Algebraic and Combinatorial Properties of t-spread Strongly Stable Ideals

by

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Abstract

In this thesis, we study t-spread strongly stable monomial ideals. It is proved that for an ideal to be t-spread strongly stable, it is sufficient for the definition criterion to be satisfied only on its minimal monomial generating set. The generators, height, Cohen-Macaulayness, and minimal free resolution of some special classes of t-spread strongly stable monomial ideals, namely, t-spread Veronese ideals and t-spread principal Borel ideals and their Alexander dual are studied. We also study the Rees algebras of t-spread principal Borel ideals, and it is shown that they have the so-called ℓ-exchange property. Consequently, the Rees algebra of a t-spread principal Borel ideal is Koszul.

t-YAYILMIŞ FAZLASIYLA KARARLI İDEALLERİN CEBİRSEL VE KOMBİNATORYAL ÖZELLİKLERİ

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Anahtar Kelimeler: t-yayılmış monomsal idealler, fazlasıyla kararlı idealler, Borel idealler, t-yayılmış Veronese idealler, t-yayılmış ideallerin Rees cebirleri

Özet

Bu tezde, t-yayılmış fazlasıyla kararlı monomsal idealleri çalıştık. Bir idealin t-yayılmış fazlasıyla kararlı olması için, tanım kriterinin sadece minimal monomsal üreteç kümesinde sağlanmasının yeterli olduğu kanıtlanmıştır. Bazı özel t-yayılmış fazlasıyla kararlı monomsal ideal sınıflarının, yani t-yayılmış Veronese ideallerin ve t-yayılmış asal Borel ideallerin ve bunların Alexander çifteşlerinin üreteçleri, yüksekliği, Cohen-Macaulaylığı ve minimal serbest çözünümleri incelenmiştir. Ayrıca t-yayılmış esas Borel ideallerin Rees cebirlerini inceledik ve bunların ℓ-takas özelliğine sahip olduğunu gösterdik. Sonuç olarak, bir t-yayılmış esas Borel idealin Rees cebiri Koszul'dur.

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Contents

Introduction

Monomial ideals are one of the fundamental objects in commutative algebra, algebraic combinatorics, and algebraic geometry, as they provide a rich framework for exploring the interplay between algebraic structures and combinatorial objects within polynomial rings. The foundational work on monomial ideals and their combinatorial aspects started to gain attention during the 1970s due to the work of Richard Stanley, which led to a new area of research within commutative algebra known as combinatorial commutative algebra. In this thesis, we studied a special type of monomial ideal, t-spread strongly stable monomial ideals, both algebraically and combinatorially.

The first chapter of this thesis is reserved for introducing the basic concepts that we will use throughout the thesis. All rings considered in this chapter are commutative Noetherian with unity. In the first chapter, we include the definitions and some fundamental facts and theorems related to the primary decomposition of ideals, graded rings and modules, free and projective resolutions, Betti numbers, etc. Ext and Tor functors are often used to define and calculate many algebraic and homological invariants in commutative algebra. For example, if M is a finitely generated graded module over $S = \mathbb{K}[x_1, \dots, x_n]$, then the Betti numbers of M are given by $\beta_i(M) = \dim_{\mathbb{K}}(\operatorname{Tor}_i^S(M, \mathbb{K}))$. We have also included concepts from dimension theory. The Stanley-Reisner correspondence is one of the most important tools in combinatorial commutative algebra. For a given simplicial complex Δ on the vertex set $[n]$, we define the Stanley-Reisner ideal I_Δ as a monomial ideal generated by the monomials obtained from the minimal non-faces of Δ. Additionally, we define the facet ideal $I(\Delta)$ as the ideal generated by the monomials obtained from the maximal faces of Δ. Using the Alexander duality of Δ, we provide a surprising formula for the primary decomposition of I_Δ. Finally, we discuss linear resolutions and ideals with linear quotients, proving that a graded ideal of the polynomial ring $\mathbb{K}[x_1, \dots, x_n]$ has a linear resolution if it has linear quotients.

In the second chapter of the thesis, we investigate the basics of t-spread strongly stable ideals. Let t be a fixed non-negative integer. A monomial $x_{i_1} \cdots x_{i_d}$ with $i_1 \leq \cdots \leq i_d$ in the polynomial ring $S = \mathbb{K}[x_1, \dots, x_n]$ is called a *t-spread* monomial if $i_j - i_{j-1} \geq t$ for all $j = 2, \dots, d$. An ideal $I \subset S$ is called a *t-spread ideal* if it is generated by t-spread monomials. Furthermore, a t-spread ideal I is called *t-spread strongly stable* if for all t-spread monomials $u \in I$, and for all $j \in \operatorname{supp}(u)$ and all $i < j$ such that $x_i(u/x_j)$ is a t-spread monomial, it follows that $x_i(u/x_j) \in I$. We first show that t-spread strongly stable ideals have linear quotients, and therefore they are componentwise linear. We also discuss the graded Betti numbers of these ideals.

The third chapter of the thesis is devoted to t-spread Borel ideals and their powers. The smallest t-spread strongly stable ideal containing a given set of t-spread monomials $u_1, \ldots, u_m$ is called a t-spread Borel ideal and is denoted by $B_t(u_1, \ldots, u_m)$. In particular, the ideal $B_t(u)$ is called a *t-spread principal Borel ideal*, and u is called the Borel generator of $B_t(u)$. The t-spread Borel ideal becomes a $t+1$-spread Borel ideal under the σ operator. In this chapter, we also discuss t-spread Veronese ideals in detail. A t-spread Veronese ideal is a special type of t-spread principal Borel ideal. They are monomial ideals of S generated by all t-spread monomials of degree d for some $d \geq 1$. To show that a t-spread Veronese ideal is a t-spread principal Borel ideal, we demonstrate that a given t-spread Veronese ideal of degree d is the smallest t-spread strongly stable monomial ideal containing the monomial $x_{n-(d-1)t}x_{n-(d-2)t}\cdots x_n$. In [11, Theorem 2.3], Ene, Herzog, and Qureshi state the height, generators of the Alexander dual, and Betti numbers of the quotient ring of a given t-spread Veronese ideal. The proof of this theorem involves finding a suitable simplicial complex that admits the t-spread Veronese ideal as its Stanley-Reisner ideal. We recall this proof in detail.

In the last chapter of this thesis, we mainly review the work of Andrei, Ene and Lajmari in [1]. We study the Rees algebras of t-spread principal Borel ideals. For given degree d monomials u, v in $\mathbb{K}[x_1, \ldots, x_n]$, we write $uv = x_{i_1}x_{i_2}\cdots x_{i_{2d}}$ with $i_1 \leq \ldots \leq i_{2d}$. The sorting operator is defined in [9] by $\mathrm{sort}(u, v) = (u', v')$ where, $u' = x_{i_1}x_{i_3}\cdots x_{i_{2d-1}}$ and $v' = x_{i_2}x_{i_4}\cdots x_{i_{2d}}$. Any subset $\mathcal{S}$ of all degree d monomials in the polynomial ring is called sortable if $\mathrm{sort}(\mathcal{S} \times \mathcal{S}) \subseteq \mathcal{S} \times \mathcal{S}$ and a pair of degree d monomials (u, v) is called sorted if $\mathrm{sort}(u, v) = (u', v')$. The minimial monomial generating set of t-spread principal Borel ideals are sortable and possess the so-called ℓ-exchange property with respect to the sorting order on monomials, as shown in [1]. The authors in [1] also prove that all powers of $B_t(u)$ have linear quotients and therefore have a linear resolution. In particular, the Rees algebra of a t-spread principal Borel ideal $\mathcal{R}(B_t(u))$ is Koszul. After studying the depth and projective dimension of the t-spread principal Borel ideals, we also give the Krull dimension of the algebra generated by the minimal monomial generating set of a t-spread principal Borel ideal. Finally, we conclude with some remarks on t-spread principal Borel ideals.

List of symbols

$\emptyset$	:	emptyset
$\mathbb{N}$	:	$\{0, 1, 2, \ldots\}$ set of natural numbers
$\mathbb{Z}$	:	$\{\ldots, -1, 0, 1, \ldots\}$ set of integers
$\mathbb{Z}_+^n$	:	$\{\mathbf{a} = (a_1, \ldots, a_n) : a_i \in \mathbb{N}\}$
$[n]$	:	$\{1, \ldots, n\} \subset \mathbb{N}$
$\mathscr{P}(X)$	:	powerset of X
$\mathbb{K}$	:	a field
Δ	:	a simplicial complex
$\Delta(i)$	:	i-th skeleton of Δ
I_Δ	:	Stanley-Reisner ideal of Δ
$I(\Delta)$	:	facet ideal of Δ
$\mathcal{F}(\Delta)$	:	maximal faces (facets) of Δ
$\mathcal{N}(\Delta)$	:	minimal non-faces of Δ
$\Delta^\vee$	:	Alexander dual of Δ
$I^\vee$	:	$I_{\Delta^\vee}$
$\mathrm{Ann}(X)$	:	annihilator of X
$\mathrm{Ass}(M)$	:	associated primes of M
$\mathbb{K}[\Delta]$	:	face ring of Δ
$I : J$	:	colon ideal with respect to the ideals I and J
$\mathrm{Min}(I)$	:	minimal primes of I
$\beta_{i,j}$	:	i-th Betti number of the j-th graded component
β_i	:	i-th total Betti number
$\mathrm{reg}(M)$	:	Castelnuovo-Mumford regularity of M
$\mathrm{depth}_I(M)$	:	I-depth of M
$\mathrm{heigth}(I)$	:	height of I

$\dim(A)$	:	Krull dimension of the ring A
$\dim_A(M)$	:	Krull dimension of the A-module M
$\operatorname{projdim}_A(M)$	:	projective dimension of the A-module M
S	:	the polynomial ring $\mathbb{K}[x_1, \dots, x_n]$
$\operatorname{Mon}(S)$	:	monomials of S
$\operatorname{Mon}(S, t)$	:	t-spread monomials of S
$G(I)$	:	minimal monomial set of generators of the monomial ideal I
$\operatorname{supp}(f)$	:	support of the polynomial f
$\mathcal{I}_{n,d,t}$	:	t-spread Veronese ideal of degree d monomials
$B_t(\mathcal{M})$	:	t-spread Borel ideal of $\mathcal{M}$
$\mathcal{R}(I)$	:	Rees algebra of I
$\mathfrak{m}$	:	graded maximal ideal of $S = \mathbb{K}[x_1, \dots, x_n]$
$\operatorname{in}_<(f)$	:	initial monomial of the polynomial f
$\operatorname{in}_<(I)$	:	initial ideal of I

Chapter 1

Algebraic and combinatorial preliminaries

In this chapter, we will recall some basic notions and results from commutative algebra that will be used in the subsequent sections. Throughout this work all rings are considered to be commutative with unity.

1.1 Basics of algebra

Let $G \neq \emptyset$ be a set, $* : G \times G \longrightarrow G$ be a binary operation, if

- $g * (h * k) = (g * h) * h$ for all $g, h, k \in G$
- there exists an $e \in G$ such that, $g * e = e * g = g$ for all $g \in G$
- for all $g \in G$, there exists a $g' \in G$ such that $g * g' = g'g = e$

then, $(G, *)$ is called a *group*. For a group, instead of saying $(G, *)$ is a group, we call G is a group for the sake of simplicity. If a group G with the operation $*$ satisfies,

- $g * h = h * g$ for all $g, h \in G$,

then G is called an *abelian group*. Abelian groups considered as additive groups, so the group operation of an abelian group is denoted by $+$ unless otherwise stated. It is well known that, the identity element of a groups is unique, thus we denote it by 1_G or 1 and if G is an abelian group, we denote it by 0_G or simply 0. Moreover, for each element g of a given group G, there exist a unique inverse of g then we denote it by g^{-1}. In particular, if G is abelian, we denote the inverse of g by $-g$ for each $g \in G$. Let $(G, *)$ and $(H, \diamond)$ be groups, a map $\varphi : G \longrightarrow H$ is called a *group homomorphism* if $\varphi(g_1 * g_2) = \varphi(g_1) \diamond \varphi(g_2)$ for all $g_1, g_2 \in G$. If $H = G$ with $\diamond = *$ then, φ is called an *endomorphism* of G. We denote all endomorphisims of a group G by $\mathrm{End}(G)$ and all endomorphisims of an abelian group G by $\mathrm{End}_{\mathbb{Z}}(G)$.

Let $(G, *)$ be a group and $\emptyset \neq H \subseteq G$. If H is a group with the restriction of $*$ on H, H is called a *subgroup* of G. If $gH = Hg$ for all $g \in G$, H is called a *normal subgroup* of G, where $gH = \{gh : h \in H\}$ and $Hg = \{hg : h \in H\}$. Each subgroup of an abelian group is normal. Let G be an abelian group and H is a subgroup of G, then the quotient group $G/H = \{g + H : g \in G\}$ is an abelian group with $(g_1 + H) + (g_2 + H) = (g_1 + g_2) + H$ for all $g_1, g_2 \in G$.

Let $R \neq \emptyset$ be a set. Let $+ : R \times R \longrightarrow R$ and $\cdot : R \times R \longrightarrow R$ be two binary operations on R. We call R is a *ring* with the opreation $+$ and $\cdot$ if

- $(R, +)$ is an abelian group
- $r \cdot (s \cdot t) = (r \cdot s) \cdot t$ for all $r, s, t \in R$
- $r \cdot (s + t) = r \cdot s + r \cdot t$ and $(r + s) \cdot t = r \cdot t + s \cdot t$ for all $r, s, t \in R$.

The ring R is called *commutative ring* if

- $rs = sr$ for all $r, s \in R$,

and R is called *ring with unity* if

- there exists a $u \in R$ such that $ur = ru = r$ for all $r \in R$.

Similarly, R is called *commutative ring with unity* if the last two conditions are satisfied. A ring with unity R is called *division ring* if

- for all $r \in R \backslash \{0\}$ there exists an $s \in R \backslash \{0\}$ such that, $rs = sr = u$.

It is well known that, if a ring R has a unity, then it is unique. Thus, we denote the unity of R as 1_R or simply 1. A field is a commutative ring with unity and a division ring with $1_R \neq 0_R$. In this thesis, if a ring R is a field, we prefer to write $\mathbb{K}$ instead of R. A *left ideal* $I \neq \emptyset$ is a subset of the ring R with $I - I \subseteq I$ and $RI \subseteq I$, similarly A *right ideal* $I \neq \emptyset$ is a subset of the ring R with $I - I \subseteq I$ and $IR \subseteq I$. If I is both left and right ideal, then I is called *ideal.* If the ring is commutative, there is no difference between the left ideal and the right ideal. Let R be a commutative ring and $I \subseteq R$ be an ideal, $R/I = \{r + I : r \in R\}$ is a commutative ring with the operations $(r+I)+(s+I)=(r+s)+I$ and $(r+I)(s+I)=(rs)+I$. Let R be a commutative ring $\emptyset \neq X \subset R$ be a subset, then ideal generated by X is $(X)=\{\sum_{i=1}^{n} r_i x_i : r_i \in R,\ x_i \in X,\ n \geq 1\}$. If an ideal generated by one single element, then it is called *principal ideal*. If $r, s \in R \backslash \{0\}$ such that $rs=0$ then r and s are called *zero divisor*s. A ring R is an *integral domain* if it is a commutative ring with unity such that R has no zero divisors. Let R be an integral domain, if every ideal of R is principal then R is called *principal ideal domain (PID)*.

Let R be a commutative ring. If for any chain of ideals of R,

$$I_1 \subseteq I_2 \subseteq \ldots \subseteq I_i \subseteq \ldots$$

there exist an n such that $I_i = I_n$ for all $i \geq n$ we say R satisfies the *ascending chaing condition (ACC)* on ideals. A ring that satisfies ACC is said to be *Noetherian*. It is equivalent to a ring being Noetherian if all ideals of that ring are generated by finitely many elements. In particular, a PID is Noetherian. Famous Hilbert basis theorem says that if R is Noetherian then so is $R[x]$. Then our main object $\mathbb{K}[x_1, \ldots, x_n]$ is Noetherian for all $n \in \mathbb{N}$.

For instance, $\mathrm{End}_{\mathbb{Z}}(G)$ is a ring with unity with the operations $(\varphi_1 + \varphi_2)(g) = \varphi_1(g) + \varphi_2(g)$ and $(\varphi_1 \circ \varphi_2)(g) = \varphi_1(\varphi_2(g))$, where $\varphi_1,\ \varphi_2 \in \mathrm{End}_{\mathbb{Z}}(G)$ and $g \in G$. The identity element of $\mathrm{End}_{\mathbb{Z}}(G)$ is the zero map and unity of $\mathrm{End}_{\mathbb{Z}}(G)$ is the identity map on G, it is clear that both of these maps are endomorphisms of G. Let $(R, +_R, \cdot_R)$ and $(S, +_S, \cdot_S)$ be two rings, a map $\varphi : R \longrightarrow S$ is called a *ring homomorphism* if $\varphi(r_1 +_R r_2) = \varphi(r_1) +_S \varphi(r_2)$ and $\varphi(r_1 \cdot_S r_2) = \varphi(r_1) \cdot_S \varphi(r_2)$ for all $r_1,\ r_2 \in R$. If R and S are rings with unity then a ring homomorphism between then is also satisfies $1_R \mapsto 1_S$ where 1_R and 1_S are denotes the unities of R and S respectively. Definition of a *field homomorphism* is same as the definition of the homomorphism between rings with unities.

Let R be a ring with unity and M is an abelian group, if there exists a ring homomorphism,

$$\begin{array}{rcl} \lambda\colon R & \longrightarrow & \mathrm{End}_{\mathbb{Z}}(M) \\ \cup & & \cup \\ r & \longmapsto & \lambda_r \end{array}$$

with $\lambda_r \circ \lambda_s = \lambda_{rs}$ for all $r, s \in R$ (and $\lambda_1 = id_M$) then M is called a *left module* over R or called an *left R-module*. Similarly, if there exists a ring homomorphism

$$\begin{array}{rcl} \rho\colon R & \longrightarrow & \mathrm{End}_{\mathbb{Z}}(M) \\ \cup & & \cup \\ r & \longmapsto & \rho_r \end{array}$$

with $\rho_r \circ \rho_s = \rho_{sr}$ for all $r, s \in R$ (and $\lambda_1 = id_M$) then M is called a *right module* over R or called a *right R-module*. If R is a commutative ring (with unity), the left and right R-module structure on the abelian group are conincide. We denote a left (or right) R-module as the triple $(M, +, \lambda : R \to \mathrm{End}_{\mathbb{Z}}(M))$. In particular, if $R = \mathbb{K}$ is a field, then M is called a *vector space* over $\mathbb{K}$ or a $\mathbb{K}$-*vector space*. Any abelian group G can be seen as a $\mathbb{Z}$-module with $\lambda_0(g) = 0$ and $\lambda_n(g) = ng = \left(\sum_{i=1}^{n-1} g\right) + g$, for all $n \in \mathbb{N}$ and $g \in G$. If $n \in \mathbb{Z}\backslash\mathbb{N}$ then, we define $\lambda_n(g) = -\lambda_{-n}(g)$ for all $g \in G$. Let $(M, +_M, \lambda)$ and $(N, +_N, \lambda')$ be left R-modules. A map $\varphi : M \longrightarrow N$ is called a *left R-module homomorphism* if $\varphi(m_1 +_M m_2) = \varphi(m_1) +_N \varphi(m_2)$ for all $m_1,\ m_2 \in M$ and $\varphi(\lambda_r(m)) = \lambda'_r(\varphi(m))$ for all $m \in M$ and for all $r \in R$. The right R-module homomorphism defines in the similar way. When $R = \mathbb{K}$ is a field, M and N are $\mathbb{K}$-vector spaces and the φ is called a *linear map* or $\mathbb{K}$-*linear map*.

Let $(M, +, \lambda)$ be an R-module (left or right) and $N \leq M$ be a subgroup of M. If restriction of λ_r to N gives a group endomorphism of N for all $r \in R$ then N is said to be a *submodule* of M. Let M be an R-module and N be a submodule of M, the abelian group $M/N : \{m + N : m \in M\}$ is an R-module with, $\bar{\lambda} : R \to \mathrm{End}_{\mathbb{Z}}(M/N)$ with $\bar{\lambda}_r(m + N) = \lambda_r(m) + N$ for all $r \in R$. Where M/N is called *quotient module.*

Let R and A be two commutative rings and let $f : R \longrightarrow A$ be a ring homomorphism. The map

$$\begin{array}{rcl} \lambda\colon R & \longrightarrow & \mathrm{End}_{\mathbb{Z}}(A) \\ \Psi & & \Psi \\ r & \longmapsto & \lambda_r \end{array}$$

with $\lambda_r(a) = f(r)a$ gives a ring homomorphism since,

$$\lambda_{r+s}(a) = f(r+s)a = (f(r)+f(s))a = f(r)a+f(s)a = \lambda_r(a)+\lambda_s(a) = (\lambda_r+\lambda_s)(a),$$

$$\lambda_{rs}(a) = f(rs)a = (f(r)f(s))a = f(r)(f(s)a) = f(r)(\lambda_s(s)) = \lambda_r(\lambda_s(a)) = (\lambda_r \circ \lambda_s)(a)$$

for all $a \in A$ then,

$$\lambda_{r+s} = \lambda(r) + \lambda(s) \quad \text{and} \quad \lambda_{rs} = \lambda_r \circ \lambda_s \quad \text{for all } r,\ s \in R.$$

Then the mapping $\lambda_r(a) = f(r)a$ makes the ring A into an R-module. Thus, A has an R-module structure as well as a ring structure, and these two structure are compatible in the sense* $\lambda_r(ab) = f(r)(ab) = (f(r)a)b = \lambda_r(a)b$ for all $a, b \in A$ and for all $r \in R$. The ring A, equipped with this R-module structure, is said to

*Compatibility with additive structure on A is done by $\lambda_r \in \mathrm{End}_{\mathbb{Z}}(A)$ for all $r \in R$.

be an R-*algebra*. In other words, an R-algebra is a ring A together with a ring homomorphism $f : R \longrightarrow A$.

Let $f : R \longrightarrow A$ and $g : R \longrightarrow B$ be homomorphisms between commutative rings. Thus, we see A and B as R-modules in the manner described above. Considering the ring structures, A and B are R-algebras. A map $\varphi : A \longrightarrow B$ is called an R-*algebra homomorphism* if φ is a ring homomorphism and $\varphi(f(r)a) = g(r)(\varphi(a))$.

Kernel of a homomorphism φ from A to B is $\text{Ker}(\varphi) = \{a \in A : \varphi(a) = 0_B\}$. The definition of kernel is exactly the same if the φ is the homomorphism between rings, modules or abelian groups. When φ is a bijection, then φ is called an *isomorphism* and we call A and B are isomorphic. We denote isomorphic groups, rings and modules by $A \simeq B$. Kernel of φ is an ideal when φ is a ring homomorphism and similarly $\text{Ker}(\varphi)$ is a submodule when φ is a module homomorphism. The first isomoprhism theorem for groups, rings and modules says $A/\text{Ker}(\varphi) \simeq \text{Im}(\varphi)$ with the isomorphism $\bar{\varphi}(a + \text{Ker}(\varphi)) = \varphi(a)$.

1.2 Primary decomposition and monomial ideals

Let A be a Noetherian ring and M be a finitely generated A-module. We denote *annihilator* of a nonempty subset S of M as $\text{Ann}_A(S)$ and define it by $\text{Ann}_A(S) = \{a \in A : as = 0_M \text{ for all } s \in S\}$. For the sake of brevity, we will write $\text{Ann}(S)$ instead of $\text{Ann}_A(S)$. If S is a singleton, say $S = \{x\}$, we'll write $\text{Ann}_A(x)$ instead of $\text{Ann}_A(\{x\})$. Let us consider the set $\Lambda = \{\text{Ann}(x) : x \in M \text{ and } x \neq 0\}$. Since A is Noetherian, all chains in Λ stabilizes. Let $\mathfrak{p} \in \Lambda$ be a maximal element of Λ with respect to the inclusion. Then, there exists an element $x \in M$ such that $\mathfrak{p} = \text{Ann}(x)$. Assume that $ab \in \mathfrak{p}$ but $b \notin \mathfrak{p}$. Then $abx = 0$ and $bx \neq 0$ and $a \in \text{Ann}(bx)$. If $c \in \text{Ann}(x)$, by the commutativity of A we have $cbx = bcx = 0$; thus, $c \in \text{Ann}(bx)$. Since $\text{Ann}(bx) \supseteq \text{Ann}(x) = \mathfrak{p}$, we get $\text{Ann}(bx) = \mathfrak{p}$ by the maximality of $\mathfrak{p}$. Hence, $a \in \mathfrak{p}$, which shows that $\mathfrak{p} \in \text{Spec}(A)$. In other words, maximal elements of $\Lambda = \{\text{Ann}(x) : x \in M \text{ and } x \neq 0\}$ are prime ideals of A. Such special prime ideals are called *associated primes* of M. Any associated prime of a module M is denoted and defined by $\text{Ass}(M) = \{\mathfrak{p} \in \text{Spec}(A) : \mathfrak{p} = \text{Ann}(x) \text{ for some } x \in M\}$.

It is a well-known fact that every ideal I of A admits a *minimal primary decomposition*, for example see [2]. Let $I = \bigcap_{i=1}^{s} Q_i$ be a minimal primary decomposition of I and for each $i = 1, \ldots, s$, set $P_i = \sqrt{Q_i}$. In other words, each Q_i is a P_i-primary ideal. Then $\text{Ass}(I) = \{P_1, \ldots, P_s\}$.

Let $\mathbb{K}$ be a field, and let $S = \mathbb{K}[x_1, \dots, x_n]$ be the polynomial ring in n variables over $\mathbb{K}$. By the *Hilbert Basis Theorem*, S is Noetherian. Set $\mathbb{Z}_+^n = \{\mathbf{a} = (a_1, \dots, a_n) : a_i \in \mathbb{N}\}$. For any $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}_+^n$, the element $x^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}$ in S is called a *monomial*. We denote the set of monomials of S as $\mathrm{Mon}(S)$. In particular, if each $a_i \in \{0, 1\}$, then $x^{\mathbf{a}}$ is called a *squarefree* monomial. An ideal $I \subset S$ is called *monomial ideal* if it is generated by monomials, and I is called *squarefree monomial ideal* if it is generated by squarefree monomials. Moreover, if I is a monomial ideal then $f \in I$ if and only if $\mathrm{supp}(f) \subset I$, for all $f \in S$, and converse of this is also true. Here, $\mathrm{supp}(f)$ is defined as the set of monomials which appears in the polynomial f with non-zero coefficients.

Proposition 1.2.1. *[12, Proposition 1.1.6.] Each monomial ideal I of $S = \mathbb{K}[x_1, \dots, x_n]$ has a minimal monomial set of generators and this generating set is unique. More precisely, let $G(I)$ denote the set of monomials in I which are minimal with respect to divisibility. Then $G(I)$ is the unique minimal set of monomial generators of the ideal $I \subset S$.*

Throughout the following text, for any given monomial ideal I of S, we will denote the unique minimal monomial set of generator of I by $G(I)$.

As mentioned above, any ideal of S admits a minimal primary decomposition. However, if I is monomial ideal, then its primary decomposition take a special form as described in the following theorem. Observe that a monomial ideal generated by pure powers of variables is a primary ideal.

Proposition 1.2.2. *[12, Theorem 1.3.1] Let I be a monomial ideal of the polynomial ring $S = \mathbb{K}[x_1, \dots, x_n]$. Then $I = \bigcap_{\ell=1}^{m} \mathcal{Q}_\ell$, where each $\mathcal{Q}_\ell$ is generated by pure powers of the variables, that is, $\mathcal{Q}_\ell$ is of the following form*

$$\mathcal{Q}_\ell = \left(x_{\ell_1}^{a_1}, \dots, x_{\ell_k}^{a_k}\right) \quad \textit{for all} \quad \ell \in \{1, \dots, m\}.$$

Moreover, an irredundant presentation of the given form above is unique.

It follows immediately from above theorem that for a monomial ideal I, its associated primes are also monomial prime ideals. Moreover, if I is a squarefree monomial ideal then $\mathrm{Min}(I) = \mathrm{Ass}(I)$. When we take $I \subset S$ as a squarefree monomial ideal, each monomial ideals appearing in the intersection presentation of I is generated by some collection of variables of the ambient polynomial ring S. Then we obtain that every squarefree monomial ideal can be seen as an intersection of some monomial prime ideals since all monomial prime ideals are generated by some collection of the variables of S.

Now let us define one of the operations on ideals that we will use a lot. Let I be an ideal and X be a subset of a given ring A. Then, the set

$$I : X \ = \{a \in A : aX \subseteq I\}$$

is an ideal, called the *colon ideal* or *ideal quotient* of I with respect to X. $I : X$ is an extension ideal of I. While X in $I : X$ can be any subset of A, we will be generally interested in the colon ideals formed by two ideals.

The following proposition is well-known in the theory of monomial ideals.

Proposition 1.2.3. *Let $S = \mathbb{K}[x_1, \dots, x_n]$ be the polynomial ring in n variables and $I, J \subset S$ be two monomial ideals. Then $I : J$ is a monomial ideal, and*

$$I : J = \bigcap_{v \in G(J)} I : (v) .$$

Moreover, $\left\{ \frac{u}{gcd(u,v)} : u \in G(I) \right\}$ is a set of generators of $I : (v)$.

1.3 Graded rings and modules

In this section, our aim is to introduce and investigate essential concepts of graded algebraic structures that we used in the thesis.

Definition 1.3.1. *Let A be a ring and $(G, *)$ be a group (or a monoid). A is called G-graded or graded ring if following conditions holds:*

(i) $A = \bigoplus_{g \in G} A_g$ where A_g is an abelian group for each $g \in G$,

(ii) $A_g A_h \subseteq A_{g*h}$ for all $g, h \in G$.

An element a of the graded ring A is called *homogeneous element* if $a \in A_g$ for some $g \in G$. Every element $a \in A$ can be written as a finite sum $a = \sum_g a_g$ of non-zero elements $a_g \in A_g$. Where each a_g is called a *homogeneous component of* a of *degree* g and each A_g is called a *homogeneous component of* A. Then, S has a direct sum decomposition of abelian groups $S = \bigoplus_{i \in \mathbb{N}} S_i$

Based on the definition of a graded ring, a graded module is defined in a natural way.

Definition 1.3.2. *Let M be a module over ring A and $(G, *)$ be a group (or a monoid). M is called G-graded if following conditions holds:*

(i) $M = \bigoplus_{g \in G} M_g$ *where M_g is an abelian group for each $g \in G$,*

(ii) $A_g M_h \subseteq M_{g*h}$ *for all $g, h \in G$.*

A graded A-*algebra* is a graded A-module that is also a graded ring. A ring is a module over itself and any ideal of the ring A can be seen as a submodule of A. We give the definition of a graded ideal by applying the definition of graded module to the ideals of A.

Definition 1.3.3. *A proper ideal I in the G-graded ring A is called graded if it satisfies $I = \bigoplus_{g \in G} I_g$ where $I_g = A_g \cap I$.*

The following well-known theorem gives the conditions equivalent to being a graded ideal.

Theorem 1.3.4. *[20] The following are equivalent:*

(1) I is a graded ideal of S.

(2) If $f \in I$, then every homogeneous component of f is an element of I.

(3) If $\widetilde{I}$ is the ideal generated by all homogeneous elements in I, then $I = \widetilde{I}$.

(4) I has a system of homogeneous generators.

In this thesis, our main object is $\mathbb{K}[x_1, \ldots, x_n]$ the polynomial ring of n-variable over a field $\mathbb{K}$. Let us denote $S = \mathbb{K}[x_1, \ldots, x_n]$ and set $\deg(x_i) = 1$ for each $i \in \{1, \ldots, n\}$ and $\deg(\alpha) = 0$ for all $\alpha \in \mathbb{K}^*$. Note that 0 is an homogeneous element with arbitrary degree since 0 is an element of each graded component of the direct sum decomposition. A monomial $\mathbf{x} = x_1^{d_1} \cdots x_n^{d_n}$ has *degree* $d = \sum_{i=1}^{n} d_i$. The polynomial ring S is an $\mathbb{N}$-graded ring since S can be written as,

$$\mathbb{K}[x_1, \ldots, x_n] = \mathbb{K} \oplus \left(\bigoplus_{i} \mathbb{K} x_i \right) \oplus \left(\bigoplus_{i,j} \mathbb{K} x_i x_j \right) \oplus \left(\bigoplus_{i,j,k} \mathbb{K} x_i x_j x_k \right) \oplus \cdots ,$$

and denote, $S_d = \bigoplus_{i_1, \ldots, i_d} \mathbb{K} x_{i_1} \cdots x_{i_d} = \{\sum \alpha_{i_1, \ldots, i_d} x_{i_1} \cdots x_{i_d} : \alpha_{i_1, \ldots, i_d} \in \mathbb{K}\}$ for each d. Thus, following properties are equivalent.

- $S_i S_j \subseteq S_{i+j}$ for all $i, j \in \mathbb{N}$.

- $\deg(uv) = \deg(u)\deg(v)$ for every two homogeneous elements $u, v \in S$.

Definition 1.3.5. *Let $A = \bigoplus_{i\in\mathbb{N}} A_i$ be a graded $\mathbb{K}$-algebra. A is called standard graded if R is a finitely generated $\mathbb{K}$ algebra and all its generators are of degree* 1.

Namely, $A = \bigoplus_{i\in\mathbb{N}} A_i$ is called standard graded K-algebra if $A = \mathbb{K}[A_1]$ and $\dim_{\mathbb{K}}(A_1) \leq \infty$. Where $A = \mathbb{K}[A_1]$ denotes the $\mathbb{K}$-algebra generated by the subset A_1. It is immediate to see that $S = \mathbb{K}[x_1, \dots, x_n]$ is a standard graded $\mathbb{K}$-algebra since $S = \mathbb{K}\left[\bigoplus_i \mathbb{K}x_i\right]$ and $\dim_{\mathbb{K}}(\bigoplus_i \mathbb{K}x_i) = n \leq \infty$. A well-known fact is that any other standard graded $\mathbb{K}$-algebra is isomorphic to the quotient of a polynomial ring by a graded ideal [12].

Let M be a graded A-module and $\mathcal{G}$ be the system of generators of M. Let $\widetilde{\mathcal{G}}$ be the set which is obtained by taking all homogeneous components of each element in $\mathcal{G}$. Then, $\widetilde{G}$ is a system of homogeneous generators of M. Moreover, by the facts that every element in M is an A-linear combination of the generators and $A_iM_j \subseteq M_{i+j}$ for all i, j; the degrees of the elements in a system of homogeneous generators determine the grading of M.

Since each graded component A_d of the graded module A is an abelian group, A_d is $\mathbb{K}$-vector space for each d. A basis of the $\mathbb{K}$-space A_d is called a *basis in degree d*. For $r \in \mathbb{Z}$ we denote A_{d-r} as $A(-r)_d$ for all d. We say that $A(-r)$ is the module A *shifted d degrees*, and r is called *shift*. Note that, $A(-r)_r = A_0$. Thus, the shifted module $A(-r)$ is the free A-module generated by one element in degree r. For instance, degree of $1 \in A(-r)$ is r since $1 \in A(-r)_0$ and $A(-r)$ is generated by $\{1\}$ as an A-module[†].

Let M and N be graded A-modules and $M \xrightarrow{\varphi} N$ be an A-module homomorphism. Then, φ has *degree i* if $\deg(\varphi(m)) = i + \deg(m)$ for each homogeneous element $m \in M \setminus \operatorname{Ker}(\varphi)$. That is, the degree of a homomorphism is a measure of how much it shifts its grade in the image of a homogeneous element. Recall that 0 has arbitrary degree, thus the condition $\deg(\varphi(m)) = i + \deg(m)$ only on the homogeneous elements of M outside the kernel of the homomorphism φ. We denote $\operatorname{Hom}_i(M, N)$ as the set of all degree i A-module homomorphisms from M to N. We define

$$\mathcal{H}(M, N) = \bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}_i(M, N)$$

and this is called as *graded* Hom from M to N. In general, $\mathcal{H}(M, N)$ is an A-submodule of $\operatorname{Hom}(M, N)$. Moreover, if U is a finitely generated A-module and T is a graded R-module, then $\mathcal{H}(U, T) = \operatorname{Hom}(U, T)$, see [20, Proposition 2.7].

[†]In the book [20], $1 \in A(-r)$ is called *1-generator* of $A(-r)$.

A graded submodule of a graded module is defined similarly to a graded ideal of a graded ring. Let M be a graded A-module and N be a submodule of M. N is called *graded* submodule if N has the direct sum decomposition $N = \bigoplus_i N_i$ where $N_i = M_i \cap N$.

Proposition 1.3.6 ([20]). *Let M be a graded A-module and N be a submodule of M. Then, the following are equivalent.*

(i) N is a graded submodule of M.

(ii) If $n \in N$, then every homogeneous component of n is in N.

(iii) If $\widetilde{N}$ is the submodule generated by all homogeneous elements in N, then $N = \widetilde{N}$.

(iv) N has a system of homogeneous generators.

The grading structure of M inherits the quotient modules of M/N if N is a graded submodule of M. That is,

$$M/N = \bigoplus_i (M/N)_i \quad \text{where} \quad (M/N)_i = M_i/N_i \ .$$

If $\varphi : M \to U$ is a homomorphism of graded A-modules, then $\mathrm{Ker}(\varphi), Im(\varphi),$ and $\mathrm{Coker}(\varphi) = U/Im(\varphi)$ are graded [20, Proposition 2.9].

The following theorem is called the *structure theorem for graded finitely generated A-modules.*

Theorem 1.3.7. *[20, Theorem 2.10] The following properties are equivalent.*

(1) U is a finitely generated graded A-module.

(2) $U \simeq W/T$, where W is a finite direct sum of shifted free A-modules, T is a graded submodule of W and T is called the module of relations, and the isomorphism has degree 0.

1.4 Free resolution

Chain complexes are one of the fundamental objects in homological algebra. In this section, we will provide the basics of chain complexes on which we base the definitions of free resolution and projective resolution.

Definition 1.4.1. *Let A be a ring. A chain complex $F_\bullet$ over A is a sequence of homomorphisms of A-modules*

$$F_\bullet : \ldots \longrightarrow F_i \longrightarrow F_{i-1} \longrightarrow \ldots \longrightarrow F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \longrightarrow \ldots\ ,$$

such that $d_{i-1} \circ d_i = 0$ for each $i \in \mathbb{Z}$. For each i the maps $\{d_i\}_{i\in\mathbb{Z}}$ is called the differential or differential map or differential homomorphism of $F_\bullet$. In the literature, the complex is also denoted by $(F_\bullet, d)$. If $F_i = 0$ for all $i < 0$, it is called a left complex, that is,

$$F_\bullet : \ldots \longrightarrow F_i \xrightarrow{d_i} F_{i-1} \longrightarrow \ldots \longrightarrow F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \longrightarrow 0$$

with $i \in \mathbb{N}$. Furthermore $(F_\bullet, d)$ is called a left complex over W (or a complex over W) if it is a left complex and we have a homomorphism $\varepsilon : F_o \longrightarrow W$, this map is called augmentation map.

The complex is called *graded* if the modules F_i are graded and each d_i is a homomorphism of degree 0 [20]. When the homomorphisms have degree $i > 0$, we shift the modules by a suitable degree to make the homomorphisms of degree 0. If a complex is graded, the module $F_\bullet$ is actually bigraded since

$$F_i = \bigoplus_{j\in\mathbb{Z}} F_{i,j} \quad \text{for all} \quad i.$$

For any chain complex $F_\bullet$, an element in $F_{i,j}$ is said to have *homological degree* i and *internal degree* j. We denote the homological degree by hdeg, and internal degree by deg. Consider $F_\bullet$ as a module and the differential as homomorphism $d : F_\bullet \to F_\bullet$. Then d has homological degree -1 and internal degree 0.

If each F_i is a finitely generated graded free module over A in a given complex $F_\bullet$, we can write

$$F_i = \bigoplus_{r\in\mathbb{Z}} A(-r)^{\beta_{i,r}} \quad \text{for all} \quad i.$$

Where, the numbers $\beta_{i,r}$ are the *graded Betti numbers* of the chain complex. Given a graded Betti number we say that $\beta_{i,r}$ is the Betti number in *homological degree* i and internal degree r, or i'th Betti number in internal degree r. Namely, $\beta_{i,r}$ denotes the number of summands in F_i of the form $A(-r)$.

The *homology* of a complex $F_\bullet$ is defined as follows

$$\mathrm{H}_i(F_\bullet) = \mathrm{Ker}(d_i)/\,\mathrm{Im}(d_{i+1}).$$

Elements in the kernel of each d_i are called *cycles* and the elements in the image of each d_{i+1} are called *boundaries.* The complex $F_\bullet$ is called *exact* at F_i (or at *step* i) if $\mathrm{H}_i(F_\bullet) = 0$. The complex is *exact* if $\mathrm{H}_i(F_\bullet) = 0$ for all i.

Example 1.4.2. Following sequence of $\mathbb{Z}$-module homomorphisms is in the form of a chain complex.

$$M_\bullet : \quad 0 \to \mathbb{Z} \xrightarrow{\binom{8}{-4}} \mathbb{Z}^2 \xrightarrow{(1 \quad 2)} \mathbb{Z} \to 0 \ .$$

Obviously the composition of consecutive maps are 0*. And homology modules are,*

$$\mathrm{H}_0(M_\bullet) = \frac{\mathbb{Z}}{\mathbb{Z}} = 0 \ , \quad H_1(M_\bullet) = \frac{\left\langle \binom{2}{-1} \right\rangle_{\mathbb{Z}}}{\left\langle \binom{8}{-4} \right\rangle_{\mathbb{Z}}} = \frac{2\mathbb{Z} \times \mathbb{Z}}{8\mathbb{Z} \times 4\mathbb{Z}} \simeq \frac{\mathbb{Z}}{4\mathbb{Z}} \ , \quad \mathrm{H}_2(M_\bullet) = \frac{0}{0} = 0.$$

The remeaning homology modules are equal to 0 *since* $M_i = 0$ *for all* $i \in \mathbb{Z} \backslash \{0, 1, 2\}$

Definition 1.4.3. *A free resolution of a finitely generated* A*-module* M *is a sequence of homomorphisms of* A*-modules*

$$F_\bullet : \ldots \longrightarrow F_i \xrightarrow{d_i} F_{i-1} \longrightarrow \ldots \longrightarrow F_1 \xrightarrow{d_1} F_0 \ ,$$

such that

(i) $F_\bullet$ *is a chain complex of finitely generated free* A*-modules* F_i*,*

(ii) the chain complex $F_\bullet$ *is exact,*

(iii) $M \simeq F_0 / \operatorname{Im}(d_1)$.

A resolution is *graded* if M is a graded module and $F_\bullet$ is a graded complex, and isomorphism between M and $F_0 / \operatorname{Im}(d_1)$ has degree 0. We recommend the reader refer to the construction in [20, Chapter 4] to see that any A-module M has a free resolution. Moreover, by the [20, Theorem 7.5] up to isomorphism, there exists a unique minimal graded free resolution of a given finitely generated module over a ring. In particular, for the finitely generated modules over a polynomial ring with finite variable we have a bound for the length of free resolution of the module.

Theorem 1.4.4. *Every finitely generated* $S = \mathbb{K}[x_1, \ldots, x_n]$*-module* M *has a finite free resolution of length at most* n*.*

Since each F_i is a free A-module in a complex $F_\bullet$. Then, $F_i \simeq A^n$ for some n for each i. Since, $\mathrm{Hom}_A(F_i, F_{i-1}) \simeq \mathrm{Hom}_A(A^n, A^m) \simeq \mathrm{Mat}_{m \times n}(A)$, the differential map $F_i \xrightarrow{d_i} F_{i-1}$ can be represented as a matrix with respect to the chosen basis for F_i and F_{i-1}. These matrices are called *differential matrices.*

Definition 1.4.5. *Let $S = \mathbb{K}[x_1, \ldots, x_n]$. A graded free resolution of a graded finitely generated S-module M is minimal if*

$$d_{i+1}(F_{i+1}) \subseteq (x_1, \ldots, x_n)\, F_i \quad \textit{for all} \quad i \geq 0.$$

Example 1.4.6. *[20] Let $A = \mathbb{K}[x, y]$ and $I = (x^3, xy, y^5)$. Then, the following free resolution of A/I over A is minimal,*

$$0 \hookrightarrow A(-4) \oplus A(-6) \xrightarrow{d_2} A(-3) \oplus A(-2) \oplus A(-5) \xrightarrow{d_1} A \xrightarrow{\pi} A/I \twoheadrightarrow 0.$$

Where, matrix representations of the homomorphisms with respect to the appropriate basis are as follows:

$$d_2 \mapsto \begin{pmatrix} y & 0 \\ -x^2 & -y^4 \\ 0 & x \end{pmatrix} \quad \textit{and} \quad d_1 \mapsto \begin{pmatrix} x^3 & xy & y^5 \end{pmatrix}.$$

Let M be a finitely generated module over a ring A and $F_\bullet$ be the a minimal graded free resolution of M over A. For $i \geq 1$ the submodule $\mathrm{Im}(d_i) = \mathrm{Ker}(d_{i-1}) \simeq \mathrm{Coker}(d_{i+1})$ of F_{i-1} is called *i-th syzygy module* of M and we denote it by $\mathrm{Syz}_i^A(M)$. Elemets of $\mathrm{Syz}_i^A(M)$ are called i-th *syzygies*. We set $M = \mathrm{Syz}_0^A(M)$.

Let us adapt what we have done so far to understand a given ideal. Let I be an ideal of A and $M = I$. Consider a minimal free resolution of I over A $\ldots \to F_2 \to F_1 \to A \to I \to 0$. Indeed, matrix representation of the map d_0 generators of the ideal I. The first syzygies are nothing but the non-trivial algebraic relations between the generators of the ideal I since image of d_1 is equal to the free module F_0. And second syzygies are non-trivial algebraic relations between the generators of the first syzygy module which is the kernel of d_1 and other syzygies are considered in similar way. In general, free resolution of an ideal has the following form

$$\ldots \xrightarrow[\text{third syzygies}]{d_3} F_2 \xrightarrow[\text{second syzygies}]{d_2 = \begin{pmatrix}\text{relations on}\\ \text{the generators}\\ \text{in } d_1\end{pmatrix}} F_1 \xrightarrow[\text{first syzygies}]{d_1 = \begin{pmatrix}\text{relations on}\\ \text{the generators}\\ \text{of } I\end{pmatrix}} F_0 \xrightarrow{d_0 = \begin{pmatrix}\text{Generators of } I\end{pmatrix}} I \longrightarrow 0.$$

For instance, if $I = (xy, xz)$ is an ideal of $A = \mathbb{K}[x, y]$, non-trivial algebraic relations on the generators of I is

$$z(xy) = y(xz) \Longrightarrow z(xy) - y(xz) = 0$$

Therefore, the non-graded minimal free resolution of I over A is

$$0 \longrightarrow A \xrightarrow[d_1]{\binom{z}{-y}} A \oplus A \xrightarrow[d_0]{(xy \quad xz)} I \longrightarrow 0.$$

Now, let us calculate the graded version of the minimal free resolution of I. Let $\{f_1, f_2\}$ be a basis of $A \oplus A$ such that the matrix corresponding to d_0 written in this basis then we have,

$$f_1 \overset{d_0}{\longmapsto} xy \quad \text{and} \quad \deg(xy) = 2$$
$$f_2 \overset{d_0}{\longmapsto} xz \quad \text{and} \quad \deg(xz) = 2$$

since we want to d_0 to be homogeneous of degree 0, we set

$$\deg(f_1) = 2, \ \deg(f_2) = 2.$$

Hence, the free A-module generated by $\{f_1, f_2\}$ is $A(-2) \oplus A(-2)$ which is identified with $A \oplus A$ in the non-graded free resolution. Furthermore, let $\{g\}$ be a basis of A such that the matrix representation of d_1 given in this basis. Since

$$g \overset{d_1}{\longmapsto} zf_1 - yf_2 \quad \text{and} \quad \deg(zf_1 - yf_2) = \deg(zf_1) = \deg(z) + \deg(f_1) = 3$$

and since we want d_1 to be homogeneous of degree 0, we set

$$\deg(g) = 3.$$

Then, the free A-module generated by $\{g\}$ is identified with $A(-3)$. Therefore, the minimal graded free resolution of I over A is

$$0 \longrightarrow A(-3) \xrightarrow[d_1]{\binom{z}{-y}} A(-2) \oplus A(-2) \xrightarrow[d_0]{(xy \quad xz)} I \longrightarrow 0.$$

As the number of variables in the polynomial ring increases and, correspondingly, the number of generators in the given ideal increases, it becomes very difficult to describe differential maps in the free resolution of the ideal. In such a case, we would like to know at least some numerical invariants of the free resolution to understand the structure of the ideal.

Definition 1.4.7. *Let M be a finitely generated module over A. The i-th Betti number $\beta_i^A(M)$ defined as rank(F_i) in the free resolution $\ldots \to F_i \xrightarrow{d_i} F_{i-1} \ldots \to F_1 \xrightarrow{d_1} F_0$.*

Instead of $\beta_i^A(M)$, we simple denote i-th Betti number as β_i if there is no confusion. It is well-known fact that, the Betti numbers are invariant numbers of the minimal graded free resolution of a module.

Now, let us define the functors Ext and Tor that are frequently used in commutative algebra. For this, define start with *projextive resolution* of a module. If each F_i is graded by $\mathbb{N}$ in the minimal free resolution of M over A given in the Definition 1.4.7, for each $j \in \mathbb{N}$ it is straightforward to see that

$$\beta_i(M) = \sum_{j \in \mathbb{N}} \beta_{i,j}(M) = \operatorname{rank}(F_i) \ .$$

This is why if we work on graded structures, i-th Betti number of a module M is also called *i-th total Betti number* of M.

Definition 1.4.8. *Let A be a ring, and let M be an A-module. A projective resolution of M over A is an exact sequance of A-module homomorphisms*

$$P_\bullet^+ : \ldots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\tau} M \to 0$$

such that P_i is a projective A-module for each $i \geq 0$. The truncated projective resolution of M associated to $P_\bullet^+$ is the chain complex over A

$$P_\bullet : \ldots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \to 0.$$

As is well known, every free module over the ring A is a projective module over A. Considering this fact, Definition 1.4.8 generalises the definition of free resolution of a module over A.

Let M be an A-module. Consider the given projective resolution of M over A

$$P_\bullet : \ldots \xrightarrow{\partial_{i+2}} P_{i+1} \xrightarrow{\partial_{i+1}} P_i \xrightarrow{\partial_i} P_{i-1} \xrightarrow{\partial_{i-1}} \ldots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \to 0.$$

For each A-module N, the following sequence is a chain complex over A.

$$P_\bullet \otimes_A N : \ldots \xrightarrow{\partial_{i+2} \otimes N} P_{i+1} \otimes_A N \xrightarrow{\partial_{i+1} \otimes N} P_i \otimes_A N \xrightarrow{\partial_i \otimes N} P_{i-1} \otimes_A N \xrightarrow{\partial_{i-1} \otimes N} \ldots$$
$$\ldots \xrightarrow{\partial_2 \otimes N} P_1 \otimes_A N \xrightarrow{\partial_1 \otimes N} P_0 \otimes_A N \to 0.$$

For each $i \in \mathbb{Z}$ we define,

$$\operatorname{Tor}_i^A(M, N) = \operatorname{H}_i(P_\bullet \otimes_A N) = \frac{\operatorname{Ker}(\partial_i \otimes N)}{\operatorname{Im}(\partial_{i+1} \otimes N)}$$

Namely, $\operatorname{Tor}_i^A(\_, N)$ is a functor from the category of A-modules to itself. To be more precise, for each $N \in \mathbf{Obj}(\mathtt{Mod}_A)$ and for each $i \in \mathbb{Z}$ we have a covariant functor $\operatorname{Tor}_i^A(\_, N) : \mathtt{Mod}_A \to \mathtt{Mod}_A$ such that

$$\begin{array}{rcl} \mathrm{Tor}_i^A(\_, N)\colon \mathbf{Obj}(\mathtt{Mod}_A) & \longrightarrow & \mathbf{Obj}(\mathtt{Mod}_A) \\ \cup & & \cup \\ M & \longmapsto & \mathrm{Tor}_i^A(M, N) \end{array} .$$

$$\begin{array}{rcl} \mathrm{Tor}_i^A(\_, N)\colon \mathbf{Mor}_{\mathtt{Mod}_A}(M, M') & \longrightarrow & \mathbf{Mor}_{\mathtt{Mod}_A}\left(\mathrm{Tor}_i^A(M, N), \mathrm{Tor}_i^A(M', N)\right) \\ \cup & & \cup \\ \varphi & \longmapsto & \mathrm{Tor}_i^A(\varphi, N) \end{array}$$

Where,

$$\mathbf{Obj}(\mathtt{Mod}_A)$$

is objects of the category which are A-modules $\mathtt{Mod}_A$ and

$$\mathbf{Mor}_{\mathtt{Mod}_A}(M, M')$$

is the collection of the morphisms from M to M', and

$$\mathbf{Mor}_{\mathtt{Mod}_A}\left(\mathrm{Tor}_i^A(M, N), \mathrm{Tor}_i^A(M', N)\right)$$

is the morphisms from $\mathrm{Tor}_i^A(M, N)$ to $\mathrm{Tor}_i^A(M', N)$. Indeed they are A-module homomorphisms, then

$$\mathbf{Mor}_{\mathtt{Mod}_A}(M, M') = \mathrm{Hom}_A(M, M')$$

and

$$\mathbf{Mor}_{\mathtt{Mod}_A}\left(\mathrm{Tor}_i^A(M, N), \mathrm{Tor}_i^A(M', N)\right) = \mathrm{Hom}_A\left(\mathrm{Tor}_i^A(M, N), \mathrm{Tor}_i^A(M', N)\right).$$

Since the tensor product along a commutative ring A satisfies $M \otimes_A N \simeq N \otimes_A M$ for all $M, N \in \mathbf{Obj}(\mathtt{Mod}_A)$, the results of $\mathrm{Tor}_i^A(\_, N)$ and $\mathrm{Tor}_i^A(N, \_)$ are same on $\mathtt{Mod}_A$[‖].

Example 1.4.9. Consider the projective resolution of $\mathbb{Z}/m\mathbb{Z}$ as given below

$$P_\bullet^+ :\ 0 \to \mathbb{Z} \xrightarrow{\mathbf{m}} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/m\mathbb{Z} \to 0 \textit{ and } P_\bullet :\ 0 \to \mathbb{Z} \xrightarrow{\mathbf{m}} \mathbb{Z} \to 0.$$

Thus,

$$P_\bullet \otimes \mathbb{Z}/n\mathbb{Z} :\ 0 \to \mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \xrightarrow{\mathbf{m} \otimes \mathbb{Z}/n\mathbb{Z}} \mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \to 0$$

[‖] Of course, this equivalence are not holds for the modules over a non-commutative ring.

since $\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \simeq \mathbb{Z}/n\mathbb{Z}$, we obtain

$$0 \to \mathbb{Z}/n\mathbb{Z} \xrightarrow{\mathbf{m}} \mathbb{Z}/n\mathbb{Z} \to 0.$$

Hence,

$$\mathrm{Tor}_0^{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = \frac{\mathrm{Ker}(\mathbb{Z}/n\mathbb{Z} \to 0)}{\mathrm{Im}(\mathbb{Z}/n\mathbb{Z} \xrightarrow{\boldsymbol{m}} \mathbb{Z}/n\mathbb{Z})} \simeq \frac{\mathbb{Z}/n\mathbb{Z}}{(\mathbb{Z}/\frac{n}{\gcd\{m,n\}}\mathbb{Z})} \simeq \frac{\mathbb{Z}}{\gcd\{m,n\}\mathbb{Z}},$$

and similarly,

$$\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = \frac{\mathrm{Ker}(\mathbb{Z}/n\mathbb{Z} \xrightarrow{\boldsymbol{m}} \mathbb{Z}/n\mathbb{Z})}{\mathrm{Im}(0 \xrightarrow{\boldsymbol{m}} \mathbb{Z}/n\mathbb{Z})} \simeq \frac{\mathbb{Z}/\gcd\{m,n\}\mathbb{Z}}{\langle 0 \rangle} \simeq \frac{\mathbb{Z}}{\gcd\{m,n\}\mathbb{Z}}\ .$$

Let $P_\bullet\ :\quad \ldots \xrightarrow{\partial_{i+2}} P_{i+1} \xrightarrow{\partial_{i+1}} P_i \xrightarrow{\partial_i} P_{i-1} \xrightarrow{\partial_{i-1}} \ldots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \to 0$ be a projective resolution of A-module M. Let fix an A-module N. If we apply $\mathrm{Hom}_A(\_, N)$ to the complex $P_\bullet$, we obtain following chain complex over A

$$\begin{aligned}\mathrm{Hom}_A(P_\bullet, N):\quad 0 &\to \mathrm{Hom}_A(P_0, N) \xrightarrow{\mathrm{Hom}_A(\partial_1, N)} \mathrm{Hom}_A(P_1, N) \xrightarrow{\mathrm{Hom}_A(\partial_2, N)} \ldots \\ \ldots &\to \mathrm{Hom}_A(P_{i-1}, N) \xrightarrow{\mathrm{Hom}_A(\partial_{i-1}, N)} \mathrm{Hom}_A(P_i, N) \xrightarrow{\mathrm{Hom}_A(\partial_i, N)} \ldots\ .\end{aligned}$$

Where, homological degrees of the modules in the chain complex $\mathrm{Hom}_A(P_\bullet, N)$ start with 0 at $\mathrm{Hom}_A(P_0, N)$ and decreasing. For instanca, homological degree of $\mathrm{Hom}_A(P_{i-1}, N)$ is $1 - i$.

For each $i \in \mathbb{Z}$ we define

$$\mathrm{Ext}_A^i(M, N) = H_{-1}(Hom_A(P_\bullet, N)) = \frac{\mathrm{Ker}(\mathrm{Hom}_A(\partial_{i+1}, N))}{\mathrm{Im}(\mathrm{Hom}_A(\partial_i, N))}.$$

Similar to what we explained for Tor functor, $\mathrm{Ext}_A^i(M, N) \in \mathtt{Mod}_A$ for each $N \in \mathtt{Mod}_A$ and for each $i \in \mathbb{Z}$. Thus, $Ext_A^i(\_, N)$ is a functor from the category $\mathtt{Mod}_A$ to itself such that the action of the functor over the objects and over the morphisms looks as follows.

$$\begin{array}{rcl}\mathrm{Ext}_A^i(\_, N) : \mathbf{Obj}(\mathtt{Mod}_A) & \longrightarrow & \mathbf{Obj}(\mathtt{Mod}_A) \\ \cup & & \cup \\ M & \longmapsto & \mathrm{Ext}_A^i(M, N)\end{array}$$

and

$$\begin{array}{ccc} \mathrm{Ext}^i_A(\_, N)\colon \mathbf{Mor}_{\mathtt{Mod}_A}(M, M') & \longrightarrow & \mathbf{Mor}_{\mathtt{Mod}_A}(\mathrm{Hom}_A(M', N), \mathrm{Hom}_A(M, N)) \\ \Cup & & \Cup \\ \partial & \longmapsto & \mathrm{Hom}_A(\partial, N) \end{array}$$

Considering the order of the indices on the chain complexes $P_\bullet$, $P_\bullet \otimes_A N$ and $\mathrm{Hom}_A(P_\bullet, N)$ it is clear that $\mathrm{Tor}^A_i(\_, N)$ preserves the direction of the morphisms but $Ext^i_A(\_, N)$ reverses the direction of the morphisms. Therefore, $\mathrm{Tor}^A_i(\_, N)$ is a covariant functor, while $Ext^i_A(\_, N)$ is a contravariant functor.

Let us now give the theorem and its corollary expressing how the Betti numbers for any module M over the polynomial ring $S = \mathbb{K}[x_1, \ldots, x_n]$ can be computed using the Tor functor.

Theorem 1.4.10. *Let M be an $\mathbb{N}$-graded module over $A = \mathbb{K}[x_1, \ldots, x_n]$. Then we have,*

$$\beta_{i,j} = \dim_{\mathbb{K}}(\mathrm{Tor}^A_i(M, \mathbb{K}))_j.$$

Proof. Let $\mathfrak{m}$ denotes the graded maximal ideal $(x_1, \ldots, x_n) \subset S$ and

$$F_\bullet : \quad \ldots \to F_i \xrightarrow{d_i} F_{i-1} \to \ldots \xrightarrow{d_1} F_0 \to 0$$

be a minimal graded free resolution of M and note that $\mathbb{K} \simeq S/\mathfrak{m}$. Consider the chain complex $F_\bullet \otimes_A \mathbb{K}$. Where, $F_i \otimes \mathbb{K} = \left(\bigoplus_{j\in\mathbb{N}} S(-j)^{\beta_{i,j}}\right) \otimes_A \mathbb{K} \simeq \bigoplus_{j\in\mathbb{N}} \mathbb{K}(-j)^{\beta_{i,j}}$ and all induced differential maps in the complex $F_\bullet \otimes_A \mathbb{K}$ are identically $\mathbf{0}$. Thus, we have the following complex,

$$\ldots \xrightarrow{\mathbf{0}} \bigoplus_{j\in\mathbb{N}} \mathbb{K}(-j)^{\beta_{i+1,j}} \xrightarrow{\mathbf{0}} \bigoplus_{j\in\mathbb{N}} \mathbb{K}(-j)^{\beta_{i,j}} \xrightarrow{\mathbf{0}} \bigoplus_{j\in\mathbb{N}} \mathbb{K}(-j)^{\beta_{i-1,j}} \xrightarrow{\mathbf{0}} \ldots \xrightarrow{\mathbf{0}} \bigoplus_{j\in\mathbb{N}} \mathbb{K}(-j)^{\beta_{0,j}} \to 0.$$

Thus,

$$\mathrm{Tor}^S_i(M, \mathbb{K})_j = \mathrm{H}_i(F_\bullet \otimes \mathbb{K})_j = \frac{\mathrm{Ker}(\mathbf{0})}{\mathrm{Im}(\mathbf{0})} = \frac{\mathbb{K}(-j)^{\beta_{i,j}}}{\langle 0 \rangle} = \mathbb{K}(-j)^{\beta_{i,j}}.$$

Therefore, $\beta_{i,j} = \dim_{\mathbb{K}}(\mathrm{Tor}^S_i(M, \mathbb{K}))_j$. □

Corollary 1.4.11. *Let M be an $\mathbb{N}$-graded module over the polynomial ring $A = \mathbb{K}[x_1, \ldots, x_n]$. Then,*

$$\beta_i = \dim_{\mathbb{K}}(\mathrm{Tor}^S_i(M, \mathbb{K})).$$

The Theorem 1.4.10 and the corollary above for modules over the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ is generalized below for any ring. In addition, the following theorem shows how the Betti numbers and the Ext functor are related.

Theorem 1.4.12. *[20, Theorem 11.2] Let M be a finitely generated graded A-module. Then,*

$$\begin{aligned}\beta_i(M) =& \textit{number of the minimal generators of } Syz_i^A(M)\\ =& \dim_{\mathbb{K}}(\operatorname{Tor}_i^A(M,\mathbb{K}))\\ =& \dim_{\mathbb{K}}(\operatorname{Ext}_A^i(M,\mathbb{K})).\end{aligned}$$

1.5 Regularity and depth

In this section we introduce Castelnuovo-Mumford regularity, regular sequence and depth. The depth is an algebraic property of modules and ideals, which serves a tool for investigating Cohen-Macaulay rings in the next section.

Definition 1.5.1. *Let M be an A-module. Castelnuovo-Mumford regularity $reg(M)$ of M is defined by*

$$reg(M) = \max\{j - i : \beta_{i,j}(M) \neq 0\}.$$

In the case of the Castelnuovo-Mumford regularity of an ideal $I \subseteq A$, we treat the ideal as a module over the ring A and use Definition 1.5.1.

Before giving the definition of depth, let us give the definition of a regular sequence.

Definition 1.5.2. *Let A be a ring and let M be a module over A. A sequence of elements in $f_1, \ldots, f_k \in A$ is called a regular sequence in A or the sequence $f_1, \ldots, f_k$ is called an M-sequence if*

(i) $(f_1, \ldots, f_k)M \neq M$, and

(ii) f_{i+1} is a non-zero divisor on $M/(f_1M + \ldots + f_iM)$ for all $0 \leq i < k$.

If M is a graded $\mathbb{K}$-algebra, Definition 1.5.2 and the following are equivalent to each other: $f_1, \ldots, f_k$ are algebraically independent over $\mathbb{K}$ and M is a free $\mathbb{K}[f_1, \ldots, f_k]$ - module [21, Definition 5.6].

Let A be a Noetherian ring, M be an A-module and I be an ideal of A such that $IM \neq M$, then each maximal M-sequences in I has the same length. In other words, $\inf\{i \geq 0 : \operatorname{Ext}_A^i(A/I, M) \geq 0\}$ (see [12, Chapter A.4]). This fact gives us the ground for the following definition.

Definition 1.5.3. *Let M be a finitely generated module over the Noetherian ring A. Let $I \subseteq A$ be an ideal with $IM \neq M$, then the I-depth of M, denoted by* $\operatorname{depth}_I(M)$, *is*

$$\operatorname{depth}_I(M) = \mathit{inf}\{i \geq 0 : \operatorname{Ext}^i_A(A/I, M) \neq 0\}.$$

If $IM = M$ we set $\operatorname{depth}_I(M) = \infty$, *and if A is a local ring with the unique maximal ideal $\mathfrak{m}$, we call depth of M instead of I-depth of M. In this case we simply denote depth of M by* $\operatorname{depth}(M)$.

Definition of I-depth of the finitely generated module over the Noetherian ring also can be given as

$$\operatorname{depth}_I(M) = \sup\{k \geq 0 : \text{there exists an } M\text{-sequence } f_1, \dots, f_k \text{ in } I\}.$$

1.6 Dimension and Cohen-Macaulayness

Let $A \neq 0$ be a ring. A finite sequence of $n+1$ prime ideals $\mathfrak{p}_n \subset \mathfrak{p}_{n-1} \subset \dots \mathfrak{p}_1 \subset \mathfrak{p}_0$ is called a *prime chain* of lenght n. Let us denote $\operatorname{Spec}(A)$ as the set of all prime ideals of A. If $\mathfrak{p} \in \operatorname{Spec}(A)$, $\sup\{n : \mathfrak{p} \supset \mathfrak{p}_1 \supset \dots \supset \mathfrak{p}_n$ is a prime chain of length $n\}$ is called *height* of $\mathfrak{p}$ and we denote it by $\operatorname{height}(\mathfrak{p})$. Note that a minimal prime has height zero. Since every proper ideal of a ring lies inside a maximal ideal, and maximal ideals are also prime, then we can define *height* of an ideal $I \subset A$ as to be the minimum of the heights of the prime ideals containing the ideal, i.e.,

$$\operatorname{height}(I) = \inf\{\operatorname{height}(\mathfrak{p}) : \mathfrak{p} \supseteq I\}.$$

The *Krull dimension* of A is defined to be the supremum of the heights of the prime ideals of A, we denote the Krull dimension of A by $\dim(A)$:

$$\dim(A) = \sup\{\operatorname{height}(\mathfrak{p}) : \mathfrak{p} \in \operatorname{Spec}(A)\}.$$

Mostly, we call *dimension* of A instead of Krull dimension of A. If $\dim(A) \leq \infty$ then it is equal to the lenght of one of the longest prime chain in A. For instance, any principal ideal domain has Krull dimension one. It is immediate to obtain from the definition that $\dim(A/I) \leq \dim(A) - \operatorname{height}(I)$ for any ideal $I \subset A$.

The annihilator of M over A is $\operatorname{Ann}_A(M) = \{f \in A : fM = 0\}$. We define the *dimension* of an A-module M by

$$
dim(M) = \begin{cases} \dim(A/\operatorname{Ann}_A(M)) & \text{if } M \neq 0 \\ -1 & \text{if } M = 0. \end{cases}
$$

Let A be a Noetherian ring and $M \neq 0$ be a finitely generated A-module, then the following conditions are equivalent [18]:

(I) M is an A-module of finite length,

(II) the ring $A/\operatorname{Ann}_A(M)$ is Artinian,

(III) $\dim(M) = 0$.

Let A be a local ring with unique maximal ideal $\mathfrak{m}$ and $I \subseteq A$ be an ideal, then

$$
\operatorname{depth}_I(A) \leq \operatorname{height}(I) \leq \operatorname{height}(\mathfrak{m}) = \dim(A).
$$

Furthermore, if M is an A-module, $\operatorname{depth}(M) \leq \dim(M)$.

Being Cohen-Macaulay is defined for both local rings and modules over a local ring. After the material we have given so far, we are now ready to give a definition of a Cohen-Macaulay ring and a Cohen-Macaulay module.

Definition 1.6.1. *Let* $(A, \mathfrak{m})$ *be a local ring. A is called Cohen-Macaulay ring if* $\operatorname{depth}(A) = \dim(A)$.

Recall where that we consider A as a module over itself and we mean $\operatorname{depth}_{\mathfrak{m}}(A) = \dim(A)$ as we stated in the Definition 1.5.3.

Definition 1.6.2. *The module* M *over a local ring* $(A, \mathfrak{m})$ *is said to be Cohen-Macaulay if* $\operatorname{depth}(M) = \dim(M)$.

Namely, a ring A is *Cohen-Macaulay ring* if A is a Cohen-Macaulay module viewed as a module over itself.

Theorem 1.6.3. *[18, Theorem 31] Let A be a Cohen-Macaulay local ring with unique maximal ideal* $\mathfrak{m}$. *Then, for every proper ideal* $I \subset A$, *we have*

$$
\operatorname{height}(I) = \operatorname{depth}_I(A) \quad \textit{and} \quad \operatorname{height}(I) + \dim(A/I) = \dim(A).
$$

Let us present the theorem that describes the relationship between the height and depth of a Cohen-Macaulay local ring. Following proposition states that, Cohen-Macaulay property on a local ring A is preserved under the ring extension by a transcendental element over A.

Proposition 1.6.4. *Let $(A, \mathfrak{m})$ be a Noetherian local ring. A is Cohen-Macaulay if and only if the polynomial ring $A[x]$ is Cohen-Macaulay.*

Let us continue by giving the definition of projective dimension to state the *Auslander-Buchsbaum* formula. We then give the theorem expressing how the graded Betti numbers obtained from the minimal graded free resolution of the graded ideal are used to calculate the projective dimension.

Definition 1.6.5. *Let M be an A-module. The projective dimension of M is*

$$\operatorname{proj\,dim}_A(M) = \inf \left\{ n \geq 0 \;\middle|\; \begin{array}{c} \textit{there exists a projective resolution} \\ 0 \to P_n \to P_{n-1} \to \ldots \to P_1 \to P_0 \to M \to 0. \end{array} \right\}.$$

For brevity, $\operatorname{proj\,dim}(I)$ is used instead of $\operatorname{proj\,dim}_A(I)$ if it does not cause confusion.

Theorem 1.6.6. *Let $S = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring over the field $\mathbb{K}$ with n-variables, and $I \subseteq$ be a graded ideal. Then,*

$$\operatorname{proj\,dim}(I) = \max\{i : \beta_{i,j} \neq 0\}.$$

Proof. Assume that the projective dimension of I is n. Then, there exist a projective resolution of I with lengt n:

$$0 \to P_n \to P_{n-1} \to \ldots \to P_1 \to P_0 \to I.$$

Since every free module is projective, then this resolution is minimal if an only if each P_i is a free module. Namely,

$$P_i \simeq \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{i,j}} \quad \text{for all } i \in \{1, \ldots, n\}.$$

Considering the minimal resolution, $\beta_{i,j} \neq 0$ for some j if and only if there are $S(-j)$ components in P_i. Hence, if $\operatorname{proj\,dim}(I) = n$, there exists $\beta_{n,j} \neq 0$ for some j and for all $i > n$ we have $\beta_{i,j} = 0$.

Suppose that $\beta_{n,j} \neq 0$ for some j, this implies that we have the following free resoluiton consisting of non-trivial free modules:

$$\bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{n,j}} \to \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{n-1,j}} \to \ldots \to I \to 0.$$

Therefore, the minimal free resolution cannot be terminate before the n-th step. Namely, $\operatorname{proj}\dim(I) \geq n$. By these observations, we have

$$\operatorname{proj}\dim(I) = \max\{i : \beta_{i,j} \neq 0 \text{ for some j }\}.$$

□

That is to say, projective dimension of a graded ideal in the polynomial ring over a field is equal to the maximum index i of the graded Betti numbers $\beta_{i,j}$ of the graded ideal for which $\beta_{i,j} \neq 0$.

Now let us give the *Auslander-Buchsbaum formula*, which is the main theorem of this section. The formula, one of the fundamental results of homological and commutative algebra, establishes the relation between projective dimension and depth.

Theorem 1.6.7. *(Auslander-Buchsbaum Formula) Let A be a local Noetherian ring and M be a finitely generated module over A with finite projective dimension. Then,*

$$\operatorname{proj}\dim(M) + \operatorname{depth}(M) = \operatorname{depth}(A).$$

1.7 Simplicial complexes and Stanley-Reisner correspondence

Now we recall the one to one correspondence between monomials ideals in S and simplicial complexes on the ground set $[n] = \{1, \ldots, n\}$. To do this, we first recall some notation and terminologies about simplicial complexes.

Let $[n] = \{1, \ldots, n\} \subseteq \mathbb{N}$ be the *vertex set*. An *abstract simplicial complex* Δ is a subset of $\mathscr{P}([n])$ such that whenever $F \in \Delta$ then all subsets of F is also an element of Δ. Each element of Δ is called a *face* of Δ. If Δ contains no faces, it is called *void* complex. If $\Delta \neq \emptyset$ is a simplicial complex; for any face $F \in \Delta$, the dimension of F is defined as $|F| - 1$, thus each verticies of Δ is a face of dimension 0. The *dimension* of Δ denoted by $\dim\Delta$ and defined as the face with the maximum cardinality that can be found in Δ, namely $\dim\Delta = \max\{|F| - 1 : F \in \Delta\}$. By the definition of the dimension of simplicial complexes, $\{\emptyset\} \subseteq \mathscr{P}([n])$ is the unique simplicial complex of dimension -1. An *edge* of Δ is a face of dimension 1 and with respect to inclusion each maximal face of Δ called a *facet* and we denote the facets of Δ as $\mathcal{F}(\Delta)$. Given $\mathcal{F}(\Delta)$, Δ can be determined as taking all subsets of each facet $F \in \mathcal{F}(\Delta)$. Thus one can see facets as generator set of simplicial complexes. When $\mathcal{F}(\Delta)$ given, we write $\Delta = \langle \mathcal{F}(\Delta) \rangle = \langle F : F \in \mathcal{F}(\Delta) \rangle$. For any Δ, the complex recognising a given collection of faces as facets is a subcomplex of Δ.

Let Δ be a simplicial complex on the given vertex set $[n]$ with $\dim \Delta = d-1$. For each i given that $0 \leq i \leq d-1$ the *i-th skeleton* of the complex Δ is the simplicial complex $\Delta^{(i)}$ on the same vertex set with $[n]$ whose faces are those faces of F of Δ with $|F| \leq i+1$. A *pure* complex is a simplicial complex with all facets of same dimension. A *nonface* of a given simplicial complex Δ is an element F of $\mathscr{P}([n])$ with $N \notin \Delta$ and we denote minimal nonfaces of Δ as $\mathcal{N}(\Delta)$. The subcomplex $\Delta(i)$ of Δ is called *pure i-th skeleton*, its faces are those faces F of Δ with $|F| = i+1$. Namely,

$$\Delta(i) = \{F \in \Delta : \dim(F) = i\} \cup \{f \subseteq F : f \in \Delta, \dim(f) = i\}.$$

Now we will give a fascinating correspondence between commutative algebra and combinatorics. The theory known as *Stanley-Reisner Theory* reveals the surprising relationship between simplicial complexes and squarefree monomial ideals.

Let $S = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring in n variables over the field $\mathbb{K}$ and Δ be a simplicial complex on $[n]$. For each $F \subset [n]$ we can attach the monomial $x_F = \prod_{i \in F} x_i$. Then, it is immediate that simplicies and squarefree monomials are in natural bijection.

The *Stanley-Reisner* ideal of a given simplicial complex Δ is the ideal $I_\Delta \subset S$ generated by nonfaces of Δ. It is obvious that, instead of taking all monomials nonfaces as generators,

Instead of taking the monomials derived from nonfaces, it is sufficient to take only the monomials described by the minimal nonfaces. It is obvious that both generates the same ideal. Then, the Stanley-Reisner ideal of the simplicial complex Δ is

$$I_\Delta = (x_F : F \in \mathcal{N}(\Delta)).$$

The other ideal defined by Δ is generated by those monomials which are determined by facets of Δ. We call the ideal $I(\Delta)$ as *facet ideal* of Δ and it is defined as follows:

$$I(\Delta) = (x_F : F \in \mathcal{F}(\Delta)).$$

Determining the minimal nonfaces of a simplicial complex is very difficult even for simple simplicial complexes, especially when attempted by hand. Therefore, we will use some other methods to calculate the Stanley-Reisner ideal described by Δ.

Now we will explain the concept called Alexander-Duality, which is very useful to calculate the Stanley-Reisner ideal of a simplicial complex.

1.8 The Alexander duality

Let Δ be a simplicial complex on the vertex set $[n]$, we denoted $\Delta^\vee$ as *Alexander Dual* of Δ and it is defined as another simplicial complex admitting the complements of nonfaces of Δ as its own faces. Let us denote $\bar{F} = [n]\backslash F$ for all faces of Δ. Then,

$$\Delta^\vee = \{\bar{F} : F \notin \Delta\}.$$

It can be easily shown that $(\Delta^\vee)^\vee = \Delta$ and facets of the Alexander dual of Δ, i.e $\mathcal{F}(\Delta^\vee)$, is the set consists of complements of minimal nonfaces of Δ, in other words

$$\mathcal{F}(\Delta^\vee) = \{\bar{F} : F \in \mathcal{N}(\Delta)\}.$$

Complement of a given simplicial complex Δ is generated by complements of facets in Δ and we denote it as follows:

$$\bar{\Delta} = \left\langle \bar{F} : F \in \mathcal{F}(\Delta) \right\rangle.$$

As a fact, $I_{\Delta^\vee} = I(\bar{\Delta})$ is a well-known fact that can be found in [12, Lemma 1.5.3]. Now, for each subset $F \subseteq [n]$ we define the squarefree monomial ideal as follows:

$$P_F = (x_i : i \in F).$$

Lemma 1.8.1. *[12] The standard primary decomposition of I_Δ is*

$$I_\Delta = \bigcap_{F \in \mathcal{F}(\Delta)} P_{\bar{F}}.$$

1.9 Linear resolution and linear quotients

Definition 1.9.1. *Let $S = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring over the field $\mathbb{K}$ with n variables, M be a graded S-module, and d be a non-negative integer. We say that M has a d-linear resolution if the graded minimal free resolution of M is the following form:*

$$0 \longrightarrow S(-d-r)^{\beta_r} \longrightarrow \cdots \longrightarrow S(-d-1)^{\beta_1} \longrightarrow S(-d)^{\beta_0} \longrightarrow M \longrightarrow 0.$$

Definition 1.9.2. *[12] Let $I \subset S$ be a graded ideal. If there exists a system of homogeneous generators $f_1, \ldots, f_m \in G(I)$ such that the colon ideal $(f_1, \ldots, f_m) : f_i$ is generated by linear forms* * *for all $i \in \{1, \ldots, m\}$, then we say that the ideal I has linear quotients.*

Definition 1.9.3. *Let Δ be a simplicial complex over the vertex set $[n]$, then face ring $\mathbb{K}[\Delta]$ defined by*

$$\mathbb{K}[\Delta] = \mathbb{K}[x_1, \ldots, x_n]/I_\Delta \ .$$

Definition 1.9.4. *Let Δ be a simplicial complex. Δ is a Cohen-Macaulay complex if the face ring $\mathbb{K}[\Delta]$ is Cohen-Macaulay.*

Let Δ be a simplicial complex on $[n]$ of dimension $d-1$. We say that Δ is *sequentially Cohen-Macaulay* simplicial complex if $\Delta(i)$ is Cohen-Macaulay for all $i \in [n]$.

Now, we are ready to state the Eagon-Reiner theorem

Theorem 1.9.5 (Eagon-Reiner)**.** *Let Δ be a simplicial complex on the vertex set $[n]$ and let $\mathbb{K}$ be a field. Then the Stanley-Reisner ideal $I_\Delta \subset \mathbb{K}[x_1, \ldots, x_n]$ has a linear resolution if and only if the face ring of $\Delta^\vee$, that is, $\mathbb{K}[\Delta^\vee]$ is Cohen-Macaulay.*

Now we talk about a larger class of ideals that encompasses the class of ideals with the linear resolution property. These are ideals with linear quotients. Each ideal in this bigger class has a d-linear resolution.

Definition 1.9.6. *Let $S = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring over $\mathbb{K}$ with n-variables and $I \subset S$ be a graded ideal. We say that I has linear quotients, if there exists a system of homogeneous generators $f_1, \ldots, f_m \in I$ such that the colon ideal $(f_1, \ldots, f_{i-1}) : f_i$ is generated by linear forms for all i.*

Example 1.9.7. Let $S = \mathbb{K}[x, y, z]$, $I = (xy, xz, yz) \subset S$. Set $f_1 = xy,\ f_2 = xz,\ f_3 = yz$. For $j = 2$ we have $(f_1) : f_2 = (xy) : xz = (y)$, for $j = 3$ we have $(f_1, f_2) : f_3 = (xy, xz) : yz = (x)$. Then I has linear quotients.

Let $I \subset S$ and $\mathfrak{m} = (x_1, \ldots, x_n)$ the graded maximal ideal of S. If I is a graded ideal then we define

$$I_{<j>} = (\{f \in I : f \text{ is homogeneous polynomial and } \deg(f) = j\})\,,$$

*Where linear forms means that $\mathbb{K}$-linear combinations of variables.

and we say that a graded ideal $I \subset S$ is *componentwise linear* if $I_{<j>}$ has a linear resolution for all j. We know that by the [12, Lemma 8.2.10], if $I \subset S$ is a graded ideal with linear resolution, then $\mathfrak{m}I$ has a linear resolution. Therefore, any ideal with linear resolution are componentwise linear.

Let us continue with the *exterior algebra* information needed to give a definition of *Koszul complex*. Let M be an A-module. Recall that the tensor algebra of M is a graded algebra and defined by $T(M) = \bigoplus_{n\in\mathbb{N}} (\bigotimes^n M)$, where we set $\bigotimes^0 M = A$, $\bigotimes^1 M = M$ and $\bigotimes^n M = \left(\bigotimes^{n-1} M\right) \otimes_A M$. Wedge product $\wedge$ of two elements is an alternating product, that is $m_1 \wedge m_2 = -m_2 \wedge m_1$ for all $m_1, m_2 \in M$. We construct exterior algebra of M by adding this alternating product relation on the tensor algebra, which does not hold in the tensor algebra. Let us denote $\mathcal{L}$ by submodule of $M \otimes_A M$ generated by all elements of the form $m \otimes m$, that is, $\mathcal{L} = \langle m \otimes m : m \in M\rangle$. We denote $m \wedge n$ by the image of $m \otimes n$ under the natural A-module epimorphism $M \otimes_A M \longrightarrow\!\!\!\!\!\rightarrow (M \otimes_A M)/\mathcal{L}$.

Definition 1.9.8. *The exterior algebra $\bigwedge(M)$ is defined to be a the quotient algebra $T(M)/\mathcal{L}$ with the wedge product.*

We define *k-th exterior power* of M to be the image of $\bigotimes^k M$ in $\bigwedge(M)$ and we denote it by $\bigwedge^k M$. Thus we have

$$\bigwedge^k M \simeq \frac{\bigotimes^k M}{\mathcal{L} \cap \bigotimes^k M} = \frac{\bigotimes^k M}{\mathcal{L}^k},$$

where $\mathcal{L}^k = \mathcal{L} \cap \bigotimes^k M$, that is, considering the $\mathbb{N}$-graded structure over $T(M)$, all homogeneous elements of degree k in $\mathcal{L}$. The exterior algebra is also an $\mathbb{N}$-graded A-algebra. Indeed, $\bigwedge(M)$ inherits its grading from the tensor algebra $T(M)$.

Lemma 1.9.9. *Let M be an A-module. Then,*

$$m_1 \wedge m_2 = -m_2 \wedge m_1$$

in $\bigwedge(M)$ for all $m_1, m_2 \in M$.

Proof. By the definition of the exterior product, $m \otimes m \mapsto 0 \in \bigwedge(M)$. By the definition of wedge product, $m \wedge m = 0$ for any $m \in M$. Thus, for all $m_1, m_2 \in M$,

$$\begin{aligned} 0 &= (m_1 + m_2) \wedge (m_2 + m_1) \\ &= (m_1 \wedge m_1) + (m_1 \wedge m_2) + (m_2 \wedge m_1) + (m_2 \wedge m_2) \\ &= m_1 \wedge m_2 + m_2 \wedge m_1. \end{aligned}$$

Hence, we obtain $m_1 \wedge m_2 = -m_2 \wedge m_1$ in $\bigwedge(M)$ for all $m_1, m_2 \in M$. $\square$

Proposition 1.9.10. *If $m \in \bigwedge^i M$ and $n \in \bigwedge^j M$, then $m \wedge n \in \bigwedge^{i+j} M$ and $m \wedge n = (-1)^{i+j} n \wedge m$.*

Proof. Suppose that $m = m_1 \wedge \ldots \wedge m_i$ and $n = n_1 \wedge \ldots \wedge n_j$. By Lemma 1.9.9, for each alternating commutation operation in $(m_1 \wedge \ldots \wedge m_i) \wedge (n_1 \wedge \ldots \wedge n_j)$ we get one multiplier (-1) in the product. Thus,

$$\begin{aligned}
m \wedge n =& m_1 \wedge \ldots \wedge m_i \wedge n_1 \wedge \ldots \wedge n_j \\
=& (-1)^i n_1 \wedge m_1 \wedge \ldots \wedge m_i \wedge n_2 \wedge \ldots \wedge n_j \\
=& (-1)^i (-1)^i n_1 \wedge n_2 \wedge m_1 \wedge \ldots \wedge m_i \wedge n_3 \wedge \ldots \wedge n_j \\
& \vdots \\
=& ((-1)^i)^j n_1 \wedge \ldots \wedge n_j \wedge m_1 \wedge \ldots \wedge m_i \\
=& (-1)^{i+j} n \wedge m.
\end{aligned}$$

Therefore, $m \wedge n = (-1)^{i+j} n \wedge m$ for all $m \in \bigwedge^i M$ and all $n \in \bigwedge^j M$. $\square$

Now, we are ready to construct the *Koszul complex*. Let $f_1, \ldots, f_r$ be a sequence of elements in the ring A. We set $\mathcal{K}_i = \bigwedge^i A^r$, the r-th exterior power of the free A-module A^r. Let $\{e_1, \ldots, e_r\}$ be the standard basis for A^r. We define differential maps as follows,

$$d_q : \bigwedge^q A^r \longrightarrow \bigwedge^{q-1} A^r$$

such that, $d_q(e_{j_1} \wedge \ldots \wedge e_{j_\ell}) = \sum_{k=1}^{r} (-1)^{k+1} f_{j_k} (e_{j_1} \wedge \ldots \wedge \widehat{e_{j_k}} \wedge \ldots \wedge e_{j_\ell})$ where $\widehat{e_{j_k}}$ means that e_{j_k} is omitted in the wedge product. Also, $d_q \circ d_{q+1} = 0$ for all $q \geq 0$ is a well-known fact, see for example [20],[6] or [7].

The chain complex $\mathcal{K}_\bullet(f_1, \ldots, f_r)$ called *Koszul complex* on the sequence $f_1, \ldots, f_r$ and defined as follows

$$\mathcal{K}_\bullet(f_1, \ldots, f_r) : \quad 0 \to \mathcal{K}_q \xrightarrow{d_q} \mathcal{K}_{q-1} \xrightarrow{d_{q-1}} \ldots \xrightarrow{d_2} \mathcal{K}_1 \xrightarrow{d_1} \mathcal{K}_0 \to 0 \ ,$$

and note that the elements

$$\{e_{j_1}, \ldots e_{j_\ell} : 1 \leq j_1 < \ldots < j_\ell \leq q\}$$

forms an A-basis for each $\mathcal{K}_i$. Therefore,

$$\operatorname{rank}_A(\mathcal{K}_i) = \binom{q}{i}.$$

Let M be a finitely generated A-module, the *Koszul homology* of M is defined by $\mathcal{K}_\bullet(f_1, \ldots, f_r; M) = \mathcal{K}_\bullet(f_1, \ldots, f_r) \otimes_A M$. Any permutation action on the sequence $f_1, \ldots, f_r$ gives us another Koszul complex by simply permuting the basis of the exterior algebra. We now have the necessary material to prove the following proposition.

Proposition 1.9.11. *[12, Proposition 8.2.1] Assume that the ideal $I \subset S$ is a graded and generated by degree d elements and that I has linear quotients. Thus the ideal I has a d-linear resolution.*

Proof. Let $f_1, \ldots, f_m$ be a system of generators of I where $\deg(f_j) = d$ for all $j \in \{1, \ldots, m\}$, and let us denote $I_k = (f_1, \ldots, f_k)$ and $L_k = (f_1, \ldots, f_{k-1}) : f_k$. Assume that L_k is generated by linear forms for all k. We show by induction on k that I_k has a d-linear resolution. It is immediate to see that the assertion is holds for $k = 1$. Suppose now that $k > 1$ and let the set of linear forms $\{\ell_1, \ldots, \ell_r\}$ generates L_k minimally[†]. It is easy to see that the sequence $\ell_1, \ldots, \ell_r$ is regular. Indeed, we can complete $\{\ell_1, \ldots, \ell_r\}$ to a $\mathbb{K}$-basis $\{\ell_1, \ldots, \ell_n\}$ for S. Thus, we have a $\mathbb{K}$-vector -space automorphism given as $\varphi : S \hookrightarrow\!\!\!\!\rightarrow S$ with $\varphi(x_i) = \ell_i$ for all $i \in \{1, \ldots, n\}$. Since $x_1, \ldots, x_r$ is a regular sequence it implies that $\ell_1 = \varphi(x_1), \ldots, \ell_r = \varphi(x_r)$ is a regular sequence as well.

The Koszul complex $K(\ell_1, \ldots, \ell_r; S)$ ensures a minimal graded free resolution of S/L_k since the sequence $\ell_1, \ldots, \ell_r$ is regular (see, [12, Theorem A.3.4.]). This implies that

$$\operatorname{Tor}_i^S((S/L_k)(-d), \mathbb{K})_{i+j} \simeq \operatorname{Tor}_i^S(S/L_k, \mathbb{K})_{i+(j-d)} = 0 \quad \text{for} \quad j \neq d.$$

Our aim is to show that $\operatorname{Tor}_i(I_k, \mathbb{K})_{i+j} = 0$ for all i and all $j \neq d$. Since I is generated in degree d, we have $I_k/I_{k-1} \simeq (S/L_k)(-d)$, so that we have the following short exact sequence

$$0 \longrightarrow I_{k-1} \longrightarrow I_k \longrightarrow (S/L_k)(-d) \longrightarrow 0 \ .$$

The short exact sequence above implies the long exact sequence below,

$$\operatorname{Tor}_i^S(I_{k-1}, \mathbb{K})_{i+j} \longrightarrow \operatorname{Tor}_i^S(I_k, \mathbb{K})_{i+j} \longrightarrow \operatorname{Tor}_i^S((S/L_k)(-d), \mathbb{K})_{i+j} \ . \qquad (1.1)$$

[†]Observe that $\{\ell_1, \ldots, \ell_r\}$ is $\mathbb{K}$-linearly independent.

By applying our induction hypothesis on k, we observe that both ends in this exact sequence vanish for $j \neq d$. Thus this also holds for the term $\mathrm{Tor}_i^S(I_k, \mathbb{K})_{i+j}$ in the sequence. This is what we desired. $\square$

Chapter 2

t-spread strongly stable monomial Ideals

In this part of the thesis, we will give a new interpretation of the squarefreeness of a monomial in the terms of the t-spreadness of the indices of the variables in its support. Below we recall the definition of t-spread monomial introduced in [11] and some related notions.

2.1 t-spread monomial ideals.

Throughout this section, let $S = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring over a field $\mathbb{K}$. Let t be a non-negative integer. A monomial $x_{i_1} \cdots x_{i_d} \in S$ with $i_1 < \cdots < i_d$ is called *t-spread* monomial if $i_j - i_{j-1} \geq t$ for all $j \in \{2, \ldots, d\}$. It is easy to see that any monomial can be regarded as a 0-spread monomial and any squarefree monomials as a 1-spread. Moreover, any t-spread monomial is also a $(t-1)$-spread monomial. A monomial ideal $I \subset S$ is called *t-spread monomial ideal* if it is generated by t-spread monomials.

Example 2.1.1. Let $I = (x_1x_5x_9, x_1x_5x_8, x_2x_6x_{10}, x_2x_8)$ be a monomial ideal in $\mathbb{K}[x_1, \ldots, x_{10}]$. Note that I is a 3-spread monomial ideal. However I is not a 4-spread monomial ideal because $x_1x_5x_9$ is not a 4-spread monomial.

We denote by $\mathrm{Mon}(S)$ the set of monomials in S. If $I \subseteq S$ is an ideal, we denote by $\mathrm{Mon}(I)$ the set of monomials in I. The set of t-spread monomials of S is denoted by $\mathrm{Mon}(S, t)$. Then we have $\mathrm{Mon}(S) = \mathrm{Mon}(S; 0)$.

Let T denotes the polynomial ring $\mathbb{K}[x_1, x_2, \ldots]$ in infinitely many variable with variables indexed by natural numbers.

Definition 2.1.2. *[11] Let $u = \prod_{j=1}^{d} x_{i_j} \in T$ with $i_1 \leq i_2 \leq \cdots \leq i_d$. Then we define the maps $\sigma : Mon(T;t) \to Mon(T;t+1)$ and $\tau : Mon(T;t) \to Mon(T;t-1)$ by*

$$\sigma(u) = \prod_{j=1}^{d} x_{i_j+(j-1)} \ ,$$

$$\tau(u) = \prod_{j=1}^{d} x_{i_j-(j-1)}.$$

It is immediate to see that σ and τ are inverse of each other. In other words, we can see σ (resp. τ) as a bijection of $\mathrm{Mon}(T)$. If we iterate the map σ, obviously $\sigma^t : \mathrm{Mon}(T) \hookrightarrow\!\!\!\!\rightarrow \mathrm{Mon}(T;t)$ becomes a bijection between all monomials and t-spread monomials of T.

Definition 2.1.3. *Let I be a t-spread monomial ideal. The ideal generated by the monomials $\sigma(u)$ with $u \in G(I)$ is denoted by I^{σ}.*

2.2 t-spread strongly stable ideals

Let u be a monomial in the polynomial ring $S = \mathbb{K}[x_1, \ldots, x_n]$, maximum index i for which x_i divides u denoted by $\max(u)$; $\min(u)$ defined similarly to $\max(u)$ that is, the minimum index for which $x_i \mid u$.

Definition 2.2.1. *A t-spread monomial ideal $I \subset S$ is called t-spread stable, if for all t-spread monomials $u \in I$ and for all $i < \max(u)$ such that $x_i(u/x_{\max(u)})$ is a t-spread monomial, it follows that $x_i(u/x_{\max(u)}) \in I$.*

Definition 2.2.2. *A t-spread monomial ideal $I \subset S$ is called t-spread strongly stable, if for all t-spread monomials $u \in I$ and for all $j \in supp(u)$ and all $i < j$ such that $x_i(u/x_j)$ is a t-spread monomial, it follows that $x_i(u/x_j) \in I$.*

Example 2.2.3. *Let us find smallest 2-spread strongly stable monomial ideal which contains $u = x_1x_4$ and $v = x_2x_4x_8$. Let I be the smallest 2-spread strongly stable monomial ideal which contains the monomials u and v. If we apply the conditions on u and v given in the Lemma 2.2.6, we need extra x_1x_3, $x_2x_4x_6$ and $x_2x_4x_7$ in $G(I)$. Then we obtain, $I = (x_1x_3, x_1x_4, x_2x_4x_6, x_2x_4x_7, x_2x_4x_8)$ as a 2-spread strongly stable monomial ideal.*

It is not difficult to observe that I^σ is actually $(t+1)$-spread ideal. Given that I is strongly stable, the most natural question after defining I^σ is whether I^σ has the strongly stability property. The next proposition guarantees us that I^σ is a strongly stable ideal.

Proposition 2.2.4. *[11, Proposition 1.9] Let $I \subset S$ be a monomial ideal. If I is a t-spread strongly stable ideal, this implies that I^σ is a $(t+1)$-spread strongly stable monomial ideal of S as well.*

To facilitate the terminology in the subsequent text, we introduce the following definition.

Definition 2.2.5. *A Borel move or Borel exchange move on a monomial u in S is an operation which maps u to $(x_i/x_j)u$ where $j \in supp(u)$ and $i < j$.*

Lemma 2.2.6. *[11, Lemma 1.2] Let $I \subset S$ be a t-spread monomial ideal. The following conditions are equivalent:*

(a) I is t-spread strongly stable.

(b) If $u \in G(I), j \in supp(u)$ and $i < j$ such that $x_i\,(u/x_j)$ is a t-spread monomial, then $x_i\,(u/x_j) \in I$.

Proof. (a) $\Rightarrow$ (b) is immediate to see by the definition of t-spread strongly stability of an ideal. Let us prove (b) $\Rightarrow$ (a). Let $u \in I$ be a t-spread monomial. Under the Borel move, we get $u' = x_i(u/x_j)$ where $i < j$. Assume that u' is a t-spread monomial. Let $v \in G(I)$ and v divides u. There are exactly two possible case, $x_j \in \text{supp}(v)$ and $x_j \notin \text{supp}(v)$. Now, let us consider these two cases separately.

If $x_j \notin \text{supp}(v)$, thus we have v divides u' and $u' \in I$. Or else, if $x_j \in \text{supp}(v)$, then $v' = x_i(v/x_j) \in I$ by our assumption and v' divides u', hence we have $u' \in I$. □

Next, we will show that t-spread strongly stable ideals are componentwise linear. To do this, we first recall following definitions.

A *monomial order* on S as a total order $<$ on $\text{Mon}(S)$ such that,

- $1 < u$ for all $1 \neq u \in \text{Mon}(S)$;
- if $u, v \in \text{Mon}(S)$ and $u < v$, then $uw < vw$ for all $w \in \text{Mon}(S)$.

We recall a special monomial order ofen used in the literature (for more details see,[11, 2.1.2]).

Definition 2.2.7. *Let* $\mathbf{c} = (c_1, \ldots, c_k)$ *and* $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{Z}_+^n$. *We define the total order* $<_{lex}$ *by setting* $\mathbf{x}^{\mathbf{c}} <_{lex} \mathbf{x}^{\mathbf{d}}$ *if the leftmost nonzero component of the vector* $\mathbf{c} - \mathbf{d}$ *is negative. By this way* $Mon(S)$ *becomes a totally ordered set with* $<_{lex}$. *Namely,* $<_{lex}$ *is a monomial order on* S, *which is called pure lexicographic order on* S *induced by the ordering* $x_1 > x_2 > \cdots > x_{k-1} > x_k$.

To prove the componentwise linearity of t-spread strongly stable ideals, we first state the following lemma.

Lemma 2.2.8. *[11, Lemma 1.3] Consider a* t*-spread strongly stable ideal* I *in the polynomial ring* $S = \mathbb{K}[x_1, \ldots, x_n]$, *and let* $\omega \in I$ *be a* t*-spread monomial. Then,* ω *can be expressed as* $\omega = \omega_1\omega_2$, *where* $\omega_1 \in G(I)$, $\omega_2 \in Mon(S)$, *and* $\max(\omega_1) < \min(\omega_2)$.

Now we give main theorem of this section.

Theorem 2.2.9. *[11, Theorem 1.4] Let* $I \subset S$ *be a* t*-spread strongly stable ideal. Then,* I *possesses linear quotients. Consequently, the ideal* I *is componentwise linear.*

Proof. Let the minimal generating set of I be $G(I) = \{\omega_1, \omega_2, \ldots, \omega_m\}$. Where, we put pure lexicographic order on $G(I)$. We set $\ell \leq m$ and $K = (\omega_1, \ldots, \omega_{\ell-1})$. In order to show that $K : \omega_\ell$ is generated by variables, it is enough to show that for all $k \in \{1, \ldots, \ell-1\}$ there exists $x_i \in K : \omega_\ell$ such that x_i divides $\omega_k/\gcd(\omega_k, \omega_\ell)$ by the [12, Proposition 1.2.2]. Let $\omega_k = x_{i_1}x_{i_2}\cdots x_{i_s}$ with $i_1 \leq i_2 \leq \cdots \leq i_s$ and $\omega_\ell = x_{j_1}x_{j_2}\cdots x_{j_t}$ with $j_1 \leq j_2 \leq \cdots \leq j_t$. There exists a δ such that $1 \leq \delta \leq t$ with $i_1 = j_1, \ldots, i_{\delta-1} = j_{\delta-1}$ and $i_\delta < j_\delta$, since $\omega_k >_{\text{lex}} \omega_\ell$. Let $\nu = x_{i_\delta}(\omega_r/x_{j_\delta})$. Then $\nu = x_{j_1}x_{j_2}\cdots x_{j_{\delta-1}}x_{i_\delta}x_{j_{\delta+1}}\cdots x_{j_t}$. Since $i_\delta - j_{\delta-1} = i_\delta - i_{\delta-1} \geq t$ and $j_{\delta+1} - i_\delta > j_{\delta+1} - j_\delta \geq t$, it follows that ν is t-spread, and so $\nu \in I$ and $\nu >_{\text{lex}} \omega_\ell$. In fact, $\nu \in K$. Indeed, by Lemma 2.2.8, there exists $\omega_h \in G(I)$ such that $\nu = \omega_h \cdot \xi$ and $\max(\omega_h) < \min(\xi)$. Suppose that $\nu \notin K$. Then $\omega_h \leq_{\text{lex}} \omega_\ell$. From the presentation of $\nu = \omega_h \cdot \xi$, it gives that $\nu \leq_{\text{lex}} \omega_h$, this is a contradiction.

Now, as we know that ν is an element of K, it implies that $x_{i_\delta} \in K : \omega_\ell$. Since x_{i_δ} divides $\omega_k/\gcd(\omega_k, \omega_\ell)$, we are done. □

The following result, which we will use for the Theorem 2.2.12, is a natural consequence of [16, Lemma 1.5].

Lemma 2.2.10. *[11] Let* I *be a monomial ideal with linear quotients. Then*

$$\beta_{i,i+j}(I) = |\{\alpha \subset set(u) : u \in G(I)_j \text{ and } |\alpha| = i\}|$$

where $G(I)_j = \{u \in G(I) : deg(u) = j\}$.

Let I be a t-spread strongly stable ideal with $G(I) = \{u_1, \ldots, u_m\}$ ordered with respect to the pure lexicographic order. We define

$$\text{set}(u_k) = \{i : x_i \in (u_1, \ldots, u_{k-1}) : u_k\}$$

as the collection of indices of the variables appearing in those colon ideals.

By the Theorem 2.2.9, $\text{set}(u_k)$ is the set of positive integers satisfying

$$i < \max(u_k),\ i \notin \text{supp}(u_k) \text{ and } i - j \geq t \text{ for all } j \in supp(u_k) \text{ with } j < i. \quad (2.1)$$

Example 2.2.11. *Let us consider the ideal which constructed in the Example 2.2.3 and compute the* $\text{set}(u)$ *and* $\text{set}(v)$*.*

$\text{set}(x_1x_4) = \{3\}$ *since* $(x_1x_3) : x_1x_4 = (x_3)$ *then* $x_3 \in (x_1x_3) : x_1x_4$ *and* $\text{set}(x_1x_4x_7) = \{1, 2, 3, 4, 5, 6, 7, 8\}$ *since* $(x_1x_3, x_1x_4, x_1x_4x_6) : x_1x_4x_7 = (1) = S$*.*

Next, we show that Betti numbers t-spread strongly stable ideals are preserved under sigma operator.

Theorem 2.2.12. *[11, Theorem 1.11] Let* I *be a* t*-spread strongly stable ideal. Then* $\beta_{i,i+j}(I) = \beta_{i,i+j}(I^\sigma)$ *for all* i *and* j*.*

Proof. Let $\omega = x_{i_1}x_{i_2}\cdots x_{i_g} \in G(I)$. Let $\text{set}(\omega) = \{c_1, \ldots, c_\rho\}$ with the ordering $c_1 < \cdots < c_\rho$ and let

$$d_i = c_i + \max\{\lambda : i_\lambda < c_i\}$$

for all $i \in \{1, \ldots, \rho\}$. To complete the proof, let us proceed by demonstrating the following:

$$d_1 < \cdots < d_\rho \text{ and } \text{set}(\sigma(\omega)) = \{d_1, \ldots, d_\rho\}. \quad (2.2)$$

Let $r < j$ and $i_\lambda < c_r < i_{\lambda+1}$ and $i_m < c_j < i_{m+1}$. Thus, we obtain immediately $m \geq \lambda$ and $d_j - d_r = c_j + m - (c_r + \lambda) = c_j - c_r + (m - \lambda) > 0$. If we combine (2.2) and the Lemma 2.2.10, the proof is complete.

Now, our aim is to show that $d_i \in \text{set}(\sigma(\omega))$. If $c_i \in \text{set}(\omega)$ and $i_\lambda < c_i < i_{\lambda+1}$, then it is immediate to obtain $c_i - i_\lambda \geq t$ by (2.1). This means that, $i_\lambda + (\lambda - 1) < c_i + \lambda < i_{\lambda+1} + \lambda$ and $c_i + \lambda - (i_\lambda + (\lambda - 1)) \geq t + 1$. Since, $d_i = c_i + \lambda$, this shows that $d_i \in \text{set}(\sigma(\omega))$.

Conversely, let e be an element in $\text{set}(\sigma(\omega))$. Hence, by 2.1, we observe that there exists an integer λ for which $i_\lambda +\ (\lambda - 1) < e < i_{\lambda+1} + \lambda$ and moreover, $e - (i_\lambda + (\lambda - 1)) \geq t + 1$. This gives us the inequalities $i_\lambda < e - \lambda < i_{\lambda+1}$ and $(e - \lambda) - i_\lambda \geq t$. Therefore, $e - \lambda = c_i$ for some integer i and $e = c_i + \lambda = d_i$. This concludes the proof. □

Corollary 2.2.13. *[11, Corollary 1.12] Let $J \subset S$ be a t-spread strongly stable ideal. Thus we have,*

$$\beta_{i,i+j}(J) = \sum_{\omega \in G(J)_j} \binom{\max(\omega) - t(j-1) - 1}{i}.$$

Proof. We know that J^{τ^t} is strongly stable. From [8], we know that

$$\beta_{i,i+j}\left(J^{\tau^t}\right) = \sum_{\omega \in G(J^t)_j} \binom{\max(\omega) - 1}{i}.$$

By Theorem 2.2.12, we have $\beta_{i,i+j}\left(J^t\right) = \beta_{i,i+j}(J)$, therefore,

$$\beta_{i,i+j}(J) = \sum_{\omega \in G(J)_j} \binom{\max\left(\tau^t(\omega)\right) - 1}{i}.$$

The proof follows, because $\max\left(\tau^t(\omega)\right) = \max(\omega) - t(\deg(\omega) - 1)$, for all $\omega \in G(J)$. □

Chapter 3

t-spread Borel ideals and their powers

In this chapter, we introduce t-spread Borel ideals, and some special subclasses of t-spread Borel ideals, namely, t-spread principal Borel ideals and t-spread Veronese ideals. Then we give the main theorem of this section which gives the exact form of the elements in the generating set of the Alexander dual of a t-spread Borel ideal and a t-spread Veronese ideal. Moreover, we will investigate some algebraic properties of t-spread Veronese ideals such as, height, Cohen-Macaulayness and Betti numbers.

3.1 t-spread Veronese ideals

Let $\mathcal{M} = \{u_1, \ldots, u_m\}$ be a set of t-spread monomials in $S = \mathbb{K}[x_1, \ldots, x_n]$. The smallest t-spread strongly stable monomial ideal containing $\mathcal{M}$ with respect to inclusion, is called *t-spread Borel ideal* and denoted by $B_t(u_1, \ldots, u_m)$ or $B_t(\mathcal{M})$. The monomials $u_1, \ldots, u_m$ are called the *t-spread Borel generators* of $B_t(u_1, \ldots, u_m)$. Moreover, if $|\mathcal{M}| = 1$, then $I = B_t(\mathcal{M})$ is called *t-spread principal Borel ideal*.

Example 3.1.1. *Let $I = B_2(u_1, u_2) \subset \mathbb{K}[x_1, \ldots, x_9]$ where the monomials are chosen as $u_1 = x_1x_3x_{11}$ and $u_2 = x_2x_5x_8$. The monomials in the minimal set of generators of I are given below:*

$x_1x_3x_{11}$	$x_1x_4x_8$	$x_1x_5x_8$	$x_2x_4x_8$	$x_2x_5x_8$
$x_1x_3x_{10}$	$x_1x_4x_7$	$x_1x_5x_7$	$x_2x_4x_7$	$x_2x_5x_7$
$x_1x_3x_9$	$x_1x_4x_6$		$x_2x_4x_6$	
$x_1x_3x_8$				
$x_1x_3x_7$				
$x_1x_3x_6$				
$x_1x_3x_5$				

As we will in see the following example, the t-spread Borel ideal described by a given collection of monomials can be represented as a principal Borel ideal.

Example 3.1.2. *Let $I = B_2(u_1, u_2) \subset \mathbb{K}[x_1, \ldots, x_9]$ where $u_1 = x_2x_4x_8$ and $u_2 = x_3x_7x_9$. Then the minimal generators of I and minimal generators of $B_2(u_2)$ are same. Indeed, with the appropriate Borel moves, u_1 can be obtained from u_2. More precisely, $u_2 = x_3x_7x_9$ gives $v_1 = x_2x_7x_9$ after exchanging x_3 with x_2. In a similar way, we obtain $v_2 = x_2x_4x_9$ from v_1 after exchanging x_7 with x_4. Lastly, we obtain $u_1 = x_2x_4x_8$ from v_2 after exchanging x_9 with x_8. Therefore, $G(B_2(u_2)) = G(B_2(u_1, u_2))$ and I is t-spread principal Borel ideal.*

Recall that if an ideal I is t-spread strongly stable, then I^σ is also t-spread strongly stable. Next, we show that the set of Borel generators of a t-spread strongly stable are preserved under the operator σ.

Proposition 3.1.3. *[11, Proposition 2.1] Let $I \subset S$ be a t-spread Borel ideal $I = B_t(\mathcal{M})$ with $\mathcal{M} = \{\nu_1, \ldots, \nu_m\}$. Then $I^\sigma = B_{t+1}(\sigma(\mathcal{M}))$ where $\sigma(\mathcal{M}) = \{\sigma(\nu_1), \ldots, \sigma(\nu_m)\}$.*

Proof. Let $\omega \in G(I)$ and $\omega = x_{\ell_1} \cdots x_{\ell_d}$. Then there exists a monomial $\nu_r = x_{k_1} \cdots x_{k_d}$ such that $\ell_s \leq k_s$ for all $s \in \{1, \ldots, d\}$ since I is a t-spread strongly stable monomial ideal. It is immediate to see that $\ell_s + (s-1) \leq i_s + (s-1)$ for all $s \in \{1, \ldots, d\}$. Thus, we have the following inclusion

$$\sigma(\omega) \in B_t(\sigma(\nu_r)) \subseteq B_{t+1}(\sigma(\mathcal{M})).$$

Since we know that I^σ is generated by the elements of type $\sigma(\omega)$ with $\omega \in G(I)$, indeed it follows that $I^\sigma \subseteq B_{t+1}(\mathcal{M})$. As well, $B_{t+1}(\mathcal{M})$ is the smallest $(t+1)$-spread strongly stable ideal containing $\mathcal{M} = \{\sigma(\nu_1), \ldots, \sigma(\nu_m)\}$. This means that, $B_{t+1}(\mathcal{M}) = B_{t+1}(\sigma(\nu_1), \ldots, \sigma(\nu_m)) = B_{t+1}(\sigma(\mathcal{M})) \subseteq I^\sigma$ since $\sigma(\nu_1), \ldots, \sigma(\nu_m) \in I^\sigma$ and I^σ is a $(t+1)$-spread strongly stable monomial ideal. □

It is not difficult to observe the following fact about the elements of the minimal generating set of a principal Borel ideal: if $\omega = x_{k_1} \cdots x_{k_d}$, necessary and sufficient conditions for $x_{\ell_1} \cdots x_{\ell_d} \in G(B_t(u))$ are

- $\ell_1 \leq k_1, \ldots, \ell_d \leq k_d$ and;
- $\ell_s - \ell_{s-1} \geq t$ for $s \in \{2, \ldots, d\}$.

Now we give the definition of a special class of t-spread principal Borel ideal which is called *t-spread Veronese* ideal.

Definition 3.1.4. *Let* $\mathrm{g} \in \mathbb{Z}$ *with* $\mathrm{g} > 0$. *If a monomial ideal in the polynomial ring* $S = \mathbb{K}[x_1, \ldots x_{\mathrm{N}}]$ *is generated by all t-spread monomials of degree* g, *then the ideal is called a t-spread* Veronese ideal of degree g, *and we denote it by* $\mathcal{I}_{\mathrm{N,g},t}$.

Below we give a straightforward proof of the fact that any t-spread Veronese ideal of degree g is in fact a t-spread principal Borel ideal.

Proposition 3.1.5. *Let $\mathcal{I}$ be an ideal of $S = \mathbb{K}[x_1, \ldots, x_{\mathrm{N}}]$. Then, $\mathcal{I}$ is a t-spread Veronese ideal $\mathcal{I} = \mathcal{I}_{\mathrm{N,g},t}$ if and only if $\mathcal{I}$ is a t-spread principal Borel ideal $\mathcal{I} = B_t(\omega)$ for $\omega = x_{\mathrm{N}-(\mathrm{g}-1)t} x_{\mathrm{N}-(\mathrm{g}-2)t} \cdots x_{\mathrm{N}}$.*

Proof. Assume that $\mathcal{I} = \mathcal{I}_{\mathrm{N,g},t}$ is a t-spread Veronese ideal. Then $\mathcal{I}$ is generated by all t-spread monomials of degree g. Let $v \in G(\mathcal{I}_{\mathrm{N,g},t})$. We will show that, v can be obtained from u by applying Borel move. If we target the smaller indexed variable in u with a Borel move, t-spreadness property of the monomial will be preserved. Then, let us define

$$u_1 = \frac{x_{min(v)}}{x_{min(u)}} u = \frac{x_{min(v)}}{x_{\mathrm{N}-(\mathrm{g}-1)t}} u.$$

In this way, the smallest indices of v and u_1 are equalised. Moreover, u_1 is a t-spread monomial since we would replace the smallest indexed variable of a t-spread monomial with a smaller indexed variable of S. Next, we will target the second smallest indexed variable of u_1. If we denote $\mathrm{supp}(v) = \{i_1 < i_2 < \ldots < i_{\mathrm{g}}\}$, then $x_{\min(v)} = x_{i_1}$ and the second smallest indexed variable of v is x_{i_2}. Let us define,

$$u_2 = \frac{x_{i_2}}{x_{\mathrm{N}-(\mathrm{g}-2)t}} u_1.$$

In this way, we guaranteed that the first two smallest indices of u_2 and v are equalized. Moreover, u_2 is also a t-spread monomial since the difference

$$\mathrm{N} - (\mathrm{g} - 2)t \; - \; i_2 \geq \; t.$$

If we continue to applying this process, one can see that at the end of the g-th step of the process, we will be obtain v. Namely, $u_{\mathrm{g}} = v$. Therefore, under the appropriate Borel moves, u will be transformed any monomial in the minimal generating set of $\mathcal{I}_{\mathrm{N},\mathrm{g},t}$. This shows that, $v \in B_t(u)$, so $\mathcal{I}_{\mathrm{N},\mathrm{g},t} \subseteq B_t(u)$.

By the definition, the converse inclusion follows immediately since Among all t-spread strongly stable ideals that contain the monomial u, $B_t(u)$ is the smallest one and, $u \in \mathcal{I}_{\mathrm{N},\mathrm{g},t}$ since it is a t-spread monomial of degree g in S. This completes the proof. □

We can find the primary decomposition of a given t-spread principal Borel ideal by viewing it as a Stanley-Reisner ideal of a suitable simplicial complex. Let $\mathcal{I} = B_t(\omega)$ be a t-spread principal Borel ideal for a given $\omega \in \mathrm{Mon}(S)$, and let Δ be the simplicial complex over [N] whose Stanley-Reisner ideal is $\mathcal{I}$. Then

$$\mathcal{I} = \bigcap_{F \in \mathcal{F}(\Delta)} P_{\bar{F}}$$

recall that where $\bar{F} = [\mathrm{N}] \backslash F$ and, $P_{\bar{F}} = \left(x_\ell : \ell \in \bar{F}\right) \in \mathrm{Spec}(S)$.

Theorem 3.1.6. *[11, Theorem 2.3] Let $t \in \mathbb{Z}$ with $0 < t$ and $\mathcal{I}_{\mathrm{N},\mathrm{g},t} \subset S$ be the t-spread Veronese ideal generated in degree* g *where, $S = \mathbb{K}[x_1, \ldots, x_{\mathrm{N}}]$. In order to ensure consistency is easily attained, we make the assumption of*

$$\bigcup_{\omega \in G\left(\mathcal{I}_{\mathrm{N},\mathrm{g},t}\right)} supp(\omega) = \{1, \ldots, \mathrm{N}\}.$$

The subsequent results are as follows:

(a) $\mathrm{height}\left(\mathcal{I}_{\mathrm{N},\mathrm{g},t}\right) = \mathrm{N} - t(\mathrm{g} - 1)$.

(b) $\mathcal{I}^{\vee}_{\mathrm{N},\mathrm{g},t}$ *is generated by the following type of monomials*

$$\prod_{\alpha=1}^{\mathrm{N}} x_\alpha / \left(\mathcal{V}_{i_1,t} \cdots \mathcal{V}_{i_{\mathrm{g}-1},t}\right) = \frac{x_1 \cdots x_{\mathrm{N}}}{\mathcal{V}_{i_1,t} \cdots \mathcal{V}_{i_{\mathrm{g}-1},t}} \text{ with } i_{\ell+1} - i_\ell \geq t, \text{ for } \ell \in \{1, \ldots, \mathrm{g}-2\}$$

where $\mathcal{V}_{i_r,t} = x_{i_r} x_{i_r+1} \cdots x_{i_r+t-1}$ *for* $r \in \{1, \ldots, \mathrm{g}-1\}$.

(c) $\mathcal{I}_{\mathrm{N},\mathrm{g},t}$ *has Cohen-Macaulay property and it has a linear resolution.*

(d) $\beta_i\left(S/\mathcal{I}_{\mathrm{N},\mathrm{g},t}\right) = \binom{\mathrm{g}+i-2}{\mathrm{g}-1}\binom{\mathrm{N}-(t-1)(\mathrm{g}-1)}{\mathrm{g}+i-1}$ *for all integer i with $i > 0$. In particular,* $\mu\left(\mathcal{I}_{\mathrm{N},\mathrm{g},t}\right) = \binom{\mathrm{N}-(t-1)(\mathrm{g}-1)}{\mathrm{g}}$.

Proof. Consider the simplicial complex Δ with Stanley-Reisner ideal of Δ is $\mathcal{I}_{\mathrm{N,g},t}$, and let $\mathcal{F}(\Delta)$ denote the set of facets of Δ. We demonstrate that each facet H of Δ can be expressed in the form provided below:

$$\begin{aligned}\mathrm{H} = \{&\ell_1, \ell_1+1, \ldots, \ell_1+(t-1), \ell_2, \ell_2+1, \ldots, \ell_2+(t-1), \ldots, \\ &\ell_{\mathrm{g}-1}, \ell_{\mathrm{g}-1}+1, \ldots, \ell_{\mathrm{g}-1}+(t-1)\}\end{aligned}$$

for some $\ell_1, \ell_2, \ldots, \ell_{\mathrm{g}-1}$ such that $\ell_l \leq i_l$ for $l \in \{1, \ldots, \mathrm{g}-1\}$ and $j_l - j_{l-1} \geq t$ for $l \in \{2, \ldots, \mathrm{g}-1\}$.

This shows that all the facets of Δ have the same cardinality, that is $|\mathrm{H}| = t(\mathrm{g}-1)$ for all $\mathrm{H} \in \mathcal{F}(\Delta)$, thus $\dim \Delta = t(\mathrm{g}-1)-1$. This means that $\dim(S/\mathcal{I}_{\mathrm{N,g},t}) = t(\mathrm{g}-1)$, thus height $(\mathcal{I}_{\mathrm{N,g},t}) = \mathrm{N} - t(\mathrm{g}-1)$ which clearly proves and supports the validity of (a). Furthermore, $\mathcal{I}_{\mathrm{N,g},t}$ has the following primary decomposition

$$\mathcal{I}_{\mathrm{N,g},t} = \bigcap_{\mathrm{H} \in \mathcal{F}(\Delta)} P_{\bar{\mathrm{H}}},$$

where $\bar{\mathrm{H}}$ denotes the complement of H, that is, $\bar{\mathrm{H}} = [\mathrm{N}] \backslash \mathrm{H}$ and $P_{\bar{\mathrm{H}}}$ is the monomial prime ideal generated by all variables of the form x_ℓ such that $\ell \in [\mathrm{N}] \backslash \mathrm{H}$ (see: [12, Corollary 1.5.5]).

We show that every set

$$\mathrm{H} = \{\ell_1, \ldots \ell_1+(t-1), \ell_2, \ldots, \ell_2+(t-1), \ldots \ell_{\mathrm{g}-1}, \ldots, \ell_{\mathrm{g}-1}+(t-1)\} \tag{3.1}$$

for some $\ell_1, \ldots, \ell_{\mathrm{g}-1}$ such that $\ell_b - \ell_{b-1} \geq t$ for $b \in \{2, \ldots, \mathrm{g}-1\}$ is a facet of Δ. We have $\mathrm{H} \in \Delta$ since $x_{\mathrm{H}} = \prod_{\alpha \in \mathrm{H}} x_\alpha \notin \mathcal{I}_\Delta$, where $\mathcal{I}_\Delta$ is the Stanley-Reisner ideal of the simplicial complex Δ. In addition, we claim that $\mathrm{H} \cup \{\ell\} \notin \Delta$ for every $\ell \in [\mathrm{N}] \backslash \mathrm{H}$. This will show that H is indeed a facet of Δ. Let $\ell \in [\mathrm{N}] \backslash \mathrm{H}$. In the situation where $\ell < \ell_1$, we also obtain

$$x_\ell x_{\ell_1+(t-1)} \cdots x_{\ell_{\mathrm{g}-1}+(t-1)} \in \mathcal{I}_\Delta$$

hence the set $\{\ell, \ell_1+(t-1), \ldots, \ell_{\mathrm{g}-1}+(t-1)\}$ is a non-face of Δ, then it is immediate to see that $\mathrm{H} \cup \{\ell\} \notin \Delta$. If $\ell \geq \ell_{\mathrm{g}-1}+t$, we get the non-face $\{\ell_1, \ldots, \ell_{\mathrm{g}-1}, \ell\}$, thus $\mathrm{H} \cup \{\ell\} \notin \Delta$. Lastly, if there exists $b \in \{2, \ldots, \mathrm{g}-1\}$ such that $\ell_{b-1}+(t-1) < \ell < \ell_b$, then $\{\ell_1, \ldots, \ell_{b-1}, \ell, \ell_b+(t-1), \ldots, \ell_{\mathrm{g}-1}+(t-1)\}$ is a nonface of Δ. As a result, $\mathrm{H} \cup \{\ell\} \notin \Delta$. Consequently, we have established that each set H, as described in the (3.1), indeed constitutes a facet of the simplicial complex Δ. This demonstrates that the sets satisfying the given conditions in (3.1) are precisely the facets of the simplicial complex Δ.

Our objective is to demonstrate that the sets described in equation (3.1) are the sole facets present in our simplicial complex. This is equivalent to proving that for every face Γ in Δ, there exists a facet H in $\mathcal{F}(\Delta)$ having the form specified in (3.1), that includes the face Γ as a subset.

Let $\Gamma \in \Delta$ and $i_1 = \min \Gamma$. Inductively, for $b \geq 2$, we set $\gamma_b = \min\{\gamma \in \Gamma : \gamma \geq \gamma_{b-1} + t\}$.

Maximum possible number of elements in the sequence $\gamma_1 < \gamma_2 < \cdots$ is $\mathrm{g} - 1$. Otherwise, $\Gamma \supseteq \{\gamma_1, \ldots, \gamma_{\mathrm{g}}\}$ with $\gamma_b \geq \gamma_{b-1} + t$ for $2 \leq b \leq \mathrm{g}$. But $\{\gamma_1, \ldots, \gamma_{\mathrm{g}}\} \notin \Delta$ since $x_{\gamma_1} \cdots x_{\gamma_{\mathrm{g}}} \in \mathcal{I}_\Delta$. Thus $\Gamma \notin \Delta$, a contradiction. Therefore, Γ has the form

$$\Gamma = \{\gamma_1, \gamma_1 + 1, \ldots, \gamma_1 + k_1, \ldots, \gamma_r, \gamma_r + 1, \ldots, \gamma_r + k_r\}$$

for some $r \leq \mathrm{g} - 1$, $k_1, \ldots, k_r \in \{0, \ldots, t-1\}$, and $\gamma_b \geq \gamma_{b-1} + t$ for $b \in \{2, \ldots, r\}$. Obviously,

$$\Gamma \subseteq \{\gamma_1, \gamma_1 + 1, \ldots, \gamma_1 + (t-1), \ldots, \gamma_{r-1}, \gamma_{r-1} + 1, \ldots, \gamma_{r-1} + (t-1), \gamma_r, \ldots, \gamma_r + k\}.$$

Let us denote Γ' as the right hand side of the inclusion above, namely,

$$\Gamma' = \{\gamma_1, \gamma_1 + 1, \ldots, \gamma_1 + (t-1), \ldots, \gamma_{r-1}, \gamma_{r-1} + 1, \ldots, \gamma_{r-1} + (t-1), \gamma_r, \ldots, \gamma_r + k\},$$

and, $\Gamma \subseteq \Gamma'$ here, we identify $k = k_r$.

To complete the proof of the theorem, we prove the following claim.

Claim. For $r \leq \mathrm{g} - 2$, there is a face $\Xi \in \Delta$, where the relationship $\Xi \supset \Gamma' \supset \Gamma$ holds true, along with the following

$$\Xi = \left\{\gamma_1', \ldots, \gamma_1' + (t-1), \ldots, \gamma_r', \gamma_r' + 1, \ldots, \gamma_r' + (t-1), \gamma_{r+1}', \ldots, \gamma_{r+1}' + k'\right\}$$

for some $0 \leq k' \leq t - 1, \gamma_1' \leq t$, and $\gamma_b' \geq \gamma_{b-1}' + t$ for $2 \leq b \leq r + 1$.

Proof. (Proof of the Claim.) If $\min \Gamma' - 1 = \gamma_1 \geq t$, then $\Gamma \subset \Gamma' \subset \Xi \in \Delta$ where $\Xi = \{1, \ldots, t\} \cup \Gamma'$ and our claim immediately follows. Assume initially that $\gamma_1 \leq t$ and let $\gamma_b = \gamma_{b-1} + t$ for each b in the set $\{2, \ldots, r\} \subset \mathbb{N}$. Then

$$\gamma_r = \gamma_1 + (r-1)t \leq rt \leq (\mathrm{g} - 2)t \leq \mathrm{N} - t - 1.$$

In the previous inequality, we incorporated the condition $\mathrm{N} \geq 1 + (\mathrm{g} - 1)t$, which establishes a lower bound for N by ensuring that it is at least one more than the product of degrees of the monomials in the generating set and our spreading t thereby providing a crucial constraint for our calculations. Since we obtain $\gamma_r + t \leq \mathrm{N} - 1$, we may consider Ξ as follows:

$$\Xi = \{\gamma_1, \gamma_1 + 1, \ldots, \gamma_1 + (t-1), \ldots, \gamma_r, \gamma_r + 1, \ldots, \gamma_r + (t-1), \gamma_{r+1} = \gamma_r + t\}.$$

To finalize the proof of the claim, we need to address the scenario where there exists an index η such that $\gamma_\eta > \gamma_{\eta-1} + t$. Let $b = \max\{\eta : \gamma_\eta > \gamma_{\eta-1} + t\}$. Then, we obtain $\gamma_r > \gamma_{b-1} + (r - b + 1)t$ and we may set Ξ as

$$\Xi = \{\gamma_1, \ldots, \gamma_1 + (t-1), \ldots, \gamma_{b-1}, \ldots, \gamma_{b-1} + (t-1), \gamma'_b, \ldots, \gamma'_b + (t-1), \ldots,$$

$$\gamma'_r, \ldots, \gamma'_r + (t-1), \gamma'_{r+1} \ldots, \gamma'_{r+1} + \sigma'\}$$

for some $\sigma' \in \mathbb{N}$, note that, where $\gamma'_b = \gamma_{b-1} + t, \gamma'_{b+1} = \gamma_{b-1} + 2t, \ldots, \gamma'_{r+1} = \gamma_{b-1} + (r - b + 2)t$.

According to our claim, it is now apparent that each face $\Gamma \in \Delta$ is contained in a larger face Ξ of the form given in below

$$\Xi = \{\gamma_1, \ldots, \gamma_1 + (t-1), \ldots, \gamma_{\mathrm{g}-2}, \ldots, \gamma_{\mathrm{g}-2} + (t-1), \gamma_{\mathrm{g}-1}, \ldots, \gamma_{\mathrm{g}-1} + \sigma\}$$

for some natural number $\sigma \in \{0, \ldots, t-1\}$, where $\gamma_1 \leq t$, and $\gamma_b \geq \gamma_{b-1} + t$ for $b \in \{2, \ldots, \mathrm{g} - 1\}$. It remains to show that there exists $\mathrm{H} \in \mathcal{F}(\Delta)$ which contains Ξ. But this follows if we show that for every $\sigma \leq t - 2$, there exists a Δ such that H is contained in a face of the Δ of the form

$$\left\{\gamma'_1, \ldots, \gamma'_1 + (t-1), \ldots, \gamma'_{\mathrm{g}-2}, \ldots, \gamma'_{\mathrm{g}-2} + (t-1), \gamma'_{\mathrm{g}-1}, \ldots, \gamma'_{\mathrm{g}-1} + (\sigma + 1)\right\}.$$

Let $\sigma \leq t-2$. Indeed, if $\gamma_{\mathrm{g}-1} + \sigma < \mathrm{N}$, then we can immediately obtain the larger face by adding the vertex $\gamma_{\mathrm{g}-1} + (\sigma + 1)$ to Ξ. Let $\gamma_{\mathrm{g}-1} + \sigma = \mathrm{N}$. If $\gamma_b = \gamma_{b-1} + t$ for all $b \in \{2, \ldots, \mathrm{g} - 1\}$, then $\gamma_{\mathrm{g}-1} = \gamma_1 + (\mathrm{g} - 2)t$, thus $\gamma_1 + (\mathrm{g} - 2)t + \sigma = \mathrm{N}$ which consequently implies the following

$$\gamma_1 = \mathrm{N} - (\mathrm{g} - 2)t - \sigma \geq 1 + (\mathrm{g} - 1)t - (\mathrm{g} - 2)t - \sigma > 2.$$

At that point, we can set

$$\Xi \subset \left\{\gamma'_1, \ldots, \gamma'_1 + (t-1), \ldots, \gamma'_{\mathrm{g}-2}, \ldots, \gamma'_{\mathrm{g}-2} + (t-1), \gamma'_{\mathrm{g}-1}, \ldots, \gamma'_{\mathrm{g}-1} + (\sigma + 1)\right\}$$

where $\gamma'_1 = \gamma_1 - 1, \gamma'_2 = \gamma_2 - 1, \ldots, \gamma'_r = \gamma_r - 1$.

Ultimately, let us pick the maximal ℓ for which $\gamma_b > \gamma_{b-1} + t$. In this part of the proof, we set Ξ as follows,

$$\Xi \subset \left\{\gamma'_1, \ldots, \gamma'_1 + (t-1), \ldots, \gamma'_{\mathrm{g}-2}, \ldots, \gamma'_{\mathrm{g}-2} + (t-1), \gamma'_{\mathrm{g}-1}, \ldots, \gamma'_{\mathrm{g}-1} + (\sigma + 1)\right\}$$

for which $\gamma'_1 = \gamma_1, \ldots, \gamma'_{b-1} = \gamma_{b-1}, \gamma'_b = \gamma_b - 1, \gamma'_{b+1} = \gamma_{b+1} - 1, \ldots, \gamma'_{\mathrm{g}-1} = \gamma_{\mathrm{g}-1} - 1$. As a natural consequence of the Theorem 2.2.9, the t-spread Veronese ideal $\mathcal{I}_{\mathrm{N},\mathrm{g},t}$ has a linear resolution. Since we know the fact that $\mathcal{I}_{\mathrm{N,g},t}$ is generated by monomials of fixed single degree, it follows that the t-spread Veronese ideal has a linear resolution.

Subsequently, we demonstrate that the ideal $\mathcal{I}_{\mathrm{N,g},t}^{\vee}$ has linear quotients. This means that there exists a particular ordering of the minimal monomial generators $\mu_1, \ldots, \mu_k$ of $\mathcal{I}_{\mathrm{N,g},t}^{\vee}$ such that, under this ordering, the following condition is satisfied: for each $\gamma < \ell$ there exists an integer $r < \gamma$ and an integer b for which x_b divides $(\mu_\gamma/\gcd(\mu_\gamma, \mu_\ell))$ and $x_b = \mu_r/\gcd(\mu_r, \mu_\ell)$. Therefore, as established by [12, Proposition 8.2.5], the simplicial complex Δ is shellable. Consequently, [12, Theorem 8.2.6] implies that the ideal $\mathcal{I}_\Delta = \mathcal{I}_{\mathrm{N,g},t}$ is Cohen-Macaulay.

Let $G(\mathcal{I}_{\mathrm{N,g},t}^{\vee}) = \{\mu_1, \ldots, \mu_k\}$ be the minimal monomial generators of the ideal t-spread Veronese ideal and let $G(\mathcal{I}_{\mathrm{N,g},t}^{\vee})$ ordered with respect to the lexicographic order such that $\mu_1, \ldots, \mu_k$ are decreasing. Let

$$\mu_i = \prod_{\alpha=1}^{\mathrm{N}} x_\alpha / \left(\mathcal{V}_{i_1} \cdots \mathcal{V}_{i_{\mathrm{g}-1}}\right) \text{ and } \mu_j = \prod_{\alpha=1}^{\mathrm{N}} x_\alpha / \left(\mathcal{V}_{j_1} \cdots \mathcal{V}_{j_{\mathrm{g}-1}}\right)$$

where $i \neq j$ and also we removed the index t that is $\mathcal{V}_{i_r}$ denotes $\mathcal{V}_{i_r,t}$ and $\mathcal{V}_{j_r}$ denotes $\mathcal{V}_{j_r,t}$ for the sake of simplifying the notation to work easily on calculations. Following equality is immediate after some easy calculations:

$$\frac{\mu_i}{\gcd(\mu_i, \mu_j)} = \frac{\mathcal{V}_{j_1} \cdots \mathcal{V}_{j_{\mathrm{g}-1}}}{\gcd\left(\mathcal{V}_{i_1} \cdots \mathcal{V}_{i_{\mathrm{g}-1}}, \mathcal{V}_{j_1} \cdots \mathcal{V}_{j_{\mathrm{g}-1}}\right)}.$$

At this point, we use the assumption $i < j$. Hence, $\mu_i >_{\mathrm{lex}} \mu_j$, that is, $\mathcal{V}_{j_1} \cdots \mathcal{V}_{j_{\mathrm{g}-1}} >_{\mathrm{lex}} \mathcal{V}_{i_1} \cdots \mathcal{V}_{i_{\mathrm{g}-1}}$ equivalently we obtain the following condition:

There exists a non-zero natural number σ such that
$j_1 = i_1, \ldots, j_{\sigma-1} = i_{\sigma-1}$ and $j_\sigma < i_\sigma$.

We first observe that x_{j_σ} divides $(\mu_i/\gcd(\mu_i, \mu_j))$ since x_{j_σ} divides $\mathcal{V}_{j_1} \cdots \mathcal{V}_{j_{\mathrm{g}-1}}$ and it does not divide the product $\mathcal{V}_{i_1} \cdots \mathcal{V}_{i_{\mathrm{g}-1}}$ since we have the inequality $i_\sigma > j_\sigma$. Let us assume that there exists a least integer $b \leq \mathrm{g} - 2$ such that $j_{b+1} > j_b + t$. Let

$$\mu_r = \prod_{\alpha=1}^{\mathrm{N}} \frac{x_\alpha}{\left(\mathcal{V}_{j_1} \cdots \mathcal{V}_{j_{\sigma-1}} \mathcal{V}_{j_\sigma+1} \mathcal{V}_{j_\sigma+2} \cdots \mathcal{V}_{j_b+1} \mathcal{V}_{j_{b+1}} \cdots \mathcal{V}_{j_{\mathrm{g}-1}}\right)}.$$

Obviously, $\mu_r >_{\mathrm{lex}} \mu_j$, thus $r < j$, and we claim that $\mu_r/\gcd(\mu_r, \mu_j) = x_{j_\sigma}$. Following equality is immediate after a simple calculation

$$\gcd\left(\mathcal{V}_{j_1} \cdots \mathcal{V}_{j_{\sigma-1}} \mathcal{V}_{j_\sigma+1} \mathcal{V}_{j_\sigma+2} \cdots \mathcal{V}_{j_b+1} \mathcal{V}_{j_{b+1}} \cdots \mathcal{V}_{j_{\mathrm{g}-1}}, \mathcal{V}_{j_1} \cdots \mathcal{V}_{j_{\mathrm{g}-1}}\right) = \frac{\prod_{\mathrm{i}=1}^{\mathrm{g}-1} \mathcal{V}_{j_{\mathrm{i}}}}{x_{j_\sigma}}.$$

Thus we obtain,

$$\frac{\mu_r}{\gcd(\mu_r,\mu_j)} = \frac{\mathcal{V}_{j_1}\cdots\mathcal{V}_{j_{\mathrm{g}-1}}}{\left(\left(\mathcal{V}_{j_1}\cdots\mathcal{V}_{j_{\mathrm{g}-1}}\right)/x_{j_\sigma}\right)} = x_{j_\sigma}.$$

If $j_{b+1} = j_b + t$ for $b \in \{2,\ldots,\mathrm{g}-2\}$, we get

$$\mathrm{N} - t + 1 \geq i_{\mathrm{g}-1} \geq i_\sigma + (\mathrm{g}-\sigma-1)t > j_\sigma + (\mathrm{g}-\sigma-1)t = j_{\mathrm{g}-1}.$$

This means that $j_{\mathrm{g}-1} + (t-1) \leq \mathrm{N}$ indeed. Thus, we may consider the monomial $\mathcal{V}_{j_{\mathrm{g}-1}+1}$. In this case we take

$$\mu_r = \frac{\prod_{\alpha=1}^{\mathrm{N}} x_\alpha}{\left(\mathcal{V}_{j_1}\cdots\mathcal{V}_{j_{\sigma-1}}\mathcal{V}_{j_\sigma+1}\mathcal{V}_{j_\sigma+2}\cdots\mathcal{V}_{j_{\mathrm{g}-1}+1}\right)}$$

and check that $\mu_r/\gcd(\mu_r,\mu_j) = x_{j_\sigma}$. In summary, to calculate the Betti numbers of $\mathcal{I}_{\mathrm{N},\mathrm{g},t}$ and $\mathcal{I}^\vee_{\mathrm{N},\mathrm{g},t}$, we engage [5, Theorem 4.1.15]. This theorem provides the Betti numbers for a Cohen-Macaulay ideal $\mathcal{I}$ in a polynomial ring S with type - $(\mathrm{g}_1,\ldots,\mathrm{g}_q)$ pure resolution. We have the following for all $i \geq 1$

$$\beta_i\left(\frac{S}{\mathcal{I}}\right) = (-1)^{i+1}\prod_{\ell\neq i}\frac{\mathrm{g}_\ell}{\mathrm{g}_\ell - \mathrm{g}_i}.$$

In our case, the type of the resolution of $S/\mathcal{I}_{\mathrm{N},\mathrm{g},t}$ is given by $\mathrm{g}_\ell = \mathrm{g}+\ell-1$ for $1 \leq \ell \leq q = \mathrm{N} - t(\mathrm{g}-1)$. Therefore,

$$\begin{aligned}\beta_i\left(\frac{S}{\mathcal{I}}\right) &= (-1)^{i+1}\prod_{\ell=1}^{i-1}\frac{\mathrm{g}+\ell-1}{\ell-i}\prod_{\ell=i+1}^{q}\frac{\mathrm{g}+\ell-1}{\ell-i}\\ &= \frac{\mathrm{g}(\mathrm{g}+1)\cdots(\mathrm{g}+i-2)}{(i-1)!}\cdot\frac{(\mathrm{g}+i)(\mathrm{g}+i+1)\cdots(\mathrm{g}+q-1)}{(q-i)!} = \\ &= \binom{\mathrm{g}+i-2}{\mathrm{g}-1}\binom{\mathrm{N}-(\mathrm{g}-1)(t-1)}{\mathrm{g}+i-1}.\end{aligned}$$

□

Example 3.1.7. In this example, we demonstrate an application of Theorem 3.1.6 by computing the generators of $\mathcal{I}^\vee$ where

$$\mathcal{I} = \mathcal{I}_{9,3,3} \subset S = \mathbb{K}[x_1,\ldots,x_9]$$

and $(\mathrm{N},\mathrm{g},t) = (9,3,3)$.

Let us first list $\mathcal{V}_{i_k,t}$, recall that $\mathcal{V}_{i_k,t} = x_{i_k}x_{i_k+1}\cdots x_{i_k+t-1}$ for $1 \leq k \leq \mathrm{g}-1$. By the inequality $i_k + t - 1 \leq \mathrm{N}$, we get $i_k \leq 7$ as suitable non-negative integer values of i_k. Then we obtain the following.

for $i_k = 1$	for $i_k = 2$	for $i_k = 3$	for $i_k = 4$
$\mathcal{V}_{1,3}\mathcal{V}_{4,3}$	$\mathcal{V}_{2,3}\mathcal{V}_{5,3}$	$\mathcal{V}_{3,3}\mathcal{V}_{6,3}$	$\mathcal{V}_{4,3}\mathcal{V}_{7,3}$
$\mathcal{V}_{1,3}\mathcal{V}_{5,3}$	$\mathcal{V}_{2,3}\mathcal{V}_{6,3}$	$\mathcal{V}_{3,3}\mathcal{V}_{7,3}$	
$\mathcal{V}_{1,3}\mathcal{V}_{6,3}$	$\mathcal{V}_{2,3}\mathcal{V}_{7,3}$		
$\mathcal{V}_{1,3}\mathcal{V}_{7,3}$			

Thus we obtain 10 monomials as minimal generators. Let us denote each of them as ξ_j:

$$\begin{array}{llll} \xi_1 = x_7x_8x_9 & \xi_5 = x_1x_8x_9 & \xi_8 = x_1x_2x_9 & \xi_{10} = x_1x_2x_3 \\ \xi_2 = x_4x_8x_9 & \xi_6 = x_1x_5x_9 & \xi_9 = x_1x_2x_6 & \\ \xi_3 = x_4x_5x_9 & \xi_7 = x_1x_5x_6 & & \\ \xi_4 = x_4x_5x_6 & & & . \end{array}$$

For instance we get $\xi_6 = \frac{x_1 \cdots x_9}{x_2x_3x_4 \cdot x_6x_7x_8} = x_1x_5x_9$ by the formula given in (b) of 3.1.6. Then, $\mathcal{I}^\vee = (G(\mathcal{I}^\vee)) = (\xi_1, \ldots, \xi_{10})$. For other part of this example, we calculate the minimal generators of the ideal itself. For $\mathcal{I} = \mathcal{I}_{9,3,3}$, by the Proposition 3.1.5, we have $\mathcal{I} = B_3(x_3x_6x_9)$. We compute the minimal generators of the smallest 3-spread ideal containing $x_3x_6x_9$, as we did in Example 3.1.1 above, we obtain the minimal generators of $B_3(x_3x_6x_9)$ as follows:

$$\begin{array}{llll} x_3x_6x_9 & x_2x_5x_9 & x_2x_5x_8 & x_1x_4x_9 \\ x_2x_6x_9 & x_1x_5x_9 & x_1x_5x_8 & x_1x_4x_8 \\ x_1x_6x_9 & & & x_1x_4x_7 \end{array} .$$

Now we calculate the Betti numbers of $\frac{S}{\mathcal{I}}$ where $\mathcal{I} = \mathcal{I}_{9,3,3} = B_3(x_3x_6x_9)$. By the formula (d) of Theorem 3.1.6, $\beta_i(\frac{S}{\mathcal{I}_{9,3,3}}) = \binom{1+i}{2}\binom{5}{2+i}$. Then non-zero Betti numbers are:

$$\beta_1(S/\mathcal{I}_{9,3,3}) = \binom{2}{2}\binom{5}{3} = 10$$

$$\beta_2(S/\mathcal{I}_{9,3,3}) = \binom{3}{2}\binom{5}{4} = 15$$

$$\beta_3(S/\mathcal{I}_{9,3,3}) = \binom{4}{2}\binom{5}{5} = 6.$$

Then minimal free resolution of $S/\mathcal{I}_{9,3,3}$ is

$$0 \longrightarrow S^6 \longrightarrow S^{15} \longrightarrow S^{10} \longrightarrow S \longrightarrow S/\mathcal{I}_{9,3,3} \longrightarrow 0.$$

Using Theorem 3.1.6, now we can compute the height of t-spread strongly stable ideals.

Theorem 3.1.8. *[11, Theorem 2.4] Let J be a t-spread strongly stable monomial ideal. Then*

$$\text{height}(J) = \max\{\min(\omega) : \omega \in G(J)\}.$$

Proof. Consider $\omega_0 \in G(J)$ with $\min(\omega_0) = \max\{\min(\omega) : \omega \in G(J)\}$, and take

$$Q = (x_j : j \leq \min(\omega_0)).$$

Then $J \subset Q$, because for all $\eta \in G(J)$ immediately we have $\min(\eta) \leq \min(\omega_0)$. This shows that $\text{height}(J) \leq \min(\omega_0)$. On the other hand, let $\omega_0 = x_{j_1} \cdots x_{j_g}$. Thus, we have the following

$$\omega_0' = x_{j_1} x_{j_1+t} \cdots x_{j_1+t(g-1)} \in J$$

given that J is a t-spread strongly stable monomial ideal. Let $J' = B_t(\omega_0')$. Then $J' \subset J$ and Theorem 3.1.6 implies that

$$\min(\omega_0) = \min(\omega_0') = j_1 = j_1 + t(g-1) - t(g-1) = \text{height}(J') \leq \text{height}(J)$$

The proof is thus complete. □

Applying the theorem above to the Example 3.1.7, it is immediate to obtain that $\text{height}(\mathcal{I}_{9,3,3}) = \max(\{1,2,3\}) = 3$.

3.2 t-spread Borel ideals

We have seen before that any t-spread Veronese ideal is a t-spread Borel ideal. So, t-spread Borel ideals are more general class of ideals than t-spread Veronese ideals. In this section, we give a description of the generators of the Alexander dual of an arbitrary t-spread principal Borel ideal $B_t(u)$ and we use this description to show that $S/B_t(u)$ is Cohen-Macaulay if $B_t(u)$ is a t-spread Veronese ideal.

Theorem 3.2.1. *[1, Theorem 1.1] Let $t \geq 1$ be an integer and $\mathcal{I} = B_t(u)$ be the t-spread principal Borel ideal of $S = \mathbb{K}[x_1, \ldots, x_N]$ where the monomial $u = x_{\ell_1} x_{\ell_2} \cdots x_{\ell_g}$ is a t-spread monomial. We set $\bigcup_{\omega \in G(\mathcal{I})} supp(\omega) = [N]$. Then, $\mathcal{I}^\vee$ is generated by the monomials of the following forms*

$$\frac{\prod_{\alpha=1}^{N} x_\alpha}{\mathcal{V}_{\ell_1} \cdots \mathcal{V}_{\ell_{g-1}}}$$

with $\ell_l \leq \gamma_l$ *for* $1 \leq l \leq \mathrm{g}-1$ *and* $\ell_l - \ell_{l-1} \geq t$ *for* $2 \leq l \leq \mathrm{g}-1$, $\mathcal{V}_{\ell_k} = x_{\ell_k} \cdots x_{\ell_k+(t-1)}$ *for* $1 \leq k \leq \mathrm{g}-1$.

$$\prod_{\alpha=1}^{\gamma_1} x_\alpha.$$

$$\left(\prod_{\alpha=1}^{\gamma_s} x_\alpha\right) / \left(\mathcal{V}_{\ell_1} \cdots \mathcal{V}_{\ell_{s-1}}\right)$$

with $2 \leq s \leq \mathrm{g}-1$, $\ell_l \leq \gamma_l$ *for* $1 \leq l \leq s-1, \ell_l - \ell_{l-1} \geq t$ *for* $2 \leq l \leq s-1$, *where* $\mathcal{V}_{\ell_k} = x_{\ell_k} \cdots x_{\ell_k+(t-1)}$ *for* $1 \leq k \leq s-1$.

Proof. Let Δ be the simplicial complex whose Stanley-Reisner ideal is I and let $\mathcal{F}(\Delta)$ be the set of the facets of Δ. We prove that every facet of Δ is of one of the forms below:

(i) $\mathrm{H}_1 = \{\ell_1, \ell_1+1, \ldots, \ell_1+(t-1), \ell_2, \ell_2+1, \ldots, \ell_2+(t-1), \ldots, \ell_{\mathrm{g}-1}, \ell_{\mathrm{g}-1}+1, \ldots, \ell_{\mathrm{g}-1}+(t-1)\}$

for some $\ell_1, \ell_2, \ldots, \ell_{\mathrm{g}-1}$ such that $\ell_l \leq \gamma_l$ for $1 \leq l \leq \mathrm{g}-1$ and $\ell_l - \ell_{l-1} \geq t$ for $2 \leq l \leq \mathrm{g}-1$.

(ii) $\mathrm{H}_2 = \{\gamma_1+1, \gamma_1+2, \ldots, n\}$.

(iii) $\mathrm{H}_3 = \{\ell_1, \ell_1+1, \ldots, \ell_1+(t-1), \ldots, \ell_{s-1}, \ell_{s-1}+1, \ldots, \ell_{s-1}+(t-1), \ell_s, \ell_s+1, \ldots, n\}$

for some $\ell_1, \ell_2, \ldots, \ell_s$ such that $2 \leq s \leq \mathrm{g}-1, \ell_l \leq \gamma_l$ for $1 \leq l \leq s-1$, $\ell_s = \gamma_s + 1$ and $\ell_l - \ell_{l-1} \geq t$ for $2 \leq l \leq s$.

The ideal $\mathcal{I} = B_t(u)$ has the primary decomposition

$$\mathcal{I} = \bigcap_{\mathrm{H} \in \mathcal{F}(\Delta)} P_{\bar{\mathrm{H}}},$$

where $\bar{\mathrm{H}}$ denotes the complement of H, that is, $\bar{\mathrm{H}} = [\mathrm{N}] \backslash \mathrm{H}$ and $P_{\bar{\mathrm{H}}}$ is the monomial prime ideal generated by all variables x_j with $j \in [\mathrm{N}] \backslash \mathrm{H}$ (see: [12, Corollary 1.5.5]). Since $(x_1, \ldots x_{\gamma_1}) \in \mathrm{Min}(\mathcal{I})$ by Theorem 3.1.8, we obtain $\mathrm{H}_2 \in \mathcal{F}(\Delta)$. We have H_1 and $\mathrm{H}_3 \in \Delta$, since $\xi_{\mathrm{H}_1}, \xi_{\mathrm{H}_3} \notin \mathcal{I}$ where $\xi_{\mathrm{H}_1} = \prod_{i \in \mathrm{H}_1} x_i$ and $\xi_{\mathrm{H}_3} = \prod_{i \in \mathrm{H}_3} x_i$. Indeed, if $\xi_{\mathrm{H}_1} \in \mathcal{I}$, then there exists $v = x_{k_1} \cdots x_{k_\mathrm{g}} \in G(\mathcal{I})$ such that $v \mid \xi_{\mathrm{H}_1}$. Since v is the product of g distinct variables and H_1 consists of $\mathrm{g}-1$ intervals of the form $[\ell_r, \ell_r+(t-1)]$, $1 \leq r \leq \mathrm{g}-1$, there exist $l \in \{2, \ldots, \mathrm{g}\}$ and $1 \leq r \leq \mathrm{g}-1$ such that $k_{l-1}, k_l \in [\ell_r, \ell_r+(t-1)]$. This implies that $k_l - k_{l-1} < t$, which is false.

If $\xi_{\mathrm{H}_3} \in \mathcal{I}$, then there exists $v = x_{k_1} \cdots x_{k_{\mathrm{g}}} \in G(\mathcal{I})$ such that $v \mid \xi_{\mathrm{H}_3}$. Since $k_l - k_{l-1} \geq t$ for $2 \leq l \leq s$ and H_3 consists of $s-1$ intervals of the form $[\ell_r, \ell_r + (t-1)], 1 \leq r \leq s-1$, and only one interval of the form $[\ell_s, \mathrm{N}]$, we have $k_s \in [\ell_s, \mathrm{N}]$. It follows that $\ell_s = \gamma_s + 1 \leq k_s$, which is false because $v \in G(\mathcal{I})$ and $k_s \leq \gamma_s$.

We claim that $\mathrm{H}_i \cup \{j\} \notin \Delta$ for every $j \in [\mathrm{N}] \backslash \mathrm{H}_i$ and $i \in \{1, 3\}$. This will prove that every set of the form (i) or (iii) is a facet of Δ.

Let $j \in [\mathrm{N}] \backslash \mathrm{H}_1$. We have exactly three cases given as below:

(A) $j < \ell_1$. Then $x_j x_{\ell_1 + (t-1)} \cdots x_{\ell_{\mathrm{g}-1} + (t-1)} \in \mathcal{I}$ because $j < \ell_1 \leq \gamma_1$ and $\ell_l + (t-1) \leq \gamma_l + (t-1) < \gamma_{l+1}$ for $1 \leq l \leq \mathrm{g} - 1$.

(B) $j \geq \ell_{\mathrm{g}-1} + t$. Then $x_{\ell_1} x_{\ell_2} \cdots x_{\ell_{\mathrm{g}-1}} x_j \in \mathcal{I}$.

(C) There exists $1 \leq l \leq \mathrm{g} - 2$ such that $\ell_l + (t-1) < j < \ell_{l+1}$. Then

$$x_{\ell_1} x_{\ell_2} \cdots x_{\ell_l} x_j x_{\ell_{l+1} + (t-1)} \cdots x_{\ell_{\mathrm{g}-1} + (t-1)} \in \mathcal{I},$$

since $\ell_k \leq \gamma_k$ for $1 \leq k \leq l, j < \ell_{l+1} \leq \gamma_{l+1}$ and $\ell_k + (t-1) \leq \gamma_k + (t-1) < \gamma_{k+1}$ for $l + 1 \leq k \leq \mathrm{g} - 1$.

Therefore, we have proved that for $j \in [\mathrm{N}] \backslash \mathrm{H}_1, \mathrm{H}_1 \cup \{j\} \notin \Delta$ which implies that H_1 is a facet in Δ. Let $j \in [\mathrm{N}] \backslash \mathrm{H}_3$. We show that $x_j \xi_{\mathrm{H}_3} \in \mathcal{I}$, thus $\mathrm{H}_3 \cup \{j\} \notin \Delta$.

(A) If $j < \ell_1$, then

$$x_j x_{\ell_1 + (t-1)} \cdots x_{\ell_{s-1} + (t-1)} x_{\ell_s + (t-1)} x_{\ell_s + (2t-1)} \cdots x_{\ell_s + (\mathrm{g}-s)t - 1} \in \mathcal{I}$$

because $j < \ell_1 \leq \gamma_1, \ell_k + (t-1) \leq \gamma_k + (t-1) < \gamma_{k+1}$ for $2 \leq k \leq s-1$ and $\ell_s + (kt-1) = \gamma_s + 1 + (kt-1) = \gamma_s + kt \leq \gamma_{s+k}$ for $1 \leq k \leq \mathrm{g} - s$.

(B) If there exists $2 \leq l \leq s$ such that $\ell_{l-1} + (t-1) < j < \ell_l$, then

$$x_{\ell_1} x_{\ell_2} \cdots x_{\ell_{l-1}} x_j x_{\ell_l + (t-1)} \cdots x_{\ell_s + (t-1)} \cdots x_{\ell_s + (\mathrm{g}-s)t - 1} \in \mathcal{I}.$$

Indeed, $\ell_k \leq \gamma_k$ for $1 \leq k \leq l$. If $l = s$, then $j < \ell_s = \gamma_s + 1$ and $\ell_s + (kt-1) = \gamma_s + 1 + (kt-1) = \gamma_s + kt \leq \gamma_{s+k}$ for $1 \leq k \leq \mathrm{g} - s$. In the case that $l < s$, we have $j < \ell_l \leq \gamma_l, \ell_k + (t-1) \leq \gamma_k + (t-1) < \gamma_{k+1}$ for $l \leq k \leq s-1$ and $\ell_s + (kt-1) = \gamma_s + 1 + (kt-1) \leq \gamma_{s+k}$ for $1 \leq k \leq \mathrm{g} - s$.

We still need to prove that the sets of forms (i)-(iii) are the exclusive facets of Δ. This is equivalent to showing that for every face $\Gamma \in \Delta$, there exists $\mathrm{H} \in \mathcal{F}(\Delta)$ of one of the forms (i)-(iii) which contains Γ. Let $\Gamma \in \Delta$ and $\ell_1 = \min\{j : j \in \Gamma\}$. Inductively, for $l \geq 2$, we set

$$\ell_l = \min\left\{j : j \in \Gamma \text{ and } j \geq \ell_{l-1} + t\right\}.$$

If $\ell_l \leq \gamma_l$ for all l, then the sequence $\ell_1 < \ell_2 < \ldots < \ell_l < \ldots$ has at most $\mathrm{g} - 1$ elements. Otherwise, $x_{\ell_1} \cdots x_{\ell_{\mathrm{g}}} \in \mathcal{I}$, which implies that $\Gamma \notin \Delta$, a contradiction. Let $\ell_1 < \ell_2 < \ldots < \ell_r$ with $r \leq \mathrm{g} - 1$. Then $\Gamma \subset \mathrm{H}_1$ where

$$\begin{aligned}\mathrm{H}_1 = \{&\ell_1, \ldots, \ell_1 + (t-1), \ldots, \ell_r, \ldots, \ell_r + (t-1), \gamma_{r+1}, \ldots, \gamma_{r+1} + (t-1), \ldots, \\ &\gamma_{\mathrm{g}-1}, \ldots, \gamma_{\mathrm{g}-1} + (t-1)\} \in \mathcal{F}(\Delta).\end{aligned}$$

If there exists $l \leq \mathrm{g}$ such that $\ell_l > \gamma_l$, then we denote by s the smallest index with this property. In the case that $s = 1$, $\ell_1 = \min\{j : j \in \Gamma\}$ and $\Gamma \subset \mathrm{H}_2 = \{\gamma_1 + 1, \gamma_1 + 2, \ldots, n\} \in \mathcal{F}(\Delta)$. If $s > 1$, then $\Gamma \subset \mathrm{H}_3$ where

$$\begin{aligned}\mathrm{H}_3 = \{&\ell_1, \ell_1 + 1, \ldots, \ell_1 + (t-1), \ldots, \ell_{s-1}, \ell_{s-1} + 1, \ldots, \ell_{s-1} + (t-1), \\ &\gamma_s + 1, \gamma_s + 2, \ldots, \mathrm{N}\} \in \mathcal{F}(\Delta).\end{aligned}$$

So the theorem is proved. □

It is clear that the last two generators given in the Theorem 3.2.1 does not appear as generators of $\mathcal{I}_{\mathrm{N},\mathrm{g},t}$, since the degrees of the monomials are less than g. The type of monomials that we can take as generators are all of the following form by the Theorem 3.2.1:

$$\prod_{\alpha=1}^{\mathrm{N}} x_\alpha / \left(\mathcal{V}_{\gamma_1,t} \cdots \mathcal{V}_{\gamma_{\mathrm{g}-1},t}\right) \text{ with } \gamma_{j+1} - \gamma_j \geq t, \text{ for } 1 \leq j \leq \mathrm{g} - 2$$

where $\mathcal{V}_{\gamma_k,t} = x_{\gamma_k} x_{\gamma_k+1} \cdots x_{\gamma_k+t-1}$ for $1 \leq k \leq \mathrm{g} - 1$. This is one of the remarkable differences between the two theorems that is particularly worth emphasizing.

It follows that $\mathcal{I}^{\vee}_{\mathrm{N},\mathrm{g},t}$ is also Cohen-Macaulay and has a linear resolution by Eagon-Reiner theorem [12, Theorem 8.1.9]. This means that, the Betti numbers of $S/\mathcal{I}^{\vee}_{\mathrm{N},\mathrm{g},t}$ may be computable via $S/\mathcal{I}_{\mathrm{N},\mathrm{g},t}$ as we did before. Note that, in this case, we have height $\left(\mathcal{I}^{\vee}_{\mathrm{N},\mathrm{g},t}\right) = \operatorname{projdim}\left(\mathrm{S}/\mathcal{I}^{\vee}_{\mathrm{N},\mathrm{g},t}\right) = \mathrm{g}$, moreover the degree of the generators of $\mathcal{I}^{\vee}_{\mathrm{N},\mathrm{g},t}$ is exactly same as the height of $\mathcal{I}_{\mathrm{N},\mathrm{g},t}$. Remaining part of the calculation of Betti numbers is quite similar to the above, so we omit the remaining part. □

Corollary 3.2.2. *[11, Corollary 2.5] Let $J \subset S = \mathbb{K}[x_1, \ldots, x_n]$ be a t-spread strongly stable monomial ideal such that $\bigcup_{\omega \in G(J)} supp(\omega) = \{x_1, \ldots, x_n\}$. Then S/J is Cohen-Macaulay if and only if there exists $\omega \in G(J)$ of degree d such that $\omega = x_{n+t(d-1)} \cdots x_{n-t}x_n$.*

In particular, if J is generated in a single degree then the quotient ring S/J is Cohen-Macaulay if and only if J is t-spread Veronese ideal.

Proof. From (2.2.13), this implies that

$$\operatorname{projdim}\left(\frac{S}{J}\right) = \max\{\max(\omega) - t(\deg(\omega) - 1) : \omega \in G(J)\}.$$

and from Theorem 3.1.6, it follows that

$$\dim\left(\frac{S}{J}\right) = n - \max\{\min(\omega) : \omega \in G(J)\}.$$

Utilizing the Auslander-Buchsbaum theorem, we deduce that S/J is Cohen-Macaulay if and only if

$$\max\{\max(\omega) - t(\deg(\omega)) : \omega \in G(J)\} = \max\{\min(\omega) : \omega \in G(J)\}. \tag{3.2}$$

Let ω_0 be a monomial in the generating set $G(J)$ such that $\min(\omega_0) = \max\{\min(\omega) : \omega \in G(J)\}$. Since

$$\min(\omega) \leq \max(\omega) - t(\deg(\omega) - 1)$$

for all $\omega \in G(J)$, equality 3.2 above holds if and only if

$$\min(\omega_0) = \max(\omega_0) - t(\deg(\omega_0) - 1).$$

Namely, S/J is a Cohen-Macaulay ring if and only if there exists a monomial $\omega_0 \in G(J)$ such that

$$\min(\omega_0) = \max\{\min(\omega) : \omega \in G(J)\} \text{ and } \omega_0 = x_{i_1}x_{i_1+t} \cdots x_{i_1t(d-1)}.$$

Since $\bigcup_{\omega \in G(J)} \operatorname{supp}(\omega) = \{x_1, \ldots, x_n\}$, there exists $\omega \in G(J)$ such that $\max(\omega) = n$ and $\min(\omega) \leq \min(\omega_0)$. Note that

$$\max(\omega) - t(\deg(\omega) - 1) \leq i_1 = \min(\omega_0).$$

Hence, this implies that $n \leq i_1 + t(\deg(\omega) - 1)$. Assume that $\deg(\omega)$ less than or equal to $\deg(\omega_0) = d$, then this gives us the equality $n = i_1 + t(d - 1)$, as required. Conversely, if $d < \deg(\omega)$, we get $\omega = x_{\ell_1} \cdots x_{\ell_{d-1}}x_{\ell_d}x_{\ell_{d+1}} \cdots x_n$ such that

$n \geq \cdots \geq \ell_d \geq \ell_{d-1} \geq \cdots \geq \ell_2 \geq \ell_1$. Let the monomial ω' is equal to $x_{\ell_1} \cdots x_{\ell_d}$. Since ω is an element in $G(J)$, we obtain $\ell_d > i_1 + t(d-1)$, and also we have

$$\max(\omega) - t(\deg(\omega) - 1) \geq \max(\omega') - t(\deg(\omega') - 1) > i_1,$$

this is a contradiction. Therefore, we are done. □

Chapter 4

The Rees algebra of t-spread principal Borel ideals

In this chapter, we investigate the Rees algebra of t-spread principal Borel ideals. To do this, we first recall some definitions and notations. We refer reader to [9] for more prospection.

Let u and v be two monomials of degree d in $S = \mathbb{K}[x_1, \ldots, x_n]$. We can write $uv = x_{i_1}x_{i_2}\cdots x_{i_{2d}}$ with $i_1 \leq \ldots \leq i_{2d}$. We set $u' = x_{i_1}x_{i_3}\cdots x_{i_{2d-1}}$ and $v' = x_{i_2}x_{i_4}\cdots x_{i_{2d}}$. We denote the set of all degree d monomials of S by S_d. The *sorting operator* on S_d is defined as follows:

$$\begin{array}{rccc} \mathrm{sort}\colon & S_d \times S_d & \longrightarrow & S_d \times S_d \\ & \cup\!\!\!\!| & & \cup\!\!\!\!| \\ & (u,v) & \longmapsto & (u',v') \end{array}.$$

For any subset $\mathcal{S} \subseteq S_d$, if $\mathrm{sort}(\mathcal{S} \times \mathcal{S}) \subseteq \mathcal{S} \times \mathcal{S}$ then $\mathcal{S}$ is called a *sortable* subset of S_d. Any pair (u, v) of S_d is called *sorted pair* or *sorted* if $\mathrm{sort}(u, v) = (u, v)$. Let $\mathbf{m} = (u_1, \ldots, u_r) \in S_d \times \ldots \times S_d$ be any r-tuple of degree d monomials. If (u_i, u_j) is sorted for all pairs (u_i, u_j) appearing in $\mathbf{m}$ with $1 \leq i < j \leq r$, then the monomial r-tuple $\mathbf{m}$ is called *sorted.* It is immediate to see that if we have $u_1 = x_{i_1}\cdots x_{i_d},\ u_2 = x_{j_1}\cdots x_{j_d},\ \ldots,\ u_r = x_{k_1}\cdots x_{k_d}$, then $\mathbf{m}$ is sorted if and only if $i_\delta \leq j_\delta \leq \ldots \leq k_\delta$ and $k_\delta \leq i_\rho$ when $\delta < \rho$, for all $\delta, \rho \in \{1, \ldots, d\}$. Moreover, by [9, Theorem 6.12] for every r-tuple $(u_1, \ldots, u_r) \in S_d \times \ldots \times S_d$, there exists a unique sorted r-tuple $(u'_1, \ldots, u'_r) \in S_d \times \ldots \times S_d$ such that $u_1 \cdots u_r = u'_1 \cdots u'_r$.

4.1 Rees algebras, toric rings and toric ideals

Let us express the fundamental material of toric rings, toric ideals and Rees algebras.

Let $S = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring over the field $\mathbb{K}$ with n variables. For each graded ideal $I = (g_1, \ldots, g_m)$ of S we define the *Rees ring* $\mathcal{R}(I)$ as follows,

$$\mathcal{R}(I) = \bigoplus_{j \in \mathbb{N}} I^j t^j = S[g_1 t, \ldots, g_m t].$$

Considering the the surjective ring homomorphism $\varphi : R = S[y_1, \ldots, y_m] \twoheadrightarrow \mathcal{R}(\mathcal{I})$ defined by $x_i \overset{\varphi}{\mapsto} x_i$ for all i and $y_j \overset{\varphi}{\mapsto} g_j t$ for all j, we have $\mathcal{R}(I) \simeq R/\operatorname{Ker}\varphi$. Kernel of the homomorphism φ is called *presentation ideal* of $\mathcal{R}(I)$. When $I = (w_1, \ldots, w_m)$ is a monomial ideal, we define L as the toric ideal of $\mathbb{K}[w_1, \ldots, w_m]$ which is the kernel of the surjective ring homomorphism $\pi : T = \mathbb{K}[y_1, \ldots, y_m] \twoheadrightarrow \mathbb{K}[w_1, \ldots, w_m]$ defined by $\pi(y_i) = w_i$ for all i. It is immediate to see that, $\mathbb{K}[w_1, \ldots, w_m] \simeq \mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I)$ where $\mathfrak{m} = (x_1, \ldots, x_n) \subseteq S$ is the graded maximal ideal. In addition, $\mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I)$ is called the fiber ring of $\mathcal{R}(I)$.

If $f \neq 0$ is a polynomial in S and $<$ be a monomial order on S, we set $\mathrm{in}_<(f)$ to be the largest monomial $u \in \mathrm{supp}(f)$ with respect to $<$, and we call $\mathrm{in}_<$ by *initial monomial* of f. The coefficient c of $\mathrm{in}_<(f)$ in f is called *leading coefficient* of f with respect to $<$, and $c\,\mathrm{in}_<(f)$ is called *leading term* of f. Note that, we set $\mathrm{in}_<(0) = 0$ and set $\mathrm{in}_<(0) < \mathrm{in}_<(f)$ for all $f \neq 0$ in S for convenience. Let $0 \neq I \subset S$ be an ideal. The *initial ideal* of I is the monomial ideal

$$in_<(I) = (\mathrm{in}_<(f) : f \in I,\ f \neq 0).$$

Let $<$ be a monomial order on the polynomial ring $T = \mathbb{K}[y_1, \ldots, y_m]$. If a monomial $\mathbf{y}^{\mathbf{a}} = y_{j_1}^{a_1} y_{j_2}^{a_2} \cdots y_{j_k}^{a_k}$, $k \in \{1, \ldots, m\}$, does not belong to $\mathrm{in}_<(L)$, $\mathbf{y}^{\mathbf{a}}$ is called a *standard monomial* of L with respect to the monomial order $<$.

Definition 4.1.1. *[9, Definition 6.23] The monomial ideal I satisfies the ℓ-exchange property with respect to the monomial order $<$ on T, if the following condition is satisfied: let $\mathbf{y}^{\mathbf{c}}$ and $\mathbf{y}^{\mathbf{d}}$ be any two standard monomials of L with respect to the monomial order $<$ of the same degree with $\omega = \pi(\mathbf{y}^{\mathbf{c}})$ and $\nu = \pi\left(\mathbf{y}^{\mathbf{d}}\right)$ satisfying*

- *$deg_{x_t}\omega = deg_{x_t}\nu$ for $t = 1, \ldots, q-1$ with $q \leq n-1$,*
- *$deg_{x_q}\omega < deg_{x_q}\nu$.*

Let $S = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring over the field $\mathbb{K}$ with n variables. A finite subset $\mathcal{U} = \{u_1, \ldots, u_m\} \subseteq \mathrm{Mon}(S)$ is called a *monomial configuration* of

S. The polynomial ring $\mathbb{K}[\mathcal{U}] = \mathbb{K}[u_1, \ldots, u_m]$, which is $\mathbb{K}$-algebra generated by $\mathcal{U}$ is called the *toric ring of* $\mathcal{U}$. Indeed, a toric ring of a given monomial configuration $\mathcal{U}$ can be seen as the ring extension of $\mathbb{K}$ with transcendental elements $\{u_1, \ldots, u_m\} = \mathcal{U}$.

Let $R = \mathbb{K}[t_1, \ldots, t_m]$ denote the polynomial ring in m variables over the field $\mathbb{K}$ and consider the surjective ring homomorphism

$$\begin{array}{rcl} \pi : R & \longrightarrow\!\!\!\!\rightarrow & \mathbb{K}[\mathcal{U}] \\ \cup\!\!\!\!\!\!\;\;\; & & \cup \\ t_i & \longmapsto & u_i \end{array}$$

for all $i \in \{1, \ldots, m\}$. We call $\operatorname{Ker}(\pi)$ as the *toric ideal* of $\mathcal{U}$. Since $R/\operatorname{Ker}(\pi) \simeq \mathbb{K}[\mathcal{U}]$, the toric ring of $\mathcal{U}$ is determined by kernel of π. To refer to this, the toric ideal $\operatorname{Ker}(\pi)$ of $\mathcal{U}$ is also called the defining ideal of the toric ring $\mathbb{K}[\mathcal{U}]$. We denote the toric ideal of $\mathcal{U}$ by $I_{\mathcal{U}}$. It is well-known fact that, every toric ideal is a prime ideal [12].

A *binomial* of R is a polynomial of the form $u - v$ such that, $u, v \in \operatorname{Mon}(R)$. A *binomial ideal* is an ideal which is generated by binomials. By the [12, Proposition 10.1.1], it is known that, $I_{\mathcal{U}}$ is a binomial ideal.

Definition 4.1.2. *[9, Definition 6.3] Let $\mathcal{F} = \{f_1, \ldots, f_s\} \subset R$ be a finite family of marked binomials in R. In other words, $f_i = \underline{m_i} - m'_i$ for some monomials $m_i, m'_i \in R$ for all $i = 1, \ldots, s$, where the monomial m_i in f_i is marked by underline. $\mathcal{F}$ is called marked coherently if there exists a monomial order $<$ on R such that in $in_<(f_i) = m_i$ for $1 \leq i \leq s$.*

4.2 Rees algebras of t-spread principal Borel ideals

Let us consider the Rees algebra $\mathcal{R}(I) = \bigoplus_{j\geq 0} I^j t^j$ of the ideal $I = B_t(u)$. Since the minimal generators of $B_t(u)$ have the same degree, the fiber ring $\mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I)$ of the Rees ring $\mathcal{R}(I)$ is isomorphic to $\mathbb{K}[G(I)]$. Let us consider the following algebra homomorphism

$$\begin{array}{rcl} \pi : T = \mathbb{K}[t_v : v \in G(I)] & \longrightarrow & \mathbb{K}[G(I)] \\ \cup\;\;\;\;\;\; & & \cup \\ t_v\;\;\;\;\;\; & \longmapsto & v \end{array}.$$

Let S_d be degree d homogeneous component of the polynomial ring S, and $B \subset S_d$ be a sortable set of monomials and $\mathbb{K}[B]$ the semigroup ring generated over $\mathbb{K}$ by B.

Let $t \geq 1$ be an integer and say $I = B_t(u)$ with some degree d monomial u. If $v, w \in I$ and $\operatorname{sort}(v, w) = (v', w')$ then, by the proposition above $v',\ w' \in G(I)$.

Proposition 4.2.1. *[11, Proposition 3.1] Let $S = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring and $u \in Mon(S)$ be any monomial. Then $G(B_t(u))$ is sortable.*

Proof. Let $u = x_{i_1} \cdots x_{i_d}$ with $i_1 \leq \cdots \leq i_d$. Let $v, w \in G(B_t(u))$ and write $wv = x_{j_1} x_{j_2} \cdots x_{j_{2d}}$. Then $w' = x_{j_1} x_{j_3} \cdots x_{j_{2d-1}}, v' = x_{j_2} x_{j_4} \cdots x_{j_{2d}}$. By [19, Lemma 2.7], we know that $j_{2k},\ j_{2k-1} \leq i_k$ for all $k \in \{1 \ldots, d\}$.

To complete the proof, it is enough to show that $j_{2\ell+1} - j_{2\ell-1} \geq t$ and $j_{2\ell+2} - j_{2\ell} \geq t$ for all $\ell \in \{1, \ldots, d-1\}$. To prove $j_{2\ell+1} - j_{2\ell-1} \geq t$: we may consider the two cases below,

(I) $x_{j_{2\ell-1}}$ and $x_{j_{2\ell+1}}$ divides same monomial, say v but they do not divide w;

(II) $x_{j_{2\ell-1}}$ divides w but does not divide v and $x_{j_{2\ell+1}}$ divides v but does not divide w.

Let us start with first case. Assume that $x_{j_{2\ell-1}}$ and $x_{j_{2\ell+1}}$ divides v but they do not divide w. Since $v \in G(B_t(u))$, v is a t-spread monomial. Then the inequality $j_{2\ell+1} - j_{2\ell-1} \geq t$ holds for all $\ell \in \{1, \ldots, d-1\}$, otherwise the t-spreadness property of v would be broken. For the second case of the proof, let us assume that, $x_{j_{2\ell+1}}$ divides v and $x_{j_{2\ell-1}}$ divides w, then $j_{2\ell+1} - j_{2\ell-1} \geq j_{2\ell} - j_{2\ell-1} \geq t$, since $w \in G(B_t(u))$. If $x_{j_{\ell\ell}} \mid v$, then $j_{2\ell+1} - j_{2\ell-1} \geq j_{2\ell+1} - j_{2\ell} \geq t$ since $v \in G(B_t(u))$. One can do similar to prove the other inequality. □

Theorem 4.2.2. *[9, Theorem 6.15] Let B be a sortable subset of monomials of S_d and*

$$\mathcal{F} = \left\{ \underline{t_u t_v} - t_{u'} t_{v'} : u, v \in B, (u, v) \text{ unsorted pair and } (u', v') = sort(u, v) \right\}.$$

Then there exists a monomial order $<$ on R which is called the sorting order such that for every $g = \underline{t_u t_v} - t_{u'} t_{v'} \in \mathcal{F}$, in $in_<(g) = t_u t_v$.

As it was proved in [11, Theorem 3.2], the set of binomials

$$\mathcal{G} = \{t_v t_w - t_{v'} t_{w'} : (v, w) \text{ unsorted }, (v', w') = \text{sort}(v, w)\}$$

is a Gröbner basis of the toric ideal $I_u = \text{Ker}(\pi)$ with respect to the sorting order on T. A monomial $t_{u_1} \cdots t_{u_N}$ is called *standard* with respect to $<$ if $t_{u_1} \cdots t_{u_N} \notin \text{in}_<(P)$. Where $\text{in}_{<_{\text{sort}}}(t_v t_w - t_{v'} t_{w'}) = t_v t_w$ for $t_v t_w - t_{v'} t_{w'} \in \mathcal{G}$.

Let $I \subset S$ be a monomial ideal and assume that the minimal generating set $G(I)$ consists of the monomials with same degree and $\mathbb{K}[t_u : u \in G(I)]$ the polynomial ring in $\mid G(I) \mid$-many variables equipped with a chosen monomial order $<$.

Proposition 4.2.3. *[1, Proposition 2.2] Let $\omega = x_{i_1} \cdots x_{i_d}$ be a t-spread monomial in S. Therefore the t-spread principal Borel ideal $B_t(\omega) \subset S$ has the ℓ-exchange property defined in 4.1.1 with respect to the sorting order which we denoted as $<_{sort}$.*

Proof. Let $T = K\left[t_\mu : \mu \in G\left(B_t(\omega)\right)\right]$ and $t_{\omega_1} \cdots t_{\omega_M}, t_{\mu_1} \cdots t_{\mu_M} \in \operatorname{Mon}(T)$ be two standard monomials with respect to the $<_{\text{sort}}$ and let be $\deg(t_{\omega_1} \cdots t_{\omega_M}) = \deg(t_{\mu_1} \cdots t_{\mu_M}) = M$ such that the all of the two conditions given in the Definition 4.1.1 above be satisfied. Since the selected monomials are standard with respect to the sorting order, it consequently follows that the products $\omega_1 \cdots \omega_M, \mu_1 \cdots \mu_M$ are sorted. Since $\deg_{x_i} \omega_1 \cdots \omega_M = \deg_{x_i} \mu_1 \cdots \mu_M$ for $1 \leq i \leq q-1$, we also have $\deg_{x_i}(\omega_\gamma) = \deg_{x_i}(\mu_\gamma)$ for all $1 \leq \gamma \leq M$ and $1 \leq i \leq q-1$. If we consider the condition $\deg_{x_q} \omega_1 \cdots \omega_M < \deg_{x_q} \mu_1 \cdots \mu_M$, then it is immediate to see that, the condition implies that there exists $1 \leq \delta \leq M$ such that $\deg_{x_q}(\omega_\delta) < \deg_{x_q}(\mu_\delta)$.

Let $\omega_\delta = x_{j_1} \cdots x_{j_d}$, $\mu_\delta = x_{\ell_1} \cdots x_{\ell_d}$, and assume that $q = \ell_\eta$ for some $1 \leq \eta < d$. It follows that $j_1 = \ell_1, \ldots, j_{\eta-1} = \ell_{\eta-1}$ and $j_\eta > \ell_\eta = q$. Now we have two cases that is $j_\eta \notin \operatorname{supp}(\mu_\delta)$ and $j_\eta \in \operatorname{supp}(\mu_\delta)$. Therefore, we proceed with the proof by considering these two cases separately.

If $j_\eta \notin \operatorname{supp}(\mu_\delta)$, then we take $j = j_\eta$. Then the monomial $x_q \omega_\delta / x_j$ is t-spread, thus it is an element of $B_t(\omega)$, since $q = \ell_\eta \geq \ell_{\eta-1} + t = j_{\eta-1} + t$ and $j_{\eta+1} \geq j_\eta + t > q + t$.

If $j_\eta \in \operatorname{supp}(\mu_\delta)$, we can pick any $j \in \operatorname{supp}(\omega_\delta)$ such that $j \notin \operatorname{supp}(\mu_\delta)$ with $j > q$. This choice of j is possible; it is because, $\deg(\omega_\delta) = \deg(\mu_\delta)$ and $\deg_{x_q}(\omega_\delta) < \deg_{x_q}(\mu_\delta)$. Since $q, j_\eta \in \operatorname{supp}(\mu_\delta)$, therefore $x_q \omega_\delta / x_j$ is a t-spread monomial. □

Before the theorem, let us recall the definition of the *Gröbner basis.*

Definition 4.2.4. *[9] Let $I \subset S = \mathbb{K}[x_1, \ldots, x_n]$ be an ideal, and let $<$ be a monomial order on S. A sequence $g_1, \ldots, g_m$ of elements in I with $in_<(I) = (in_<(g_1), \ldots, in_<(g_m))$ is called a Gröbner basis of I with respect to the monomial order I.*

Theorem 4.2.5. *[1] Let $I = B_t(u)$ be a t-spread principal Borel ideal. Gröbner basis of the toric ideal J with respect to $<$ consists of the monomials $t_v t_w - t_{v'} t_{w'}$ where (v, w) is unsorted and $(v', w') = sort(v, w)$, together with the binomials of the form $x_i t_v - x_j t_w$ where $i < j$, $x_i v = x_j w$ and j is the largest integer for which $x_i v / x_j \in G(I)$.*

Proof. We know that any principal Borel ideal satisfies the ℓ-exchange property with respect to the sorting order, by the Proposition 4.2.3. Then we finish the proof by applying [14, Theorem 5.1]. □

Proposition 4.2.6. *[1, Proposition 2.4] All the powers of $B_t(u)$ have linear quotients. In particular, all the powers of $B_t(u)$ have a linear resolution.*

Corollary 4.2.7. *[1, Corollary 2.5] The Rees algebra $\mathcal{R}(B_t(u))$ is Koszul.*

4.3 Asympototic behavior of Borel ideals

In this section, we investigate the asymptotic behavior of the depth of the t-spread principal Borel ideals and we give some remarks for when the case a t-spread Borel ideal is a t-spread Veronese ideal.

Let $S = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring over the field $\mathbb{K}$ with n variables. Let us take the ideal $B_t(u) \subset S$ with $u = x_{i_1}x_{i_2}\cdots x_{i_d}$. Our aim is to compute that the limit depth of S/I^k. Since, by Borel moves, we cannot obtain a monomial with x_j in its support such that $j > i_d$, so we can restrict ourselves to the case when $i_d = n$.

Theorem 4.3.1. *[1] Let $t \geq 1$ be an integer and $I = B_t(u) \subset S$ the t-spread principal Borel ideal generated by $u = x_{i_1}\cdots x_{i_d}$ where $t+1 \leq i_1 < i_2 < \ldots < i_{d-1} < i_d = n$. Then we have,*

$$\operatorname{depth}\left(\frac{S}{I^k}\right) = 0, \ \textit{for } k \geq d.$$

Proof. By the Proposition 4.2.3, we know that I satisfies the ℓ-exchange property. Then, [10, Theorem 3.6] implies that I^k has linear quotients with respect to $>_{lex}$ for all $k \geq 1$. Let us say $G(I^k) = \{w_1, \ldots, w_m\}$ with $w_1 >_{lex} \ldots >_{lex} w_m$ for $1 \leq j \leq m$. We define r_j by the number of variables appearing in the colon ideal $(w_1, \ldots, w_j - 1) : w_j$ as a generator. Then by the [12, Corollary 8.2.2] we have

$$\operatorname{projdim}\left(\frac{S}{I^k}\right) = \max\{r_1, \ldots, r_m\} + 1. \tag{4.1}$$

By the Auslander - Buchsbaum formula, we have

$$\operatorname{projdim}\left(\frac{S}{I^k}\right) + \operatorname{depth}\left(\frac{S}{I^k}\right) = \operatorname{depth}(S) = n$$

then, we can write

$$\operatorname{depth}\left(\frac{S}{I^k}\right) = n - \operatorname{projdim}\left(\frac{S}{I^k}\right).$$

Therefore by the composing Auslander-Buchsbaum formula with (4.1), to prove the first statement of the theorem, it is enough to show that $\max\{r_1, \ldots, r_m\} + 1 = n$ for $k \geq n$ or it is also enough to show that $\operatorname{projdim}\left(\frac{S}{I^k}\right) = n$.

Let us denote $w_0 = v_1 v_2 \cdots v_d$, here

$$\begin{aligned} v_1 &= x_1 x_{t+1} \ldots x_{(d-3)t+1} x_{(d-2)t+1} x_n \\ v_2 &= x_1 x_{t+1} \ldots x_{(d-3)t+1} x_{i_{d-1}} x_n \\ &\vdots \\ v_{d-1} &= x_1 x_{i_2} \ldots x_{i_{d-2}} x_{i_{d-1}} x_n \\ v_d &= u = x_{i_1} x_{i_2} \ldots x_{i_{d-2}} x_{i_{d-1}} x_n. \end{aligned}$$

It is immediate to see that $v_i \in G(I)$ for all $i \in \{1, \ldots, d\}$. Let $k \geq d$ be an integer and let us define w as follows

$$w = w_0 u^{k-d} \in G(I^k).$$

If is shown to be

$$\left(w' \in G(I^k) : w' >_{lex} w\right) : w \supseteq (x_1, \ldots, x_{n-1}),$$

the equality $\operatorname{projdim}\left(\frac{S}{I^k}\right) = n$ given in just above to be proved.

Consider the ideal defined as $J = \left(w' \in G\left(I^k\right) : w' >_{\text{lex}} w\right)$. Our aim is to show that $x_j w \in J$ for all $1 \leq j \leq n-1$. Let be $1 \leq s \leq d$ such that $i_{s-1} \leq j < i_s$, where we set $i_0 = 1$. We consider the monomial

$$v'_{d-s+1} = \frac{x_j v_{d-s+1}}{x_{i_s}} = x_1 x_{t+1} \cdots x_{(s-2)t+1} x_j x_{i_{s+1}} \cdots x_{i_{d-1}} x_n.$$

Since we have $j < i_s$ and v'_{d-s+1} is a t-spread monomial in S, it is immediate to see that $v'_{d-s+1} \in G(I)$. Indeed, we have

$$\begin{aligned} & i_{s+1} - j > i_{s+1} - i_s \geq t, \\ & j - (s-2)t - 1 \geq i_{s-1} - (s-2)t - 1 \geq (i_1 + (s-2)t) \\ & -(s-2)t - 1 = i_1 - 1 \geq t \end{aligned}$$

Let w'_0 be monomial obtained from w_0 by the Borel move $\frac{v'_{d-s+1}}{v_{d-s+1}}$ and let $w' = w'_0 u^{k-d}$. Then $w' >_{\text{lex}} w$, thus $w' \in J$ and since $x_j w = x_{i_s} w'$, we have $x_j w \in J$, which implies that $x_j \in J : w$. □ □

If the interested reader would also like to see the results obtained on analytic spread of a principal Borel ideal, they can see the [1, Chapter 3]. However, analytic spread of an ideal is not discussed in this thesis.

By the last theorem which we just proved, we give the exact number of the Krull dimension of the $\mathbb{K}$-algebra generated by the minimal generating set of a given principal Borel ideal. Let us give this result in the following corollary.

Corollary 4.3.2. *[1] Let $t \geq 1$ be an integer and $B_t(u) \subset S$ the t-spread principal Borel ideal generated by $u = x_{i_1} \cdots x_{i_d}$ where $t+1 \leq i_1 < i_2 < \cdots < i_{d-1} < i_d = n$. Then $dim\Big(\mathbb{K}\left[G\left(B_t(u)\right)\right]\Big) = n$.*

4.4 Remarks on t-spread principal Borel ideals

Let I be a monomial ideal in $S = \mathbb{K}[x_1, \ldots, x_n]$ and let T be the polynomial ring over $\mathbb{K}$ in $(n + |G(I)|)$-many variables. We assign a new variable to each monomial in I, such that

$$G(I) \ni u \longmapsto y_u \in T.$$

In this way, we are able to denote T as $T = \mathbb{K}[x_1, \ldots, x_n, \{y_u\}_{u \in G(I)}]$ the polynomial ring in $(n+ \mid G(I) \mid)$-many variables.

Let us denote $<_{\text{lex}}$ denote the lexicographic order on S such that $<_{\text{lex}}$ comes from $x_1 > x_2 > \cdots > x_n$. Fix an arbitrary monomial order $<^{\#}$ on $K\left[\{y_u\}_{u \in G(I)}\right]$. Then, we are able to define a new monomial order denoted by $<^{\#}_{\text{lex}}$ on R and defined as below:

For monomials $\left(\prod_{i=1}^n x_i^{a_i}\right)\left(\prod_{u \in G(I)} y_u^{a_u}\right)$ and $\left(\prod_{i=1}^n x_i^{b_i}\right)\left(\prod_{u \in G(I)} y_u^{b_u}\right)$ belonging to R, one has

$$\left(\prod_{i=1}^{n} x_i^{a_i}\right)\left(\prod_{u \in G(I)} y_u^{a_u}\right) <^{\#}_{\text{lex}} \left(\prod_{i=1}^{n} x_i^{b_i}\right)\left(\prod_{u \in G(I)} y_u^{b_u}\right)$$

if either

(i) $\prod_{u \in G(I)} y_u^{a_u} <^{\#} \prod_{u \in G(I)} y_u^{b_u}$ or,

(ii) $\prod_{u \in G(I)} y_u^{a_u} = \prod_{u \in G(I)} y_u^{b_u}$ and $\prod_{i=1}^n x_i^{a_i} <_{lex} \prod_{i=1}^n x_i^{b_i}$.

Now, we give an important remark for distinction of the monomial orders which we used below. The following difference in ordering is essential for understanding the techniques and results in each case.

Remark 4.4.1. *[1] Note that the ordering used to establish that all powers of the principal Borel ideal $B_t(u)$ have linear quotients in Proposition 4.2.6 is distinct from the decreasing lexicographic order applied in the proof of Theorem 4.3.1.*

Actually, the toric ideal of $\mathbb{K}[B_t(u)]$ does not have a quadratic Gröbner basis in terms of the lexicographic order. Let us give an example for this:

Example 4.4.2. *[1] Let $S = \mathbb{K}[x_1, \ldots, x_{10}]$ and $u = x_6x_8x_{10} \in Mon(S)$ and $I = B_2(u)$. Let $f = t_{u_1}t_{u_2}t_{u_3} - t_{v_1}t_{v_2}t_{v_3}$ with $u_1 = x_1x_3x_8, u_2 = x_1x_7x_9$, $u_3 = x_2x_4x_6$ and $v_1 = x_1x_3x_9, v_2 = x_1x_6x_8, v_3 = x_2x_4x_7$. As a result, f can be identified as a binomial within the toric ideal of $K\left[B_2(u)\right]$. The leading monomial of f, when considered under the lexicographic order, is $t_{u_1}t_{u_2}t_{u_3}$.It is immediate to obtain that there is no quadratic monomial in the initial ideal of the toric ideal that divides $t_{u_1}t_{u_2}t_{u_3}$. Consequently, in terms of the lexicographic order, the reduced Gröbner basis of the toric ideal of $K\left[B_2(u)\right]$ is not quadratic.*

Lastly we state a remark on the maximal number of variables which generate the colon ideals of I^k (see [1]). In the proof of Theorem 4.3.1, we consider the following monomials

$$v_1 = x_1x_{t+1}\cdots x_{(d-2)t+1}x_{dt},$$
$$\vdots$$
$$v_{d-1} = x_1x_{2t}\cdots x_{dt},$$
$$v_d = u = x_tx_{2t}\cdots x_{dt}\ ;$$

and the one shows that, for $k \geq d$,

$$\left(w' \in G\left(I^k\right) : w' >_{\text{lex}} w\right) : w \supseteq \left(x_j : j \in [n]\backslash \operatorname{supp}(u)\right),$$

where $w = v_1 \cdots v_d u^{k-d}$. Nonetheless, we show that if an index $j \in \operatorname{supp}(u)$, then x_j does not appear among the variables in the minimal generating set of the colon ideals of I^k. Let us assume that there exists $k \geq 1$ and a monomial $\xi = \xi_1 \cdots \xi_k \in I^k$ with $\xi_1, \ldots, \xi_k \in I$, such that $x_j \in \left(\xi' \in I^k : \xi' >_{\text{lex}} \xi\right)$ for some $j \in \operatorname{supp}(u)$. This follows that there exists $\xi' = \xi'_1 \cdots \xi'_k \in I^k$ and an integer $s > j$ such that $x_j\xi = \xi' x_s$.

Let $j = qt$ for some $1 \leq q \leq d$. Then

$$\begin{aligned}\sum_{\ell=(q-1)t+1}^{qt} \deg_{x_\ell}(x_j\xi) &= \sum_{\ell=(q-1)t+1}^{qt} \deg_{x_\ell}(x_j\xi_1\cdots\xi_k) \\ &= k+1 > k \\ &= \sum_{\ell=(q-1)t+1}^{qt} \deg_{x_\ell}(\xi'_1\cdots\xi'_k x_s) \\ &= \sum_{\ell=(q-1)t+1}^{qt} \deg_{x_\ell}(\xi' x_s),\end{aligned}$$

and then we obtain the following inequality:

$$\sum_{\ell=(q-1)t+1}^{qt} \deg_{x_\ell}(x_j\xi) > \sum_{\ell=(q-1)t+1}^{qt} \deg_{x_\ell}(\xi' x_s).$$

This is a contradiction. Therefore, the maximal number of variables which generate the colon ideals of I^k for $k \geq d$ is $n-d$. Hence, $\mathrm{projdim}\frac{S}{I^k} = n-d+1$, for all $k \geq d$.

Bibliography

[1] C. Andrei, V. Ene, B. Lajmiri, *Powers of t-spread principal Borel ideals*, Arch. Math. **112** (2019), 587–597.

[2] M. Atiyah, I. MacDonald, Introduction to Commutative Algebra. Addison-Wesley-Longman, (1969).

[3] M. Broadmann, *The asymptotic nature of the analytic spread*, Math. Proc. Cambridge Philos. Soc. **86** (1979), 35-39.

[4] M. Broadmann, *Asymptotic stability of* $\operatorname{Ass}(M/I^nM)$, Proc. Am. Math. Soc. **74** (1979), 16-18.

[5] W. Bruns, J. Herzog, Cohen-Macaulay Rings. Cambridge Studies in Advanced Mathematics, Cambridge University Press, second edition, (1998).

[6] D. Eisenbud, Commutative Algebra with a View Toward Algebraic Geometry. Graduate Texts in Mathematics **150**, Springer-Verlag New York, (1999).

[7] D. Eisenbud, The Geometry of Syzygies, A Second Course in Commutative Algebra and Algebraic Geometry. Springer, (2005).

[8] S. Eliahou, M. Kervaire, *Minimal resolutions of some monomial ideals*, J. Algebra **129**(1) (1990), 1–25.

[9] V. Ene, J. Herzog, Gröbner Bases in Commutative Algebra. Graduate Studies in Mathematics **130**, American Mathematical Society, Providence, RI (2012).

[10] V. Ene, O. Olteanu, *Powers of lexsegment ideals with linear resolution*, Ill. J. Math. **56**(2) (2012), 533–549.

[11] V. Ene, J. Herzog, A. A. Qureshi, *t-spread strongly stable monomial ideals*, Commun. Algebra **47**(12) (2019), 5303–5316.

[12] J. Herzog, T. Hibi, Monomial ideals, Graduate text in Mathmetaics **260**, Springer, (2010).

[13] J. Herzog, T. Hibi, *The depth of powers of an ideal*, J. Algebra **291** (2005), 534-550.

[14] H. Herzog, T. Hibi, *Ideals of fibre type and polymatroids*, Osaka J. Math **42** (2005), 807-829.

[15] J. Herzog, A. A. Qureshi, *Persistence and stability properties of powers of ideals*, J. Pure Appl. Algebra **219**(3) (2015), 530-542.

[16] J. Herzog, Y. Takayama, *Resolutions by mapping cones*, Homology, Homotopy Appl. **4**(2) (2002), 277-294.

[17] H. Matsumura, Commutative Ring Theory. Cambridge University Press, (1986).

[18] H. Matsumura, Commutative Algebra. Mathematics Lecture Note Series, The Benjamin Cummings Publishing Company, Second Edition, (1980).

[19] E. D. Negri, *Toric rings generated by special stable sets of monomials*, Math. Nachr. **203**(1) (1999), 31–45.

[20] I. Peeva, Graded Syzygies. Springer-Verlag London Limited, (2011).

[21] R. P. Stanley, Combinatorics and Commutative Algebra. Second Edition, Birkhäuser, (1996).