RATIONAL PREPERIODIC POINTS OF RATIONAL MAPS

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ABSTRACT

RATIONAL PREPERDIODIC POINTS OF RATIONAL MAPS

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Thesis Supervisor: Assoc. Prof. Dr. Mohammad Sadek

Keywords: Arithmetic dynamics, elliptic curves, hyperelliptic curves, abelian varieties, preperiodic points, periodic points

In this thesis we focus on the preperiodic and periodic points of the dynamical systems associated to rational maps of degree 2. We discuss the state of the art regarding rational preperiodic points of polynomials of degree 2 defined over the rational field with an emphasis on the Uniform Boundedness Conjecture on the number of such points. We then study the known classification results of rational preperiodic points of rational maps of degree 2, these maps include rational maps of degree 2 with abelian automorphism groups and rational maps of degree 2 with a rational periodic critical point of period 2.

ÖZET

RASYONEL FONKSİYONLARIN RASYONEL PREPERİYODİK NOKTALARI

KADER BULUT

Matematik, Yüksek Lisans Tezi, Aralık 2025

Tez Danışmanı: Assoc. Prof. Dr. Mohammad Sadek

Anahtar Kelimeler: Aritmetik dinamik, eliptik eğriler, hipereliptik eğriler, abelyen değişkenler, preperiodic noktalar, periodic noktalar

Bu tezde, derecesi 2 olan rasyonel fonksiyonlarla ilişkilendirilen dinamik sistemlerin preperiyodik ve periyodik noktalarını inceliyoruz. Rasyonel sayı cismi üzerinde tanımlı derecesi 2 olan polinomların rasyonel preperiyodik noktalarına ilişkin güncel durumu, bu noktaların sayısı üzerindeki rational field with an emphasis on the Uniform Boundedness Conjecture'na odaklanarak ele alıyoruz. Ardından, derecesi 2 olan rasyonel fonksiyonların rasyonel preperiyodik noktalarının bilinen sınıflandırma sonuçlarını inceliyoruz. Bu inceleme, özellikle şu iki sınıfı kapsamaktadır: abelyen otomorfizm gruplarına sahip derecesi 2 olan rasyonel fonksiyonlar ve periyodu 2 olan rasyonel bir periyodik kritik noktaya sahip derecesi 2 olan rasyonel fonksiyonlar.

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To my brother

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1. INTRODUCTION

The study of arithmetic dynamics, which brings together number theory and dynamical systems, has emerged as an important field in modern mathematics, especially within the last few decades. This area focuses on the behavior of orbits under repeated applications of functions defined over number fields or other arithmetic rings. When these maps are iterated, particularly polynomials or rational functions, they reveal complex structures within number fields. This thesis focuses on the behavior of periodic and preperiodic rational points under iteration, particularly through the view of the *Uniform Boundedness Conjecture* in arithmetic dynamics.

One of the important question in arithmetic dynamics involves periodic points, which return to their starting point after a fixed number of iterations, and preperiodic points, which eventually enter a repeating cycle. Classical theorems, such as Northcott's Theorem (1950), provide some of the earliest foundational results in this area. Northcott proved that, for any morphism of a number field and a given bound on the height of points, there are finitely many periodic points. This finiteness is crucial for understanding how periodic structures form under iteration, particularly for functions with integer or rational coefficients. Building upon this foundational work, Morton, Poonen, and Silverman have advanced the study of periodicity and boundedness within specific families of dynamical systems.

In 1994, Morton and Silverman introduced a seminal conjecture, often referred to as the Uniform Boundedness Conjecture for preperiodic points of rational functions. This conjecture offers that for any given number field K and degree $d \geq 2$, there should exist a uniform upper bound on the number of preperiodic points that can exist for a rational function defined over K with degree d. This conjecture, inspired by the analogous Uniform Boundedness Conjecture in the context of elliptic curves, has substantial research, with recent studies attempting to prove or provide bounds for specific cases. The conjecture remains open in its most general form, though substantial advances have been made.

Morton and Silverman's conjecture has significant implications, as it suggests a uniform constraint on the periodic and preperiodic structure within arithmetic dynamical systems across families of functions. This uniformity would imply a level of predictability and regularity within the behavior of rational points under iteration, which is an exciting prospect in the study of dynamical systems on algebraic varieties.

Bjorn Poonen has made considerable contributions to the understanding of periodic points, particularly in the context of quadratic polynomials. One of Poonen's key results involves showing the finiteness of rational points in certain cases. For instance, in his work on quadratic rational functions, Poonen investigated conditions under which rational points exhibit periodic orbits, contributing to a deeper understanding of the possible periodic structures within this family of functions. His approach often involves constructing explicit examples and examining how constraints on the field affect periodicity, helping to frame specific instances where the Uniform Boundedness Conjecture holds.

In addition to proving particular cases of boundedness, Poonen's work is a foundstion for further research in higher degree cases and in different number fields. His methods have inspired new approaches to understanding the relationship between function degree, number field characteristics, and the potential periodic and preperiodic structure.

Michael Stoll has also contributed significantly to arithmetic dynamics, particularly through his work on modular curves and their application to dynamical systems. Dynamical modular curves, analogous to classical modular curves, allow researchers to parametrize families of dynamical systems with specified periodic behaviors. Stoll's research has focused on leveraging these curves to understand constraints on periodic points, providing tools to estimate bounds on periodic and preperiodic points across certain classes of maps. By studying these modular structures, Stoll has offered a powerful method to examine boundedness properties and periodic behavior in greater depth, particularly for quadratic and cubic polynomial maps.

Michelle Manes has explored several key questions in arithmetic dynamics, particularly focusing on the distribution and structure of preperiodic points. Manes has conducted extensive research on the field-specific characteristics that influence boundedness, especially over finite fields. One notable area of Manes' work involves the relationship between the degree of the map and the density of periodic points, which provides insight into how rational points distribute over dynamical systems. Her findings have contributed to a deeper understanding of the periodic structures that can arise within these systems, particularly through explicit constructions and computational models that reveal possible configurations of preperiodic points.

Canci's work has included proving finiteness results in certain cases for rational periodic points, often examining how the arithmetic properties of the base field influence the bounds on periodicity. These contributions offer extensions to the general conjectures within arithmetic dynamics, providing valuable insights into the range and limitations of periodic structures across different arithmetic settings.

The aim of this thesis is to further investigate the boundedness of periodic and preperiodic points within arithmetic dynamical systems, particularly under polynomial maps. By examining specific families of functions and recent results from Morton, Silverman, Stoll, Manes, and others, this study seeks to clarify the limitations and structures imposed by the Uniform Boundedness Conjecture. This work contributes to the broader understanding of periodicity and preperiodicity within rational points, offering new approaches and potentially tighter bounds on the behavior of points within dynamical systems defined over number fields.

This thesis is organized as follows: the first chapter for algebraic curves provides an overview of foundational concepts in arithmetic dynamics, including definitions and preliminary theorems. Subsequent chapters delve into specific results and techniques developed by leading researchers in the field, focusing on the behavior of rational periodic points under iteration. The concluding chapter synthesizes these findings, offering insights into potential directions for future research, including possible extensions to higher-degree maps and applications to open problems in Diophantine geometry.

2. Affine Spaces

Our purpose in this chapter is to give an introduction to algebraic geometry. We work over a fixed algebraically closed field K. We define the material we need for the main objects of the study, which are algebraic varieties, abelian varieties, singular and non-singular algebraic curves in affine and projective spaces.

Let K be a fixed algebraically closed field. We define the *affine n-space* over K, denoted $\mathbb{A}^n_K$ or simply $\mathbb{A}^n$, to be the set of all n-tuples of elements of K.

$$\mathbb{A}^n_K = \{P = (a_1, \ldots, a_n) : a_i \in K\}$$

An element $P \in \mathbb{A}^n$ will be called a *point*, and if $P = (a_1, \ldots, a_n)$ with $a_i \in K$, then the a_i's will be called the *coordinates* of P.

Let $A = K[x_1, \ldots, x_n]$ be the polynomial ring in n variables over K. We will interpret the elements of A as functions from the affine n-space to K, by defining $f(P) = f(a_1, \ldots, a_n)$, where $f \in A$ and $P \in \mathbb{A}^n$. Thus if $f \in A$ is a polynomial, we can talk about *the set of zeros* of f, namely

$$Z(f) = \{P \in \mathbb{A}^n : f(P) = 0\}.$$

More generally, if T is any subset of A, we define the zero set of T to be the common zeros of all the elements of T, namely

$$Z(T) = \{P \in \mathbb{A}^n : f(P) = 0, \ \textit{for all } f \in T\} = \bigcap_{f \in T} Z(f).$$

Since A is a noetherian ring, any ideal $\underline{a}$ has a finite set of generators $f_1, \ldots, f_r$. If $\underline{a}$ is the ideal of A generated by T, then $Z(T) = Z(\underline{a}) = \bigcap_{i=1}^r Z(f_i)$. Hence, $Z(T)$ can be expressed as the common zeros of the finite set of polynomials $f_1, \ldots, f_r$.

Definition 2.1. *A subset Y of $\mathbb{A}^n$ is an algebraic set if there exists a subset $T \subseteq A$ such that $Y = Z(T)$.*

$$Y \text{ is an algebraic set} \Leftrightarrow \exists\ (f_1,\ldots,f_r) \in K[x_1,\ldots x_n] \text{ such that } Y = \bigcap_{i=1}^{r} Z(f_i)$$

An algebraic set is defined over K if its ideal $Z(T)$ can be generated by polynomials in K[X]. We denote this by T/K. If T is defined over K, then the set of K-rational points of T is the set

$$T(K) = T \cap \mathbb{A}^n(K).$$

One of the fundamental problems in the subject of Diophantine geometry is finding the solution of polynomial equations in rational numbers. This is equivalent to describing sets of the form $T(K)$ when K is a number field.

Example 2.2. *Let $I = \langle x,y \rangle \subset \mathbb{A}^3_K$;*

$$I = \langle x,y \rangle = Z(x,y) = \{p \in \mathbb{A}^3 : f(p) = 0, \text{ for all } f \in K[x,y]\},$$

$$I\langle x,y \rangle = \{gx+hy : g,h \in K[x,y,z]\},$$

$$\Rightarrow Z(I) = \{p \in \mathbb{A}^3 : (gx+hy)(p) = 0, \text{ for all } g,h \in K[x,y,z]\}$$

a line in the 3-space.

Proposition 2.3. *The union of by two algebraic sets is an algebraic set. The intersection of any family of algebraic sets is an algebraic set. The empty set and the whole affine space are algebraic sets.*

Proof. If $Y_1 = Z(T_1)$ and $Y_2 = Z(T_2)$, then $Y_1 \cup Y_2 = Z(T_1T_2)$, where T_1T_2 denotes the products of polynomials, i.e., $T_1T_2 = \{f_1f_2 : f_1 \in T_1, f_2 \in T_2\}$. If $p \in Y_1 \cup Y_2$, then either $p \in Y_1 = Z(T_1)$ or $p \in Y_2 = Z(T_2)$, so either p is a zero of every polynomial in T_1 or T_2. Take any $f \in T_1T_2$, which is $f = f_1f_2$ where $f_1 \in T_1,\ f_2 \in T_2$. Then, $f(p) = f_1(p)f_2(p) = 0$. Hence, $p \in Z(T_1T_2)$. So, $Y_1 \cup Y_2 \subseteq Z(T_1T_2)$.
Conversely, let $p \in Z(T_1T_2) = \{p \in \mathbb{A}^n : fg(p) = 0,\ f \in T_1,\ g \in T_2\}$. So $f(p)g(p) = 0,$ *for all* $f \in T_1, g \in T_2$. Then $p \in Z(f)$ or $p \in Z(g)$. Hence, $p \in Y_1 \cup Y_2$, so $Z(T_1T_2) \subseteq Y_1 \cup Y_2$.

Let $p \in \cap Y_\alpha$. Then by the definition $p \in \bigcup_{\alpha \in I} Z(T_\alpha)$. So, $p \in Z(T_\alpha)$ for all $\alpha \in I$. This means that for each $f_\alpha \in T_\alpha,\ f_\alpha(p) = 0$. Hence $p \in Z(\bigcup_{\alpha \in I} T_\alpha)$. So, $\bigcap Y_\alpha$ is also an algebraic set. $\square$

Finally, the empty set $\emptyset = Z(1)$, and the whole space $\mathbb{A}^n = Z(0)$.

Example 2.4. *A set containing a single point $P = (a_1, \dots a_n) \in \mathbb{A}^n$ is algebraic set because,*

$$f(x_1, \dots, x_n) = (x_1 - a_1)(x_2 - a_2)\dots(x_n - a_n)$$

$$f(a_1, \dots a_n) = 0 \Rightarrow \{(a_1, \dots a_n)\} = Z(f)$$

Example 2.5. *The algebraic set*

$$T : Y^2 = X^3 + 17$$

has many $\mathbb{Q}$-rational points, for example $(-2,3)$, $(5234, 378661)$, $(\frac{137}{64}, \frac{2651}{512})$.
In fact, the set $Z(T)$ is infinite.

Definition 2.6. *We define the Zariski topology on $\mathbb{A}^n$ by taking the open subsets to be the complements of the algebraic sets. This is a topology, because according to the proposition, the intersection of two open sets is open, and the union of any family of open sets is open. Furthermore, the empty set and the whole space are both open.*

Example 2.7. *Let us consider the Zariski topology on the affine line over $\mathbb{A}^1_K$. Every ideal in $A = K[x]$ is principal, since $K[x]$ is noetherian and K is an algebraically closed field. So every algebraic set is the set of zeros of a single polynomial. Since K is algebraically closed, every nonzero polynomial $f(x)$ can be written $f(x) = c(x - a_1)\cdots(x - a_n)$ with $c, a_1, \dots, a_n \in K$. Then $Z(f) = (a_1, \dots, a_n)$ $\cdot$ Thus the algebraic sets in $\mathbb{A}^1$ are just the finite subsets (including the empty set) and the whole space (corresponding to $f = 0$). Thus the open sets are the empty set and the complements of finite subsets.*

Definition 2.8. *A Hausdorff space is a topological space in which any two distinct points can be separated by disjoint open sets. In other words, for any $x \neq y$, there exist open sets U and V such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.*

Notice in particular that Zariski topology is not Hausdorff because in this topology, the open sets are either empty or the complements of finite sets. Given two distinct points $x, y \in \mathbb{A}^1_K$, it is impossible to find disjoint open sets that separate them. So no pair of disjoint open sets can isolate distinct points.

Definition 2.9. *A nonempty subset Y of a topological space X is irreducible if it cannot be expressed as the union $Y = Y_1 \cup Y_2$ of two proper subsets, each one of which is closed in Y. The empty set is not considered to be irreducible.*

Example 2.10. *$\mathbb{A}^1$ is irreducible, because its only proper closed subsets are finite, yet it is infinite (because K is algebraically closed, hence infinite).*

Definition 2.11. *An affine algebraic set T is called an (affine) variety if $Z(T)$ is a prime ideal in $\bar{K}[X]$. Note that if T is defined over K, it is not enough to check that $Z(T/K)$ is prime in $K[X]$.*

To see this, consider the ideal $(X_1^2 - 2X_2^2) \in \mathbb{Q}[X_1, X_2]$.

Definition 2.12. *Let V be a variety, $P \in V$ and $(f_1, \ldots, f_m) \in K[X]$ a set of generators for $Z(T)$. Then V is nonsingular (or smooth) at P if the $m \times n$ matrix*

$$\Big(\frac{\partial f_i}{\partial X_j}(P)\Big)_{1 \leq i \leq m, 1 \leq j \leq n}$$

has rank $n - \dim(V)$. If V is nonsingular at every point, then we say that V is nonsingular (or smooth).

2.1 Projective Varieties

To define projective varieties, we proceed in a manner analogous to the definition of affine varieties, except that we work in projective space. Let K be our fixed algebraically closed field. We define projective n-space over K, denoted $\mathbb{P}^n_K$, or simply $\mathbb{P}^n$, to be the set of equivalence classes of $(n+1)$-*tuples* $(a_0, \ldots, a_n)$ of elements of K, not all zero, under the equivalence relation given by $(a_0, \ldots, a_n) \sim (\lambda a_0, \ldots, \lambda a_n)$ for all $\lambda \in K,\ \lambda \neq 0$. Another way of saying this is that $\mathbb{P}^n$ as a set is the quotient of the set $\mathbb{A}^{n+1} - (0, ..., 0)$ under the equivalence relation which identifies points lying on the same line through the origin.

Definition 2.13. *An element of $\mathbb{P}^n$ is called a point. If p is a point, then any*

$(n+1)-$tuple $(a_0,\dots,a_n)$ in the equivalence class p is called a set of homogeneous coordinates for p.

Let S be the polynomial ring $K[x_0,\dots,x_n]$. We want to regard S as a graded ring, so we recall briefly the notion of a graded ring.
A graded ring is a ring S, together with a decomposition $S=\oplus_{d\geq S_d}$ of S into a direct sum of abelian groups S_d, such that for any $d,e\geq 0$, $S_dS_e\subseteq S_{d+e}$. An element of S_d is called a *homogeneous element of degree d*. Thus any element of S can be written uniquely as a (finite) sum of homogeneous elements.

Example 2.14. *Let $A=K[x_1,\dots,x_n]$. A_n is the set of all homogeneous polynomials of degree n.*

Definition 2.15. *An ideal $\underline{a}\subseteq S$ is a homogeneous ideal if $\underline{a}=\oplus_{d\geq 0}(\underline{a}\cap S_d)$.*

An ideal is homogeneous if and only if it can be generated by homgeneous elements. The sum, product, intersection, and radical of homogeneous ideals are homogeneous.
To test whether a homogeneous ideal is prime, it is sufficient to show for any two homogeneous elements f,g, that $fg\in\underline{a}$ implies $f\in\underline{a}$ or $g\in\underline{a}$.

We make the polynomial ring $S=K[x_0,\dots,x_n]$ into a graded ring by taking S_d to be the set of all linear combinations of monomials of total weight d in $x_0,\dots,x_n$. If $f\in S$ is a polynomial, we cannot use it to define a function on $\mathbb{P}^n_K$, because of the non-uniqueness of the homogeneous coordinates. However, if f is a homogeneous polynomial of degree d, then $f(\lambda a_0,\dots,\lambda a_n)=\lambda^d f(a_0,\dots,a_n)$, so that the property of f being zero or not depends only on the equivalence class of $(a_0,\dots,a_n)$. Thus f gives a function from $\mathbb{P}^n$ to $\{0,1\}$ by $f(p)=0$ if $f(a_0,\dots,a_n)=0$, and $f(p)=1$ if $(a_0,\dots,a_n)\neq 0$. Thus we can talk about the zeros of a homogeneous polynomial, namely

$$Z(f)=\{p\in\mathbb{P}^n_K : f(p)=0\}.$$

If T is any set of homogeneous elements of S, we define the zero set of T to be

$$Z(T)=\{p\in\mathbb{P}^n_K : f(p)=0,\ \textit{for all}\ f\in T\}=\bigcap_{f\in T}Z(f).$$

Definition 2.16. *A subset Y of $\mathbb{P}^n$ is an algebraic set if there exists a set T of homogeneous elements of S such that $Y=Z(T)$.*

Definition 2.17. *A projective algebraic variety (or simply projective variety) is an irreducible algebraic set in $\mathbb{P}^n$, with the induced topology.*

An open subset of a projective variety is a quasi-projective variety. The dimension of a projective or quasi-projective variety is its dimension as a topological space.

If Y is any subset of $\mathbb{P}^n$, we define the *homogeneous ideal* of Y in S, denoted $I(Y)$, to be the ideal generated by

$$\{f \in S : f \text{ is homogeneous and } f(p) = 0, \textit{ for all } p \in Y\}.$$

If Y is an algebraic set, we define the *homogeneous coordinate ring* of Y to be $S(Y) = S/I(Y)$.

Our next objective is to show that projective n-space has an open covering by affine n-spaces, and hence that every projective (respectively, quasiprojective) variety has an open covering by affine (respectively, quasi-affine) varieties. First we introduce some notation. If $f \in S$ is a linear homogeneous polynomial, then the zero set of f is called a *hyperplane*. In particular we denote the zero set of x_i by H_i, for $i = 0, \dots, n$. Let U_i be the open set $\mathbb{P}^n - H_i$. Then $\mathbb{P}^n$ is covered by the open sets U_i, because if $p = (a_0, \dots, a_n)$ is a point, then at least one $a_i \neq 0$, hence $p \in U_i$. We define a mapping $\Phi_i : U_i \to \mathbb{A}^n$ as follows: if $p = (a_0, \dots, a_n) \in U_i$, then $\Phi_i(P) = Q$, where Q is the point with affine coordinates

$$(\frac{a_0}{a_i}, \dots, \frac{a_n}{a_i}),$$

with $\frac{a_i}{a_i}$ omitted. Note that Φ_i is well defined since the ratios $\frac{a_j}{a_i}$ are independent of the choice of hemogeneous coordinates.

2.2 Elliptic Curves

Let K be a field. We define an elliptic curve as a non-singular abelian variety of dimension 1 with a K-rational point $\mathcal{O}$ called the point at infinity. It has an equation of the form,

$$F(x,y) = ax^3 + bx^2y + cxy^2 + dy^3 + ex^2 + fxy + gy^2 + hx + iy + j = 0$$

where the coefficients $a, b, ..., j \in K$, and the non-singularity means that for each point on the curve, considered in the projective plane $\mathbb{P}^2(\bar{K})$ over the algebraic closure of K, at least one partial derivative of F is non-zero.

2.2.1 Weierstrass Equations

After applying certain birational transformation, one may express any elliptic curve with a Weierstrass equation of the form,

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

with $a_1, \ldots, a_6 \in K$ together with the point $\mathcal{O} = (0:1:0)$.

A projective plane $\mathbb{P}^2(K)$ is obtained by introducing on the set $K^3 - (0,0,0)$ the equivalence relation $(X,Y,Z) \sim (kX,kY,kZ)$, $k \in K$, $k \neq 0$. A point at infinity appears naturally if we represent an elliptic curve in a projective plane. By substituting $x = \frac{X}{Y}, y = \frac{Y}{Z}$ in the affine equation E, we obtain the projective equation,

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3a_2X^2Z + a_4XZ^2 + a_6Z^3.$$

If $Z \neq 0$, then the equivalence class of (X,Y,Z) has the representative $(x,y,1)$, so we can identify that class by (x,y). However, there is also an equivalence class which contains points with $Z = 0$. It has the representative $(0:1:0)$ and we identify that class with the point at infinity $\mathcal{O}$.

Also, if $\mathrm{char}(\bar{K}) \neq 2$, then we can simplify the equation by completing the square.

Thus the substitution,

$$y \to \frac{1}{2}(y - a_1, x - a_3)$$

gives us an equation of the form

$$E : y^2 = 4x^3 + b_2x^2 + 2b_4x + b_6$$

where,

$$b_2 = {a_1}^2 + 4a_4, \ \ b_4 = 2a_4 + a_1a_3, \ \ b_6 = {a_3}^2 + 4b_6$$

We also define quantities

$$b_8 = {a_1}^2a_6 + 4a_2a_6 - a_1a_3a_4 + a_2{a_3}^2 - {a_4}^2,$$

$$c_4 = {b_2}^2 - 24b_4,$$

$$c_6 = -{b_2}^3 + 36b_2b_4 - 216b_6,$$

$$\Delta = -{b_2}^2b_8 - 8{b_4}^3 - 27{b_6}^2 + 9b_2b_4b_6.$$

If the characteristic of the field K is different from 2 and 3, then this equation can be transformed into the form

$$y^2 = x^3 + ax + b$$

which is called the short Weierstrass equation. The condition of non- singularity now means that the cubic polynomial $f(x) = x^3 + ax + b$ does not have multiple roots (in the algebraic closure $\bar{K}$), which is equivalent to the condition that the discriminant $\Delta = -16(4a^3 + 27b^2)$ is non-zero.

2.2.2 The Group Law

Let E be an elliptic curve given by a Weierstrass equation. Thus $E \subset \mathbb{P}^2$ consists of the points $P = (x, y)$ satisfying the Weierstrass equation, along with the point at infinity $\mathcal{O} = (0 : 1 : 0)$.

Now, let $L \subset \mathbb{P}^2$ be a line. Since the Weierstrass equation is of degree three, the line L intersects E at exactly three points, denoted P, Q, and R. Note that if L is tangent to E at one or more points, the points P, Q, and R may coincide, with appropriate multiplicities.

This result—that $L \cap E$, counted with multiplicities, always consists of exactly three points—is a special case of Bézout's theorem [19].

2.2.3 Composition Law

Let $P,Q \in E$, and let L denote the line passing through P and Q. If $P = Q$, L is taken as the tangent line to E at P. Let R represent the third point of intersection of L with E. Next, consider the line L' passing through R and the point at infinity $\mathcal{O}$. The line L' intersects E at R, $\mathcal{O}$, and a third point, which we denote by $P \oplus Q$.

Proposition 2.18. *The composition law has the following properties:*

- *If a line L intersects E at the (not necessarily distinct) points P,Q,R, then $(P \bigoplus Q) \bigoplus R = \mathcal{O}$.*
- *$P \bigoplus \mathcal{O} = P$ for all $P \in E$.*
- *$P \bigoplus Q = Q \bigoplus P$ for all $P,Q \in E$.*
- *Let $P \in E$. There is a point of E, denoted by $\ominus P$, satisfying $P \bigoplus (\ominus P) = \mathcal{O}$.*
- *Let $P,Q,R \in E$. Then $(P \bigoplus Q) \bigoplus R = P \bigoplus (Q \bigoplus R)$.*

  In other words, the composition law makes E into an abelian group with identity element $\mathcal{O}$. Further:

- *Suppose that E is defined over K. Then $E(K) = \{(x,y) \in K^2 : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6\} \cup \{\mathcal{O}\}$ is a subgroup of E.*

One of the most important properties of elliptic curves is that on the set $E(K)$, of its K-rational points, we can, in a natural way, introduce an operation with which it will become an Abelian group. In order to explain that, let us take that $K = \mathbb{R}$. Then the elliptic curve $E(\mathbb{R})$ (without the point at infinity) can be represented as a subset of the plane. The polynomial $f(x)$ can either have one (if $\Delta < 0$) or three (if $\Delta > 0$) real roots. Depending on that, the graph of the corresponding elliptic curve has one or two components.

Definition 2.19. *Let E be an elliptic curve over K. The subgroup $E(K)_{tor}$ of $E(K)$ which consists of all points of finite order is called the torsion group of E, and the non-negative integer r is called the rank of E and it is denoted by rank(E) (or more precisely by rank(E(K))).*

Given an elliptic curve E over a number field K, Mordell-Weil Theorem asserts that the set of K-rational points $E(K)$ of E is a finitely generated abelian group.

Theorem 2.20. ***(The Mordell-Weil Theorem)*** *[35] The group $E(K)$ is a finitely generated abelian group.*

In particular, $E(K) \cong \mathbb{Z}^r \bigoplus T$, where T is the torsion subgroup of $E(K)$ and rank r is a non-negative integer.

The Mordell-Weil theorem states that there is a finite set of rational points $\{P_1, \ldots, P_k\}$ on E from which all other rational points on E can be obtained by using the secant-tangent construction. With the knowledge that, each finitely generated abelian group is isomorphic to the product of cyclic groups we obtain the followings.

This states that there are r rational independent points $P_1, \ldots, P_r$ of infinite order on curve E such that each rational point P on E can be represented in the form $P = T + m_1P_1 + \cdots + m_rP_r$, where T is a point of finite order and $m_1, \ldots, m_r$ are integers. Here m_1P_1 denotes the sum $P_1 + \cdots + P_1$ of m_1 summands, which is often also denoted by $[m_1]P_1$.

Theorem 2.21. ***(Mazur)*** *[26] Let $E/\mathbb{Q}$ be an elliptic curve. Then the torsion subgroup $E_{tor}(\mathbb{Q})$ of $E(\mathbb{Q})$ is isomorphic to one of the following fifteen groups:*

$$\mathbb{Z}/k\mathbb{Z} \text{ for } k = 1,2,3,4,5,6,7,8,9,10,12$$

$$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/k\mathbb{Z} \text{ for } k = 2,4,6,8.$$

Further, each of these groups occurs as $E_{tor}(\mathbb{Q})$ for some elliptic curve $E/\mathbb{Q}$.

Example 2.22. *Consider the elliptic curve given by the equation:*

$$E : y^2 = x^3 - x.$$

This elliptic curve has a torsion subgroup $E(\mathbb{Q})_{tors}$ which is isomorphic to $\mathbb{Z}/4\mathbb{Z}$.

- *Infinity Point*: The point at infinity, denoted by $\mathcal{O}$, acts as the identity element in the group $E(\mathbb{Q})$ and is part of the torsion subgroup.

- *Other Torsion Points*: To understand the structure of the torsion subgroup, we need to find points of finite order in $E(\mathbb{Q})$. For this curve, we have the following torsion points:

$$P = (0,0), \ Q = (1,0), \ R = (-1,0).$$

- *Addition of Points in the Group*: The point $P=(0,0)$ has order 2 because $2P=\mathcal{O}$, the identity element.
Similarly, the points $Q=(1,0)$ and $R=(-1,0)$ also have order 2.

- *Structure of the Torsion Subgroup*: Now, we can determine the full structure of $E(\mathbb{Q})_{\text{tors}}$. This torsion subgroup is generated by the points $\{\mathcal{O},P,Q,P+Q\}$, and the structure of this set is isomorphic to $\mathbb{Z}/4\mathbb{Z}$.

Here, P acts as a generator for the group since all elements of the torsion subgroup can be expressed in terms of P (for instance, $P,\ 2P=\mathcal{O}$).

Therefore, we conclude that $E(\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/4\mathbb{Z}$, which is one of the allowed structures according to Mazur's Theorem.

The following theorem provides a comprehensive classification of the possible torsion points on elliptic curves over quadratic fields.

Theorem 2.23. *Let K be a quadratic field and E an elliptic curve over K. Then the torsion subgroup $E(K)_{tor}$ of $E(K)$ is isomorphic to one of the following 26 groups:*

$$\mathbb{Z}/m\mathbb{Z} \ \textit{for}\ 1\leq m\leq 18, m\neq 17,$$

$$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2m\mathbb{Z} \ \textit{for}\ 1\leq m\leq 6,$$

$$\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/3m\mathbb{Z},\ \textit{for}\ m=1,2,$$

$$\mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}.$$

Example 2.24. *Consider the quadratic field $K=\mathbb{Q}(\sqrt{-1})$ and the elliptic curve defined by:*

$$E: y^2 = x^3 + x.$$

This elliptic curve E has a torsion subgroup over K given by

$$E(K)_{tors} \cong \mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z},$$

which is one of the allowed structures under the theorem above over quadratic fields.

- Quadratic Field: Here, $K=\mathbb{Q}(\sqrt{-1})$ is a quadratic extension of $\mathbb{Q}$.

- Elliptic Curve: The elliptic curve $E: y^2=x^3+x$ over K includes points that are

rational over K but may not be rational over $\mathbb{Q}$.

- Torsion Structure: The torsion subgroup $E(K)_{tors}$ contains points that form the group structure

$$\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z},$$

which satisfies the conditions in the quadratic field K.

The following theorem completes the classification of torsion over cubic number fields.

Theorem 2.25. *Let $K/\mathbb{Q}$ be a cubic extension and E/K be an elliptic curve. Then $E(K)$ is isomorphic to one of the following 26 groups:*

$$\mathbb{Z}/N_1\mathbb{Z} \text{ with } N_1 = 1, \dots, 16, 18, 20, 21,$$

$$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2N_2\mathbb{Z} \text{ with } N_2 = 1, \dots, 7.$$

There exist finitely many $\bar{\mathbb{Q}}$- isomorphism classes for each torsion subgroup except for $\mathbb{Z}/21\mathbb{Z}$.

The following theorem, which is about a complete classification for torsion points of elliptic curves defined over Galois quartic fields.

Theorem 2.26. *Let $E/\mathbb{Q}$ be an elliptic curve, and let K be a quartic Galois extension of $\mathbb{Q}$. Then $E(K)_{tor}$ is isomorphic to one of the following groups:*

$$\mathbb{Z}/N_1\mathbb{Z} \text{ for } N_1 = 1, \dots, 16, \ N_1 \neq 11, 14,$$

$$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2N_2\mathbb{Z} \text{ for } N_2 = 1, \dots, 6, 8,$$

$$\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3N_3\mathbb{Z} \text{ for } N_3 = 1, 2,$$

$$\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4N_4\mathbb{Z} \text{ for } N_4 = 1, 2,$$

$$\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z},$$

$$\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6Z.$$

Each of these groups, except for $\mathbb{Z}/15\mathbb{Z}$, appears as the torsion structure over some quartic Galois field for infinitely many (non-isomorphic) elliptic curves defined over $\mathbb{Q}$.

Example 2.27. *Consider the elliptic curve*

$$E : y^2 = x^3 + 2x + 1$$

defined over $\mathbb{Q}$. *Now let* $K = \mathbb{Q}(\sqrt[4]{2})$, *which is a quartic Galois extension of* $\mathbb{Q}$.

For this curve E *over* K, *the torsion subgroup* $E(K)_{tors}$ *is isomorphic to* $\mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$, *which is one of the groups given in the theorem. This group structure arises from the additional points in* $E(K)$ *that satisfy the torsion conditions over the extended field* K, *which are not necessarily rational over* $\mathbb{Q}$ *but become rational in the quartic extension* K.

- *Here,* $K = \mathbb{Q}(\sqrt[4]{2})$ *is a quartic Galois extension of* $\mathbb{Q}$, *that is a degree 4 extension with Galois group symmetry over* $\mathbb{Q}$.

- *The elliptic curve* $E : y^2 = x^3 + 2x + 1$ *over* K *includes points that are rational over* K *but not necessarily over* $\mathbb{Q}$. *These points form part of the torsion structure in* $E(K)$.

- *In* $E(K)$, *the torsion subgroup* $E(K)_{tors}$ *includes points that form the group structure* $\mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$, *which satisfies the conditions stated in the theorem. Specifically, the points in* $E(K)$ *that generate this torsion structure are possible due to the additional symmetry and solutions in* K, *which a quartic field provides.*

Let K be an algebraically closed field. Let g be a positive integer. Let $h(x), f(x) \in K[x]$ such that $\deg f = 2g+1$ and $\deg h \leq g$. Suppose that f is monic. The curve C given by the equation

$$C : y^2 + h(x)y = f(x)$$

is called a hyperelliptic curve of genus g if it is nonsingular for all $x, y \in K$. When $g = 1$, we obtain an elliptic curve in generalized Weierstrass form. For a curve of genus greater than one, we have the following theorem.

Theorem 2.28. ***(Faltings)*** *[17] Let* K *be a number field. A curve of genus* $g > 1$ *over* K *has only finitely many rational points.*

Example 2.29. *Consider the curve defined by the equation:*

$$y^2 = x^5 - x + 1.$$

This is an example of a hyperelliptic curve, a type of algebraic curve with a specific form $y^2 = f(x)$, *where* $f(x)$ *is a polynomial of degree at least 5 for higher-genus cases. This particular curve has genus* $g = 2$, *which can be computed based on the* *degree of the polynomial $f(x)$. In this example we have genus $g > 1$ curve and it is defined over $\mathbb{Q}$, so we can apply Faltings' Theorem directly. According to the theorem, since the curve $y^2 = x^5 - x + 1$ has genus 2, it can has only finitely many rational points. In explicitly, there are only finitely many pairs (x, y), where $x, y \in \mathbb{Q}$ that satisfy this equation.*

Example 2.30. *Fermat curves are a family of algebraic curves defined by equations of the form:*

$$x^n + y^n = z^n$$

for some integer $n \geq 3$. Fermat's Last Theorem states that the equation $x^n + y^n = z^n$ has no non-trivial integer solutions for $n > 2$ when x, y and z are positive integers. z are positive integers. However, Fermat curves themselves, viewed as algebraic curves, are interesting objects of study in number theory and algebraic geometry.

The genus of a Fermat curve $x^n + y^n = z^n$ depends on the exponent n. For $n = 3$, the curve has genus $g = 1$, however, for $n > 3$, the genus g of these curves is greater than 1. By a classical result in algebraic geometry, the genus g of the Fermat curve with $n > 3$ is:

$$g = \frac{(n-1)(n-2)}{2}.$$

For example, the curve $x^4 + y^4 = z^4$ has genus $g = 3$, and any Fermat curve for $n > 3$ will similarly have $g > 1$.

Again by Faltings' Theorem, Fermat curves with $n > 3$ have only finitely many rational points.

3. Arithmetic Dynamics

Throughout this thesis K will be a field and $K[x_1,\ldots,x_n]$ denotes the polynomial ring in n variables over K.

Let $f(x)$ be a rational map over the field K. The *n-fold composition* of the polynomial $f(x)$ with itself is denoted by $f^n(x) = f \circ f^{n-1}(x)$ for all $n \geq 1$, where f^0 is defined as the identity map.

The set $O_f(p) := \{f^n(p) : n \geq 0\}$ is called the *orbit* of the point p under f.
Generally, we define K as a number field and construct $f(x) = A(x)/B(x)$ as a rational function defined over K. In this construction, $A(x)$ and $B(x)$ are coprime polynomials with coefficients in K.
We say a point $\alpha \in K$ is a *periodic point* of $f(x)$ if $f^n(\alpha) = \alpha$ for some $n \geq 1$. The smallest positive integer n with this property is called the period of α. If there is no positive integer $m < n$ such that $f^m(\alpha) = \alpha$ then α is a periodic point of *exact period* n. A point $\beta \in K$ is called a *preperiodic point* of $f(x)$ if $f^n(\beta) = f^m(\beta)$ for some positive integers such that $n \neq m$.
We will denote the K-rational preperiodic points of a polynomial $f(x)$ as the set

$$PrePer(f,K) := \{\alpha \in K : \alpha \text{ is preperiodic under } f\}.$$

Example 3.1. *[3] The function* $f(x) = \frac{176z^2+1397z-1573}{176z^2+500z-1144}$ *has an orbit set consisting of 8 elements, as shown below, which correspond to the preperiodic points of* $f(x)$.

$$PrePer(f,K) := \{\infty, 1, 0, \frac{11}{8}, \frac{-11}{2}, \frac{-11}{4}, \frac{55}{16}, 2\}.$$

Throughout this study the degree of the rational map f is given by, $\deg(f(x)) = \deg(A(x)/B(x))$, is defined to be the integer $d := max\{\deg A, \deg B\}$ if $\gcd(A(x),B(x)) = 1$.

By a classical theorem of Northcott [29], the set $PrePer(\phi, K)$ is finite over a number field K; it can therefore be given a finite directed graph structure, the preperiodicity graph of ϕ, denoted by G_ϕ, by drawing an arrow from P to $\phi(P)$ for each $P \in PrePer(\phi, K)$. It is a natural question to ask which types of graphs (up to graph isomorphism) can be obtained from such maps. It is not known whether the list of possible graphs is finite; this is equivalent to the (1-dimensional) Uniform Boundedness Conjecture of Morton and Silverman [28] that says $\#PrePer(\phi, K)$ is bounded by a bound depending only on the degrees of $K/\mathbb{Q}$ and ϕ.

Morton and Silverman stated the following conjecture about the size of the orbits of rational functions;

Conjecture 3.0.1. *(Uniform Boundedness Conjecture) [28] For fixed integers $n \geq 1$ and $d \geq 2$ there exists a constant $M(n,d)$ such that for every number field K of degree n, and every rational function $f(x) \in K(x)$ of degree d,*

$$\#PrePer(f, K) \leq M(n,d).$$

Currently there is limited knowledge about this conjecture. Notably, the existence of a constant $M(n,d)$ has not been proved, even in the more straightforward structure, where $K = \mathbb{Q}$ and $f(x) \in K[x]$ is a quadratic polynomial.

Poonen proposed an upper bound for this case and furthermore offered a conjectural complete list of all potential graph structures that may arise in this context.

Theorem 3.2. *(Poonen) [32] Assume that there is no quadratic polynomial over $\mathbb{Q}$ that has a rational periodic point of period greater than 3. Then, for every quadratic polynomial f with rational coefficients,*

$$\#PrePer(f, \mathbb{Q}) \leq 9.$$

Graphs of Dynamical Systems

In the context of a dynamical system, the graph of a function $f(x) \in \mathbb{Q}$ can be defined as a directed graph where:

- Each point $x \in \mathbb{Q}$ is represented as a vertex in the graph.
- An edge is drawn from x to $f(x)$, indicating the image of x under the map.

The **orbit** of a point x under $f(x)$ is the sequence:

$$\mathcal{O}_f(x) = \{x, f(x), f^2(x), f^3(x), \dots\}$$

where $f^n(x) = f(f^{n-1}(x))$ for $n \geq 1$.

Types of Points on Graphs

- **Fixed points**: A point $x \in \mathbb{Q}$ is a fixed point if $f(x) = x$. In the graph, this is represented as a vertex with a self-loop.
- **Periodic points**: A point $x \in \mathbb{Q}$ is periodic with period n if $f^n(x) = x$ for the smallest such $n \geq 1$. This forms a cycle in the graph.
- **Preperiodic points**: A point $x \in \mathbb{Q}$ is preperiodic if there exists some $m \geq 1$ such that $f^m(x)$ is periodic. In the graph, such points have directed edges leading into a periodic cycle.
- **Critical points**: The orbits of critical points (where $f'(x) = 0$) are crucial for understanding the global dynamics.

The simplest case of the graph classification question is for quadratic polynomials defined over $\mathbb{Q}$. Flynn, Poonen and Schaefer [18] conjectured that for any integer $N > 3$ there is no quadratic polynomial with coefficients in $\mathbb{Q}$ with a $\mathbb{Q}$-periodic point of period N. Assuming this conjecture is true, Poonen [32] provided a complete classification of 12 possible preperiodicity graphs for quadratic polynomials defined over $\mathbb{Q}$. Another consequence of Poonen's classification is that the number of $\mathbb{Q}$-preperiodic points of a quadratic polynomial is at most 9.

Usage in Arithmetic Dynamics

In arithmetic dynamics, understanding the idea of functions behavior of rational points under iteration of rational maps becomes easy using graphs.

Poonen, Morton, and Silverman have explored the structure of these graphs in arithmetic dynamics to study:

- **Growth of orbits**: Understanding how the number of distinct points in an orbit increases as the function is iterated.
- **Arithmetic properties**: Investigating the behavior of orbits over different fields, such as $\mathbb{Q}$ or finite fields.

Flynn, Poonen and Schaefer conjecture was proved for the special cases of a periodic point of period N = 4 (Morton [27]), N = 5 (Flynn, Poonen and Schaefer [18]) and N = 6 (Stoll [36], depending on the Birch and Swinnerton-Dyer conjecture); experimental results by Hutz and Ingram [20] and Benedetto et al. [3] provide further evidence for it.

3.1 Rational Periodic Points of the Quadratic Function $f(x) = x^2 + c$

Definition 3.3. *Let ϕ and γ be two rational maps. These maps are linearly conjugate if there is some $f \in \mathrm{PGL}_2(\bar{K})$ such that $f^{-1}\gamma f = \phi$. They are linearly conjugate over K if there is some $f \in \mathrm{PGL}_2(K)$ such that $f^{-1}\gamma f = \phi$. Linearly conjugate maps have the same dynamical behaviour as*

$$\alpha^n(P) = \alpha^m(P) \text{ if and only if } \beta^n(f^{-1}(P)) = \beta^m(f^{-1}(P)),$$

where $\beta = f^{-1}\alpha f$.

Let $\phi(x) = Ax^2 + Bx + C \in K[x]$ be a conjugacy quadratic polynomial map, where $A \in K^{\times}$. Such a map is either linearly conjugate over K to a map of the form $f(x) = x^2 + c$ for some $c \in K$. Therefore, we focus on quadratic polynomial maps of this form.

By the results of Northcott [29], it is known that these maps have only finitely many rational periodic points. A complete classification of rational quadratic polynomials $f(x)$ with periodic points of exact period 1, 2, or 3 has been provided by Walde and Russo [37], and Poonen [31].

Theorem 3.4. *[32] Let $f(x) = x^2 + c$, where $c \in \mathbb{Q}$.*

1) *$f(x)$ has a rational fixed point if and only if $c = 1/4 - \rho^2$ for some $\rho \in \mathbb{Q}$. In this case, there are exactly two, $1/2 + \rho$ and $1/2 - \rho$, unless $\rho = 0$, in which case they coincide.*

2) $f(x)$ has a rational point of period 2 if and only if $c = -3/4 - \sigma^2$ for some $\sigma \in \mathbb{Q}^*$. In this case, there are exactly two, $-1/2+\sigma$ and $-1/2-\sigma$.

3) $f(x)$ has a rational point of period 3 if and only if

$$c = -\frac{\tau^6 + 2\tau^5 + 4\tau^4 + 8\tau^3 + 9\tau^2 + 4\tau + 1}{4\tau^2(\tau+1)^2}$$

for some $\tau \in \mathbb{Q}, \tau \neq -1, 0$. In this case, there are exactly three,

$$x_1 = \frac{\tau^3 + 2\tau^2 + \tau + 1}{2\tau(\tau+1)}$$

$$x_2 = \frac{\tau^3 - \tau - 1}{2\tau(\tau+1)}$$

$$x_3 = -\frac{\tau^3 + 2\tau^2 + 3\tau + 1}{2\tau(\tau+1)}$$

and these are cyclically permuted by $f(x)$.

Proof. Taking c as above and using the corresponding periodic points we get one implication. Let p be a rational periodic point of $f(x)$ with period length 1. Then, we get

$$c = p - p^2.$$

On the other hand, roots of the equality

$$x^2 + p - p^2 = x,$$

are p and $1-p$. Hence $f(x)$ has a rational fixed point if and only if $c = p - p^2$ for some $p \in \mathbb{Q}$. In this case, rational fixed points are p and $1-p$. If we substitute

$$p = \frac{1}{2} + \rho,$$

this completes the proof of first part.

Let p be a rational periodic point of $f(x)$ with exact period length 2. So we want p to be a root of the polynomial

$$f^2(x) - x = x^4 + 2cx^2 - x + c^2 + c.$$

If we factor this polynomial, we get

$$(x^2+c-x)(x^2+x+c+1).$$

We only want p to be the root of

$$(x^2+z+x+1),$$

since we do not want to have $f(p)=p$. Hence, $c=-1-p-p^2$. Now, other root of the polynomial $(x^2+x-p-p^2)$ is $-1-p$. So $f(x)$ has a rational periodic point of exact period 2 if and only if $c=-1-p-p^2$ for some $p\in\mathbb{Q}$. In this case, rational period 2 points of the map are p and $-1-p$. If we substitute

$$p=-\frac{1}{2}+\sigma,$$

this completes the second part of theorem. Let $\zeta\in\mathbb{Q}$ be an exact period 3 point $f(x)$. Assume that $\omega:=f(\zeta)=\zeta^2+c$ is not equal to ζ, that is ζ is not a rational fixed point of $f(x)$. Since we have $c=\omega-\zeta^2$, we get

$$f(x)=x^2+\omega-\zeta^2.$$

Now,

$$\begin{aligned}\zeta&=f^3(\zeta)\\&=f^2(\omega)\\&=f(\omega^2+\omega-\zeta^2)\\&=\omega^4+2\omega^3+\omega^2-2\omega^2\zeta^2-2\omega\zeta^2+\zeta^4+\omega-\zeta^2.\end{aligned}$$

Now, if we rearrange the last equation and divide it by $\omega-\zeta$, we get an equivalent form of

$$(\omega+\zeta)^3+(2-2\zeta)(\omega+\zeta)^2+(1-2\zeta)(\omega+\zeta)+1=0.$$

Let $\tau=(\omega+\zeta)$. Using this equality in the previous one, we get

$$\zeta=\frac{\tau^3+2\tau^2+\tau+1}{2\tau(\tau+1)},$$

where $\tau\in\mathbb{Q}\setminus\{-1,0\}$. One can see directly, this is formula of x_1. So we get the following equalities,

$$x_2 = f(\zeta) = \omega = \tau - x_1,$$

$$c = \omega - \zeta^2 = x_2 - x_1^2,$$

$$x_3 = f(x_2).$$

This completes the proof of theorem. $\square$

We have some examples of $f(x)$ with both rational fixed and period two point. We examine a few of them with their graphs.

Example 3.5. *Let $f(x) = x^2 + \frac{-21}{16} \in \mathbb{Q}[x]$. This function has two fixed point and period 2 points. We have the following graphs of these points,*

$$\begin{tikzcd} \frac{7}{4} \arrow[loop above] & -\frac{3}{4} \arrow[loop above] \end{tikzcd}$$

$$\textit{Fixed Points: } \frac{7}{4}, -\frac{3}{4}$$

$$\begin{tikzcd} \frac{1}{4} \arrow[r, bend left, "f"] & -\frac{5}{4} \arrow[l, bend left, "f"] \end{tikzcd}$$

$$\textit{Period-Two Points: } \frac{1}{4} \leftrightarrow -\frac{5}{4}$$

Example 3.6. *Let $f(x) = x^2 + \frac{-1849}{576} \in \mathbb{Q}[x]$. This function has period 3 orbit as seen below,*

Let start with periodic point $\frac{49}{24}$:

$$f\left(\frac{49}{24}\right) = \frac{23}{24},\ f\left(\frac{23}{24}\right) = -\frac{55}{24},\ f\left(-\frac{55}{24}\right) = \frac{49}{24}$$

As a consequence, we have following sequence of images for $f(x)$,

$$\frac{49}{24} \to \frac{23}{24} \to -\frac{55}{24} \to \frac{49}{24}$$

which has the following graph,

$$\begin{tikzcd} \frac{49}{24} \arrow[r, bend left] & \frac{23}{24} \arrow[r, bend left] & -\frac{55}{24} \arrow[ll, bend left] \end{tikzcd}$$

Theorem 3.7. *[32] Let $f(x) = x^2 + c$ with $c \in \mathbb{Q}$. Then,*

1) $f(x)$ has rational points of period 1 and rational points of period 2 if and only if

$$c = -\frac{3\mu^4 + 10\mu^2 + 3}{4(\mu^2 - 1)^2}$$

for some $\mu \in \mathbb{Q}$, $\mu \neq -1, 0, 1$. In this case the parameters ρ and σ of the previous theorem is

$$\rho = -\frac{\mu^2 + 1}{\mu^2 - 1} \quad \sigma = \frac{2\mu}{\mu^2 - 1}.$$

2) If $f(x)$ has rational points of period 3, it cannot have any rational points of period 1 or 2.

Proof. By Theorem 3.4, $x^2 + c$ has rational points of period 1 and 2 if and only if $c = 1/4 - \rho^2 = -3/4 - \rho^2$, where $\rho, \sigma \in \mathbb{Q}$ with $\sigma \neq 0$. The curve in the (ρ, σ)-plane described by this equation is a conic with a rational point $(1, 0)$, so it is birational to $\mathbb{P}^1$ over $\mathbb{Q}$, with the rational function $\mu = (1 - \rho)/\sigma$ giving the birational map. Solving for c, ρ, and σ in terms of μ gives the result. The values of μ not allowed are $-1, 1, 0, \infty$, because these correspond to the two points at infinity on the conic and the two points where $\sigma = 0$.
If $x^2 + c$ has rational points of period 1 and 3, then

$$c = 1/4 - \sigma^2 = -\frac{\tau^6 + 2\tau^5 + 4\tau^4 + 8\tau^3 + 9\tau^2 + 4\tau + 1}{4\tau^2(\tau + 1)^2},$$

so $(\tau, 2\tau(\tau + 1)\rho)$ is a point on the hyperelliptic curve,

$$C : y^2 = x^6 + 2x^5 + 5x^4 + 10x^3 + 10x^2 + 4x + 1.$$

Also this is an equation for the modular curve $X_1(18)$. But $X_1(18)$ has only six rational points, so the only rational points on C besides the two points at infinity are $(-1, 1)$, $(-1, -1), (0, 1)$, and $(0, -1)$. These do not give rise to a valid pair (τ, ρ), since τ is not allowed to be 0 or -1 in Theorem 3.4. Hence it is impossible for the maps to exist rational periodic points of period 1 and 3.

Similarly, if x^2+c has rational points of period 2 and 3, then

$$c = -3/4 - \sigma^2 = -\frac{\tau^6+2\tau^5+4\tau^4+8\tau^3+9\tau^2+4\tau+1}{4\tau^2(\tau+1)^2},$$

so $(\tau, 2\tau(\tau+1)\rho)$ is a point on

$$C^{'} : y^2 = x^6+2x^5+x^4+2x^3+6x^2+4x+1.$$

This curve is $X_1(13)$, since if we dehomogenize the model

$$x_1^2x_2^2 - x_1x_2^3 - x_1x_2x_3^2 + x_1x_3^3 + x_2^3x_3 - x_2^2x_3^2$$

of $X_1(13)$ in [6] by setting $x_3 = 1$, we find the discriminant of the resulting quadratic in x_1 is

$$x_2^6+2x_2^5+x_2^4+2x_2^3+6x_2^2+4x_2+1.$$

The curve $X_1(13)$ also has exactly six rational points, so the only rational points on $C^{'}$ besides the two points at infinity are (-1, 1), (-1, -1), (0, 1), and (0, -1). Again this implies that x^2+c cannot have both rational points of period 2 and 3, since τ is not allowed to be 0 or -1. $\square$

If m and n are positive integers, then a point of type m_n for $f(x)$ is a preperiodic point that enters an m-cycle after n iterations. For example, $\frac{3}{4}$ is a point of type 3_2 for $f(x) = x^2 - \frac{29}{16}$.

Theorem 3.8. *Let $f(z) = z^2+c$ with $c \in \mathbb{Q}$. Then,*
1. For each $m \geq 1$, x is a rational point of type m_1 for $f(z)$ if and only if $-x$ is a nonzero rational point of period m. The number of rational points of type m_1 equals the number of rational points of period m, except when $c = -1,\ m = 2$, or $c = 0,\ m = 1$, in which case there is one less.
2. $f(z)$ has rational points of type 1_2 if and only if

$$c = \frac{-2(\eta^2+1)}{(\eta^2-1)^2}$$

for some $\eta \in \mathbb{Q},\ \eta \neq -1, 1$. In this case, there are exactly 2 such points, $\frac{2\eta}{\eta^2-1}$ and $-\frac{2\eta}{\eta^2-1}$ unless $\eta = 0$ $(c = -2)$, in which they coincide. The parameter ρ is

$$\rho = -\frac{-2(\eta^2+1)}{(\eta^2-1)}$$

3. $f(z)$ *has rational points of type* 2_2 *if and only if*

$$c = \frac{-\nu^4 - 2\nu^3 - 2\nu^2 + 2\nu - 1}{(\nu^2 - 1)^2}$$

for some $\nu \in \mathbb{Q}$, $\nu \neq -1, 0, 1$. *In this case, there are exactly 2 such points,* $\frac{\nu^2+1}{\nu^2-1}$ *and* $-\frac{\nu^2+1}{\nu^2-1}$. *The parameter* σ *is*

$$\sigma = \frac{\nu^2 + 4\nu - 1}{2(\nu^2 - 1)}$$

4. $f(z)$ *has rational points of type* 3_2 *if and only if* $c = -29/16$. *For* $c = -29/16$, *the rational points of type* 3_2 *are 3/4 and -3/4.*
5. If $f(z)$ *has rational points of type* m_2 *with* $1 \leq m \leq 3$, *then there are no rational points of period* $b \leq 3$ *unless* $b = a$.
6. $f(z)$ *cannot have rational points of type* $1_n, 2_n$, *or* 3_n *for any* $n \geq 3$.

Moreover, there are exactly 12 graphs that arise from $PrePer(f, \mathbb{Q})$ as f varies over all quadratic polynomials with rational coefficients.

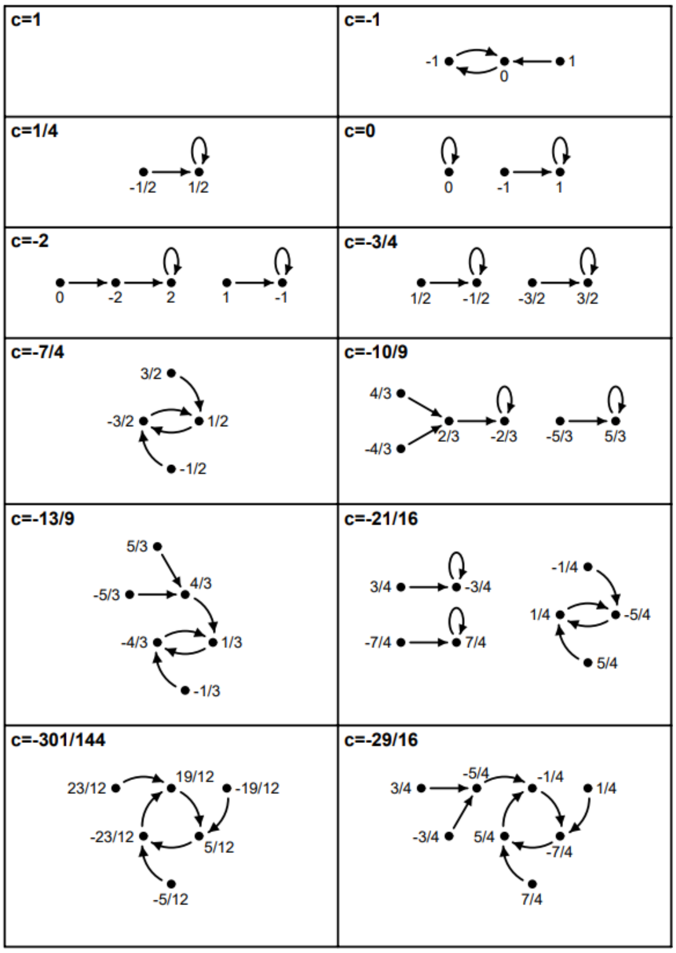


Figure 3.1 [32] Finite rational preperiodic points of $z^2 + c$ for selected values of c.

Theorem 3.9. *[27] There is no quadratic polynomial $f(x) \in \mathbb{Q}[x]$ with a rational point of exact period 4.*

E. V. Flynn, Bjorn Poonen, and Edward F. Schaefer established that no quadratic polynomial over $\mathbb{Q}$ has a rational periodic point with a period of $m = 5$.

Theorem 3.10. *[18] There is no quadratic polynomial $g(x) \in \mathbb{Q}[x]$ with a rational point of exact period 5.*

Proof. (Sketch)
Flynn, Poonen, and Schaefer's proof addresses the nonexistence of rational periodic points of exact period $N = 5$ for quadratic polynomials over $\mathbb{Q}$ by analyzing the curve $C_0(5)$, which parametrizes these rational maps.

Goal. They start by defining a quadratic polynomial $f(x) = x^2 + c$ over $\mathbb{Q}$. The objective is to determine whether any value of $c \in \mathbb{Q}$ allows $f(x)$ to have a rational periodic point of exact period 5. A point $x_0 \in \mathbb{Q}$ is said to have period 5 if $f^5(x_0) = x_0$ and $f^k(x_0) \neq x_0$ for $k < 5$, ensuring it does not belong to a smaller cycle.

Setting Up the Curve $C_0(5)$. The existence of such a periodic point implies a set of polynomial relations that x_0 and c must satisfy. These relations define an algebraic curve in the (x_0, c)-plane, known as $C_0(5)$. This curve is designed to capture all possible quadratic polynomials with periodic points of exact period 5 by parametrizing the values of c that yield such cycles. Importantly, $C_0(5)$ is a curve of genus 2, which is crucial to the proof.

Genus Constraints and Faltings' Theorem. The genus of $C_0(5)$ is significant due to Faltings' theorem (formerly the Mordell conjecture), which states that a curve of genus $g \geq 2$ over $\mathbb{Q}$ has only finitely many rational points. Thus, if $C_0(5)$ has genus 2 and lacks sufficient rational points that could correspond to exact period-5 cycles, then it follows that no such quadratic polynomial exists with a rational point of period 5.

Analysis of Rational Points on $C_0(5)$. By explicitly computing rational points on $C_0(5)$, the authors confirm that none of these points correspond to an actual period-5 cycle under a quadratic polynomial over $\mathbb{Q}$. This involves a detailed analysis of the possible solutions, where each candidate either fails to meet the period-5 condition or does not satisfy the requirements for $f(x) = x^2 + c \in \mathbb{Q}$.

Conclusion and Nonexistence. The absence of suitable rational points on $C_0(5)$ thus implies that there cannot be a quadratic polynomial over $\mathbb{Q}$ with a rational periodic point of period $N = 5$. This approach leverages both the geometric properties of the curve and arithmetic constraints on rational points, resulting in a proof by contradiction.

□

This proof shows how advanced tools in arithmetic geometry, such as arithmetic of elliptic curves and Faltings' theorem, help answer questions in dynamical systems and periodic points for polynomials.

They also list in [18] (see Table 2) all quadratic polynomials in $\mathbb{Q}[x]$ (up to linear conjugacy) with a $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$- stable 5-cycle.
A Galois-stable n-cycle refers to a set of elements structured as an n-cycle that remains invariant under the action of a Galois group. This stability under the Galois group action essentially means that each element in the cycle, when mapped by any Galois transformation, maps to another element within the same cycle, preserving the structure as a whole.

Definition 3.11. *If α is an element of the Galois group $Gal(L/K)$ (where L is a field extension of K) and $(a_1, a_2, \ldots, a_n)$ is an n-cycle of elements in L, then $\alpha(a_i) = a_{i+k} \pmod{n}$ for all i, where k is a spesific to α.*

Each point in such a cycle generates a degree 5 cyclic extension of $\mathbb{Q}$, which we describe.

Theorem 3.12. *[36] Under the assumption of standard conjectures on L-series of curves, the same conclusion applies for $m = 6$.*

4. Rational Maps of Degree 2 With an Automorphism

This chapter focuses on the behavior of morphisms of degree 2 defined on the projective line $\mathbb{P}^1$ and the properties of their periodic points. Let $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be a morphism defined over a field K, and denote by ϕ^n the n-th iterate of ϕ under composition, $\phi^n = \phi \circ ... \circ \phi$ (n times). A point $P \in \mathbb{P}^1$ is said to be **periodic** if there exists an integer $n > 0$ such that $\phi^n(P) = P$. A point P is called **preperiodic** if there exist integers $n > m \geq 0$ such that $\phi^n(P) = \phi^m(P)$. If $\phi^n(P) = P$, and n is the smallest positive such integer, then P has **period** n. For P to have **formal period** n, it must be a root of the n-th dynatomic polynomial.

In this chapter, the focus is specifically on morphisms on $\mathbb{P}^1$, where every rational map is in fact a morphism. Further, if we write $\phi(z) \in K(z)$ as a rational map $\phi(z) = F(z)/G(z)$, $F(z), G(z) \in K[z]$, then $\phi = \max\{\deg(F(z)),\ \deg(G(z))\}$, which corresponds to the usual notion of the degree of a morphism of projective curves.

4.1 Rational Maps of Degree 2 With $\mathrm{Aut}(\phi) \cong \mathbb{Z}/2\mathbb{Z}$

Theorem 4.1. *[24] Let $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be a morphism of degree 2 defined over $\mathbb{Q}$, and suppose that $\mathrm{Aut}(\phi) \cong \mathbb{Z}/2\mathbb{Z}$, the cyclic group of order 2. Then we have the following:*
(a) The map ϕ has at least one rational fixed point.
(b) There is a one-parameter family of maps such that $\phi(z)$ has exactly three rational fixed points. No such map has exactly two rational fixed points.
(c) There is another one-parameter family of maps such that ϕ has a rational point of primitive period 2.
(d) No such rational maps have a rational point of primitive period 3.
(e) There is a one-parameter family of maps such that ϕ has a rational point of period 4.

(f) These maps have exactly four such rational points. No such rational maps have more than four rational points of primitive period 4.

Conjecture 4.1.1. *If $\phi(z) \in \mathbb{Q}(z)$ is a degree-2 rational map with* $\mathrm{Aut}(\phi) \cong \mathbb{Z}/2\mathbb{Z}$, *then ϕ has no rational point of exact period $N > 4$.*

Theorem 4.2. *[24] Let $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be a morphism of degree 2 defined over $\mathbb{Q}$, and suppose that $Aut(\phi) \cong \mathbb{Z}/2\mathbb{Z}$. Then*
$\#\{P \in \mathbb{Q} | P$ is preperiodic and lands on a cycle of length at most $4\} \leq 12$.

These results contrast with the case of quadratic polynomials, where it is known that there exists a one-parameter family of maps having rational points of period 3, and there are no $\mathbb{Q}$-rational points of primitive period 4, [24].

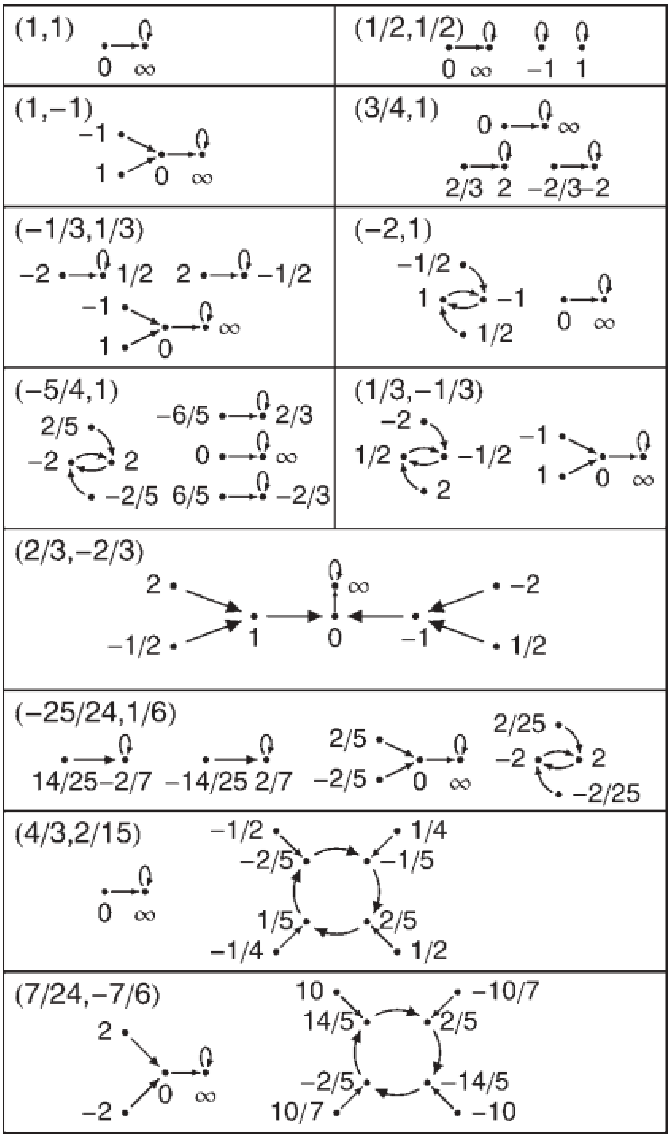


Figure 4.1 [24] The complete list of possible directed graphs of rational maps with automorphism group $\mathbb{Z}/2\mathbb{Z}$

As Poonen explains in [32], these directed graphs serve as analogs to the possible torsion subgroups of elliptic curves over $\mathbb{Q}$, as classified by Mazur's theorem [25]. Consider a rational map $\phi(z)$ of degree 2 with $\mathrm{Aut}(\phi) \cong \mathbb{Z}/2\mathbb{Z}$ (up to linear conjugacy), and let G denote a specific graph of rational preperiodic points. The pairs $(\phi(z), G)$ are parametrized by points on an algebraic curve, just as elliptic curves with prescribed level structures correspond to points on modular curves. Determining whether a given graph is possible thus translates to identifying rational points on the associated algebraic curves.

The specific curves whose rational points we need to identify have genus 0, 1, or 3. The genus 0 and 1 curves have rational points at infinity, so they are, respectively, the projective line $\mathbb{P}^1$ and elliptic curves. All the elliptic curves involved have a small conductor and rank 0, allowing us to enumerate their rational points fully. For the genus 3 curve, it covers an elliptic curve which, unfortunately, has rank 1, preventing a complete listing of its (necessarily finitely many) rational points. However, it also covers a genus 2 curve. By determining all the rational points on the genus 2 curve, we are able to identify all the rational points on the original genus 3 curve.

Conjecture 4.1.2. *Let $K/\mathbb{Q}$ be a number field of degree D, and let $\phi : \mathbb{P}^N \to \mathbb{P}^N$ be a morphism of degree $d \geq 2$ defined over K. There is a constant $\kappa(D,N,d)$ such that*

$$\#PrePer(\phi, K) \leq \kappa(D,N,d).$$

This conjecture, for instance, implies uniform boundedness for torsion points on abelian varieties over number fields. On elliptic curves, torsion points correspond precisely to preperiodic points under the multiplication-by-2 map on the curve. Points on the elliptic curve are mapped to $\mathbb{P}^1$ via their x-coordinates, and the multiplication-by-2 map induces a degree-4 rational map $\phi : \mathbb{P}^1 \to \mathbb{P}^1$, where the x-coordinates of the torsion points map to the preperiodic points of ϕ.

4.2 Preperiodic Points

Given a morphism $\phi : \mathbb{P}^1 \to \mathbb{P}^1$, we may write $\phi(z) \in K(z)$ as a rational map

$$\phi(z) = F(z)/G(z), \quad F(z), G(z) \in K[z],$$

and $\deg\phi = max\{\deg F, \deg G\}$. For any such $\phi(z)$, we create a homogeneous polynomial $\Phi_{n,\phi} \in K[x,y]$ with roots that are precisely points of period n for ϕ. If we homogenize $\phi(z) = \phi(x,y) = [F(x,y) : G(x,y)]$ and write $\phi^n(x,y) = [F_n(x,y) : G_n(x,y)]$, then

$$\Phi_{n,\phi}(x,y) = yF_n(x,y) - xG_n(x,y).$$

If $P = [x : y] \in \mathbb{P}^1$ is a root of this polynomial, then by construction $\phi^n(P) = P$.

The polynomial Φ_n has as its roots all points of period n, including those of primitive period $k < n$ but satisfying $k|n$. We would like to examine points of primitive period n, so we define the n-th dynatomic polynomial for ϕ by

$$\Phi^*_{n,\phi}(x,y) = \prod_{k|n}(\Phi_{k,\phi}(x,y))^{\mu(n/k)} = \prod_{k|n}(yF_k(x,y) - xG_k(x,y))^{\mu(n/k)},$$

where μ is the Moebius function. It is not clear that $\Phi^*_n(x,y)$ is a polynomial, but this is in fact the case. The roots of $\Phi^*_n(x,y)$ are points of formal period n, which include all points of primitive period n.

Definition 4.3. *We say that two rational maps ϕ and ψ are linearly conjugate if there is some $f \in PGL_2(\bar{K})$ such that $\phi^f = \psi$.*

If P is a point of primitive period n for ϕ, then $f^{-1}(P)$ has the same property for ϕ^f, and the same holds for preperiodic points. Consequently, linearly conjugate maps exhibit essentially identical dynamical behavior.

Let a rational map $\phi(z)$, a point P, and $f \in PGL_2(K)$ all be defined over K, with $\phi^n(P) = P$. Then both $\psi(z) = \phi^f(z)$ and $Q = f^{-1}(P)$ are defined over K, and $\psi^n(Q) = Q$. However, if f is instead defined over some finite extension of K, it is possible that ψ remains defined over K, but the periodic point Q is not.

Example 4.4. *Consider the following two rational maps, both defined over the rational field $\mathbb{Q}$:*

$$\phi(z) = 2z + \frac{5}{z}, \;\; \psi(z) = \frac{z^2 - 3z}{3z - 1}$$

Both $\phi(z)$ and $\psi(z)$ have a fixed point at infinity. The finite fixed points of $\psi(z)$ are rational numbers, namely $z = -1$ and $z = 0$. However, the finite fixed points of $\phi(z)$ are not rational; they are the complex numbers $\pm i\sqrt{5}$.

Despite this apparent difference, the rational maps $\phi(z)$ and $\psi(z)$ are conjugate over $\mathbb{Q}$. In other words, there exists a linear fractional transformation $f(z) \in PGL_2(\mathbb{Q})$ such that $\phi^f(z) = \psi(z)$, where $\phi^f(z) = f^{-1}(\phi(f(z)))$ denotes the conjugation of $\phi(z)$ by $f(z)$. This conjugating map $f(z)$ is given by:

$$f(z) = \frac{i\sqrt{5}(z-1)}{1+z}$$

One can verify that $\phi^f(z) = \psi(z)$ by direct computation. This conjugacy implies that $\phi(z)$ and $\psi(z)$ have the same dynamical behavior, despite having different fixed point sets over $\mathbb{Q}$.

The significance of this example is that it demonstrates how two rational maps defined over $\mathbb{Q}$, with seemingly different fixed point structures, can be conjugate and hence have equivalent dynamical properties. The existence of a conjugating map $f(z) \in \mathrm{PGL}_2(\mathbb{Q})$ guarantees that the dynamics of $\phi(z)$ and $\psi(z)$ are essentially the same, just expressed in different coordinate systems.

Usually, $\phi^h(z) \neq \phi(z)$ for rational maps, but this is not always the case. For example, the map $\phi(z) = 2z + \frac{5}{z}$ defined above has a nontrivial PGL_2 automorphism $h(z) = -z$. We can easily verify that $h^{-1}(z) = -z$. Therefore, we have:

$$\phi^h(z) = h^{-1}(\phi(h(z))) = -\phi(-z) = -\left(2(-z) + \frac{5}{-z}\right) = 2z + \frac{5}{z} = \phi(z)$$

Thus, $\phi^h(z) = \phi(z)$ for this particular rational map $\phi(z)$ and the automorphism $h(z) = -z$. In other words, $\phi(z)$ is invariant under the automorphism $h(z) = -z$, which is a symmetry of the map.

Definition 4.5. *The stabilizer group (or automorphism group) of a map ϕ is defined as*

$$\mathrm{Aut}(\phi) = \{f \in \mathrm{PGL}_2(\bar{K}) \mid \phi^f = \phi\},$$

where ϕ^f denotes the map obtained by conjugating ϕ with f.

If $h \in \mathrm{Aut}(\phi)$, then then $f^{-1}hf \in \mathrm{Aut}(\phi^f)$. Thus, linearly conjugate maps have isomorphic stabilizer groups.

Lemma 4.6. *[24] Let K be a field with char $K \neq 2, 3$, and let ϕ be a rational map of degree 2 defined over K. Then $Aut(\phi) \cong \mathbb{Z}/2\mathbb{Z}$ if and only if ϕ is linearly conjugate over K to some map of the form $\phi_{k,b} = kz + \frac{b}{z}$, with $k \in K\{0, -1/2\}$ and $b \in K^*$.*

Furthermore, two such maps $\phi_{k,b}$ and $\phi_{k',b'}$ are linearly conjugate over K if and only if $k = k^{'}$ and $b/b^{'} \in (K^*)^2$. The map $\phi_{k,b}$ has the automorphism $z \to -z$.

Remark 4.7. *Note that for a fixed k, all maps of the form $\phi_{k,b}(z)$ are linearly conjugate over $\bar{K}$. Conjugate by $f_b(z) = (z/\sqrt{b})$ to see that*

$$\phi_{k,b}^{f_b}(z) = kz + b/z = \phi_{k,b}(z).$$

Now consider a rational map $\phi(z) \in K(z)$ of degree 2, satisfying $\mathrm{Aut}(\phi) \cong \mathbb{Z}/2\mathbb{Z}$. To examine the rational periodic points of ϕ, it suffices, by Lemma 4.6, to focus on the case $\phi(z) = kz + b/z$, where $k \in K \setminus \{0, -1/2\}$ and $b \in K^*/(K^*)^2$. Throughout, we will use the expressions $\phi(z) = kz + b/z$ and its homogeneous form $\phi(x, y) = [kx^2 + by^2 : xy]$ interchangeably.

Proposition 4.8. *[24] Let $\phi(z) = kz + b/z$, with $k \in K^*$ and $b \in K^*/(K^*)^2$, with K a field of characteristic different from 2. Then we have the following.*
(a) For all k and b, the point at infinity is a K-rational fixed point for $\phi(z)$.
(b) If $b \equiv 1 - k$ modulo squares, then $\phi(z)$ has two finite K-rational fixed points; otherwise, $\phi(z)$ has no finite K-rational fixed points.

Proof. Consider the dynatomic polynomial $\Phi_1^*(x, y) = (k-1)x^2y + by^3$. The roots of this polynomial correspond to the fixed points of $\phi(x, y)$. Notably, the point at infinity $P = [1 : 0]$ is always a root. When $k = 1$, P becomes a triple root of the polynomial, which implies that there are no finite fixed points in this case.

To determine the finite fixed points for $k \neq 1$, we dehomogenize the polynomial by setting $y = 1$, obtaining

$$\Phi_1^*(z) = (k-1)z^2 + b.$$

Solving $\Phi_1^*(z) = 0$, we find

$$z = \pm\sqrt{\frac{b}{1-k}}.$$

These roots are K-rational if and only if $\frac{b}{1-k} \in (K^*)^2$. Since $b \neq 0$, the two finite fixed points are distinct.

□

Remark 4.9. *Proposition 4.8 and Lemma 4.6 together imply that every degree-2 rational map defined over K with automorphism group $\mathbb{Z}/2\mathbb{Z}$ has at least one K-rational fixed point. This follows from the fact that ϕ must be linearly conjugate over K to a map of the form $\phi_{k,b}$, and the fixed point at infinity is mapped to a K-rational fixed point of ϕ.*

Suppose $\mathrm{Aut}(\phi) = \langle f \rangle \cong \mathbb{Z}/2\mathbb{Z}$. Then $f \in \mathrm{PGL}_2(K)$, as shown below. Let $\sigma \in \mathrm{Gal}(\bar{K}/K)$. Since ϕ is defined over K, it follows that $\phi = \phi^\sigma$. Moreover, we have

$$\phi^\sigma = (\phi^f)^\sigma = (\phi^\sigma)^{f^\sigma} = \phi^{f^\sigma}.$$

This implies that $f^\sigma \in \mathrm{Aut}(\phi)$, so $f^\sigma \in \{\mathrm{id}, f\}$. Because f^σ must have order 2, we conclude that $f = f^\sigma$. Therefore, f is also defined over K.

Now, f must permute the fixed points of ϕ. If ϕ has only one fixed point, it is evident that f must fix that point. If ϕ has three fixed points, f must interchange two of them while leaving the third fixed, since f has order 2 (recall that ϕ cannot have exactly two fixed points, as stated in Proposition 4.8).

In either case, there exists exactly one point P that is fixed by both f and ϕ. We claim that P is a K-rational point.

To see this, let $\sigma \in \mathrm{Gal}(\bar{K}/K)$. Since f is defined over K, we have

$$f(P^\sigma) = (f(P))^\sigma = P^\sigma.$$

Similarly, the same calculation applies to $\phi(P^\sigma)$, which shows that P^σ is the common fixed point of f and ϕ. Therefore, $P^\sigma = P$, implying that P is K-rational.

Again, let $\sigma \in \mathrm{Gal}(\bar{K}/K)$. Then, $f(P^\sigma) = (f(P))^\sigma = P^\sigma$ since f is defined over K. The same calculation works for $\phi(P^\sigma)$, so P^σ is the common fixed point of f and ϕ. In other words, $P^\sigma = P$.

Proposition 4.10. *[24] Let $\phi(z) = kz + b/z$, with $k \in K^*$ and $b \in K^*/(K^*)^2$, with K a field of characteristic different from 2. Then $\phi(z)$ has a K-rational point of primitive period 2 if and only if $b \equiv -(k+1)$ modulo squares.*

Proof. We begin with the dynatomic polynomial

$$\phi_2^*(x, y) = k((k+1)x^2 + by^2).$$

The roots of this polynomial correspond to the points of formal period 2 for $\phi(x,y)$. First, consider the point at infinity, $P=[1:0]$. Substituting P into the polynomial, we see that P is a root if and only if $k=-1$.

However, $P=[1:0]$ is already a fixed point of ϕ, not part of a 2-cycle. Thus, when $k=-1$, there are no 2-cycles. Furthermore, for a fixed point to have formal period 2, it must have a multiplier of -1. This condition is only satisfied when $k=-1$, confirming the absence of 2-cycles in this case.

Now, let us determine the points of period 2 for $k \neq -1$. Dehomogenizing $\phi_2^*(x,y)$ by setting $y=1$, we obtain

$$\phi_2^*(z) = k((k+1)z^2+b).$$

Solving $\phi_2^*(z)=0$, we find

$$z = \pm\sqrt{\frac{-b}{k+1}}.$$

These points are rational if and only if $-b/(k+1) \in (K^*)^2$. Since $b \neq 0$, the two roots are distinct, giving exactly two points of period 2.

$\square$

Theorem 4.11. *[24] Let $\phi(z)=kz+b/z$ with $k,b \in \mathbb{Q}^*$. Then $\phi(z)$ has no rational point of primitive period 3.*

Theorem 4.12. *[24] Let $\phi(z)=kz+b/z$ with $k \in \mathbb{Q}^*$ and $b \in \mathbb{Q}^*/(\mathbb{Q}^*)^2$.*
(a) There is a one-parameter family of such maps

$$\phi_m(z) = \frac{2mz}{m^2-1} - \frac{m}{z(m^4-1)}, \quad m \in \mathbb{Q}-\{0,\pm 1\},$$

with a rational point of primitive period 4. In this case, $\phi(z)$ has exactly four points of primitive period 4.
(b) The map $\phi(z)$ cannot have more than four points of primitive period 4.

Proof. First calculate the fourth dynatomic polynomial for ϕ.

$$\Phi_4^*(b,k,z) = \Psi_4^*(b,k,z)\Lambda_4^*(b,k,z),$$

where $\Psi_4^*(b,k,z) = k^3z^4+2bk^2z^2+2bz^2+b^2k$ and

$$\begin{aligned}\Lambda_4^*(b,k,z) = {} & z^8k^9+4bz^6k^8+bz^6k^4+2bz^6k^6+6b^2z^4k^7+4b^2z^4k^5 \\ & +3b^2z^4k^3+b^2z^4k+4b^3z^2k^6+b^3z^2k^4+2b^3z^2k^2+b^3z^2+b^4k^5.\end{aligned}$$

We will demonstrate that there is a one-parameter family of k- and b-values for which $\Psi_4^*(b,k,z)$ has four rational z-roots, and that $\Lambda_4^*(b,k,z)$ has no rational z-roots.

The polynomial $\Psi_4^*(b,k,z)$ is even in z, so we substitute $z^2 = x$, leading to the family of curves

$$k^3x^2 + kx^2 + 2bk^2x + 2bx + b^2k = 0.$$

(Again, we start with the family of curves $\Psi_4^* = 0$ and form the quotient by the automorphism $z \mapsto -z$.)

Next, we apply a change of coordinates by setting $x = bx_1$ and dividing by b^2, yielding the expression

$$k^3x_1^2 + kx_1^2 + 2k^2x_1 + 2x_1 + k = 0.$$

Then, we make another substitution $x_2 = \frac{x_1}{k}$, and multiplying by k simplifies the equation to

$$k^2x_2^2 + x_2^2 + 2k^2x_2 + 2x_2 + k^2 = 0.$$

This equation is quadratic in x_2, and the discriminant is given by

$$\Delta = 4(1+k^2).$$

The curve $\Psi_4^*(b,k,z)$ is birational to the rational curve

$$C : d^2 = (1+k^2).$$

The curve C is parametrized by $k = 2m/(m^2-1)$ and $d = (m^2+1)/(m^2-1)$. Tracing back the change of coordinates, we find that

$$x_2 = \frac{-2(1+k^2) \pm 2d}{2(1+k^2)} = -1 \pm \frac{d}{1+k^2}.$$

Therefore, let

$$x_2 = -1 + \frac{d}{1+k^2} = -\frac{2}{m^2+1},$$

thus,

$$x_1 = kx_2 = -\frac{4m}{m^4-1}, \quad bx = -\frac{4m}{m^4-1}.$$

We now define

$$b = -\frac{m}{m^4-1}.$$

This gives the rational map

$$\phi_m(z) = kz + \frac{b}{z} = \frac{2mz}{m^2-1} - \frac{m}{z(m^4-1)}.$$

Now, we recalculate the fourth dynatomic polynomial for the map $\phi_m(z)$. After performing the necessary algebraic manipulations, we obtain the fourth dynatomic polynomial:

$$\begin{aligned}\Phi_4^* = &\left(\frac{-2m^4}{(m-1)^{12}(m+1)^{12}(m^2+1)^6}\right)\\ &(m^2z+z-1)(m^2z+z+1)(m^2z-z-m)(m^2z+z-m)\\ &\Big(-512z^8m^{14} - 2048z^8m^{12} - 3072z^8m^{10} - 2048z^8m^8 - 512z^8m^6 + 16z^6m^{16}\\ &+ 112z^6m^{14} + 1104z^6m^{12} + 2864z^6m^{10} + 2864z^6m^8 + 1104z^6m^6\\ &+ 112z^6m^4 + 16z^6m^2 - 2z^4m^{16} - 16z^4m^{14} - 88z^4m^{12} - 752z^4m^{10}\\ &- 1356z^4m^8 - 752z^4m^6 - 88z^4m^4 - 16z^4m^2 - 2z^4 + z^2m^{14} - z^2m^{12}\\ &+ 292z^2m^{10} + 227z^2m^8 + 29z^2m^4 - z^2m^2 - 32m^6\Big).\end{aligned}$$

For $m \notin \{-1,0,1\}$, the rational points of period 4 are given by,

$$\frac{1}{m^2+1} \xrightarrow{\phi_m} \frac{-m}{m^2+1} \xrightarrow{\phi_m} \frac{-1}{m^2+1} \xrightarrow{\phi_m} \frac{m}{m^2+1} \xrightarrow{\phi_m} \frac{1}{m^2+1} \xrightarrow{\phi_m} \cdots$$

Note that the condition on m ensures that all four points in the cycle are distinct.

If $\phi(z) = kz + \frac{b}{z}$ were to have more than four rational points of primitive period 4, then there must be rational roots of $\Lambda_4^*(b,k,z) = 0$. Following Morton's method, we define the trace of an n-cycle in $\mathbb{C}$ for $\phi(z) = kz + \frac{b}{z}$ as the sum of the elements in the cycle. The polynomial $\tau_n(z,b,k)$ has roots that correspond to the traces of all the n-cycles.

Rational solutions to $\Lambda_4^*(b,k,z) = 0$ lead to rational solutions to $\tau_4^*(b,k,z) = 0$. However, $\tau_4^*(b,k,z) = 0$ has infinitely many rational solutions, indicating that there are infinitely many rational values of b and k such that $\phi_{k,b}$ has three Galois-stable 4-cycles (See Remark 4.14 following this proof).

Let α be a root of $\Lambda_4^*(b,k,z)$. The other roots of $\Lambda_4^*(b,k,z) = 0$ are

$$\phi(\alpha), \phi^2(\alpha), \phi^3(\alpha), -\alpha, -\phi(\alpha), -\phi^2(\alpha), -\phi^3(\alpha).$$

Let

$$t_1 = \alpha + \phi^2(\alpha) \quad t_2 = \phi(\alpha) + \phi^3(\alpha)$$
$$t_3 = -\alpha - \phi^2(\alpha) \quad t_4 = -\phi(\alpha) - \phi^3(\alpha).$$

We let $\tau_{4,2}(z,b,k) \in \mathbb{Q}(b,k)[z]$ be the polynomial with roots for generic b,k that are the t_i for $i = 1,\ldots,4$. We see that $\tau_{4,2}(z,b,k)$ must have degree 4 in z, and that the coefficients of the linear and cubic terms both vanish because the set of roots is invariant under $z \mapsto -z$. Let $\tau_{4,2}(z,b,k) = z^4 + Uz^2 + V$ and solve for U and V so that this polynomial in z vanishes identically modulo $\Lambda_4^*(z,b,k)$. We find that

$$\tau_{4,2}(z,b,k) = z^4 + \frac{4bk^4 + 4bk^2 + b}{k^5} z^2 + \frac{4b^2k^4 + 4b^2k^2 + b^2}{k^8}.$$

Since $\tau_{4,2}$ is even in z, we may substitute $x = z^2$ and consider instead the roots of

$$x^2 + \frac{4bk^4 + 4bk^2 + b}{k^5} x + \frac{4b^2k^4 + 4b^2k^2 + b^2}{k^8} = 0.$$

A change of variables $x \to bx$ and dividing by b^2 removes dependence on b:

$$x^2 + \frac{4k^4 + 4k^2 + 1}{k^5} x + \frac{4k^4 + 4k^2 + 1}{k^8} = 0.$$

This curve is already quadratic in x, and the discriminant is

$$d^2 = \frac{(2k^2 - 2k + 1)(2k^2 + 2k + 1)(1 + 2k^2)^2}{k^{10}}.$$

Therefore we may search for rational points on the genus 1 curve

$$y^2 = (2k^2 - 2k + 1)(2k^2 + 2k + 1).$$

Letting $k \to 2k/y$ and $y \to -1 + 2k^3/y^2$ and multiplying both sides by y^4/k^3 puts the curve in Weierstrass form

$$y^2 = k^3 - 16k,$$

which has the minimal model

$$y^2 = k^3 - k.$$

This corresponds to curve 32a2 in Cremona's tables [12]. It has rank 0 and a torsion subgroup of order 4. therefore the curve in equation must have exactly four rational points. It has a double point at infinity (since it is of the form $y^2 =$ quartic) and the obvious points $(k, y) = (0, \pm 1)$. They have found all of the rational points, so the only possible finite value for k is 0, which does not yield a valid rational map of degree 2. $\square$

Example 4.13. *Let $m = 2$. By computing k and b from equations, we obtain the rational map*

$$\phi_2(z) = \frac{4z}{3} - \frac{2}{15z}.$$

This map has the following 4-cycle:

$$\frac{1}{5} \xrightarrow{\phi_2} -\frac{2}{5} \xrightarrow{\phi_2} -\frac{1}{5} \xrightarrow{\phi_2} \frac{2}{5} \xrightarrow{\phi_2} \frac{1}{5} \xrightarrow{\phi_2} \cdots.$$

Remark 4.14. *Let $\phi_{m_1} = k_1 z + b_1/z$ and $\phi_{m_2} = k_2 z + b_2/z$ with $k_1, k_2, b_1,$ and b_2 as given in equation (4). By Lemma 4.6, these rational maps are linearly conjugate over $\mathbb{Q}$ if and only if $k_1 = k_2$ and $b_1/b_2 \in (\mathbb{Q}^*)^2$.*

If $m_1 \neq m_2$, then $k_1 = k_2$ if and only if $m_1 = -1/m_2$. We can now compute the relevant b_1 and b_2 in this case:

$$b_1 = -\frac{m_1}{m_1^4 - 1}, \quad b_2 = -\frac{m_2}{m_2^4 - 1} = -\frac{(-1/m_1)}{(-1/m_1)^4 - 1} = -\frac{m_1^3}{m_1^4 - 1} = m_1^2 b_1.$$

We see, therefore, that ϕ_{m_1} is linearly conjugate over K to ϕ_{m_2} if and only if $m_1 = m_2$ or $m_1 = -1/m_2$.

Remark 4.15. *For a fixed b, the curve $C_b : \Lambda_4^*(b, k, z) = 0$ has an automorphism $(b, k, z) \mapsto (b, k, \phi(z))$. The curve $\tau_{4,2}(b, k, z) = 0$ represents $C_b/\sim$, the quotient of C_b by this automorphism. Each curve C_b also has an automorphism $(b, k, z) \mapsto (b, k, \phi^2(z))$, and the quotient of C_b by this second automorphism is $\tau_{4,2}(b, k, z) = 0$.*

Examining $\tau_4(b, k, z)$, we can show that for any k-value, we may choose b so that $\phi_{k,b}$ has three Galois-stable 4-cycles, but no rational points of primitive period 4.

If α is a root of Ψ_4^*, then so is $-\alpha$. In fact, Ψ_4^* corresponds to a single 4-cycle with trace 0, which clearly results in a Galois-stable 4-cycle. For a detailed description of the polynomials Ψ_n^* and the justification of these claims, see Chapter 3 of [23].

To compute τ_4, we need the traces of the cycles given by $\Lambda_4^*(b, k, z)$. As a result, we expect τ_4 to be a quadratic polynomial. Additionally, we can observe that the sum of the two traces will vanish: if α belongs to one 4-cycle, then $-\alpha$ must belong to the other. Therefore, the irreducible polynomial for the traces will have the form

$$\tau_4(z,b,k) = z^2 + V,$$

for some $V \in \mathbb{Q}(b,k)$. Using computational tools such as Mathematica, we can compute

$$\tau_4(z,b,k) = z^2 + \frac{b\left(4k^4+4k^3+4k^2+2k+1\right)}{k^5}.$$

Since τ_4 is even in z, let $x = z^2$ and consider instead the polynomial

$$x + \frac{b\left(4k^4+4k^3+4k^2+2k+1\right)}{k^5} = 0.$$

After a change of variables $x \to bx/k^5$ and then multiplying by k^5/b, we have

$$x + \left(4k^4+4k^3+4k^2+2k+1\right) = 0.$$

Any rational k yields a rational x. We may then choose b so that $\frac{k^5 x}{b}$ is a square. In fact, it is sufficient to let $b = -\left(4k^4+4k^3+4k^2+2k+1\right)$. These choices guarantee that $\phi_{k,b}$ will have three Galois-stable 4-cycles.

However, if this relationship between k and b, along with the one in the equations, is satisfied by rational numbers, we would have a rational root of

$$m^{10}+4m^9+13m^8+23m^7+50m^6+3m^5+50m^4-27m^3+13m^2-3m+1 = 0.$$

This is impossible; thus, if $\phi_{k,b}$ has three Galois-stable 4-cycles, it does not have any rational points of primitive period 4.

Torsion points on elliptic curves correspond precisely to preperiodic points under the multiplication-by-2 map on the curve. Points on the elliptic curve are mapped to $\mathbb{P}^1$ via their x-coordinates, and this multiplication-by-2 map induces a degree-4 rational map $\phi : \mathbb{P}^1 \to \mathbb{P}^1$, with the x-coordinates of the torsion points mapping to the preperiodic points of ϕ.

Example 4.16. *Let* $f(x) = \frac{330x^2-187x-143}{330x^2+1217x+429} \in \mathbb{Q}[x]$*. We have the following orbit and graph for* $f(x)$*,*

$$f(\infty) = 1, \quad f(1) = 0, \quad f(0) = -\frac{1}{3}, \quad f\left(-\frac{1}{3}\right) = -\frac{11}{15}$$

$$f\left(-\frac{11}{15}\right) = \frac{-3}{5}, \quad f\left(-\frac{3}{5}\right) = \frac{-55}{114}, \quad f\left(-\frac{55}{114}\right) = \frac{-13}{44}, \quad f\left(-\frac{13}{44}\right) = \frac{-3}{5}$$

$$\begin{tikzcd}
\infty \arrow[r, bend left] & 1 \arrow[r, bend left] & 0 \arrow[r, bend left] & -\frac{1}{3} \arrow[r, bend left] & -\frac{11}{15} \arrow[r, bend left] & -\frac{3}{5} \arrow[r, bend left] & -\frac{55}{114} \arrow[r, bend left] & -\frac{13}{44} \arrow[ll, bend left]
\end{tikzcd}$$

In this study, our goal is to determine an upper bound for periodic points of the rational function $f(x)$ with $\mathbb{Q}$-coefficients. Specifically, we focus on the following conjecture;

Conjecture 4.2.1. *[3] For a rational function* $f(x) \in \mathbb{Q}(x)$ *with deg(f)* $= 2$*, and any* $x \in \mathbb{P}^1(\mathbb{Q})$*, the followings hold;*

1. $\#PrePer(f, \mathbb{Q}) \leq 14$.
2. *If* x *is preperiodic, then* $\#O_f(x) \leq 8$.
3. *If* x *is periodic, then* $\#O_f(x) \leq 7$.

Benedetto, Chen, Hyde, Yordanka Kovacheva, and White give a list of some degree 2 rational polynomials with preperiodic points 8 in [3]. Also they found many degree-2 polynomails with 14 $\mathbb{Q}$-rational preperiodic points. One known example of rational polynomials with exact period 7 is:

$$f(x) = \frac{4655z^2 - 4826z + 171}{4655z^2 - 8071z + 798}$$

with orbit,

$\infty \to 1 \to 0 \to \frac{3}{14} \to \frac{19}{21} \to \frac{1}{7} \to \frac{57}{35} \to \infty$ [3].

In fact, when K is any quadratic field, it becomes easier to generalize rational functions that produce preperiodic points with a specified orbit size. For example, an algorithm presented in [15] computes all the preperiodic points of a given quadratic polynomial defined over a number field. This algorithm can be applied to quadratic polynomials over any number field. Using this approach, the authors were able to compute the set $PrePer(f, K)$ for approximately 250,000 pairs (K, f), where K is a quadratic field and f is a quadratic polynomial with coefficients in K. As a key result of [15], they identified a total of 46 non-isomorphic graphs (listed in the Appendix), with the maximum number of preperiodic points for the considered polynomials being 15.

Theorem 4.17. *[15] Suppose that there exists a constant N such that $\#PrePer(f,K) \leq N$ for every quadratic number field K and quadratic polynomial f with coefficients in K. Then $N \geq 15$. Moreover, there are at least 46 directed graphs that arise from the set $PrePer(f,K)$ for such a field K and polynomial f.*

Theorem 4.18. *[15] For each of the graphs of type; 8(1,1)a, 8(1,1)b, 8(2)a, 8(2)b, 8(4), 10(2,1,1)a, 10(2,1,1)b there exist infinitely many pairs (K,c) consisting of an imaginary quadratic field and an element $c \in K$ for which $G(f_c,K)$ contains a graph of this type. The same holds for the graphs 10(3,1,1) and 10(3,2), but these occur only over real quadratic fields.*

Showing the existence of infinitely many pairs for which $G(f_c,K)$ not only contains a graph of a given type but is, in fact, itself of this type is a more difficult problem. Such a result was achieved in the article [16] for several of the graphs $G(f_c,\mathbb{Q})$ with $c \in \mathbb{Q}$.

In most cases, the difficulties do not appear excessively complex, but the methods required to analyze these graphs can differ substantially from those previously employed. Moreover, for some of the graphs, only partial results have been obtained concerning the quadratic points on the parameterizing curve. The primary obstacle lies in the difficulty of determining all rational points on certain hyperelliptic curves.

An open question remains regarding functions with period 5. Current research strongly suggests that no quadratic polynomial f defined over a quadratic field K possesses a K-rational point of period 5. Such a polynomial did not appear in our computations, or identified in related searches.

Through their extensive search for periodic points with large periods over quadratic fields, Hutz and Ingram [20] provide compelling evidence supporting the conjecture that 6 is the maximum possible cycle length in this context. Furthermore, they identified precisely one instance of a 6-cycle over a quadratic field, which aligns with the results of our computations and matches the example previously discovered by Flynn, Poonen, and Schaefer. While proving that 6 is the absolute maximum cycle length may be an ambitious objective, our focus is on studying and classifying all instances of 6-cycles over quadratic fields.

In [11], Canci and Solomon Vishkautsan investigate rational maps with a critical periodic point of period 2. They provide a comprehensive classification of the possible graphs of rational preperiodic points for degree-2 endomorphisms of the projective line, defined over the rationals, with a rational periodic critical point of period 2. This classification is made under the assumption that these maps have no periodic points of period 7 or greater. Their work extends the earlier results of Poonen on quadratic polynomials. Specifically, they identify exactly 13 possible graphs and demonstrate that such maps can have at most 9 rational preperiodic points.

We define $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ as an endomorphism over a field K. A periodic point of ϕ corresponds to $P \in \mathbb{P}$ such that $\phi^n(P) = P$ for some $n \geq 1$, and the minimal such n is called the period of P.

A point $P \in \mathbb{P}^1$ is called preperiodic if some iterate of P is periodic, i.e. there exists an $m \geq 0$ such that $\phi^m(P)$ is periodic. We denote by $PrePer(\phi, K)$ the set of preperiodic points for ϕ in $\mathbb{P}^1(K)$ (similarly $Per(\phi, K)$ is the set of K-periodic points and $Per_n(\phi, K)$ is the set of K-periodic points of period n).
For a quadratic map defined over $\mathbb{Q}$ with a $\mathbb{Q}$-rational periodic critical point of period 2, we have a complete classification of possible preperiodicity graphs, assuming a conjecture similar to that of Flynn, Poonen and Schaefer. [18]

Conjecture 4.2.2. *[11] Let ϕ be a quadratic map defined over $\mathbb{Q}$ with a $\mathbb{Q}$-rational periodic critical point of period 2, then ϕ has no $\mathbb{Q}$ -periodic point of period greater or equal to 3.*

Theorem 4.19. *Assuming the Conjecture, there are exactly 13 possible preperiodicity graphs for quadratic maps defined over $\mathbb{Q}$ with a $\mathbb{Q}$-rational periodic critical point of period 2 (see in [11]). Moreover, the number of preperiodic points of such maps is at most 9 (as in the quadratic polynomial case).*

Next theorem provide an evidence for the previous Conjecture;

Theorem 4.20. *Let ϕ be a quadratic map defined over $\mathbb{Q}$ with a Q-rational periodic critical point of period 2, then ϕ has no $\mathbb{Q}$-periodic point of period 3, 4, 5 or 6.*

An endomorphism of $\mathbb{P}^1$ is called *post-critically finite* (PCF) if all of its critical points are preperiodic. Lukas, Manes, and Yap [22] provided a complete classification, over $\mathbb{Q}$, of the possible graphs realized by PCF quadratic maps defined over $\mathbb{Q}$. Here, we present a comprehensive list of all preperiodicity graphs associated with PCF quadratic maps over $\mathbb{Q}$ that have a $\mathbb{Q}$-rational periodic critical point of period 2.

Two rational functions $\phi, \psi : \mathbb{P}^1 \to \mathbb{P}^1$ are said to be linearly conjugate if there exists an $f \in \mathrm{PGL}_2$, acting as a projective automorphism of $\mathbb{P}^1$, such that $\phi = \psi^f = f^{-1}\phi f$.

Conjugate rational functions exhibit the same dynamical behavior. Specifically, if P is a periodic point for ϕ with period n, then $f^{-1}(P)$ is a periodic point for ψ with the same period n (the same applies to preperiodic points). Moreover, when ϕ, ψ, and f are all defined over the same number field K, it follows that $G_\phi = G_\psi$. Thus, in the classification of realizable graphs, we focus on the conjugacy classes of quadratic maps rather than on individual maps.

They establish Theorem 1 by demonstrating that the graphs listed as 5.5 in [11] are inadmissible. This result is sufficient to conclude that no graph other than those presented in Tables 5.1 and 5.2 of [11] can be realized by a quadratic map defined over $\mathbb{Q}$ with a $\mathbb{Q}$-rational periodic critical point of period 2.

The approach involves constructing an affine curve $C(G)$ for each graph G in Table 5.5, where the points of $C(G)$ parametrize $\mathrm{PGL}_2(\mathbb{Q})$-conjugacy classes of quadratic maps with a $\mathbb{Q}$-rational periodic critical point of period 2 that admit G. They then demonstrate that these curves have no $\mathbb{Q}$-rational points.

To generate the graphs in the tables, they employed a recursive algorithm described in Lemma 3.2 of [11], producing a list of approximately eighty potentially realizable graphs with up to 14 vertices. Initially, they assumed the conjecture of Benedetto et al. [3], which states that a quadratic map defined over $\mathbb{Q}$ has at most 14 $\mathbb{Q}$-preperiodic points. However, this conjecture is not used in the proofs.

Question: Except for graph R2P4 in Table 5.2 which has a unique class realizing it (see Proposition 3.7 in [11]), are there infinitely many conjugacy classes realizing each of the graphs in Table 5.2?

Assuming Conjecture 4.2.2, it is evident that the graphs $R2P5$, $R2P6$, and $R2P7$ in Table 5.2 are realized by infinitely many classes. This follows from the fact that the associated curves $C(G)$ have infinitely many rational points, as they are genus 0 curves with a rational point (see Corollary 3.11 in [11]). Moreover, these graphs are maximal with respect to the isomorphic subgraph relation among the realizable graphs (refer to the Hasse diagram in Proposition 4.2 of [11]).

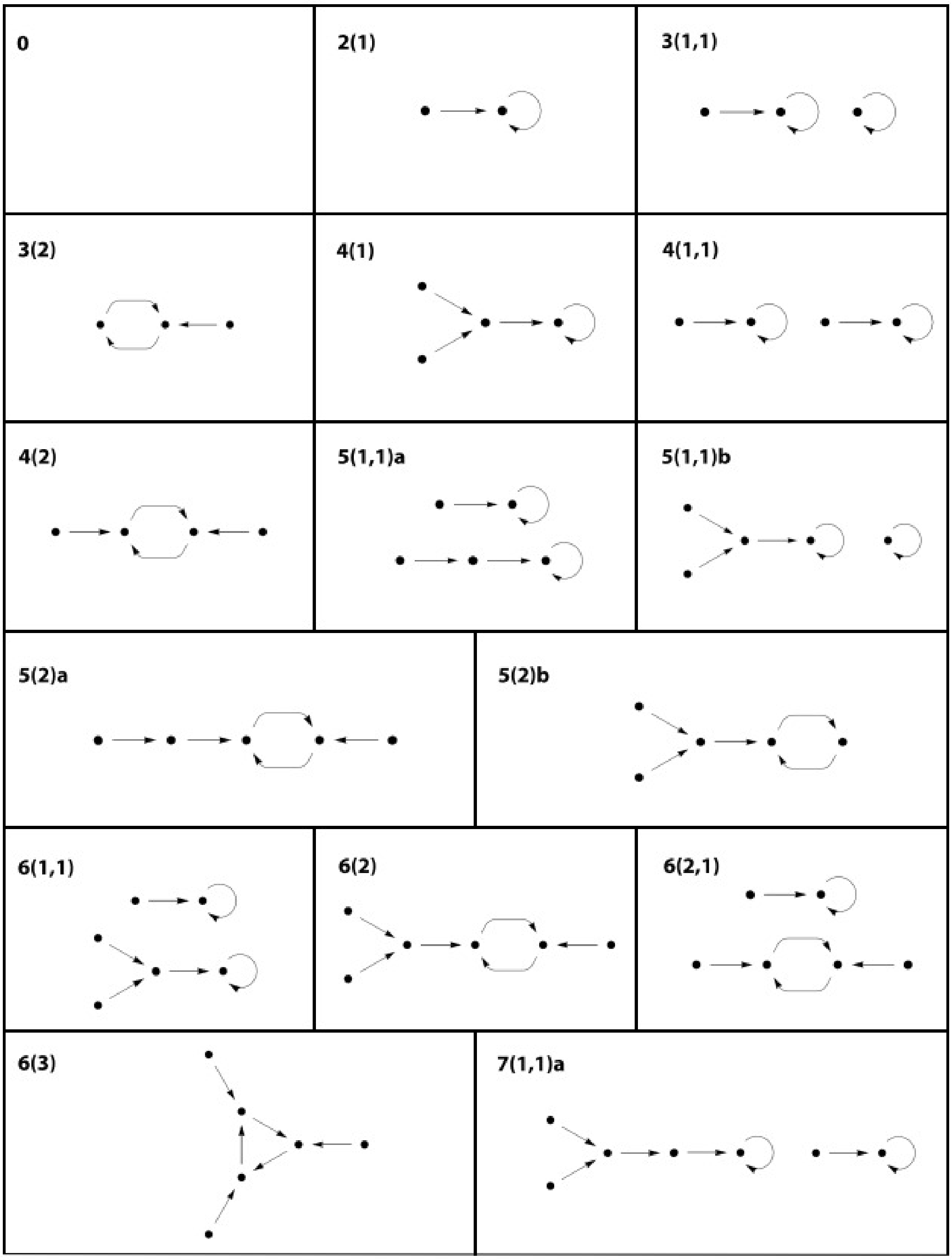
However, it may be possible to demonstrate that all graphs, except for $R2P4$, are realizable by infinitely many classes without relying on this conjecture. This could be achieved by adapting the method of Faber in [16], who proved a similar result for quadratic polynomials.

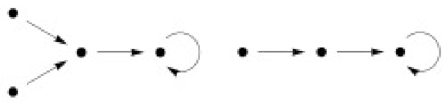
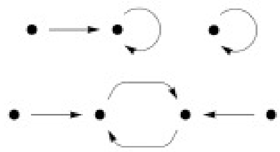
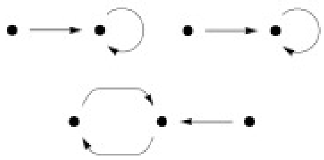
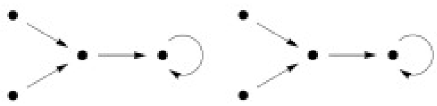
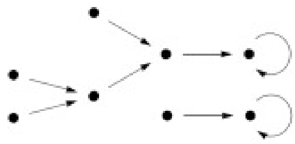
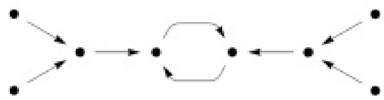
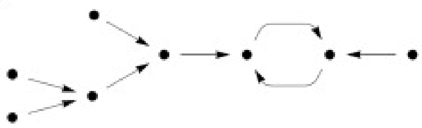
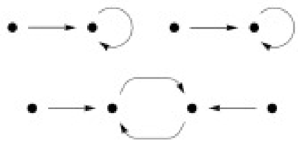
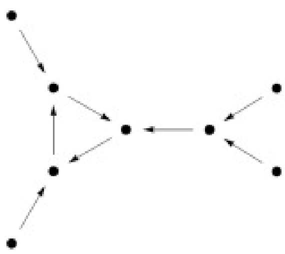
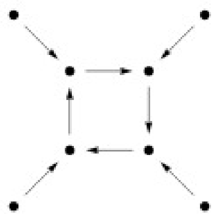
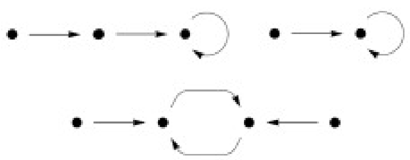
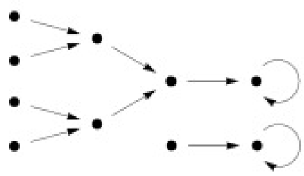
Arithmetic dynamics is a relatively new and rapidly evolving field, characterized by its significant potential for further exploration and cross-disciplinary research. Its foundations lie at the intersection of number theory and dynamical systems, offering a fertile ground for uncovering deep mathematical structures and connections.

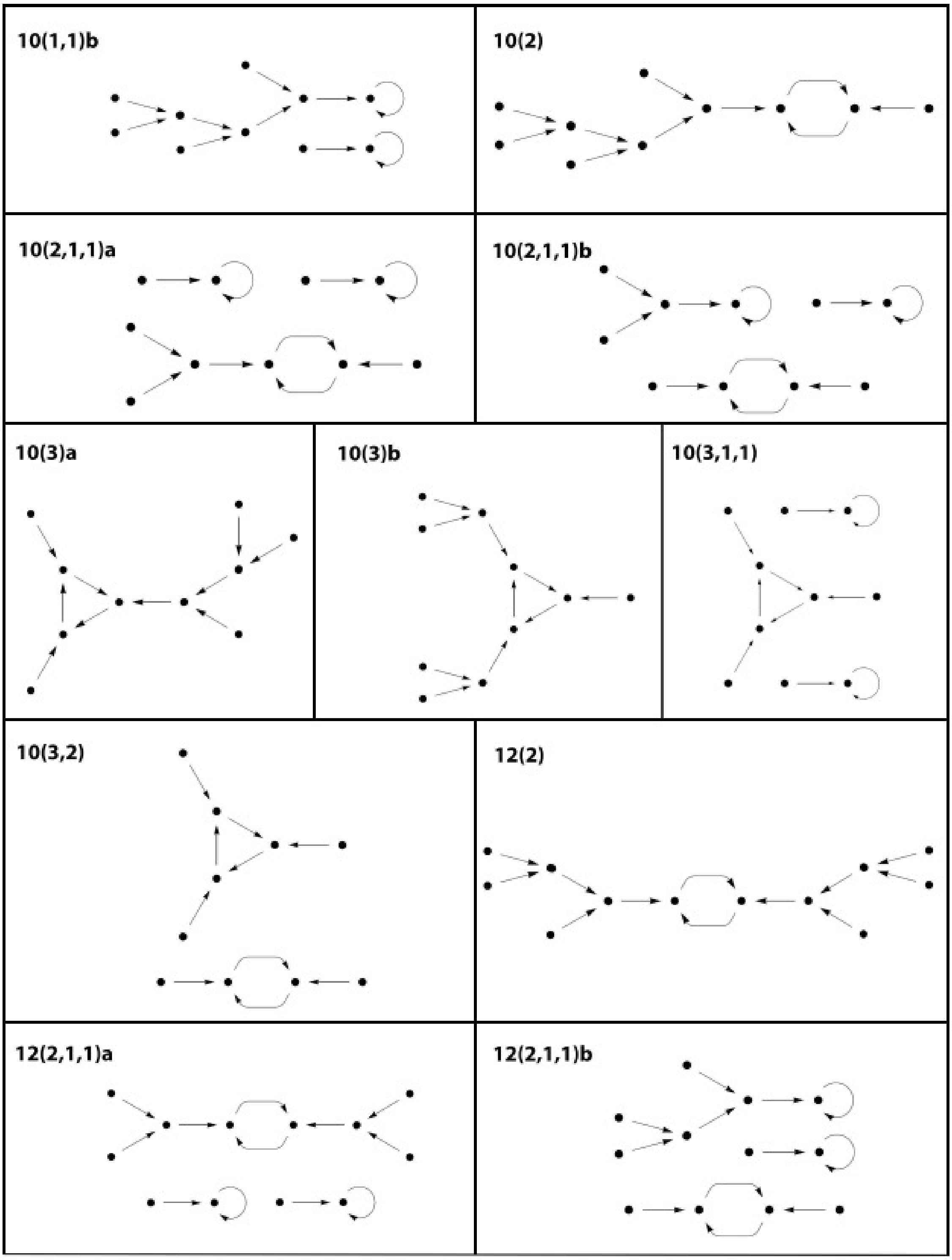
Arithmetic dynamics, being a relatively young and evolving field, offers a vast array of open questions that remain to be explored. Many of these questions are fundamental to advancing our understanding of the intricate connections between number theory and dynamical systems. The field's interdisciplinary nature and its foundational aspects make it a fertile ground for uncovering new mathematical insights and addressing long-standing problems.

For those interested in delving deeper into the subject, [4] provides a comprehensive overview, including a detailed discussion of the key challenges and unresolved problems currently shaping the field. Additionally, it highlights several promising directions for future research, offering valuable guidance for researchers aiming to contribute to this rapidly growing area of study.

APPENDIX



7(1,1)b 	**7(2,1,1)a** 
7(2,1,1)b 	**8(1,1)a** 
8(1,1)b 	**8(2)a** 
8(2)b 	**8(2,1,1)** 
8(3) 	**8(4)** 
9(2,1,1) 	**10(1,1)a** 


10(1,1)b
10(2)
10(2,1,1)a
10(2,1,1)b
10(3)a
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10(3,2)
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12(2,1,1)a
12(2,1,1)b

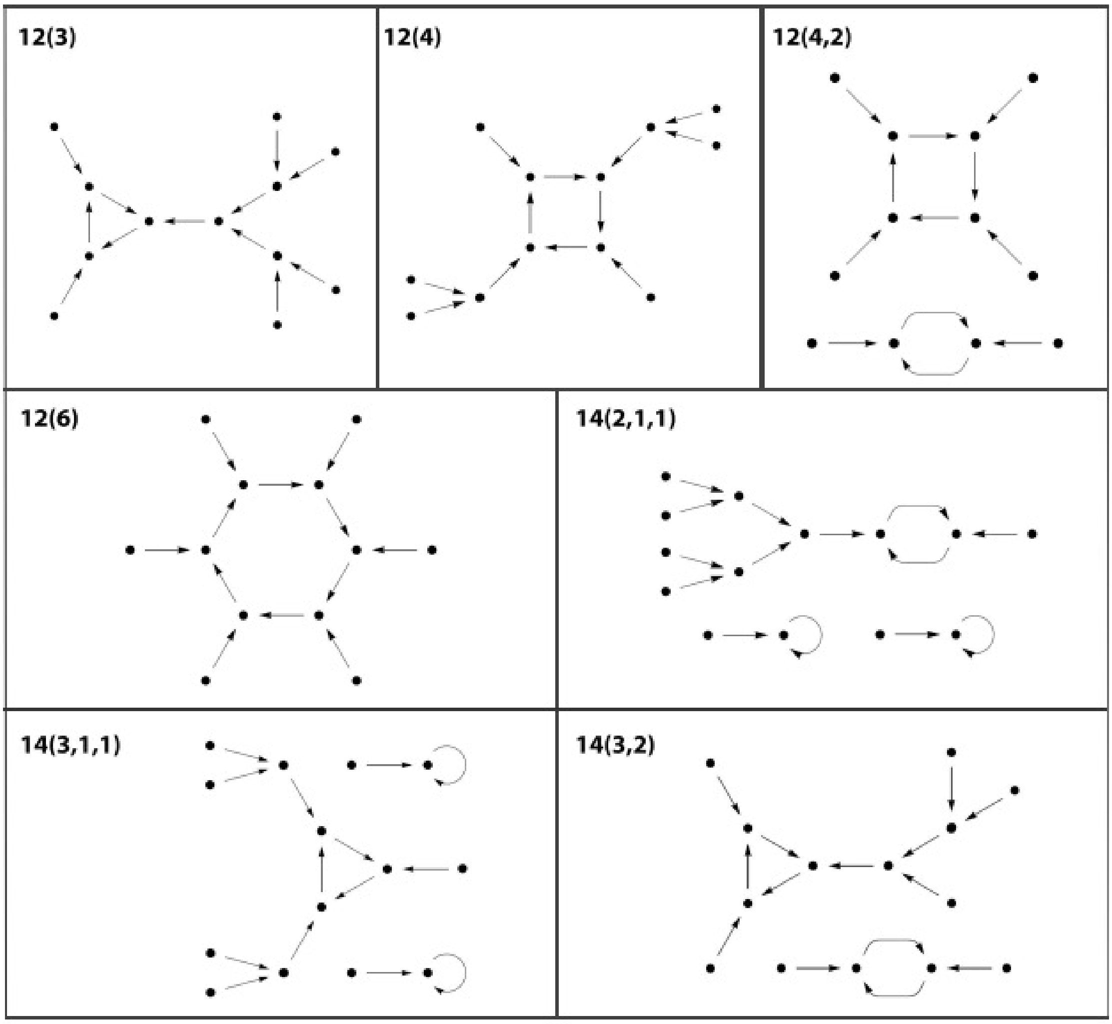

Figure 4.2 Total of 46 non-isomorphic graphs over a quadratic field K

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