\| \sum_j b_j e_j \right\|_b \leq \left\| \sum_j a_j e_j \right\|_b \quad \text{if } |b_j| \leq |a_j| \text{ for each } j.$$

Corollary 2.3 (James (1982))**.** *Let $(\mathcal{E}, \|\cdot\|_b)$ be a real Banach space with unconditional Schauder basis $\{e_j\}$. If we define the product xy as $\sum_j x_j y_j e_j$ where $x = \sum_j x_j e_j$ and $y = \sum_j y_j e_j$, then $(\mathcal{E}, \|\cdot\|_b)$ is a commutative Banach algebra. If we define $x \preceq y$ as $x_j \leq y_j$ for each j, $x \curlyvee y = \sum_j (\max\{x_j, y_j\}) e_j$ and $x \curlywedge y =$*

$\sum_j(\min\{x_j, y_j\})e_j$, *then* $|x| = \sum_j |x_j|e_j$ *and* $(\mathcal{E}, \preceq, \|\cdot\|_b)$ *is a Banach lattice.*

Let $\mathcal{E}$ be a complex Banach space with a Schauder basis $\{e'_j : j \in \mathbb{N}\}$, that is, for each $x \in \mathcal{E}$, there exist uniquely determined $\zeta_j \in \mathbb{C}$ so that

$$x = \sum_{j=1}^{\infty} \zeta_j e'_j.$$

For $\zeta_j = \alpha_j + i\beta_j$, where $\alpha_j \in \mathbb{R}$ and $\beta_j \in \mathbb{R}$, and a vector $x \in \mathcal{E}$,

$$x = \sum_{j=1}^{\infty} (\alpha_j + i\beta_j)e'_j = \sum_{j=1}^{\infty} [\alpha_j e'_j + \beta_j(ie'_j)] = \sum_{k=1}^{\infty} c_k e_k,$$

where $c_k \in \mathbb{R}$ in the last equality, we have rearranged the basis vectors e'_j and ie'_j so that $e_k = e'_j$ if $k = 2j-1$ and $e_k = ie'_j$ if $k = 2j$. That is, $\{e_j : j \in \mathbb{N}\}$ is a Schauder basis for the real space $\mathcal{E}_{\mathbb{R}}$ which is nothing but the space $\mathcal{E}$ when thought of as a real Banach space. We normalize the basis elements and assume that $\|e_j\| = 1$ for each j for the rest of the subsection.

As we mentioned before, with the usual order on the class $S(\mathcal{H})$ of self-adjoint linear operators on a complex Hilbert space $\mathcal{H}$, defined by positive definiteness, $S(\mathcal{H})$ turns out to be a partially ordered vector space that is not a vector lattice. Sherman (1951) proved that a subalgebra $\mathcal{A}$ of a C^*-algebra of self-adjoint operators on a complex Hilbert space has a lattice structure if and only if $\mathcal{A}$ is commutative. In particular, a subalgebra $\mathcal{A}$ of $S(\mathcal{H})$ is a lattice if and only if it is commutative. Olson (1971) defined the spectral order on $S(\mathcal{H})$ and proved that $S(\mathcal{H})$ becomes a conditionally complete lattice under the spectral order which is not a vector lattice. Furthermore, Olson (1971) proved that if $\mathcal{Y}$ is a commutative subalgebra of $S(\mathcal{H})$, then the usual order and the spectral order on $\mathcal{Y}$ are equivalent to each other, and so $\mathcal{Y}$ with the usual order becomes a conditionally complete vector lattice.

Let ℓ^2 be the sequence space of square summable complex sequences. Let $\mathcal{D}$ be the class of bounded, self-adjoint diagonal operators on ℓ^2 and $K(\ell^2)$ the subspace of compact operators in $B(\ell^2)$. Let I_{ℓ^2} be the identity operator on ℓ^2. If E_j denotes the diagonal operator which has 1 on the j-th place and all other entries equal to 0, then

$$\mathcal{D} = \left\{ \sum_{j=1}^{\infty} c_j E_j : \sup_j |c_j| < \infty \right\}, \ \mathcal{D}^+ = \left\{ \sum_{j=1}^{\infty} c_j E_j \in \mathcal{D} : c_j \geq 0 \right\},$$

and

$$\mathcal{D} \cap K(\ell^2) = \left\{ \sum_{j=1}^{\infty} c_j E_j \in \mathcal{D} : \lim_{j\to\infty} c_j = 0 \right\}.$$

Let $(\mathcal{E}, \|\cdot\|)$ be Banach space with an unconditional Schauder basis $\{e_j\}$. Let $\Gamma : \mathcal{E} \to \mathcal{D} \cap K(\ell^2)$ be the linear map given by

$$\Gamma(x) = \sum_{j=1}^{\infty} c_j E_j, \;\; x = \sum_{j=1}^{\infty} c_j e_j \in \mathcal{E}. \tag{2.2}$$

Proposition 2.4. *(Göğüş, 2024, Theorem 1)* Γ *is a well-defined, real linear, norm-continuous (when* $\mathcal{Y}$ *equipped with the operator norm), injective map. Moreover,* $\mathcal{Y} = \Gamma(\mathcal{E})$ *is a conditionally complete vector lattice with the usual order* $\leq$ *if the basis* $\{e_j\}$ *is unconditional.*

Consider the subspace $\mathcal{Y} = \Gamma(\mathcal{E})$ of $\mathcal{D} \cap K(\ell^2)$ and the positive cones

$$\mathcal{E}^+ = \left\{ x = \sum_{j=1}^{\infty} c_j e_j \in \mathcal{E} : c_j \geq 0 \text{ for each } j \right\}$$

and

$$\mathcal{Y}^+ = \left\{ A = \sum_{j=1}^{\infty} c_j E_j \in \Gamma(\mathcal{E}) : c_j \geq 0 \text{ for each } j \right\}.$$

Then Γ maps $\mathcal{E}^+$ onto $\mathcal{Y}^+$. If $\mathcal{E}$ has an unconditional basis $\{e_j\}$, then, both $\mathcal{E}$ and $\mathcal{Y}$ become a conditionally complete vector lattices with the natural ordering from the positive cones above. If $\mathcal{Y} = \Gamma(\mathcal{E})$ is given the norm induced from $\mathcal{E}$, that is, $\|\Gamma(x)\|_{\mathcal{Y}} = \|x\|, x \in \mathcal{E}$, then Γ is obviously an isometric isomorphism. Thus, $\mathcal{Y}$ equipped with the norm $\|\cdot\|_{\mathcal{Y}}$ is a (real) Banach space. As a result, $B(\mathcal{Y})$, the space of continuous linear operators from $\mathcal{Y}$ to itself, is a Banach space with the norm

$$\|T\|_{B(\mathcal{Y})} = \sup_{\substack{y \in \mathcal{Y} \\ y \neq 0}} \frac{\|Ty\|_{\mathcal{Y}}}{\|y\|_{\mathcal{Y}}}.$$

It is worth to mention that $\{E_j\}$ provides an unconditional Schauder basis to $\mathcal{Y}$, hence both $\mathcal{E}$, $\mathcal{Y}$ are separable Banach spaces. Moreover, if $(\mathcal{E}, \preceq, \|\cdot\|)$ is a Banach lattice, then $(\mathcal{Y}, \leq, \|\cdot\|_{\mathcal{Y}})$ is a Banach lattice.

For a Banach space $\mathcal{E}$ with an unconditional Schauder basis $\{e_j\}$ and a numerical sequence $z = \{z_j\}$, where z_j are either -1 or 1, we define the operator P_z on $\mathcal{E}$ by

$$P_z(\sum_j x_j e_j) = \sum_j z_j x_j e_j,$$

and the unconditional Schauder basis constant of $\{e_j\}$ as $\sup_z \|P_z\|$. It is known that a real Banach space $\mathcal{E}$ with unconditional Schauder basis $\{e_j\}$ of unconditonal Schauder basis constant 1 is a Banach lattice with order continuous norm (cf. (Wnuk, 1999, p.91)).

By using the map $\Gamma : \mathcal{E} \to \mathcal{Y}$, we define the Γ-order that is equivalent to the order relation $\preceq$ given in Corollary 2.3. Then, we observe the following by Theorems 2.1 and 2.4:

Proposition 2.5. *If $\mathcal{E}$ is a Banach space with an unconditional Schauder basis $\{e_j\}$, then $(\mathcal{E}, \preceq, \|\cdot\|_b)$ is a conditionally complete, separable Banach lattice and hence it has an order continuous norm.*

2.4.2 Non-commutative Matrix Spaces

2.4.2.1 Simultaneous Diagonalization

In this subsection, we summarize some of the well-known results on simultaneous diagonalization of matrices. We refer to Horn & Johnson (2013) for further details on the topic. We denote by $M(n,m)$ the space of $n \times m$ matrices with complex entries and by $H(n)$ the space of $n \times n$ Hermitian matrices. For a matrix $A \in H(n)$, $A \geq 0$ means that A is positive-definite. The usual order $\leq$ on $H(n)$ is given by the relation $A \leq B$ if $B - A$ is positive-definite.

Definition 2.22. *We say that a collection $\mathcal{Y}$ in $M(N)$ is simultaneously diagonalizable if there exists a unique nonsingular matrix $S \in M(N)$ so that $S^{-1}AS$ is diagonal for every $A \in \mathcal{Y}$, and we say that S simultaneously diagonalizes $\mathcal{Y}$.*

A family $\mathcal{Y}$ of diagonalizable matrices is simultaneously diagonalizable if and only if $\mathcal{Y}$ is commutative. In addition, if $\mathcal{Y}$ is a commutative family in $H(N)$, there exists a unique (up to a unimodular constant) $N \times N$ unitary matrix U so that U simultaneously diagonalizes $\mathcal{Y}$. Note that a commutative subspace of $H(N)$ is at most N real dimensional.

Let $\mathcal{Y}$ be a commutative subspace of $H(N)$ of real dimension N. In other words, the vector space $\tilde{\mathcal{Y}} = \{U^*AU : A \in \mathcal{Y}\}$ has a vector space basis $\{E_j : j = 1,2,\dots,N\}$ where E_j is the $N \times N$ canonical diagonal matrix

$$E_j = diag(0,0,\dots,0,1,0,\dots,0)$$

with 1 as its j-th diagonal entry and U is the unitary matrix that simultaneously diagonalizes $\mathcal{Y}$. The reason for taking such a $\mathcal{Y}$ is clear from the simple observation below:

Lemma 2.19. *(Erdur & Göğüş, 2025, Lemma 2.1) Let $\mathcal{Y}$ be a commuting real subspace in $H(N)$ with N real dimension so that $\tilde{\mathcal{Y}} = span\{E_j : j = 1,2,\dots,N\}$.*

(i) Let $\{A_\alpha = [a_{ij,\alpha}] : \alpha \in \Lambda\}$ be a collection in $\mathcal{Y}$ such that $A_\alpha \leq A$ for some $A \in \mathcal{Y}$. Then, the matrix $\tilde{A} = \sup_\alpha A_\alpha$ belongs to the family $\mathcal{Y}$.

(ii) Let $\{A_\alpha = [a_{ij,\alpha}] : \alpha \in \Lambda\}$ be a collection in $\mathcal{Y}$ such that $A_\alpha \geq A$ for some $A \in \mathcal{Y}$. Then, the matrix $\hat{A} = inf_\alpha A_\alpha$ belongs to the family $\mathcal{Y}$.

Proof. We will only prove (i). Let us denote $D_\alpha = U^*A_\alpha U = diag(\lambda_{j,\alpha})_{1\leq j\leq N}$, where $\lambda_{j,\alpha}$'s are eigenvalues of A_α. Note that $\sup_\alpha \lambda_{j,\alpha}$, $j = 1,2,\dots,N$, are finite real numbers. Then, $D = diag(\sup_\alpha \lambda_{j,\alpha})_{1\leq j\leq N} = \sup_\alpha U^*A_\alpha U \leq U^*AU$ and so $UDU^* \leq A$. Hence, $\tilde{A}$ exists and $\tilde{A} = UDU^* \in \mathcal{Y}$. □

Therefore $\mathcal{Y}$ is a conditionally complete vector lattice under the usual order $\leq$.

Let $\mathcal{E}$ be a nontrivial real subspace of $M(n,m)$ with real dimension N, and $\Gamma : \mathcal{E} \to \mathcal{Y}$ a bijective, real linear map. Then, the continuity of Γ as $\mathcal{E}$ is finite dimensional. Let us define the relation $\preceq$ on $\mathcal{E}$ given by

$$A \preceq B \Longleftrightarrow \Gamma(A) \leq \Gamma(B),\ A,B \in \mathcal{E}.$$

Then, the supremum of any nonempty order bounded above family $\{A_\alpha\}$ in $\mathcal{E}$ belongs to $\mathcal{E}$ itself. In other words, by Theorem 2.1, $(\mathcal{E},\preceq)$ is a conditionally complete vector lattice.

2.4.2.2 Circulant Matrices

For a given vector $c = (c_0,c_1,\dots,c_{n-1}) \in \mathbb{C}^n$, we define a circulant matrix $C \in M(n)$ by

$$C = circ(c) = \begin{bmatrix} c_0 & c_1 & \dots & c_{n-1} \\ c_{n-1} & c_0 & \dots & c_{n-2} \\ c_{n-2} & c_{n-1} & \dots & c_{n-3} \\ \vdots & \vdots & \vdots & \vdots \\ c_1 & c_2 & \dots & c_0 \end{bmatrix}$$

We refer to Gray (2006) for details on circulant matrices. Let us denote by $Circ(n)$ the class of $n \times n$ circulant matrices, and its subclass of Hermitian circulant matrices by $CircH(n)$. It is a known fact that $Circ(n)$ is a commutative algebra. The eigenvalues of a $C = circ(c_0, c_1, \dots, c_{n-1}) \in Circ(n)$ are obtained by

$$\lambda_j = c_0 + c_1\omega^j + c_2\omega^{2j} + \dots + c_{n-1}\omega^{(n-1)j}$$

where $\omega = \exp(2i\pi/n)$ is a primitive n^{th} root of unity. The normalized eigenvectors corresponding λ_j is

$$x_j = \frac{1}{\sqrt{n}}(1, \omega^j, \omega^{2j} \dots, \omega^{(n-1)j})^\top, j = 0, 1, \dots, n-1.$$

The n-dimensional Fast Fourier Transform (FFT) matrix,

$$U = (x_0|x_1|\dots|x_{n-1}),$$

which is unitary matrix, simultaneously diagonalizes $Circ(n)$. Let us emphasize that $CircH(n)$ is a nontrivial example for a commutative subspace $\mathcal{Y}$ of $H(n)$ so that $\tilde{\mathcal{Y}}$ is generated by all diagonal matrices D_k

$$D_k = diag\Big(0, 0, \dots, 0, 1, 0 \dots, 0, \dots, 0\Big), k = 1, 2, \dots, n.$$

where 0 as all entries except its k^{th} main diagonal entry and 1 as its k^{th} main diagonal entry. One can easily show that $CircH(n)$ is an algebra.

2.4.2.3 Examples in Non-commutative Matrix Spaces

We will provide several examples of Γ-order in this subsection depends on two basic methods of constructing examples of conditionally complete vector lattices of matrix spaces, one via using simultaneous diagonalization and one via using a map $\Gamma_0$defined as follows:

For a given matrix $A = [a_{ij}] \in M(n,m)$, let us define the circulant Hermitian matrix $\Gamma_0 A$ by

$$\Gamma_0 A = \begin{bmatrix} 0 & a_{11} & a_{12}\dots & a_{nm} & \overline{a_{nm}} & \overline{a_{n(m-1)}} & \dots & \overline{a_{12}} & \overline{a_{11}} \\ \overline{a_{11}} & 0 & a_{11} & a_{12} & \dots & a_{nm} & \overline{a_{nm}} & \dots & \overline{a_{12}} \\ \overline{a_{12}} & \overline{a_{11}} & 0 & \dots & \dots & \dots & \dots & \dots & \overline{a_{13}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{11} & a_{12}\dots & a_{nm} & \overline{a_{nm}} & \overline{a_{n(m-1)}} & \dots & \overline{a_{12}} & \overline{a_{11}} & 0 \end{bmatrix}$$

We note that by this construction, we define an injective, continuous, real-linear operator $\Gamma_0 : M(n,m) \to CircH(N)$, $N = 2nm+1$.

Example 2.5. *(Erdur & Göğüş (2025))*

(i) *Let $\mathcal{E}$ be a nontrivial real vector subspace of $M(n,m)$ of real dimension k and $\{\hat{E}_j : j = 1,2,\dots k\}$ a basis for $\mathcal{E}$. Let U be the N-dimensional FFT matrix and $\mathcal{Y} = span\{UE_jU^* : j = 1,2,\dots,k\}$ where $k \le N$, E_j are the $N \times N$ canonical diagonal matrices. $\mathcal{Y}$ is a commutative subalgebra of $CircH(N)$ as well as a conditionally complete vector lattice with the usual order. Define the map $\Gamma : \mathcal{E} \to \mathcal{Y}$ as $\Gamma\hat{E}_j = UE_jU^*, j = 1,2,\dots,k$. The map Γ is a bijective, real linear map, and hence, $(\mathcal{E},\preceq)$ is a conditionally complete vector lattice.*

(ii) *Let $T(n) = \{A \in M(n) : tr(A) \in \mathbb{R}\}$ where $tr(A)$ is the trace of $A \in M(n)$. $T(n)$ is a real subspace of $M(n)$ with real dimension $2n^2 - n$. Let $\{\hat{E}_j : j = 1,2,\dots,2n^2 - n\}$ be a basis of $T(n)$. The operator $\Gamma : T(n) \to \mathcal{Y}$ defined by $\Gamma A = \Gamma_0 A + tr(A)I_N$, $A \in T(n)$ has the image $\mathcal{Y} = span\{\tilde{E}_j = \Gamma\hat{E}_j : j = 1,2,\dots,2n^2 - n\}$. It is clear Γ is a bijective real-linear operator and $(T(n),\preceq)$ is a conditionally complete vector lattice. Notice that any eigenvalue $\lambda_j(\Gamma A)$ of ΓA is equal to $\lambda_j(\Gamma_0 A) + tr(A)$ and $\|\Gamma_0\| = 1$. For a positive definite matrix $A \in H(n)$, $\|\Gamma_0 A\| \le \|A\| \le tr(A)$ holds. Since $\Gamma_0 A$ is Hermitian, $\Gamma_0 A + \|\Gamma_0 A\| I_N$ is positive definite and $\Gamma_0 A + \|\Gamma_0 A\| I_N \le \Gamma A$. Hence, $A \in H(n)$ is positive definite, then ΓA is positive definite.*

(iii) *Let $s,t > 0$ be given scalars and $\{\hat{E}_j : j = 1,2,\dots,2n^2\}$ a real vector basis of $M(n)$. Let us define the map $\Gamma_{s,t} : M(n) \to \mathcal{Y}$ as $\Gamma_{s,t}(A) = \Gamma_0 A + s(tr(ReA))I_N + t(tr(ImA))I_N$ where $ReA = (A + A^*)/2$, $ImA = (A - A^*)/2i$ and $\mathcal{Y} = span\{\Gamma_{s,t}\hat{E}_j : j = 1,2,\dots,2n^2\} \subset CircH(N)$. Then, $\Gamma_{s,t}$ is a bijective real linear operator. We also see that $(M(n),\le_{\Gamma_{s,t}})$ is a conditionally complete vector lattice. For a positive definite matrix $A \in H(n)$, $\Gamma_{s,t}(A) = \Gamma_0 A + str(A)I_N$. If $s \ge 1$, we can show that $\Gamma_{s,t}(A)$ is positive definite.*

3. SEMI-CONTINUITY AND TRUNCATIONS FOR VECTOR VALUED FUNCTIONS

In this chapter, we introduce the notions of order semi-continuity, order continuity and semi-continuity for vector lattice valued functions and investigate their connections with norm continuity for functions with values in a Banach lattice with order continuous norm. To achieve stated goal, we employ truncations of vector valued functions and several approximation functions.

3.1 Order Semi-continuity for Vector Valued Functions

Let (M,ρ) be a metric space, $(\mathcal{E},\preceq,)$ a conditionally complete vector lattice. Let $B(z_0,r)=\{z\in M:\ \rho(z,z_0)<r\}$ denote the open ball in M centered at $z_0\in M$ with radius $r>0$. We denote $\curlyvee$ and $\curlywedge$ by taking supremum and infimum operations in the vector lattice $\mathcal{E}$.

Definition 3.1. *A function $u:M\to\mathcal{E}$ is said to be locally order bounded above (below) near a point $z_0\in M$ if for some number $r>0$, the set $\{u(z):z\in B(z_0,r)\}$ has its supremum (infimum, respectively) in $\mathcal{E}$.*

Definition 3.2. *A function $u:M\to\mathcal{E}$ is said to be order bounded above if there exist some $A\in\mathcal{E}$ so that $u\preceq A$ on M, and is said to be order bounded below if $-u$ is order bounded above. If u is both order bounded above and below, then we say u is order bounded.*

We define $M_r(u,z_0)$ for a given function $u:M\mathcal{E}$ that is locally order bounded above near $z_0\in M$ as

$$M_r(u,z_0)=\curlyvee_{z\in B(z_0,r)}u(z).$$

Similarly, for a function u that is locally order bounded below near $z_0 \in M$, we define

$$m_r(u, z_0) = \curlywedge_{z \in B(z_0,r)} u(z), z_0 \in M.$$

We note that $\{M_r(u,z_0)\}/\{m_r(u,z_0)\}$ (whenever they exist) is a monotone decreasing/increasing net in $\mathcal{E}$ indexed by $r \in \mathbb{R}^+$.

Definition 3.3. *For a given function $u : M \to \mathcal{E}$ which is locally order bounded above near $z_0 \in M$, we define its upper semi-continuous regularization at $z_0 \in M$ as*

$$u^*(z_0) = \limsup_{z \to z_0} u(z) = \curlywedge_{r>0} \curlyvee_{z \in B(z_0,r)} u(z).$$

Since $u(z_0) \preceq \curlyvee_{z \in B(z_0,r)} u(z)$, the operation of taking limit superior in the last line is meaningful within the settings of $\mathcal{E}$. One can see that $M_r(u,z_0) \downarrow u^*(z_0)$ as $r \to 0^+$ and $u(z_0) \preceq u^*(z_0)$.

Definition 3.4. *For a given function $u : M \to \mathcal{E}$ which is locally order bounded below near $z_0 \in M$, we define its lower semi-continuous regularization at $z_0 \in M$ as*

$$u_*(z_0) = \liminf_{z \to z_0} u(z) = \curlyvee_{r>0} \curlywedge_{z \in B(z_0,r)} u(z).$$

One easily shows that $m_r(u,z_0) \uparrow u_*(z_0)$ as $r \to 0^+$ and $u_*(z_0) \preceq u(z_0)$.

Lemma 3.1. *Let $(\mathcal{E}, \preceq)$ be a conditionally complete Riesz space and (K, ρ) a compact metric space. Then, any order semi-continuous function $u : K \to \mathcal{E}$ is order bounded above.*

Proof. Let $u : M \to \mathcal{E}$ be order upper semi-continuous function and $z_0 \in K$ an arbitrary point. Then, for some $r = r(z_0) > 0$, $M_r(u,z_0)$ exists in $\mathcal{E}$. For any $z \in B(z_0,r)$, $u(z) \preceq M_r(u,z_0)$. Hence, for any $w \in K$, we can find some $r(w) > 0$ so that the set $\{u(z) : z \in B(w, r(w))\}$ has the supremum $M_{r(w)}(u,w)$. Since K is compact, we can find finitely many open balls $B(z_j,r_j), j = 1,2,\ldots,n$, so that $K = \cup_{j=1}^{n} B(z_j,r_j)$. Let $S = \curlyvee\{M_{r_j}(u,z_j) : j = 1,2,\ldots,n\} \in \mathcal{E}$. It is clear that $u(z) \preceq S$ for every $z \in K$. □

Proposition 3.1. *Let (K,ρ) be a compact metric space, $(\mathcal{E}, \preceq, \|\cdot\|)$ a Banach lattice with order continuous norm and $u \in o\text{-}USC(K;\mathcal{E})$. Then, there exists a sequence of Borel simple functions $\psi_n : K \to \mathcal{E}$ that converges in norm to u pointwise on K.*

Proof. Let $u : K \to \mathcal{E}$ be an order upper semi-continuous function and $\{r_n\}$ a nonnegative sequence in $\mathbb{R}$ such that $r_n \downarrow 0$ as $n \to \infty$. Since (K,ρ) is a compact metric space, for each n one can find a finite open cover $\{B(z_j,r_n)\}_{j=1}^{N}, B(z_j,r_n) = \{z \in K : \rho(z_j,z) < r_n\}, z_j = z_j(n) \in K, N = N(n)$, for K. Let us consider the sets $B_{n,1} = B(z_1,r_n)$ and $B_{n,j} = B(z_j,r_n) \setminus (\bigcup_{k=1}^{j-1} B(z_j,r_n)), j \geq 2$ and the vectors

$s_{n,j} = \curlyvee_{z \in B_{n,j}} u(z) \in \mathcal{E}$. Define the functions $\psi_n : K \to \mathcal{E}$ given by

$$\psi_n(z) = \sum_{j=1}^{N} \chi_{B_{n,j}}(z) s_{n,j}, z \in K.$$

Let $z_0 \in K$ be a fixed point. Since $B_{n,j}$ is a finite cover for K, for each n, there exists a unique $j_0 \in \{1, 2, \ldots, N = N(n)\}$ such that $z_0 \in B_{n,j_0}$. Clearly, $B_{n,j_0} \subset B(z_0, 2r_n)$,

$$\curlyvee_{z \in B(z_0, 2r_n)} u(z) \downarrow u(z_0) \text{ as } n \to \infty,$$

and

$$u(z_0) \preceq s_{n,j_0} \preceq \curlyvee_{z \in B(z_0, 2r_n)} u(z).$$

Thus,

$$0 \preceq s_{n,j_0} - u(z_0) \preceq \curlyvee_{z \in B(z_0, 2r_n)} u(z) - u(z_0).$$

Since $\| \curlyvee_{z \in B(z_0, 2r_n)} u(z) - u(z_0) \| \to 0$ as $n \to \infty$, we conclude that $\|u(z_0) - \psi_n(z_0)\| = \|u(z_0) - s_{n,j_0}\| \to 0$ as $n \to \infty$. □

Under these circumstances, we observe that order upper semi-continuous functions on a compact metric space with values in a Banach lattice with order continuous norm are Borel functions with separable range (cf. (Dinculeanu, 2000, Chapter 1, Section 1, Theorem 8)).

Lemma 3.2. *Let $\{u_k\}$ be a decreasing sequence of order upper semi-continuous functions on M with values in $\mathcal{E}$. If the pointwise infimum $u = \curlywedge_k u_k$ (i.e. $u_k \downarrow u$) exists, then it is order upper semi-continuous on M.*

Proof. The inequality $u(z) \preceq u_k(z) \preceq M_r(u_k, z_0)$ holds for any given point $z_0 \in M$ and $z \in B(z_0, r) \subset M$. Then, $u^*(z_0) \preceq M_r(u_k, z_0)$, and $u^*(z_0) \preceq u_k^*(z_0) = u_k(z_0)$ for every k. This relation implies that $u^*(z_0) \preceq u(z_0)$. Therefore, the function u is order upper semi-continuous on M. □

Lemma 3.3. *Let u, v be two functions of the class o-$USC(M; \mathcal{E})$. Then, the functions $u \curlyvee v$ and $u \curlywedge v$ are of the class o-$USC(M; \mathcal{E})$.*

Proof. We first note that $u \curlyvee v$ is well-defined on M. For given $z_0 \in M$, there exists some $r > 0$ so that $M_r(u, z_0) = \curlyvee_{z \in B(z_0, r)} u(z)$ and $M_r(v, z_0) = \curlyvee_{z \in B(z_0, r)} v(z)$ exist in $\mathcal{E}$. Hence, $u \curlyvee v$ is order bounded above by $M_r(u, z_0) \curlyvee M_r(v, z_0)$ on $B(z_0, r)$, so $(u \curlyvee v)^*(z_0)$ exists in $\mathcal{E}$ and satisfies $(u \curlyvee v)^*(z_0) \preceq M_r(u, z_0) \curlyvee M_r(v, z_0)$. By letting $r \to 0^+$ in the last inequality, we deduce that $(u \curlyvee v)^*(z_0) \preceq u(z_0) \curlyvee v(z_0)$ and thus $u \curlyvee v$ is order upper semi-continuous at z_0. It is similar to show that $u \curlywedge v$ is of the class *o*-$USC(M; \mathcal{E})$. □

As one might expect, a function $u: M \to \mathcal{E}$ is order lower semi-continuous at z_0 if $-u$ is order upper semi-continuous at z_0.

We denote by $C(M;\mathbb{R})$ the space of continuous real valued functions on M, and by $C(M;\mathcal{E})$ the space of continuous functions $u : M \to \mathcal{E}$ whenever $(\mathcal{E}, \|\cdot\|)$ is a normed space. Similarly, we denote by $USC(M;\mathbb{R})$ the cone of real valued upper semi-continuous functions on M.

We will extensively use the cones

$$\begin{aligned} C_{oba}(M;\mathcal{E}) &= \{u \in C(M;\mathcal{E}) : u \text{ is order bounded above on } M\}, \\ C_{obb}(M;\mathcal{E}) &= \{u \in C(M;\mathcal{E}) : u \text{ is order bounded below on } M\}, \end{aligned}$$

and the subspace $C_{ob}(M;\mathcal{E}) = C_{oba}(M;\mathcal{E}) \cap C_{obb}(M;\mathcal{E})$, where $(\mathcal{E}, \preceq, \|\cdot\|)$ is a normed Riesz space.

Definition 3.5. *If $u : M \to \mathcal{E}$ is a function that is both order upper and lower semi-continuous at a point $z_0 \in M$, we say that u is order continuous at z_0. We will denote by o-$C(M;\mathcal{E})$ the set of order continuous functions $u : M \to \mathcal{E}$.*

If a function $u : M \to \mathcal{E}$ is order continuous at a point $z_0 \in M$, then $\{M_r(u,z_0)\}$ and $\{m_r(u,z_0)\}$ are nets in $\mathcal{E}$ so that $M_r(u,z_0) \downarrow u(z_0)$ and $m_r(u,z_0) \uparrow u(z_0)$ as $r \to 0^+$. Let $y_r = M_r(u,z_0) - m_r(u,z_0)$. For any $z \in B(z_0,r), r > 0$,

$$\begin{aligned} |u(z) - u(z_0)| &= (u(z) - u(z_0))^+ + (u(z) - u(z_0))^- \\ &\preceq (M_r(u,z_0) - u(z_0))^+ + (u(z_0) - m_r(u,z_0))^+ \\ &= \curlyvee\{M_r(u,z_0), u(z_0)\} - \curlywedge\{m_r(u,z_0), u(z_0)\} \\ &= M_r(u,z_0) - m_r(u,z_0) = y_r. \end{aligned}$$

This shows that $u(z) \xrightarrow{o} u(z_0)$ as $z \to z_0$. By Lemma 2.9, we also show that if $u : M \to \mathcal{E}$ is a function so that $u(z) \xrightarrow{o} u(z_0)$ as $z \to z_0$, then u is order continuous at z_0. If $(\mathcal{E}, \preceq, \|\cdot\|)$ is a normed Riesz space with order continuous norm, then $u(z) \xrightarrow{o} u(z_0)$ as $z \to z_0$ implies that $\|u(z) - u(z_0)\| \to 0$ as $z \to z_0$. By (Meyer-Nieberg, 1991, Proposition 1.1.11), one shows that o-$C(M;\mathcal{E})$ is a vector space.

Corollary 3.1. *Let $(\mathcal{E}, \preceq, \|\cdot\|)$ be a normed Riesz space with order continuous norm. Then, o-$C(M;\mathcal{E})$ is a subspace of $C(M;\mathcal{E})$.*

Let $(\mathcal{E}, \preceq, \|\cdot\|)$ be a Banach lattice with order continuous norm, $u : M \to \mathcal{E}$ a function locally order bounded above near a point $z_0 \in M$. For given $x' \in \mathcal{E}'_+$,

$$x'\left(M_r(u,z_0)\right) = x'\left(\curlyvee_{z\in B(z_0,r)} u(z)\right) = \sup_{z\in B(z_0,r)} x' \circ u(z),$$
$$x'(u^*(z_0)) = x'\left(\curlywedge_{r>0} M_r(u,z_0)\right) = \inf_{r>0} \sup_{z\in B(z_0,r)} x' \circ u(z) = (x' \circ u)^*(z_0).$$

This implies that a locally order bounded above function u is order upper semi-continuous if and only if the composition $x' \circ u : M \to \mathbb{R}$ is upper semi-continuous for every $x' \in \mathcal{E}'_+$. However, in the observation above, we cannot remove the locally order boundedness above condition, as we will see in Example 3.2 that a continuous function $u : M \to \mathcal{E}$ does not need to be locally order bounded above while being bounded in the norm. If $u \in C(M;\mathcal{E})$ is a function locally order bounded above in M, then for every $x' \in \mathcal{E}'_+$, the composition $x' \circ u$ is upper semi-continuous. Thus, if $(\mathcal{E}, \preceq, \|\cdot\|)$ is a Banach lattice with order continuous norm, one can prove easily that any function $u \in C(M;\mathcal{E})$ that is locally order bounded is also order continuous.
Corollary 3.2. *If $(\mathcal{E}, \preceq, \|\cdot\|)$ is a Banach lattice with order continuous norm, o-$C(M;\mathcal{E})$ exactly consists of locally order bounded functions $u \in C(M;\mathcal{E})$. If (M,ρ) is compact as well, then o-$C(M;\mathcal{E})$ coincides with the space $C_{ob}(M;\mathcal{E})$.*

3.2 Truncations of Functions

We start this section with the following example:
Example 3.1. *(Kreuter, 2018, Remark 8.9) Let $x_n = e_n/n \in \ell^1$ where e_n are the canonical basis elements. The sequence $\{x_n\}$ is norm bounded in ℓ^1,$\|x_n\|_{\ell^1} \leq 1$, yet $\curlyvee\{x_n\} = (1/n)_{n\in\mathbb{N}}$ does not belong to ℓ^1. Then, $\{x_n\}$ is not order bounded.*

Hence, a norm bounded set in a normed Riesz space may not be order bounded. A result for continuous functions corresponding this observation is also true. The following is an example to this observation. For sake of being self-contained, we first present a continuous extension result.
Theorem 3.1 (Dugundji's Extension Theorem)**.** *(Dugundji, 1951, Theorem 4.1) Let (M,ρ) be a metric space, A a closed subset of M, $\mathcal{V}$ a locally convex topological vector space, $f : A \to \mathcal{V}$ a continuous map. Then, there exists a continuous extension $F : M \to \mathcal{V}$ of f so that $F(X)$ is a subset of the convex hull of $f(A)$.*
Example 3.2. *(Kreuter, 2020, Example 2.2) Let M be a bounded open set in $\mathbb{R}^n$. Let $(z_n)_{n\in\mathbb{N}}$be a pairwise distinct sequence of points in ∂M converging to z. Define*

f on $\{z_n : n \in \mathbb{N}\} \cup \{z\}$ via $f(z_n) = \frac{e_n}{n} \in \ell^1$ and $f(z) = 0$ where e_n are the canonical vectors in ℓ^1. Then f is continuous on its domain. Using Dugundji's Extension Theorem, we may extend f to a function $F \in C(\partial M, \ell^1)$ attaining values in the convex hull $\{\sum_{j=1}^{N}(c_j e_j/j) : c_j \geq 0, \sum_{j=1}^{N} c_j = 1, N \in \mathbb{N}\}$. If F was order bounded above by some $x = (x_n) \in \ell^1$, we would have $e_n/n \preceq x$ for each n. But , this proves that x cannot be an element of ℓ^1. Hence, F is not order bounded above in the order of ℓ^1. By the same reasoning , one also shows that F is not locally order bounded above near z.

Let (M,ρ) be a metric space, $(\mathcal{E},\preceq)$ a vector lattice and $f : M \to \mathcal{E}$ a function. For given fixed $s \in \mathcal{E}$, define the function $f_s : M \to \mathcal{E}$ given by $f_s(\cdot) = f(\cdot) \curlyvee s$ that is order bounded below by s. Similarly, we can define the function ${}_sf : M \to \mathcal{E}$, ${}_sf(\cdot) = f(\cdot) \curlywedge s$ and it is clear that ${}_sf$ is order bounded above by s.

A classical Dini's theorem (cf. (Rudin, 1976, Theorem 7.13)) states that if (K,ρ) is a compact metric space, $\{f_j\}$ is a pointwise monotone increasing/decreasing sequence in $C(K;\mathbb{R})$ and the pointwise limit f of f_j is in $C(K;\mathbb{R})$, then f_j uniformly converges to f on K. The following is a Dini type result for functions with values in normed Riesz spaces.

Lemma 3.4. *Let (K,ρ) be a compact metric space, $(\mathcal{E},\preceq,\|\cdot\|)$ a normed Riesz space, $\{f_j\}$ a monotone increasing sequence in $C(K;\mathcal{E})$ so that f_j converges in the norm to a function $f \in C(K;\mathcal{E})$ pointwise on K. Then, f_j uniformly converges to f on K.*

Proof. For each $j \in \mathbb{N}$, $0 \preceq f - f_{j+1} \preceq f - f_j$ on K. Since $\|\cdot\|$ is a lattice norm, the sequence $\{\|f(\cdot) - f_j(\cdot)\|\}$ is a monotone decreasing sequence in $C(K;\mathbb{R})$ that converges to 0 pointwise on K. Thus, by the recalled Dini's theorem, f_j uniformly converges to f on K. □

Proposition 3.2. *Let (M,ρ) be a metric space and $(\mathcal{E},\preceq,\|\cdot\|)$ a normed Riesz space.*

(i) For given $s \in \mathcal{E}$ and $f \in C(M;\mathcal{E})$, the functions ${}_sf$ and f_s are in $C(M;\mathcal{E})$.

(ii) Suppose that $(\mathcal{E},\preceq,\|\cdot\|)$ is a conditionally complete Banach lattice and $f : M \to \mathcal{E}$ a function with range separable $f(M)$. Let $\mathcal{S}$ be a countable dense subset of $f(M)$. Then,

$$f(z_0) = \curlyvee_{s\in\mathcal{S}}\, {}_sf(z_0) = \curlywedge_{s\in\mathcal{S}} f_s(z_0), z_0 \in M. \tag{3.1}$$

(iii) Let $(\mathcal{E},\preceq,\|\cdot\|)$ be a conditionally complete separable Banach lattice. Then, for given $f \in C(M;\mathcal{E})$, there exists a monotone increasing sequence $\{f_j\}$ in

$C_{oba}(M;\mathcal{E})$ *so that* f_j *converges in the norm to* f *pointwisely. In addition,* f_j *uniformly converges to* f *on compact subsets of* M*.*

Proof. *(i)* By Birkhoff's inequalities (Lemma 2.4), we have for given $s \in \mathcal{E}$ that

$$|\,{}_s f(z) - {}_s f(z_0)| = |f(z) \curlywedge s - f(z_0) \curlywedge s| \preceq |f(z) - f(z_0)|,$$

$z, z_0 \in M$. Since $\|\cdot\|$ is a lattice norm,

$$\|\,{}_s f(z) - {}_s f(z_0)\| \leq \|f(z) - f(z_0)\|,$$

$z, z_0 \in M$. This shows that ${}_s f$ is of the class $C(M;\mathcal{E})$. It is analogue to show that f_s is of the class $C(M;\mathcal{E})$.

(ii) Let $\{s_j\}$ be a sequence in $\mathcal{S}$ so that s_j converges in the norm to $f(z_0)$. By Birkhoff's inequalities,

$$|\,{}_{s_j} f(z_0) - f(z_0)| \preceq |s_j - f(z_0)|.$$

Then,

$$\|\,{}_{s_j} f(z_0) - f(z_0)\| \leq \|s_j - f(z_0)\|.$$

Thus, the sequence $\{{}_{s_j} f(z_0)\}$ converges in the norm to $f(z_0)$ as $j \to \infty$. For $k \in \mathbb{N}$, the inequalities

$$\curlyvee_{j=1}^{k} {}_{s_j} f(z_0) \succeq {}_{s_1} f(z_0),$$
$$\curlyvee_{j=1}^{k+1} {}_{s_j} f(z_0) \succeq \curlyvee_{j=1}^{k} {}_{s_j} f(z_0)$$

hold. For $1 \leq j \leq k$, ${}_{s_j} f(z_0) \preceq \curlyvee_{i=1}^{k} {}_{s_i} f(z_0) \preceq f(z_0)$,

$$|\curlyvee_{i=1}^{k} {}_{s_i} f(z_0) - f(z_0)| \preceq |\,{}_{s_j} f(z_0) - f(z_0)|,$$

and hence,

$$\|\curlyvee_{i=1}^{k} {}_{s_i} f(z_0) - f(z_0)\| \leq \|\,{}_{s_j} f(z_0) - f(z_0)\|.$$

This implies that $\{\curlyvee_{i=1}^{k} {}_{s_i} f(z_0)\}$ converges in the norm to $f(z_0)$. This further brings out that the sequence

$$\left\{(\curlyvee_{i=1}^{k} {}_{s_i} f(z_0)) - {}_{s_1} f(z_0)\right\}_{k \in \mathbb{N}}$$

is a monotone increasing, Cauchy sequence in $\mathcal{E}$ with positive terms that is order bounded above by $f(z_0) - {}_{s_1}f(z_0)$ and

$$\left\| (\curlyvee_{i=1}^{k} {}_{s_i}f(z_0)) - {}_{s_1}f(z_0)(z_0) - \left(f(z_0) - {}_{s_1}f(z_0) \right) \right\| \to 0$$

as $k \to \infty$. As a consequence of Lemma 2.10,

$$(\curlyvee_{i=1}^{\infty} {}_{s_i}f(z_0)) - {}_{s_1}f(z_0),$$

the supremum of the sequence

$$\left\{ (\curlyvee_{i=1}^{k} {}_{s_i}f(z_0)) - {}_{s_1}f(z_0) \right\}$$

exists in $\mathcal{E}$ and equals $f(z_0) - {}_{s_1}f(z_0)$. Therefore, $\curlyvee_{i=1}^{\infty} {}_{s_i}f(z_0)$ exists in $\mathcal{E}$ and equals $f(z_0)$. In addition, since

$$f(z_0) = \curlyvee_{i=1}^{\infty} {}_{s_i}f(z_0) \preceq \curlyvee_{s \in \mathcal{S}} {}_{s}f(z_0) \preceq f(z_0),$$

we conclude that $\curlyvee_{s \in \mathcal{S}} {}_{s}f(z_0) = f(z_0)$. Since $-f_s(\cdot) = {}_{-s}(-f)(\cdot)$, the rest of the identity (3.1) follows immediately.

(iii) Let $\mathcal{S}$ be a countable dense subset of $\mathcal{E}$, $\{s'_j\}$ an enumeration of $\mathcal{S}$. For fixed $k \in \mathbb{N}$, define the function f_k given by $f_k(z) = \curlyvee_{j=1}^{k} {}_{s'_j}f(z), z \in M$. Trivially, $f_k(\cdot) = f(\cdot) \curlywedge (\curlyvee_{j=1}^{k} s'_j) \preceq \curlyvee_{j=1}^{k} s'_j$ in M. Since f is of the class $C(M;\mathcal{E})$, by part (i), we conclude that f_k belongs to $C_{oba}(M;\mathcal{E})$. Consider the monotone increasing sequence $\{f_k\}_{k\in\mathbb{N}}$. It follows from part (ii) that $f_k(\cdot) \uparrow \curlyvee_{j=1}^{\infty} f_j(\cdot) = f(\cdot)$. Since the norm $\|\cdot\|$ is order continuous, we conclude that f_k converges in the norm to f pointwisely. The part on uniform convergence follows from Lemma 3.4.

□

Remark 3.1. *In part (iii) of Propositon 3.2, one can show that the monotone decreasing sequence* $\{\tilde{f}_k\}, \tilde{f}_k(\cdot) = \curlywedge_{j=1}^{k} f_{s'_j}(\cdot)$, *in* $C_{obb}(M;\mathcal{E})$ *converges in the norm to* f *pointwise. Moreover, replacing the conditions on* f *and* $(\mathcal{E}, \preceq, \|\cdot\|)$ *in part (iii) by the conditions that* $f(M)$ *is separable and* $(\mathcal{E}, \preceq, \|\cdot\|)$ *is a Banach lattice with order continuous norm , respectively, we still obtain the same conclusion.*

As a consequence, we obtain the following;

Corollary 3.3. *Let* (K,ρ) *be compact metric space and* $(\mathcal{E}, \preceq, \|\cdot\|)$ *a Banach lattice with order continuous norm. Then, the closure of* $C_{ob}(K;\mathcal{E})$ *with respect to the topology of uniform convergence on* K *is* $C(K;\mathcal{E})$.

Proposition 3.3. *Let (K,ρ) be a compact metric space and $(\mathcal{E},\preceq,\|\cdot\|)$ a Banach lattice with order continuous norm. Then, a function $u : K \to \mathcal{E}$ is in o-$USC(K;\mathcal{E})$ if and only if there exists a monotone decreasing sequence $\{\varphi_n\}$ in o-$C(K;\mathcal{E})$ such that $\varphi_n \downarrow u$ pointwise on K.*

Proof. Suppose first that there exists a sequence $\{\varphi_n\}$ in o-$C(K;\mathcal{E})$ so that $\varphi_n \downarrow u$ pointwise on K. It is trivial by Lemma 3.2 that $u : K \to \mathcal{E}$ is an order upper semicontinuous function.

Now, let $u : K \to \mathcal{E}$ be an order bounded function of the class o-$USC(K;\mathcal{E})$. Recall that the range of u is separable (Lemma 3.1). Let $\tilde{\mathcal{E}} = \overline{(span(u(K)))^{\curlyvee\curlywedge}}$ be the separable Banach sublattice of $\mathcal{E}$ generated by $u(K)$. Naturally, $\tilde{\mathcal{E}}$ inherits the order continuous norm $\|\cdot\|$ from $\mathcal{E}$. Let $S_{\tilde{\mathcal{E}}} = \{x \in \tilde{\mathcal{E}} : \|x\| = 1, x \succeq 0\}$ and let $\{s_j\} \subset S_{\tilde{\mathcal{E}}}$ be a dense sequence in $S_{\tilde{\mathcal{E}}}$. Let $x = \sum_{j=1}^{\infty} 2^{-j} s_j \in \tilde{\mathcal{E}}^{+}$. Since for $z, a \in K$ and $n \in \mathbb{N}$,

$$[u(a) - n\rho(z,a)x] \preceq \curlyvee_{a\in K} u(a),$$

we can define the functions $\varphi_n : K \to \tilde{\mathcal{E}}, n \in \mathbb{N}$, by

$$\varphi_n(z) = \curlyvee_{a\in K}[u(a) - n\rho(z,a)x] \in \tilde{\mathcal{E}},$$

$z \in K$. Then, $\varphi_n(z) \preceq \curlyvee_{a\in K} u(a)$. For $a, z, z' \in K$,

$$|u(a) - n\rho(z,a)x - [u(a) - n\rho(z',a)x]| = n|\rho(z,a) - \rho(z',a)|x \preceq n\rho(z,z')x,$$
$$u(a) - n\rho(z,a)x \preceq n\rho(z,z')x + u(a) - n\rho(z',a)x.$$

By replacing z by z' in the last inequality, we obtain that

$$u(a) - n\rho(z,a)x \preceq n\rho(z,z')x + u(a) - n\rho(z',a)x.$$

By taking supremum over all $a \in K$, we have that

$$|\varphi_n(z) - \varphi_n(z')| \preceq n\rho(z,z')x.$$

This means that each φ_n is of the class o-$C(K;\mathcal{E})$. Moreover, the relation $u \preceq \varphi_{n+1} \preceq \varphi_n, n \in \mathbb{N}$, holds on K. In particular, $\lim_{n\to\infty} \varphi_n(z)$ exists in $\tilde{\mathcal{E}}$ and $\lim_{n\to\infty} \varphi_n(z) \succeq u(z)$ for given $z \in K$. Let $\ell \in \tilde{\mathcal{E}}'_{+} \setminus \{0\}$ be arbitrary. Then, there exists some $j_0 \in \mathbb{N}$ so that $\ell(s_{j_0}) \geq \alpha > 0$ for some real number $\alpha > 0$ (otherwise, $\ell(s_j) = 0$ for each j implies $\ell(s) = 0$ for every $s \in \tilde{\mathcal{E}}$ and hence $\ell = 0$). Thus, $\ell(x) \geq 2^{-j_0}\alpha > 0$ holds.

Since $\ell \circ u : K \to \mathbb{R}$ is upper semi-continuous,

$$\begin{aligned}\ell \circ \varphi_n(z) &= \sup_{a \in K} \{\ell \circ u(a) - n\rho(z,a)\ell(x)\} \\ &\leq \max\left\{\sup_{a \in B(z,r)} \ell \circ u(a), \sup_{a \in K \setminus B(z,r)} (\ell \circ u(a) - n\rho(z,a)\ell(x))\right\} \\ &\leq \max\left\{sup_{a \in B(z,r)} \ell \circ u(a), (\sup_K \ell \circ u) - nr\ell(x)\right\}, z \in K, r > 0,\end{aligned}$$

which implies that $\lim_{n\to\infty} \ell \circ \varphi_n(z) \leq \sup_{B(z,r)} \ell \circ u$ for $z \in K$ and $r > 0$. Since $\ell \circ u$ is upper semi-continuous, $\sup_{B(z,r)} \ell \circ u \downarrow u(z)$ as $r \downarrow 0$. Hence,

$$\ell(\lim_{n\to\infty} \varphi_n(z)) = \lim_{n\to\infty} \ell \circ \varphi_n(z) \leq \ell \circ u(z), z \in K,$$

and as $\ell \in \tilde{\mathcal{E}}'_+$ is arbitrary, we further infer that $\lim_{n\to\infty} \varphi_n(z) \preceq u(z), z \in K$. Therefore, we conclude that $\lim_{n\to\infty} \varphi_n(z) = u(z), z \in K$.

Let $u : K \to \mathcal{E} \in o\text{-}USC(K;\mathcal{E})$. The functions $\varphi_n(z) = \curlyvee_{a\in K}[u(a) - n\rho(z,a)x], z \in K$, are of the class $o\text{-}C(K;\mathcal{E})$ so that $u \preceq \varphi_{n+1} \preceq \varphi_n, n \in \mathbb{N}$ on K. Recall again that u has separable range. Let $\{s'_j\}$ be a dense sequence in $u(K)$. Define the functions

$$\begin{aligned}u_k(z) &= u(z) \curlyvee (\curlywedge_{j=1}^k s'_j), \\ \varphi_{n,k}(z) &= \curlyvee_{a\in K}[u_k(a) - n\rho(z,a)x] = \varphi_n(z) \curlyvee (\curlywedge_{j=1}^k s'_j),\end{aligned}$$

$z \in K$, as a result of the Infinite Distributive Law (Lemma 2.1). By our previous argument, we observe that for fixed k, $\{\varphi_{n,k}\} \subset o\text{-}C(K;\mathcal{E})$ is a monotone decreasing sequence that converges pointwise to u_k on K. Notice also that $u_k(z) \downarrow u(z)$ and $\varphi_{n,k}(z) \downarrow \varphi_n(z), z \in K$ as $k \to \infty$. Then, for given $z \in K$, the inequality

$$\|u(z) - \varphi_n(z)\| \leq \|u(z) - u_k(z)\| + \|u_k(z) - \varphi_{n,k}(z)\| + \|\varphi_{n,k}(z) - \varphi_n(z)\|$$

implies that $\lim_{n\to\infty} \varphi_n(z) = u(z)$. □

3.3 Semi-continuity for Vector Valued Functions

A well known fact for scalar valued functions is that upper semi-continuity is preserved under uniform convergence. This fact inspires us to give the following definition of upper semi-continuity for vector valued functions;

Definition 3.6. *Let $(\mathcal{E}, \preceq, \|\cdot\|)$ be a conditionally complete Banach lattice. A function $u : M \to \mathcal{E}$ is said to be upper semi-continuous if there exists a sequence $\{u_k\}$ in o-$USC(M;\mathcal{E})$ that uniformly converges to u on compact subsets of M, that is, on every compact subset K of M,*

$$\sup_{z\in K} \|u_k(z) - u(z)\| \to 0, \quad \textit{as } k \to \infty.$$

We say that $u : M \to \mathcal{E}$ is lower semi-continuous if $-u$ is upper semi-continuous. We denote by $USC(M;\mathcal{E})$ the cone of upper semi-continuous functions $u : M \to \mathcal{E}$.

Let $(\mathcal{E}, \preceq, \|\cdot\|)$ be a Banach lattice with order continuous norm and $x' \in \mathcal{E}'_+$. For $u \in USC(M;\mathcal{E})$, one easily shows that $x' \circ u : M \to \mathbb{R}$ is upper semi-continuous. If u is locally order bounded above as well, then u is order upper semi-continuous. It is also straightforward to show that an upper/lower semi-continuous function defined on a compact metric space with values in a Banach lattice with order continuous norm is a Borel function with separable range (cf. (Dinculeanu, 2000, Theorem 10, p. 6)). Whenever (M,ρ) is a separable metric space and $(\mathcal{E}, \preceq, \|\cdot\|)$ is a Banach lattice lattice with order continuous norm, we infer from Proposition 3.2 (iii) that, every function $f \in C(M;\mathcal{E})$ is both upper and lower semi-continuous on M. In addition, we emphasize that one should not confuse upper semi-continuity with order upper semi-continuity, as it is evident that the function F in Example 3.2 is upper semi-continuous that is not order upper semi-continuous.

Proposition 3.4. *Let (M,ρ) be a metric space and $(\mathcal{E}, \preceq, \|\cdot\|)$ is a conditionally complete Banach lattice. For any $u, v \in USC(M;\mathcal{E})$, the functions $u \curlyvee v$ and $u \curlywedge v$ belong to the class $USC(M;\mathcal{E})$.*

Proof. Let $\{u_k\}$ and $\{v_k\}$ be two sequences in o-$USC(M;\mathcal{E})$ such that on every compact subset K of M,

$$\sup_{z\in K} \|u_k(z) - u(z)\| \to 0 \text{ and } \sup_{z\in K} \|u_k(z) - u(z)\| \to 0 \text{ as } k \to \infty.$$

Lemma 3.3 implies that $u_k \curlyvee v_k$ and $u_k \curlywedge v_k$ belong to o-$USC(M;\mathcal{E})$. By Lemma

2.5, we have

$$
\begin{aligned}
&|u_k(z) \curlyvee v_k(z) - u(z) \curlyvee v(z)| \preceq |u_k(z) - u(z)| + |v_k(z) - v(z)|, \\
&|u_k(z) \curlywedge v_k(z) - u(z) \curlywedge v(z)| \preceq |u_k(z) - u(z)| + |v_k(z) - v(z)|, z \in K.
\end{aligned}
$$

The rest of the proof follows immediately from the fact that the norm on $\mathcal{E}$ is a lattice norm.

□

4. JENSEN MESAURES AND EDWARDS' THEOREM

4.1 Jensen Measures

Let $(\mathcal{E}, \preceq, \|\cdot\|)$ be a Banach lattice with order continuous norm and $\mathcal{F} \subset USC(K;\mathcal{E})$ a cone of functions on a compact metric space (K,ρ). We say that $\mathcal{F}$ has the constant function property if all functions $z \mapsto x, z \in K$, for given $x \in \mathcal{E}$, belongs to the cone $\mathcal{F}$.

Definition 4.1. *Let (K,ρ) be a compact metric space and $\mathcal{F} \subset USC(K;\mathcal{E})$ a cone which has the constant function property. A positive measure $\mu \in \mathcal{M}(K; B(\mathcal{E},\mathcal{E}''))$ is said to be a Jensen measure for $\mathcal{F}$ with barycenter $z \in K$ if*

$$u(z) \preceq \int_K u d\mu$$

for any $u \in \mathcal{F} \cap L_{\mathcal{E}}(\mu)$. We denote by $\mathcal{J}_z^{\mathcal{F}}$ the class of Jensen measure for $\mathcal{F}$ with barycenter z.

Analogue to the fact for scalar valued Jensen measures, the measure $\delta_z I_{\mathcal{E}}$ belongs to $\mathcal{J}_z^{\mathcal{F}}$. We also note that if a measure μ belongs to $\mathcal{J}_z^{\mathcal{F}}$, then the support of μ, which is the smallest closed set $C \subset K$ so that $|\mu|(K \setminus C) = 0$, contains the barycenter z.

For given $x \in \mathcal{E}$ and $\mu \in \mathcal{J}_z^{\mathcal{F}}$, $\int_K x d\mu - x$ and $\int_K -x d\mu + x$ belong to the positive cone $\mathcal{E}^+$ and so $\int_K x d\mu = x$. In other words, $\mu(K) = I_{\mathcal{E}}$ and then $|\mu|(K) \geq 1$. Let $\{A_i : i \in \mathcal{I}\}$ be a finite collection of pairwise disjoint Borel subsets of K so that $\cup_{i\in\mathcal{I}} A_i = K$. Since A_i are pairwise disjoint, there exists only one $i_0 \in \mathcal{I}$ so that $z \in A_{i_0}$. Then, $x = \int_K x d\mu = \sum_i \int_K \chi_{A_i} x d\mu = \mu(A_{i_0})x$ which implies that $\mu(K) = \mu(A_{i_0}) = I_{\mathcal{E}}$. Considering the definition of variation of μ over K, we conclude that $|\mu|(K) = 1$.

4.2 Envelope Functions

Let $\varphi : K \to \mathcal{E}$ be a function with $\mathcal{T}[\varphi] = \{u \in \mathcal{F} : u \preceq \varphi \text{ on } K\} \neq \varnothing$. We define the upper envelope of φ at $z \in K$ by

$$S^{\mathcal{F}}\varphi(z) = \curlyvee\{u(z) : \ u \in \mathcal{F},\ u \preceq \varphi \text{ on } K\},$$

and in addition if φ is a Borel function, we define its lower envelope at $z \in K$ by

$$I^{\mathcal{F}}\varphi(z) = \curlywedge\left\{ \int_K \varphi d\mu : \mu \in \mathcal{J}_z^{\mathcal{F}} \right\}.$$

where the infimum is taken over all $\mu \in \mathcal{J}_z^{\mathcal{F}}$ so that φ is μ-integrable.

As $\mathcal{E}$ is conditionally complete, $S^{\mathcal{F}}\varphi(z)$ exists in $\mathcal{E}$, and by its definition, $S^{\mathcal{F}}\varphi(z) \preceq \varphi(z)$ at any given $z \in K$. We also note that for two functions φ and ψ with $\varphi \preceq \psi$ on K and $\mathcal{T}[\varphi] \neq \varnothing$, their upper envelopes satisfy $S^{\mathcal{F}}\varphi \preceq S^{\mathcal{F}}\psi$ on K. Recall that φ is integrable with respect to $\delta_z I_{\mathcal{E}} \in \mathcal{J}_z^{\mathcal{F}}$. Notice also that $u(z) \preceq \int_K u d\mu \preceq \int_K \varphi d\mu$ holds for all measures $\mu \in \mathcal{J}_z$ so that φ is μ-integrable and for all $u \in \mathcal{T}[\varphi] \cap L_{\mathcal{E}}(\mu)$. Hence, the lower envelope of φ evaluated at a given point$z \in K$, $I^{\mathcal{F}}\varphi(z)$, always exists in $\mathcal{E}$ and we have the relation $S^{\mathcal{F}}\varphi(z) \preceq I^{\mathcal{F}}\varphi(z) \preceq \varphi(z), z \in K$. Lastly, for any $\varphi, \psi : K \to \mathcal{E}$ such that $\mathcal{T}[\varphi] \neq \varnothing$ and $\mathcal{T}[\psi] \neq \varnothing$, $S^{\mathcal{F}}\varphi + S^{\mathcal{F}}\psi \preceq S^{\mathcal{F}}(\varphi + \psi)$ and $S^{\mathcal{F}}(\alpha\varphi) = \alpha S^{\mathcal{F}}\varphi, \alpha \geq 0$, on K .

4.3 Edwards' Theorem in Banach Lattices

The classical Edwards' Theorem states that if $\mathcal{F}$ is cone of real valued upper semi-continuous functions on a compact metric space K which contains real constant functions, and $\varphi : K \to \mathbb{R}$ is a bounded lower semi-continuous function, then $S^{\mathcal{F}}\varphi(z) = I^{\mathcal{F}}\varphi(z)$ for any point $z \in K$. In regards to the classical Edwards' Theorem, we refer to Edwards (1966), Gamelin (1978) and Wikström (2001).

Proof of Theorem 1.1. Let us assume that $\mathcal{F} \subset o\text{-}USC(K;\mathcal{E})$ is a cone of functions with the constant function property.

We first consider the case where $\varphi \in o\text{-}C(K;\mathcal{E}) = C_{ob}(K;\mathcal{E})$. Without loss of generality, assume that $\varphi \preceq x \precneqq 0$ for some $x \in -\mathcal{E}^+$. Let z be an arbitrary point in K. Let us define the linear map $S : \mathcal{Z} = span\{\varphi\} \to \mathcal{E}$ given by $t\varphi \mapsto tS^{\mathcal{F}}\varphi(z), t \in \mathbb{R}$, which satisfies $S^{\mathcal{F}}(t\varphi)(z) \preceq S(t\varphi), t \in \mathbb{R}$. For $t_1, t_2 \in \mathbb{R}$, the function $t_1\varphi \curlyvee t_2\varphi = \min\{t_1, t_2\}\varphi \in \mathcal{Z}$. Thus, $\mathcal{Z}$ is a Banach sublattice of $C(K;\mathcal{E})$. Note that the positive cone of $\mathcal{Z}$ is $\mathcal{Z}^+ = \{-t\varphi : t \geq 0\}$. Hence for $t > 0$, $S(-t\varphi) = -tS\varphi \succeq 0$, and so $S : \mathcal{Z} \to \mathcal{E}$ is a positive linear operator. By Proposition 2.1, we can extend S to a linear operator $\tilde{S}$ to $C(K;\mathcal{E})$ so that $\tilde{S}\psi \succeq S^{\mathcal{F}}\psi(z)$ for every $\psi \in C_{ob}(K;\mathcal{E})$. Consequently, $\tilde{S} : C(K;\mathcal{E}) \to \mathcal{E}$ is a positive operator. By Proposition 2.2, there exists a unique positive measure $\mu \in \mathcal{M}(K;B(\mathcal{E}))$ such that

$$\tilde{S}\psi = \int_K \psi d\mu, \, \psi \in C(K;\mathcal{E}).$$

We now show that $\mu \in \mathcal{J}_z^{\mathcal{F}}$. Let $u \in \mathcal{F} \cap L_{\mathcal{E}}(\mu)$ be given. By Proposition 3.3, there exists a monotone decreasing sequence $\{u_k\}$ in $C_{ob}(K;\mathcal{E})$ such that $u_k \downarrow u$. Note that

$$\text{(4.1)} \qquad \int_K u_1 d\mu \succeq \tilde{S}u_k = \int_K u_k d\mu \succeq S^{\mathcal{F}}u_k(z) \succeq S^{\mathcal{F}}u(z) = u(z), k \in \mathbb{N}.$$

Since $\{\int_K u_k d\mu\}$ is an order bounded monotone decreasing sequence in $\mathcal{E}$, by Lemma 2.7 and Lemma 2.10, we deduce that the sequence is convergent in the norm in $\mathcal{E}$. Since $u \in L_{\mathcal{E}}(\mu)$, the function $\Phi(\cdot) = \sup\{\|u(\cdot)\|, \|u_1(\cdot)\|\}$ is $|\mu|$-integrable on K, and by Propososition 2.3, $\lim_k \int_K u_k d\mu = \int_K u d\mu$. Thus, $\int_K u d\mu \succeq u(z)$ and $\mu \in \mathcal{J}_z^{\mathcal{F}}$. This result further shows that $S^{\mathcal{F}}\varphi(z) = \tilde{S}\varphi = \int_K \varphi d\mu = I^{\mathcal{F}}\varphi(z)$.

Let us assume that $\mathcal{F} \subset o\text{-}USC(K;\mathcal{E})$ is a cone with the constant function property and $\widetilde{\mathcal{F}}$ is the cone obtained by taking the closure of $\mathcal{F}$ with respect to the topology of uniform convergence on K. By the argument above, we already know that $S^{\mathcal{F}}\varphi = I^{\mathcal{F}}\varphi$ on K. Trivially, $S^{\mathcal{F}}\varphi \preceq S^{\widetilde{\mathcal{F}}}\varphi$, $\mathcal{J}_z^{\widetilde{\mathcal{F}}} \subset \mathcal{J}_z^{\mathcal{F}}$ for any $z \in K$ and $I^{\mathcal{F}}\varphi \preceq I^{\widetilde{\mathcal{F}}}\varphi$ on K. Let $\mu \in \mathcal{J}_z^{\mathcal{F}}$ be arbitrary. For any fixed $u \in \widetilde{\mathcal{F}} \cap L_{\mathcal{E}}(\mu)$, there exists a sequence $\{u_k\}$ in $\mathcal{F}$ converging u uniformly on K. We recall that by Proposition 3.1, u_k are μ-measurable. The sequence $\{u_k - u\}$ is uniformly bounded on K, that is,

$$C = \sup_{k \in \mathbb{N}} \sup_{z \in K} \|u_k(z) - u(z)\| < \infty.$$

Then, for each k, the inequality $\|u_k(\cdot)\| \leq \|u(\cdot)\| + \|u_k(\cdot) - u(\cdot)\| \leq \|u(\cdot)\| + C$ implies that $u_k \in L_{\mathcal{E}}(\mu)$. Since $\mathcal{E}^+$ is closed with respect to the norm topology on $\mathcal{E}$ (cf. (Meyer-Nieberg, 1991, Proposition 1.1.6)) , $0 \preceq \int_K u_k d\mu - u_k(z)$ implies that $0 \preceq$

$\int_K u d\mu - u(z)$. This means that the Jensen measure classes $\mathcal{J}_z^{\widetilde{\mathcal{F}}}$ and $\mathcal{J}_z^{\mathcal{F}}$ coincide for every $z \in K$, and therefore $I^{\mathcal{F}}\varphi = I^{\widetilde{\mathcal{F}}}\varphi$ on K. In conclusion, $S^{\widetilde{\mathcal{F}}}\varphi \preceq I^{\widetilde{\mathcal{F}}}\varphi = I^{\mathcal{F}}\varphi = S^{\mathcal{F}}\varphi$ on K.

□

Remark 4.1. *The proof of the classical Edwards' Theorem for scalar valued functions employs the fact that Jensen measures for a given cone with barycenter z is a weak-$*$ compact set in order to extend the result of the theorem for bounded lower semi-continuous functions. One may extend Edwards' Theorem in Banach lattices by introducing a weak-$*$ convergence notion for the space $\mathcal{M}(K;B(\mathcal{E}))$. Note that for a given sequence $\{\mu_i\}$ in $\mathcal{J}_z^{\mathcal{F}}$ and given $x' \in \mathcal{E}'_+$, there corresponds a sequence $\{\mu_{i,x}\}$ in $\mathcal{M}(K;\mathcal{E}'_+)$ with a weak-$*$ convergent subsequence. If one can prove that there exists a subsequence independent of the choice of x', then we can extend our proof for Edwards' Theorem in Banach lattices for lower semi-continuous functions φ.*

5. VECTOR VALUED SUBHARMONIC FUNCTIONS AND A DIRICHLET PROBLEM

5.1 Harmonic Functions and Dirichlet Problem

Let $(\mathcal{E}, \|\cdot\|)$ be a real Banach space and Ω an open subset of $\mathbb{R}^n, n \geq 1$.

Definition 5.1. *We denote by $C^1(\Omega;\mathcal{E})$ the space of all functions $u : \Omega \to \mathcal{E}$ so that the partial derivatives*

$$\partial_j f(x) = \lim_{h \to 0,\, h \in \mathbb{R}} \frac{f(x + hi_j) - f(x)}{h}$$

exist (in $\mathcal{E}$) for every $x \in \Omega$ and $\partial_j f : \Omega \to \mathcal{E}$ is continuous for each $j = 1, 2, \ldots, n$. Here $i_j \in \mathbb{R}^n$ is the j-th canonical vector. We let

$$C^2(\Omega;\mathcal{E}) = \{f \in C^1(\Omega;\mathcal{E}) : \partial_j f \in C^1(\Omega;\mathcal{E}), j = 1, 2, \ldots, n\}.$$

For $f \in C^2(\Omega;\mathcal{E})$, we define the Laplacian of f by

$$\Delta f = \sum_{j=1}^{n} \partial_j^2 f.$$

For given $x' \in \mathcal{E}'$, $x'(\Delta f) = \Delta(x' \circ f)$. A function $f \in C^2(\Omega;\mathcal{E})$ is said to be harmonic if $\Delta f = 0$ in Ω. We also say that a function $f : \Omega \to \mathcal{E}$ is weakly harmonic if $x' \circ f : \Omega \to \mathbb{R}$ is harmonic for all $x' \in \mathcal{E}'$.

We denote by $H(\Omega;\mathcal{E})$ and $H_b(\Omega;\mathcal{E})$ the classes of harmonic and norm bounded harmonic functions on Ω with values in $\mathcal{E}$, respectively.

Lemma 5.1. *(Arendt, 2016, Lemma 5.1, Proposition 5.3)*

(i) If $u \in \mathcal{H}(\Omega;\mathcal{E})$ *then Poisson's Integral Formula*

$$u(x) = \frac{1}{r_0\sigma_{n-1}(B(x_0,r_0))}\int_{\partial B(x_0,r_0)} \frac{r_0^2 - |x-x_0|^2}{|x-s|^d} u(s)\, d\sigma_{n-1}(s)$$

holds for all $x_0 \in \Omega$ *and* $r_0 > 0$ *such that* $B(x_0,r_0) \Subset \Omega$ *where* σ_{n-1} *is the surface measure.*

(ii) If bounded net $\{u_i\}_{i\in I} \subset H(\Omega;\mathcal{E})$ *converges pointwise to a function* $u : \Omega \to \mathcal{E}$*, then* $\{u_i\}_{i\in I}$ *converges uniformly on compact subsets of* Ω *and the limit function* u *is a harmonic function.*

As a result of (i), a function $u : \Omega \to \mathcal{E}$ is harmonic if and only if it is weakly harmonic (c.f (Arendt, 2016, Theorem 5.2)). Moreover, by (ii), we observe that $H_b(\Omega;\mathcal{E})$ with the topology of uniform convergence on compact subsets of Ω is a Banach space.

Definition 5.2. *Let* Ω *be a domain in* $\mathbb{R}^n$ *and* $\zeta \in \partial\Omega$. *A barrier function* b *at* ζ *is a (scalar valued) subharmonic function defined on* $\Omega \cap B$*, where* B *is an open neighborhood of* ζ*, such that* $b < 0$ *on* $\Omega \cap B$ *and* $\lim_{z\to\zeta} b(z) = 0$. *If there exists a barrier function at a point* $\zeta \in \partial\Omega$*, then we say that* ζ *is a (Dirichlet) regular boundary point. We denote by* $\partial_r\Omega$ *the set of all regular boundary points of* Ω.

Lemma 5.2. *(Kreuter, 2020, Theorem 1.1) Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain and* $(\mathcal{E}, \|\cdot\|)$ *a real Banach space. Then, for any given given boundary data* $f \in C(\partial\Omega;\mathcal{E})$*, there exists a unique solution* H_f *of the Dirichlet problem with*

$$\begin{cases} u \in H_b(\Omega;\mathcal{E}), \\ \lim_{\Omega\ni z\to\zeta} u(z) = f(\zeta) \text{ for all } \zeta \in \partial_r\Omega. \end{cases} \tag{5.1}$$

Kreuter (2020) proposes the following definition of subharmonicity for vector valued functions:

Definition 5.3. *(Kreuter, 2020, p. 733) Let* $(\mathcal{E}, \preceq, \|\cdot\|)$ *be a Banach lattice and* $\Omega \subset \mathbb{R}^n$ *an open set. A function* $u \in C(\Omega;\mathcal{E})$ *is called subharmonic in the Kreuter sense if for all* $z_0 \in \Omega$*, there exists some* $R > 0$ *so that* $B(z_0,R) \Subset \Omega$ *and*

$$u(z_0) \preceq \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{it})dt$$

holds for every $r \in (0,R)$. *Let* $SH_K(\Omega;\mathcal{E})$ *be the cone of functions* $u : \Omega \to \mathcal{E}$ *subharmonic in the Kreuter sense.*

Definition 5.4. *For given* $f \in C(\partial\Omega;\mathcal{E})$*, a continuous subsolution (supersolution) of the Dirichlet problem* (5.1) *with boundary data* f *is a function* $v \in SH_K(\Omega;\mathcal{E}) \cap C(\overline{\Omega};\mathcal{E})$ $(-SH_K(\Omega;\mathcal{E}) \cap C(\overline{\Omega};\mathcal{E})$*, resp.)* *such that* $v \preceq f$ $(v \succeq f$*, resp.)* *on* $\partial\Omega$.

We will denote the set of continuous sub/supersolutions for f by $\mathcal{CS}_f^-$ and $\mathcal{CS}_f^+$ respectively,

$$\mathcal{CS}_f^- = \{v \in SH_K(\Omega;\mathcal{E}) \cap C(\overline{\Omega};\mathcal{E}) : v \preceq f \text{ on } \partial\Omega\},$$
$$\mathcal{CS}_f^+ = \{v \in -SH_K(\Omega;\mathcal{E}) \cap C(\overline{\Omega};\mathcal{E}) : v \succeq f \text{ on } \partial\Omega\}.$$

In the real valued case, $\mathcal{CS}_f^-$ and $\mathcal{CS}_f^+$ are always nonempty since $\min_{\partial\Omega} f \in \mathcal{CS}_f^-$ and $\max_{\partial\Omega} f \in \mathcal{CS}_f^+$. We recall, as we have seen in Example 3.2, that a norm continuous function on a compact set may or may not be order bounded above/ below while being norm bounded. This leads us to the following open question that is first asked in Kreuter (2020):

Question 5.1. *Is there a function $f \in C(\partial\Omega;\mathcal{E})$ for some open and bounded $\Omega \subset \mathbb{R}^n$ and some Banach lattice $\mathcal{E}$ so that both $\mathcal{CS}_f^-$ and $\mathcal{CS}_f^+$ are empty?*

Kreuter (2020) made the assumption that both $\mathcal{CS}_f^-$ and $\mathcal{CS}_f^+$ are nonempty in order to develop a Perron method, and paid particular attention to the case of order bounded boundary data functions. For a function $f \in C(\partial\Omega;\mathcal{E})$ with $\mathcal{CS}_f^{\pm} \neq \varnothing$, if the pointwise supremum $\curlyvee\{u(\cdot) : u \in \mathcal{CS}_f^-\}$ and the pointwise infimum $\curlywedge\{u(\cdot) : u \in \mathcal{CS}_f^+\}$ exists in $\mathcal{E}$, Kreuter (2020) gives a Perron solution of (5.1) under suitable conditions via the functions

$$\underline{H}_f(\cdot) = \curlyvee\{u(\cdot) : u \in \mathcal{CS}_f^-\}, \; \overline{H}_f(\cdot) = \curlywedge\{u(\cdot) : u \in \mathcal{CS}_f^+\}.$$

5.2 Vector valued Subharmonic Functions and Perron Method

The Perron method for real valued Dirichlet Problems is to define a solution as the pointwise supremum/infimum of sub/supersolutions and it is clear that the method relies of the usual order on $\mathbb{R}$. An analogue method for vector valued Dirichlet problems may be possible if $\mathcal{E}$ is equipped with a suitable order relation and a topological structure on the space $\mathcal{E}$ compatible with the the order relation. Hence, we assume that $(\mathcal{E}, \preceq, \|\cdot\|)$ is a Banach lattice with order continuous norm. Let us give the following definitions of subharmonicity for vector valued functions.

Definition 5.5. *Let Ω be an open subset of $\mathbb{R}^n$, $(\mathcal{E}, \preceq, \|\cdot\|)$ a Banach lattice with order continuous norm.*

(i) A function $u \in o\text{-}USC(\Omega;\mathcal{E})$ is said to be order subharmonic if it satisfies the

submean inequality property, that is for all $z_0 \in \Omega$, *there exists some* $R > 0$ *so that* $B(z_0, R) \Subset \Omega$ *so that the following conditions hold;*

(a) The Bochner integrals

$$\int_0^{2\pi} u(z_0 + re^{it})dt$$

exist for every $r \in (0, R)$;

(b)

$$u(z_0) \preceq \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{it})dt$$

holds for every $r \in (0, R)$.

We denote by o-$SH(\Omega;\mathcal{E})$ *the cone of order subharmonic functions.*

(ii) We say that a function $u : \Omega \to \mathcal{E}$ *is subharmonic if there exists a sequence* $\{u_k\} \subset$ *o-*$SH(\Omega;\mathcal{E})$ *that converges to* u *uniformly on compact subsets of* Ω. *The cone of subharmonic functions is denoted by* $SH(\Omega;\mathcal{E})$.

We note that *o-*$SH(\Omega;\mathcal{E})$ and $SH(\Omega;\mathcal{E})$ are cones of upper semi-continuous functions with the constant function property. We denote by $SH(\Omega;\mathbb{R})$ the cone of real valued subharmonic functions on Ω. The results below have proofs analogue to the ones for plurisubharmonic functions. Thus, we omit their proofs.

Definition 5.6. *Let* Ω *be a bounded domain in* $\mathbb{K}^n$ *(*$\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$*) and* $u : \Omega \to \mathcal{E}$ *a function order bounded above. We define* u^*, *the upper semi-continuous regularization of* u *on* $\overline{\Omega}$, *by the following*

$$u^*(w) = \limsup_{\Omega \ni z \to w} u(z), w \in \overline{\Omega},$$

whenever it is possible.

Lemma 5.3. *Let* Ω *be an open subset of* $\mathbb{R}^n$ *and* $(\mathcal{E}, \preceq, \|\cdot\|)$ *a Banach lattice with order continuous norm. Then, the following results hold;*

(i) Subharmonicity is preserved under uniform convergence on compact subsets of Ω.

(ii) Let Ω *be a bounded domain in* $\mathbb{R}^n$ *and* $\mathcal{F}$ *the cone o-*$USC(\overline{\Omega};\mathcal{E}) \cap$ *o-*$SH(\Omega;\mathcal{E})$. *For every* $\varphi \in C(\overline{\Omega};\mathcal{E})$, *the family* $\mathcal{T}[\varphi] = \{u \in \mathcal{F} : u \preceq \varphi$ *on* $\overline{\Omega}\}$ *is non-empty. If* $\varphi \in C_{oba}(\overline{\Omega};\mathcal{E})$, *the function* $S^{\mathcal{F}}\varphi$ *belongs to* $\mathcal{T}[\varphi]$. *For any* $\varphi \in$ *o-*$C(\overline{\Omega};\mathcal{E})$, $S^{\mathcal{F}}\varphi = I^{\mathcal{F}}\varphi$ *on* $\overline{\Omega}$.

(iii) Let Ω *be a bounded domain in* $\mathbb{R}^n$. *Then, for every* $\varphi \in C(\partial\Omega;\mathcal{E})$, *the family* $\mathcal{U}[\varphi] = \{u \in \mathcal{F} : u \preceq \varphi$ *on* $\partial\Omega\}$ *is non-empty. If* $\varphi \in C_{oba}(\partial\Omega;\mathcal{E})$, *the Perron envelope*

$$\mathbf{H}\varphi(z) = \curlyvee\{u(z) : u \in \mathcal{U}[\varphi]\}, z \in \Omega,$$

is well-defined and $(\mathbf{H}\varphi)^*$ *belongs to* $\mathcal{U}[\varphi]$.

Let Ω be a bounded domain in $\mathbb{R}^n$. Let $\varphi, \psi \in C_{ob}(\partial\Omega;\mathcal{E})$ and

$$\alpha = \curlyvee_{z\in\partial\Omega}|\varphi(z)-\psi(z)|.$$

Hence, both $\mathbf{H}\varphi$ and $\mathbf{H}\psi$ are well defined functions on Ω. Since $|\varphi-\psi| \preceq \alpha$ on $\partial\Omega$, we have $|\mathbf{H}\varphi(z)-\mathbf{H}\psi(z)| \preceq \alpha$ for $z\in\Omega$. Then,

$$\|\mathbf{H}\varphi-\mathbf{H}\psi\|_{\mathcal{E},\Omega} \leq \|\alpha\| = \|\varphi-\psi\|_{\mathcal{E},\partial\Omega}. \tag{5.2}$$

The following lemma is a slight adaptation of (Kreuter, 2020, Lemma 1.7) to Banach lattice settings. Hence, we omit its proof.

Lemma 5.4. *Let K be a compact space, $(\mathcal{E},\preceq,\|\cdot\|)$ a Banach lattice. If $W \subset C(K;\mathbb{R}^+)$ and $X \subset \mathcal{E}^+$ are dense in $C(K;\mathbb{R}^+)$ and $\mathcal{E}^+$, respectively, then the set $\{f \in C(K;\mathcal{E}^+) : f = \sum_{j=1}^n f_j x_j, f_j \in W, x_j \in X\}$ is dense in $C(K;\mathcal{E}^+)$.*

We now prove part (i) of Theorem 1.4.

Proof of (i) of Theorem 1.4. We first note that the uniqueness of the solution follows from Lemma 5.2.

Suppose first that φ is of the form

$$\varphi = \sum_{n=1}^{N} f_n x_n \tag{5.3}$$

where $f_n \in C(\partial\Omega;\mathbb{R}^+)$ and $x_n \in \mathcal{E}^+$. Notice that φ is order bounded. Consider the function $\mathbf{\Phi} = \sum_{n=1}^{N}\mathbf{H}f_n x_n$, where

$$\mathbf{H}f_n(z) = \sup\{u(z) : u \in SH(\Omega;\mathbb{R})\cap USC(\overline{\Omega};\mathbb{R}), u \leq f_n \text{ on } \partial\Omega\},$$

$z\in\Omega$. $\mathbf{\Phi}^*$ is a function that belongs to $\mathcal{U}[\varphi]$. Notice also that $\mathbf{\Phi} \in H_b(\Omega;\mathcal{E})$ is the unique solution of the Dirichlet problem (1.2). Then, $\mathbf{\Phi}^* \preceq (\mathbf{H}\varphi)^*$ on $\overline{\Omega}$ and $(\mathbf{H}\varphi)^*$ satisfies the boundary condition $(\mathbf{H}\varphi)^*(\zeta) = \text{o-}\lim_{\Omega\ni z\to\zeta}\mathbf{H}\varphi(z) = \varphi(\zeta), \zeta \in \partial_r\Omega$. For any $u \in \mathcal{U}[\varphi]$, $u \preceq \mathbf{\Phi}$ on Ω. Then, $\mathbf{H}\varphi \preceq \mathbf{\Phi}$ on Ω and so $\mathbf{H}\varphi = \mathbf{\Phi}$ on Ω. This means that $\mathbf{H}\varphi$ is the unique solution of the problem (1.2).

Let us consider the case $\varphi \in C_{ob}(\partial\Omega;\mathcal{E})$. Without loss of generality, assume that $\varphi \succeq 0$. Then, by Lemma 5.4, there exists a sequence $\{\varphi_n\}$ in $C(\partial\Omega;\hat{\mathcal{E}}^+)$ of the form (5.3) converging φ uniformly on $\partial\Omega$ where $\hat{\mathcal{E}} = \overline{(span\varphi(\partial\Omega))^{\curlyvee\curlywedge}}$. By the inequality (5.2),

$$\|\mathbf{H}\varphi-\mathbf{H}\varphi_n\|_{\mathcal{E},\Omega} \leq \|\varphi-\varphi_n\|_{\mathcal{E},\partial\Omega}.$$

Then, $\mathbf{H}\varphi$ is in $H_b(\Omega;\mathcal{E})$. For given $\zeta \in \partial_r\Omega$ and $z \in \Omega$,

$$\|\mathbf{H}\varphi(z)-\varphi(\zeta)\| \leq \|\mathbf{H}\varphi(z)-\mathbf{H}\varphi_n(z)\|+\|\mathbf{H}\varphi_n(z)-\varphi_n(\zeta)\|+\|\varphi_n(\zeta)-\varphi(\zeta)\|$$

holds for $n \in \mathbb{N}$. Therefore, $\lim_{\Omega \ni z \to \zeta} \mathbf{H}\varphi(z) = \varphi(\zeta)$ for every $\zeta \in \partial_r\Omega$. By Remark 2.1, $\text{o-}\lim_{\Omega \ni z \to \zeta} \mathbf{H}\varphi(z) = \varphi(\zeta)$ for every $\zeta \in \partial_r\Omega$. Thus, $\mathbf{H}\varphi$ is the unique solution of (1.2) with the boundary data φ.

We now examine the case where $\varphi \in C_{oba}(\partial\Omega;\mathcal{E})$. By Lemma 3.4 and Proposition 3.2, there exists a monotone decreasing sequence $\{\varphi_n\}$ in $C_{ob}(\partial\Omega;\mathcal{E})$ that uniformly converges in the norm to φ on $\partial\Omega$. Since each φ_n is of the class $C_{ob}(\partial\Omega;\mathcal{E})$, each corresponding Dirichlet problem (1.2) has the unique solution $\mathbf{H}\varphi_n$. Then, by the inequality (5.2), we have for $n, m \in \mathbb{N}$ that

$$\|\mathbf{H}\varphi_n(z)-\mathbf{H}\varphi_m(z)\| \leq \sup_{w\in\partial\Omega} \|\varphi_n(w)-\varphi_m(w)\|, z \in \Omega.$$

This implies that $\{\mathbf{H}\varphi_n\}$ is a monotone decreasing, uniformly Cauchy sequence of functions in $H_b(\Omega;\mathcal{E})$. We prove that

$$\mathbf{H}\varphi(z) = \lim_{n\to\infty} \mathbf{H}\varphi_n(z) = \curlywedge_n \mathbf{H}\varphi_n(z),$$

$z \in \Omega$, in a way identical to the proof of Theorem 1.3. Hence, $\mathbf{H}\varphi$ is in $H_b(\Omega;\mathcal{E})$. For $n \in \mathbb{N}$,

$$|\mathbf{H}\varphi(z)-\varphi(\zeta)| \preceq \mathbf{H}\varphi_n(z)-\mathbf{H}\varphi(z)+|\mathbf{H}\varphi_n(z)-\varphi_n(\zeta)|+\varphi_n(\zeta)-\varphi(\zeta).$$

This means that $\mathbf{H}\varphi$ is the unique solution of (1.2) with the boundary data φ.

If $\varphi \in C(\partial\Omega;\mathcal{E}) \setminus C_{oba}(\partial\Omega;\mathcal{E})$, there exists a monotone increasing sequence $\{\varphi_n\} \subset C_{oba}(\partial\Omega;\mathcal{E})$ so that φ_n uniformly converges to φ on $\partial\Omega$. Since

$$\|\mathbf{H}\varphi_n-\mathbf{H}\varphi_m\|_{\mathcal{E},\Omega} \leq \|\varphi_n-\varphi_m\|_{\mathcal{E},\partial\Omega},$$

the function $\mathbf{H} = \curlyvee_n \mathbf{H}\varphi_n$, which is also the uniform limit of $\mathbf{H}\varphi_n$ on Ω, belongs to $H_b(\Omega;\mathcal{E})$. For given $\zeta \in \partial_r\Omega$ and $z \in \Omega$,

$$|\mathbf{H}(z)-\varphi(\zeta)| \preceq |\mathbf{H}(z)-\mathbf{H}\varphi_n(z)|+|\mathbf{H}\varphi_n(z)-\varphi_n(\zeta)|+|\varphi_n(\zeta)-\varphi(\zeta)|$$

holds for $n \in \mathbb{N}$. Thus, $\text{o-}\lim_{\Omega \ni z \to \zeta} \curlyvee_n \mathbf{H}\varphi_n(z) = \varphi(\zeta)$. In conclusion, $\curlyvee_n \mathbf{H}\varphi_n$ is the unique solution of (1.2) with the boundary data φ.

Since $\hat{\mathcal{E}}$ is a Banach lattice with order continuous norm, by following the steps above, we prove that the function

$$\begin{cases} \hat{\mathbf{H}}\varphi, \text{ if } \varphi \in C_{oba}(\partial\Omega;\mathcal{E}), \\ \curlyvee_n \hat{\mathbf{H}}\tilde{\varphi}_n, \text{ if } \varphi \in C(\partial\Omega;\mathcal{E}) \setminus C_{oba}(\partial\Omega;\mathcal{E}), \end{cases}$$

is a solution of (1.2). By Lemma 5.2, it follows that these two solutions of (1.2) coincide. □

6. VECTOR VALUED PLURISUBHARMONIC FUNCTIONS AND A DIRICHLET PROBLEM

6.1 Plurisubharmonic Functions

In this section, we will introduce plurisubharmonic vector valued functions. We refer the reader to Klimek Klimek (1991) for all the corresponding notions and results in the classical potential/pluripotential theory.

Definition 6.1. *Let Ω be an open subset of $\mathbb{C}^n$, $(\mathcal{E}, \preceq, \|\cdot\|)$ a Banach lattice with order continuous norm.*

(i) A function $u \in o\text{-}USC(\Omega; \mathcal{E})$ is said to be order plurisubharmonic on Ω if

(a) The Bochner integrals

$$\int_0^{2\pi} u(z_0 + e^{it}\omega)dt$$

exists for every $z_0 \in \Omega$ and $\omega \in \mathbb{C}^n$ such that $\{z_0 + \lambda\omega : \lambda \in \mathbb{C}, |\lambda| \leq 1\} \subset \Omega$;

(b)

$$u(z_0) \preceq \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + e^{it}\omega)dt$$

holds for every $z_0 \in \Omega$ and $\omega \in \mathbb{C}^n$ such that $\{z_0 + \lambda\omega : \lambda \in \mathbb{C}, |\lambda| \leq 1\} \subset \Omega$.

We denote by $o\text{-}PSH(\Omega; \mathcal{E})$ the class of $\mathcal{E}$-valued order plurisubharmonic functions on Ω.

(ii) A function $u : \Omega \to \mathcal{E}$ is said to be plurisubharmonic if it is the uniform limit of a sequence $\{u_k\}$ in $o\text{-}PSH(\Omega; \mathcal{E})$ on compact subsets of Ω. We denote by $PSH(\Omega; \mathcal{E})$ the class of $\mathcal{E}$-valued plurisubharmonic functions on Ω.

One observes that a function $u : \Omega \to \mathcal{E}$ is plurisubharmonic if the mapping $\lambda \mapsto u(z_0 + \lambda\omega)$ is subharmonic on $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$. As in the classical theory, plurisubharmonicity coincides with subharmonicity in single complex variable and also plurisubharmonicity of a vector valued function is a local property. We note that o-$PSH(\Omega;\mathcal{E})$ and $PSH(\Omega;\mathcal{E})$ are cones of upper semi-continuous functions with the constant function property. Similarly, we denote by $PSH(\Omega;\mathbb{R})$ the cone of real valued plurisubharmonic functions on Ω.

Lemma 6.1. *Let Ω be an open subset of $\mathbb{C}^n$ and $(\mathcal{E},\preceq,\|\cdot\|)$ a Banach lattice with order continuous norm. Let $\{u_j\}$ be a sequence in $PSH(\Omega;\mathcal{E})$ so that u_j converges to a function $u:\Omega\to\mathcal{E}$ uniformly on compact subsets of Ω. Then $u\in PSH(\Omega;\mathcal{E})$.*

Proof. The function u is upper semi-continuous on Ω, since $USC(\Omega;\mathcal{E})$ is the closure of o-$USC(\Omega;\mathcal{E})$ with respect to the uniform convergence on compact subsets of Ω. By assumption, for each $j\in\mathbb{N}$,

$$u_j(z_0)\preceq\frac{1}{2\pi}\int_0^{2\pi}u_j(z_0+e^{it}\omega)dt$$

holds for every $z_0\in\Omega$ and $w\in\mathbb{C}^n$ so that $\{z_0+\lambda\omega:\lambda\in\mathbb{C},|\lambda|\leq 1\}\subset\Omega$. Since u_j uniformly converges to u on compact subsets of Ω, the Bochner integral

$$\int_0^{2\pi}u(z_0+e^{it}\omega)dt$$

exists for every $z_0\in\Omega$ and $w\in\mathbb{C}^n$ so that $\{z_0+\lambda\omega:\lambda\in\mathbb{C},|\lambda|\leq 1\}\subset\Omega$. As $\mathcal{E}^+$ is closed with respect to the norm topology on $\mathcal{E}$ (cf. (Meyer-Nieberg, 1991, Proposition 1.1.6)), by the uniform convergence argument, we conclude that

$$u(z_0)\preceq\frac{1}{2\pi}\int_0^{2\pi}u(z_0+e^{it}\omega)dt$$

holds for every $z_0\in\Omega$ and $w\in\mathbb{C}^n$ so that $\{z_0+\lambda\omega:\lambda\in\mathbb{C},|\lambda|\leq 1\}\subset\Omega$. Hence, u is in $PSH(\Omega;\mathcal{E})$. □

Let Ω be a bounded open subset of $\mathbb{C}^n$. Denote by $\mathcal{F}=o\text{-}PSH(\Omega;\mathcal{E})\cap o\text{-}USC(\overline{\Omega};\mathcal{E})$ and by $\widetilde{\mathcal{F}}$ its closure with respect to the topology of uniform convergence on $\overline{\Omega}$. For $\varphi:\overline{\Omega}\to\mathcal{E}$ such that $\mathcal{T}[\varphi]=\{u\in\mathcal{F}:u\preceq\varphi\text{ on }\overline{\Omega}\}$, we may further consider the family of functions $\widetilde{\mathcal{T}}[\varphi]=\{u\in\widetilde{\mathcal{F}}:u\preceq\varphi\text{ on }\overline{\Omega}\}$ that clearly contains $\mathcal{T}[\varphi]$.

Theorem 1.2 suggests a method for generating new plurisubharmonic function via a given function.

Proof of Theorem 1.2. First we consider the case that $\varphi\in C_{ob}(\overline{\Omega};\mathcal{E})$. Obviously, the upper envelope $S^{\mathcal{F}}\varphi(z)=\curlyvee\{u(z):u\in\mathcal{T}[\varphi]\}, z\in\overline{\Omega}$, is well-defined and

$$(S^{\mathcal{F}}\varphi)^*(z)=\curlywedge_{r>0}\curlyvee_{w\in B(z,r)}S^{\mathcal{F}}\varphi(w)\preceq\curlywedge_{r>0}\curlyvee_{w\in B(z,r)}\varphi(w)=\varphi(z)$$

for every $z \in \overline{\Omega}$, where $\mathcal{F} = o\text{-}PSH(\Omega;\mathcal{E}) \cap o\text{-}USC(\overline{\Omega};\mathcal{E})$. For any $z \in \overline{\Omega}$, $(S^{\mathcal{F}}\varphi)^*(z)$ exists and hence $(S^{\mathcal{F}}\varphi)^*$ is order upper semi-continuous on $\overline{\Omega}$. Since φ is order bounded, we have $\curlywedge_{z\in\overline{\Omega}}\varphi(z) \preceq (S^{\mathcal{F}}\varphi)^* \preceq \curlyvee_{z\in\overline{\Omega}}\varphi(z)$ on $\overline{\Omega}$. Therefore the Bochner integral

$$\frac{1}{2\pi}\int_0^{2\pi}(S^{\mathcal{F}}\varphi)^*(z_0+e^{it}\omega)dt$$

exists for every $z_0 \in \Omega$ and $\omega \in \mathbb{C}^n$ such that $\{z_0+\lambda\omega : \lambda \in \mathbb{C}, |\lambda| \leq 1\} \subset \Omega$. By (Klimek, 1991, Theorem 2.9.14) and the fact $x' \circ (S^{\mathcal{F}}\varphi)^* = (\sup\{x' \circ u : u \in \mathcal{T}[\varphi]\})^* \in PSH(\Omega;\mathbb{R})$ for every $x' \in \mathcal{E}'_+$, we infer that the function $(S^{\mathcal{F}}\varphi)^*$ is in $o\text{-}PSH(\Omega;\mathcal{E})$. Since $(S^{\mathcal{F}}\varphi)^* \preceq \varphi$ on $\overline{\Omega}$, $(S^{\mathcal{F}}\varphi)^*$ is in the collection $\mathcal{T}[\varphi]$ and so $(S^{\mathcal{F}}\varphi)^* = S^{\mathcal{F}}\varphi$. Hence, the upper envelope $S^{\mathcal{F}}\varphi$ is in the collection $\mathcal{T}[\varphi]$. Moreover, Theorem 1.1 shows that $S^{\mathcal{F}}\varphi$ coincides with the lower envelope $I^{\mathcal{F}}\varphi$ on $\overline{\Omega}$ if $\varphi \in C_{ob}(\overline{\Omega};\mathcal{E})$.

Let $\varphi,\psi \in C_{ob}(\overline{\Omega};\mathcal{E})$ and $\alpha = \curlyvee_{z\in\overline{\Omega}}|\varphi(z)-\psi(z)|$. We note that $\|\varphi-\psi\|_{\mathcal{E},\overline{\Omega}} = \|\alpha\|$. Thus, both the inequalities

$$S^{\mathcal{F}}(\varphi-\alpha)(z) = S^{\mathcal{F}}\varphi(z)-\alpha \preceq S^{\mathcal{F}}\psi(z),$$
$$S^{\mathcal{F}}(\psi-\alpha)(z) = S^{\mathcal{F}}\psi(z)-\alpha \preceq S^{\mathcal{F}}\varphi(z)$$

hold for every $z \in \overline{\Omega}$. Therefore, $|S^{\mathcal{F}}\varphi(z)-S^{\mathcal{F}}\psi(z)| \preceq \alpha$ for $z \in \overline{\Omega}$, and,

$$\|S^{\mathcal{F}}\varphi - S^{\mathcal{F}}\psi\|_{\mathcal{E},\overline{\Omega}} = \left\||S^{\mathcal{F}}\varphi - S^{\mathcal{F}}\psi|\right\|_{\mathcal{E},\overline{\Omega}} \leq \|\alpha\| = \|\varphi-\psi\|_{\mathcal{E},\overline{\Omega}}. \tag{6.1}$$

Let us consider the case where $\varphi \in C_{oba}(\overline{\Omega};\mathcal{E})$. Let $\mathcal{S} = \{s_j\}$ be a countable dense subset of $\hat{\mathcal{E}} = \overline{(span(\varphi(\overline{\Omega})))^{\curlyvee\curlywedge}}$, $x_m = \curlywedge_{j=1}^m s_j, m \in \mathbb{N}$. We know from our previous observations that φ is order upper semi-continuous on $\overline{\Omega}$. The monotone decreasing sequence $\{\varphi_m\}$ in $C_{ob}(\overline{\Omega};\mathcal{E})$, $\varphi_m(\cdot) = \varphi(\cdot) \curlyvee x_m$, uniformly converges in the norm to φ on $\overline{\Omega}$. By (6.1), we see that $\|S^{\mathcal{F}}\varphi_n - S^{\mathcal{F}}\varphi_m\|_{\mathcal{E},\overline{\Omega}} \leq \|\varphi_n - \varphi_m\|_{\mathcal{E},\overline{\Omega}}$ for every $n,m \in \mathbb{N}$ and as a result, $S^{\mathcal{F}}\varphi_n$ uniformly converges to the function $u = \curlywedge_n S^{\mathcal{F}}\varphi_n$ on $\overline{\Omega}$. It is obvious that $S^{\mathcal{F}}\varphi_n \preceq \varphi_n$ on $\overline{\Omega}$ and $\{(S^{\mathcal{F}}\varphi_n)\}$ is a monotone decreasing sequence of functions. Then, by Lemma 2.10, we deduce that $u \preceq \varphi$ on $\overline{\Omega}$. Lemma 3.2 shows that u is of the class $o\text{-}USC(\overline{\Omega};\mathcal{E})$. Moreover, as u is the uniform limit of $\{S^{\mathcal{F}}\varphi_n\}$, we also deduce that u belongs to $o\text{-}PSH(\Omega;\mathcal{E})$. Hence, the function u belongs to $\mathcal{F}$, and we further conclude that $u \in \mathcal{T}[\varphi]$. Therefore, the upper envelope $S^{\mathcal{F}}\varphi$ is well-defined on $\overline{\Omega}$. It is trivial that for any $n \in \mathbb{N}$, $\mathcal{T}[\varphi] \subset \mathcal{T}[\varphi_n]$. Then, for any $v \in \mathcal{T}[\varphi]$ and $n \in \mathbb{N}$, we have $v \preceq S^{\mathcal{F}}\varphi_n$ which further implies that $v \preceq u$ on $\overline{\Omega}$. Since v was arbitrary, we infer that $S^{\mathcal{F}}\varphi \preceq u$ and thus $S^{\mathcal{F}}\varphi = u$ on $\overline{\Omega}$.

Lastly, we study the case where $\varphi \in C(\overline{\Omega},\mathcal{E})$. Let $\mathcal{S} = \{s_j\}$ be a countable dense subset of $\hat{\mathcal{E}} = \overline{(span(\varphi(\overline{\Omega})))^{\curlyvee\curlywedge}}$, $\tilde{x}_m = \curlyvee_{j=1}^m s_j, m \in \mathbb{N}$. Define the functions $\tilde{\varphi}_m =$

$\varphi \curlywedge \tilde{x}_m \in C_{oba}(\overline{\Omega};\mathcal{E}), m \in \mathbb{N}$. Note that $\{\widetilde{\varphi}_m\}$ is a monotone increasing sequence that uniformly converges to φ on $\overline{\Omega}$ and each $\widetilde{\varphi}_m$ is order upper semi-continuous. As we proved in the previous case, $\mathcal{T}[\widetilde{\varphi}_m] \neq \varnothing$ and $(S^{\mathcal{F}}\widetilde{\varphi}_m)^* = S^{\mathcal{F}}\widetilde{\varphi}_m$ for each m. Clearly, $S^{\mathcal{F}}\widetilde{\varphi}_m \preceq \widetilde{\varphi}_m \preceq \varphi$ on $\overline{\Omega}$ for each m. Therefore, the family $\mathcal{T}[\varphi] \neq \varnothing$ and the upper envelope $S^{\mathcal{F}}\varphi$ is well-defined on $\overline{\Omega}$. □

6.1.1 Envelope Functions Generated by boundary data

Definition 6.2. *Let Ω be a domain in $\mathbb{C}^n$. We say that Ω is hyperconvex if there exists a negative (scalar valued) plurisubharmonic function u so that for every $c > 0$, the set $\{z \in \Omega : u(z) < -c\}$ is relatively compact in Ω.*
Definition 6.3. *Let Ω be a domain in $\mathbb{C}^n$. The Shilov boundary $S(\Omega)$ of Ω is the smallest closed subset of $\partial\Omega$ such that for every function f holomorphic in Ω and continuous in $\overline{\Omega}$,*

$$|f(\cdot)| \leq M \text{ on } S(\Omega) \Rightarrow |f(\cdot)| \leq M \text{ on } \Omega.$$

Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$. Let $\varphi : S(\Omega) \to \mathcal{E}$ be a function so that

$$\check{\mathcal{U}}[\varphi] = \{u \in \mathcal{F} : u \preceq \varphi \text{ on } S(\Omega)\}$$

is a non-empty class of functions, where $\mathcal{F} = o\text{-}PSH(\Omega;\mathcal{E}) \cap o\text{-}USC(\overline{\Omega};\mathcal{E})$. Whenever it is possible, we define $\check{\mathbf{P}}\varphi$, the Perron envelope of φ, as

$$\check{\mathbf{P}}\varphi(z) = \curlyvee \left\{u(z) : u \in \check{\mathcal{U}}[\varphi]\right\}, z \in \Omega.$$

We note that $\check{\mathbf{P}}\varphi + \check{\mathbf{P}}\psi \preceq \check{\mathbf{P}}(\varphi + \psi)$ for any functions $\varphi, \psi : S(\Omega) \to \mathcal{E}$ such that $\check{\mathcal{U}}[\varphi] \neq \varnothing$ and $\check{\mathcal{U}}[\psi] \neq \varnothing$ and whenever these Perron envelopes are well-defined.
Lemma 6.2. *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$ and $(\mathcal{E}, \preceq, \|\cdot\|)$ a Banach lattice with order continuous norm. Then, for any $\varphi \in C(S(\Omega);\mathcal{E})$, the family $\check{\mathcal{U}}[\varphi]$ is non-empty. If φ is order bounded above, then $\check{\mathbf{P}}\varphi$ is well-defined on Ω and it satisfies*

$$\check{\mathbf{P}}\varphi(z) \preceq \curlyvee_{w \in S(\Omega)} \varphi(w), z \in \Omega.$$

Proof. We first recall that the Shilov boundary $S(\Omega)$ is a closed subset of $\overline{\Omega}$. Then, by Proposition 3.1, we can find a function $\Phi \in C(\overline{\Omega};\mathcal{E})$ so that $\Phi = \varphi$ on $S(\Omega)$. By Theorem 1.2, we conclude that the family $\mathcal{T}[\Phi] = \{u \in \mathcal{F} : u \preceq \Phi \text{ on } \overline{\Omega}\} \neq \varnothing$ which is clearly a subset of $\check{\mathcal{U}}[\varphi]$.

By (Bremermann, 1959, Theorem 7.1), for given $\varphi \in C_{oba}(S(\Omega);\mathcal{E})$, $x' \in \mathcal{E}'_+$ and $u \in \check{\mathcal{U}}[\varphi]$, the inequality

$$x' \circ u(z) \leq \sup_{w \in S(\Omega)} x' \circ \varphi(w),$$

$z \in S(\Omega)$, implies that

$$x' \circ u(z) \leq \sup_{w \in S(\Omega)} x' \circ \varphi(w),$$

$z \in \Omega$. We recall that the functional $x' \in \mathcal{E}'_+$ is order continuous. Hence,

$$u(z) \preceq \curlyvee_{w \in S(\Omega)} \varphi(w),$$

$z \in \Omega$, and $\check{\mathbf{P}}\varphi$ is well-defined on Ω. Trivially,

$$\check{\mathbf{P}}\varphi(z) \preceq \curlyvee_{w \in S(\Omega)} \varphi(w), z \in \Omega.$$

□

Let $\varphi, \psi \in C_{ob}(S(\Omega);\mathcal{E})$ and $\alpha = \curlyvee_{z \in S(\Omega)} |\varphi(z) - \psi(z)|$. Both $(\check{\mathbf{P}}\varphi)^*$ and $(\check{\mathbf{P}}\psi)^*$ are well defined functions on $\overline{\Omega}$.

$$(\check{\mathbf{P}}(\varphi - \alpha))^*(z) = (\check{\mathbf{P}}\varphi)^*(z) - \alpha \preceq (\check{\mathbf{P}}\psi)^*(z),$$
$$(\check{\mathbf{P}}(\psi - \alpha))^*(z) = (\check{\mathbf{P}}\psi)^*(z) - \alpha \preceq (\check{\mathbf{P}}\psi)^*(z),$$

for every $z \in \overline{\Omega}$. Therefore, $|(\check{\mathbf{P}}\varphi)^*(z) - (\check{\mathbf{P}}\psi)^*(z)| \preceq \alpha$ for $z \in \overline{\Omega}$. Then,

$$\|(\check{\mathbf{P}}\varphi)^* - (\check{\mathbf{P}}\psi)^*\|_{\mathcal{E},\overline{\Omega}} = \left\| |(\check{\mathbf{P}}\varphi)^* - (\check{\mathbf{P}}\psi)^*| \right\|_{\mathcal{E},\overline{\Omega}} \leq \|\alpha\| = \|\varphi - \psi\|_{\mathcal{E},S(\Omega)}. \tag{6.2}$$

With this estimate at our disposal, we can now prove Theorem 1.3.

Proof of Theorem 1.3. Let $\varphi \in C_{ob}(S(\Omega);\mathcal{E})$ be given. Then, $\check{\mathbf{P}}\varphi$ is order bounded on $\overline{\Omega}$. By this result, its upper semi-continuous regularization $(\check{\mathbf{P}}\varphi)^*$ exists and order bounded on $\overline{\Omega}$. Since $(\check{\mathbf{P}}\varphi)^*$ is order upper semi-continuous and order bounded on $\overline{\Omega}$, the Bochner integral

$$\frac{1}{2\pi}\int_0^{2\pi} (\check{\mathbf{P}}\varphi)^*(z_0 + e^{it}w)dt$$

exists for every $z_0 \in \Omega$ and $w \in \mathbb{C}^n$ such that $\{z_0 + \lambda w : |\lambda| \leq 1\} \subset \Omega$. (Klimek, 1991, Theorem 2.9.14) and the fact that $x' \circ (\check{\mathbf{P}}\varphi)^* = (\sup\{x' \circ u : u \in \check{\mathcal{U}}[\varphi]\})^* \in PSH(\Omega;\mathbb{R})$ for every $x' \in \mathcal{E}'_+$ show that

$$(\check{\mathbf{P}}\varphi)^*(z_0) \preceq \frac{1}{2\pi}\int_0^{2\pi} (\check{\mathbf{P}}\varphi)^*(z_0 + e^{it}w)dt$$

for every $z_0 \in \Omega$ and $w \in \mathbb{C}^n$ such that $\{z_0 + \lambda w : |\lambda| \leq 1\} \subset \Omega$. With these results, we conclude that $(\check{\mathbf{P}}\varphi)^*$ is in $\mathcal{F}$. For given $u \in \check{\mathcal{U}}[\varphi]$ and given $x' \in \mathcal{E}'_+$, $x' \circ u \in PSH(\Omega;\mathbb{R})$ and $x' \circ u \leq x' \circ \varphi$ on $S(\Omega)$. Then,

$$\begin{aligned} x' \circ u(z) \leq & \check{\mathbf{P}}(x' \circ \varphi)(z) \\ & = \sup\{f(z) : f \in PSH(\Omega;\mathbb{R}) \cap USC(\overline{\Omega};\mathbb{R}), f \leq x' \circ \varphi \text{ on } S(\Omega)\} \end{aligned}$$

for $z \in \Omega$, and hence,

$$\begin{aligned} x'(\check{\mathbf{P}}\varphi)(z) = x'(\curlyvee\{u(z) : u \in \check{\mathcal{U}}[\varphi]\}) &= \sup\{x' \circ u(z) : u \in \check{\mathcal{U}}[\varphi]\} \\ &\leq \check{\mathbf{P}}(x' \circ \varphi)(z), \end{aligned}$$

$z \in \Omega$. For arbitrary $\zeta \in S(\Omega)$, we have that

$$x' \circ (\check{\mathbf{P}}\varphi)^*(\zeta) = x'(\limsup_{\Omega \ni z \to \zeta} \check{\mathbf{P}}\varphi(z)) = \limsup_{\Omega \ni z \to \zeta} x' \circ \check{\mathbf{P}}(z) \leq \limsup_{\Omega \ni z \to \zeta} \check{\mathbf{P}}(x' \circ \varphi)(z) = x' \circ \varphi(\zeta).$$

Since, $x' \in \mathcal{E}'_+$ was arbitrary, we deduce that

$$(\check{\mathbf{P}}\varphi)^*(\zeta) \preceq \varphi(\zeta) \tag{6.3}$$

for any given $\zeta \in S(\Omega)$. Thus, the function $(\check{\mathbf{P}}\varphi)^*$ is in the collection $\check{\mathcal{U}}[\varphi]$, and eventually $(\check{\mathbf{P}}\varphi)^* = \check{\mathbf{P}}\varphi$ on Ω.

Now let $\varphi \in C_{oba}(S(\Omega);\mathcal{E})$ be given. Lemma 6.2 shows that the Perron envelope $\check{\mathbf{P}}\varphi$ is order bounded above on Ω. By Proposition 3.2, there exists a monotone decreasing sequence $\{\varphi_n\}$ in $C_{ob}(S(\Omega);\mathcal{E})$ so that $\varphi_n \downarrow \varphi$, and eventually by Lemma 3.4, φ_n uniformly converges to φ on $S(\Omega)$. We note that for each fixed n, $(\check{\mathbf{P}}\varphi_n)^*$ is in the family $\check{\mathcal{U}}[\varphi_n]$. We also have the inequality $\|(\check{\mathbf{P}}\varphi_n)^* - (\check{\mathbf{P}}\varphi_m)^*\|_{\mathcal{E},\overline{\Omega}} \leq \|\varphi_n - \varphi_m\|_{\mathcal{E},S(\Omega)}$ for every n,m. This means that $\{(\check{\mathbf{P}}\varphi_n)^*\}$ is a uniformly Cauchy sequence of functions. Since $\{(\check{\mathbf{P}}\varphi_n)^*\}$ is a monotone decreasing sequence of functions, by Lemma 2.7 and Lemma 2.10, the pointwise limit $\lim_{n\to\infty}(\check{\mathbf{P}}\varphi_n)^*(z) = \curlywedge_n(\check{\mathbf{P}}\varphi_n)^*(z), z \in \overline{\Omega}$. Then, $\{(\check{\mathbf{P}}\varphi_n)^*\}$ uniformly converges to $\curlywedge_n(\check{\mathbf{P}}\varphi_n)^*$ on $\overline{\Omega}$. We further observe by Lemma 3.2 that $\curlywedge_n(\check{\mathbf{P}}\varphi_n)^*$ belongs to o-$USC(\overline{\Omega};\mathcal{E})$. Moreover, Lemma 6.1 shows that $\curlywedge_n(\check{\mathbf{P}}\varphi_n)^*$ is order plurisubharmonic on Ω. Thus, $\curlywedge_n(\check{\mathbf{P}}\varphi_n)^*$ belongs to the family $\mathcal{F}$. For given $k \in \mathbb{N}$, $\curlywedge_n(\check{\mathbf{P}}\varphi_n)^* \preceq (\check{\mathbf{P}}\varphi_k)^* \preceq \varphi_k$ on $S(\Omega)$. Since k is arbitrary, we deduce that $\curlywedge_n(\check{\mathbf{P}}\varphi_n)^* \preceq \varphi$ on $S(\Omega)$ and so $\curlywedge_n(\check{\mathbf{P}}\varphi_n)^* \in \check{\mathcal{U}}[\varphi]$. This implies that $\curlywedge_n \check{\mathbf{P}}\varphi_n \preceq \check{\mathbf{P}}\varphi$ on Ω. On the other hand, since $\check{\mathcal{U}}[\varphi] \subset \check{\mathcal{U}}[\varphi_n]$, we have $\check{\mathbf{P}}\varphi \preceq \check{\mathbf{P}}\varphi_n = (\check{\mathbf{P}}\varphi_n)^*$ on Ω for every $n \in \mathbb{N}$. Then, $\check{\mathbf{P}}\varphi \preceq \curlywedge_n \check{\mathbf{P}}\varphi_n$ on Ω and eventually $\check{\mathbf{P}}\varphi = \curlywedge_n \check{\mathbf{P}}\varphi_n$ on Ω. For given $w \in \overline{\Omega}$ and $r > 0$, the inequality

$$\curlywedge_n(\check{\mathbf{P}}\varphi_n)^*(w) \preceq \curlyvee_{z \in B(w,r)\cap\Omega}(\curlywedge_n \check{\mathbf{P}}\varphi_n(z)) = \curlyvee_{z \in B(w,r)\cap\Omega} \check{\mathbf{P}}\varphi(z)$$

holds. Hence, $(\check{\mathbf{P}}\varphi)^*$ is well-defined on $\overline{\Omega}$ and $(\check{\mathbf{P}}\varphi)^* \preceq \varphi$ on $S(\Omega)$. Clearly, $(\check{\mathbf{P}}\varphi)^* \preceq \curlywedge_n(\check{\mathbf{P}}\varphi_n)^*$ on $\overline{\Omega}$. As $\curlywedge_n(\check{\mathbf{P}}\varphi_n)^* \in \check{\mathcal{U}}[\varphi]$ and $\curlywedge_n(\check{\mathbf{P}}\varphi_n)^* = \check{\mathbf{P}}\varphi$ on Ω, we conclude that $\curlywedge_n(\check{\mathbf{P}}\varphi_n)^* = (\check{\mathbf{P}}\varphi)^*$ on $\overline{\Omega}$. Therefore, $(\check{\mathbf{P}}\varphi)^* = \check{\mathbf{P}}\varphi$ on Ω.

□

For a relatively compact open subset G of Ω, denote $\mathcal{F}(G) = o\text{-}PSH(G;\mathcal{E}) \cap o\text{-}USC(\overline{G};\mathcal{E})$ and $\widetilde{\mathcal{F}}(G)$ the closure of $\mathcal{F}(G)$ with respect to the uniform convergence topology on $\overline{G}$.

Definition 6.4. *A function $u \in PSH(\Omega;\mathcal{E})$ is said to be maximal if for every relatively compact open subset G of Ω and for every $v \in \widetilde{\mathcal{F}}(G)$ so that $v \preceq u$ on ∂G, we have $v \preceq u$ in G. We denote $MPSH(\Omega;\mathcal{E})$ by the family of all maximal $\mathcal{E}$-valued plurisubharmonic functions on Ω.*

Lemma 6.3. *Let Ω be a domain in $\mathbb{C}^n$ and $u \in PSH(\Omega;\mathcal{E})$. If u is the pointwise limit of a monotone decreasing net of functions $\{u_i\}$ (i.e. $u = \curlywedge_i u_i$) in $MPSH(\Omega;\mathcal{E})$, then u is maximal.*

Proof. Now, let G be a relatively compact open subset of Ω and $v \in \widetilde{\mathcal{F}}(G)$ such that $v \preceq u$ on ∂G. Then, $v \preceq u_i$ on ∂G, and hence, $v \preceq u_i$ in G for each i. Thus, we observe that $v \preceq u$ in G, which proves that the function u is maximal. □

Lemma 6.4. *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$ and $\varphi \in C_{ob}(S(\Omega);\mathcal{E})$. Then, $\check{\mathbf{P}}\varphi$ is maximal.*

Proof. Lemma 6.2 shows that $\check{\mathbf{P}}\varphi$ is order bounded above by $\curlyvee_{z\in S(\Omega)}\varphi(z)$ on Ω. Also, $\check{\mathbf{P}}\varphi$ is order bounded below by $\curlywedge_{z\in S(\Omega)}\varphi(z)$. Then,

$$\curlywedge_{z\in S(\Omega)}\varphi(z) \preceq \check{\mathbf{P}}\varphi \preceq \curlyvee_{z\in S(\Omega)}\varphi(z)$$

on Ω. Let G be a relatively compact open subset of Ω and $v \in \widetilde{\mathcal{F}}(G)$ with $v \preceq \check{\mathbf{P}}\varphi$ on ∂G. Then, $v \preceq \curlyvee_{\Omega}\check{\mathbf{P}}\varphi$ on ∂G. As a result of (Klimek, 1991, Corollary 2.9.9), for every $x' \in \mathcal{E}'_+$, $x' \circ v \leq x'(\curlyvee_{\Omega}\check{\mathbf{P}}\varphi)$ in G. Thus, v is order bounded above, and hence $v \in \mathcal{F}(G)$. Let $w : \Omega \to \mathcal{E}$ be the function defined as

$$w(z) = \begin{cases} v(z) \curlyvee \check{\mathbf{P}}\varphi(z), z \in G, \\ \check{\mathbf{P}}\varphi(z), z \in \Omega \setminus G. \end{cases}$$

Note that $\curlywedge_{z\in S(\Omega)}\varphi(z) \preceq w(z) \preceq \curlyvee_{z\in S(\Omega)}\varphi(z), z \in \Omega$. For every $y \in \partial G$,

$$\limsup_{G\ni z\to y} v(z) \preceq w(y) = \check{\mathbf{P}}\varphi(y).$$

These two facts ensure that w is order upper semi-continuous on Ω and the Bochner integral $\int_0^{2\pi} w(z_0 + e^{it}\beta)dt$ exists for all $z_0 \in \Omega$ and $\beta \in \mathbb{C}^n$ so that $\{z_0 + \lambda\beta : \lambda \in \mathbb{C}, |\lambda| \leq 1\} \subset \Omega$. Let $x' \in \mathcal{E}'_+$ be arbitrary. Then, by (Klimek, 1991, Corollary 2.9.15), it immediately follows that $x' \circ w$ is plurisubharmonic in Ω. As x' was arbitrary, we deduce that w is plurisubharmonic on Ω, particularly $w^* \in \check{\mathcal{U}}[\varphi]$. Then, $\check{\mathbf{P}}\varphi$ is maximal, since $v \preceq w^* \preceq \check{\mathbf{P}}\varphi$ holds in G as well. □

6.2 Perron Method for Plurisubharmonic Functions

For a bounded hyperconvex domain Ω, Bremermann (1959) introduced the family

$$\mathcal{V}[\varphi] = \left\{ u \in PSH(\Omega;\mathbb{R}) \cap USC(\overline{\Omega};\mathbb{R}) : u(\zeta) \leq \varphi(\zeta), \zeta \in S(\Omega) \right\}$$

for a function $\varphi \in C(S(\Omega);\mathbb{R})$, and under the condition that Ω is strictly pseudoconvex where $S(\Omega) = \partial\Omega$, also proved that envelope $\mathbf{P}\varphi(z) = \sup\{u(z) : u \in \mathcal{V}[\varphi]\}, z \in \Omega$, attains the boundary data function φ continuously. Walsh (1969) further demonstrated that $\mathbf{P}\varphi$ is continuous whenever it is continuous on $\partial\Omega$. In order to consider Perron solution for the Dirichlet problem for maximal plurisubharmonic functions on a bounded hyperconvex domain Ω, Bremermann proved that the continuous boundary data function φ needed to be defined on and only on the Shilov boundary $S(\Omega)$ of Ω (cf. (Bremermann, 1959, Theorem 7.2)).

Proof of Theorem 1.4 (ii). Suppose first that φ is of the form

$$\varphi = \sum_{n=1}^{N} f_n x_n \tag{6.4}$$

where $f_n \in C(S(\Omega);\mathbb{R}^+)$ and $x_n \in \mathcal{E}^+$. By its form, we observe that φ is an order bounded function. Consider the function $\mathbf{\Phi} = \sum_{n=1}^{N} (\check{\mathbf{P}} f_n) x_n$, where

$$\check{\mathbf{P}} f_n(z) = \sup\{u(z) : u \in PSH(\Omega;\mathbb{R}) \cap USC(\overline{\Omega};\mathbb{R}), u \leq f_n \text{ on } S(\Omega)\},$$

$z \in \Omega$. The function $\mathbf{\Phi}^*$ belongs to $\check{\mathcal{U}}[\varphi]$ so that $\mathbf{\Phi}^* = \varphi$ on $S(\Omega)$. As $\mathbf{\Phi} \preceq \check{\mathbf{P}}\varphi$ on Ω, we have the inequality $\varphi(\zeta) = \mathbf{\Phi}^*(\zeta) \preceq (\check{\mathbf{P}}\varphi)^*(\zeta)$ for every $\zeta \in S(\Omega)$. As we have shown in (6.3), $(\check{\mathbf{P}}\varphi)^* \preceq \varphi$ on $S(\Omega)$. Then, $(\check{\mathbf{P}}\varphi)^* = \varphi$ on $S(\Omega)$. By Lemma 6.4, $\check{\mathbf{P}}\varphi$

is a solution of the Dirichlet problem (1.3) with the boundary data φ.

Let us consider the case $\varphi \in C_{ob}(S(\Omega);\mathcal{E})$. Without loss of generality, assume that $\varphi \succeq 0$. Then, there exists a sequence $\{\varphi_n\}$ in $C(S(\Omega);\hat{\mathcal{E}}^+)$ of the form (6.4) converging φ uniformly on $S(\Omega)$ where $\hat{\mathcal{E}} = \overline{(span\varphi(S(\Omega)))^{\curlyvee\curlywedge}}$. By the inequality (6.2),

$$\|(\check{\mathbf{P}}\varphi)^* - (\check{\mathbf{P}}\varphi_n)^*\|_{\mathcal{E},\overline{\Omega}} \leq \|\varphi - \varphi_n\|_{\mathcal{E},S(\Omega)}.$$

Then, $(\check{\mathbf{P}}\varphi)^* \in \widetilde{\mathcal{F}}$. Since φ is order bounded, we conclude that $(\check{\mathbf{P}}\varphi)^*$ is precisely in $\mathcal{F}$. By Lemma 6.4, $\check{\mathbf{P}}\varphi$ maximal. For given $\zeta \in S(\Omega)$,

$$\begin{aligned}\|(\check{\mathbf{P}}\varphi)^*(\zeta) - \varphi(\zeta)\| \leq \|(\check{\mathbf{P}}\varphi)^*(\zeta) - (\check{\mathbf{P}}\varphi_n)^*(\zeta)\| + \|(\check{\mathbf{P}}\varphi_n)^*(\zeta) - \varphi_n(\zeta)\| \\ + \|\varphi_n(\zeta) - \varphi(\zeta)\|.\end{aligned}$$

This inequality shows that $(\check{\mathbf{P}}\varphi)^*(\zeta) = \varphi(\zeta)$. Thus, $\check{\mathbf{P}}\varphi$ is a solution of (1.3) with the boundary data φ.

We now examine the case where $\varphi \in C_{oba}(S(\Omega);\mathcal{E})$. As we have shown in Lemma 3.4 and Proposition 3.2, since $\mathcal{E}$ is a Banach lattice with order continuous norm, there exists a monotone decreasing sequence $\{\varphi_n\}$ in $C_{ob}(S(\Omega);\mathcal{E})$ that uniformly converges in the norm to φ on $S(\Omega)$. Since each φ_n is of the class $C_{ob}(S(\Omega);\mathcal{E})$, each corresponding Dirichlet problem (1.3) has a solution $\check{\mathbf{P}}\varphi_n$. Then, by inequality (6.2), we have for $n,m \in \mathbb{N}$ that

$$\|(\check{\mathbf{P}}\varphi_n)^*(z) - (\check{\mathbf{P}}\varphi_m)^*(z)\| \leq \sup_{w\in S(\Omega)} \|\varphi_n(w) - \varphi_m(w)\|, z \in \overline{\Omega}.$$

This implies that $\{(\check{\mathbf{P}}\varphi_n)^*(z)\}$ is a monotone decreasing Cauchy sequence in $\mathcal{E}$ for each fixed $z \in \overline{\Omega}$. As we have shown in the proof of Theorem 1.3, $\check{\mathbf{P}}\varphi(z) = \lim_{n\to\infty} \check{\mathbf{P}}\varphi_n(z) = \curlywedge_n \check{\mathbf{P}}\varphi_n(z), z \in \Omega$. Lemma 3.2 and (Klimek, 1991, Theorem 2.9.14) bring out that $(\check{\mathbf{P}}\varphi)^* \in \mathcal{F}$ satisfying $(\check{\mathbf{P}}\varphi)^* \preceq \varphi$ on $S(\Omega)$. Lemma 6.3 shows that $\check{\mathbf{P}}\varphi$ is maximal. For $\zeta \in S(\Omega)$,

$$\begin{aligned}\|(\check{\mathbf{P}}\varphi)^*(\zeta) - \varphi(\zeta)\| \leq \|(\check{\mathbf{P}}\varphi)^*(\zeta) - (\check{\mathbf{P}}\varphi_n)^*(\zeta)\| + \|(\check{\mathbf{P}}\varphi_n)^*(\zeta) - \varphi_n(\zeta)\| \\ + \|\varphi_n(\zeta) - \varphi(\zeta)\|\end{aligned}$$

holds for $n \in \mathbb{N}$. Hence, $(\check{\mathbf{P}}\varphi)^*(\zeta) = \varphi(\zeta)$ for every $\zeta \in S(\Omega)$. This means that $\check{\mathbf{P}}\varphi$ is a solution of (1.3) with the given boundary data φ.

□

"Don't Stop Me Now!"
Song by Queen

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