Anonymous Implementation*

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Abstract

We consider Nash implementation under complete information with the additional feature that planners are restricted by anonymity when designing mechanisms and shaping individuals' unilateral deviation opportunities. We identify the necessary and (almost) sufficient condition for (full) implementation of social choice correspondence. We show that there are collective goals that are implementable under anonymity but not in Nash equilibrium. Thus, our observations justify that anonymity provides society with additional decentralizable social choice correspondences that are otherwise not implementable. Unfortunately, anonymity imposes a heavy burden when implementing efficiency: The Pareto social choice correspondence is not implementable under anonymity in the full domain.

Keywords: Nash Implementation; Behavioral Implementation; Anonymity; Maskin Monotonicity; Consistent Collections; Efficiency.

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1 Introduction

Implementation of collective goals in Nash equilibrium (NE) involves designing mechanisms that incentivize society members to chose outcomes aligned with the desired goal.[1] The seminal works Maskin (1999), Saijo (1988), Moore and Repullo (1990), Dutta and Sen (1991), and de Clippel (2014) establish that designing mechanisms that provide incentives aligned with the collective goal involves the identification of choice sets corresponding to opportunities individuals can sustain through unilateral deviations within the mechanism. Following in the footsteps of Hurwicz (1986) and Moore and Repullo (1990), de Clippel (2014) shows that Nash implementation of collective goals is almost fully characterized by the existence of a collection of choice sets providing individuals incentives consistent with the goal at hand. Indeed, a consistent collection of sets of alternatives is a family of choice sets indexed for each individual, each state, and each socially optimal alternative at that state such that the following hold: A socially optimal alternative at a state is chosen by every individual at that state from the corresponding choice set; if an alternative socially optimal at the first state but not at the second, then there is an individual who does not choose this alternative at the second state from her choice set corresponding to this alternative and the first state.

On the other hand, the nearly complete characterization of Nash implementable collective goals based on consistency reveals that planners have significant flexibility when designing mechanisms and shaping individuals' opportunity sets. However, in many interesting economic environments, planners often face restrictions for many reasons. These limitations may arise due to legal considerations, such as the requirement for gender-neutral mechanisms. Alternatively, it may not be realistic to consider a small trustees meeting of a major conglomerate with choice sets exclusively custom-tailored to each member's preferences. These limitations may also arise due to practical considerations, e.g., when the design of individual specific choice sets and implementation administration are somewhat complex and costly.

In this study, we analyze Nash implementation in complete information environments with the feature that the planners are restricted exogenously when shaping individuals' opportunity sets. To address the issue more directly, we ask, what if the planner were restricted by anonymity and has

[1] For more on Nash implementation, please see Maskin and Sjöström (2002), Palfrey (2002), and Serrano (2004).

to offer each individual the same set of opportunities when designing mechanisms?

We propose an equilibrium notion, *anonymous Nash equilibrium* (ANE), to restrict attention to anonymity while allowing the use of any mechanism: A message profile is an ANE of a mechanism at a state if it is a NE and the opportunity sets of all individuals equal one another. Next, we provide a necessary and (almost) sufficient condition for (full) implementation of social choice correspondence in ANE, namely, *anonymous consistency*. This condition coincides with de Clippel (2014)'s consistency with the restriction that choice sets are independent of individuals' identities. We prove that (*necessity*) if a social goal is implementable in ANE, then there exists a collection of choice sets anonymous consistent with the goal at hand; (*sufficiency*) if a unanimous social goal possesses an anonymous consistent profile of choice sets, then it is implementable in ANE.

We show that implementation in ANE does not necessarily restrict the set of social goals under consideration: In our example in Section 3, we describe an environment and a social choice correspondence that is implementable in ANE but not in NE. Indeed, anonymity enlarges society's opportunities by allowing society to decentralize social choice correspondences that are otherwise not implementable in NE.

On the other hand, we show that when dealing with efficiency, anonymity imposes a heavy burden: We identify a domain description which, if allowed for, implies that the Pareto social choice correspondence is not implementable in ANE. As the full domain of preferences includes this particular instance, we observe that the Pareto social choice correspondence is not implementable in ANE on the full domain.

Our results cover both the rational and behavioral environments.

In the rest of the paper, Section 2 provides the preliminaries and 3 the example discussed above. In Section 4, we deal with the necessity and sufficiency of implementation in ANE, while Section 5 provides our results concerning efficiency. Finally, Section 6 presents our concluding remarks.

2 Preliminaries

Let $N = \{1, ..., n\}$ denote a *society* with at least two individuals, X a set of *alternatives*, 2^X the set of all subsets of X, and $\mathcal{X}$ the set of all non-empty subsets of X.

We denote by Ω the set of all *possible states* of the world, capturing all the payoff-relevant characteristics of the environment. In *behavioral environments*, the choice correspondence of individual $i \in N$ at state $\omega \in \Omega$ maps $\mathcal{X}$ to itself so that for all $S \in \mathcal{X}$, $C_i^{\omega}(S)$ is a non-empty subset of S. In *rational environments*, every individual's choice correspondence at every state satisfies Chernoff's α and Sen's β and are represented by *preferences* of individual $i \in N$ at state $\omega \in \Omega$ captured by a complete and transitive binary relation, a ranking, $R_i^{\omega} \subseteq X \times X$, while P_i^{ω} represents its strict counterpart.[2] Given $i \in N$, $\omega \in \Omega$, and $x \in X$, $L_i^{\omega}(x) := \{y \in X \mid xR_i^{\omega}y\}$ denotes the *lower contour set of individual i at state ω of alternative x*. For all $i \in N$, all $\omega \in \Omega$, and all $S \in \mathcal{X}$, define $C_i^{\omega}(S) := \{x \in S \mid xR_i^{\omega}y, \forall y \in S\}$.

We refer to $\tilde{\Omega} \subset \Omega$ as a *domain*. A *social choice correspondence* (SCC) defined on a domain $\tilde{\Omega}$ is $f : \tilde{\Omega} \to \mathcal{X}$, a non-empty valued correspondence mapping $\tilde{\Omega}$ into X. Given $\omega \in \tilde{\Omega}$, $f(\omega)$, the set of *f-optimal* alternatives at ω, consists of alternatives that the planner desires to sustain at ω. SCC f on domain $\tilde{\Omega}$ is *unanimous* if for any $\omega \in \tilde{\Omega}$, $x \in \cap_{i\in N} C_i^{\omega}(X)$ implies $x \in f(\omega)$.

A mechanism $\mu = (M, g)$ assigns each individual $i \in N$ a non-empty *message space* M_i and specifies an *outcome function* $g : M \to X$ where $M = \times_{j\in N} M_j$. $\mathcal{M}$ denotes the set of all mechanisms. Given $\mu \in \mathcal{M}$ and $m_{-i} \in M_{-i} := \times_{j\neq i} M_j$, the *opportunity set* of individual i pertaining to others' message profile m_{-i} in mechanism μ is $O_i^{\mu}(m_{-i}) := g(M_i, m_{-i}) = \{g(m_i, m_{-i}) \mid m_i \in M_i\}$.

A message profile $m^* \in M$ is a ***Nash equilibrium of mechanism μ at state $\omega \in \Omega$*** if $g(m^*) \in \cap_{i\in N} C_i^{\omega}(O_i^{\mu}(m^*_{-i}))$.[3] Given $\mu \in \mathcal{M}$, the correspondence $NE^{\mu} : \Omega \twoheadrightarrow 2^X$ identifies ***Nash equilibrium outcomes of mechanism μ at state $\omega \in \Omega$*** and is defined by $NE^{\mu}(\omega) := \{x \in X \mid \exists m^* \in M \text{ s.t. } g(m^*) \in \cap_{i\in N} C_i^{\omega}(O_i^{\mu}(m^*_{-i})) \text{ and } g(m^*) = x\}$. We say that a ***mechanism μ implements SCC f on domain $\tilde{\Omega}$, $f : \tilde{\Omega} \to \mathcal{X}$, in Nash equilibrium*** if $NE^{\mu}(\omega) = f(\omega)$ for all $\omega \in \tilde{\Omega}$.

Thanks to the necessity result for Nash implementability of an SCC by to Maskin (1999), we know that if $f : \Theta \to \mathcal{X}$ is Nash implementable, then it is ***Maskin-monotonic***: $x \in f(\omega)$ and $L_i^{\omega}(x) \subset L_i^{\tilde{\omega}}(x)$ for all $i \in N$ implies $x \in f(\tilde{\omega})$. de Clippel (2014) generalizes Maskin's results on Nash implementation to behavioral domains. The resulting necessary condition behavioral implementation is equivalent to Maskin-monotonicity in the rational domain (Barlo & Dalkıran, 2022) and calls for the existence of a profile of sets that are *consistent* with this SCC at hand: We

[2]A choice correspondence C defined on $\mathcal{X}$ satisfies Chernoff's α if for all $S, T \in \mathcal{X}$ with $S \subset T$, $x \in C(T) \cap S$ implies $x \in C(S)$, and Sen's β if for all $S, T \in \mathcal{X}$ with $S \subset T$, $x, y \in C(S)$ implies $x \in C(T)$ if and only if $y \in C(T)$. A binary relation $R \subseteq X \times X$ is *complete* if for all $x, y \in X$ either xRy or yRx or both; *transitive* if for all $x, y, z \in X$ with xRy and yRz implies xRz.

[3]The notion of NE in behavioral domains, the behavioral Nash equilibrium, is introduced by Korpela (2012).

say that a profile of sets $\mathbf{S} := (S_i(x,\theta))_{i\in N,\omega\in\tilde{\Omega},x\in f(\theta)}$ is ***consistent*** with a given SCC $f : \tilde{\Omega} \to \mathcal{X}$ if for any $\omega, \tilde{\omega} \in \tilde{\Omega}$,

(i) if $x \in f(\omega)$, then $x \in \cap_{i\in N} C_i^{\omega}(S_i(x,\omega))$, and

(ii) if $x \in f(\omega) \setminus f(\tilde{\omega})$, then $x \notin \cap_{i\in N} C_i^{\tilde{\omega}}(S_i(x,\omega))$.

The current study aims to restrict the planner to anonymity when designing the mechanism and its choice sets. That is why we introduce the notion of anonymous implementation in NE:

Definition 1. *We say that a mechanism μ* ***anonymously implements SCC f on domain $\tilde{\Omega}$, $f : \tilde{\Omega} \to \mathcal{X}$, in Nash equilibrium*** *if*

(i) *for all $\omega \in \tilde{\Omega}$ and all $x \in f(\omega)$, there is $m^{(x,\omega)} \in M$ such that $g(m^{(x,\omega)}) = x \in \cap_{i\in N} C_i^{\omega}(O_i^{\mu}(m_{-i}^{(x,\omega)}))$, and $O_i^{\mu}(m_{-i}^{(x,\omega)}) = O_j^{\mu}(m_{-j}^{(x,\omega)})$ for all $i, j \in N$; and*

(ii) *if $m^* \in M$ is such that $g(m^*) \in \cap_{i\in N} C_i^{\tilde{\omega}}(O_i^{\mu}(m_{-i}^*))$ and $O_i^{\mu}(m_{-i}^*) = O_j^{\mu}(m_{-j}^*)$ for all $i, j \in N$, then $g(m^*) \in f(\tilde{\omega})$.*

A practical shortcut to formalizing anonymous implementation in NE involves the introduction of the following refinement of NE:[4] A message profile $m^* \in M$ is an ***anonymous Nash equilibrium of mechanism μ at state*** $\omega \in \Omega$ if $g(m^*) \in \cap_{i\in N} C_i^{\omega}(O_i^{\mu}(m_{-i}^*))$ and $O_i^{\mu}(m_{-i}^*) = O_j^{\mu}(m_{-j}^*)$ for all $i, j \in N$. So, a mechanism μ anonymously implements SCC f on domain $\tilde{\Omega}$ in NE if and only if $ANE^{\mu}(\omega) = f(\omega)$ for all $\omega \in \tilde{\Omega}$, where $ANE^{\mu} : \Omega \twoheadrightarrow 2^X$, the set of ANE outcomes of mechanism μ at state $\omega \in \Omega$ and is given by $ANE^{\mu}(\omega) := \{x \in X \mid \exists m^* \in M \text{ s.t. } m^* \in M \text{ is an } ANE \text{ of } \mu \text{ at } \omega\}$.

3 An Example

In what follows, we present an example in the rational domain involving an SCC that is anonymously implementable in NE but not implementable in NE.

We have two agents, Ann and Bob, and three alternatives, a, b, c. The domain $\tilde{\Omega}$ includes two

[4]We thank Kemal Yıldız for suggesting this approach.

states, $\omega^{(1)}$ and $\omega^{(2)}$, and agents' rankings depend on states and are as follows:

$\omega^{(1)}$		$\omega^{(2)}$	
$R_A^{\omega^{(1)}}$	$R_B^{\omega^{(1)}}$	$R_A^{\omega^{(2)}}$	$R_2^{\omega^{(2)}}$
b	a	a, b	c
a	b	c	a, b
c	c		

The planner aims to implement SCC $f : \tilde{\Omega} \to \mathcal{X}$ given by $f(\omega^{(1)}) = \{a\}$ and $f(\omega^{(2)}) = \{b\}$.

Now, consider the following mechanism:

		Bob		
		L	M	R
Ann	U	a	c	a
	M	c	b	a
	D	a	a	$\boxed{b}$

We note that $a \in C_A^{\omega^{(1)}}(O_A^{\mu}(L)) \cap C_B^{\omega^{(1)}}(O_B^{\mu}(U))$ and $O_A^{\mu}(L) = O_B^{\mu}(U) = \{a, c\}$ implies that (U, L) (shown as circled) is an ANE of μ at state $\omega^{(1)}$. Thus, $ANE^{\mu}(\omega^{(1)}) = \{a\} = f(\omega^{(1)})$. On the other hand, $b \in C_A^{\omega^{(2)}}(O_A^{\mu}(R)) \cap C_B^{\omega^{(2)}}(O_B^{\mu}(D))$ and $O_A^{\mu}(R) = O_B^{\mu}(D) = \{a, b\}$ enables us to conclude that (D, R) (depicted with a square around it) is an ANE of μ at state $\omega^{(2)}$. So, $ANE^{\mu}(\omega^{(2)}) = \{b\} = f(\omega^{(2)})$. Therefore, μ anonymously implements SCC f in NE in this example.

Notwithstanding, (D, M) is a NE of μ at state $\omega^{(2)}$ as $a \in C_A^{\omega^{(2)}}(O_A^{\mu}(M)) \cap C_B^{\omega^{(2)}}(O_B^{\mu}(D))$, $O_A^{\mu}(M) = \{a, b, c\}$, and $O_B^{\mu}(D) = \{a, b\}$. Yet, $O_A^{\mu}(M) \neq O_B^{\mu}(D)$ implies (D, M) is not an ANE of μ at state $\omega^{(2)}$.

The message profile (D, M), the NE at $\omega^{(2)}$, is unappealing if the following is a legitimate concern: The equilibrium behavior in the mechanism ends up resulting in discriminating between Ann and Bob in terms of opportunities provided even though the mechanism itself is symmetric.

Meanwhile, (D, M) being NE at $\omega^{(2)}$ also shows that μ does not implement f in NE since $NE^{\mu}(\omega^{(2)}) = \{a, b\} \neq \{b\} = f(\omega^{(2)})$.

Naturally, one may wonder whether or not there may be other mechanisms that implement SCC f in NE. In what follows, we establish that in this example, f is not implementable in NE.

To achieve a contradiction, suppose that SCC $f : \tilde{\Omega} \to \mathcal{X}$ were implementable in NE. Then, thanks to de Clippel's necessity result, we know that there is a profile of sets $\mathbf{S} = (S_i(x,\omega))_{i\in N,\omega\in\tilde{\Omega},x\in f(\omega)}$ consistent with f. In particular, for any $i \in N$, $\omega \in \tilde{\Omega}$, and $x \in f(\omega)$, $S_i(x,\omega)$ is given by $O_i^{\mu}(m_{-i}^{(x,\omega)})$ where $m^{(x,\omega)} \in M$ is a NE sustaining x, i.e., $g(m^{(x,\omega)}) = x \in \cap_{i\in N} C_i^{\omega}(O_i^{\mu}(m_{-i}^{(x,\omega)}))$. So, $f(\omega^{(2)}) = \{b\}$ and (i) of consistency implies $S_B(b,\omega^{(2)})$ equals either $\{b\}$ or $\{a,b\}$. If $S_B(b,\omega^{(2)}) = \{b\}$, then the mechanism μ has a NE $m^{(b,\omega^{(2)})} \in M$ such that $O_B^{\mu}(m_A^{(b,\omega^{(2)})}) = \{b\}$ (i.e., b constitutes Bob's only choice) and hence for all messages $m_B \in M_B$ we have $g(m_A^{(b,\omega^{(2)})}, m_B) = b$. So, $b \in O_A^{\mu}(m_B)$ for all $m_B \in M_B$. As b is Ann's top-ranked alternative at $\omega^{(1)}$ and $O_B^{\mu}(m_A^{(b,\omega^{(2)})}) = \{b\}$, we observe that $(m_A^{(b,\omega^{(2)})}, m_B)$ is a NE of μ at $\theta^{(1)}$ since $b \in C_A^{\omega^{(1)}}(O_A^{\mu}(m_B)) \cap C_B^{\omega^{(1)}}(O_B^{\mu}(m_A^{(b,\omega^{(2)})}))$. But, $b \notin f(\omega^{(1)}) = \{a\}$. Thus, $S_B(b,\omega^{(2)}) = \{a,b\}$ and $S_B(b,\omega^{(2)})$ cannot equal $\{b\}$. So, $S_B(b,\omega^{(2)}) = O_B^{\mu}(m_A^{(b,\omega^{(2)})}) = \{a,b\}$ and hence there exists $\tilde{m}_B \in M_B$ such that $g(m_A^{(b,\omega^{(2)})}, \tilde{m}_B) = a$; ergo, $a \in O_A^{\mu}(\tilde{m}_B)$. Then, because $a \in C_B^{\omega^{(2)}}(S_B(b,\omega^{(2)})) = C_B^{\omega^{(2)}}(\{a,b\}) = \{a,b\}$ and a is Ann's top-ranked alternative at $\omega^{(2)}$, a emerges as a Nash outcome (and message profile $(m_A^{(b,\omega^{(2)})}, \tilde{m}_B)$ as a NE) at $\omega^{(2)}$ since $a \in C_A^{\omega^{(2)}}(O_A^{\mu}(\tilde{m}_B)) \cap C_B^{\omega^{(1)}}(O_B^{\mu}(m_A^{(b,\omega^{(2)})}))$. But, $a \notin f(\omega^{(2)}) = \{b\}$. This finishes the proof as all the possibilities for $S_B(b,\omega^{(2)})$ are considered.

4 Necessity and Sufficiency

We start with our main condition for anonymous implementation in NE. In what follows, we show that this condition is necessary and (almost) sufficient for anonymous implementation of SCCs in NE. We note that the following condition is specified both for the rational and the behavioral domains.

Definition 2. *Given an environment $\langle N, X, \Omega, (C_i^{\omega})_{i\in N}\rangle$ and SCC f on domain $\tilde{\Omega}$, $f : \tilde{\Omega} \to \mathcal{X}$, a profile of sets $\mathbf{S} := (S(x,\omega))_{\omega\in\tilde{\Omega},\ x\in f(\omega)}$ is* ***anonymous consistent*** *with f on domain $\tilde{\Omega}$ if*

(i) for all $\omega \in \tilde{\Omega}$ and all $x \in f(\omega)$, $x \in \bigcap_{i\in N} C_i^{\omega}(S(x,\omega))$; and

(ii) $x \in f(\omega) \setminus f(\tilde{\omega})$ for any $\omega, \tilde{\omega} \in \tilde{\Omega}$ implies that $x \notin \bigcap_{i\in N} C_i^{\tilde{\omega}}(S(x,\omega))$.

Before proceeding further with the necessity and sufficiency results, we wish to display the relation of anonymous consistency with Maskin-monotonicity in the following lemma:

Lemma 1. *Given a rational environment $\langle N, X, \Omega, (C_i^{\omega})_{i\in N}\rangle$ and SCC $f : \tilde{\Omega} \to \mathcal{X}$, there exists a profile of sets anonymous consistent with f on domain $\tilde{\Omega}$ if and only if f satisfies the following* *(anonymous Maskin-monotonicity) condition on domain $\tilde{\Omega}$: For any $\omega, \tilde{\omega} \in \tilde{\Omega}$,*

$$x \in f(\omega) \text{ and } \cap_{i \in N} L_i^{\omega}(x) \subset \cap_{i \in N} L_i^{\tilde{\omega}}(x) \text{ implies } x \in f(\tilde{\omega}).$$

Proof of Lemma 1. Suppose that the environment $\langle N, X, \Omega, (C_i^{\omega})_{i \in N} \rangle$ is rational and SCC f defined on domain $\tilde{\Omega}$ is given by $f : \tilde{\Omega} \to \mathcal{X}$.

For the necessity direction of the lemma, suppose that $\mathbf{S} := (S(x, \omega))_{\omega \in \tilde{\Omega},\ x \in f(\omega)}$ is anonymous consistent with f on domain $\tilde{\Omega}$ and adopt the hypothesis that $\omega, \tilde{\omega} \in \tilde{\Omega}$, $x \in f(\omega)$, and $\cap_{i \in N} L_i^{\omega}(x) \subset \cap_{i \in N} L_i^{\tilde{\omega}}(x)$. Hence, by (i) of anonymous consistency, we see that $S(x, \omega) \subset \cap_{i \in N} L_i^{\omega}(x)$. Ergo, $S(x, \omega) \subset \cap_{i \in N} L^{\tilde{\omega}}(x)$. If $x \notin f(\tilde{\omega})$, then by (ii) of anonymous consistency, there is $j \in N$ such that $x \notin C_i^{\tilde{\omega}}(S(x, \omega))$. So, there is $j \in N$ and $y^* \in S(x, \omega)$ such that $y^* P_j^{\tilde{\omega}} x$; i.e., $y^* \notin L_j^{\tilde{\omega}}(x)$. But, $y^* \in S(x, \omega)$ and $y^* \notin L_j^{\tilde{\omega}}(x)$ contradicts $S(x, \omega) \subset \cap_{i \in N} L_i^{\tilde{\omega}}(x)$.

To establish the sufficiency direction, define $\mathbf{S}$ so that for any $\omega \in \tilde{\Omega}$ and $x \in f(\omega)$, we have $S(x, \omega) := \cap_{i \in N} L_i^{\omega}(x)$. Then, $\mathbf{S}$ satisfies (i) of anonymous consistency trivially due to the definition of lower contour sets. To obtain (ii) of anonymous consistency, suppose that $x \in f(\omega) \setminus f(\tilde{\omega})$ for some $\omega, \tilde{\omega} \in \tilde{\Omega}$. So, $S(x, \omega) = \cap_{i \in N} L_i^{\omega}$ is not a subset of $\cap_{i \in N} L_i^{\tilde{\omega}}(x)$. Thus, there is $j \in N$ and $y^* \in S(x, \omega)$ with $y^* \notin L_j^{\tilde{\omega}} x$; i.e. $y^* P_j^{\tilde{\omega}} x$. Ergo, $x \notin C_j^{\tilde{\omega}}(S(x, \omega))$. ■

Next, we present our result, providing a full characterization of SCCs that are anonymously implementable in NE (both in the rational and behavioral domains):

Theorem 1. *Given an environment $\langle N, X, \Omega, (C_i^{\theta})_{i \in N} \rangle$,*

(i) *if SCC $f : \tilde{\Omega} \to \mathcal{X}$ is anonymously implementable in NE on domain $\tilde{\Omega}$, then there is a profile of sets anonymous consistent with f on domain $\tilde{\Omega}$.*

(ii) *if there is a profile of sets anonymous consistent with a unanimous SCC $f : \tilde{\Omega} \to \mathcal{X}$, then f is anonymously implementable in NE on domain $\tilde{\Omega}$.*

Proof of (i) of Theorem 1. To prove (i) of Theorem 1, suppose that $f : \tilde{\Omega} \to \mathcal{X}$ is anonymously implementable in NE on domain $\tilde{\Omega}$. So, for all ω and all $x \in f(\omega)$, there is $m^{x,\omega} \in M$ s.t. $O_i^{\mu}(m_{-i}^{x,\omega}) = O_j^{\mu}(m_{-j}^{x,\omega})$ for all $i, j \in N$ and $g(m^{x,\omega}) = x \in \cap_{i \in N} C_i^{\omega}(O^{\mu}(m_{-i}^{x,\omega}))$. Define $\mathbf{S}$ as follows: for all ω and $x \in f(\omega)$, let $S(x, \omega) := O_i^{\mu}(m_{-i}^{x,\omega})$ for any $i \in N$. Then $\mathbf{S}$ satisfies (i) of anonymous consistency as for all $\omega \in \tilde{\Omega}$, and $x \in f(\omega)$, $g(m^{x,\omega}) = x \in \cap_{i \in N} C_i^{\omega}(O^{\mu}(m_{-i}^{x,\omega}))$ and $O_i^{\mu}(m_{-i}^{x,\omega}) = O_j^{\mu}(m_{-j}^{x,\omega})$ for all $i, j \in N$.

To show that $\mathbf{S}$ satisfies (ii) of anonymous consistency, suppose for some $\omega, \tilde{\omega} \in \tilde{\Omega}$, $x \in f(\omega) \setminus f(\tilde{\omega})$ and $x \in \cap_{i \in N} C_i^{\tilde{\omega}}(S(x, \omega))$. Then, $x \in \cap_{i \in N} C_i^{\tilde{\omega}}(O^{\mu}(m_{-i}^{x,\omega}))$. Since, $O_i^{\mu}(m_{-i}^{x,\omega}) = S(x, \omega) = O_j^{\mu}(m_{-j}^{x,\omega})$ for all $i, j \in N$, $m^{x,\omega}$ is an ANE at $\tilde{\omega}$ as $x = g(m^{x,\omega})$. Thus, we obtain the desired contradiction as $x \in f(\tilde{\omega})$ (thanks to μ implementing f anonymously in NE on domain $\tilde{\Omega}$). ■

Proof of (ii) **of Theorem 1.** Suppose SCC $f : \tilde{\Omega} \to \mathcal{X}$ is unanimous and the profile $\mathbf{S} = (S(x, \omega))_{\omega \in \Omega,\ x \in f(\omega)}$ is anonymous consistent with f on domain $\tilde{\Omega}$. Consider the canonical mechanism given as follows: $M_i = X \times \tilde{\Omega} \times X \times \mathbb{N}$ where $m_i = (x^i, \omega^i, y^i, k^i)$ with $x^i \in f(\omega^i)$, $y^i \in X$, $\omega^i \in \tilde{\Omega}$, and $k^i \in \mathbb{N}$ for all $i \in N$; the outcome function $g : M \to X$ defined by

Rule 1: If $m_i = (x, \omega, \cdot, \cdot)$ for all $i \in N$, then $g(m) = x$;

Rule 2: If $m_i = (x, \omega, \cdot, \cdot)$ for all $i \in N \setminus \{j\}$ for some $j \in N$ and $m_j \neq m_i$ with $m_j = (x', \omega', y', \cdot)$, then

$$g(m) = \begin{cases} x & \text{if } y' \notin S(x, \omega), \\ y' & \text{if } y' \in S(x, \omega). \end{cases}$$

Rule 3: In all other cases, $g(m) = y^{i^*}$ where $i^* = \max\{i \in N \mid k^i \geq k^j\ \forall j \in N\}$.

The result holds thanks to the following two claims.

Claim 1. *For all $\omega \in \tilde{\Omega}$ and $x \in f(\omega)$, $m^{(x,\omega)}$ defined by $m_i^{(x,\omega)} = (x, \omega, x, 1)$ is an ANE of μ at ω s.t. $g(m^{(x,\omega)}) = x$.*

Proof. Let $\omega \in \tilde{\Omega}$, $x \in f(\omega)$, and $m^{(x,\omega)}$ be as in the statement of the claim. Then, Rule 1 holds under $m^{(x,\omega)}$. So, $g(m^{(x,\omega)}) = x$, and due to Rules 1 and 2, $O_i^{\mu}(m_{-i}^{(x,\omega)}) = S(x, \omega)$ for all $i \in N$. By (i) of anonymous consistency, $g(m^{(x,\omega)}) = x \in \cap_{i \in N} C_i^{\omega}(S(x, \omega))$. So, $m^{x,\omega}$ is an ANE of μ at ω. ■

Claim 2. *If m^* is an ANE of μ at $\omega \in \tilde{\Omega}$, then $g(m^*) \in f(\omega)$.*

Proof. Suppose m^* is an ANE of μ at $\omega \in \tilde{\Omega}$.

Suppose additionally that Rule 1 holds under m^*. So, let $m_i^* = (x', \omega', \cdot, \cdot)$ with $\omega' \in \tilde{\Omega}$ and $x' \in f(\omega')$ for all $i \in N$. By Rules 1 and 2, $O_i^{\mu}(m_{-i}^*) = S(x', \omega')$ for all $i \in N$ and $g(m^*) = x'$. If $x' \notin f(\omega)$, then $x' \notin \cap_{i \in N} C_i^{\omega}(S(x', \omega'))$ (by (ii) of anonymous consistency); this is equivalent to $x' \notin \cap_{i \in N} C_i^{\omega}(O_i^{\mu}(m_{-i}^*))$ thanks to Rule 1; i.e., m^* is not an ANE of μ at ω. This delivers the desired contradiction and establishes that $g(m^*) = x' \in f(\omega)$ when Rule 1 holds under m^*.

If Rule 2 holds under m^*, then (by Rules 1,2, and 3) for all $i \in N \setminus \{j\}$ for some $j \in N$, $O_i^\mu(m^*_{-i}) = X$ and $O_j^\mu(m^*_{-j}) = S(x, \omega)$. Thus, $S(x, \omega) = X$ as m^* is an ANE. Then, as f is unanimous, $g(m^*) \in \cap_{i \in N} C_i^\omega(X)$ implies $g(m^*) \in f(\omega)$.

On the other hand, if Rule 3 holds under m^*, then for all $i \in N$, $O_i^\mu(m^*_{-i}) = X$. As m^* is an ANE, $g(m^*) \in \cap_{i \in N} C_i^\omega(X)$. This implies that $g(m^*) = f(\omega)$ since f is unanimous. ■ ■

5 Efficiency

In rational environments, the Pareto SCC on the full domain Ω is $PO : \Omega \to \mathcal{X}$ defined by

$$PO(\omega) := \{x \in X \mid \nexists y^* \in X \text{ s.t. } y^* P_i^\omega x \ \forall i \in N\}$$

for any $\omega \in \Omega$. On the other hand, in behavioral environments, the efficiency SCC we consider is defined on the full domain by $E^{\text{eff}} : \Omega \to \mathcal{X}$ with

$$E^{\text{eff}}(\omega) := \{x \in X \mid \exists (S_i)_{i \in N} \in \mathcal{X}^N \text{ s.t. } x \in \cap_{i \in N} C_i^\omega(S_i) \text{ and } \cup_{i \in N} S_i = X\}$$

for any $\omega \in \Omega$. We know that when $\tilde{\Omega}$ is a subset of the rational domain, then these two notions coincide, and hence efficiency SCC is an extension of the Pareto SCC to behavioral domains (de Clippel, 2014). Moreover, as choices are nonempty-valued, so are these SCCs: We observe that for all ω (in rational or behavioral domains) $x \in C_1^\omega(X)$ implies $x \in E^{\text{eff}}(\omega)$ by setting $S_1 = X$ and $S_j = \{x\}$ for all $j \neq 1$.

Below, we report bad news about the anonymous implementation of these efficiency notions in NE.

Definition 3. *Given environment $\langle N, X, \Omega, (C_i^\theta)_{i \in N} \rangle$, we say that the domain $\tilde{\Omega} \subset \Omega$ satisfies condition NI if there are $\omega, \tilde{\omega} \in \tilde{\Omega}$ such that for some $x \in E^{eff}(\omega) \setminus E^{eff}(\tilde{\omega})$, $x \in \cap_{i \in N} C_i^\omega(S)$ implies $x \in \cap_{i \in N} C_i^{\tilde{\omega}}(S)$.*

In words, regardless of whether or not the environment is behavioral or rational, condition NI demands the existence of two states ω and $\tilde{\omega}$ in the domain $\tilde{\Omega}$ on which efficiency SCC is defined such that there is an alternative x that is efficient at ω but not at $\tilde{\omega}$ while the following holds: If x is chosen from a set S at ω by all individuals, then x must be chosen from S at $\tilde{\omega}$ by all agents. That is why condition NI holds if there are $\omega, \tilde{\omega} \in \tilde{\Omega}$ such that for some $i \in N$ and $x \in E^{\text{eff}}(\omega) \setminus E^{\text{eff}}(\tilde{\omega})$, $x \in C_i^{\omega}(S)$ implies $S = \{x\}$. Consequently, we observe that the full rational domain satisfies condition NI, let $\omega, \tilde{\omega}$ be such that $L_1^{\omega}(x) = X$, $L_2^{\omega}(x) = \{x\}$, and $\cup_{i \in N} L_i^{\tilde{\omega}}(x) \neq X$.

The following result holds both in the rational and behavioral domains:

Proposition 1. *Given an environment $\langle N, X, \Omega, (C_i^{\theta})_{i \in N} \rangle$, efficiency SCC $E^{\mathit{eff}} : \tilde{\Omega} \to \mathcal{X}$ defined on domain $\tilde{\Omega}$ is not anonymously implementable in NE on domain $\tilde{\Omega}$ whenever this domain satisfies condition NI.*

Proof of Proposition 1. Let $\tilde{\Omega} \subset \Omega$ be a domain that satisfies condition NI, and assume that efficiency SCC $E^{\text{eff}} : \tilde{\Omega} \to \mathcal{X}$ is defined on domain $\tilde{\Omega}$. To achieve a contradiction, suppose SCC E^{eff} is anonymously implementable in NE on domain $\tilde{\Omega}$ by mechanism μ. Then, for all $\omega \in \tilde{\Omega}$ and $x \in E^{\text{eff}}(\omega)$, there is $m^{x,\omega} \in M$ such that $g(m^{x,\omega}) = x$, $O_i^{\mu}(m_{-i}^{x,\omega}) = O_j^{\mu}(m_{-j}^{x,\omega})$ for all $i, j \in N$, and $x \in \cap_{i \in N} C_i^{\omega}(O_i^{\mu}(m_{-i}^{x,\omega}))$.

As $\tilde{\Omega}$ satisfies condition NI, there are $\omega^{(1)}, \omega^{(2)} \in \tilde{\Omega}$ such that for some $x^* \in E^{\text{eff}}(\omega^{(1)}) \setminus E^{\text{eff}}(\tilde{\omega}^{(2)})$, $x^* \in \cap_{i \in N} C_i^{\omega^{(1)}}(S)$ implies $x^* \in \cap_{i \in N} C_i^{\omega^{(2)}}(S)$. Then, from the above we know that $m^{x^*,\omega^{(1)}}$ is such that $g(m^{x^*,\omega^{(1)}}) = x^*$ and $O_i^{\mu}(m_i^{x^*,\omega^{(1)}}) = S^*$ for all $i \in N$; and $x^* \in \cap_{i \in N} C_i^{\omega^{(2)}}(S^*)$. But then, $m^{x^*,\omega^{(1)}}$ is also an ANE of μ at $\omega^{(2)}$ which implies (thanks to μ anonymouysly implementing E^{eff} in NE) $x^* \in E^{\text{eff}}(\omega^{(2)})$, a contradiction. ■

An immediate corollary of our result is as follows:

Corollary 1. *Given a rational environment $\langle N, X, \Omega, (C_i^{\theta})_{i \in N} \rangle$, Pareto SCC $PO : \Omega \to \mathcal{X}$ defined on the full domain not anonymously implementable in NE.*

Notwithstanding, anonymous implementation of the Pareto SCC in NE on subdomains can be achieved as the following example demonstrates: The two agents are Ann and Bob, $X = \{a, b, c\}$,

$\tilde{\Omega} = \{\omega^{(1)}, \omega^{(2)}\}$, and agents' rankings are as follows:

$\omega^{(1)}$		$\omega^{(2)}$	
$R_A^{\omega^{(1)}}$	$R_B^{\omega^{(1)}}$	$R_A^{\omega^{(2)}}$	$R_2^{\omega^{(2)}}$
a	b	b	c
b	a	c	b
c	c	a	a

Pareto SCC PO on $\tilde{\Omega}$ is given by $PO(\omega^{(1)}) = \{a, b\}$ and $PO(\omega^{(2)}) = \{b, c\}$. One can verify that the following mechanism implements the Pareto SCC in ANE on domain $\tilde{\Omega}$ (where ANE at $\omega^{(1)}$ are depicted by circling the corresponding cells and those at $\omega^{(2)}$ by indicating them with squares):

		Bob			
		L	M_1	M_2	R
	U	a	c	c	a
Ann	C_1	c	b	c	b
	C_2	c	c	$\boxed{c}$	a
	D	a	b	a	$\boxed{b}$

6 Concluding Remarks

Employing the notion of ANE allows us to restrict attention to anonymity while not limiting the mechanisms under consideration as an ANE of a mechanism at a state is a NE, and the opportunity sets of all individuals equal. We identify the necessary and (almost) sufficient condition for (full) implementation of social choice correspondence in ANE, namely, *anonymous consistency*. This condition parallels de Clippel (2014)'s consistency with the new restriction that choice sets are independent of individuals' identities. We establish that implementation in ANE does not necessarily restrict the set of social goals under consideration: In our example in Section 3, we describe an environment and a social choice correspondence that is implementable in ANE but not in NE. Our observation justifies that anonymity provides society with additional ANE-decentralizable social choice correspondences that are otherwise not implementable in NE. Notwithstanding, we show that anonymity imposes a heavy burden when dealing with efficiency: The Pareto social choice correspondence is not implementable in ANE on the full domain.

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