A SEMI-CONSTRUCTIVE APPROACH TO EVIDENTLY POSITIVE PARTITION GENERATING FUNCTIONS

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ABSTRACT

A SEMI-CONSTRUCTIVE APPROACH TO EVIDENTLY POSITIVE PARTITION GENERATING FUNCTIONS

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In this work, we start with the celebrated Rogers–Ramanujan identities, which are fundamental results in the theory of integer partitions. Then, we continue by presenting certain combinatorial generalizations related to these theorems, called Rogers-Ramanujan type identities. These identities involve two types of constraints: modulus constraints and difference constraints. Although generating functions on the modulus side are relatively straightforward to construct, manipulate, and interpret, addressing the difference side requires a more nuanced approach.

To this end, we introduce a general framework, termed the moves framework, for interpreting evidently positive series arising from a specific form of two-variable generating functions. This framework is applicable under certain algebraic conditions on the exponents of the generating functions. For cases where these conditions are not satisfied, we propose an alternative method. This involves deriving a system of functional equations satisfied by the series and translating this information into a recursive combinatorial construction, which allows us to provide a combinatorial interpretation of the series.

The thesis concludes with a discussion of potential directions for future research, highlighting open problems and areas for further exploration.

ÖZET

A SEMI-CONSTRUCTIVE APPROACH TO EVIDENTLY POSITIVE PARTITION GENERATING FUNCTIONS

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Matematik Doktora Tezi, Aralık 2024

Tez Danışmanı: Doç. Dr. Kağan Kurşungöz

Anahtar Kelimeler: tamsayı parçalanışları, parçalanış üreteç fonksiyonları, açıkca pozitif seriler, Andrew-Gordon serileri, fonksiyonel denklem

Bu teze, parçalanış teorisindeki en temel sonuçlardan olan Rogers-Ramanujan özdeşlikleri ile başlıyoruz. Devamında, bu özdeşliklerin belli kombinatorik genelleştirmelerini, Rogers-Ramanujan türü özdeşlikler, sunuyoruz. Bu özdeşlikler iki tür kısıt içeriyorlar: Kalan sınıfları kısıtları ve fark kısıtları. Kalan sınıfları kısıtlarına karşılık gelen üreteç fonksiyonlarını yazmak, manipüle etmek ve kombinatorik olarak yorumlamak oldukça kolay. Ancak, fark kısıtları için bu çok daha çetrefilli bir iş.

Bu nedenle, fark kısıtlarına karşılık gelen, belli bir cebirsel formdaki üreteç fonksiyonlarını açıklamak için, hamlelerle inşa adını verdiğimiz bir kombinatorik yorum oluşturuyoruz. Bu inşa, ancak üreteç fonksiyonlarındaki üsler belli cebirsel koşulları sağlarsa işe yarıyor. Bu koşulları sağlamayan durumlar için farklı bir kombinatorik yorum öneriyoruz. Bu yorumda, verilen bir üreteç fonksiyonundan, o üreteç fonksiyonunun sağladığı bir fonksiyonel denklem sistemine geçiyoruz. Bu fonksiyonel denklem sistemi, üreteç fonksiyonunun katsayıları üzerinde bir özyineleme ilişkisi veriyor, buradan da özyinelemeli kombinatorik bir inşa oluşturuyoruz. Böylelikle, verilen üreteç fonksiyonunu kombinatorik olarak yorumlamış oluyoruz.

Tezi, bazı açık problemler ve olası araştırma önerileri ile noktalıyoruz.

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İsmail Serhat Oğuz'a,

O iyi insanlar, o güzel atlara binip çekip gittiler.

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1. Introduction

In this chapter, we provide basic notation, give fundamental theorems in the theory of partitions, and introduce the main problem of this thesis.

1.1 Definition and Examples

Here, we discuss the basic definitions and fundamental theorems of partition theory. For more information, one can check [4], [9].

Definition 1. An **integer partition** λ of a natural number n is $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 1$. We usually denote λ by $\lambda_1 + \lambda_2 + \cdots + \lambda_m$. We refer n as the **weight** of partition λ. The number of parts m is called the **length** of λ, each λ_i is called a **part** of λ. We use $l(\lambda)$ to denote the length of partition λ.

For example, listing all integer partitions of 4, we get the following list:

$$4,\ 3+1,\ 2+2,\ 2+1+1,\ 1+1+1+1.$$

Let $p(n)$ denote the number of partitions of n. The above example shows that $p(4) = 5$. By convention, we let $p(0) = 1$ and the unique partition of 0 is called the **empty partition**.

Usually, rather than looking at all partitions of a number, we look at certain subsets of that. In other words, we choose a condition and look for partitions that satisfy that condition.

Example 1. 1. All partitions of 5 where each part is odd:

$$5,\ 3+1+1,\ 1+1+1+1+1.$$

2. All partitions of 5 where each part appears at most one time:

$$5,\ 4+1,\ 3+2.$$

3. All partitions of 9, where the difference between each consecutive part is at least two:

$$9,\ 8+1,\ 7+2,\ 6+3,\ 5+3+1.$$

In this thesis, the set of all partitions is denoted by $\mathcal{P}$ and the set of all partitions where parts cannot repeat is denoted by $\mathcal{D}$. The set of all partitions where the consecutive difference between the parts is at least $s \geq 1$ is denoted by $\mathcal{D}_s$. Thus, $\mathcal{D} = \mathcal{D}_1$. Such script letters in this thesis denote a subset of partitions.

Remark 1. There are different ways to represent partitions:

1. *Ferrers Graph:* It is a diagram where the mth row has the same number of dots as the mth term of the partition. For instance, the Ferrers Graph of $6+4+1$ as:

$$\begin{array}{llllll} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & & \\ \bullet & & & & & \end{array}$$

2. *Frequency notation:* We can denote a partition λ by $(f_1, f_2, f_3, \dots)$ where f_i represents the number of times part i appears in λ. In this notation, the set of all partitions becomes

$$\mathcal{P} = \{(f_1, f_2, f_3, \dots) : f_i \in \mathbb{Z}\,,\ f_i \geq 0\,,\ f_i = 0 \,\text{for all but finitely many } i\}.$$

   For example, the frequency notation of $6+4+1$ is $(1, 0, 0, 1, 0, 1, 0, 0, \dots)$, i.e., writing the number of repetitions of each part.

We use the following notation throughout: $p(n \,|\, \text{condition})$ is the number of partitions of n that satisfy the "condition".

Therefore, Example 1 shows that $p(5 \,|\, \text{parts are odd}) = 3$, $p(5 \,|\, \text{parts are distinct}) = 3$, and $p(9 \,|\, \text{the difference between consecutive parts are at least two}) = 5$.

Remark 2. There are some generalizations of partitions. We mention two of them:

1. *Overpartitions,* [13] : Each part is overlined or not. An overlined part can appear at most 1 time.

2. *Colored Partitions:* We have k colors, and each part is colored by one of these k colors. An order is fixed on these colors. For instance, if $k = 2$, and the colors are blue and red, then, we may assume that $b \geq b$. For more information on colored partitions, one can look at [2] and [14].

Example 2. 1. All overpartitions of 3 are:

$$3,\ \bar{3},\ 2+1,\ 2+\bar{1},\ \bar{2}+1,\ \bar{2}+\bar{1},\ 1+1+1,\ \bar{1}+1+1.$$

2. All ten 2-colored partitions of 3 are:

$$3,\ 3,\ 2+1,\ 2+1,\ 2+1,\ 2+1,\ 1+1+1,\ 1+1+1,\ 1+1+1,\ 1+1+1.$$

Remark 3. Ordinary partitions can be seen as a special case of colored partitions where $k = 1$, i.e. each part has the same color or they are uncolored. Similarly, overpartitions can be seen as a special case of a colored partition where $k = 2$ and one of the color parts can appear at most 1 time.

1.2 Generating Functions

In combinatorics, given a sequence of numbers $\{a_n\}_{n=0}^{\infty}$, we usually put them in a compact object and study that object instead of the sequence itself. We divide this section into two subsections: one variable generating functions and two variable generating functions.

1.2.1 One Variable Generating Functions

Definition 2. A **(ordinary) generating function** of $\{a_n\}_{n=0}^{\infty}$ is given by

$$A(q) = \sum_{n=0}^{\infty} a_n q^n.$$

Also, without any numbers, we can define the generating function of a subset of partitions as follows: Let S be any subset of partitions, then a generating function of S is given by: $\sum_{\lambda \in S} q^{|\lambda|}$.

For any generating function $A(q)$, we use $[q^n]A(q)$ to denote the coefficient of the term q^n in the generating function $A(q)$.

Euler found the generating function of $p(n)$ in [15]:

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)}. \tag{1.1}$$

Remark 4. 1. Firstly, we can interpret the right-hand side of (1.1) as follows: Expand each factor as a geometric series and then decide how many times you use each part. This gives you the set of all partitions. For more details on this interpretation, see [9].

2. Secondly, (1.1) gives our first *sum-product identity.* As the name suggests, this refers to an identity where the left-hand side is an infinite sum and the right-hand side is an infinite product.

3. Third, (1.1) allows us to represent partitions in frequency notation as explained in Remark 1.

4. We define $(a\,;\,b)_m := (1-a)(1-ab)(1-ab^2)\cdots(1-ab^{m-1})$. This means that a is the first factor we start with, b is the expansion between each factor, and m is the number of factors. Also, $(a\,;\,b)_0 := 1$. We can extend this definition to the negative m and $m=\infty$ as follows: Let m be a negative integer, define $(a\,;\,b)_m := \frac{1}{(aq^{-n}\,;\,b)_{-m}}$. Also, define $(a\,;\,b)_\infty$ as $\lim_{m\to\infty}(a\,;\,b)_m$. Thus, the generating function of partitions can be written as $\frac{1}{(q\,;\,q)_\infty}$. This gives us a more compact representation of the right-hand side of (1.1). A much more detailed account on the q-series is [16].

We now look at some examples of generating functions of subsets of partitions.

Example 3. 1. Distinct partitions, i.e. the partitions where no part can repeat:

$$\prod_{n=1}^{\infty}(1+q^n) = (-q\,;\,q)_\infty.$$

2. Odd Partitions, i.e. the partitions where each part is odd:

$$\prod_{n=1}^{\infty} \frac{1}{1-q^{2n-1}} = \frac{1}{(q\,;\,q^2)_\infty}.$$

3. Partitions into parts congruent to $\pm 1 \pmod 5$:

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-4})(1-q^{5n-1})} = \frac{1}{(q\,;\,q^5)_\infty(q^4\,;\,q^5)_\infty}.$$

We refer to [4] and [9] for more examples.

Let $cp_k(n)$ be the number of k-colored partitions of n and $\bar{p}(n)$ be the number of overpartitions of n. Similarly, the generating function for k colored partitions and overpartitions are given respectively as:

$$\sum_{n=0}^{\infty} cp_k(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^k} \text{ and } \sum_{n=0}^{\infty} \bar{p}(n)q^n = \prod_{n=1}^{\infty} \frac{(1+q^n)}{(1-q^n)}.$$

Remark 5. If we have a generating function of some class of partitions, obviously all the coefficients of the sequence would be nonnegative. However, sometimes this is evident from the form of the generating function without any computations, and sometimes we need to perform some computations. The first case type of generating functions is called **evidently positive generating functions**. From now on, we will call them **positive generating functions** in short.

Example 4. Let's look at an identity from [7] where one side is evidently positive, the other side is not:

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n} q^{\frac{n(5n+1)}{2}}(1-x^2q^{4n+2})}{(q;q)_j(xq^{n+1};q)_\infty} = \sum_{n=0}^{\infty} \frac{x^n q^{n^2}}{(q;q)_n}.$$

The right-hand side is evidently positive, however by inspection one cannot infer that the left hand side is evidently positive.

Remark 6. It is possible to see q as a complex variable and talk about the analytic behaviors of the series, asymptotics of the coefficients and so on. However, in this thesis, we will not explore this. For more details, see [16], Chapter 5 and Chapter 6 of [4] and for the asymptotic techniques in combinatorics in general, check [28].

Remark 7. Almost without exception, we use the exponent of q to count the weight of a partition.

1.2.2 Two Variable Generating Functions

If the only thing we care about is the weight of a partition, then, we can use a one-variable generating function. However, if we care about the length of the partition or some other *statistic* (any function from partitions to integers) of the partition as well, then we need to use more variables to keep track of them.

Next, we give some examples and refer to [9], [4] for more.

Example 5. 1. A generating function of distinct partitions:

$$\sum_{\lambda\in\mathcal{D}} q^{|\lambda|}x^{l(\lambda)} = \sum_{n\geq 0} \frac{q^{\binom{n+1}{2}}x^n}{(q;q)_n}.$$

2. A generating function of 2-distinct partitions :

$$\sum_{\lambda\in\mathcal{D}_2} q^{|\lambda|}x^{l(\lambda)} = \sum_{n\geq 0} \frac{q^{n^2}x^n}{(q;q)_n}.$$

   where $\mathcal{D}_2$ is set of all partitions where the consecutive difference between each part is at least 2.

3. A generating function for partitions with no conditions:

$$\sum_{\lambda\in\mathcal{P}} q^{|\lambda|}x^{\lambda_1} = \sum_{n\geq 0} \frac{q^n x^n}{(q;q)_n}.$$

Remark 8. For any subset of partitions, $\mathcal{S}$, we denote by $S(x)$ the two variable generating function of $\mathcal{S}$, and we use $s(m,n)$ as the coefficient of a two-variable generating function of $\mathcal{S}$, i.e.

$$S(x) := \sum_{\lambda\in\mathcal{S}} q^{|\lambda|}x^{l(\lambda)} = \sum_{m,n\geq 0} s(m,n)x^m q^n.$$

Equivalently, $s(m,n) := [x^m q^n]S(x)$. In addition, we use the notation $\mathcal{S}(n)$ to denote the partitions in $\mathcal{S}$ with weight n.

1.3 Partition Identites

Any equality of the form $p(n\,|\,\text{CONDITION 1}) = p(n\,|\,\text{CONDITION 2})$ is called a **partition identity**.

The first partition identity is found by Euler, [15].

Theorem 1. [Euler's Theorem , [4]] The number of partitions of n into odd parts is equal to the number of partitions of n where the parts are distinct.

Proof. We give two different proofs of this theorem.

1. Firstly, if we can show that the generating functions of both sides are the same,

then the identity immediately follows.

$$\begin{aligned}\sum_{n=0}^{\infty} p(n \mid \text{parts are odd}) q^n &= \frac{1}{(q;q^2)_\infty} \\ &= \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty (q^2;q^2)_\infty} \\ &= \frac{(q^2;q^2)_\infty}{(q;q)_\infty} \\ &= (-q;q)_\infty \\ &= \sum_{n=0}^{\infty} p(n \mid \text{parts are distinct}) q^n.\end{aligned}$$

2. Secondly, we present a bijection between these two classes of partitions, found by Glaisher. Let λ be a partition into odd parts. We want to obtain a partition π into distinct parts. The idea is as follows: If λ has no repeated parts, then π is exactly the same as λ, else merge the parts of λ until no part is repeated. We look at an example and then generalize the example.

   Suppose that we are given $\lambda = 7+7+5+5+5+1$. We want to obtain a distinct partition from λ. To do that, we merge 7's and 5's to get the distinct partition $\pi = 14+10+5+1$. From the other direction, we start with $\pi = 14+10+5+1$ which is a distinct partition. We want to obtain an odd partition λ, from π. The idea is as follows: Consider parts one by one; if the part is odd, just keep it as is, otherwise replace the part by the sum of its halves, that is, $2a \to a+a$. In our example, we have 14 and 10 as even parts and we replace them by $7+7$ and $5+5$ respectively. As a result, we get the partition $\lambda = 7+7+5+5+5+1$. For more details on this bijection, one can check Chapter 2 of [9]. □

Remark 9. There are two main ways to prove a partition identity, either showing that the generating functions of both sides are the same or finding a bijection between them, as seen in the proof of Theorem 1. However, it is rare to find bijective proofs of partition identities. Thus, Glaisher's proof of Euler's identity is an exception.

Remark 10. In some other combinatorial problems, it is possible to explicitly count both sides and conclude that they have the same number of elements. However, such proofs are very rare in partition theory, since, it is very hard to count the classes of partitions.

Remark 11. We can use statistics to refine the bijections or look for bijections. For more details, see [9], [30].

After Euler's theorem, the most famous partition identities are Rogers-Ramanujan identities. We present Schur's version of these identities:

Theorem 2 (Rogers-Ramanujan 1, [4] , [26])**.** Let n be any natural number. Then,

$$p(n \,|\, \text{parts} \equiv \pm 1 \pmod 5) = p(n \,|\, \text{No repeated or consecutive parts}).$$

Theorem 3 (Rogers-Ramanujan 2, [4] , [26])**.** Let n be any natural number. Then,

$$p(n \,|\, \text{parts} \equiv \pm 2 \pmod 5) = p(n \,|\, \text{No repeated or consecutive parts, 1 excluded}).$$

Example 6.
1. Let $n = 9$. Then, on the left-hand side of Rogers-Ramanujan 1 we have: $9\,,\, 8+1\,,\, 7+2\,,\, 6+3\,,\, 5+3+1$. On the right-hand side, we have: $9\,,\, 6+1+1+1, 4+4+1\,,\, 4+1+1+1+1+1\,,\, 1+1+1+1+1+1+1+1+1$.
2. Let $n = 12$. Then, on the left-hand side of Rogers-Ramanujan 2 we have: 12, $10+2$, $9+3$, $8+4$, $7+5$, $6+4+2$ and on the right-hand side we have: 12, $8+2+2$, $7+3+2$, $3+3+3+3$, $3+3+2+2$, $2+2+2+2+2+2$.

The big picture of Euler's theorem and the Rogers-Ramanujan identities contains two sides: the difference side and the modulus side. In other words, one side concerns about the differences on the parts and the other side has a condition about the residue classes. We call them **Rogers-Ramanujan type identities**.

1.3.1 Rogers-Ramanujan Type Identities

Remark 12. There are partition identities which are not Rogers-Ramanujan type identities. For instance,

$$p(n \,|\, \text{parts at most } m) = p(n \,|\, \text{at most } m \text{ parts}).$$

However, in this thesis, almost all identities that we consider would be Rogers-Ramanujan type identities. For more identities which are not of Rogers-Ramanujan type, one can look at [9].

Gordon generalizes the Rogers-Ramanujan identities as follows.

Theorem 4 (Rogers-Ramanujan-Gordon, [18])**.** Let a and k be natural numbers such that $1 \leq a \leq k$. Then, the number of partitions of n into parts not equivalent to 0, $\pm a \pmod{2k+1}$ is equal to the number of partitions of $n = \lambda_1 + \lambda_2 + \cdots + \lambda_k$ where $\lambda_i \geq \lambda_{i+k-1} + 2$ and the number of ones is at most $a-1$.

Example 7. 1. Let $n = 8$, $k = 4$, $a = 3$. Then, on the difference side we have 8 , $7+1$, $6+2$, $6+1+1$, $5+3$, $5+2+1$, $4+4$, $4+3+1$, $4+2+2$, $4+2+1+1$, $3+3+2$, $3+3+1+1$, $3+2+2+1$. On the modulus side, we have 8 , $7+1$, $5+2+1$, $5+1+1+1$, $4+4$, $4+2+2$, $4+2+1+1$, $4+1+1+1+1$, $2+2+2+2$, $2+2+2+1+1$, $2+2+1+1+1+1$, $2+1+1+1+1+1+1$, $1+1+1+1+1+1+1+1$.

Remark 13. Note that the Rogers-Ramanujan identities are special cases of the Rogers-Ramanujan-Gordon identity. More precisely, letting $k=2$, $a=2$ in Theorem 4 gives Rogers-Ramanujan 1 and letting $k=2$, $a=1$ in Theorem 4 gives Rogers-Ramanujan 2 identities.

We use $\mathcal{RRG}_{k,a}$ to denote the set of all partitions which satisfy the difference conditions of the Rogers-Ramanujan-Gordon theorem. More explicitly, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathcal{RRG}_{k,a}$ if and only if

1. λ contains at most $a-1$ ones as a part.
2. $\lambda_i - \lambda_{i+k-1} \geq 2$.

Andrews found an evidently positive generating function for the difference side of the Rogers-Ramanujan-Gordon identity.

Theorem 5 (Andrews' Evidently Positive Series for Rogers-Ramanujan-Gordon Theorem, [8])**.** For $1 \leq i \leq k$, and $k \geq 2$, we have

$$\sum_{n_1, n_2, \ldots, n_{k-1} \geq 0} \frac{q^{N_1^2 + N_2^2 + \cdots + N_{k-1}^2 + N_i + N_{i+1} + \cdots + N_{k-1}}}{(q;q)_{n_1}(q;q)_{n_2} \cdots (q;q)_{n_{k-1}}} = \prod_{\substack{n=1 \\ n \not\equiv 0, \pm a \pmod{2k+1}}}^{\infty} \frac{1}{1-q^n}. \tag{1.2}$$

where $N_j := n_j + n_{j+1} + \cdots + n_{k-1}$.

Bressoud found the even modulus analog of the Rogers-Ramanujan-Gordon Theorem.

Theorem 6 (Bressoud's Theorem, [10])**.** Let k be a fixed positive integer, and let $1 \leq a \leq k$. Then, the number of partitions into parts not congruent to 0 , $\pm a \pmod{2k}$ is equal to the number of partitions of $n = \lambda_1 + \lambda_2 + \cdots + \lambda_m \in \mathcal{RRG}_{k,a}$

with the property that

$$\lambda_j - \lambda_{j+k-2} \leq 1 \implies \sum_{h=0}^{k-2} \lambda_{j+h} \equiv (a-1) \pmod{2}.$$

Example 8. Let $n = 11$, $k = 3$ and $a = 1$. Then, on the difference side we have: $11,\ \ 9+2,\ \ 8+3,\ \ 7+4,\ \ 7+2+2,\ \ 5+3+3$. Note that the modulus condition eliminates the following partitions: $6+5,\ \ 6+3+2,\ \ 5+4+2$.

On the modulus side we have: $9+2,\ \ 8+3,\ \ 4+4+3\ ,\ \ 4+3+2+2,\ \ 3+3+3+2,\ \ 3+2+2+2+2$.

Remark 14. Note that all of the above theorems are generalizations of Rogers-Ramanujan identities.

1.3.2 Alternative Generalizations of Rogers-Ramanujan Identities

We can look at the Rogers-Ramanujan theorems from another perspective.

Recall that any partition can be written via frequency notation: $(f_1, f_2, \dots)$ where f_i is the number of times part i occurs in the partition. We can rewrite Euler's Theorem and Rogers-Ramanujan's First Identity as:

Euler's Theorem: $p(n \,|\, f_i \leq 1) = p(n \,|\, \text{parts} \equiv \pm 1 \pmod{4})$

Rogers-Ramanujan First Identity: $p(n \,|\, f_i + f_{i+1} \leq 1) = p(n \,|\, \text{parts} \equiv \pm 1 \pmod{5})$

Thus, it is tempting to conjecture that $p(n \,|\, f_i + f_{i+1} + f_{i+2} \leq 1) = p(n \,|\, \text{parts} \equiv \pm 1 \pmod{6})$. However, this is wrong! The correct generalization is found by Schur:

Theorem 7 (Schur's Theorem, [27])**.** The number of partitions of n where the consecutive difference between the parts is at least 3 and the consecutive multiples of 3 do not occur is equal to the number of partitions of n into parts that are congruent to $\pm 1 \pmod{6}$.

Remark 15. In fact, more is correct, we cannot find $d \geq 3$ such that

$$p(n \,|\, f_i + f_{i+1} + \cdots + f_{i+d-1} \leq 1) = p(n, |, \text{parts} \equiv \pm 1 \pmod{d+3}),$$

as proved by Lehmer, [25]. Then, Alder conjecture that

$$p(n \,|\, f_i + f_{i+1} + \cdots + f_{i+d-1} \leq 1) \geq p(n \,|\, \text{parts} \equiv \pm 1 \pmod{d+3}),$$

which is proved by [3], [31], [1]. For more details, see [9].

Göllnitz and Gordon found Schur-like identities:

Theorem 8 (First Göllnitz-Gordon, [19], [17])**.** The number of partitions of n into parts in which the consecutive difference is at least 2, and the difference is at least 3 unless both parts are odd, is equal to the number of partitions of n into parts $\equiv 1, 4, 7 \pmod 8$.

Theorem 9 (Second Göllnitz-Gordon, [19], [17])**.** The number of partitions of n into parts in which the consecutive difference is at least 2, and the difference is at least 3 unless both parts are odd and the smallest part is at least 3, is equal to the number of partitions of n into parts $\equiv 3, 4, 5 \pmod 8$.

Remark 16. The direct generalization of Schur's theorem is not true. In other words, the number of partitions of n into parts in which the consecutive difference between parts is at least 4 and consecutive multiples of 4 cannot occur is not the same as the number of partitions in which parts are equivalent to $\pm 1 \pmod 7$.

Remark 17. If we look at any Rogers-Ramanujan type identity the product (the modulus) sides are very easy to write, manipulate and interpret. However, this is not true for the sum sides (difference sides).

The above remark suggests a natural question: Is there a general way to interpret sum-sides? Obviously, this question is too general to answer. Thus, we fix a form of the sum side and then try to give a framework to interpret this sum side.

In other words, in the following chapters, our main aim is the following: Given a sum side of a particular form, is it possible to interpret it in a general framework?

Remark 18. Another avenue to look at Rogers-Ramanujan type identites is partition ideals. For more details, please check [4], [6], [12].

The rest of this thesis is organized as follows. The second chapter is devoted to a special form of a two-variable generating function and its combinatorial interpretation. In the third chapter, we study a more general form of a two-variable generating function and give another framework to extract combinatorial explanation from the generating function. In the last chapter, we conclude the thesis with a discussion and future research directions.

2. Algebraic Generalization and Moves Framework

If we look at the theorems we discussed in the introduction and their generating functions, it seems that the generating functions are of the form:

$$\sum_{m,n\geq 0} \frac{q^{Q(m,n)+L_1(m,n)}x^{L_2(m,n)}}{(q^K;q^K)_m(q^L;q^L)_n}. \tag{2.1}$$

where $Q(m,n)$ is a quadratic form of m and n, $L_1(m,n)$ and $L_2(m,n)$ are linear forms of m and n.

In the next example, we give a generating function of the form (2.1) and another one that is not of that form.

Example 9.

1. A generating function of Rogers-Ramanujan Gordon where $k = 3$ and $i = 3$ in (1.2) is given by:

   $$\sum_{m,n=0}^{\infty} \frac{q^{(m+n)^2+m^2}}{(q;q)_m(q;q)_n}.$$

   This is an evidently positive generating function.

2. A generating function for Rogers-Ramanujan 1 is given by:

   $$\frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{\frac{1}{2}n(5n-1)}(1+q^n). \tag{2.2}$$

   This is not an evidently positive generating function due to $(-1)^n$ term in the summand [7].

Remark 19. Our generating function, (2.1), is an evidently positive generating function.

So far, in the 2-variable generating functions considered in the thesis, we use n to keep track of the weight of the partition and m is the length of the partition. Now, we are given another interpretation for the sum of the form (2.1).

We want to study combinatorial interpretations of such sums. Before introducing a framework for that, we mention some articles using moves to interpret generating functions of the form (2.1). The moves framework is used in the following articles: [11], [23] and [24].

Setting: We have two *types* of parts, where m counts the number of *Type 1* parts and n counts the number of *Type 2* parts. The numerator of the summand in (2.1) can be thought of the *base partition* and the denominator of the summand in (2.1) can be interpreted as the *moves* on *Type 1* and *Type 2* parts. (We will formalize these concepts later.)

The basic idea is that, you pick m and n which determines number of *Type 1* and *Type 2* parts you can use. Then, you find the smallest weight partition (called the base partition, in some sources it is called the minimal partition) which is of this form. Then, using the moves on the parts, you can get any partition from that.

First, we look at one example given in Theorem 10. Then, we generalize this example to get an abstract setup. After that, we give various small examples and one more detailed example. We conclude the chapter by mentioning the advantages and disadvantages of the moves framework.

2.1 The Connection Between Combinatorial Parameters and Generating Function Parameters

Throughout this section, we work with a running example to help us understand the concepts we are going to explain.

Theorem 10 (Caparelli's Theorem)**.** The number of partitions of n where 1 does not appear as a part, the difference between each consecutive part is at least 2, and at least 4, unless the sum of successive parts is a multiple of three; is the same as the number of partitions of n into distinct parts which are not $\pm 1 \pmod 6$.

There are two types of parts: The parts where the difference between the consecutive parts is less than 4, they are called *pairs*, the parts where the difference between the consecutive parts is greater than or equal to 4, they are called *singletons*. (In general, we refer to them as Type 1 and Type 2).

Now, we want to divide all partitions satisfied by the difference side of Theorem 10, denoted by $\mathcal{C}$, into buckets depending on the number of pairs and singletons they contain. Thus, we have

$$\sum_{\lambda \in \mathcal{C}} q^{|\lambda|} x^{l(\lambda)} = \sum_{m,n \geq 0} T_{m,n}(x).$$

where $T_{m,n}(x)$ is the generating function of partitions in $\mathcal{C}$ that contains m singletons and n pairs.

To find $T_{m,n}(x)$ we need to find the smallest weight partition (*base partition*) in $\mathcal{C}$ which contains m singletons and n pairs first. The base partition is:

$$[2+4]+[8+10]+\cdots+[6n-4,6n-2]+(6n+2)+(6n+6)+\cdots+(6n+4m-2).$$

Thus, the smallest part is 2 in the pairs, the difference between each consecutive pair is 6, each part appears 1 time, the difference between the largest pair and the smallest singleton is 4, the difference between each singleton is 4, and each singleton appears 1 time.

We can *push* the pairs forward; however, we do not want them to be too close (due to the statement of the Theorem) to each other. Thus, to keep their sum as a multiple of 3, we push them 3 by 3 (i.e., after each push the weight of the pairs increases by 3). For instance, $[2,4] \to [3,6]$. (So, the difference between the pairs can be 2 or 3). Similarly, we can push the singletons 1 by 1. As an example, $(7) \to (8)$.

The weight of the base partition is: $6n^2 + 6mn + 2m^2$

The length of the base partition is: $2n + m$

The interpretation of the moves we applied to the pairs: $\frac{1}{(q^3;q^3)_n}$

The interpretation of the moves we applied to the singletons: $\frac{1}{(q;q)_m}$

As a result,

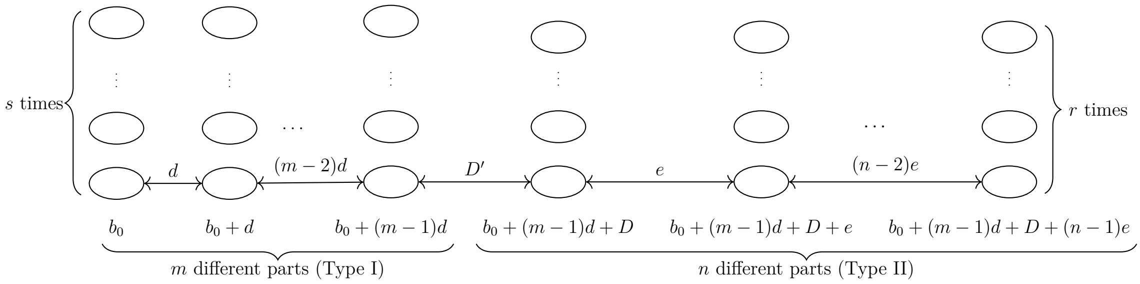
$$T_{m,n}(x) = \frac{q^{6n^2+6mn+2m^2} x^{2n+m}}{(q^3;q^3)_n (q;q)_m}.$$

Remark 20. One needs to define the moves properly and show that any partition in $\mathcal{C}$ with n pairs and m singletons can be obtained from the base partition via moves. The details are in [24].

Now, we will generalize the above example.

We start with a base partition: We have m different Type 1 parts where each Type 1 part appears s times, the smallest of them is b_0 and the difference between each Type 1 part is d. The difference between the largest Type 1 part and the smallest Type 2 part is D. Each Type 2 part appears r times, the difference between each

Type 2 part is e. Now, we can compute the weight of that partition and the length of that partition. This allows us to connect the combinatorial parameters to algebraic parameters.



Then, the weight of the base partition is:

$$ds\binom{m+1}{2}+er\binom{n+1}{2}+rd(mn)+m(sb_0-ds)+n(rb_0-rd+rD'-er). \tag{2.3}$$

The length of the base partition is: $ms+nr$.

Now, we will explain the connection between the algebraic framework and the combinatorial framework.

Name all the coefficients in the exponents of (2.1) as follows:

$$T_{D,E}(x)=\sum_{m,n\geq 0}\frac{q^{A\binom{m+1}{2}+B\binom{n+1}{2}+Cmn+Dm+En}x^{Fm+Gn}}{(q^K;q^K)_m(q^L;q^L)_n}. \tag{2.4}$$

Remark 21. Although at the right hand side of (2.4) there are a lot of parameters, we use only D and E to name the left hand side. The reason is that, in the functional equations, only the parameters D and E may change, the other parameters are fixed.

Thus, combining (2.3) and the length of the base partition with (2.4) gives:

$$\begin{aligned}A&=ds\\B&=er\\C&=rd\\D&=sb_0-ds\\E&=rb_0-rd+rD'-er\\b_0&=d\\e&=D'.\end{aligned} \tag{2.5}$$

Obviously, all d,e,r,s,b_0,D' must be nonnegative integers. As a result, if the conditions in (2.5) are not satisfied, then we cannot interpret the given sum via the moves framework we described.

Remark 22. There is a symmetry on the parameters. Basically, the generating function corresponds to the parameters $(A, B, C, D, E, F, G, K, L)$ is exactly the same as $(B, A, C, E, D, G, F, L, K)$. Thus, the conditions in (2.5) can change accordingly.

Observe that comparing (2.1) and (2.4) leads to a natural question: Why are we using the binomial coefficient $\binom{m+1}{2}$ instead of the quadratic m^2? The reason is that we want to use integer parameters in our exponents and with m^2 we need to use half-integers as well, for example, can be seen from (2.2).

We conclude this section with multiple examples.

Example 10.

1. We start with partitions which have only one part type. For instance, the distinct partitions can be thought as: Start with a partition $(n, n-1, n-2, \ldots, 2, 1)$ and move forward each part where the forward moves on all parts gives a partition, i.e we push n at least as many times as we push $n-1$ etc. Thus, the difference between consecutive parts would be at least 1. This gives us:
$$\sum_{\lambda \in \mathcal{D}} q^{|\lambda|} = \sum_{n=0}^{\infty} \frac{q^{\binom{n+1}{2}}}{(q;q)_n}.$$
where $\mathcal{D}$ is the set of all distinct partitions.

   Note that this generating function can be interpret as:

   Numerator: Minimum weight distinct partition which contains n parts

   Denominator: "Forward Moves" applied on these parts

2. Similarly, any 2-distinct partition can be seen as: Start with the partition $(2n-1, 2n-3, \ldots, 3, 1)$ and push larger parts at least as much as smaller parts. As a result, we get:
$$\sum_{\lambda \in \mathcal{D}_2} q^{|\lambda|} = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n}.$$
where $\mathcal{D}_2$ is the set of all partitions where the consecutive difference between each part is at least 2.

3. We can generalize the above two examples by considering any s-distinct partition: We start with a partition $(n-1)s + 1 + (n-2)s + 1 + \cdots + s + 1 + 1$. We can push a part forward by d units, as long the consecutive differences are

at least s. This gives us:

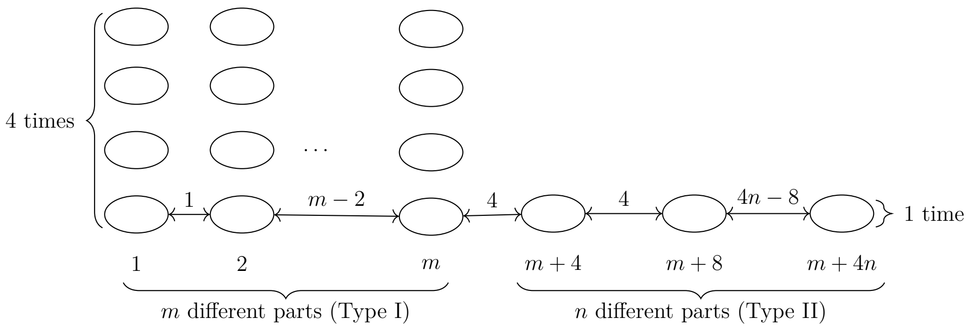
$$\sum_{\lambda\in\mathcal{D}_s} q^{|\lambda|} = \sum_{n=0}^{\infty} \frac{q^{s\binom{n+1}{2}-(s-1)n}}{(q^d;q^d)_n}.$$

Remark 23. One can keep track of the number of parts, via putting an x into the generating function.

4. Let $s = 1$, $d = 1$, $r = 4$, $e = 4$, $K = 5$, $L = 2$. This corresponds to the generating function:

$$\sum_{m,n\geq 0} \frac{q^{4\binom{m+1}{2}+4mn+4\binom{n+1}{2}}x^{4m+n}}{(q^5;q^5)_m(q^2;q^2)_n}.$$

We hope to find a proper definition of moves that explains this generating function via moves framework. More precisely, we need to define moves on Type 1 parts and Type 2 parts which increases the weight of the partition via 5 and 2, respectively.



5. Consider Rogers-Ramanujan-Gordon partitions where $k = a = 3$ in Theorem 4. In this case, a part can repeat 1 or 2 times. Thus, a natural candidate for the types is: The parts that appear once (we will call them singletons), the parts that appear twice (we will call them pairs). Thus any partition in $\mathcal{RRG}_{3,3}$ can be constructed via starting with a base partition of the form $[1,1]+[3,3]+[5,5]+\cdots+[2m-1,2m-1]+(2m+1)+(2m+2)+\cdots+(2m+n)$. Then, we push forward any part one by one:

$$\sum_{\lambda\in\mathcal{RRG}_{3,3}} q^{\lambda}x^{|\lambda|} = \sum_{m,n\geq 0} \frac{q^{4\binom{m+1}{2}-2m+2mn+\binom{n+1}{2}-n}x^{2m+n}}{(q;q)_m(q;q)_n}.$$

The details are presented in [23].

6. Same example as above, however the moves on the parts are defined differently. This time, we push the pairs together. We will explain this example in great

detail below. This example is given here to show that it is possible to have different type and moves combinations for the same partition class.

$$\sum_{\lambda \in \mathcal{RRG}_{3,3}} q^{\lambda} x^{|\lambda|} = \sum_{m,n \geq 0} \frac{q^{4\binom{m+1}{2}-2m+2mn+\binom{n+1}{2}-2n} x^{2m+n}}{(q^2;q^2)_m (q;q)_n}.$$

In the next section, we prove the following theorem, which brings a moves framework interpretation of Rogers-Ramanujan Gordon partitions where $k = a = 3$.

Remark 24. In principle, if you have a s-fold sum, you expect to have s different types of parts for $s \geq 1$.

2.2 Moves Interpretation of Rogers Ramanujan Gordon $k = a = 3$

This section is submitted to a journal, we give the arxiv link : [21]

Before explaining the moves in full generality, we start with an example:

Example 11. Consider the partition $\lambda = 14+14+11+10+7+7+5+5+2+1$. Note that this partition satisfies the conditions of $\mathcal{RRG}_{3,3}$. We will use this partition to construct a partition triple: (β, μ, ν) where β is the base partition, μ is the partition which contains *backward moves* applied on the pairs (the parts that repeat) of λ and ν is the partition which contains backward moves applied on the singletons (the parts that do not repeat) of λ.

Step 1: Firstly, divide the parts into two: The ones that repeat and those that do not repeat. Thus, our partition is of the form:

$$[14, 14] + (11) + (10) + [7, 7] + [5, 5] + (2) + (1).$$

Now, we are looking for a partition which contains $m = 3$ pairs and $n = 4$ singletons.

Step 2: We will answer the following question: What is the smallest weight partition that contains 3 pairs and 4 singletons that satisfies the $\mathcal{RRG}_{3,3}$ conditions?

It can be seen that this is the following partition:

$$\beta = (10) + (9) + (8) + (7) + [5, 5] + [3, 3] + [1, 1].$$

Thus, this would our base partition.

Step 3: We want to reach β from λ using backward moves: Firstly, we will obtain $[1,1]$, using backward moves:

$$[14,14]+(11)+(10)+[7,7]+[5,5]+(2)+(1) \rightarrow$$
$$[14,14]+(11)+(10)+[7,7]+[\mathbf{4},\mathbf{4}]+(2)+(1).$$

Now we want to pull $[4,4]$ back and get $[3,3]$. However, the resulting partition, $[14,14]+(11)+(10)+[7,7]+[\mathbf{3},\mathbf{3}]+(2)+(1)$, will not satisfy the $\mathcal{RRG}_{3,3}$ conditions. Thus, we need to do adjustments on the nearby parts (the details are given in the proof of Theorem 11).

$$[14,14]+(11)+(10)+[7,7]+[\mathbf{4},\mathbf{4}]+(2)+(1) \rightarrow$$
$$[14,14]+(11)+(10)+[7,7]+(4)+(3)+[\mathbf{1},\mathbf{1}].$$

We obtained $[1,1]$, as desired. Note that at each backward move on pairs, the weight of the partition decreases by 2. We can operate inductively, i.e $[1,1]$ will not interfere with our moves anymore. Now, we want to get $[3,3]$, the next smallest part in β:

$$[14,14]+(11)+(10)+[7,7]+(4)+(3)+[1,1] \rightarrow$$
$$[14,14]+(11)+(10)+[\mathbf{6},\mathbf{6}]+(4)+(3)+[1,1] \rightarrow$$
$$[14,14]+(11)+(10)+(6)+(5)+[\mathbf{3},\mathbf{3}]+[1,1].$$

Note that a similar adjustment is done in the backward move to get $[3,3]$. Similarly, we continue with obtaining $[5,5]$:

$$[14,14]+(11)+(10)+(6)+(5)+[3,3]+[1,1] \rightarrow$$
$$[\mathbf{13},\mathbf{13}]+(11)+(10)+(6)+(5)+[3,3]+[1,1] \rightarrow$$
$$(13)+(12)+[\mathbf{10},\mathbf{10}]+(6)+(5)+[3,3]+[1,1] \rightarrow$$
$$(13)+(12)+[\mathbf{9},\mathbf{9}]+(6)+(5)+[3,3]+[1,1] \rightarrow$$
$$(13)+(12)+[\mathbf{8},\mathbf{8}]+(6)+(5)+[3,3]+[1,1] \rightarrow$$
$$(13)+(12)+(8)+(7)+[\mathbf{5},\mathbf{5}]+[3,3]+[1,1].$$

We now got the pairs of β. We continue with singletons. Note that since in the base partition the pairs come before the singletons, we do not need to do any adjustment on backward moves on singletons. Our first aim is to obtain (7). As (7) is already contained in the partition at hand, we do not perform any moves. Similar arguments applies for (8). Therefore, we continue with (9):

$$
\begin{gathered}
(13)+(12)+(8)+(7)+[5,5]+[3,3]+[1,1] \rightarrow \\
(13)+(\mathbf{11})+(8)+(7)+[5,5]+[3,3]+[1,1] \rightarrow \\
(13)+(\mathbf{10})+(8)+(7)+[5,5]+[3,3]+[1,1] \rightarrow \\
(13)+(\mathbf{9})+(8)+(7)+[5,5]+[3,3]+[1,1].
\end{gathered}
$$

Note that each backward move on singletons decreases the weight by 1. Lastly, we need to get (10):

$$
\begin{gathered}
(13)+(9)+(8)+(7)+[5,5]+[3,3]+[1,1] \rightarrow \\
(\mathbf{12})+(9)+(8)+(7)+[5,5]+[3,3]+[1,1] \rightarrow \\
(\mathbf{11})+(9)+(8)+(7)+[5,5]+[3,3]+[1,1] \rightarrow \\
(\mathbf{10})+(9)+(8)+(7)+[5,5]+[3,3]+[1,1].
\end{gathered}
$$

This yields the base partition β. We are left to construct the partitions that contains backward moves applied on the pairs, and singletons, namely μ and ν.

Step 4: We applied five backward moves on $[14,14]$, two backward moves on $[7,7]$, and two backward moves on $[5,5]$ where each move decreases the weight by 2. This corresponds to the partition $\mu=(10,4,4)$. Similarly, we applied three backward moves on (11), three backward moves on (10), and no backward moves on (2) and (1). This corresponds to the partition $\nu=(3,3,0,0)$. As a small remark, we allow μ and ν to contain 0 as a part. Thus, we have

$$
14{+}14{+}11{+}10{+}7{+}7{+}5{+}5{+}2{+}1 \rightarrow (10{+}9{+}8{+}7{+}5{+}5{+}3{+}3{+}1{+}1,\ 10{+}4{+}4,\ 3{+}3{+}0{+}0).
$$

As a result, this is one direction of the bijection between

$$
\mathcal{RRG}_{3,3} \rightarrow (\text{Base Partition, Backward Moves on Pairs, Backward Moves on Singletons}).
$$

For the other direction of the same example, i.e. given $\beta=(10)+(9)+(8)+(7)+[5,5]+[3,3]+[1,1]$, $\mu=10+4+4$ and $\nu=3+3+0+0$, we want to obtain $\lambda=[14,14]+(11)+(10)+[7,7]+[5,5]+2+1$. This time, we will apply *forward moves* on β, the base partition, to get λ.

Step 1: This time, we will start with the moves on the singletons, as we applied the moves on the singletons last in the other direction. Given $\nu=3+3+0+0$, we need to push (10) and (9) three times, and we do not push (8) and (7). More explicitly,

$$
\begin{aligned}
(10) + (9) + (8) + (7) + [5,5] + [3,3] + [1,1] &\to \\
(\mathbf{11}) + (9) + (8) + (7) + [5,5] + [3,3] + [1,1] &\to \\
(\mathbf{12}) + (9) + (8) + (7) + [5,5] + [3,3] + [1,1] &\to \\
(\mathbf{13}) + (9) + (8) + (7) + [5,5] + [3,3] + [1,1] &\to \\
(13) + (\mathbf{10}) + (8) + (7) + [5,5] + [3,3] + [1,1] &\to \\
(13) + (\mathbf{11}) + (8) + (7) + [5,5] + [3,3] + [1,1] &\to \\
(13) + (\mathbf{12}) + (8) + (7) + [5,5] + [3,3] + [1,1]&.
\end{aligned}
$$

We now have the singletons of λ.

Step 2: We continue with the pairs. Since $\mu = 10 + 4 + 4$, we need to apply five forward moves on $[5,5]$, two forward moves on $[3,3]$ and two forward moves on $[1,1]$. As a result,

$$
\begin{aligned}
(13) + (12) + (8) + (7) + [5,5] + [3,3] + [1,1] &\to \\
(13) + (12) + [\mathbf{8},\mathbf{8}] + (6) + (5) + [3,3] + [1,1] &\to \\
(13) + (12) + [\mathbf{9},\mathbf{9}] + (6) + (5) + [3,3] + [1,1] &\to \\
(13) + (12) + [\mathbf{10},\mathbf{10}] + (6) + (5) + [3,3] + [1,1] &\to \\
[\mathbf{13},\mathbf{13}] + (11) + (10) + (6) + (5) + [3,3] + [1,1] &\to \\
[\mathbf{14},\mathbf{14}] + (11) + (10) + (6) + (5) + [3,3] + [1,1] &\to \\
[14,14] + (11) + (10) + [\mathbf{6},\mathbf{6}] + (4) + (3) + [1,1] &\to \\
[14,14] + (11) + (10) + [\mathbf{7},\mathbf{7}] + (4) + (3) + [1,1] &\to \\
[14,14] + (11) + (10) + [7,7] + [\mathbf{4},\mathbf{4}] + (2) + (1) &\to \\
[14,14] + (11) + (10) + [7,7] + [\mathbf{5},\mathbf{5}] + (2) + (1)&.
\end{aligned}
$$

Thus, we recover λ. Observe that the forward moves and the backward moves are exactly opposite to each other in this example. This yields the other direction of the bijection

(Base Partition, Forward Moves On Pairs, Forward Moves On Singletons) $\to \mathcal{RRG}_{3,3}$.

Going back to the example, we have

$(10{+}9{+}8{+}7{+}5{+}5{+}3{+}3{+}1{+}1,\ 10{+}4{+}4,\ 3{+}3{+}0{+}0) \to 14{+}14{+}11{+}10{+}7{+}7{+}5{+}5{+}2{+}1.$

Now, we will look at another example where $a = 2$ instead of $a = 3$. In this case, the order of the moves and the definition of the moves are different. The main reason is that the form of the base partition is different in this case.

Example 12. Let $a = 2$, instead of $a = 3$ as in Example 11, and $\lambda = 17 + 13 + 9 + 6 + 6 + 4 + 4 + 1$.

Firstly, the number of pairs is 2, i.e $m = 2$ and the number of singletons is 4, i.e $n = 4$. We are looking for the smallest weight partition, the base partition, which satisfies $\mathcal{RRG}_{3,2}$ conditions. Moreover, it should contain 2 pairs and 4 singletons. So, the base partition is:

$$\beta = [8, 8] + [6, 6] + (4) + (3) + (2) + (1).$$

Note that since $a = 2$, we are not allowed to use $[1, 1]$ in the base partition. We note here that the uniqueness and the particular form of β will become more evident by Lemma 12. Our aim is to reach from λ to β using backward moves.

Step 1: Firstly, we want to obtain (1). Since we already have (1) in λ, there is no need to apply any backward moves. We continue with (2):

$$\begin{gathered}(17) + (13) + (9) + [6, 6] + [4, 4] + (1) \rightarrow \\ (17) + (13) + (\mathbf{8}) + [6, 6] + [4, 4] + (1) \rightarrow \\ (17) + (13) + [7, 7] + [5, 5] + (\mathbf{3}) + (1) \rightarrow \\ (17) + (13) + [7, 7] + [5, 5] + (\mathbf{2}) + (1).\end{gathered}$$

Note that, at the backward move which makes (8), (3) we need to make some adjustments to the nearby parts. Next, we get (3):

$$\begin{gathered}(17) + (13) + [7, 7] + [5, 5] + (2) + (1) \rightarrow \\ (17) + (\mathbf{12}) + [7, 7] + [5, 5] + (2) + (1) \rightarrow \\ (17) + (\mathbf{11}) + [7, 7] + [5, 5] + (2) + (1) \rightarrow \\ (17) + (\mathbf{10}) + [7, 7] + [5, 5] + (2) + (1) \rightarrow \\ (17) + (\mathbf{9}) + [7, 7] + [5, 5] + (2) + (1) \rightarrow \\ (17) + [8, 8] + [6, 6] + (\mathbf{4}) + (2) + (1) \rightarrow \\ (17) + [8, 8] + [6, 6] + (\mathbf{3}) + (2) + (1).\end{gathered}$$

Lastly, we obtain (4):

$$
\begin{aligned}
&(17) + [8,8] + [6,6] + (3) + (2) + (1) \rightarrow \\
&(\mathbf{16}) + [8,8] + [6,6] + (3) + (2) + (1) \rightarrow \\
&(\mathbf{15}) + [8,8] + [6,6] + (3) + (2) + (1) \rightarrow \\
&(\mathbf{14}) + [8,8] + [6,6] + (3) + (2) + (1) \rightarrow \\
&(\mathbf{13}) + [8,8] + [6,6] + (3) + (2) + (1) \rightarrow \\
&(\mathbf{12}) + [8,8] + [6,6] + (3) + (2) + (1) \rightarrow \\
&(\mathbf{11}) + [8,8] + [6,6] + (3) + (2) + (1) \rightarrow \\
&(\mathbf{10}) + [8,8] + [6,6] + (3) + (2) + (1) \rightarrow \\
&[9,9] + [7,7] + (\mathbf{5}) + (3) + (2) + (1) \rightarrow \\
&[9,9] + [7,7] + (\mathbf{4}) + (3) + (2) + (1).
\end{aligned}
$$

Now, we obtained all singletons in the base partition β. Note that each backward move on the singletons decreases the weight of the partition by 1.

Step 2: We continue with the pairs. First, we will obtain $[6,6]$:

$$[9,9] + [7,7] + (4) + (3) + (2) + (1) \rightarrow [9,9] + [\mathbf{6},\mathbf{6}] + (4) + (3) + (2) + (1).$$

To conclude, we need to obtain $[8,8]$:

$$[9,9] + [6,6] + (4) + (3) + (2) + (1) \rightarrow [\mathbf{8},\mathbf{8}] + [6,6] + (4) + (3) + (2) + (1).$$

We arrived at β. Note that each backward move on pairs decreases the weight by 2.

Step 3: We are left to construct the partitions μ and ν. Counting the total number of backward moves: nine backward moves on (17), six backward moves on (13), three backward moves on (9), no backward moves on (1), one backward move on $[6,6]$, and one backward move on $[4,4]$ leads to $\nu = (9,6,3,0)$ and $\mu = (2,2)$. As a result, what we establish is the correspondence:

$$17+13+9+6+6+4+4+1 \rightarrow (8+8+6+6+4+3+2+1\,,\ 2+2\,,\ 9+6+3+0).$$

For the other direction, we start from the triple partition

$$(\beta,\mu,\nu) = ([8,8] + [6,6] + (4) + (3) + (2) + (1),\ 2+2,\ 9+6+3+0).$$

We want to obtain $\lambda = 17+13+9+6+6+4+4+1$ using forward moves on β.

Step 1: Similar to Example 11, we will reverse the order of the moves we applied.

More precisely, we ended with pairs for the backward moves; now we will start with forward moves on pairs. Since $\mu = 2 + 2$, this means that we need to apply one forward move on $[8, 8]$ and one forward move on $[6, 6]$.

$$\begin{aligned} &[8,8] + [6,6] + (4) + (3) + (2) + (1) \to \\ &[\mathbf{9},\mathbf{9}] + [6,6] + (4) + (3) + (2) + (1) \to \\ &[9,9] + [\mathbf{7},\mathbf{7}] + (4) + (3) + (2) + (1). \end{aligned}$$

This concludes the part with the pairs.

Step 2: Next, we continue with the singletons. Firstly, we apply nine forward moves on (4):

$$\begin{aligned} &[9,9] + [7,7] + (4) + (3) + (2) + (1) \to \\ &[9,9] + [7,7] + (\mathbf{5}) + (3) + (2) + (1) \to \\ &(\mathbf{10}) + [8,8] + [6,6] + (3) + (2) + (1) \to \\ &(\mathbf{11}) + [8,8] + [6,6] + (3) + (2) + (1) \to \\ &(\mathbf{12}) + [8,8] + [6,6] + (3) + (2) + (1) \to \\ &(\mathbf{13}) + [8,8] + [6,6] + (3) + (2) + (1) \to \\ &(\mathbf{14}) + [8,8] + [6,6] + (3) + (2) + (1) \to \\ &(\mathbf{15}) + [8,8] + [6,6] + (3) + (2) + (1) \to \\ &(\mathbf{16}) + [8,8] + [6,6] + (3) + (2) + (1) \to \\ &(\mathbf{17}) + [8,8] + [6,6] + (3) + (2) + (1). \end{aligned}$$

Similarly, we continue with six moves on (3):

$$\begin{aligned} &(17) + [8,8] + [6,6] + (3) + (2) + (1) \to \\ &(17) + [8,8] + [6,6] + (\mathbf{4}) + (2) + (1) \to \\ &(17) + (\mathbf{9}) + [7,7] + [5,5] + (2) + (1) \to \\ &(17) + (\mathbf{10}) + [7,7] + [5,5] + (2) + (1) \to \\ &(17) + (\mathbf{11}) + [7,7] + [5,5] + (2) + (1) \to \\ &(17) + (\mathbf{12}) + [7,7] + [5,5] + (2) + (1) \to \\ &(17) + (\mathbf{13}) + [7,7] + [5,5] + (2) + (1). \end{aligned}$$

Lastly, we need to apply three forward moves on (2) in order to get λ back:

$$
\begin{aligned}
&(17)+(13)+[7,7]+[5,5]+(2)+(1)\to\\
&(17)+(13)+[7,7]+[5,5]+(\mathbf{3})+(1)\to\\
&(17)+(13)+(\mathbf{8})+[6,6]+[4,4]+(1)\to\\
&(17)+(13)+(\mathbf{9})+[6,6]+[4,4]+(1).
\end{aligned}
$$

Hence, we got the correspondence:

$$(8+8+6+6+4+3+2+1\,,\ 2+2\,,\ 9+6+3+0)\to 17+13+9+6+6+4+4+1,$$

as desired.

For the rest of the section, we state the main theorem and give the proof that the above procedures always work.

Recall that we can use 1 at most $a-1$ times as a part of $\mathcal{RRG}_{k,a}$ partitions, and we formally define the "forward moves" and "backward moves". Then, we show that we get the bijection introduced in Section 2.2 using these moves. This allows us to write a series for $\mathcal{RRG}_{3,a}$ partitions. The next theorem treats the case $a=3$. Moreover, at the end of the section we explain what happens if $a=2$ or $a=1$. We directly state our main theorem:

Theorem 11. Let $rrg_{3,3}(m,n)$ denote the number of partitions of n with exactly m parts which satisfies the $\mathcal{RRG}_{3,3}$ conditions. Then,

$$RRG_{3,3}(x)=\sum_{m,n\geq 0} rrg_{3,3}(m,n)x^m q^n=\sum_{m,n\geq 0}\frac{q^{4\binom{m+1}{2}+2mn+\binom{n+1}{2}-2m}x^{2m+n}}{(q^2;q^2)_m(q;q)_n}.$$

To prove this theorem, first of all we need a lemma which shows that the base partition is really what we claimed, i.e. it is of the form $(2m+n)+\cdots+(2m+3)+(2m+1)+[2m-1,2m-1]+\cdots+[3,3]+[1,1]$:

Lemma 12. The base partition for $\mathcal{RRG}_{3,3}$ with m pairs and n singletons, i.e the partition which satisfies the $\mathcal{RRG}_{3,3}$ conditions with m pairs and n singletons and has the smallest weight is the following unique partition:

$$(2m+n)+(2m+n-1)+\cdots+(2m+1)+[2m-1,2m-1]+[2m-3,2m-3]+\cdots+[3,3]+[1,1].$$

Proof. We need to prove the following:

1. The form of the base partition is as claimed
2. Moreover it is unique, i.e it is justified to call it "the" base partition.

If it is not the smallest weight partition, then there must exist a backward move on one of the parts so that the weight can be decreased. Either we can move a singleton or a pair backward. We will show that neither a backward singleton move nor a backward pair move is possible.

Suppose a backward move on a singleton is possible. If we want to move a singleton backward, we need to move $(2m+1)$. This is not allowed due to the presence of the pairs $[2m-1, 2m-1] + [2m-3, 2m-3] + \cdots + [1, 1]$ as this would lead to a partition of the form:

$$\text{Larger Parts} + (2m+2) + [2m, 2m] + [2m-2, 2m-2] + \cdots + [2, 2] + (1).$$

However, $[2, 2] + (1)$ violates the difference conditions. Thus, the backward move on singletons is not a possibility.

Similarly, we cannot apply a backward move on pairs, since the pairs already start from the smallest number, $[1, 1]$, and the difference between consecutive pairs is the smallest as well. Thus, a backward move on pairs is not possible as well. As a result, this must be the smallest weight partition. Moreover, the form is unique, since each consecutive difference is the smallest possible that satisfies $\mathcal{RRG}_{3,3}$ conditions. $\square$

Proof of the main theorem. We will do the proof in three steps: First, we will show that if forward and backward moves are defined properly, we get the claimed series. Second, the proper definition of forward moves is given. Third, we give the proper definition of backward moves.

Step 1: We show that each partition, λ , counted by $rrg_{3,3}(m, n)$ corresponds to a triple of partitions (β, μ, ν) where β is the corresponding base partition, μ is the partition which keeps the forward moves on pairs and ν is the partition which keeps the forward moves on singletons. Let n_1 be the number of pairs in λ and n_2 be the number of singletons in λ. Then, by Lemma 12, $\beta = (2n_1 + n_2 - 1) + \cdots + (2n_1 + 1) + (2n_1) + [2n_1 - 1, 2n_1 - 1] + \cdots + [3, 3] + [1, 1]$. Note that β has $2n_1 + n_2$ parts and

$$|\beta| = 4\binom{n_1+1}{2} + 2n_1n_2 + \binom{n_2+1}{2} - 2n_1.$$

Also, since there are n_1 pairs and n_2 singletons in λ, the partition μ has n_1 even parts (because each forward move on pairs increases the weight of the partition by

2) and ν has n_2 parts. Hence, we obtain

$$\sum_{\substack{n_1,n_2\geq 0\\ \mu,\nu \text{ are partitions as above}}} q^{|\beta|+|\mu|+|\nu|}x^{2n_1+n_2} = \sum_{n_1,n_2\geq 0} \frac{q^{4\binom{n_1+1}{2}+2n_1n_2+\binom{n_2+1}{2}-2n_1}x^{2n_1+n_2}}{(q^2;q^2)_{n_1}(q;q)_{n_2}}.$$

Using (β,μ,ν) we will get a unique partition λ via forward moves, and given λ we will get a unique (β,μ,ν) via backward moves. As a result, we will prove that

$$\sum_{n,m\geq 0} rrg_{3,3}(m,n)x^mq^n = \sum_{n_1,n_2\geq 0} \frac{q^{4\binom{n_1+1}{2}+2n_1n_2+\binom{n_2+1}{2}-2n_1}x^{2n_1+n_2}}{(q^2;q^2)_{n_1}(q;q)_{n_2}}.$$

Step 2: We will investigate the forward moves now: Given (β,μ,ν), we want to push the parts of the base partition β to obtain λ, a $\mathcal{RRG}_{3,3}$ partition. We will apply the forward moves on the singletons. More precisely, we push the largest singleton in the base partition ν_1 times, the second largest singleton ν_2 times, $\ldots$, and the smallest singleton ν_{n_2} times. Then, we apply the forward moves on the pairs. Again, we push the largest pair μ_1 times, the second largest pair μ_2 times, $\ldots$, and the smallest pair μ_{n_1} times. We now investigate the details of the moves:

1. Forward moves on singletons: Since in the base partition the pairs come before the singletons, the forward moves on singletons are straightforward. Pushing a singleton (a) gives us $(a+1)$, which increases the weight of the partition by 1.

2. Forward moves on pairs: If there is a pair $[b,b]$ we want to push, there are two cases to consider:

   (a) The pair becomes $[b+1,b+1]$ and the resulting partition satisfies $\mathcal{RRG}_{3,3}$ conditions. In this case, the move is defined as: $[b,b]\to[\mathbf{b+1},\mathbf{b+1}]$.

   (b) We cannot make it $[b+1,b+1]$ because it violates the conditions of $\mathcal{RRG}_{3,3}$. The reason is the existence of the singleton $(b+2)$, and the possible existence of other singletons $(b+3),(b+4),\ldots,(b+s)$ for some integer $s\geq 2$. Then, the forward move on the pair is defined as

$$\begin{aligned}(b+s)+(b+s-1)+\cdots+(b+3)+(b+2)+[b,b] &\to\\ [\mathbf{b+s},\mathbf{b+s}]+(b+s-2)+\cdots+(b+1)+(b).&\end{aligned}$$

   Note that in either case, the weight of the partition increases by 2.

As a result, given (β,μ,ν) we get a unique λ which satisfies $\mathcal{RRG}_{3,3}$ conditions.

Step 3: Lastly, we need to investigate the backward moves: Given a λ which satisfies $\mathcal{RRG}_{3,3}$ conditions, first we look at the number of pairs and singletons in λ. This allows us to find the corresponding base partition, β. Our aim is to pull parts of λ and get β. We start with the smallest pair in λ and apply backward moves on it until it becomes $[1,1]$ and continue with the next smallest pair of λ and apply backward moves on it until it becomes $[3,3]$ and so on. Once we obtain all pairs of the base partition β, we will continue with the singletons, again from the smallest singletons to the largest singleton. In other words, we will apply backward moves on the smallest singletons in λ until we have the smallest singleton in β and so on. More precisely, the backward moves are defined as:

1. For the backward moves on pairs, there are two cases to consider:

   (a) A backward move on $[b+1,b+1]$ is defined as : $[b+1,b+1] \to [\mathbf{b},\mathbf{b}]$, if it does not violate the $\mathcal{RRG}_{3,3}$ conditions.

   (b) We cannot turn $[b+1,b+1]$ into $[b,b]$, i.e., there is a singleton $(b-1)$ and possibly $(b-2),(b-3),\ldots,(b-s)$, where s in an integer $2 \leq s < b$. In this case, we define the backward move as $[b+1,b+1]+(b-1)+(b-2)+\cdots+(b-s+1)+(b-s) \to (b+1)+(b)+\cdots+(b-s+2)+[\mathbf{b-s},\mathbf{b-s}]$. Note that in either case, the weight of the partition decreases by 2.

2. Backward moves on singletons: Since, we will pull the pairs back first, the backward moves on singletons are straightforward: $(b+1)$ becomes (b). Note that the weight of the partition decreases by 1.

Observe that the backward moves are exactly the opposite of the forward moves. For each pair, we keep track of the backward moves we applied to that pair. We put this into μ. Similarly, each backward move on singletons is stored in ν. More precisely, μ would be a vector of length n_1 where μ_i is the number of backward moves applied to the ith largest pair in λ. Similarly, ν would be a vector of length n_2 where ν_i is the number of backward moves applied to the ith largest singleton in λ. As a result, given λ satisfying $\mathcal{RRG}_{3,3}$ conditions, we get a unique (β,μ,ν). This concludes the proof. $\square$

In Theorem 11, we only consider the case $a=3$. It is also possible to take $a=1$ or $a=2$. We will discuss the details in the following remarks.

Remark 25. The proof of Theorem 11 works for $a=1$ as well, the only difference is that the corresponding base partition is $\beta=(2n_1+n_2)+\cdots+(2n_1+2)+(2n_1+1)+[2n_1,2n_1]+\cdots+[4,4]+[2,2]$ instead of $(2n_1+n_2-1)+\cdots+(2n_1+1)+(2n_1)+[2n_1-1,2n_1-1]+\cdots+[3,3]+[1,1]$. The reason is that the form of the base partition is exactly the same in the $a = 3$ case. The corresponding generating function is

$$RRG_{3,1}(x) = \sum_{n,m\geq 0} \frac{q^{4\binom{m+1}{2}+2mn+\binom{n+1}{2}+n} x^{2m+n}}{(q^2;q^2)_m(q;q)_n}.$$

Remark 26. The proof of Theorem 11 works for $a = 2$ if the definition of the moves are changed as well as the base partition:

1. Our base partition changes since we cannot use $[1, 1]$ anymore. As a result, our base partition would be the following:

   $$[n+2m, n+2m]+[n+2m-2, n+2m-2]+\cdots+[n+2, n+2]+(n)+(n-1)+\cdots+(2)+(1).$$

   We can use a similar arguments as in Lemma 12 to show that this is indeed the base partition. In particular, it is clear that we cannot perform a backward move on singletons of this base partition. Similarly, backward move on pairs is not possible as well. The main reason is that this base partition is constructed so that the difference between each consecutive part is as small as possible. One example which shows why we cannot use the pairs first: Suppose we have one pair and one singleton, then $(4) + [2, 2]$ has more weight that $[3, 3] + (1)$. Thus, in this case, the smallest weight partition is $[3, 3] + (1)$ instead of $(4) + [2, 2]$.

   Since, now, in the base partition the singletons comes before the pairs, we need to redefine the moves as well:

2. We will look at the forward moves first:

   (a) Forward moves on pairs: Since pairs comes after the singletons in the base partition, the move is defined as $[b, b] \to [b + 1, b + 1]$. Note that, the weight of the partition increases by 2.

   (b) Forward moves on singletons. We have two cases to consider:

      i. If you can push them, without violating the conditions, the move is defined as $b \to b + 1$. The weight of the partition increases by 1.

      ii. Otherwise, there are some pairs which prevents our move, namely we are in the situation $[b+2s, b+2s]+\cdots+[b+4, b+4]+[b+2, b+2]+(b)$. Then, the forward move on (b) is $(\mathbf{b + 2s + 1}) + [b + 2s - 1, b + 2s - 1] + \cdots + [b + 3, b + 3], [b + 1, b + 1]$. Note that the weight of the partition increases by 1.

   (c) We will look at the details of backward moves now:

      i. Backward moves on pairs: Since there cannot be any restriction,

$[b+1, b+1]$ becomes $[b, b]$. Note that the weight of the partition decreases by 2.

ii. Backward moves on singletons:

A. If you can pull them, without violating the conditions, then the backward move is $(b+1)$, becomes (b).

B. However, it can be the case that there are some restrictions due to nearby pairs: $(b+1)+[b-1, b-1]+[b-3, b-3]+\cdots+[b-2s+1, b-2s+1]$ for some integer $1 \leq s < b/2$, then the backward move is defined as $[b, b]+[b-2, b-2]+\cdots+[b-2s+2, b-2s+2]+(\mathbf{b-2s})$.

3. Unlike the $a = 3$ case, first we will apply forward moves on pairs first and then singletons. On the other hand, we will apply backward moves on singletons first and then the pairs.

4. As a result, we get the generating function

$$RRG_{3,2}(x) = \sum_{n,m \geq 0} \frac{q^{4\binom{m+1}{2}+2mn+\binom{n+1}{2}} x^{2m+n}}{(q^2; q^2)_m (q; q)_n}.$$

Remark 27. As a side note, it is possible to construct our series via adding and removing staircases. For the adding/removing staircase method, one can check [5] and [20]. Note that this method is called *adding/subtracting triangles* in the former article.

2.3 Advantages and Disadvantages of Moves Framework

Advantages of the method: Normally, the combinatorial interpretations are made example by example; in other words, for each new identity or series, one needs to come up with a new interpretation from scratch. This framework allows us to combine a lot of different-looking series under one roof.

The disadvantages of the approach: The types of parts and definition of the moves should be defined in each case differently. Another problem is that the moves on each type should work together properly as well. In other words, maybe individually they work fine, but the interplay between them is not well-behaved.

Also, this method is not continuous or recursive. By noncontinuity, what we mean is that if you make a small change in the parameters, then you may not get a small change in the moves. Hence, a small change in the parameters may require a brand new definition of moves, and-or a new definition of types of parts. By nonrecursiveness, what we mean is that if you found an interpretation with 2 different types and want to extend this to 3 different types, you may not be able to use the moves in the 2 different type case.

What happens if the generating function we start with does not satisfy the moves framework conditions? In other words, if (2.5) does not hold. How can we try to find a combinatorial interpretation? In the following chapter, we give a partial answer to this question. For example, the parameters $(A, B, C, D, E, F, G, K, L, \gamma) = (1, 2, 1, 0, 0, 1, 1, 1, 1, 1)$ do not satisfy the constraints in (2.5). We look at these parameters in detail in the next chapter.

3. Maple Program

In this section, we give another methodology to interpret a generating function of the form (2.1):

$$\sum_{m,n\geq 0} \frac{q^{Q(m,n)+L(m,n)}x^{L'(m,n)}}{(q^K;q^K)_m(q^L;q^L)_n}.$$

Also, for the sake of completeness, recall that we name the exponents in the following way, (see (2.4)) :

$$T_{D,E}(x) = \sum_{m,n\geq 0} \frac{q^{A\binom{m+1}{2}+B\binom{n+1}{2}+Cmn+Dm+En}x^{Fm+Gn}}{(q^K;q^K)_m(q^L;q^L)_n}.$$

The functional equations may change only the parameters D and E, therefore the left hand side of (2.4) contains only D and E.

Remark 28. For the remainder of the chapter, we assume that $D = E = 0$ and $A, B, C, F, G, K, L \geq 1$. Note that this assumption is very natural, since changing D and E corresponds to a linear shift of parts. We want all other parameters to be nonzero, primarily because this ensures that the sum cannot be expressed as a product of two single summations.

In Chapter 2, we see that if the parameters satisfy certain conditions, then, maybe, it is possible to interpret the generating function using the moves framework. However, there are two obstacles:

1. If conditions (2.5) are not satisfied, then, certainly, we cannot use the moves framework to explain the generating function.

2. Even if the conditions (2.5) are satisfied, we may not be able to produce an explanation with generating functions. In other words, we may not be able to properly define the moves.

In the next section, we discuss two examples to illustrate our new interpretation.

3.1 Two Examples

We start directly with our first example.

Example 13. Let $(A, B, C, D, E, F, G, K, L, \gamma) = (2, 1, 1, 0, 0, 2, 1, 1, 1, 1)$. In other words, we start with the following generating functions:

$$T_1(x) = \sum_{m,n\geq 0} \frac{q^{2\binom{m+1}{2}+mn+\binom{n+1}{2}} x^{2m+n}}{(q;q)_m (q;q)_n},$$

$$T_2(x) = \sum_{m,n\geq 0} \frac{q^{2\binom{m+1}{2}+mn+\binom{n+1}{2}+m} x^{2m+n}}{(q;q)_m (q;q)_n}.$$

Remark 29. In general, T_1 and T_2 would be the same except they can have different D, E values.

Step 1: Using the Maple program, [22], we get the following system of functional equations:

$T_1(x) = (1 + x^2q^2)T_1(xq) + (xq + x^2q^3)T_2(xq),$

$T_2(x) = T_1(xq) + (xq + x^2q^3)T_2(xq).$

This means that we have the following equations:

$$\begin{aligned}\sum_{m,n\geq 0} t_1(m,n)x^m q^n &= \sum_{m,n\geq 0} t_1(m,n)x^m q^{m+n} + \sum_{m,n\geq 0} t_1(m,n)x^{m+2}q^{m+n+2} \\ &+ \sum_{m,n\geq 0} t_2(m,n)x^{m+1}q^{m+n+1} + \sum_{m,n\geq 0} t_2(m,n)x^{m+2}q^{m+n+3},\end{aligned} \tag{3.1}$$

$$\begin{aligned}\sum_{m,n\geq 0} t_2(m,n)x^m q^n &= \sum_{m,n\geq 0} t_1(m,n)x^m q^{m+n} + \sum_{m,n\geq 0} t_2(m,n)x^{m+1}q^{m+n+1} \\ &+ \sum_{m,n\geq 0} t_2(m,n)x^{m+2}q^{m+n+3}.\end{aligned} \tag{3.2}$$

Now, we equate the coefficients of the term $x^m q^n$ on both sides of (3.1) and (3.2). On the left-hand side of (3.1), we have $t_1(m,n)$. On the right-hand side of (3.1) we need to perform simple manipulations on the terms to get the coefficient of $x^m q^n$ in each term. We explain the procedure the get the coefficient of $x^m q^n$ in detail:

$$\begin{aligned}
[x^m q^n] \sum_{m,n\geq 0} t_1(m,n) x^m q^{m+n} &= [x^m q^n] \sum_{m,n} t_1(m,n) x^m q^{m+n} \\
&= [x^m q^n] \sum_{m,n} t_1(m,n-m) x^m q^n \\
&= t_1(m,n-m).
\end{aligned}$$

Thus, we get $t_1(m,n-m)$. Using similar arguments, we get the following recurrence relations:

1. For $t_1(m,n)$ we get:

$$\begin{aligned}
t_1(m,n) = t_1(m,n-m) + t_1(m-2,n-m-2) + \\
t_2(m-1,n-m-1) + t_2(m-2,n-m-3).
\end{aligned}$$

2. For $t_2(m,n)$ we get:

$$t_2(m,n) = t_1(m,n-m) + t_2(m-1,n-m-1) + t_2(m-2,n-m-3).$$

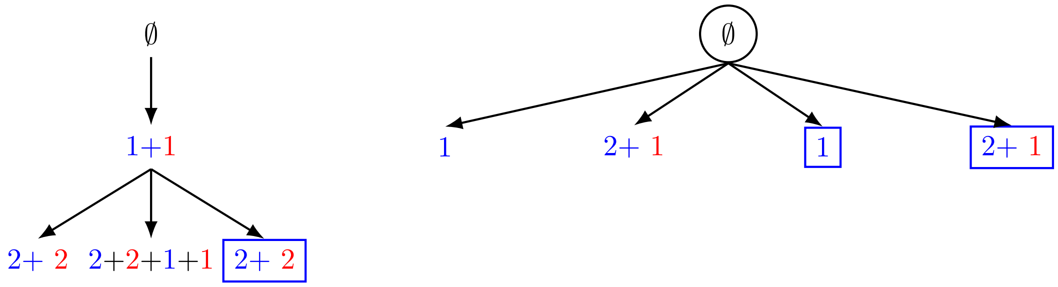
Note that $t_1(0,0) = t_2(0,0) = 1$ and $t_1(m,n) = t_2(m,n) = 0$ if $m > n$.

Step 2: Then we can interpret this as follows:

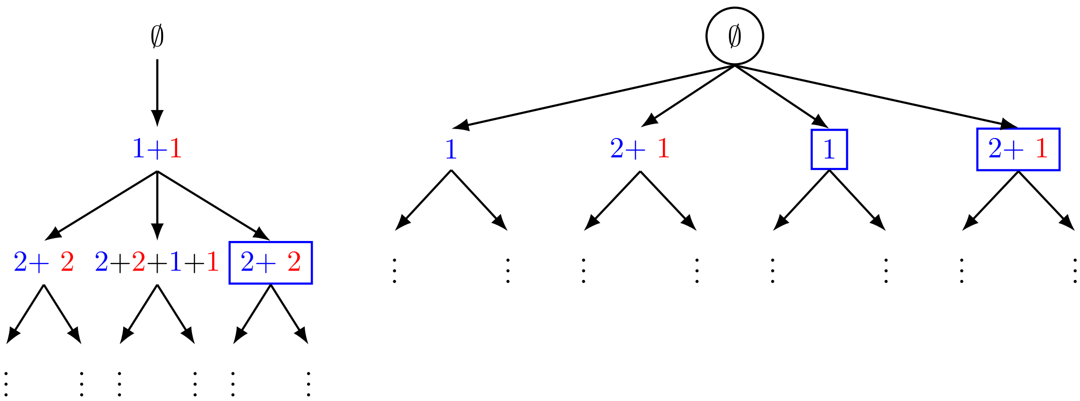
The first equation implies that if $(\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathcal{T}_1$, where $\mathcal{T}_1$ is the set of partitions enumerated by T_1, then $(\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_m + 1) \in \mathcal{T}_1$ and $(\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_m + 1, 1, 1) \in \mathcal{T}_1$. More explicitly, multiplying $T_1(xq)$ by x^2q^2 means that we are adding $1 + 1$ at the end of the partition. Similarly, using the first equation, we see that if $(\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathcal{T}_2$ where $\mathcal{T}_2$ is the set of partitions that is enumerated by T_2, then $(\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_m + 1, 1) \in \mathcal{T}_1$ and $(\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_m + 1, 2, 1) \in \mathcal{T}_1$.

Likewise, from the second equation, if $(\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathcal{T}_1$, then $(\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_m + 1) \in \mathcal{T}_2$ and if $(\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathcal{T}_2$, then $(\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_m + 1, 1) \in \mathcal{T}_2$ and $(\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_m + 1, 2, 1) \in \mathcal{T}_2$.

Note that these rules can be used as a recursive algorithm to construct the partitions enumerated by $\mathcal{T}_1$ and $\mathcal{T}_2$.



In the above picture, we showed the construction tree with height 1. The enclosed partitions are in $\mathcal{T}_2$. We now explain why we use colors in the figure above. If we assume that $\mathcal{T}_1$ and $\mathcal{T}_2$ are sets of ordinary partitions, going deeper (as shown in the next figure) in the tree creates more than one occurrence of the same partition in $\mathcal{T}_1$ and $\mathcal{T}_2$. To distinguish multiple occurrences of the same ordinary partition, we switch to colored partitions. We formalize this in Theorem 13.



Remark 30. The moral of the story is that functional equations give us recurrence relations on the coefficients of the generating functions, which, in turn, give us combinatorial constructions of $\mathcal{T}_1$ and $\mathcal{T}_2$.

Theorem 13. T_1 and T_2 count the following classes of partitions: $\mathcal{T}_1$:= the set of 2 colored partitions such that

1. Both blue and red parts appears at most 1 time

2. If we go through the parts of the partition from the smallest part to largest part, for each red part b, we should find a blue part b or $b+1$. Moreover, these blue parts should be distinct for each red part b.

$\mathcal{T}_2$:= same as $\mathcal{T}_1$, except that we cannot use a blue 1 and a red 1 together.

Example 14. Before proving the theorem, let us look at some examples of partitions which are in $\mathcal{T}_1$ and $\mathcal{T}_2$

1. $5 + 5 \in \mathcal{T}_1$

2. $5 + 5 + 4 \notin \mathcal{T}_1$

3. $14 + 5 + 2 + 1 \in \mathcal{T}_1$

4. $9 + 6 + 6 + 1 \notin \mathcal{T}_1$.

Proof. We will use structural induction.

Base Case: When, the largest part of the partition is 1, this definitely holds, since from construction tree we that $\mathcal{T}_1(1) = \{1\}$ and $\mathcal{T}_2(1) = \{1\}$

Suppose it holds for all the partitions with largest part $\leq \lambda_1$. We want to show that it holds for all the partitions with largest part $\lambda_1 + 1$ as well. We have the following moves for the construction: $(\lambda_1, \lambda_2, \dots, \lambda_m) \in \mathcal{T}_1$ implies that

A1. $(\lambda_1 + 1, \lambda_2 + 1, \dots, \lambda_m + 1) \in \mathcal{T}_1$

A2. $(\lambda_1 + 1, \lambda_2 + 1, \dots, \lambda_m + 1, 1, 1) \in \mathcal{T}_1$

A3. $(\lambda_1 + 1, \lambda_2 + 1, \dots, \lambda_m + 1) \in \mathcal{T}_2$.

Similarly, if $(\lambda_1, \lambda_2, \dots, \lambda_m) \in \mathcal{T}_2$, then

B1. $(\lambda_1 + 1, \lambda_2 + 1, \dots, \lambda_m + 1, 1) \in \mathcal{T}_1$

B2. $(\lambda_1 + 1, \lambda_2 + 1, \dots, \lambda_m + 1, 2, 1) \in \mathcal{T}_1$.

B3. $(\lambda_1 + 1, \lambda_2 + 1, \dots, \lambda_m + 1, 1) \in \mathcal{T}_2$

B4. $(\lambda_1 + 1, \lambda_2 + 1, \dots, \lambda_m + 1, 2, 1) \in \mathcal{T}_2$.

We are using green colors to indicate that for different cases the colors can be different.

First of all, we need to show that the rules do not violate the conditions.

A1, A2, A3, B1 and B3 clearly do not violate any conditions. In other words, if the partition is in $\mathcal{T}_i$ before we apply the move, it will be in the indicated set ($\mathcal{T}_1$ or $\mathcal{T}_2$) after the move.

For B2 and B4 we need to be more careful, since we cannot have two copies of 2 or two copies of 2. We have three cases to consider:

1. $\lambda_m \geq 2$ (regardless of its color), then B2 and B4 take the form $(\lambda_1, \lambda_2, \dots, \lambda_m + 1, 2, 1)$.

2. $\lambda_m = 1$. Then, B2 and B4 would be applied as $(\lambda_1, \lambda_2, \dots, \lambda_m + 1, 2, 1)$. We cannot have another 2, since the smallest part λ_m is 1.

3. $\lambda_m = 1$. Then, B2 and B4 would be applied as $(\lambda_1, \lambda_2, \ldots, \lambda_m + 1, 2, 1)$. This is not a problem, since 1 and 1 cannot exist in λ together due to $\mathcal{T}_2$ conditions.

As a result, all of these moves preserve $\mathcal{T}_1$ and $\mathcal{T}_2$.

Now, we know that this construction creates a subset of $\mathcal{T}_1$ and $\mathcal{T}_2$. To conclude the proof, it remains to show that, this construction can create all partitions in $\mathcal{T}_1$ and $\mathcal{T}_2$ in a unique way.

Let $\lambda = (\lambda_1 + 1, \lambda_2, \lambda_3, \ldots, \lambda_m)$ be a partition in $\mathcal{T}_i$. We want to show that, λ can be obtained uniquely via above moves. We have different cases to consider:

1. If λ does not contain any 1's. Then, λ is obtained from the move $A1$ applied on the partition $(\lambda_1, \lambda_2 - 1, \ldots, \lambda_m - 1) \in \mathcal{T}_1$ if $\lambda \in \mathcal{T}_1$, or from the move $A3$ applied on the partition $(\lambda_1, \lambda_2 - 1, \ldots, \lambda_m - 1) \in \mathcal{T}_1$ for $\lambda \in \mathcal{T}_2$. Note that, these are the unique moves to obtain λ since all other moves gives us 1 (red or blue) as a part.

2. λ contains 1, but not a 1, then we have two cases to consider:

   (a) If λ does not contain 2, then it is obtained from the move B1 applied on the partition $(\lambda_1, \lambda_2 - 1, \ldots, \lambda_{m-1} - 1) \in \mathcal{T}_2$ if $\lambda \in \mathcal{T}_1$, and from the move B3 applied on the partition $(\lambda_1, \lambda_2 - 1, \ldots, \lambda_{m-1} - 1) \in \mathcal{T}_2$ if $\lambda \in \mathcal{T}_2$. These moves are unique, since we cannot use A1 and A3, they do not give 1. We cannot use A2, this gives 1. We cannot use B2 or B4, either they give 1 or 2.

   (b) If λ contains 2, then λ must contain 2 or 3:

      i. If λ contains 2, then we cannot use B1 or B3 to obtain λ, since these require applying B1 or B3 to a partition in $\mathcal{T}_2$ which contains both 1 and 1 which is impossible. Thus, λ is obtained via move B2 applied on $(\lambda_1, \lambda_2 - 1 \ldots, \lambda_{m-2} - 1) \in \mathcal{T}_1$ where $\lambda_{m-2} - 1 = 1$, if $\lambda \in \mathcal{T}_1$. Similarly, λ is obtained via move B4 applied on $(\lambda_1, \lambda_2 - 1 \ldots, \lambda_{m-2} - 1) \in \mathcal{T}_1$ where $\lambda_{m-2} - 1 = 1$, if $\lambda \in \mathcal{T}_2$.

      ii. If λ does not contain 2, then λ is obtained via B1 applied on $(\lambda_1, \lambda_2 - 1, \ldots, \lambda_{m-2} - 1, \lambda_{m-1} - 1)$ where $\lambda_{m-2} - 1 = 2$ and $\lambda_{m-1} - 1 = 1$, if $\lambda \in \mathcal{T}_1$. Similarly, if $\lambda \in \mathcal{T}_2$, we obtain λ via applying B3 on $(\lambda_1, \lambda_2 - 1, \ldots, \lambda_{m-2} - 1, \lambda_{m-1} - 1)$ where $\lambda_{m-2} - 1 = 2$ and $\lambda_{m-1} - 1 = 1$, if $\lambda \in \mathcal{T}_2$.

3. λ contains 1 and 1 then λ must be in $\mathcal{T}_1$ and not in $\mathcal{T}_2$. There is only one way to obtain λ, use A2 on $(\lambda_1, \lambda_2 - 1, \ldots, \lambda_{m-2} - 1)$, uniqueness follows from the

fact that there is only one move which gives both 1 and 1 together.

4. If λ contains 1, but not a 1. Then, we must have 2 as well. Thus, if $\lambda \in \mathcal{T}_1$ we must use B2 on $(\lambda_1, \lambda_2 - 1, \ldots, \lambda_{m-2} - 1)$ where $\lambda_{m-2} - 1 \neq 1$, and if $\lambda \in \mathcal{T}_2$, we must use B4 on $(\lambda_1, \lambda_2 - 1, \ldots, \lambda_{m-2} - 1)$ where $\lambda_{m-2} - 1 \neq 1$, to obtain $\lambda \in \mathcal{T}_2$. Uniqueness is obvious, since there is only one move which gives 1 without 1.

This concludes the proof. $\square$

Step 3: Now, we show that the starting generating function counts the partitions where each part appears at most 3 times.

$$\sum_{m,n\geq 0} \frac{q^{2\binom{m+1}{2}+\binom{n+1}{2}+mn}}{(q;q)_m(q;q)_n} = \sum_{m\geq 0} \frac{q^{2\binom{m+1}{2}}}{(q;q)_m} \sum_{n\geq 0} \frac{q^{\binom{n+1}{2}}(q^m)^n}{(q;q)_n} = \sum_{m\geq 0} \frac{q^{2\binom{m+1}{2}}}{(q;q)_m} \left(-q^{m+1};q\right)_\infty$$
$$(-q;q)_\infty \sum_{m\geq 0} \frac{q^{2\binom{m+1}{2}}}{(q^2;q^2)_m} = (-q;q)_\infty(-q^2;q^2)_\infty = \frac{(q^4;q^4)_\infty}{(q;q)_\infty}$$

As a result, $T_1(1)$ generates partitions into parts that are not divisible by 4. In addition, by conjugation, $T_1(1)$ generates partitions into parts that are repeated at most 3 times.

As a result, we start with a well-known generating function and via a system of functional equations discover another interpretation of it which is not clear from the generating function.

Remark 31. It is possible to interpret the generating function we start with, namely $T_1(x)$ via (colored) moves framework.

Base Partition:

$$[1+1]+[2+2]+\cdots+[m+m]+(m+1)+(m+2)+\cdots+(m+n).$$

The types of parts:

Singletons: b which is not paired to another red part.

Pairs: $[b, b-1]$ or $[b, b]$.

The moves: We start with the backward moves.

Backward moves on pairs: We have two cases to consider:

1. The pair is $[b, b-1]$. Then, the backward move on the pair is defined as $[b, b-1]$ becomes $[b-1, b-1]$. Since, it is not possible to have the singleton $(b-1)$ at the beginning. (Otherwise, we will have the pair $[b-1, b-1]$, not $[b, b-1]$).

2. The pair is $[b, b]$. Then, the backward move on the pair is defined as $[b, b-1]$, i.e. $[b, b] \to [b, b-1]$. If there is a singleton $(b-1)$, then $[b, b] + (b-1) \to (b) + [b-1, b-1]$.

In either case, the weight of the partition decreases by 1, and the number of pairs and the number of singletons do not change.

Backward moves on singletons: We just pull (b) as $(b-1)$. Since, in the base partition the singletons are in front of the pairs, we do not need to perform any adjustments. It decreases the weight of the partition by 1, and the number of pairs and the number of singletons do not change.

Forward moves on pairs: Again, we have two cases to consider:

1. The pair is of the form $[b, b-1]$. Then, we just push it as $[b, b]$. Since, if we have b as a part, we must have $b+1$, i.e. we have $[b+1, b]$, then we will not push $[b, b-1]$ in the first place.

2. The pair is of the form $[b, b]$. We have two different cases:

   (a) If there is no singleton $(b+1)$, we define the move as $[b, b] \to [b+1, b]$.

   (b) If there is a singleton $(b+1)$, then we define the move as $(b+1)+[b, b] \to [b+1, b+1] + (b)$

   In both cases, the weight of the partition increases by 1 and the number of pairs and singletons remain constant.

Forward moves on singletons: We just push (b) as $(b+1)$. Since, we are pulling singletons last, we are pushing singletons first. In the base partition, the singletons are already in front of the pairs, so we do not need to perform any adjustments. This increases the weight of the partition by 1, and the number of pairs and the number of singletons do not change.

Now, we look at an example where the parameters do not satisfy the moves framework conditions, (2.5).

Errata: We found a combinatorial explanation for it through the moves framework using colored partitions.

Example 15. Consider the parameters

$$(A, B, C, D, E, F, G, K, L, \gamma) = (2, 1, 1, 0, 0, 1, 1, 1, 1, 1).$$

More explicitly, the generating functions we start with are as follows:

$$T_1(x) = \sum_{m,n\geq 0} \frac{q^{2\binom{m+1}{2}+mn+\binom{n+1}{2}} x^{m+n}}{(q;q)_m (q;q)_n},$$

$$T_2(x) = \sum_{m,n\geq 0} \frac{q^{2\binom{m+1}{2}+mn+\binom{n+1}{2}+m} x^{m+n}}{(q;q)_m (q;q)_n}.$$

The Maple program gives us the following system of functional equations:

$T_1(x) = T_1(xq) + (xq + xq^2)T_2(xq)$

$T_2(x) = T_1(xq) + (xq)T_2(xq)$

We can interpret this as, if $(\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathcal{T}_1$, then

C1. $(\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_m + 1) \in \mathcal{T}_1$

C2. $(\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_m + 1) \in \mathcal{T}_2$.

Similarly, if $(\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathcal{T}_2$, then

D1. $(\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_m + 1, 1) \in \mathcal{T}_1$

D2. $(\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_m + 1, 2) \in \mathcal{T}_1$

D3. $(\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_m + 1, 1) \in \mathcal{T}_2$.

Theorem 14. $\mathcal{T}_1$ and $\mathcal{T}_2$ are the following classes of partitions:

$\mathcal{T}_1$:= the set of 2 colored partitions such that

1. Both blue and red parts appear at most 1 time
2. 1 does not appear as a part.
3. For any b part, we cannot have $b + 1$ or $b + 2$.
4. For any b, we cannot have the differences $[2^*, 1]$ (where the notation 2^* means that any number of apperances of 2, including 0) with consecutive red parts. Thus, we cannot have $b+(b-1)$ or $b+(b-2)+(b-3)$ or $b+(b-2)+(b-4)+(b-5)$ etc.

$\mathcal{T}_2 :=$ {same as $\mathcal{T}_1$ except that we cannot use a 2}.

Proof. Again, we prove this via structural induction.

Base Case: If the largest part ≤ 1, we are done since $\mathcal{T}_1(1) = \mathcal{T}_2(1) = \{1\}$.

First of all, we need to show that the rules preserve $\mathcal{T}_i$. Suppose they preserve $\mathcal{T}_i$ for all partitions with largest part $\leq \lambda_1$. We need to show that the rules preserves all partitions with largest part $\leq \lambda_1 + 1$ as well.

Let us recall that the moves, if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathcal{T}_1$, then

C1. $(\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_m + 1) \in \mathcal{T}_1$

C2. $(\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_m + 1) \in \mathcal{T}_2$

Similarly, if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathcal{T}_2$, then

D1. $(\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_m + 1, 1) \in \mathcal{T}_1$

D2. $(\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_m + 1, 2) \in \mathcal{T}_1$

D3. $(\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_m + 1, 1) \in \mathcal{T}_2$

Clearly, C1 and C2 preserve $\mathcal{T}_i$ since they respect the consecutive differences of the parts.

For D1 and D3, we have different cases to consider:

1. $\lambda_m \geq 3$ (regardless of its color), there is no problem. Since, we have enough space between newly added 1 and $\lambda_m + 1$.
2. $\lambda_m = 2$, then there is no problem, since we cannot have 2 or 3 in λ.
3. $\lambda_m = 2$. This is not possible since $\lambda \in \mathcal{T}_2$.
4. $\lambda_m = 1$. Then, due to the rules, λ cannot have 2 or 3, thus we are good.
5. $\lambda_m = 1$ is not possible due to the rules.

As a result, both D1 and D3 preserves $\mathcal{T}_i$.

For D2, since, $\lambda_m \neq 2$, $\lambda_m + 1 \neq 3$. Thus, we cannot create a new $[2^*, 1]$ difference pattern. Thus, D2 preserves $\mathcal{T}_i$ as well.

Now, we need to show that any partition in $\mathcal{T}_i$ can be uniquely obtained from the moves. Suppose this is true for all partitions in $\mathcal{T}_i$ with largest part $\leq \lambda_1$. Consider a partition $\lambda = (\lambda_1 + 1, \lambda_2, \ldots, \lambda_m)$ in $\mathcal{T}_i$, we want to show that this can be obtained uniquely from the above moves. We have different cases to consider:

1. If $\lambda_m \geq 3$ (regardless of its color) or $\lambda_m = 2$, then we must use C1 on $(\lambda_1, \lambda_2 - 1, \ldots, \lambda_m - 1)$ if $\lambda \in \mathcal{T}_1$ and C2 on $(\lambda_1, \lambda_2 - 1, \ldots, \lambda_m - 1)$ if $\lambda \in \mathcal{T}_2$. Clearly, these are unique choices since the other moves give 1 or 2.

2. If $\lambda_m = 2$, then λ must be in $\mathcal{T}_1$ and it cannot be in $\mathcal{T}_2$. Then, we must use D2 on $(\lambda_1, \lambda_2 - 1, \ldots, \lambda_{m-1} - 1)$. No other move is available since we cannot have a 1, before the move.

3. If $\lambda_m = 1$, then we must use D1 on $(\lambda_1, \lambda_2 - 1, \ldots, \lambda_{m-1} - 1)$ or D3 on $(\lambda_1, \lambda_2 - 1, \ldots, \lambda_{m-1} - 1)$ depending on whether λ in $\mathcal{T}_1$ or $\mathcal{T}_2$. Again, these are unique moves since there is no other move which creates 1.

This concludes the proof. □

As in the above example, the generating function counts the ordinary partitions where each part appears at most 3 times. (Since the generating functions become the same when $x = 1$).

Combining Example 13 and Example 15, we proved the following theorem:

Theorem 15. Let n be a non-negative integer. Then, the following sets have the same cardinalities:

1. The set of ordinary partitions of weight n, where each part appears at most 3 times.

2. The set of 2 colored partitions of weight n where

   (a) Both blue and red parts appear at most 1 time

   (b) If we go through the parts of the partition from the smallest part to largest part, for each b, we should find a b or $b + 1$. Moreover, these blue parts should be distinct for each b.

3. The set of 2-colored partitions with weight n in which

   (a) Both blue and red parts appears at most 1 time

   (b) 1 does not appear as a part.

   (c) For any b part, we cannot have $b + 1$ or $b + 2$.

   (d) For any b, we cannot have the differences $[2^*, 1]$ with consecutive red parts. Thus, we cannot have $b + (b - 1)$ or $b + (b - 2) + (b - 3)$ or $b + (b - 2) + (b - 4) + (b - 5)$ etc.

We have a Maple program which does the following: Given parameters $(A, B, C, F, G, K, L, \gamma)$, it finds a functional equation satisfied by $T_{D,E}(x)$.

Remark 32. Given a series, we can find one functional equation satisfied by it and interpret it combinatorially. However, the system of functional equations is usually easier to interpret. For example, the parameters $(A, B, C, F, G, K, L, \gamma) = (2, 1, 1, 2, 1, 1, 1)$ give us

$$S(x) = (1 + xq + x^2q^2 + x^2q^3)S(xq) - (1 + xq^2)x^3q^5S(xq^2).$$

This is much harder to interpret than the functional system in Example 13.

Step 1: Given the parameters $(A, B, C, D, E, F, G, K, L, \gamma)$ (we will explain γ below), find a functional equation satisfied by $T_{D,E}(x)$:

(3.3)

$$\begin{aligned}
T_{D_1,E_1}(x) &= R_{1,1}(x)T_{D_1,E_1}(xq^\gamma) + R_{1,2}(x)T_{D_2,E_2}(xq^\gamma) + \cdots + R_{1,m}(x)T_{D_m,E_m}(xq^\gamma)\\
T_{D_2,E_2}(x) &= R_{2,1}(x)T_{D_1,E_1}(xq^\gamma) + R_{2,2}(x)T_{D_2,E_2}(xq^\gamma) + \cdots + R_{2,m}(x)T_{D_m,E_m}(xq^\gamma)\\
&\cdots\\
T_{D_m,E_m}(x) &= R_{m,1}(x)T_{D_1,E_1}(xq^\gamma) + R_{m,2}(x)T_{D_2,E_2}(xq^\gamma) + \cdots + R_{m,m}(x)T_{D_m,E_m}(xq^\gamma)
\end{aligned}$$

where each $R_{i,j}(x)$ is a rational function of q and x.

Remark 33. The article [29], guarantees that such a functional equation exists.

Remark 34. If we do not care about the D_i, E_i indices on the left-hand side of (3.3), instead of writing T_{D_i,E_i} we just write $T_i(x)$, for $i \in \{1, \ldots, m\}$.

Remark 35. If some of the coefficients in (3.3) are nonpositive, then it is harder to find a combinatorial interpretation of the functional equations.

Step 2: Interpret the functional equation from a combinatorial perspective to obtain a recursive definition for the partition classes counted by $T_i(x)$'s.

More explicitly, each functional equation can be turned into a recurrence equation of the coefficients of the generating functions, which can be converted into a recursive algorithm to construct the partitions in $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_m$.

Our combinatorial interpretation depends on the following assumptions:

1. $T_i(x)$ generates some partition classes

2. The exponent of x must count the number of parts.

Remark 36. Usually, in partition theory, one uses functional equations to find generating functions rather than the other way around. Thus, the usual pipeline is as follows. First, we start with a partition identity conjecture. Then, we get a functional equation of generating functions. Then, solving this system gives a generating function. Here, we start with a (possible) generating function and get a functional equation from that, this allows us to see a combinatorial interpretation.

Step 3: We can use the recursive definition obtained in Step 2 to conjecture the partition classes counted by $T_i(x)$'s. Then, prove this using structural induction.

Note that this recursive definition gives us a tree-type structure to generate the partition classes counted by $T_i(x)$. Thus, we can induct on the depth of the tree which can be related to the largest part of the partition.

The proof should contain two things: $\mathcal{T}_i$ contains a certain class of partitions, $\mathcal{T}_i$ contains nothing else.

Remark 37. If $\mathcal{T}_i$ contains some partitions more than once, this indicates that it is a good idea to use colored partitions to interpret T_i combinatorially.

Step 4: If we know that the starting generating function already generates a particular class of partitions, then we can try to find a bijection between the class and the class we found in Step 3.

4. Discussion and Future Research

In this section, we talk about possible further directions of research.

1. We try to interpret sums of the form (2.1). Can we add a real character term in the summand and interpret the generating function with the moves framework?

2. Can we generalize the approach in Section 2.2 to $k = 4$? Or further, for any k? We try to generalize our approach to $k = 4$, The case $k = 4$ seems to be more challenging since we have triples as well as singletons and pairs. We could not define the moves properly to get a similar result. As a general observation, once the number of types of parts increases, defining moves becomes more and more difficult.

3. The best case scenario is to generalize this to an arbitrary k and find series of the form
$$\sum_{m,n\geq 0} rrg_{k,a}(m,n)x^m q^n = \sum_{n_1,n_2,n_3,\ldots,n_{k-1}\geq 0} \frac{q^{\text{QUADRATIC}} x^{\text{LINEAR}}}{(q;q)_{n_1}(q^2;q^2)_{n_2}\cdots(q^{k-1};q^{k-1})_{n_{k-1}}}.$$
Is it possible to combinatorially interpret this series using base partition and moves, as we discussed in this paper? Let us write our ultimate hope as well: Given a series of the form
$$\sum_{n_1,n_2,n_3,\ldots,n_{k-1}\geq 0} \frac{q^{\text{QUADRATIC}} x^{\text{LINEAR}}}{(q^{\alpha_1};q^{\alpha_1})_{n_1}(q^{\alpha_2};q^{\alpha_2})_{n_2}\cdots(q^{\alpha_{k-1}};q^{\alpha_{k-1}})_{n_{k-1}}}$$
is it possible to **automatically** interpret this combinatorially? We emphasize "automatically" here, since the hardest part of this approach is to define part types (singletons and pairs, in our case) and moves. Thus, it would be good to have a combinatorial framework which works for all such series without any thinking.

4. Can we fully automate the process explained in Chapter 3? More explicitly, given parameters $(A, B, C, D, E, F, G, K, L, \gamma)$, can we interpret this generat-

ing function combinatorially?

5. Is it possible to use x to count another statistic than number of parts in Chapter 3? We should use a statistic that does not change under transformation $x \to xq$.

6. Can we find a bijective proof of Theorem 15?

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