ORBITS OF TENSORS OVER FINITE FIELDS

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ABSTRACT

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Keywords: Tensors, Ranks, Segre Variety, Veronese Surface, Linear Systems of Conics

This thesis forms part of a project aiming to classify subspaces of $\mathrm{PG}(5,q)$ under the action of the subgroup $K < \mathrm{PGL}(6,q)$ stabilising the Veronese surface $\mathcal{V}(\mathbb{F}_q)$, where $\mathbb{F}_q$ is the finite field of order q. Firstly, we determine the K-orbits of solids of $\mathrm{PG}(5,q)$ in the case where q is even. We compute as well two useful combinatorial invariants of each type of solids, namely their point-orbit and hyperplane-orbit distributions. Additionally, we calculate the stabiliser of each orbit representative, and thereby obtain the size of each orbit. The classification of solids in $\mathrm{PG}(5,q)$ corresponds to the classification of pencils of conics in $\mathrm{PG}(2,q)$, q even. The latter classification was incompletely obtained by Campbell in 1927. Our results complete Campbell's work and correct two of his claims. Moreover, we give a partial classification of planes in $\mathrm{PG}(5,q)$, q even. Specifically, we determine the K-orbits of planes intersecting the Veronese surface in at least one point. Our proof is geometric based on studying the different types of points that are incident with a plane $\pi \subset \mathrm{PG}(5,q)$. In some cases, point orbit-distributions are not sufficient to characterise each orbit, and we tend to determine stronger geometric-combinatorial invariants such as

line-orbit distributions and inflexion points. Finally, we introduce the GAP package, *T233*, which uses some functionality from the FinInG package to determine G-orbits and ranks of points in $\mathrm{PG}(\mathbb{F}_q^2 \otimes \mathbb{F}_q^3 \otimes \mathbb{F}_q^3) \cong \mathrm{PG}(17, q)$, where G is the group stabilising the Segre variety $S_{1,2,2}(\mathbb{F}_q)$. Note that, the algorithms defined in *T233* and the combinatorial tools introduced earlier can be generalised to higher-ordered tensor product spaces, and thus one may extend these implementation tools and classifications to higher-ordered tensor product spaces.

ÖZET

TENSÖRLERİN SONLU CİSİMLER ÜZERİNDEKİ YÖRÜNGELERİ

NOUR ALNAJJARINE

MATEMATİK DOKTORA TEZİ, Temmuz 2022

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Anahtar Kelimeler: Tensörler, Sıralamalar, Segre Variety, Veronese Yüzeyini, Lineer Konik Sistemler

Bu tez, Veronese yüzeyini $\mathcal{V}(\mathbb{F}_q)$ dengeleyen $K < \mathrm{PGL}(6, q)$ alt grubunun etkisi altında $\mathrm{PG}(5, q)$ alt uzaylarını sınıflandırmayı amaçlayan bir projenin parçasıdır, burda $\mathbb{F}_q$, q dereceli sonlu cisimdir. İlk olarak, q çift iken $\mathrm{PG}(5, q)$ solidlerinin K-yörüngenin belirliyoruz. Her solid tipi için nokta yörünge ve hiper düzlem yörünge dağımları olmak üzere iki kullanışlı kombinatoryal değişmezi de hesaplıyoruz. Ek olarak her yörünge temsilcisinin dengeleyicisini hesaplıyoruz ve böylece her yörüngenin boyutunu elde ediyoruz $\mathrm{PG}(5, q)$ üzerinde solidlerin sınıflandırılması, q çift iken $\mathrm{PG}(2, q)$ üzerinde konik kalemlerin sınıflandırılmasını karşılık gelir. İkinci sınıflandırma bilinmektedir, fakat litaratürde hiç bir kanıtın kaydedilmediği, genellikle Campbell'in yalnızca tamamlanmamış bir sınıflandırma içeren 1927 tarihli bir makalesine işaret edilmektedir. Yaklaşımımız, Campbell'in düzeltip tamamladığımız çalışmasından farklı ve bağımsızdır. Ayrıca, q çift iken, $\mathrm{PG}(5, q)$'de düzlemlerin kısmı bir sınıflandırmasını veriyoruz. Özellikle, Veronese yüzeyini en az bir noktada kesen düzlemlerin K-orbitlerini belirliyoruz. Kanıtımız geometrik olarak bir $\pi \subset \mathrm{PG}(5, q)$ düzlemi ile ilişkili olan farklı nokta tiplerini incelemeye dayanmaktadır. Bazı durm-

larda, nokta dağılımları her bir yörüngeyi karakterize etmek için yeterli değildir. Ve çizgi yörünge dağılımları ve bükülme noktaları gibi daha güçlü geometrik kombinatoryal değişmezleri belirleme eğilimindeyiz. Son olarak, $\mathrm{PG}(\mathbb{F}_q^2 \otimes \mathbb{F}_q^3 \otimes \mathbb{F}_q^3) \cong \mathrm{PG}(17, q)$ içindeki G-yörüngelerini ve nokta sınıflarını belirlemek için FinInG paketinden bazı işlevleri kullanan *T233* paketini tanıtıyoruz, burada G Segre çeşidi $S_{1,2,2}(\mathbb{F}_q)$ 'yü dengeleyen gruptur. *T233* 'te tanımlanan algoritmaların ve daha önce tanıtılan kombinatoryal araçların daha yüksek sıralı tensör çarpım uzaylarına genelleştirilebileceğini ve bu uygulama araçlarının ve sınıflandırmaların daha yüksek sıralı alanlara genelleştirilmesi olasılığını önerdiğini unutmayın.

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To my dear family

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1 INTRODUCTION

Tensors are fundamental in mathematics and physics with numerous applications in complexity theory (Landsberg, 2011), representation theory (Kloda & Bader, 2009), signal processing (De Lathauwer & De Moor, 1998) and numerical linear algebra (De Lathauwer, De Moor & Vandewalle, 2000). For instance, the problem of determining the complexity of matrix multiplication can be rephrased as the problem of determining the minimum number of arithmetic operations needed to multiply two square matrices. This problem is equivalent to determining the rank of a particular tensor (the matrix multiplication operator), and it has been only solved for 2×2-matrices (Strassen, 1969; Winograd, 1971).

Many applications of tensors are concerned with the following types of questions. Let $V = \mathbb{F}^{m_1} \otimes ... \otimes \mathbb{F}^{m_t}$ be a tensor product space defined over a field $\mathbb{F}$ and $A \in V$.

1.1 **Decomposition:** Can we write A as the sum of k fundamental tensors (tensors of the form: $v_1 \otimes ... \otimes v_t$); $k \in \mathbb{N} \setminus \{0\}$?

1.2 **Uniqueness:** If such a writing exists, is it unique?

1.3 **Algorithms:** Do we have algorithms to determine the rank of A and to decompose A as the sum of fundamental tensors?

1.4 **Classification:** Can we classify tensors in V under the action of some natural groups such as the group stabilising fundamental tensors or its subgroup defined by $\mathrm{GL}(\mathbb{F}^{m_1}) \times ... \times \mathrm{GL}(\mathbb{F}^{m_t})$?

In most tensor decomposition problems the first issue to resolve is to determine the rank of the tensor, which is not always an easy task (Håstad, 1990). In general, most of the known results on tensors are considered over the complex field or algebraically closed fields (Kloda & Bader, 2009; Landsberg, 2011). However, we are interested in tensors over finite fields, and we focus particularly on the algorithmic and the classification types of questions.

The group $H = \mathrm{GL}(\mathbb{F}^{m_1}) \times ... \times \mathrm{GL}(\mathbb{F}^{m_t})$ acts on the set of fundamental tensors in V via $(v_1 \otimes \cdots \otimes v_m)^{(g_1,\ldots,g_m)} = v_1^{g_1} \otimes \ldots v_m^{g_m}$, and on all V by linearity. If some of the m_i's are equal, then we can extend H by a subgroup of the symmetric group Sym_m to obtain the group G defined as the setwise stabiliser of fundamental tensors in V. One may seek then to classify the G-orbits of tensors in V. This is an elementary problem when $t = 2$ and becomes more difficult, depending on the field and the m_i's, when $t \geq 3$. For instance, Lavrauw and Sheekey classified in (Lavrauw & Sheekey, 2015) G-orbits of tensors in $V = \mathbb{F}^2 \otimes \mathbb{F}^3 \otimes \mathbb{F}^3$. Precisely, they proved the existence of 15, 17, or 18 such G-orbits depending on the field being algebraically closed, the real space, or a finite field respectively.

Indeed, one may also look at the G-orbits of *subspaces* of a given tensor product space. For instance, Lavrauw and Sheekey classified the 2-dimensional subspaces of $\mathbb{F}^3 \otimes \mathbb{F}^3$ under the action of $\mathrm{GL}(3,q)^2 \wr \mathrm{Sym}_2$ by suitably contracting tensors in $V = \mathbb{F}^2 \otimes \mathbb{F}^3 \otimes \mathbb{F}^3$ (Lavrauw & Sheekey, 2015, pp. 136–137). Basically, this classification was done as a part of studying the different types of tensors in V (Lavrauw & Sheekey, 2015).

Similar questions arise when considering the space $W = S^n\mathbb{F}^m$ of symmetric tensors in $V = \mathbb{F}^m \otimes ... \otimes \mathbb{F}^m$ and the action of $G = \mathrm{GL}(V)$ on W defined by $(v \otimes \cdots \otimes v)^g = v^g \otimes \cdots \otimes v^g$ and expanding linearly. In this case, fundamental tensors in W correspond to points of the Veronese surface in $\mathrm{PG}(W)$, and one may use this connection to extract information from tensors in W. We draw a particular attention to the case where $n = 2$ and $m = 3$. Under this setting, rank-1 tensors in W correspond to points of the Veronese surface $\mathcal{V}(\mathbb{F}) \subset \mathrm{PG}(5,\mathbb{F})$, and G induces a subgroup of $\mathrm{PGL}(6,q)$, $K \cong \mathrm{PGL}(3,\mathbb{F})$, leaving $\mathcal{V}(\mathbb{F})$ invariant. Moreover, subspaces of $\mathrm{PG}(5,\mathbb{F})$ correspond to *linear systems of conics* in $\mathrm{PG}(2,\mathbb{F})$. In particular, lines, planes and solids in $\mathrm{PG}(5,\mathbb{F})$ correspond to 3-, 2- and 1-dimensional linear systems, respectively, namely: *webs*, *nets* and *pencils* of conics. Therefore, classifying K-orbits of subspaces in $\mathrm{PG}(5,\mathbb{F})$ correspond to classifying linear systems of conics in $\mathrm{PG}(2,\mathbb{F})$ up to projective equivalence.

This problem is completely determined over $\mathbb{R}$ and $\mathbb{C}$ by Jordan and Wall who classified pencils and nets of conics respectively over these fields ((Jordan, 1906), (Jordan, 1907) , (Wall, 1977)). More precisely, pencils of conics correspond to solids of $\mathrm{PG}(5,\mathbb{F})$, which correspond in turn to lines of $\mathrm{PG}(5,\mathbb{F})$ through a particular polarity α of $\mathrm{PG}(5,\mathbb{F})$ defined over non-characteristic 2 fields. Similarly, one can obtain the classification of planes of $\mathrm{PG}(5,\mathbb{F})$ from that of nets of conics in $\mathrm{PG}(2,\mathbb{F})$. In general, K-orbits of points, which correspond to K-orbits of hyperplanes through α, are easily obtained yielding to the complete classification of subspaces of $\mathrm{PG}(5,\mathbb{F})$; $\mathbb{F} \in \{\mathbb{R},\mathbb{C}\}$.

As mentioned earlier, we are interested in working over finite fields. Let $\mathbb{F} = \mathbb{F}_q$ for some prime power q. In this case, the subgroup $K \cong \mathrm{PGL}(3,q)$ is the setwise stabiliser of $\mathcal{V}(\mathbb{F}_q)$, unless $q = 2$. If $q = 2$, then $\mathrm{PGL}(3,2)$ is strictly contained in the setwise stabiliser of $\mathcal{V}(\mathbb{F}_2) \cong \mathrm{Sym}_7$. For q odd, points, lines, solids and hyperplanes are completely classified in $\mathrm{PG}(5,q)$. Indeed, K-orbits of lines and solids can be deduced from the classification of pencil of conics in (Dickson, 1908). Moreover, planes in $\mathrm{PG}(5,q)$, q odd, are partially classified by Lavrauw et al. in (Lavrauw, Popiel & Sheekey, 2020,2). For q even, K-orbits of points and hyperplanes are easily determined, and K-orbits of lines are given in (Lavrauw & Popiel, 2020). In principle, K-orbits of solids can be deduced from the classification of pencils of conics over finite fields of even characteristic, which is recorded in (Hirschfeld, 1998, Theorem 7.31). However, to the best of our knowledge, there is no proof in the literature for the latter classification, which is attributed to Campbell (Campbell, 1927), who provided only an incomplete classification.

In this thesis, we classify and characterise solids in $\mathrm{PG}(5,q)$, q even, and thus we obtain an independent proof of the classification of pencils of conics over characteristic two fields. Our proof, which shows the existence of 15 K-orbits of solids, relies on studying some combinatorial invariants such as point-orbit and hyperplane-orbit distributions, which measure the number of different types of points and hyperplanes in $\mathrm{PG}(5,q)$ incident with a solid $S \subseteq \mathrm{PG}(5,q)$. Note that hyperplane-orbit distributions can be interpreted in the setting of pencils of conics as counting the number of double lines, pairs of real lines, pairs of conjugate imaginary lines, and nonsingular conics contained in each type of pencil. Our work is structured as follows. We start by considering for an arbitrary solid $S \subseteq \mathrm{PG}(5,q)$ the possible hyperplane-orbit distributions. Then, we discuss if solids having the same hyperplane-orbit distribution split under the action of $K \cong \mathrm{PGL}(3,q)$ or not. Sometimes, the distribution of points and hyperplanes are not sufficient to distinguish between orbits. In such cases, we tend to study some further combinatorial invariants such as line-orbit distributions. Additionally, we calculate the stabiliser in K of each orbit representative, and thereby determine the size of each orbit. Finally, we compare our classification with Campbell's work (Campbell, 1927). We note that our arguments intentionally exploit the connection between solids in $\mathrm{PG}(5,q)$ and pencils of conics in $\mathrm{PG}(2,q)$. By this we mean that we generally aim to use each point of view to its advantage. For instance, there seems to be no obvious way to calculate the point-orbit distribution of a solid by working directly with the associated pencil of conics. On the other hand, stabilisers are sometimes significantly easier to compute by working with pencils of conics, since we can appeal to well-known transitivity properties of the natural action of $\mathrm{PGL}(3,q)$ on $\mathrm{PG}(2,q)$.

This is only one part of our aim. We also classify planes in $\mathrm{PG}(5,q)$, q even, which intersect the Veronese surface in at least one point. In particular, we prove that we have exactly 15 such orbits defined under the action of the group K stabilising $\mathcal{V}(\mathbb{F}_q)$. We change our perspective when classifying planes to study the possible point-orbit distributions instead of hyperplane-orbit distributions. Namely, the four-tuple $[r_1, r_{2n}, r_{2s}, r_3]$, where r_i is the number of rank-i points in a plane $\pi \subseteq \mathrm{PG}(5,q)$ for $i \in \{1,3\}$, r_{2n} is the number of rank-2 points in π meeting the nucleus plane and r_{2s} is the number of the remaining rank-2 points in π. Note that, unlike fields of odd characteristic, planes with at least one rank-1 point over characteristic-2 fields do not correspond to *rank-1* nets of conics, namely nets with at least one double line. In general, determining the point orbit-distributions is not sufficient to distinguish between the 15 orbits. For this reason, we use stronger geometric-combinatorial tools such as line-orbit distributions and inflexion points to completely characterise each orbit. We believe that these combinatorial tools can be generalised to higher-ordered tensor product spaces, and thus one may look at the classification problem in the generalised sense.

Finally, we introduce the GAP-package, *T233*, which uses some functionality from the FinInG package to determine orbits and ranks of points in $\mathrm{PG}(\mathbb{F}_q^2 \otimes \mathbb{F}_q^3 \otimes \mathbb{F}_q^3) \cong \mathrm{PG}(17,q)$. Our algorithms are based on the classification of tensors in $V = \mathbb{F}_q^2 \otimes \mathbb{F}_q^3 \otimes \mathbb{F}_q^3$ under the action of the subgroup of $\mathrm{GL}(V)$ stabilising the set of fundamental tensors in V (Lavrauw & Sheekey, 2015). We illustrate the importance of *T233* by Example 5.1 which shows how hard it would be to compute ranks of tensors in $\mathrm{PG}(17,q)$ without this package.

1.1 Thesis Organization

In Chapter 2, we collect some definitions and theory needed in our main results. We start with an overview of projective spaces over finite fields and some basic definitions in group theory. We recall as well some algebraic sets that are strongly related to our work. We discuss then solutions of quadratic and cubic equations over finite fields. Later, we give a detailed review about tensors, their representations and properties over finite fields. Lastly, we introduce the problem of classifying subspaces of $\mathrm{PG}(5,q)$ under the action of the group stabilising the Veronese surface and we explain its connection with linear systems of conics.

In Chapter 3, we present our results from (Alnajjarine, Lavrauw & Popiel, 2022) published in the journal of Finite Fields and Their Applications. In particular, we classify orbits of solids of $\mathrm{PG}(5,q)$, q even, under the action of the subgroup K of $\mathrm{PGL}(6,q)$ stabilising the Veronese surface. We also determine two useful combinatorial invariants of each type of solid, namely their *point-orbit* and *hyperplane-orbit distributions*. Additionally, we calculate the stabiliser in $\mathrm{PGL}(3,q)$ of each (type of) solid S, and thereby determine the size of each orbit. Finally, we compare our work with Campbell's partial classification of pencils of conics.

In Chapter 4, we present our results from (Alnajjarine & Lavrauw, 2022). Particularly, we determine the K-orbits of planes having at least one rank-1 point in $\mathrm{PG}(5,q)$, q even. Specifically, unless $q=2$, we prove the existence of 15 such orbits. In general, we distinguish between orbits using point-orbit distributions, line-orbit distributions and inflexion points. Our discussion is structured as follows. We start by considering planes intersecting the Veronese surface $\mathcal{V}(\mathbb{F}_q)$ in at least three points. We then classify planes meeting $\mathcal{V}(\mathbb{F}_q)$ in exactly two points. Finally, we deal with planes having a unique intersection with $\mathcal{V}(\mathbb{F}_q)$.

In Chapter 5, we introduce the GAP-package *T233* which is concerned with finding orbits and ranks of points in $\mathrm{PG}(17,q)$. We start by explaining the implementation of our main and auxiliary codes. We then find representatives of the orbits o_{10}, o_{15} and o_{17}. At the end, we give an example showing the importance of *T233* while computing ranks of tensors over finite fields with large orders. For a detailed description of the codes in *T233* and for more examples, we refer to the webpage (Alnajjarine & Lavrauw, 2020) and to our paper (Alnajjarine & Lavrauw, 2020) published in the proceedings of MACIS 2019, Lecture Notes in Computer Science.

2 PRELIMINARIES

In this chapter, we collect some preliminary definitions, notations and results that we use throughout the study.

2.1 Projective spaces over finite fields

Throughout the thesis, let $\mathbb{F}_q$ denotes a finite field of order q where $q = p^h$ for some prime p and positive integer h.

Definition 2.1. *Let U be an (n+1)-dimensional vector space defined over $\mathbb{F}_q$. The n-dimensional Desarguesian projective space,* $\mathrm{PG}(U)$ *or* $\mathrm{PG}(n,q)$*, is the quotient space of $U \setminus \{0\}$ by the equivalence relation $\sim$ defined by: $x \sim y \iff y = \lambda x$, for some $\lambda \in \mathbb{F}_q \setminus \{0\}$.*

The m-subspaces of $\mathrm{PG}(n,q)$ are the $(m+1)$-dimensional subspaces of U. In particular, *points*, *lines*, *planes*, *solids* and *hyperplanes* of $\mathrm{PG}(n,q)$ are the 1-dimensional, 2-dimensional, 3-dimensional, 4-dimensional and n-dimensional subspaces of U respectively. The homogeneous coordinates of a point P in $\mathrm{PG}(n,q)$ are usually denoted by $(x_0 : ... : x_n) = \lambda(x_0, ..., x_n)$, however, for simplicity, we will use the notation $(x_0, ..., x_n)$.

Alternatively, we may define a Desarguesian projective space $\mathrm{PG}(n,q)$ by starting with an affine space $\mathrm{AG}(n,q)$, which is simply $\mathbb{F}_q^n$ with its lattice of subspaces and their translates, and add a hyperplane at infinity defined by parallel classes. More specifically, the m-dimensional subspaces of the hyperplane at infinity are the parallel classes of the $(m+1)$-dimensional subspaces of $\mathrm{AG}(n,q)$. Conversely, given a projective space $\mathrm{PG}(n,q)$, we can obtain an affine space by deleting a hyperplane with its subspaces.

Sometimes we refer to $\mathrm{PG}(U)$ as the *projective geometry associated with* U. Let U_1, U_2 be two vector subspaces of U. The dimension of $\langle \mathrm{PG}(U_1), \mathrm{PG}(U_2)\rangle$ is given by

$$dim(\langle \mathrm{PG}(U_1), \mathrm{PG}(U_2)\rangle) = dim(\mathrm{PG}(U_1)) + dim(\mathrm{PG}(U_2)) - dim(\mathrm{PG}(U_1 \cap U_2)), \tag{2.1}$$

which follows from the Grassmann dimension formula for vector spaces. Note that $\mathrm{PG}(U_1+U_2) = \langle \mathrm{PG}(U_1), \mathrm{PG}(U_2)\rangle$ and $\mathrm{PG}(U_1 \cap U_2) = \mathrm{PG}(U_1) \cap \mathrm{PG}(U_2)$.

The following two theorems are direct applications of (2.1) and its generalisation to a finite set of subspaces.

Theorem 2.1. *Two distinct hyperplanes of* $\mathrm{PG}(n,q)$ *intersect in a subspace of dimension* $n-2$.

Theorem 2.2. *A* k*-dimensional subspace of* $\mathrm{PG}(n,q)$ *is the intersection of* $n-k$ *hyperplanes of* $\mathrm{PG}(n,q)$.

In particular, planes and solids of $\mathrm{PG}(5,q)$ are the intersection of three and two hyperplanes respectively. Recall that a hyperplane $\mathcal{H}$ in $\mathrm{PG}(n,q)$ is defined by a linear form, $f = a_0X_0 + a_1X_1 + ... + a_nX_n \in \mathbb{F}_q[X_0, ..., X_n]$, where

$$\mathcal{H} = \mathcal{Z}(f) := \{(x_0, ..., x_n) \in \mathrm{PG}(n,q) : f(x_0, ..., x_n) = 0\} \tag{2.2}$$

and $[a_0, ..., a_n]$ are the *dual coordinates* of $\mathcal{H}$.

The next proposition is a collection of some known combinatorial properties of subspaces of $\mathrm{PG}(n,q)$.

Proposition 2.1.

- *The number of points of* $\mathrm{PG}(n,q)$ *is*

$$\frac{q^{n+1}-1}{q-1} = q^n + q^{n-1} + ... + q + 1.$$

- *The number of* m*-dimensional subspaces of* $\mathrm{PG}(n,q)$ *is*

$$\binom{n+1}{m+1}_q = \frac{(q^{n+1}-1)(q^{n+1}-q)...(q^{n+1}-q^m)}{(q^{m+1}-1)(q^{m+1}-q)...(q^{m+1}-q^m)} = \frac{(q^{n+1}-1)(q^n-1)...(q^{n-m+1}-1)}{(q^{m+1}-1)(q^m-1)...(q-1)}.$$

- *The number of k-dimensional subspaces through a given m-dimensional subspace of* $\mathrm{PG}(n,q)$*; $k \geq m$, is*

$$\binom{n-m}{k-m}_q = \frac{(q^{n-m}-1)(q^{n-m-1}-1)...(q^{n-k+1}-1)}{(q^{k-m}-1)(q^{k-m-1}-1)...(q-1)}.$$

Example 2.1. *The projective plane* $\mathrm{PG}(2,q)$ *has q^2+q+1 points. The number of lines in* $\mathrm{PG}(2,q)$ *is also q^2+q+1, which can be deduced from Proposition 2.1 or by applying the* principle of duality*. Each two lines intersect in a unique point and each line has $q+1$ points. Dualizing the last statement, gives the following further properties: each 2 points lie on a unique line and each point lies on $q+1$ lines of* $\mathrm{PG}(2,q)$*.*

We end this section by recalling some special components of $\mathrm{PG}(n,q)$. A *frame* of $\mathrm{PG}(n,q)$ is an ordered tuple of $n+2$ points, having no $n+1$ points contained in a hyperplane. A well-known example of a frame is the *standard frame* defined by the canonical basis $\{e_0,...e_n\}$ of $\mathbb{F}_q^{n+1}$ as $(P_0,P_1,...,P_{n+1})$ where $P_i = \langle e_i \rangle$; $0 \leq i \leq n$, and $P_{n+1} = \langle e_0 + ... + e_n \rangle$. A *flag* Γ of a projective space is a chain of subspaces of distinct dimensions

$$\mathrm{PG}(U_0) \subset \mathrm{PG}(U_1)... \subset \mathrm{PG}(U_r), \tag{2.3}$$

whose length is the number of nontrivial subspaces in (2.3), i.e, subspaces different from $\mathrm{PG}(n,q)$ and the empty space. Lastly, we define an *antiflag* in $\mathrm{PG}(2,q)$ as a non-incident point-line pair.

2.2 Group theory

In this section, we recall some known group-theoretic notations and theorems that we use frequently in our results.

2.2.1 Group actions

Definition 2.2. *The action of a group G on a non-empty set X is defined by the map: $G \times X \to X$, $(g,x) \mapsto x^g$ satisfying:*

(i) $x^1 = x$, *for all* $x \in X$.

(ii) $(x^{g_1})^{g_2} = x^{g_1 g_2}$, *for all* $x \in X$ *and* $g_1, g_2 \in G$.

For the group-action (G,X), the stabiliser of $x \in X$ is the subgroup of G defined as

$$G_x = \{g \in G : x^g = x\}.$$

The orbit of x is the subset of X defined as

$$x^G = \{x^g : g \in G\}.$$

For $x \neq y \in X$, we have either $x^G = y^G$ or $x^G \cap y^G = \emptyset$. Moreover, the set $\{x^G\}_{x \in X}$ forms a partition of X.

Theorem 2.3. *(The Orbit-Stabiliser Theorem)*
Consider the group action (G,X). There exists a 1-to-1 correspondence between x^G and cosets of G_x in G. Furthermore, if G is finite, then $|x^G| = [G : G_x]$.

A group-action is *transitive* if for all $x \neq y \in X$, there exists $g \in G$ such that $x^g = y$. In particular, if such a "g" is unique for all pairs $(x,y) \in X^2$, then the action is called *regular* or *sharply transitive*.

2.2.2 Direct and semidirect products

Definition 2.3. *Let $(G,\circ)$ and $(H,\star)$ be two groups.*

(i) The direct product of G and H, $G \times H$, is the group defined by $(g_1,h_1) * (g_2,h_2) = (g_1 \circ g_2, h_1 \star h_2)$. The direct product of m-copies of G is denoted by G^m.

(ii) Let ϕ be a group homomorphism from G to the group of automorphisms of H, $Aut(H)$, defined by $g\phi = \phi_g$. The semidirect product of G by H, $G \rtimes H$ or $G \rtimes_\phi H$, is the group $(G \times H, *)$ defined by $(g_1,h_1) * (g_2,h_2) = (g_1 \circ g_2, (h_1(g_2\phi)) \star h_2)$.

Theorem 2.4. *(Recognition Theorem for Direct Products)*
Let H and K be two subgroups of a group G, such that:

(i) $H, K \trianglelefteq G$,

(ii) $H \cap K = \{1\}$, *and,*

(iii) $G = HK = \{hk : h \in H, k \in K\}$.

Then, $G = H \times K$.

Theorem 2.5. *(Recognition Theorem for Semidirect Products)*
Let H *and* K *be two subgroups of a group* G*, such that:*

(i) $H \trianglelefteq G$,

(ii) $H \cap K = \{1\}$, *and,*

(iii) $G = HK$.

Then, $G = H \rtimes K$ *with respect to the homomorphism* ϕ *from* K *to* $Aut(H)$ *defined by* $\phi_k(h) = k^{-1}hk$.

Definition 2.4. *The wreath product of a finite group* G *by the symmetric group* Sym_m*,* $G \wr \mathrm{Sym}_m$*, is the semidirect product* $G^m \rtimes \mathrm{Sym}_m$ *defined by the action:* $(g_1, .., g_m)^\sigma = (g_{\sigma(1)}, ..., g_{\sigma(m)})$.

2.2.3 Group-theoretic notations

Throughout the thesis, C_k denotes the cyclic group of order k, D_k denotes the dihedral group of order k, Sym_k denotes the symmetric group on k letters, $\mathrm{GL}(n,q)$ denotes the general linear group of order n over $\mathbb{F}_q$, E_q denotes an elementary abelian group of order q, and E_q^{1+2} denotes a group with centre $Z \cong E_q$ such that $E_q^{1+2}/Z \cong E_q^2$ (e.g. the group of upper-unitriangular 3×3 matrices over $\mathbb{F}_q$).

2.3 Collineations, polarities and perspectivities

A *collineation* (or *isomorphism*) between two $\mathbb{F}_q$-projective spaces $\mathrm{PG}(U)$ and $\mathrm{PG}(W)$ having the same dimension $n \geq 3$ is a bijection from the set of subspaces of $\mathrm{PG}(U)$ to the set of subspaces of $\mathrm{PG}(W)$ that is incidence-preserving and type-preserving, where the type of a projective subspace is its (projective) dimension, and two projective subspaces are incident if and only if one contains the other. The set of collineations from $\mathrm{PG}(U)$ to itself forms a group with the composition operation, denoted by $Aut(\mathrm{PG}(U))$. The *dual space of* $\mathrm{PG}(U)$ is the projective geometry of the dual vector space of U denoted by $\mathrm{PG}(U^\vee)$. The m-dimensional subspaces of $\mathrm{PG}(U^\vee)$ are the $(n-m)$-dimensional subspaces of $\mathrm{PG}(U)$. The *standard duality* of $\mathrm{PG}(U)$ is the collineation from the set of subspaces of $\mathrm{PG}(U)$ to the set of subspaces of $\mathrm{PG}(U^\vee)$, defined by mapping a point with homogeneous coordinates $(a_0, ..., a_n)$ to the hyperplane with dual coordinates $[a_0, ..., a_n]$ and expanding linearly.

Definition 2.5. *A correlation γ is a collineation of the form: $\gamma = \delta \circ \tau$, where δ is the standard duality of $\mathrm{PG}(W)$ and τ is a collineation from $\mathrm{PG}(U)$ to $\mathrm{PG}(W)$. If $W = U^\vee$, then γ is a correlation of $\mathrm{PG}(U)$. A polarity of $\mathrm{PG}(U)$ is a correlation of $\mathrm{PG}(U)$ of order 2.*

A semilinear map between two $(n+1)$-dimensional $\mathbb{F}_q$-vector spaces is a map ϕ satisfying: (i) $\phi(u+w) = \phi(u)+\phi(w)$ and (ii) $\phi(\lambda u) = \lambda^\sigma \phi(u)$, where $\sigma \in Aut(\mathbb{F}_q)$, $\lambda \in \mathbb{F}_q$ and $u, w \in \mathbb{F}_q^{n+1}$. The set of nonsingular semilinear transformations of $\mathbb{F}_q^{n+1}$ is the group $\mathrm{GL}(n+1,q) \rtimes Aut(\mathbb{F}_q)$, denoted by $\Gamma\mathrm{L}(n+1,q)$. It is well known, that every $\phi \in \Gamma\mathrm{L}(n+1,q)$ induces a collineation $\overline{\phi}$ of $\mathrm{PG}(n,q)$ which acts on the projective geometry of a subspace U of $\mathbb{F}_q^{n+1}$ by: $\mathrm{PG}(U)^{\overline{\phi}} = \mathrm{PG}(U^\phi)$. Particularly, for $\phi = (A,\sigma)$ and $u \in \mathbb{F}_q^{n+1}$, we have $\langle u \rangle^{\overline{\phi}} = \langle w \rangle$, where $w^T = Au^{\sigma T}$. The group of collineations induced by $\Gamma\mathrm{L}(n+1,q)$ is denoted by $\mathrm{P}\Gamma\mathrm{L}(n+1,q)$. A collineation in $\mathrm{P}\Gamma\mathrm{L}(n+1,q)$ induced from a nonsingular linear transformation of $\mathbb{F}_q^{n+1}$ is a *projectivity* of $\mathrm{PG}(n,q)$. The set of all projectivities of $\mathrm{PG}(n,q)$ forms a group denoted by $\mathrm{PGL}(n+1,q)$.

Theorem 2.6. *(Fundamental Theorem of Projective Geometry)*
Every collineation of $\mathrm{PG}(n,q)$, $n \geq 3$, is a collineation induced from $\Gamma\mathrm{L}(n+1,q)$, i.e, $Aut(\mathrm{PG}(n,q)) \cong \mathrm{P}\Gamma\mathrm{L}(n+1,q)$.

Theorem 2.7. *The projectivity group $\mathrm{PGL}(n+1,q)$ acts sharply transitive on frames of $\mathrm{PG}(n,q)$.*

A collineation ϕ of $\mathrm{PG}(n,q)$, $n \geq 3$, is *axial* if there exists a hyperplane in $\mathrm{PG}(n,q)$ fixed by ϕ pointwise, and it is *central* if there exists a point of $\mathrm{PG}(n,q)$ where ϕ fixes (setwise) any hyperplane through it. Every axial collineation is central and vice versa. Moreover, the set of collineations having an axis H and a centre P forms a group of *perspectivities* denoted by $\mathrm{Pers}(P,H)$. A perspectivity ϕ is an *elation* if $P \in H$, otherwise it is a *homology*. The set of elations (resp. homologies) of centre P and axis H, $E(P,H)$, form a group called the *elation* (resp. *homology*) group.

Theorem 2.8. *The set of all perspectivities of $\mathrm{PG}(n,q)$ generates the projectivity group $\mathrm{PGL}(n+1,q)$.*

We end this section by recalling that a Desarguesian projective plane is a *translation plane* which has a line L such that for every $P \in L$ and every line $L' \neq L$ passing through P, the elation group with centre P and axis L acts transitively on points of $L' \setminus \{P\}$. For more details about projective spaces, collineations, projectivity groups and their properties we refer to (Lavrauw, 2019) and (Coxeter, 2003).

2.4 Algebraic sets

A *form of degree* $n+1$ over $\mathbb{F}_q$ is a homogeneous polynomial whose nonzero terms are all of degree $n+1$. An *algebraic set* in $\mathrm{PG}(n,q)$ is the zero set of a finite collection of forms $\mathcal{A} \subseteq \mathbb{F}_q[X_0,...,X_n]$ defined by

$$\mathcal{Z}(\mathcal{A}) = \{P \in \mathrm{PG}(n,q) : f(P) = 0;\ f \in A\},$$

where the finiteness of $\mathcal{A}$ is guaranteed by the Hilbert basis Theorem. A *hypersurface* in $\mathrm{PG}(n,q)$ is an algebraic set defined by a single form in $\mathbb{F}_q[X_0,...,X_n]$. For instance, every hyperplane is a hypersurface defined by a linear form.

Lemma 2.1. *Points of a subspace of* $\mathrm{PG}(n,q)$ *define an algebraic set.*

Proof. This follows from Theorem 2.2 and the property: $\mathcal{Z}(f_1,..,f_r) = \mathcal{Z}(f_1) \cap ... \cap \mathcal{Z}(f_r)$, $f_j \in \mathbb{F}_q[X_0,...,X_n]$. □

An algebraic set $\mathcal{X}$ in $\mathrm{PG}(n,q)$ is *reducible* if it can be written as $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2$, where $\mathcal{X}_1, \mathcal{X}_2 \subset \mathcal{X}$ are algebraic sets in $\mathrm{PG}(n,q)$. Otherwise, $\mathcal{X}$ is called *irreducible*. The algebraic set $\mathcal{X}$ in $\mathrm{PG}(n,q)$ is *absolutely irreducible* if it is irreducible in $\mathrm{PG}(\overline{\mathbb{F}}_q^{n+1})$, where $\overline{\mathbb{F}}_q$ denotes a finite extension of $\mathbb{F}_q$. The *dimension* of an algebraic set $\mathcal{X}$ is the maximal length d of chains of distinct nonempty irreducible subvarieties of $\mathcal{X}$. The tangent of a point P in $\mathcal{X} = \mathcal{Z}(f)$, $f \in \mathbb{F}_q[X_0,...,X_n]$, is the hyperplane defined by

$$T_P(\mathcal{X}) : \sum_{i=0}^{n} \frac{\partial f}{\partial X_i}(P)\, X_i = 0.$$

This notion can be expended to a point $P \in \mathcal{X} = \mathcal{Z}(f_1,...,f_r)$, $f_j \in \mathbb{F}_q[X_0,...,X_n]$, by defining the tangent as

$$T_P(\mathcal{X}) = \bigcap_{j=1}^{n} T_P(\mathcal{Z}(f_j)).$$

The point $P \in \mathcal{X}$ is *singular* if $dim(T_P(\mathcal{X})) > dim(\mathcal{X})$. If $\mathcal{X}$ has no singular points, then $\mathcal{X}$ is called *nonsingular*.

2.4.1 Conics

A conic in $\mathrm{PG}(2,q)$ is an algebraic set defined by $\mathcal{C} = \mathcal{Z}(f)$, where

$$f = \sum_{0 \leqslant i \leqslant j \leqslant 2} a_{ij} X_i X_j \tag{2.4}$$

and $a_{ij} \in \mathbb{F}_q$, for $0 \leqslant i \leqslant j \leqslant 2$. Tangent, secant and external lines to $\mathcal{C}$ are lines of $\mathrm{PG}(2,q)$ meeting $\mathcal{C}$ in one, two and zero points respectively. If q is odd, then every point of $\mathrm{PG}(2,q)$ lies on exactly two tangents, $\frac{q-1}{2}$ secant and $\frac{q-1}{2}$ external lines to $\mathcal{C}$. If q is even, then every point of $\mathrm{PG}(2,q)$ lies on a unique tangent, $\frac{q}{2}$ secant and $\frac{q}{2}$ external lines to $\mathcal{C}$. Furthermore, tangents to $\mathcal{C}$ in $\mathrm{PG}(2,q)$, q even, are concurrent meeting at the *nucleus point* (Hirschfeld, 1998, Chapter 7).

Up to projective equivalence, there are 4 types of conics in $\mathrm{PG}(2,q)$:

(i) a unique nonsingular conic, and

(ii) three classes of singular conics, namely:

- (ii-a) double lines,
- (ii-b) pairs of real lines, and
- (ii-c) pairs of (conjugate) imaginary lines, i.e, lines defined in $\mathrm{PG}(2,q^2)$.

The following criterion determines when a conic is nonsingular.

Lemma 2.2. *(Hirschfeld, 1998, Theorem 7.16)*
A conic $\mathcal{C}$ in $\mathrm{PG}(2,q)$, q even, is absolutely irreducible (or, equivalently, nonsingular) if and only if $a_{00}a_{12}^2 + a_{11}a_{02}^2 + a_{22}a_{01}^2 + a_{01}a_{02}a_{12} \neq 0$.

The group of the conic: Consider the Veronese map defined by

$$\nu_{1,1} : \mathrm{PG}(1,q) \to \mathrm{PG}(2,q)$$

$$(x_0, x_1) \mapsto (x_0^2, x_0x_1, x_1^2),$$

Then, the image of the projective line under $\nu_{1,1}$ is the conic $\mathcal{C}$ defined by

$$Y_0Y_2 - Y_1^2 = 0. \tag{2.5}$$

The subgroup $G(\mathcal{C})$ of $\mathrm{PGL}(3,q)$ stabilising $\mathcal{C}$ is equivalent to $\mathrm{PGL}(2,q)$ through the bijection Φ defined by:

$$\Phi : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a^2 & ab & b^2 \\ 2ac & ad+bc & 2bd \\ c^2 & cd & d^2 \end{bmatrix}.$$

Therefore, $G(\mathcal{C})$ acts on $\mathcal{C}$ as $\mathrm{PGL}(2,q)$ acts on $\mathrm{PG}(1,q)$, i.e, 3-transitively. This group is known as the *group of the conic* which acts transitively on secant, tangent and external lines to $\mathcal{C}$. The proof of these properties can be found in (Hirschfeld, 1998, Chapter 7).

2.4.2 Cubic curves and surfaces

Cubic curves and surfaces are algebraic sets in $\mathrm{PG}(2,q)$ and $\mathrm{PG}(3,q)$ defined by $C = \mathcal{Z}(f)$ and $S = \mathcal{Z}(g)$ respectively, where

$$f = \sum_{0\leqslant i\leqslant j\leqslant k\leqslant 2} a_{ijk}X_iX_jX_k \quad \text{and} \quad g = \sum_{0\leqslant i\leqslant j\leqslant k\leqslant 3} a_{ijk}X_iX_jX_k. \tag{2.6}$$

Cubic curves over finite fields have many familiar properties with the classical theory over $\mathbb{R}$ and $\mathbb{C}$. In particular, when $q \equiv 1 \ (mod\ 3)$ their properties are more a like the complex case, while their properties are more similar to the real case when $q \equiv -1 \ (mod\ 3)$. However, when $q \equiv 0 \ (mod\ 3)$, no suitable classical model is available. In general, many properties of cubic curves over finite fields are known. Particularly, we refer the reader to (Hirschfeld, 1998, Chapter 11) for a complete review of these properties and the classifications of singular and nonsingular cubic curves.

Notation 2.1. *Let C be a cubic curve defined by $\mathcal{Z}(f)$, where f is as in* (2.6). *Then, C can be represented by $C(A, a_{012})$ where*

$$A = \begin{bmatrix} a_{000} & a_{011} & a_{022} \\ a_{100} & a_{111} & a_{122} \\ a_{200} & a_{211} & a_{222} \end{bmatrix}. \tag{2.7}$$

Equivalently, C can be defined as the set of points $P(x_0,x_1,x_2)$ of $\mathrm{PG}(2,q)$ satisfying

$$(x_0 \quad x_1 \quad x_2)\,A\,(x_0^2 \quad x_1^2 \quad x_2^2)^T + a_{012}\,(x_0 \quad x_1 \quad x_2)\,(x_1x_2 \quad x_0x_2 \quad x_0x_1)^T = 0. \tag{2.8}$$

Definition 2.1. *Let $C(A,a_{012})$ be a cubic curve defined over a finite field of characteristic 2. The set of tangent lines to $C(A,a_{012})$, known as the cubic envelope of $C(A,a_{012})$, is the dual cubic curve defined by*

$$C(\Phi(A),a_{012}^2)^T,$$

where

$$\Phi(A) = Adj(A)^T + a_{012}\begin{bmatrix} 0 & a_{022} & a_{011} \\ a_{122} & 0 & a_{100} \\ a_{211} & a_{200} & 0 \end{bmatrix} \tag{2.9}$$

and $Adj(A)^T$ is the transpose of the adjoint matrix of A.

Definition 2.2. *An inflexion point of a cubic curve $C(A,a_{012})$ over a finite field of characteristic 2 is a point of the curve whose tangent meets the curve algebraically in a triple intersection.*

Lemma 2.1. *(Glynn, 1998, Theorem 3.5)*
Let $C(A,a_{012})$ be a cubic curve defined over a finite field of characteristic 2 such that $a_{012} \neq 0$. Then, points of inflexion of $C(A,a_{012})$ are the nonsingular points of $C(A,a_{012})$ which lie on the cubic curve $C(\Phi^2(A),a_{012}^4)$. The curve $C(\Phi^2(A),a_{012}^4)$ is also known as the Hessian of $C(A,a_{012})$.

Remark 2.1. *In other words, $C(\Phi^2(A),a_{012}^4)$ is the set of tangent points of the set of tangent lines to the cubic curve $C(A,a_{012})$.*

Remark 2.2. *For none characteristic two fields, points of inflexion are defined as points of the intersection of the cubic with the classical Hessian (the determinant of the 3×3 matrix of second derivatives), which is zero over characteristic two fields.*

Cubic surfaces over finite fields are also well-studied objects. For instance, it is known that a nonsingular cubic surface over $\mathbb{F}_q$ has q^2+nq+1 points where $2 \le n \le 7$ and $n \neq 6$ (Manin, 1986). In 1915, Dickson showed that a nonsingular cubic surface over $\mathbb{F}_2$ can have i lines where $i \in I$; $I = \{0,1,2,3,5,9,15\}$ (Dickson, 1915). He classified as well all projectively inequivalent nonsingular cubic surfaces over $\mathbb{F}_2$ (Dickson, 1915). Segre considered counting the number of lines in a nonsingular cubic surface $\mathcal{S}$ over $\mathbb{F}_q$, when q is odd. In particular, he showed that $\mathcal{S}$ can have j lines where $j \in I \cup \{7, 27\}$ (Segre, 1942). Recently, cubic surfaces with 27 lines were classified over small finite fields and interesting computational and geometric algorithms were introduced. For more information, we refer the reader to (Betten & Karaoglu, 2019).

2.4.3 Segre variety

The *Segre variety*, $S_{n_1,..,n_t}(\mathbb{F}_q)$, is an algebraic set in $\mathrm{PG}(\prod_{i=1}^{t}(n_i+1)-1, q)$ defined as the image of the *Segre embedding*, $\sigma_{n_1,..,n_t}$, given by

$$\sigma_{n_1,..,n_t} : \mathrm{PG}(n_1,q) \times \ldots \times \mathrm{PG}(n_t,q) \to \mathrm{PG}(\prod_{i=1}^{t}(n_i+1)-1, q)$$
$$((v_{1_1}, \ldots, v_{1_{n_1+1}}), \ldots, (v_{t_1}, \ldots, v_{t_{n_t+1}})) \mapsto (\prod_{i=1}^{t} v_{i_1}, \ldots, \prod_{i=1}^{t} v_{i_{n_i+1}}).$$

It is a nonsingular absolutely irreducible variety whose dimension is $n_1+\ldots+n_t$. It is an example of a determinantal variety. For instance, the Segre variety $S_{n_1,n_2}(\mathbb{F}_q)$ is the zero set of the quadratic forms: $X_{i,j}X_{k,l} - X_{i,l}X_{k,j}$, where the $X_{r,s}$'s denote the coordinates in $\mathrm{PG}((n_1+1)(n_2+1)-1, q)$.

Examples 2.1.

- *The variety $S_{2,2}(\mathbb{F}_q)$ is defined by*

$$\sigma_{2,2} : ((u_1,u_2,u_3),(v_1,v_2,v_3)) \mapsto (u_1v_1, u_1v_2, u_1v_3, u_2v_1, u_2v_2, u_2v_3, u_3v_1, u_3v_2, u_3v_3),$$

  and has dimension 4.

- *The map $\sigma_{1,1}$ defines an embedding of the product of the projective line* $\mathrm{PG}(1,q)$ *with itself in* $\mathrm{PG}(3,q)$, *whose image is a quadric defined by* $\mathcal{Z}(X_{0,0}X_{1,1} - X_{0,1}X_{1,0})$.

- *The image of the diagonal* $\Delta \subset \mathrm{PG}(n,q) \times \mathrm{PG}(n,q)$ *under the Segre embedding* $\sigma_{n,n}$ *defines the Veronese surface of degree 2,* $\mathcal{V}_2(\mathbb{F}_q)$ *(see Section 2.4.4).*

Remark 2.3. *If we represent points of* $\mathrm{PG}(n_i,q)$ *as* $\langle u_i \rangle$, *then we can alternatively define* $\sigma_{n_1,..,n_t}(\langle u_1 \rangle, \ldots, \langle u_t \rangle)$ *as* $\langle u_1 \otimes \ldots \otimes u_t \rangle$.

Theorem 2.9. *(Hirschfeld & Thas, 1991, Theorem 4.100)*
The Segre variety, $S_{n_1,n_2}(\mathbb{F}_q)$, *is not contained in any hyperplane of* $\mathrm{PG}((n_1+1)(n_2+1)-1, q)$.

Remark 2.4. *The Segre variety $S_{n_1,n_2}(\mathbb{F}_q)$ consists of all points*

$$(x_{1,1},x_{1,2},...,x_{1,n_2+1},x_{2,1},...,x_{2,n_2+1},x_{n_1+1,1},...,x_{n_1+1,n_2+1}) \in \mathrm{PG}((n_1+1)(n_2+1)-1,q)$$

for which the rank of the matrix $[x_{i,j}]$ is 1.

Example 2.2. *Points of $S_{2,2}(\mathbb{F}_q)$ are rank-1 points of* $\mathrm{PG}(8,q)$*, where the rank of a point in* $\mathrm{PG}(8,q)$ *is the rank of its associated matrix of size* 3×3 *defined in Remark 2.4.*

For further properties and examples related to Segre varieties defined over finite fields, we refer the reader to Section 4.5 in (Hirschfeld & Thas, 1991).

2.4.4 Veronese variety

The *Veronese variety* of all quadrics of $\mathrm{PG}(n,q)$ is the algebraic set $\mathcal{V}_n(\mathbb{F}_q)$ defined as the image of the map

$$\nu_n : \mathrm{PG}(n,q) \to \mathrm{PG}(\binom{n+2}{2}-1,q)$$

sending the coordinates of $\mathrm{PG}(n,q)$ to monomials of degree 2. Namely,

$$\mathcal{V}_n(\mathbb{F}_q) := \{(x_0^2, x_0x_1,...,x_0x_n,x_1^2,x_1x_2,...,x_1x_n,x_2^2,...,x_{n-1}x_n,x_n^2) : (x_0,...,x_n) \in \mathrm{PG}(n,q)\}.$$

It is also known as the quadric Veronesean of $\mathrm{PG}(n,q)$, which is a nonsingular absolutely irreducible variety of dimension n.

Lemma 2.3. *(Hirschfeld & Thas, 1991, Lemma 4.1)*
The variety $\mathcal{V}_n(\mathbb{F}_q)$ is the intersection of the $(n+1)n^2/2$ quadrics $\mathcal{Z}(F_{ij})$ and $\mathcal{Z}(F_{abc})$, where

$$F_{ij} = X_{ij}^2 - X_{ii}X_{jj}, \;\; F_{abc} = X_{aa}X_{bc} - X_{ab}X_{ac},$$

and $i,j,a,b,c \in \{0,..,n\}$ such that $i \neq j$ and a,b,c are distinct.

Remark 2.5. *The variety $\mathcal{V}_n(\mathbb{F}_q)$ consists of all points*

$$(x_{0,0},x_{0,1},...,x_{0,n},...,x_{n-1,n},x_{n,n}) \in \mathrm{PG}(n(n+3)/2,q)$$

for which the rank of the symmetric matrix defined by $[x_{i,j}]$ *is* 1*.*

Example 2.3. *The Veronese surface* $\mathcal{V}_2(\mathbb{F}_q)$ *(or simply* $\mathcal{V}(\mathbb{F}_q)$*) is a 2-dimensional algebraic set in* $\mathrm{PG}(5,q)$ *defined as the image of the* Veronese embedding

$$\nu : \mathrm{PG}(2,q) \to \mathrm{PG}(5,q)$$

$$(u_0,u_1,u_2) \mapsto (u_0^2, u_0u_1, u_0u_2, u_1^2, u_1u_2, u_2^2).$$

In particular, we have $\mathcal{V}(\mathbb{F}_q) = \mathcal{Z}(2\times 2 \text{ minors of } M)$*, where*

$$M = \begin{bmatrix} X_{00} & X_{01} & X_{02} \\ X_{01} & X_{11} & X_{12} \\ X_{02} & X_{12} & X_{22} \end{bmatrix}$$

and the X_{ij}*'s denote the coordinates in* $\mathrm{PG}(5,q)$*.*

Theorem 2.10. *(Hirschfeld & Thas, 1991, Theorem 4.3)*
The quadrics of $\mathrm{PG}(n,q)$ *are mapped by* ν_n *onto the hyperplane sections of* $\mathcal{V}_n(\mathbb{F}_q)$*.*

Corollary 2.1. *(Hirschfeld & Thas, 1991, Corollary 4.4)*
The variety $\mathcal{V}_n(\mathbb{F}_q)$ *is not contained in any hyperplane of* $\mathrm{PG}(n(n+3)/2,q)$*.*

Theorem 2.11. *(Hirschfeld & Thas, 1991, Theorem 4.11)*
The variety $\mathcal{V}_n(\mathbb{F}_q)$ *is a cap of* $\mathrm{PG}(n(n+3)/2,q)$*, i.e., no three points of* $\mathcal{V}_n(\mathbb{F}_q)$ *are collinear.*

Theorem 2.12. *(Hirschfeld & Thas, 1991, Corollary 4.13)*
For $q \neq 2$*, any two points of* $\mathcal{V}_n(\mathbb{F}_q)$ *are contained in a unique conic of* $\mathcal{V}_n(\mathbb{F}_q)$*.*

Theorem 2.13. *(Hirschfeld & Thas, 1991, Corollary 4.16)*
For $(q,n) \neq (2,2)$*, the group stabilising* $\mathcal{V}_n(\mathbb{F}_q)$ *in* $\mathrm{PGL}(\frac{n(n+3)}{2}+1,q)$ *is isomorphic to the projectivity group* $\mathrm{PGL}(n+1,q)$*.*

We focus now on the properties of the Veronese surface $\mathcal{V}(\mathbb{F}_q)$ in $\mathrm{PG}(5,q)$, for more interesting properties we refer the reader to (Havlicek, 2003; Hirschfeld & Thas, 1991). We start by recalling the *normal rational curve.*

Remark 2.6. *A normal rational curve is an algebraic set defined as the image of the map*

$$\nu' : \mathrm{PG}(1,q) \to \mathrm{PG}(n,q)$$

$$(u_0,u_1) \mapsto (u_0^n, u_0^{n-1}u_1, ..., u_0u_1^{n-1}, u_1^n).$$

It is an example of a Veronese variety of degree n.

2.4.4.1 Properties of $\mathcal{V}(\mathbb{F}_q)$ in PG$(5,q)$

The Veronese surface $\mathcal{V}(\mathbb{F}_q)$ contains q^2+q+1 conics, defined as the image of lines in PG$(2,q)$ via ν, where any two points P, Q of $\mathcal{V}(\mathbb{F}_q)$ lie on one of these conics given by

$$\mathcal{C}(P,Q) := \nu(\langle \nu^{-1}(P), \nu^{-1}(Q)\rangle).$$

Since the conics of $\mathcal{V}(\mathbb{F}_q)$ correspond to the lines of PG$(2,q)$ via ν (see Example 2.3), any two of these conics have a unique point in common. The quadrics of PG$(2,q)$ correspond to the hyperplane sections of $\mathcal{V}(\mathbb{F}_q)$. If the quadric $\mathcal{C}$ is a repeated line, then the corresponding hyperplane of PG(5, q) meets $\mathcal{V}(\mathbb{F}_q)$ in a conic, if $\mathcal{C}$ is a pair of real lines, then the corresponding hyperplane meets $\mathcal{V}(\mathbb{F}_q)$ in two conics, if $\mathcal{C}$ is a pair of conjugate imaginary lines, then the corresponding hyperplane meets $\mathcal{V}(\mathbb{F}_q)$ in a point, if $\mathcal{C}$ is a nonsingular conic, then the corresponding hyperplane meets $\mathcal{V}(\mathbb{F}_q)$ in a *normal rational curve.* For $q \neq 2$, planes of PG$(5,q)$ which meet $\mathcal{V}(\mathbb{F}_q)$ in a conic are called the conic planes.

Remark 2.7. Technically, a point of $\mathcal{V}(\mathbb{F}_q)$ is also a conic, and if $q=2$, all triples of pairwise non-collinear points are conics. However, for simplicity, we will not consider these as "conics in $\mathcal{V}(\mathbb{F}_q)$". That is to say, by a "conic in $\mathcal{V}(\mathbb{F}_q)$", we will mean the image of a line of PG$(2,q)$ under the Veronese map.

Theorem 2.14. *(Hirschfeld & Thas, 1991, Theorem 4.17)*
Any two distinct conic planes of $\mathcal{V}(\mathbb{F}_q)$ meet in a unique point.

Lemma 2.4. *(Hirschfeld & Thas, 1991, Lemma 4.20)*
If q is even, then $\mathcal{V}(\mathbb{F}_q)$ is the intersection of the quadrics $\mathcal{Z}(F_{01})$, $\mathcal{Z}(F_{02})$, $\mathcal{Z}(F_{12})$, where

$$F_{01} = X_{01}^2 + X_{00}X_{11},\ F_{02} = X_{02}^2 + X_{00}X_{22},\ \textit{and}\ F_{12} = X_{12}^2 + X_{11}X_{22}.$$

Definition 2.6. *The tangent lines of $\mathcal{V}(\mathbb{F}_q)$ are the tangent lines to the conics in $\mathcal{V}(\mathbb{F}_q)$. Since $\mathcal{V}(\mathbb{F}_q)$ has no singular points, it follows that all tangent lines of $\mathcal{V}(\mathbb{F}_q)$ at a point $P \in \mathcal{V}(\mathbb{F}_q)$ are contained in a plane. This plane is known as the tangent plane of $\mathcal{V}(\mathbb{F}_q)$ at P.*

Theorem 2.15. *(Hirschfeld & Thas, 1991, Theorem 4.22)*
The tangent planes of two distinct points in $\mathcal{V}(\mathbb{F}_q)$ meet in exactly one point.

Remark 2.8. *If q is even, then all tangent lines to a conic $\mathcal{C}$ in $\mathcal{V}(\mathbb{F}_q)$ are concurrent, meeting at the* nucleus *of $\mathcal{C}$.*

Theorem 2.16. *(Hirschfeld & Thas, 1991, Theorem 4.23)*
If q is even, the set of all nuclei of conics in $\mathcal{V}(\mathbb{F}_q)$ coincides with the set of points of a plane in $\mathrm{PG}(5,q)$ *known as the* nucleus plane *of $\mathcal{V}(\mathbb{F}_q)$.*

Theorem 2.17. *(Hirschfeld & Thas, 1991, Theorem 4.25)*
If q is odd, then $\mathrm{PG}(5,q)$ *has a polarity which maps the set of conic planes of $\mathcal{V}(\mathbb{F}_q)$ onto the set of tangent planes of $\mathcal{V}(\mathbb{F}_q)$.*

Theorem 2.18. *(Hirschfeld & Thas, 1991, Theorem 4.42)*
If q is even, then the subspace $\mathcal{Z}(X_{00}, X_{11}, X_{22})$ of $\mathrm{PG}(5,q)$ *is the nucleus plane of $\mathcal{V}(\mathbb{F}_q)$.*

2.5 Quadratic and cubic equations over $\mathbb{F}_q$

2.5.0.1 Quadratic equations

Consider the quadratic equation $f(X) = 0$ where

$$f(X) = \alpha X^2 + \beta X + \gamma \in \mathbb{F}_q[X],$$

and $\alpha \neq 0$. Over a finite field of odd characteristic, solutions of f depend on the discriminant $\Delta = \beta^2 - 4\alpha\gamma$. In particular, f has one solution if $\Delta = 0$, two solutions if Δ is a square, and no solutions if Δ is a non-square (Hirschfeld, 1998, Section 1.4).

Over a finite field of characteristic 2, the square root defines an automorphism. To study roots of f over $\mathbb{F}_{2^h}$, we need first to introduce the trace (or absolute trace) map, Tr, defined from $\mathbb{F}_{p^h}$ to $\mathbb{F}_p$ by $\mathrm{Tr}(x) = x + x^p + x^{p^2} + \ldots + x^{p^{h-1}}$, which is a linear surjective map. In particular, if $p = 2$ then Tr is a $\frac{q}{2}$-to-1 map.

Lemma 2.5. *(Berlekamp, Rumsey & Solomon, 1967)*
The polynomial $f(X) = \alpha X^2 + \beta X + \gamma \in \mathbb{F}_{2^h}[X]$ with $\alpha \neq 0$ has exactly one root in $\mathbb{F}_{2^h}$ if and only if $\beta = 0$, two distinct roots in $\mathbb{F}_{2^h}$ if and only if $\beta \neq 0$ and $\mathrm{Tr}(\frac{\alpha\gamma}{\beta^2}) = 0$, and no roots in $\mathbb{F}_{2^h}$ otherwise.

Remark 2.9. *If $\alpha^{-1} f(X)$ has no roots in $\mathbb{F}_q$, then $\alpha^{-1} f(X)$ has two conjugate roots* *in the quadratic extension of* $\mathbb{F}_q$.

2.5.0.2 Cubic equations

Consider the cubic equation $c(X) = 0$ where

$$c(X) = X^3 + a_1X^2 + a_2X + a_3 \in \mathbb{F}_q[X].$$

Solutions of $c(X)$ can be retrieved by solving a cubic equation of the form $g(\theta) = 0$, where

$$g(\theta) = \theta^3 + b\theta + a. \tag{2.10}$$

For instance, if $q = 3^h$ and $a_1 \neq 0$, we can work with $X^3\,c(1/X + a_2/a_1)$. On the other hand, if $q = 2^h$ and $a_2 \neq a_1^2$, we can apply the substitution

$$X = (a_2 + a_1^2)^{\frac{1}{2}}\theta + a_1,$$

to obtain the cubic polynomial g with $b = 1$ and

$$a = \frac{a_3 + a_2a_1}{(a_2 + a_1^2)^{\frac{3}{2}}}.$$

Notice that, as the product of the three roots of g is $a \in \mathbb{F}_q$, it follows that g is either irreducible over $\mathbb{F}_q$, have all its roots in $\mathbb{F}_q$ or exactly one root in $\mathbb{F}_q$.

Over finite fields of characteristic 2, solutions of g were independently studied by Berlekamp et. al in (Berlekamp, Rumsey & Solomon, 1966) and by Williams in (Williams, 1975). Particularly, they proved the following theorems.

Theorem 2.19. *(Berlekamp, Rumsey & Solomon, 1966, Lemma)*
Let $q = 2^h > 2$ *and* $a \neq 0$. *The cubic equation* $\theta^3 + \theta + a = 0$ *over* $\mathbb{F}_q$ *has*

- *three solutions in* $\mathbb{F}_q$ *if and only if* $q \neq 4$, $\mathrm{Tr}(a^{-1}) = \mathrm{Tr}(1)$ *and*

$$a = \frac{v + v^{-1}}{(1 + v + v^{-1})^3}$$

  for some $v \in \mathbb{F}_q \setminus \mathbb{F}_4$. *In this case, "a" is called admissible,*

- *a unique solution in* $\mathbb{F}_q$ *if and only if* $\mathrm{Tr}(a^{-1}) \neq \mathrm{Tr}(1)$,

- *no solutions in* $\mathbb{F}_q$ *if and only if* $\mathrm{Tr}(a^{-1}) = \mathrm{Tr}(1)$ *and* a *is not admissible.*

Theorem 2.20. *(Williams, 1975, Theorem 1)*
The cubic equation $\theta^3+\theta+a=0$ *with* $q=2^h>2$ *and* $a\in\mathbb{F}_q\setminus\{0\}$ *has*

- *three solutions in* $\mathbb{F}_q$ *if and only if* $q\neq 4$, $\mathrm{Tr}(a^{-1}) = \mathrm{Tr}(1)$ *and the roots of* t^2+at+1 *are both cubes in* $\mathbb{F}_q$ *(h even) or* $\mathbb{F}_{q^2}$ *(h odd),*
- *a unique solution in* $\mathbb{F}_q$ *if and only if* $\mathrm{Tr}(a^{-1}) \neq \mathrm{Tr}(1)$,
- *no solutions in* $\mathbb{F}_q$ *if and only if* $\mathrm{Tr}(a^{-1}) = \mathrm{Tr}(1)$ *and the roots of* t^2+at+1 *are both not cubes in* $\mathbb{F}_q$ *(h even) or* $\mathbb{F}_{q^2}$ *(h odd).*

Over finite fields of odd characteristic, solutions of $g(\theta)=0$ where studied by Dickson and Williams. Particularly, they proved the following theorems.

Theorem 2.21. *(Dickson, 1906, Theorem)*
If $q=p^h$, $p>3$ *and* $-4b^3-27a^2\neq 0$, *then the equation* $\theta^3+b\theta+a=0$, *with* $a,b\in\mathbb{F}_q$, *has*

- *three solutions in* $\mathbb{F}_q$ *if and only if* $-4b^3-27a^2$ *is a square in* $\mathbb{F}_q$, *say* $-4b^3-27a^2=81e^2$, *and* $1/2(-a+e\sqrt{-3})$ *is a cube in* $\mathbb{F}_q$ *if* $q\equiv 1 \ (mod\ 3)$, *or in* $\mathbb{F}_{q^2}$ *if* $q\equiv 2\ (mod\ 3)$,
- *a unique solution in* $\mathbb{F}_q$ *if and only if* $-4b^3-27a^2$ *is not a square in* $\mathbb{F}_q$,
- *no solutions in* $\mathbb{F}_q$ *if and only if* $-4b^3-27a^2$ *is a square in* $\mathbb{F}_q$, *say* $-4b^3-27a^2=81e^2$, *and* $1/2(-a+e\sqrt{-3})$ *is not a cube in* $\mathbb{F}_q$ *if* $q\equiv 1\ (mod\ 3)$, *or in* $\mathbb{F}_{q^2}$ *if* $q\equiv 2\ (mod\ 3)$,

Theorem 2.22. *(Williams, 1975, Theorem 2)*
If $q=3^h$, *then the equation* $\theta^3+b\theta+a=0$, *with* $a,b\in\mathbb{F}_q$, *has*

- *three solutions in* $\mathbb{F}_q$ *if and only if* $-b$ *is a square in* $\mathbb{F}_q$, *say* $-b=e^2$, *and* $Tr(a/e^3)=0$,
- *a unique solution in* $\mathbb{F}_q$ *if and only if* $-b$ *is not a square in* $\mathbb{F}_q$,
- *no solutions in* $\mathbb{F}_q$ *if and only if* $-b$ *is a square in* $\mathbb{F}_q$, *say* $-b=e^2$, *and* $Tr(a/e^3)\neq 0$.

Remark 2.1. *If* g *has no roots in* $\mathbb{F}_q$, *then* g *has three roots in the cubic extension of* $\mathbb{F}_q$. *On the other hand, if* g *has exactly one root in* $\mathbb{F}_q$, *then* g *has two conjugate roots in the quadratic extension of* $\mathbb{F}_q$. *Also, notice that the two roots of* t^2+at+1 *in* $\mathbb{F}_q$ *or* $\mathbb{F}_{q^2}$ *in Theorem 2.20 should both be cubes or non-cubes as their product is* 1.

2.6 Tensor products

Let $V_1,...,V_t$ be finite dimensional vector spaces defined over the field $\mathbb{F}_q$; $dim(V_i) = m_i$. The *t-fold (t-ordered) tensor product space* $V = V_1 \otimes ... \otimes V_t$ is the space of multilinear functions defined from $V_1^\vee \times ... \times V_t^\vee$ to $\mathbb{F}_q$, where $V_i^\vee$ is the dual space of V_i.

Example 2.4. *If $t = 2$, then V becomes the set of matrices of size $m_1 \times m_2$ over $\mathbb{F}_q$.*

Note that, this is not the only way to define tensors. We prefer to see them as multilinear functions from the direct product of the dual spaces to the field $\mathbb{F}_q$, however, tensors can be viewed in alternative ways. For more equivalent definitions and properties of tensors, we refer to chapter 2 in (Landsberg, 2011).

The set of symmetric tensors in $V' = V_1 \otimes ... \otimes V_t$, where $V_i \cong \mathbb{F}_q^m$, $1 \leq i \leq t$, defines a subspace denoted by $S^t(\mathbb{F}_q^m)$. Alternatively, we may define $S^t(\mathbb{F}_q^m)$ as the subspace of V' whose elements are invariant under the action of the symmetric group Sym_t on V' defined by

$$(v_1 \otimes ... \otimes v_t)^\sigma = (v_{\sigma(1)} \otimes ... \otimes v_{\sigma(t)}),$$

and expanding linearly.

2.6.1 Ranks

Fundamental (pure or rank-1) tensors of V are tensors of the form $v_1 \otimes ... \otimes v_t$. Clearly, not every tensor $A \in V$ is fundamental, however A can be written as the sum of fundamental tensors. The smallest integer r for which such a writing exists is called the *rank of A* and is denoted by $rank(A)$. For instance, the rank of a 2-fold tensor is the rank of its associated matrix.

Example 2.5. *Let $V = \mathbb{F}_q^2 \otimes \mathbb{F}_q^3 \otimes \mathbb{F}_q^3$ and $\{e_1,\dots,e_\ell\}$ be the canonical basis of $\mathbb{F}_q^\ell$, for $\ell = 2,3$. The rank of $A = e_1 \otimes e_3 \otimes e_1 + e_2 \otimes (e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3)$ in V is 4, i.e, we cannot write A as the sum of 3 or less fundamental tensors.*

In general, determining the rank of tensors when $t \geq 3$ is a hard problem, and no algorithms are available. Sometimes, this problem is generalised to finding bounds on ranks of tensors in V. For instance, Ja'Ja' bounded from above ranks of tensors in $\mathbb{F}^n \otimes \mathbb{F}^n \otimes \mathbb{F}^n$ by

$$r = \frac{3n}{2}\left\lceil \frac{n}{2} \right\rceil,$$

(Ja'Ja', 1979). Later, this bound was improved to 6 in (Lavrauw, Pavan & Zanella, 2013), when $n = 3$.

Complexity of matrix multiplications: The problem of determining the complexity of matrix multiplications is the problem of finding the minimal number of arithmetic operations needed to multiply two $n \times n$ matrices. Note that, as the total number of arithmetic operations is bounded by the number of multiplications, it follows that counting multiplications is a reasonable measure of complexity. Therefore, finding the complexity of matrix multiplications is equivalent to determining the rank of the matrix multiplication operator, $M_{n,n,n}$. Since the standard algorithm for multiplication uses n^3 (multiplicative) operations, it follows that $rank(M_{n,n,n}) \leq n^3$. For instance, if $n = 2$, we can expand this operator as

(2.11)

$$\begin{aligned} M_{2,2,2} =& e_1^* \otimes e_1^* \otimes e_1 + e_2^* \otimes e_3^* \otimes e_1 + e_1^* \otimes e_2^* \otimes e_2 + e_2^* \otimes e_4^* \otimes e_2 + e_3^* \otimes e_1^* \otimes e_3 + \\ & e_4^* \otimes e_3^* \otimes e_3 + e_3^* \otimes e_2^* \otimes e_4 + e_4^* \otimes e_4^* \otimes e_4, \end{aligned}$$

where $\{e_1, ..., e_4\}$ is the standard basis of the set of 2×2-matrices and $\{e_1^*, ..., e_4^*\}$ is its associated dual basis. In 1969, Strassen improved this bound by writing $M_{2,2,2}$ as the sum of 7 fundamental tensors instead of 8:

(2.12)

$$\begin{aligned} \mathrm{M}_{2,2,2} = & (e_1^* + e_4^*) \otimes (e_1^* + e_4^*) \otimes (e_1 + e_4) + (e_3^* + e_4^*) \otimes e_1^* \otimes (e_3 + e_4) + e_1^* \otimes (e_2^* + e_4^*) \otimes \\ & (e_2 - e_4) + e_4^* \otimes (e_3^* - e_1^*) \otimes (e_3 + e_1) + (e_1^* + e_2^*) \otimes e_4^* \otimes (e_2 - e_1) + (e_3^* - e_1^*) \otimes \\ & (e_1^* + e_2^*) \otimes e_4 + (e_2^* - e_4^*) \otimes (e_3^* + e_4^*) \otimes e_1. \end{aligned}$$

Later, Winograd proved that $Rank(M_{2,2,2}) = 7$, i.e, we cannot multiply 2×2-matrices using less than 7 multiplications. For more details on this topic, we refer to (Strassen, 1969; Winograd, 1971).

2.6.2 Contraction spaces and rank distributions

The *j-th contraction space* of a tensor A in $V = V_1 \otimes ... \otimes V_t$ is a subspace of $V_j^* = V_1 \otimes ... V_{j-1} \otimes V_{j+1} ... \otimes V_t$ defined as

$$A_j = \langle u_j^\vee(A) : u_j^\vee \in {V_j}^\vee \rangle,$$

where the $u_j^\vee(A)$'s are the j-th contractions of A defined by $u_j^\vee(u_1 \otimes ... \otimes u_t) = u_j^\vee(u_j)\, u_1 \otimes ... u_{j-1} \otimes u_{j+1} ... \otimes u_t$ and expanding linearly.

Example 2.6. *The first contraction space of a tensor A in $\mathbb{F}_q^2 \otimes \mathbb{F}_q^3 \otimes \mathbb{F}_q^3$ is the subspace of $\mathbb{F}_q^3 \otimes \mathbb{F}_q^3$ defined as $A_1 = \langle u_1^\vee(A) : u_1^\vee \in \mathbb{F}_q^{2\vee} \rangle$. The second and the third contraction spaces, A_2 and A_3, are defined analogously as subspaces of $\mathbb{F}_q^2 \otimes \mathbb{F}_q^3$. Projectively, $\mathrm{PG}(A_1)$ and $\mathrm{PG}(A_i)$ for $i = 2, 3$ are subspaces of $\mathrm{PG}(8, q)$ and $\mathrm{PG}(5, q)$ respectively.*

Definition 2.7. *The rank of the j-th contraction space of A is defined as the minimum number of rank-1 tensors needed to span a subspace containing A_j.*

In general, contraction spaces are useful tools to study tensors. For example, the following proposition can be helpful in determining the rank of tensors.

Proposition 2.2. *(Lavrauw & Sheekey, 2014, Proposition 2.1)*
Let $A \in V_1 \otimes ... \otimes V_t$ and $j \in \{1, ..., t\}$. Then, $rank(A) = rank(A_j)$.

Definition 2.8. *The* j-th rank distribution *of $A \in V$ is an m-tuple whose i-th coordinate represents the number of rank-i tensors in A_j.*

Example 2.7. *The first, second and third rank-distributions of $A \in \mathbb{F}_q^2 \otimes \mathbb{F}_q^3 \otimes \mathbb{F}_q^3$ are the 3-tuples $R_i = [a_{i1}, a_{i2}, a_{i3}]$, $1 \le i \le 3$, where a_{ij} is the number of rank-j points in A_i, $1 \le j \le 3$. Note that, the rank of a contraction in this case is the usual matrix rank.*

2.6.3 Natural actions on V

The group $H = \mathrm{GL}(\mathbb{F}_q^{m_1}) \times ... \times \mathrm{GL}(\mathbb{F}_q^{m_t})$ acts on the set of fundamental tensors in V via

$$(v_1 \otimes \cdots \otimes v_m)^{(g_1, \ldots, g_m)} = v_1^{g_1} \otimes \ldots v_m^{g_m},$$

and on all V by linearity. If some of the m_i's are equal, then we can extend H by a subgroup of the symmetric group Sym_m to obtain the group G defined as the setwise stabiliser of fundamental tensors in V. One may seek then to classify the G-orbits and the H-orbits of tensors in V.

Example 2.8. *If $t = 2$, the number of G-orbits of tensors in V is the $min(m_1, m_2)$ as tensors in $\mathbb{F}_q^{m_1} \otimes \mathbb{F}_q^{m_2}$ are totally characterised by their ranks.*

Theorem 2.23. *(Lavrauw & Sheekey, 2014, Theorem 3.5)*
There are 5 *G-orbits of tensors in* $\mathbb{F}_q^2 \otimes \mathbb{F}_q^2 \otimes \mathbb{F}_q^2$.

Theorem 2.24. *(Lavrauw & Sheekey, 2017, Theorems 5.2 and 5.3)*
There are 18 *G-orbits and* 21 *H-orbits of tensors in* $\mathbb{F}_q^2 \otimes \mathbb{F}_q^3 \otimes \mathbb{F}_q^3$.

Remark 2.2. *Ranks, contraction spaces and rank distributions of tensors are invariants under the actions of G and H on V.*

2.6.4 Tensors in $\mathbb{F}_q^2 \otimes \mathbb{F}_q^3 \otimes \mathbb{F}_q^3$

Tensors in $V = \mathbb{F}_q^2 \otimes \mathbb{F}_q^3 \otimes \mathbb{F}_q^3$ were classified by Lavrauw and Sheekey in (Lavrauw & Sheekey, 2015) by studying their associated contraction spaces. Particularly, they proved the existence of 18 G-orbits of tensors under the action of $G \cong \mathrm{GL}(\mathbb{F}_q^2) \times (\mathrm{GL}(\mathbb{F}_q^3) \wr Sym(2))$, as a subgroup of $\mathrm{GL}(V)$ stabilising the set of fundamental tensors in V. We collect in Table A.1 some information about these G-orbits of tensors and their contraction spaces, which we use to define our main algorithms in Chapter 5.

Remark 2.10. *Since ranks are not affected by multiplications with scalars, it makes more sense to consider the problem of classifying tensors and determining their ranks in the space* $\mathrm{PG}(V)$. *Projectively, nonzero tensors of rank 1 in V correspond to points of the Segre variety $S_{1,2,2}(\mathbb{F}_q)$ defined in 2.4.3.*

2.6.5 Representations of tensors in $\mathbb{F}_q^2 \otimes \mathbb{F}_q^3 \otimes \mathbb{F}_q^3$

Let $\{e_1, \ldots, e_\ell\}$ be the canonical basis of $\mathbb{F}_q^\ell$, for $\ell = 2, 3$, and define the canonical basis of V as $\{e_i \otimes e_j \otimes e_k : 1 \leq i \leq 2 \textit{ and } 1 \leq j, k \leq 3\}$. By decomposing $A \in V$ as $A = \sum A_{i,j,k} e_i \otimes e_j \otimes e_k$, we can view A as a rectangular cube whose entries are defined by the $A_{i,j,k}$'s. This cube can be partitioned into slices that completely determine A. For instance, we may view A as a set of two 3×3 matrices: $(A_{1,j,k}), (A_{2,j,k})$, called the *horizontal slices of* A, or a set of three 2×3 matrices $(A_{i,1,k}), (A_{i,2,k}), (A_{i,3,k})$, called the *lateral slices of* A, or a set of three 2×3 matrices $(A_{i,j,1}), (A_{i,j,2}), (A_{i,j,3})$, called the *frontal slices of* A. Note that these representations can be extended to any vector space of the form $V_1 \otimes \ldots \otimes V_t$.

Example 2.9. *Let $\{e_1, \ldots, e_\ell\}$ be the canonical basis of $\mathbb{F}_q^\ell$, for $\ell = 2, 3$, and consider*

$A \in \mathbb{F}_q^2 \otimes \mathbb{F}_q^3 \otimes \mathbb{F}_q^3$ *defined by*

$$e_1 \otimes (e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3) + e_2 \otimes (e_1 \otimes e_2 + e_2 \otimes e_3).$$

The horizontal, lateral and frontal slices of A *are defined by*

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right\},$$

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right\}, \textit{ and}$$

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right\},$$

respectively.

2.7 Subspaces of PG$(5,q)$

We summarize in this section orbits of points, lines and hyperplanes of PG$(5,q)$ under the action of the group stabilising the Veronese surface. We define as well some useful combinatorial invariants that we use to classify solids and planes in Chapters 3 and 4.

2.7.1 The group action

We are interested in the action on subspaces of PG$(5,q)$ of the group $K \leqslant \mathrm{PGL}(6,q)$ defined as the lift of PGL$(3,q)$ through the Veronese map ν (see 2.4.4). Explicitly, if $\phi_A \in \mathrm{PGL}(3,q)$ is represented by the matrix $A \in \mathrm{GL}(3,q)$ then we define the corresponding projectivity $\alpha(\phi_A) \in \mathrm{PGL}(6,q)$ through its action on the points of PG$(5,q)$ by

$$\alpha(\phi_A) : P \mapsto Q \quad \text{where} \quad M_Q = AM_PA^T,$$

where M_Q and M_P are the matrix representations of Q and P defined in (2.13). Then $K := \alpha(\mathrm{PGL}(3,q))$ is isomorphic to $\mathrm{PGL}(3,q)$ and leaves $\mathcal{V}(\mathbb{F}_q)$ invariant.

Remark 2.11. If $q > 2$ then $K \cong \mathrm{PGL}(3,q)$ is the *full* setwise stabiliser of $\mathcal{V}(\mathbb{F}_q)$ in $\mathrm{PGL}(6,q)$. If $q = 2$ then the full setwise stabiliser of $\mathcal{V}(\mathbb{F}_q)$ is Sym_7, as the kernel of this action stabilising $\mathcal{V}(\mathbb{F}_q)$ pointwise, is trivial.

2.7.2 Points, lines and hyperplanes of $\mathrm{PG}(5,q)$

A point $P = (y_0, y_1, y_2, y_3, y_4, y_5)$ of $\mathrm{PG}(5,q)$ can be represented by a symmetric 3×3 matrix

$$M_P = \begin{bmatrix} y_0 & y_1 & y_2 \\ y_1 & y_3 & y_4 \\ y_2 & y_4 & y_5 \end{bmatrix}. \tag{2.13}$$

This representation can be extended to any subspace of $\mathrm{PG}(5,q)$. For example, the solid spanned by the first four points of the standard frame of $\mathrm{PG}(5,q)$ is represented by

$$\begin{bmatrix} x & y & z \\ y & t & \cdot \\ z & \cdot & \cdot \end{bmatrix} := \left\{ \begin{bmatrix} x & y & z \\ y & t & 0 \\ z & 0 & 0 \end{bmatrix} : (x,y,z,t) \in \mathrm{PG}(3,q) \right\}, \tag{2.14}$$

where the notation on the left is introduced for convenience (that is, $\cdot$ represents 0, and the 4-tuple (x,y,z,t) is understood to range over all non-zero elements of $\mathbb{F}_q^4$).

In general, we define the *rank distribution* of a subspace U of $\mathrm{PG}(5,q)$ to be the 3-tuple $[r_1, r_2, r_3]$, where r_i is the number of points of rank i in U.

The *rank* of a point P of $\mathrm{PG}(5,q)$ is defined to be the rank of the matrix M_P. The points of rank 1 are (therefore precisely) those belonging to $\mathcal{V}(\mathbb{F}_q)$. Points of $\mathrm{PG}(5,q)$ of rank at most 2 are the points of the *secant variety* of $\mathcal{V}(\mathbb{F}_q)$, which we denote by $\mathcal{V}(\mathbb{F}_q)^2$.

In the above representation, points contained in the nucleus plane correspond to symmetric 3×3 matrices with zeros on the main diagonal (see Theorem 2.18). Each rank-2 point R of $\mathrm{PG}(5,q)$ defines a unique conic $\mathcal{C}(R)$ in $\mathcal{V}(\mathbb{F}_q)$. If R lies on the secant $\langle P, Q\rangle$ with $P, Q \in \mathcal{V}(\mathbb{F}_q)$ then $\mathcal{C}(R) = \mathcal{C}(P,Q)$. If q is even and R is contained in the nucleus plane, then R is the nucleus of $\mathcal{C}(R)$.

Definition 2.9. *(Alnajjarine, Lavrauw & Popiel , 2022, Definition 2.3)*
Let $G \leqslant \mathrm{P\Gamma L}(n+1,q)$ and let $U_1, U_2, \ldots, U_m$ denote (a chosen ordering of) the distinct G-orbits of r-spaces in $\mathrm{PG}(n,q)$. The *r-space G-orbit distribution* of a subspace U of $\mathrm{PG}(n,q)$ is the list

$$\mathrm{OD}_{G,r}(U) := [u_1, u_2, \ldots, u_m],$$

where u_i is the number of elements of U_i incident with U.

The rank distribution of a subspace of $\mathrm{PG}(5,q)$ is related to its 0-space K-orbit distribution as follows. There are four K-orbits of 0-spaces, i.e. points, in $\mathrm{PG}(5,q)$: the orbit $\mathcal{V}(\mathbb{F}_q)$ of rank-1 points, which has size q^2+q+1, the orbit of rank-3 points, which has size q^5-q^2, and two orbits of rank-2 points. For q even, the orbits of rank-2 points comprise the q^2+q+1 points of the nucleus plane π_n, and the $(q^2-1)(q^2+q+1)$ points contained in conic planes but not in $\pi_n \cup \mathcal{V}(\mathbb{F}_q)$. Therefore, the orbit distribution $\mathrm{OD}_{K,0}(U)$ of a subspace U of $\mathrm{PG}(5,q)$, q even, is the 4-tuple $[r_1, r_{2n}, r_{2s}, r_3]$, where r_i, $i \in \{1,3\}$, is the number of rank-i points in U, r_{2n} is the number of rank-2 points in $U \cap \pi_n$, and r_{2s} is the number of rank-2 points in $U \setminus \pi_n$.

For brevity, we also call $\mathrm{OD}_{K,r}(U)$ with $r=0$ the *point-orbit distribution* of a subspace U of $\mathrm{PG}(5,q)$. Similarly, we obtain the *line-*, *plane-*, *solid-*, and *hyperplane-orbit distributions* of U for $r = 1,2,3,4$ respectively. These data serve as useful invariants for studying K-orbits of subspaces of $\mathrm{PG}(5,q)$. For example, if q is odd and U is a plane containing at least one point of $\mathcal{V}(\mathbb{F}_q)$, then the line-orbit distribution of U completely determines its K-orbit (Lavrauw, Popiel & Sheekey, 2020). The line orbits themselves were determined (for all q) in (Lavrauw & Popiel, 2020), as a consequence of the classification of the first contraction spaces of points in $\mathrm{PG}(17,q)$ in (Lavrauw & Sheekey, 2015).

Theorem 2.25. *(Lavrauw & Popiel, 2020, Table 2)*
There are 15 K-orbits of lines in $\mathrm{PG}(5,q)$ as described in Tables 2.1 and 2.2.

Remark 2.12. *Lines in $o_{15,1}$ and o_{16} in $\mathrm{PG}(5,q)$, q odd, can be distinguished using Lemma 5.1 . Similarly, we can distinguish lines in o_{15} and $o_{16,2}$ when q is even.*

Hyperplanes of $\mathrm{PG}(5,q)$ correspond to conics of $\mathrm{PG}(2,q)$ through the Veronese map ν. We make this correspondence explicit via the following map δ between conics of $\mathrm{PG}(2,q)$ and hyperplanes of $\mathrm{PG}(5,q)$:

$$\delta : \mathcal{Z}\Big(\sum_{0\leqslant i\leqslant j\leqslant 2} a_{ij}X_iX_j\Big) \mapsto \mathcal{Z}(a_{00}Y_0 + a_{01}Y_1 + a_{02}Y_2 + a_{11}Y_3 + a_{12}Y_4 + a_{22}Y_5).$$

Here, and throughout the thesis, the homogeneous coordinates in the domain $\mathrm{PG}(2,q)$ of ν are denoted by (X_0, X_1, X_2), the homogeneous coordinates in $\mathrm{PG}(5,q)$

Orbits	Point-OD's
o_5	$[2, \frac{q-1}{2}, \frac{q-1}{2}, 0]$
o_6	$[1, q, 0, 0]$
$o_{8,1}$	$[1, 1, 0, q-1]$
$o_{8,2}$	$[1, 0, 1, q-1]$
o_9	$[1, 0, 0, q]$
o_{10}	$[0, \frac{q+1}{2}, \frac{q+1}{2}, 0]$
o_{12}	$[0, q+1, 0, 0]$
$o_{13,1}$	$[0, 2, 0, q-1]$
$o_{13,2}$	$[0, 1, 1, q-1]$
$o_{14,1}$	$[0, 3, 0, q-2]$
$o_{14,2}$	$[0, 1, 2, q-2]$
$o_{15,1}$	$[0, 1, 0, q]$
$o_{15,2}$	$[0, 0, 1, q]$
o_{16}	$[0, 1, 0, q]$
o_{17}	$[0, 0, 0, q+1]$

Table 2.1 The K-orbits of lines in $\mathrm{PG}(5,q)$, q odd.

Orbits	Point-OD's
o_5	$[2, 0, q-1, 0]$
o_6	$[1, 1, q-1, 0]$
$o_{8,1}$	$[1, 0, 1, q-1]$
$o_{8,2}$	$[1, 1, 0, q-1]$
o_9	$[1, 0, 0, q]$
o_{10}	$[0, 0, q+1, 0]$
$o_{12,1}$	$[0, q+1, 0, 0]$
$o_{12,2}$	$[0, 1, q, 0]$
$o_{13,1}$	$[0, 1, 1, q-1]$
$o_{13,2}$	$[0, 0, 2, q-1]$
o_{14}	$[0, 0, 3, q-2]$
o_{15}	$[0, 0, 1, q]$
$o_{16,1}$	$[0, 1, 0, q]$
$o_{16,2}$	$[0, 0, 1, q]$
o_{17}	$[0, 0, 0, q+1]$

Table 2.2 The K-orbits of lines in $\mathrm{PG}(5,q)$, q even.

are denoted by $(Y_0, \ldots, Y_5)$, and $\mathcal{Z}(f)$ denotes the zero locus of a form f. Note that a point P in $\mathrm{PG}(2,q)$ lies on a (given) conic $\mathcal{C}$ if and only if $\nu(P)$ lies in the hyperplane $\delta(\mathcal{C})$. The definition of δ extends to a set $\mathcal{S}$ of conics in the obvious way:

$$\delta(\mathcal{S}) = \bigcap_{\mathcal{C} \in \mathcal{S}} \delta(\mathcal{C}).$$

Up to projective equivalence, there is a unique nonsingular conic in $\mathrm{PG}(2,q)$, and three classes of singular conics, namely (i) double lines, (ii) pairs of real lines, and (iii) pairs of (conjugate) imaginary lines. We denote the corresponding K-orbits of hyperplanes (obtained via δ) as follows: $\mathcal{H}_1$, $\mathcal{H}_{2r}$ and $\mathcal{H}_{2i}$ denote the K-orbits of hyperplanes corresponding to the $\mathrm{PGL}(3,q)$-orbits of singular conics of types (i), (ii) and (iii) respectively, and $\mathcal{H}_3$ denotes the K-orbit of hyperplanes corresponding to the $\mathrm{PGL}(3,q)$-orbit of nonsingular conics.

2.8 Linear systems of conics

Let W be the space of 2-forms defined in $\mathrm{PG}(2,q)$. *Linear systems of conics* are subspaces of the projective geometry associated with W. In particular, 1, 2 and 3-dimensional subspaces are *pencils*, *nets* and *webs* of conics.

Subspaces of $\mathrm{PG}(5,q)$ correspond to linear systems of conics in $\mathrm{PG}(2,q)$ via ν (defined in 2.4.4): lines correspond to webs, planes to nets, and solids to pencils of conics in $\mathrm{PG}(2,q)$. By Remark 2.11, the classifications of K-orbits of subspaces of $\mathrm{PG}(5,q)$ correspond to the classifications of linear systems of conics in $\mathrm{PG}(2,q)$ up to projective equivalence. In particular, the classification of webs of conics over finite fields is equivalent to Theorem 2.25. The *base* (or set of *base points*) of a linear system of conics is the intersection of the conics in the system. We end this section with the following observation.

Lemma 2.6. *Let Q be a point and $\mathcal{P}$ a pencil of conics in $\mathrm{PG}(2,q)$. Then Q is a base point of $\mathcal{P}$ if and only if $\nu(Q)$ lies in the solid $S=\delta(\mathcal{P})$ of $\mathrm{PG}(5,q)$.*

Proof. Let Q be a base point of a pencil $\mathcal{P}=\langle\mathcal{C},\mathcal{C}'\rangle$. Then $\nu(Q)$ lies on the two hyperplane sections of $\mathcal{V}(\mathbb{F}_q)$ defined by $\mathcal{H}$ and $\mathcal{H}'$ whose dual coordinates are the coefficients of $\mathcal{C}$ and $\mathcal{C}'$ respectively. Therefore, $\nu(Q)$ lies in the solid $\delta(\mathcal{P})$ defined by $\mathcal{H}\cap\mathcal{H}'$. The inverse implication follows similarly. □

In other words, the points of rank 1 in S are precisely the images under the Veronese map of the base points of $\mathcal{P}$.

Example 2.10. *The pencil of conics generated by $\mathcal{C}_1=\mathcal{Z}(X_1X_2)$ and $\mathcal{C}_2=\mathcal{Z}(X_2^2)$ has $q+1$ base points. Furthermore, the point $\nu(1,0,0)=(1,0,0,0,0,0)$ lies in the associated solid S represented by*

$$\begin{bmatrix} x & y & z \\ y & t & \cdot \\ z & \cdot & \cdot \end{bmatrix}.$$

2.9 Dual subspaces of $\mathrm{PG}(5,q)$

For any finite field $\mathbb{F}_q$ of odd characteristic, there exists a polarity α of $\mathrm{PG}(5,q)$ that maps the set of conic planes of $\mathcal{V}(\mathbb{F}_q)$ onto the set of tangent planes of $\mathcal{V}(\mathbb{F}_q)$. This is Theorem 4.25. in (Hirschfeld & Thas, 1991), which implies the correspondence between K-orbits of subspaces of $\mathrm{PG}(5,q)$ and K-orbits of their associated dual spaces when q is odd. Therefore, one can deduce the classification of solids in $\mathrm{PG}(5,q)$, q odd, from that of lines in (Lavrauw & Popiel, 2020). Moreover, by α we get a correspondence between rank-1 nets of conics in $\mathrm{PG}(2,q)$, namely nets with at least one double line, and planes in $\mathrm{PG}(5,q)$ meeting $\mathcal{V}(\mathbb{F}_q)$ in at least one point, q odd (Lavrauw, Popiel & Sheekey, 2021). However, as such a polarity does not necessary exist when q is even, we cannot conclude the K-orbits of solids from those of lines in (Lavrauw & Popiel, 2020). Furthermore, the equivalence between planes in $\mathrm{PG}(5,q)$ meeting $\mathcal{V}(\mathbb{F}_q)$ in at least one point and rank-1 nets of conics in $\mathrm{PG}(2,q)$ fails when q is even as we will see later in Chapter 4.

2.10 Solids of $\mathrm{PG}(5,q)$, q odd

The correspondence between K-orbits of lines in $\mathrm{PG}(5,q)$, and pencils of conics in $\mathrm{PG}(2,q)$, q odd, can be found in Table 5 in (Lavrauw & Popiel, 2020). We used this correspondence to conclude the representatives of the 15 K-orbits of solids summarized in Table 2.3. We computed as well their rank-distributions and hyperplane-orbit distributions, summarized in Table 2.4.

K-orbits of solids	Representatives	Conditions
o_5	$\begin{bmatrix} . & x & y \\ x & . & z \\ y & z & t \end{bmatrix}$	
o_6	$\begin{bmatrix} . & . & x \\ . & y & z \\ x & z & t \end{bmatrix}$	
$o_{8,1}$	$\begin{bmatrix} . & x & y \\ x & z & t \\ y & t & z \end{bmatrix}$	
$o_{8,2}$	$\begin{bmatrix} . & x & y \\ x & \gamma z & t \\ y & t & z \end{bmatrix}$	$\gamma \notin \square$
o_9	$\begin{bmatrix} . & x & y \\ x & -2y & z \\ y & z & t \end{bmatrix}$	
o_{10}	$\begin{bmatrix} x & \frac{uv}{2}x & y \\ \frac{uv}{2}x & -vx & z \\ y & z & t \end{bmatrix}$	$(*)$
o_{12}	$\begin{bmatrix} x & . & y \\ . & z & . \\ y & . & t \end{bmatrix}$	
$o_{13,1}$	$\begin{bmatrix} x & . & y \\ . & z & t \\ y & t & -z \end{bmatrix}$	
$o_{13,2}$	$\begin{bmatrix} x & . & y \\ . & -\gamma t & z \\ y & z & y \end{bmatrix}$	$\gamma \notin \square$
$o_{14,1}$	$\begin{bmatrix} x & y & z \\ y & -x & t \\ z & t & x \end{bmatrix}$	
$o_{14,2}$	$\begin{bmatrix} -\gamma x & y & z \\ y & x & t \\ z & t & -\gamma x \end{bmatrix}$	$\gamma \notin \square$
$o_{15,1}$	$\begin{bmatrix} x & y & z \\ y & -v_1 x & t \\ z & t & -2y + uv_1 x \end{bmatrix}$	$(*)$, $-v_1 \notin \square$
$o_{15,2}$	$\begin{bmatrix} x & y & z \\ y & -v_2 x & t \\ z & t & -2y + uv_2 x \end{bmatrix}$	$(*)$, $-v_1 \notin \square$
o_{16}	$\begin{bmatrix} x & y & z \\ y & -2z & . \\ z & . & t \end{bmatrix}$	
o_{17}	$\begin{bmatrix} \alpha\gamma z - 2\alpha t & x & y \\ x & z & t \\ y & t & -2x - \beta z \end{bmatrix}$	$(**)$

Table 2.3 The K-orbits of solids in $\mathrm{PG}(5, q)$, q odd, and their representatives.

K-orbits of solids	Rank distributions	Hyperplane-orbit distributions
o_5	$[1, 2q^2+q, q^3-q^2]$	$[2, \frac{q-1}{2}, \frac{q-1}{2}, 0]$
o_6	$[q+1, 2q^2, q^3-q^2]$	$[1, q, 0, 0]$
$o_{8,1}$	$[2, q^2+2q-1, q^3-q]$	$[1, 1, 0, q-1]$
$o_{8,2}$	$[0, q^2+2q+1, q^3-q]$	$[1, 0, 1, q-1]$
o_9	$[1, q^2+q, q^3]$	$[1, 0, 0, q]$
o_{10}	$[1, 2q^2+q, q^3-q^2]$	$[0, \frac{q+1}{2}, \frac{q+1}{2}, 0]$
o_{12}	$[q+2, 2q^2-1, q^3-q^2]$	$[0, q+1, 0, 0]$
$o_{13,1}$	$[3, q^2+2q-2, q^3-q]$	$[0, 2, 0, q-1]$
$o_{13,2}$	$[1, q^2+2q, q^3-q]$	$[0, 1, 1, q-1]$
$o_{14,1}$	$[4, q^2+3q-3, q^3-2q]$	$[0, 3, 0, q-2]$
$o_{14,2}$	$[0, q^2+3q+1, q^3-2q]$	$[0, 1, 2, q-2]$
$o_{15,1}$	$[2, q^2+q-1, q^3]$	$[0, 1, 0, q]$
$o_{15,2}$	$[0, q^2+q+1, q^3]$	$[0, 0, 1, q]$
o_{16}	$[2, q^2+q-1, q^3]$	$[0, 1, 0, q]$
o_{17}	$[1, q^2, q^3+q]$	$[0, 0, 0, q+1]$

Table 2.4 Rank distributions and hyperplane-orbit distributions of the K-orbits of solids in $\mathrm{PG}(5, q)$, q odd.

3 SOLIDS IN $\mathrm{PG}(5,q)$, q EVEN

In this chapter, we present our results from (Alnajjarine, Lavrauw & Popiel , 2022). In particular, we determine orbits of solids of $\mathrm{PG}(5,q)$, q even, under the action of the subgroup K of $\mathrm{PGL}(6,q)$ stabilising the Veronese surface. We also determine two useful combinatorial invariants of each type of solid, namely their *point-orbit* and *hyperplane-orbit distributions* (see Section 2.7). Additionally, we calculate the stabiliser in $\mathrm{PGL}(3,q)$ of each type of solid S, and thereby determine the size of each orbit.

Our main results are Theorem 3.1 and Corollary 3.1, where we prove the existence of 15 K-orbits of solids in $\mathrm{PG}(5,q)$ and deduce the classification of pencils of conics in $\mathrm{PG}(2,q)$ up to projective equivalence.

Theorem 3.1. *(Alnajjarine, Lavrauw & Popiel , 2022, Theorem 1.1)*
Let q be an even prime power. There are exactly 15 orbits of solids in $\mathrm{PG}(5,q)$ under the induced action of $\mathrm{PGL}(3,q) \leqslant \mathrm{PGL}(6,q)$ defined in Section 2.7.1. Representatives of these orbits are given in Table 3.1, the notation of which is defined in Section 2.2.3.

Corollary 3.1. *Let q be an even prime power. There are 15 pencils of conics in $\mathrm{PG}(2,q)$ up to projective equivalence. Representatives of these pencils are given in Table 3.1.*

Corollary 3.2. *(Alnajjarine, Lavrauw & Popiel , 2022, Corollary 1.2)*
Let S and S' be solids in $\mathrm{PGL}(5,q)$, q even. Suppose that the point-orbit distributions of S and S' are equal, and that the hyperplane-orbit distributions of S and S' are equal. Then either

(i) *S and S' belong to the same K-orbit,*

(ii) *S and S' belong to the union of the orbits Ω_{11} and Ω_{12}, or*

(iii) *$q = 2$ and S and S' belong to the union of the orbits Ω_4 and Ω_9.*

Remark 3.1. *(Alnajjarine, Lavrauw & Popiel , 2022, Remark 1.3)*
In cases (ii) and (iii) of Corollary 3.2, one can determine whether S and S' belong to the same orbit by checking whether they intersect a certain orbit of *lines* in $\mathrm{PG}(5,q)$, specifically the orbit labelled "o_6" in (Lavrauw & Popiel, 2020). In (ii), a solid of type Ω_{11} contains a line of type o_6, but a solid of type Ω_{12} does not. In (iii), a solid of type Ω_4 contains a line of type o_6, but a solid of type Ω_9 does not. (See Remark 3.5 and Section 3.4.)

This chapter is structured as follows. The proofs of Theorem 3.1 and the associated data in Table 3.2 are given in Sections 3.1–3.3 for $q \neq 2$. The case $q = 2$ requires special treatment, and is handled in Section 3.4. In Section 3.5, we compare our results with the aforementioned partial classification of pencils of conics in $\mathrm{PG}(2,q)$, q even (Campbell, 1927). We note that, our arguments intentionally exploit the connection between solids in $\mathrm{PG}(5,q)$ and pencils of conics in $\mathrm{PG}(2,q)$. For example, point-orbit distributions cannot be obtained by working directly with the associated pencil of conics. On the other hand, stabilisers are easier to compute by working with pencils of conics, as we can appeal to well-known transitivity properties of the action of $\mathrm{PGL}(3,q)$ on $\mathrm{PG}(2,q)$ (see e.g. the proof of Lemma 3.9).

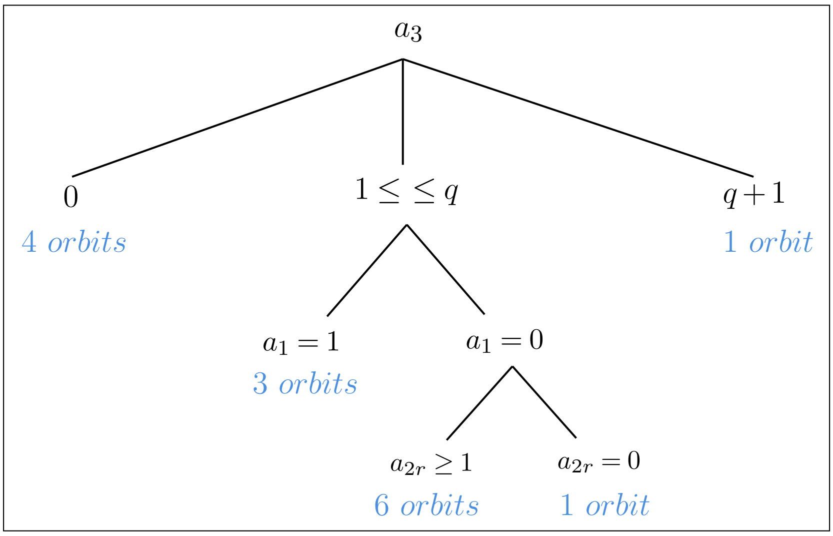

Figure 3.1 The discussion structure of Chapter 3.

Orbits	Representatives	Generating conics	Conditions
Ω_1	$\begin{bmatrix} x & y & z \\ y & t & \cdot \\ z & \cdot & t \end{bmatrix}$	$(X_1+X_2)^2$ X_1X_2	
Ω_2	$\begin{bmatrix} x & y & z \\ y & t & \cdot \\ z & \cdot & \cdot \end{bmatrix}$	X_2^2 X_1X_2	
Ω_3	$\begin{bmatrix} x & y & z \\ y & \cdot & t \\ z & t & \cdot \end{bmatrix}$	X_1^2 X_2^2	
Ω_4	$\begin{bmatrix} x & \cdot & y \\ \cdot & z & \cdot \\ y & \cdot & t \end{bmatrix}$	X_0X_1 X_1X_2	
Ω_5	$\begin{bmatrix} \cdot & x & y \\ x & z & t \\ y & t & x \end{bmatrix}$	$X_0X_1+X_2^2$ X_0^2	
Ω_6	$\begin{bmatrix} x & \cdot & y \\ \cdot & z & t \\ y & t & \cdot \end{bmatrix}$	$X_0X_1+X_2^2$ X_2^2	
Ω_7	$\begin{bmatrix} x & y & z \\ y & x+\gamma y & t \\ z & t & y \end{bmatrix}$	$X_0X_1+X_2^2$ $(X_0+X_1+\gamma X_2)^2$	$\mathrm{Tr}(\gamma^{-1})=1$
Ω_8	$\begin{bmatrix} x & y & z \\ y & t & z \\ z & z & y \end{bmatrix}$	$X_0X_1+X_2^2$ $(X_0+X_2)(X_1+X_2)$	
Ω_9	$\begin{bmatrix} x & x & y \\ x & z & t \\ y & t & t \end{bmatrix}$	$X_0(X_0+X_1)$ $X_2(X_1+X_2)$	
Ω_{10}	$\begin{bmatrix} x & y & z \\ y & y+\gamma t & t \\ z & t & y \end{bmatrix}$	$X_0X_1+X_2^2$ $X_1(X_0+X_1+\gamma X_2)$	$\mathrm{Tr}(\gamma^{-1})=1$
Ω_{11}	$\begin{bmatrix} x & y & z \\ y & t & \cdot \\ z & \cdot & y \end{bmatrix}$	$X_0X_1+X_2^2$ X_1X_2	
Ω_{12}	$\begin{bmatrix} x & y & z \\ y & t & \gamma y+z \\ z & \gamma y+z & y \end{bmatrix}$	$X_0X_1+X_2^2$ $X_2(X_0+X_1+\gamma X_2)$	$\mathrm{Tr}(\gamma^{-1})=1$
Ω_{13}	$\begin{bmatrix} x & y & z \\ y & \gamma x+y & t \\ z & t & \gamma x+z \end{bmatrix}$	$\gamma X_0^2+X_0X_1+X_1^2$ $\gamma X_0^2+X_0X_2+X_2^2$	$\mathrm{Tr}(\gamma)=1$
Ω_{14}	$\begin{bmatrix} x & y & \gamma x+y+\gamma t \\ y & \gamma x+y & z \\ \gamma x+y+\gamma t & z & t \end{bmatrix}$	$X_1^2+X_0X_2+\gamma X_2^2$ $\gamma X_0^2+X_0X_1+X_1^2$	$\mathrm{Tr}(\gamma)=1$
Ω_{15}	$\begin{bmatrix} x & y & bz+cy \\ y & z & t \\ bz+cy & t & y \end{bmatrix}$	$X_0X_1+X_2^2$ $X_0X_2+bX_1^2+cX_2^2$	$b\lambda^3+c\lambda+1$ irreducible over $\mathbb{F}_q$

Table 3.1 The K-orbits of solids in $\mathrm{PG}(5,q)$ and pencils of conics in $\mathrm{PG}(2,q)$, q even.

In what follows, let S be a solid in $\mathrm{PG}(5,q)$ and denote by $\Psi(S)$ the cubic surface defined by setting the determinant of the matrix representing S to zero (see Section 2.7). For example, for the solid S spanned by the first four points of the standard

Orbit	Point OD	Hyperplane OD	Stabiliser	Orbit size
Ω_1	$[1,q+1,2q^2-1,q^3-q^2]$	$[1,q/2,q/2,0]$	$E_q^2 \rtimes (E_q \times C_{q-1})$	$(q^3-1)(q+1)$
Ω_2	$[q+1,q+1,2q^2-q-1,q^3-q^2]$	$[1,q,0,0]$	$E_q^{1+2} \rtimes C_{q-1}^2$	$(q^2+q+1)(q+1)$
Ω_3	$[1,q^2+q+1,q^2-1,q^3-q^2]$	$[q+1,0,0,0]$	$E_q^2 \rtimes \mathrm{GL}(2,q)$	q^2+q+1
Ω_4	$[q+2,1,2q^2-2,q^3-q^2]$	$[0,q+1,0,0]$	$\mathrm{GL}(2,q)$	$q^2(q^2+q+1)$
Ω_5	$[1,q+1,q^2-1,q^3]$	$[1,0,0,q]$	$E_q^2 \rtimes C_{q-1}$	$q(q^3-1)(q+1)$
Ω_6	$[2,q+1,q^2+q-2,q^3-q]$	$[1,1,0,q-1]$	$C_{q-1}^2 \rtimes C_2$	$\frac{1}{2}q^3(q^2+q+1)(q+1)$
Ω_7	$[0,q+1,q^2+q,q^3-q]$	$[1,0,1,q-1]$	$D_{2(q+1)} \times C_{q-1}$	$\frac{1}{2}q^3(q^3-1)$
Ω_8	$[3,1,q^2+2q-3,q^3-q]$	$[0,2,0,q-1]$	$C_{q-1} \times C_2$	$\frac{1}{2}q^3(q^3-1)(q+1)$
Ω_9	$[4,1,q^2+3q-4,q^3-2q]$	$[0,3,0,q-2]$	Sym_4	$\frac{1}{24}q^3(q^3-1)(q^2-1)$
Ω_{10}	$[1,1,q^2+2q-1,q^3-q]$	$[0,1,1,q-1]$	$C_{q-1} \times C_2$	$\frac{1}{2}q^3(q^3-1)(q+1)$
Ω_{11}	$[2,1,q^2+q-2,q^3]$	$[0,1,0,q]$	$E_q \rtimes C_{q-1}$	$q^2(q^3-1)(q+1)$
Ω_{12}	$[2,1,q^2+q-2,q^3]$	$[0,1,0,q]$	C_2^2	$\frac{1}{4}q^3(q^3-1)(q^2-1)$
Ω_{13}	$[0,1,q^2+3q,q^3-2q]$	$[0,1,2,q-2]$	$C_2^2 \rtimes C_2$	$\frac{1}{8}q^3(q^3-1)(q^2-1)$
Ω_{14}	$[0,1,q^2+q,q^3]$	$[0,0,1,q]$	C_4	$\frac{1}{4}q^3(q^3-1)(q^2-1)$
Ω_{15}	$[1,1,q^2-1,q^3+q]$	$[0,0,0,q+1]$	C_3	$\frac{1}{3}q^3(q^3-1)(q^2-1)$

Table 3.2 Invariants of K-orbits of solids in $\mathrm{PG}(5,q)$, q even.

frame of $\mathrm{PG}(5,q)$:

$$S = \begin{bmatrix} x & y & z \\ y & t & \cdot \\ z & \cdot & \cdot \end{bmatrix}, \tag{3.1}$$

$\Psi(S)$ is the cubic surface comprising points as in (3.1) with $Z^2T=0$. In particular, we see that S has *rank distribution* $[q+1,2q^2,q^3-q^2]$, meaning that it contains $q+1$ points of rank 1, $2q^2$ points of rank 2, and q^3-q^2 points of rank 3. (The points of rank 1 comprise the nonsingular conic given by $Z=0$ and $XT=Y^2$.) The rank distribution is related to a particular case of what we call an *orbit distribution*. For more information about the terminology used in this chapter and the connection between solids of $\mathrm{PG}(5,q)$ and pencils of conics in $\mathrm{PG}(2,q)$, we refer to Chapter 2.

Remark 3.2. *As we will see later, studying cubic surfaces associated with solids in* $\mathrm{PG}(5,q)$ *can be useful to differentiate between non-equivalent solids, but it is not sufficient to completely characterize each orbit. For instance, by suitably reordering the variables x,y,z and t, we can represent Ω_2 by*

$$S_2 = \begin{bmatrix} z & y & t \\ y & x & \cdot \\ t & \cdot & \cdot \end{bmatrix},$$

which has the same cubic surface as Ω_3 defined by $XT^2=0$, however the two orbits are distinct by their intersection with the Veronese surface.

Before proceeding, we mention the following lemma concerning the hyperplane-orbit distribution $\mathrm{OD}_{K,4}(S) = [a_1, a_{2r}, a_{2i}, a_3]$ of a solid in $\mathrm{PG}(5,q)$, q even. Here a_j denotes the number of hyperplanes of type $\mathcal{H}_j$ incident with U for each of the symbols $j \in \{1, 2r, 2i, 3\}$ (see Section 2.7).

Lemma 3.1. *(Alnajjarine, Lavrauw & Popiel , 2022, Lemma 2.9)*
Let S be a solid of $\mathrm{PG}(5,q)$, where $q = 2^h$ with $h > 1$, and let b denote the number of points of S contained in $\mathcal{V}(\mathbb{F}_q)$. Then the hyperplane-orbit distribution $\mathrm{OD}_{K,4}(S) = [a_1, a_{2r}, a_{2i}, a_3]$ of S satisfies:

(i) $a_1 + 2a_{2r} + a_3 = q + b$.

(ii) $a_{2r} - a_{2i} + 1 = b$.

Proof. First note that $(q+1)a_1 + (2q+1)a_{2r} + a_{2i} + (q+1)a_3 - bq = q^2 + q + 1$. This follows from the fact that each point on $\mathcal{V}(\mathbb{F}_q)$ either lies in S and belongs to $q+1$ hyperplanes through S, or belongs to exactly one hyperplane of $\mathrm{PG}(5,q)$ through S, and the fact that the hyperplanes in the orbits $\mathcal{H}_1$, $\mathcal{H}_{2r}$, $\mathcal{H}_{2i}$, $\mathcal{H}_3$ intersect $\mathcal{V}(\mathbb{F}_q)$ in $q+1$, $2q+1$, 1 and $q+1$ points, respectively. Now use the fact that $a_1 + a_{2r} + a_{2i} + a_3 = q+1$ and divide by q to get (i). Substitution of $a_1 + a_{2r} + a_3$ by $q + 1 - a_{2i}$ gives (ii). □

3.1 Solids not contained in any hyperplane of type $\mathcal{H}_3$

We begin by classifying the K-orbits of solids that are not contained in any hyperplane of type $\mathcal{H}_3$, namely, those for which the corresponding pencil of conics contains no nonsingular conics. It is straightforward to list the possible configurations of pairs of conics that can occur. However, since we are interested in K-orbits of solids, i.e. pencils of conics as opposed to pairs of conics, we need to understand when two different types of pairs of conics give rise to the same pencil up to projective equivalence.

Here, and in subsequent sections, homogeneous coordinates in a solid S of $\mathrm{PG}(5,q)$ are generally denoted by (X,Y,Z,T), where the solid is represented as in (3.1). The pencil of conics in $\mathrm{PG}(2,q)$ corresponding to S is denoted by $\mathcal{P}(S)$, and the cubic surface obtained as the intersection of S with the secant variety $\mathcal{V}(\mathbb{F}_q)^2$ of $\mathcal{V}(\mathbb{F}_q)$ is denoted by $\Psi(S)$. As before, the homogeneous coordinates in the domain $\mathrm{PG}(2,q)$ of the Veronese map ν are denoted by (X_0, X_1, X_2), and those in $\mathrm{PG}(5,q)$ are denoted by $(Y_0, \dots, Y_5)$.

3.1.1 Solids contained in a hyperplane of type $\mathcal{H}_1$

We first treat the K-orbits of solids S corresponding to pencils $\mathcal{P}(S)$ that contain at least one double line, namely those whose hyperplane-orbit distribution $\mathrm{OD}_{K,4}(S) = [a_1, a_{2r}, a_{2i}, a_3]$ has $a_1 > 0$. If $\mathcal{P}(S)$ contains exactly one double line (i.e. $a_1 = 1$) and no pair of (distinct) real lines ($a_{2r} = 0$), then the orbit of S will arise later in our analysis, since any such pencil contains a nonsingular conic, by the following lemma.

Lemma 3.2. *(Alnajjarine, Lavrauw & Popiel , 2022, Lemma 3.1) If* $\mathrm{OD}_{K,4}(S) = [1, 0, a_{2i}, a_3]$ *with* $a_{2i} > 0$*, then* $a_3 > 0$*.*

Proof. Putting $a_1 = 1$ and $a_{2r} = 0$ into Lemma 3.1(ii) gives $b = 1 - a_{2i}$, which implies that $a_{2i} \leqslant 1$ since $b \geqslant 0$. Therefore, $a_{2i} = 1$ and so $a_3 = (q+1) - 2 > 0$. □

We may therefore assume that if $\mathcal{P}(S)$ contains exactly one double line, say $\mathcal{L}_1^2$, then it contains at least one pair of distinct real lines, say $\mathcal{L}_2\mathcal{L}_3$. We then have the following possibilities:

(i) the three lines are distinct and concurrent,

(ii) $\mathcal{L}_1$ coincides with one of $\mathcal{L}_2$ or $\mathcal{L}_3$, or

(iii) the three lines are distinct and not concurrent.

Since $\mathrm{PGL}(3, q)$ acts transitively on each of these configurations of lines, two solids corresponding to the same configuration belong to the same K-orbit. In case (iii), $\mathcal{P}(S)$ has exactly two base points, so the following lemma implies, together with Lemma 2.6, that S is contained in a hyperplane of type $\mathcal{H}_3$.

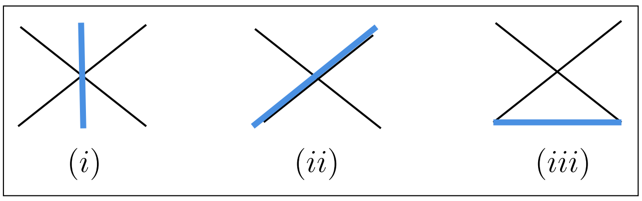

Figure 3.2 Pencils of conics generated by a double line $\mathcal{L}_1$ and a pair of real lines $\mathcal{L}_2 \cup \mathcal{L}_3$.

Lemma 3.3. *(Alnajjarine, Lavrauw & Popiel , 2022, Lemma 3.2)*
If $\mathrm{OD}_{K,4}(S) = [1, a_{2r}, a_{2i}, a_3]$ *and* S *meets* $\mathcal{V}(\mathbb{F}_q)$ *in two points, then* $a_{2i} = 0$ *and* $a_3 > 0$.

Proof. Since a hyperplane of type $\mathcal{H}_{2i}$ meets $\mathcal{V}(\mathbb{F}_q)$ in one point, it follows that $a_{2i} = 0$. Putting $a_1 = 1$ and $b = 2$ into Lemma 3.1(i) gives $a_3 = (q+1) - 2a_{2r}$. Since q is even, $(q+1) - 2a_{2r}$ is odd, so $a_3 \geqslant 1$. □

If follows that we are left with at most two K-orbits, corresponding to the cases (i) and (ii). We label these orbits as Ω_1 and Ω_2, respectively, and choose representatives for them as

$$\Omega_1 : \begin{bmatrix} x & y & z \\ y & t & \cdot \\ z & \cdot & t \end{bmatrix}, \quad \Omega_2 : \begin{bmatrix} x & y & z \\ y & t & \cdot \\ z & \cdot & \cdot \end{bmatrix}, \tag{3.2}$$

obtained by taking $\mathcal{L}_2 = \mathcal{Z}(X_1)$ and $\mathcal{L}_3 = \mathcal{Z}(X_2)$ in both cases, $\mathcal{L}_1 = \mathcal{Z}((X_1 + X_2)^2)$ for Ω_1 and $\mathcal{L}_1 = \mathcal{Z}(X_2^2)$ for Ω_2. We now calculate the point-orbit distributions, hyperplane-orbit distributions, and stabilisers of the solids in these K-orbits. We may use the representatives given in (3.2) for these calculations, since all of the aforementioned data are K-invariant. We begin with the hyperplane-orbit distributions, verifying in particular the desired condition that each solid lies in a unique hyperplane of type $\mathcal{H}_1$, and that the orbits Ω_1 and Ω_2 are indeed distinct (because their hyperplane-orbit distributions are distinct).

Lemma 3.4. *(Alnajjarine, Lavrauw & Popiel , 2022, Lemma 3.3)*
The hyperplane-orbit distribution of a solid of type Ω_1 *is* $[1, q/2, q/2, 0]$. *The hyperplane-orbit distribution of a solid of type* Ω_2 *is* $[1, q, 0, 0]$. *In particular,* $\Omega_1 \neq \Omega_2$.

Proof. Let S_i denote the representative of Ω_i defined in (3.2), for $i \in \{1,2\}$. Lemma 2.2 implies that each of the pencils $\mathcal{P}(S_i)$ does indeed contain a unique double line (namely $\mathcal{L}_1^2$) and no nonsingular conics. Hence, in the notation of Lemma 3.1, the hyperplane-orbit distribution of S_i has the form $[1, a_{2r}, a_{2i}, 0]$ in both cases, i.e. $a_1 = 1$ and $a_3 = 0$. The pencil $\mathcal{P}(S_1)$ has a unique base point (the unique point of concurrency of the three lines $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$), so putting $b = 1$ into Lemma 3.1 yields $a_{2r} = a_{2i} = q/2$. On the other hand, $\mathcal{P}(S_2)$ has $q+1$ base points (those on the line $\mathcal{L}_1$), so $a_{2r} = q$ and $a_{2i} = 0$. □

Lemma 3.5. *(Alnajjarine, Lavrauw & Popiel , 2022, Lemma 3.4)*
The point-orbit distribution of a solid of type Ω_1 *is* $[1, q+1, 2q^2 - 1, q^3 - q^2]$. *The point-orbit distribution of a solid of type* Ω_2 *is* $[q+1, q+1, 2q^2 - q - 1, q^3 - q^2]$.

Proof. Consider again the representatives S_1 and S_2 in (3.2). Points of rank at most 2 in S_1 correspond to points on the cubic surface $\Psi(S_1) = \mathcal{Z}(XT^2 + Y^2T + Z^2T)$. There are $2q^2+q+1$ such points, exactly one of which has rank 1, namely the point with homogeneous coordinates $(X,Y,Z,T) = (1,0,0,0)$, which is the image under ν of the unique base point $(X_0,X_1,X_2) = (1,0,0)$ of the pencil $\mathcal{P}(S_1)$ (cf. Lemma 2.6). Hence, the rank distribution of S_1 is $[1, 2q^2+q, q^3-q^2]$. The points of S_1 contained in the nucleus plane are those on the line $\mathcal{Z}(X,T)$, so the point-orbit distribution of S_1 is $[1, q+1, 2q^2-1, q^3-q^2]$. The cubic surface $\Psi(S_2)$ is $\mathcal{Z}(Z^2T)$, which contains $2q^2+q+1$ points, being the union of two planes meeting in a line. It intersects $\mathcal{V}(\mathbb{F}_q)$ in the conic $\mathcal{Z}(Z, XT+Y^2)$, and the nucleus plane in the line $\mathcal{Z}(X,T)$. □

We now calculate the stabiliser $K_{S_i} \leqslant K$ of $S_i \in \Omega_i$ for $i \in \{1,2\}$. Recall the group-theoretic notation established in Notation 2.2.3.

Lemma 3.6. *(Alnajjarine, Lavrauw & Popiel , 2022, Lemma 3.5)*
If $S_1 \in \Omega_1$ then $K_{S_1} \cong E_q^2 \rtimes (E_q \times C_{q-1})$. If $S_2 \in \Omega_2$ then $K_{S_2} \cong E_q^{1+2} \rtimes C_{q-1}^2$.

Proof. If S_1 is the representative of Ω_1 given in (3.2) then $K_{S_1} \leqslant K_P$, where $P = (1,0,0,0)$ is the unique point of rank 1 in S_1. Notice that K_P is equal to the stabiliser of the plane $\pi = \mathcal{Z}(T)$, because π is the tangent plane to $\mathcal{V}(\mathbb{F}_q)$ at P. An element of K_P therefore fixes S_1 if and only if it maps the point $Q = (0,0,0,1)$ into S_1, since $S_1 = \langle \pi, Q\rangle$. Elements of $K_P \cong E_q^2 \rtimes \mathrm{GL}(2,q)$ are represented by matrices $g = (g_{ij}) \in \mathrm{GL}(3,q)$ with $g_{21} = g_{31} = 0$. The subgroup $H \cong \mathrm{GL}(2,q)$ of K_P obtained by setting $g_{12} = g_{13} = 0$ fixes the conic $\mathcal{Z}(Y_3^2 + Y_4Y_5)$ in the plane $\pi' = \mathcal{Z}(Y_0,Y_1,Y_2)$. Since Q is a point external to this conic and distinct from its nucleus, it follows by considering the quotient space of π' that $K_{S_1} \cong E_q^2 \rtimes (E_q \times C_{q-1})$. The solid $S_2 \in \Omega_2$ given in (3.2) meets $\mathcal{V}(\mathbb{F}_q)$ in a conic which spans the plane $\pi : \mathcal{Z}(Z)$, so $K_{S_2} \leqslant K_\pi$. An element of K represented by a matrix $(g_{ij}) \in \mathrm{GL}(3,q)$ belongs to K_π if and only if $g_{31} = g_{32} = 0$. It fixes S_2 if and only if it also fixes the line $\mathcal{Z}(X,T)$ in which S_2 intersects the nucleus plane. This occurs if and only if g_{21} is also 0. Upon factoring out scalars we therefore obtain $K_{S_2} \cong E_q^{1+2} \rtimes C_{q-1}^2$. □

If $\mathcal{P}(S)$ contains more than one double line, then it is a pencil of lines. There is one K-orbit of such solids, which we call Ω_3. Generating $\mathcal{P}(S)$ by the double lines $\mathcal{Z}(X_1^2)$ and $\mathcal{Z}(X_2^2)$ gives the representative

$$\Omega_3 : \begin{bmatrix} x & y & z \\ y & \cdot & t \\ z & t & \cdot \end{bmatrix}.$$

Lemma 3.7. *(Alnajjarine, Lavrauw & Popiel , 2022, Lemma 3.6)*
A solid $S_3 \in \Omega_3$ *has point-orbit distribution* $[1, q^2+q+1, q^2-1, q^3-q^2]$*, hyperplane-orbit distribution* $[q+1,0,0,0]$*, and stabiliser* $K_{S_3} \cong E_q^2 \rtimes \mathrm{GL}(2,q)$*. In particular,* $\Omega_3 \notin \{\Omega_1, \Omega_2\}$*.*

Proof. Let S_3 denote the above representative of Ω_3. Since all conics in the pencil $\mathcal{P}(S_3)$ are double lines, the hyperplane-orbit distribution of S_3 is $[q+1,0,0,0]$. This implies that $\Omega_3 \notin \{\Omega_1, \Omega_2\}$ (upon comparing with Lemma 3.4). The cubic surface $\Psi(S_3)$ is the union of the nucleus plane $\mathcal{Z}(X)$ and the plane $\mathcal{Z}(T)$. It contains exactly one point of rank 1, namely the point $P = (1,0,0,0)$. Therefore, we obtain the asserted point-orbit distribution. The stabiliser is immediate from the hyperplane-orbit distribution. □

3.1.2 Solids not contained in a hyperplane of type $\mathcal{H}_1$

Next we classify the solids contained neither in hyperplanes of type $\mathcal{H}_3$, nor in hyperplanes of type $\mathcal{H}_1$. Let S be such a solid, namely one with $\mathrm{OD}_{K,4}(S) = [0, a_{2r}, a_{2i}, 0]$. Since we are assuming that $q > 2$, it follows from Lemma 3.1(i) that $a_{2r} \geqslant 2$. Hence, there exist two pairs $\mathcal{L}_1\mathcal{L}_2$ and $\mathcal{L}_3\mathcal{L}_4$ of distinct real lines generating $\mathcal{P}(S)$. There are a number of possible configurations of the lines $\mathcal{L}_1, \ldots, \mathcal{L}_4$ (see Figure 3.3), but it turns out that only one of these gives a K-orbit with the assumed hyperplane-orbit distribution.

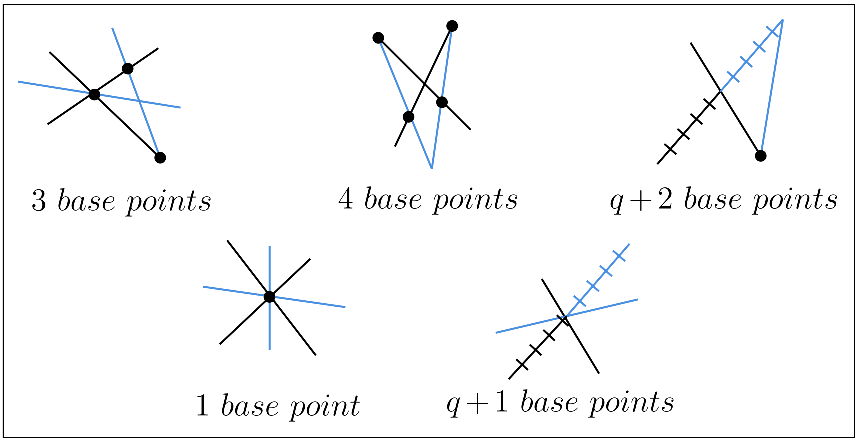

Figure 3.3 The possible configurations of the lines $\mathcal{L}_1, \ldots, \mathcal{L}_4$; $q \neq 2$.

Lemma 3.8. *(Alnajjarine, Lavrauw & Popiel , 2022, Lemma 3.7)*
There is a unique K*-orbit of solids with hyperplane-orbit distribution* $[0, a_{2r}, a_{2i}, 0]$*.*

Proof. If the four lines $\mathcal{L}_1,\ldots,\mathcal{L}_4$ are concurrent, then $S \in \Omega_1$. If $\mathcal{P}(S)$ has one of the lines as its base, then $S \in \Omega_2$ since $\mathcal{P}(S)$ then also contains that base as a double line. If the two pairs $\mathcal{L}_1\mathcal{L}_2$ and $\mathcal{L}_3\mathcal{L}_4$ meet in either three or four points, then $\mathcal{P}(S)$ contains at least one nonsingular conic (and so $a_3 \neq 0$): this can be verified by a direct computation, and also follows from the treatment of the orbits Ω_8 and Ω_9 in Section 3.2.2.1. The only remaining possibility is that the two pairs share a line and do not meet in the same point, in which case the base of $\mathcal{P}(S)$ is an antiflag (a non-incident point–line pair), consisting of the shared line and one extra point. Since $\mathrm{PGL}(3,q)$ acts transitively on antiflags, there is one such K-orbit of solids. □

The K-orbit of solids arising as above is denoted Ω_4. Taking $\mathcal{P}(S)$ generated by the pairs of real lines $\mathcal{Z}(X_0X_1)$ and $\mathcal{Z}(X_1X_2)$ gives the representative

$$\Omega_4 : \begin{bmatrix} x & \cdot & y \\ \cdot & z & \cdot \\ y & \cdot & t \end{bmatrix}.$$

Lemma 3.9. *(Alnajjarine, Lavrauw & Popiel , 2022, Lemma 3.8)*
A solid $S_4 \in \Omega_4$ has point-orbit distribution $[q+2, 1, 2q^2-2, q^3-q^2]$, hyperplane-orbit distribution $[0, q+1, 0, 0]$, and stabiliser $K_{S_4} \cong \mathrm{GL}(2,q)$. In particular, $\Omega_4 \notin \{\Omega_1, \Omega_2, \Omega_3\}$.

Proof. Let S_4 be the solid defined above. Every conic in the pencil $\mathcal{P}(S_4)$ has the form $\mathcal{Z}(X_1(\lambda X_0 + \mu X_2))$ for some λ, μ, i.e. every conic in $\mathcal{P}(S_4)$ is a pair of real lines, so the hyperplane-orbit distribution is $[0, q+1, 0, 0]$, and this implies that $\Omega_4 \notin \{\Omega_1, \Omega_2, \Omega_3\}$. The cubic surface $\Psi(S_4) = \mathcal{Z}(Z(XT + Y^2))$ is the union of a plane and a quadratic cone with vertex $P = (0,0,1,0)$, meeting in a conic $\mathcal{C} = \mathcal{Z}(Y_0Y_5 + Y_2^2)$. It intersects S_4 in $P \cup \mathcal{C}$, so S_4 contains $q+2$ points of rank 1. The nucleus of $\mathcal{C}$ is the unique point of S_4 in the nucleus plane. The pencil $\mathcal{P}(S_4)$ is fixed by an element of $\mathrm{PGL}(3,q)$ if and only if the antiflag comprising its base is fixed, so K_{S_4} is isomorphic to the stabiliser of an antiflag, i.e. $K_{S_4} \cong \mathrm{GL}(2,q)$. □

This completes the classification of solids contained in no hyperplane of type $\mathcal{H}_3$, or equivalently, of pencils of conics containing no nonsingular conics. We make the following observation for reference.

Corollary 3.3. *(Alnajjarine, Lavrauw & Popiel , 2022, Corollary 3.9)*
There is no pencil of conics in $\mathrm{PG}(2,q)$, q even, with $q+1$ singular conics and empty base.

Proof. If $q > 2$ then a pencil $\mathcal{P}$ with $q+1$ singular conics corresponds to a solid $S \in \Omega_1 \cup \ldots \cup \Omega_4$. By the point-orbit distributions calculated above, S meets $\mathcal{V}(\mathbb{F}_q)$ in at least one point, so Lemma 2.6 implies that $\mathcal{P}$ has at least one base point. By Section 3.4, the result holds also for $q = 2$. □

3.2 Solids contained in at least one and at most q hyperplanes of type $\mathcal{H}_3$

In this section we classify the K-orbits of solids contained in at least one hyperplane of type $\mathcal{H}_3$ and at most q such hyperplanes. That is, we treat the solids S with hyperplane-orbit distribution $\mathrm{OD}_{K,4}(S) = [a_1, a_{2r}, a_{2i}, a_3]$ where $1 \leqslant a_3 \leqslant q$. The cases (i) $a_1 \neq 0$, (ii) $a_1 = 0$ and $a_{2r} \neq 0$, and (iii) $a_1 = a_{2r} = 0$ and $a_{2i} \neq 0$ are analysed separately in Sections 3.2.1, 3.2.2 and 3.2.3, respectively. The following observation implies that $a_1 + a_{2r} + a_{2i} \leqslant 3$ (and hence $a_3 \geqslant q-2$) in all cases.

Lemma 3.10. *(Alnajjarine, Lavrauw & Popiel , 2022, Lemma 4.1)*
A pencil containing a nonsingular conic contains at most three singular conics.

Proof. A pencil generated by $\mathcal{Z}(f)$ and $\mathcal{Z}(g)$, with $\mathcal{Z}(g)$ nonsingular, contains a singular conic $\mathcal{Z}(f+\lambda g)$ if and only if λ is a root of a (certain) cubic in $\mathbb{F}_q[X]$ (cf. Lemma 2.2). □

3.2.1 Solids contained in a hyperplane of type $\mathcal{H}_1$

The stabiliser of a nonsingular conic $\mathcal{C}$ in $\mathrm{PG}(2,q)$ has three orbits on lines, namely tangents to $\mathcal{C}$, secants to $\mathcal{C}$, and lines external to $\mathcal{C}$. Hence, there are at most three K-orbits of solids contained both in a hyperplane of type $\mathcal{H}_3$ (which corresponds to a nonsingular conic) and a hyperplane of type $\mathcal{H}_1$ (which corresponds to a double line). Since the corresponding types of pencils have different numbers of base points, there are exactly three K-orbits. The following representatives are obtained using the nonsingular conic $\mathcal{C} = \mathcal{Z}(X_0X_1 + X_2^2)$ and the double lines corresponding to the tangent $\mathcal{Z}(X_0)$, the secant $\mathcal{Z}(X_2)$ and the external line $\mathcal{Z}(X_0 + X_1 + \sqrt{\gamma}X_2)$, where

Orbit	Point-orbit distribution	Hyperplane-orbit distribution	Stabiliser
Ω_5	$[1,q+1,q^2-1,q^3]$	$[1,0,0,q]$	$E_q^2 \rtimes C_{q-1}$
Ω_6	$[2,q+1,q^2+q-2,q^3-q]$	$[1,1,0,q-1]$	$C_{q-1}^2 \rtimes C_2$
Ω_7	$[0,q+1,q^2+q,q^3-q]$	$[1,0,1,q-1]$	$D_{2(q+1)} \times C_{q-1}$

Table 3.3 Data for Lemma 3.11.

γ is some fixed element of $\mathbb{F}_q$ with $\mathrm{Tr}(\gamma^{-1})=1$ (cf. Lemma 2.5):

$$(3.3)\quad \Omega_5 : \begin{bmatrix} \cdot & x & y \\ x & z & t \\ y & t & x \end{bmatrix}, \quad \Omega_6 : \begin{bmatrix} x & \cdot & y \\ \cdot & z & t \\ y & t & \cdot \end{bmatrix}, \quad \Omega_7 : \begin{bmatrix} x & y & z \\ y & x+\gamma y & t \\ z & t & y \end{bmatrix} \text{ where } \mathrm{Tr}(\gamma^{-1})=1.$$

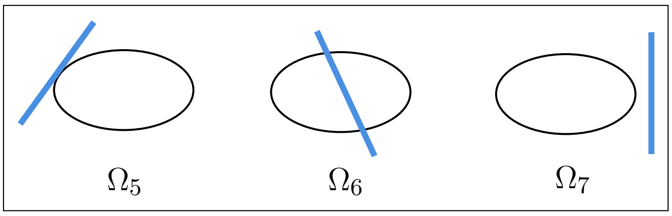

Figure 3.4 Pencils of conics associated with Ω_5, Ω_6 and Ω_7.

Lemma 3.11. *(Alnajjarine, Lavrauw & Popiel , 2022, Lemma 4.2)*
The point-orbit distributions, hyperplane-orbit distributions, and stabilisers of solids of types Ω_5, Ω_6 and Ω_7 are as in Table 3.3. In particular, these orbits are distinct from each other and from $\Omega_1,\dots,\Omega_4$.

Proof. Let $S_i \in \Omega_i$, $i \in \{5,6,7\}$, be the representatives given in (3.3). The hyperplane-orbit distribution of S_5 is an immediate consequence of Lemma 2.2, which implies that a conic $\mathcal{Z}(\lambda X_0^2+X_0X_1+X_2^2)$ in the pencil $\mathcal{P}(S_5)$ cannot be singular. Similarly, a conic $\mathcal{Z}(\lambda X_2^2+X_0X_1+X_2^2)$ in $\mathcal{P}(S_6)$ is singular if and only if $\lambda=1$, in which case one obtains the pair of real lines $\mathcal{Z}(X_0X_1)$, both of which are tangents to the conic $\mathcal{Z}(X_0X_1+X_2^2)$. Finally, a conic $\mathcal{Z}(\lambda(X_0^2+X_1^2+\gamma X_2^2)+X_0X_1+X_2^2)$ in $\mathcal{P}(S_7)$ is singular if and only if $\lambda=\gamma^{-1}$, in which case one obtains the pair of conjugate imaginary lines $\mathcal{Z}(X_0^2+\gamma X_0X_1+X_1^2)$. The hyperplane-orbit distributions imply that Ω_5, Ω_6 and Ω_7 are distinct and do not belong to $\{\Omega_1,\dots,\Omega_4\}$.

Next, we calculate the point-orbit distributions. The cubic surface $\Psi(S_5)=\mathcal{Z}(X^3+Y^2Z)$ consists of q^2+q+1 points, being a cone with vertex a point and base a planar rational cubic curve. It meets the nucleus plane π_n in the line $\ell:\mathcal{Z}(X,Z)$, and $\mathcal{V}(\mathbb{F}_q)$ in its unique singular point $P=(0,0,1,0)$, i.e. the image of the base point of $\mathcal{P}(S_5)$ under ν. The cubic surface $\Psi(S_6)=\mathcal{Z}(XT^2+Y^2Z)$ consists of q^2+2q+1 points, since its point set is in one-to-one correspondence with the points on the hyperbolic quadric $\mathcal{Z}(XT+YZ)$. It meets π_n in the line $\mathcal{Z}(X,Z)$, and $\mathcal{V}(\mathbb{F}_q)$ in the images of the two base points of $\mathcal{P}(S_6)$. Finally, $\Psi(S_7) = \mathcal{Z}(\gamma XY^2 + \gamma YZ^2 + XT^2 + X^2Y + XZ^2 + Y^3)$ consists of two lines in the plane $\mathcal{Z}(Y)$ and q^2 additional points. It is disjoint from $\mathcal{V}(\mathbb{F}_q)$ and intersects π_n in the line $\mathcal{Z}(X,Y)$.

It remains to calculate the stabilisers. If an element of K represented by a matrix $(g_{ij}) \in \mathrm{GL}(3,q)$ fixes S_5 then it must fix the point $P = S_5 \cap \mathcal{V}(\mathbb{F}_q)$ and the line $\ell = S_5 \cap \pi_n$ (both calculated above). This occurs if and only if $g_{12} = g_{13} = g_{23} = g_{32} = 0$. An element of $K_P \cap K_\ell$ fixes S_5 if and only if it also maps the point $Q = (1,0,0,0)$ into S_5, since $S_5 = \langle P, Q, \ell\rangle$. This occurs if and only if also $g_{33}^2 = g_{11}g_{22}$. Factoring out scalars therefore gives $K_{S_5} \cong E_q^2 \rtimes C_{q-1}$. Since $\mathcal{P}(S_6)$ contains a unique double line $\mathcal{L}_1^2$ and a unique pair of real lines $\mathcal{L}_2\mathcal{L}_3$, its stabiliser in $\mathrm{PGL}(3,q)$ is equal to the stabiliser of $\mathcal{L}_1$ inside the stabiliser $C_{q-1}^2 \rtimes \mathrm{Sym}_3$ of $\{\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3\}$. Hence, $K_{S_6} \cong C_{q-1}^2 \rtimes C_2$. Finally, a solid $S_7 \in \Omega_7$ is contained in a unique hyperplane H_1 of type $\mathcal{H}_1$, which meets $\mathcal{V}(\mathbb{F}_q)$ in a conic $\mathcal{C}$, and in a unique hyperplane H_2 of type $\mathcal{H}_{2i}$, which meets $\mathcal{V}(\mathbb{F}_q)$ in a point $P \notin H_1$. Therefore, K_{S_7} is a subgroup of $K_\mathcal{C} \cap K_P \cong \mathrm{GL}(2,q)$. Since S_7 is disjoint from $\mathcal{V}(\mathbb{F}_q)$, it meets the conic plane $\pi = \langle \mathcal{C}\rangle$ in a line ℓ external to $\mathcal{C}$. By considering the action of K_{S_7} on π, we therefore deduce that K_{S_7} is a subgroup of the stabiliser of ℓ in $K_\mathcal{C} \cap K_P$, which is isomorphic to $D_{2(q+1)} \times C_{q-1}$. The fact that K_{S_7} is equal to this group follows from the one-to-one correspondence between the hyperplanes of type $\mathcal{H}_{2i}$ through P and the lines external to $\mathcal{C}$ in π. (Over the quadratic extension of $\mathrm{PG}(5,q)$, ℓ meets $\mathcal{C}$ in a pair of conjugate points, and H_2 meets the Veronese surface in two conjugate conics which pass through P and meet $\mathcal{C}$ in those points, so H_2 is uniquely determined by ℓ.) □

Remark 3.3. *(Alnajjarine, Lavrauw & Popiel , 2022, Remark 4.3)*
It follows from the first part of the proof of Lemma 3.11 that Ω_6 can also be obtained by considering either (i) a pencil spanned by a nonsingular conic $\mathcal{C}$ and a pair of two real lines tangent to $\mathcal{C}$, or (ii) a pencil spanned by a pair of real lines and a double line meeting the pair in two distinct points.

3.2.2 Solids contained in a hyperplane of type $\mathcal{H}_{2r}$ and no hyperplane of type $\mathcal{H}_1$

If S is a solid with hyperplane-orbit distribution $[0, a_{2r}, a_{2i}, a_3]$ where $a_{2r} > 0$ and $1 \leqslant a_3 \leqslant q$, then we may assume without loss of generality that $\mathcal{P}(S)$ is generated by a nonsingular conic $\mathcal{C}$ and a pair of real lines $\mathcal{L}_1\mathcal{L}_2$. Let us encode the configuration

$(\mathcal{C},\mathcal{L}_1,\mathcal{L}_2)$ by the pair of integers (k_1,k_2) where k_i denotes the number of points in $\mathcal{L}_i \cap \mathcal{C}$. The possible configurations are $(k_1,k_2) \in \{(2,2),\ (2,1),\ (2,0),\ (1,1),\ (1,0),\ (0,0)\}$. By Remark 3.3, we may ignore the case $(k_1,k_2) = (1,1)$.

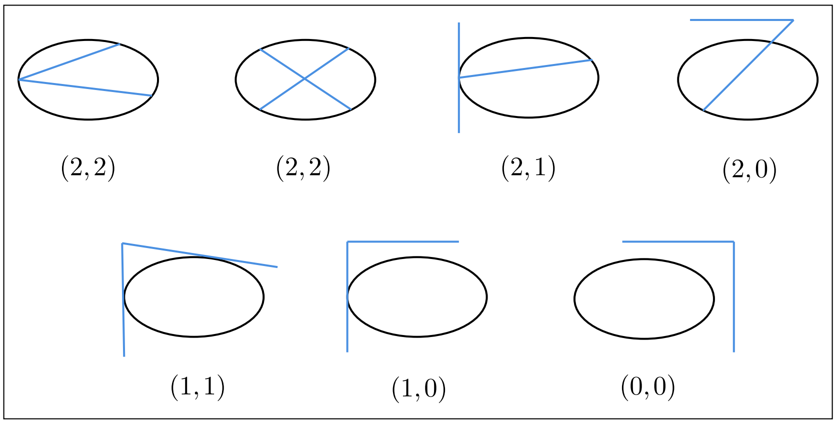

Figure 3.5 The possible configurations of pencils of conics generated by a nonsingular conic $\mathcal{C}$ and a pair of real lines $\mathcal{L}_1 \cup \mathcal{L}_2$, where (k_1,k_2) denote the number of points in $\mathcal{L}_i \cap \mathcal{C}$.

3.2.2.1 $(k_1,k_2) = (2,2)$

If $(k_1,k_2) = (2,2)$ then $\mathcal{P}(S)$ has either three or four base points. Exactly one K-orbit arises from each of these two cases. In the case of three base points, this follows from the fact that the stabiliser of a nonsingular conic acts 3-transitively on its points; in the case of four base points, it follows from the fact that the image of a frame of PG$(2,q)$ under ν spans a solid. The resulting orbits are

$$\Omega_8 : \begin{bmatrix} x & y & z \\ y & t & z \\ z & z & y \end{bmatrix}, \quad \Omega_9 : \begin{bmatrix} x & x & y \\ x & z & t \\ y & t & t \end{bmatrix}. \tag{3.4}$$

Here the representative for Ω_8 is obtained from the pencil generated by $\mathcal{C} = \mathcal{Z}(X_0X_1 + X_2^2)$ and the pair of real lines $\mathcal{L}_1 = \mathcal{Z}(X_0 + X_2)$ and $\mathcal{L}_2 = \mathcal{Z}(X_1 + X_2)$, which meet in the point $(1,1,1)$ on $\mathcal{C}$. To obtain the representative for Ω_9, note that the conic $\mathcal{C} = \mathcal{Z}(X_0(X_0+X_1)+\lambda X_2(X_1+X_2))$ is nonsingular for all $\lambda \notin \{0,1\}$, by Lemma 2.2. Fix $\mathcal{C}$ by choosing such a λ, and then take the pair of real lines $\mathcal{L}_1\mathcal{L}_2 = \mathcal{Z}(X_0(X_0+X_1))$, which meets $\mathcal{C}$ in the four points

$$P_1 = (0,1,0),\ P_2 = (1,1,0),\ P_3 = (0,1,1) \text{ and } P_4 = (1,1,1). \tag{3.5}$$

Lemma 3.12. *(Alnajjarine, Lavrauw & Popiel , 2022, Lemma 4.4)*
The hyperplane-orbit distribution of a solid of type Ω_8*, respectively* Ω_9*, is* $[0,2,0,q-1]$*, respectively* $[0,3,0,q-2]$*. In particular, these orbits are distinct and do not belong to* $\{\Omega_1,\dots,\Omega_7\}$*.*

Proof. Let S_8 and S_9 denote the representatives in (3.4). A conic $\mathcal{Z}(X_0X_1+X_2^2+\lambda((X_0+X_2)(X_1+X_2)))$ in the pencil $\mathcal{P}(S_8)$ is singular if and only if $\lambda=1$, by Lemma 2.2, and setting $\lambda=1$ yields a pair of real lines. As noted above, a conic $\mathcal{Z}(X_0(X_0+X_1)+\lambda X_2(X_1+X_2))$ in $\mathcal{P}(S_9)$ is singular if and only if $\lambda\in\{0,1\}$, and both values produce pairs of real lines distinct from the chosen generator $\mathcal{Z}(X_0(X_0+X_1))$. □

Remark 3.4. *(Alnajjarine, Lavrauw & Popiel , 2022, Remark 4.5)*
If $S_8\in\Omega_8$ then the second pair of real lines in $\mathcal{P}(S_8)$ has $(k_1,k_2)=(2,1)$: it comprises the secant $\mathcal{Z}(X_2)$ and the tangent $\mathcal{Z}(X_0+X_1)$ to the generating nonsingular conic $\mathcal{Z}(X_0X_1+X_2^2)$. Since the stabiliser of a nonsingular conic $\mathcal{C}$ acts 3-transitively on the points of $\mathcal{C}$, this implies that Ω_8 is the only K-orbit obtained from a pencil generated by a nonsingular conic $\mathcal{C}$ and a real line pair consisting of a secant and a tangent to $\mathcal{C}$ meeting at a point not on $\mathcal{C}$. (Note that the above lines meet in the point $(1,1,0)$, which is not on $\mathcal{Z}(X_0X_1+X_2^2)$.) On the other hand, the three pairs of real lines in $\mathcal{P}(S_9)$ all have $(k_1,k_2)=(2,2)$.

Lemma 3.13. *(Alnajjarine, Lavrauw & Popiel , 2022, Lemma 4.6)*
The point-orbit distribution of a solid of type Ω_8 *is* $[3,1,q^2+2q-3,q^3-q]$*. The point-orbit distribution of a solid of type* Ω_9 *is* $[4,1,q^2+3q-4,q^3-2q]$*.*

Proof. Consider again the solids S_8 and S_9 in (3.4). The cubic surface $\Psi(S_8)=\mathcal{Z}(XYT+XZ^2+Y^3+Z^2T)$ intersects the plane $\mathcal{Z}(X)$ in a rational cubic curve with $q+1$ points, and the points of $\Psi(S_8)\setminus\mathcal{Z}(X)$ comprise the set $\{(1,0,0,t):t\in\mathbb{F}_q\}\cup\{(1,1,1,t):t\in\mathbb{F}_q\}\cup\{(1,y,z,f(y,z)):y,z\in\mathbb{F}_q;\ y\neq z^2\}$, where $f(y,z)=(z^2+y^3)/(y+z^2)$, which has size q^2+q. It meets $\mathcal{V}(\mathbb{F}_q)$ in the image of the base of $\mathcal{P}(S_8)$, and the nucleus plane in a unique point. The cubic surface $\Psi(S_9)=\mathcal{Z}(Z(XT+Y^2)+XT^2+X^2T)$ meets the plane $\mathcal{Z}(X)$ in two lines and contains q^2+q additional points, namely those comprising the set $\{(1,0,z,0):z\in\mathbb{F}_q\}\cup\{(1,1,z,1):z\in\mathbb{F}_q\}\cup\{(1,y,g(y,t),t):y,t\in\mathbb{F}_q;t\neq y^2\}$ where $g(y,t)=(t+t^2)/(t+y^2)$. It meets $\mathcal{V}(\mathbb{F}_q)$ in the image of the base of $\mathcal{P}(S_9)$, and the nucleus plane in a point. □

Lemma 3.14. *(Alnajjarine, Lavrauw & Popiel , 2022, Lemma 4.7)*
If $S_8\in\Omega_8$ *then* $K_{S_8}\cong C_{q-1}\times C_2$*. If* $S_9\in\Omega_9$ *then* $K_{S_9}\cong \mathrm{Sym}_4$*.*

Proof. The solid $S_8 \in \Omega_8$ given in (3.4) contains exactly two pairs of real lines, namely $\mathcal{L}_1\mathcal{L}_2$ and $\mathcal{L}_1'\mathcal{L}_2'$ where $\mathcal{L}_1 = \mathcal{Z}(X_1+X_2)$, $\mathcal{L}_2 = \mathcal{Z}(X_0+X_2)$, $\mathcal{L}_1' = \mathcal{Z}(X_2)$ and $\mathcal{L}_2' = \mathcal{Z}(X_0+X_1)$. Note that $\mathcal{L}_1$ and $\mathcal{L}_2$ meet in a point $P = (1,1,1)$ which also lies on $\mathcal{L}_2'$, while $\mathcal{L}_1'$ and $\mathcal{L}_2'$ meet in a point $P' = (1,1,0)$ disjoint from $\mathcal{L}_1\mathcal{L}_2$. The stabiliser $G \leqslant \mathrm{PGL}(3,q)$ of $\mathcal{P}(S_8)$ therefore fixes both of $\mathcal{L}_1'$ and $\mathcal{L}_2'$, because $\mathcal{L}_1'$ meets $\mathcal{L}_1\mathcal{L}_2$ in the unique point P while $\mathcal{L}_2'$ meets $\mathcal{L}_1\mathcal{L}_2$ in two points, $Q = (1,0,0)$ and $R = (0,1,0)$. Hence, it also fixes $\mathcal{L}_1\mathcal{L}_2$ and therefore P. That is, G is equal to the stabiliser of P, P' and $\{Q,R\}$. Since P', Q and R are collinear, $G \cong C_{q-1} \times C_2$. Explicitly, $K_{S_8} \cong G$ is generated by the elements of K represented by the matrices

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & \omega+1 \\ 0 & 1 & \omega+1 \\ 0 & 0 & \omega \end{bmatrix}, \text{ where } \langle \omega \rangle = \mathbb{F}_q^\times. \tag{3.6}$$

If $S_9 \in \Omega_9$ then the base of $\mathcal{P}(S_9)$ is the frame of $\mathrm{PG}(2,q)$ given in (3.5), so $K_{S_4} \cong \mathrm{Sym}_4$. □

3.2.2.2 $(k_1,k_2) = (1,0)$

To prove that the configuration $(k_1,k_2) = (1,0)$ leads to a unique K-orbit, we consider extending the nonsingular conic $\mathcal{C}$ to a conic in the quadratic extension $\mathrm{PG}(2,q^2)$ of $\mathrm{PG}(2,q)$. For clarity, we write $\overline{\mathcal{C}}$ for the extension of $\mathcal{C}$ to $\mathrm{PG}(2,q^2)$, and use the same 'bar' notation for the corresponding extensions of other objects, in particular $\overline{\mathcal{L}_1}$ and $\overline{\mathcal{L}_2}$ for the pair of real lines $\mathcal{L}_1$ and $\mathcal{L}_2$. Let $\sigma \in \mathrm{P\Gamma L}(3,q^2)$ be the Frobenius collineation of $\mathrm{PG}(2,q^2)$ induced by the automorphism $a \mapsto a^q$ of $\mathbb{F}_{q^2}$. Since $\mathcal{L}_2$ is external to $\mathcal{C}$ (i.e. $k_2 = 0$), $\overline{\mathcal{L}_2}$ intersects $\overline{\mathcal{C}}$ in a pair of conjugate points $(\overline{P_2}, \overline{P_2^\sigma})$. Let P_1 denote the unique point in which $\mathcal{L}_1$ meets $\mathcal{C}$, and let $G_{\overline{\mathcal{C}}} \cong \mathrm{PGL}(2,q^2)$ denote the stabiliser of $\overline{\mathcal{C}}$ in $\mathrm{PGL}(3,q^2)$. Consider another real point R_1 and pair of conjugate points $\overline{R_2}$ and $\overline{R_2^\sigma}$, associated with a second pair of real lines $\mathcal{L}_1'\mathcal{L}_2'$ with $(k_1,k_2) = (1,0)$. Let α denote the unique projectivity in $G_{\overline{\mathcal{C}}}$ mapping the triple $(P_1, \overline{P}_2, \overline{P_2^\sigma})$ to $(R_1, \overline{R}_2, \overline{R_2^\sigma})$. Since $G_{\overline{\mathcal{C}}}$ acts sharply 3-transitively on the points of $\overline{\mathcal{C}}$ and $\alpha\sigma\alpha^{-1}\sigma$ fixes the triple $(P_1, \overline{P}_2, \overline{P_2^\sigma})$ pointwise, α commutes with σ and therefore belongs to $\mathrm{PGL}(3,q)$. In other words, the stabiliser of $\mathcal{C}$ in $\mathrm{PGL}(3,q)$ acts transitively on pairs of real lines meeting $\mathcal{C}$ in the configuration $(k_1,k_2) = (1,0)$, so there is a unique K-orbit of solids arising from this configuration. We denote this orbit by Ω_{10} and choose the representative

$$\Omega_{10}:\begin{bmatrix} x & y & z \\ y & y+\gamma t & t \\ z & t & y \end{bmatrix}, \quad \text{where} \quad \mathrm{Tr}(\gamma^{-1})=1,$$

obtained by taking $\mathcal{C}=\mathcal{Z}(X_0X_1+X_2^2)$, $\mathcal{L}_1=\mathcal{Z}(X_1)$ and $\mathcal{L}_2=\mathcal{Z}(X_0+X_1+\gamma X_2)$.

Lemma 3.15. *(Alnajjarine, Lavrauw & Popiel , 2022, Lemma 4.8)*
A solid $S_{10}\in\Omega_{10}$ *has point-orbit distribution* $[1,1,q^2+2q-1,q^3-q]$*, hyperplane-orbit distribution* $[0,1,1,q-1]$*, and stabiliser* $K_{S_{10}}\cong C_{q-1}\times C_2$*. In particular,* $\Omega_{10}\notin\{\Omega_1,\ldots,\Omega_9\}$.

Proof. Let S_{10} be the solid given above. The hyperplane-orbit distribution is calculated via Lemma 2.2, and implies that Ω_{10} is distinct from all previously considered K-orbits. Explicitly, the only singular conic in $\mathcal{P}(S_{10})$ other than $\mathcal{L}_1\mathcal{L}_2$ is the pair of imaginary lines $\mathcal{L}_1'\mathcal{L}_2'=\mathcal{Z}(X_1^2+\gamma X_1X_2+X_2^2)$. The cubic surface $\Psi(S_{10})=\mathcal{Z}(T^2X+\gamma TXY+\gamma TZ^2+XY^2+T^3+TZ^2)$ meets the plane $\mathcal{Z}(Y)$ in the union of the nonsingular conic $\mathcal{C}'=\mathcal{Z}(Y,TX+\gamma Z^2+T^2+Z^2)$ and the line $\mathcal{Z}(Y,T)$, which is tangent to $\mathcal{C}'$. The remaining points of $\Psi(S_{10})$ comprise the set $\{(f(z,t),1,z,t):z,t\in\mathbb{F}_q\}$, where $f(z,t)=(z^2(1+\gamma t)+1)/(t^2+\gamma t+1)$, which has size q^2. Moreover, $\Psi(S_{10})$ meets $\mathcal{V}(\mathbb{F}_q)$ in the (unique) point $(1,0,0,0)$, and the nucleus plane in the point $(0,0,1,0)$. To calculate the stabiliser, note that $\mathcal{L}_1'$ and $\mathcal{L}_2'$ meet in a point $P'=(1,0,0)$ which also lies on $\mathcal{L}_1$, while $\mathcal{L}_1$ and $\mathcal{L}_2$ meet in a point $P=(\gamma,0,1)$ disjoint from $\mathcal{L}_1'\mathcal{L}_2'$. Extending to $\mathrm{PG}(2,q^2)$, we therefore obtain a pencil $\overline{\mathcal{P}(S_{10})}$ of type Ω_8. In particular, the stabiliser $G\leqslant\mathrm{PGL}(3,q^2)$ of $\overline{\mathcal{P}(S_{10})}$ is equal to the stabiliser of $\overline{P}$, $\overline{P}'$ and $\{\overline{Q},\overline{R}\}=\overline{\mathcal{L}}_2\cap\overline{\mathcal{L}_1'\mathcal{L}_2'}$. Hence, $G\cong C_{q^2-1}\times C_2$ by Lemma 3.14, and comparing with (3.6) we see that over $\mathbb{F}_q$ we obtain $K_{S_{10}}\cong C_{q-1}\times C_2$. □

3.2.2.3 $(k_1,k_2)=(1,2)$

Next we consider the configuration $(k_1,k_2)=(1,2)$, namely the case in which $\mathcal{L}_1$ is a tangent to $\mathcal{C}$ and $\mathcal{L}_2$ is a secant to $\mathcal{C}$. If the point $P=\mathcal{L}_1\cap\mathcal{L}_2$ is not on $\mathcal{C}$ then, by Remark 3.4, we obtain the K-orbit Ω_8. Hence, we may assume that P is on $\mathcal{C}$, and since the stabiliser of a nonsingular conic acts 3-transitively on the points of the conic, a unique K-orbit arises in this way. (Indeed, 2-transitivity is sufficient to guarantee this.) We denote this K-orbit by Ω_{11} and choose the representative

$$\Omega_{11}: \begin{bmatrix} x & y & z \\ y & t & \cdot \\ z & \cdot & y \end{bmatrix},$$

obtained by taking $\mathcal{C} = \mathcal{Z}(X_0X_1 + X_2^2)$, $\mathcal{L}_1 = \mathcal{Z}(X_1)$ and $\mathcal{L}_2 = \mathcal{Z}(X_2)$.

Lemma 3.16. *(Alnajjarine, Lavrauw & Popiel , 2022, Lemma 4.9)*
A solid $S_{11} \in \Omega_{11}$ has point-orbit distribution $[2,1,q^2+q-2,q^3]$, hyperplane-orbit distribution $[0,1,0,q]$, and stabiliser $K_{S_{11}} \cong E_q \rtimes C_{q-1}$. In particular, $\Omega_{11} \notin \{\Omega_1,\ldots,\Omega_{10}\}$.

Proof. Let S_{11} be the solid given above. Lemma 2.2 implies that the pair of real lines $\mathcal{L}_1\mathcal{L}_2$ is the only singular conic in the pencil $\mathcal{P}(S_{11})$, so the hyperplane-orbit distribution of S_{11} is $[0,1,0,q]$. In particular, Ω_{11} is distinct from all of $\Omega_1,\ldots,\Omega_{10}$. The cubic surface $\Psi(\Omega_{11}) = \mathcal{Z}(XYT + Y^3 + Z^2T)$ intersects the plane $\mathcal{Z}(Y)$ in the two lines $\mathcal{Z}(Y,Z)$ and $\mathcal{Z}(Y,T)$ and contains q^2-q additional points, comprising the set $\{(x,1,z,(x+z^2)^{-1}) : x,z \in \mathbb{F}_q;\ x \neq z^2\}$. There are two points in $S_{11} \cap \mathcal{V}(\mathbb{F}_q)$, namely $P_1 = (1,0,0,0)$ and $P_2 = (0,0,0,1)$, and one point $Q = (0,0,1,0)$ in which S_{11} meets the nucleus plane. The stabiliser $K_{S_{11}}$ certainly fixes Q and $\{P_1,P_2\}$. However, P_1 is the image under ν of the point of intersection of $\mathcal{L}_1\mathcal{L}_2$, so $K_{S_{11}}$ must fix P_1 and P_2 pointwise. An element of $K_{P_1} \cap K_{P_2} \cap K_Q$ is represented by a matrix $(g_{ij}) \in \mathrm{GL}(3,q)$ with $g_{12} = g_{21} = g_{23} = g_{31} = g_{32} = 0$. It fixes S_{11} if and only if it also maps the point $R = (0,1,0,0)$ into S_{11}. This occurs if and only if also $g_{11}g_{22} = g_{33}^2$, so $K_{S_{11}} \cong E_q \rtimes C_{q-1}$. □

3.2.2.4 $(k_1,k_2) = (2,0)$

We now show that the configuration $(k_1,k_2) = (2,0)$ also produces exactly one new K-orbit. As in the case $(k_1,k_2) = (1,0)$, consider the extension $\overline{\mathcal{C}}$ of the nonsingular conic $\mathcal{C}$ to $\mathrm{PG}(2,q^2)$. The extension $\overline{\mathcal{L}_1}$ of the secant line $\mathcal{C}$ meets $\overline{\mathcal{C}}$ in two $\mathbb{F}_q$-rational points, and the extension $\overline{\mathcal{L}_2}$ of the external line $\mathcal{L}_2$ meets $\mathcal{C}$ in two $\mathbb{F}_{q^2}$-rational points which are conjugate under the Frobenius collineation σ induced by the automorphism $a \mapsto a^q$ of $\mathbb{F}_{q^2}$. These four points form a frame of $\mathrm{PG}(2,q^2)$, since they lie on $\overline{\mathcal{C}}$. Any two such configurations are therefore $\mathrm{PGL}(3,q^2)$-equivalent, via some $\alpha \in \mathrm{PGL}(3,q^2)$. Verifying that $\alpha\sigma\alpha^{-1}\sigma$ fixes the frame obtained from $\mathcal{L}_1\mathcal{L}_2$ implies that $\alpha \in \mathrm{PGL}(3,q)$, cf. the case $(k_1,k_2) = (1,0)$. Hence, we obtain at most one K-orbit from the configuration $(k_1,k_2)=(2,0)$. We verify below that this orbit is distinct from all previously considered orbits, and therefore label it Ω_{12} and choose the representative

$$\Omega_{12}: \begin{bmatrix} x & y & z \\ y & t & \gamma y+z \\ z & \gamma y+z & y \end{bmatrix}, \quad \text{where} \quad \mathrm{Tr}(\gamma^{-1})=1,$$

obtained by taking $\mathcal{C}=\mathcal{Z}(X_0X_1+X_2^2)$, $\mathcal{L}_1=\mathcal{Z}(X_2)$ and $\mathcal{L}_2=\mathcal{Z}(X_0+X_1+\gamma X_2)$.

Lemma 3.17. *(Alnajjarine, Lavrauw & Popiel , 2022, Lemma 4.10)*
A solid of type Ω_{12} has point-orbit distribution $[2,1,q^2+q-2,q^3]$, hyperplane-orbit distribution $[0,1,0,q]$, and stabiliser $K_{S_{12}}\cong C_2^2$. In particular, $\Omega_{12}\notin\{\Omega_1,\dots,\Omega_{11}\}$.

Proof. The proof is similar to that of Lemma 3.16 (for Ω_{11}). Taking S_{12} to be the solid defined above, Lemma 2.2 yields the hyperplane-orbit distribution. The cubic surface $\Psi(S_{12})$ meets the plane $\mathcal{Z}(Y)$ in two lines and contains q^2-q further points. It meets $\mathcal{V}(\mathbb{F}_q)$ in the two points $P_1=(1,0,0,0)$ and $P_2=(0,0,0,1)$, and the nucleus plane in the point $Q=(0,0,1,0)$. The stabiliser $K_{S_{12}}$ must fix Q and $\{P_1,P_2\}$. It induces a permutation group of order 2 on $\{P_1,P_2\}$ because e.g. the element of K represented by the matrix obtained by swapping the first and second columns of the identity fixes S_{12} and swaps P_1 and P_2. An element of $K_{P_1}\cap K_{P_2}\cap K_Q$ is represented by a matrix $(g_{ij})\in\mathrm{GL}(3,q)$ with $g_{12}=g_{21}=g_{31}=g_{32}=0$, $g_{22}=g_{11}$ and $g_{23}=g_{13}$. It fixes S_{12} if and only if it also maps the point $(0,1,0,0)$ into S_{12}, which occurs if and only if $g_{33}=g_{11}$ and $g_{13}\in\{0,\gamma g_{11}\}$. Factoring out scalars, we see that the kernel of the action of $K_{S_{12}}$ on $\{P_1,P_2\}$ also has order 2. Therefore, $K_{S_{12}}\cong C_2^2$. The point- and hyperplane-orbit distributions of S_{12} imply that Ω_{12} is distinct from all previously considered K-orbits, with the possible exception of Ω_{11}. However, $K_{S_{12}}\cong C_2^2$ is not isomorphic to $K_{S_{11}}\cong E_q\rtimes C_{q-1}$ (for any q), so also $\Omega_{12}\neq\Omega_{11}$. □

Remark 3.5. *(Alnajjarine, Lavrauw & Popiel , 2022, Remark 4.11)*
It is also possible to distinguish between the K-orbits Ω_{11} and Ω_{12} using their line-orbit distributions, rather than their stabilisers, as follows. As per Lavrauw & Popiel (2020), a line of type "o_6" is characterised by having point-orbit distribution $[1,1,q-1,0]$. Considering again the solids $S_i\in\Omega_i$, $i\in\{11,12\}$, used above, we therefore see that in each case the only candidates for lines of type o_6 are the two lines $\langle Q,P_1\rangle$ and $\langle Q,P_2\rangle$, where $Q=(0,0,1,0)$ is the unique point in which S_i meets the nucleus plane, and $P_1=(1,0,0,0)$ and $P_2=(0,0,0,1)$ are the two points of rank 1 in S_i. Only one of these four lines has type o_6, namely $\langle Q,P_1\rangle$ in the case $i=11$. Therefore, S_{11} and S_{12} have different line-orbit distributions, and so $\Omega_{11}\neq\Omega_{12}$. (We

note also that there is a typo in (Lavrauw & Popiel, 2020, Table 4): the fifth column should say that a line of type o_6 contains *one* point of the nucleus plane. This is, however, clear from the representative given in (Lavrauw & Popiel, 2020, Table 2).)

3.2.2.5 $(k_1,k_2)=(0,0)$

Finally, we show that the configuration $(k_1,k_2)=(0,0)$ also produces a unique K-orbit. It suffices to use an argument similar to the one used in the case $(k_1,k_2)=(2,0)$. This time, both $\mathcal{L}_1$ and $\mathcal{L}_2$ are external to $\mathcal{C}$ and so both give rise to pairs of conjugate points (with respect to the Frobenius collineation σ). The four points again form a frame, so the same argument as before shows that at most one K-orbit arises. We denote this orbit by Ω_{13} and choose the representative

$$\Omega_{13}:\begin{bmatrix} x & y & z \\ y & \gamma x+y & t \\ z & t & \gamma x+z \end{bmatrix}, \quad \text{where} \quad \mathrm{Tr}(\gamma)=1,$$

obtained as follows. Consider the two pairs of imaginary lines $\mathcal{C}_i=\mathcal{Z}(f_i)$ where $f_1=\gamma X_0^2+X_0X_i+X_i^2$, $i\in\{1,2\}$. Then the pencil $\mathcal{P}(S_{13})$ corresponding to the solid S_{13} defined above is generated by $\mathcal{C}_1$ and $\mathcal{C}_2$. We must show that $\mathcal{P}(S_{13})$ contains a nonsingular conic $\mathcal{C}$ and a pair of real lines external to $\mathcal{C}$. By Lemma 2.2, the conic $\mathcal{Z}(\lambda_1 f_1+\lambda_2 f_2)$ is singular if and only if $\lambda_1=0$, $\lambda_2=0$ or $\lambda_1=\lambda_2$. Setting $\lambda_1=\lambda_2$ yields the pair of real lines $\mathcal{L}_1=\mathcal{Z}(X_1+X_2)$ and $\mathcal{L}_2=\mathcal{Z}(X_0+X_1+X_2)$, both of which are external to every nonsingular conic in the pencil, by Lemma 2.5.

Lemma 3.18. *(Alnajjarine, Lavrauw & Popiel , 2022, Lemma 4.12)*
A solid $S_{13}\in\Omega_{13}$ *has point-orbit distribution* $[0,1,q^2+3q,q^3-2q]$*, hyperplane-orbit distribution* $[0,1,2,q-2]$*, and stabiliser* $K_{S_{13}}\cong C_2^2\rtimes C_2$*. In particular,* $\Omega_{13}\notin\{\Omega_1,\ldots,\Omega_{12}\}$.

Proof. Let S_{13} be the solid defined above. The preceding discussion gives the hyperplane-orbit distribution, which implies that $\Omega_{13}\notin\{\Omega_1,\ldots,\Omega_{12}\}$. The cubic surface $\Psi(S_{13})$ intersects the plane $\mathcal{Z}(X)$ in three concurrent lines $\mathcal{Z}(X,Y)$, $\mathcal{Z}(X,Z)$ and $\mathcal{Z}(X,Y+Z)$, and contains a further q^2 points, parameterised as $(1,y,z,f(y,z))$ where $f(y,z)=(\gamma+\gamma y+\gamma z+\gamma y^2+\gamma z^2+yz+y^2z+yz^2)^{1/2}$. It is disjoint from $\mathcal{V}(\mathbb{F}_q)$ and meets the nucleus plane in a unique point, so the point-orbit distribution of S_{13} is $[0,1,q^2+3q,q^3-2q]$. It remains to calculate the stabiliser. As per the discussion preceding the lemma, if we extend $\mathcal{P}(S_{13})$ to $\mathrm{PG}(2,q^2)$ we obtain a pencil with four base points comprising a frame $B = \{P_1, P_2, P_3, P_4\}$, say. This pencil has type Ω_9, so its stabiliser $\overline{G} \leqslant \mathrm{PGL}(3, q^2)$ is isomorphic to Sym_4, by Lemma 3.14. The stabiliser $G = \overline{G} \cap \mathrm{PGL}(3, q)$ of $\mathcal{P}(S_{13})$ is therefore a subgroup of Sym_4. Now, $\mathcal{P}(S_{13})$ also contains a unique pair of real lines $\mathcal{L}_1\mathcal{L}_2$, and over $\mathbb{F}_q^2$ each of these lines meets two points of B, say $\overline{\mathcal{L}}_1 = \langle P_1, P_2\rangle$ and $\overline{\mathcal{L}}_2 = \langle P_3, P_4\rangle$. Since G fixes $\mathcal{L}_1\mathcal{L}_2$, it fixes $\{\{P_1, P_2\}, \{P_3, P_4\}\}$ over $\mathbb{F}_q^2$, and therefore induces a subgroup of the permutation group $H = \langle (P_1, P_2), (P_3, P_4), (P_1, P_3)(P_2, P_4)\rangle \cong C_2^2 \rtimes C_2$ on B. Conversely, a calculation shows that $K_{S_{13}}$ contains the group generated by the elements of K represented by the matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

which is isomorphic to H. We therefore conclude that $K_{S_{13}} \cong C_2^2 \rtimes C_2$. □

3.2.3 Solids contained in no hyperplanes of type $\mathcal{H}_1$ or $\mathcal{H}_{2r}$

Of the solids S with hyperplane-orbit distribution $\mathrm{OD}_{K,4}(S) = [a_1, a_{2r}, a_{2i}, a_3]$ where $1 \leqslant a_3 \leqslant q$, we have now classified those for which at most one of a_1 and a_{2r} is 0. It therefore remains to consider the case in which $\mathrm{OD}_{K,4}(S) = [0, 0, a_{2i}, a_3]$. This assumption implies, of course, that $a_{2i} \geqslant 1$, since $a_3 \leqslant q$. On the other hand, Lemma 3.1(ii) implies that $a_{2i} \leqslant 1$, since $a_{2r} = 0$ and $b \geqslant 0$. Therefore, we must have $\mathrm{OD}_{K,4}(S) = [0, 0, 1, q]$. Note that this then forces $b = 0$ in Lemma 3.1, so that $\mathcal{P}(S)$ must have empty base. We claim that the hyperplane-orbit distribution $[0, 0, 1, q]$ gives rise to a unique K-orbit, with representative

$$\text{(3.7)} \qquad \Omega_{14}: \begin{bmatrix} x & y & \gamma x + y + \gamma t \\ y & \gamma x + y & z \\ \gamma x + y + \gamma t & z & t \end{bmatrix}, \quad \text{where} \quad \mathrm{Tr}(\gamma) = 1.$$

This solid, call it S_{14}, is obtained from the pencil generated by the nonsingular conic $\mathcal{Z}(X_1^2 + X_0X_2 + \gamma X_2^2)$ and the pair of imaginary lines $\mathcal{L}_1\mathcal{L}_2 = \mathcal{Z}(\gamma X_0^2 + X_0X_1 + X_1^2)$. Lemma 2.2 confirms that $\mathcal{P}(S_{14})$ contains no other singular conics, and so S_{14} has the desired hyperplane-orbit distribution; it also has empty base, since the unique real point $(0, 0, 1)$ on $\mathcal{L}_1\mathcal{L}_2$ does not lie on any of the nonsingular conics.

Lemma 3.19. *(Alnajjarine, Lavrauw & Popiel , 2022, Lemma 4.13)*
A solid of type Ω_{14} *has point-orbit distribution* $[0,1,q^2+q,q^3]$ *and hyperplane-orbit distribution* $[0,0,1,q]$*. In particular,* $\Omega_{14} \notin \{\Omega_1,\ldots,\Omega_{13}\}$*.*

Proof. It remains to calculate the point-orbit distribution. Taking $S_{14} \in \Omega_{14}$ as above, we find that the cubic surface $\Psi(S_{14})$ meets the plane $\mathcal{Z}(X)$ in the line $\mathcal{Z}(X,Y)$ and contains a further q^2 points, parameterised as $(1,y,f(y,t),t)$ with $f(y,t) = (\gamma^2t^2+\gamma yt^2+\gamma+\gamma y+\gamma t+\gamma y^2+ty+ty^2+y^3)^{1/2}$. It is disjoint from $\mathcal{V}(\mathbb{F}_q)$ (since $\mathcal{P}(S_{14})$ has empty base) and meets the nucleus plane in one point. □

We now show that all solids with hyperplane-orbit distribution $[0,0,1,q]$ belong to the K-orbit Ω_{14}, before finally calculating the stabiliser of such a solid.

Lemma 3.20. *(Alnajjarine, Lavrauw & Popiel , 2022, Lemma 4.14)*
The solids with hyperplane-orbit distribution $[0,0,1,q]$ *form one* K*-orbit.*

Proof. Let S be a solid with hyperplane-orbit distribution $[0,0,1,q]$, and let $\mathcal{L}_1\mathcal{L}_2$ be the unique pair of imaginary lines in the pencil $\mathcal{P}(S)$. To prove the result, we consider the extension of $\mathcal{P}(S)$ to $\mathrm{PG}(2,q^2)$. Since $\mathcal{L}_1$ and $\mathcal{L}_2$ are conjugate with respect to the Frobenius collineation σ induced by the automorphism $a \mapsto a^q$ of $\mathbb{F}_{q^2}$, let us relabel them as ℓ and ℓ^σ. Choose a nonsingular conic $\mathcal{C}$ in $\mathcal{P}(S)$, and denote the extensions of $\mathcal{P}(S)$, $\mathcal{C}$, ℓ and ℓ^σ to $\mathrm{PG}(2,q^2)$ using a 'bar' (as in previous such arguments). Recall from the discussion preceding Lemma 3.19 that ℓ and ℓ^σ are external to $\mathcal{C}$, since $\mathcal{P}(S)$ necessarily has empty base. We claim that $\overline{\ell}$ and $\overline{\ell^\sigma}$ are likewise external to $\overline{\mathcal{C}}$. If $\overline{\ell}$ is a tangent to $\overline{\mathcal{C}}$, meeting $\overline{\mathcal{C}}$ in a point P, then $\overline{\ell^\sigma}$ is the tangent to $\overline{\mathcal{C}}$ at the point P^σ. By the classification in Section 3.2.1, specifically Remark 3.3, the pencil $\overline{\mathcal{P}(S)}$ then has type Ω_6 (over $\mathbb{F}_{q^2}$). In particular, $\{P,P^\sigma\}$ is the base of $\overline{\mathcal{P}(S)}$, and the line $\langle P,P^\sigma\rangle$ is its unique double line. However, this line is fixed by σ, so we have a contradiction. If $\overline{\ell}$ is a secant to $\overline{\mathcal{C}}$ then it meets $\overline{\mathcal{C}}$ in a pair of conjugate points $\{P,P^\sigma\}$, and $\overline{\ell^\sigma}$ is also a secant, meeting $\overline{\mathcal{C}}$ in another pair of conjugate points $\{Q,Q^\sigma\}$. These four points are distinct because the point of intersection of ℓ and ℓ^σ does not belong to $\mathcal{C}$, so it follows from Section 3.2.2.1 that $\overline{\mathcal{P}(S)}$ has type Ω_9. However, the conic comprising the pair of lines $\langle P,Q\rangle$ and $\langle P^\sigma,Q^\sigma\rangle$ then belongs to $\overline{\mathcal{P}(S)}$, a contradiction since this line pair is fixed by σ. Hence, $\overline{\ell}$ and $\overline{\ell^\sigma}$ are external to $\overline{\mathcal{C}}$ as claimed. Section 3.2.2.5 therefore implies that $\overline{\mathcal{P}(S)}$ has type Ω_{13}. Now suppose that S' is a second solid with hyperplane-orbit distribution $[0,0,1,q]$, and let m, m^σ be the unique imaginary line pair in $\mathcal{P}(S')$. Since $\overline{\mathcal{P}(S')}$ also has type Ω_{13}, there exists a projectivity $\alpha \in \mathrm{PGL}(3,q^2)$ mapping S to S'. Choose two points R_1 and R_2 on $\overline{\ell}$ that do not belong to $\overline{\ell^\sigma}$. Then $\Lambda = (R_1,R_2,R_1^\sigma,R_2^\sigma)$ is a frame of $\mathrm{PG}(2,q^2)$, mapped by α to a frame $(W_1,W_2,W_1^\sigma,W_2^\sigma)$, where without loss of generality the points W_1 and W_2 are on $\overline{m} \setminus \overline{m^\sigma}$. The projectivity $\alpha\sigma\alpha^{-1}\sigma$ fixes Λ pointwise, and so is equal to the identity element of $\mathrm{PGL}(3,q^2)$. Hence, α commutes with σ, and therefore belongs to $\mathrm{PGL}(3,q)$. In other words, there exists an element of $\mathrm{PGL}(3,q)$ mapping $\mathcal{P}(S)$ to $\mathcal{P}(S')$, and so the solids S and S' belong to the same K-orbit. □

Lemma 3.21. *(Alnajjarine, Lavrauw & Popiel , 2022, Lemma 4.15)*
If $S_{14} \in \Omega_{14}$ then $K_{S_{14}} \cong C_4$.

Proof. Let ℓ and ℓ^σ be the unique pair of imaginary lines in $\mathcal{P}(S_{14})$, where σ is the Frobenius collineation of $\mathrm{PG}(2,q^2)$ induced by the automorphism $a \mapsto a^q$ of $\mathbb{F}_{q^2}$. As explained above, the extension $\overline{\mathcal{P}(S_{14})}$ of the pencil $\mathcal{P}(S_{14})$ to $\mathrm{PG}(2,q^2)$ has type Ω_{13}. The base B of $\overline{\mathcal{P}(S_{14})}$ comprises two distinct points P and Q on the line $\overline{\ell}$ and their conjugates P^σ and Q^σ on $\overline{\ell^\sigma}$. By the proof of Lemma 3.18, the stabiliser of $\overline{\mathcal{P}(S_{14})}$ in $\mathrm{PGL}(3,q^2)$ is isomorphic to the permutation group $H = \langle (P,Q),(P^\sigma,Q^\sigma),(P,P^\sigma)(Q,Q^\sigma)\rangle \leqslant \mathrm{Sym}(B)$, which has order 8. Now, observe that the projectivity inducing the permutation (P,Q) does not belong to $\mathrm{PGL}(3,q)$, because if an element of $\mathrm{PGL}(3,q)$ swaps P and Q then it must also swap P^σ and Q^σ. (Indeed, none of the given generators of H are realised over $\mathbb{F}_q$.) Therefore, the stabiliser of $\mathcal{P}(S_{14})$ in $\mathrm{PGL}(3,q)$ has order at most 4. Conversely, if we take S_{14} to be the solid defined in (3.7) then a calculation shows that S_{14} is fixed by the subgroup of K generated by the element of order 4 represented by the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & \gamma^{-1} & 1 \end{bmatrix}$$

We therefore conclude that $K_{S_{14}} \cong C_4$, as claimed. □

3.3 Solids contained in $q+1$ hyperplanes of type $\mathcal{H}_3$

It remains to consider the possibility that a solid S of $\mathrm{PG}(5,q)$ is contained in $q+1$ hyperplanes of type $\mathcal{H}_3$, or, equivalently, that the associated pencil of conics $\mathcal{P}(S)$ contains $q+1$ nonsingular conics. We first establish the existence of such solids. Choose $b,c \in \mathbb{F}_q$ such that the cubic $b\lambda^3 + c\lambda + 1$ has no roots over $\mathbb{F}_q$. (For example, take the minimal polynomial of a primitive element α of the field extension $\mathbb{F}_{q^3}/\mathbb{F}_q$, scale it to make the constant term 1, and then apply a coordinate transformation to eliminate the λ^2 term.) Lemma 2.2 shows that the pencil generated by $\mathcal{Z}(X_0X_1+X_2^2)$ and $\mathcal{Z}(X_0X_2+bX_1^2+cX_2^2)$ contains $q+1$ nonsingular conics, and so we obtain the desired orbit of solids with hyperplane-orbit distribution $[0,0,0,q+1]$,

$$(3.8)\qquad \Omega_{15}:\begin{bmatrix} x & y & bz+cy \\ y & z & t \\ bz+cy & t & y \end{bmatrix}, \quad \text{where} \quad b\lambda^3+c\lambda+1 \text{ is irreducible over } \mathbb{F}_q.$$

By Lemma 3.1, a pencil of conics corresponding to a solid in this orbit has a unique base point.

Lemma 3.22. *(Alnajjarine, Lavrauw & Popiel , 2022, Lemma 5.1)*
A solid of type Ω_{15} *has point-orbit distribution* $[1,1,q^2-1,q^3+q]$ *and hyperplane-orbit distribution* $[0,0,0,q+1]$.

Proof. Let S_{15} be the solid defined in (3.8), for some fixed $b,c\in\mathbb{F}_q$ such that $b\lambda^3+c\lambda+1$ is irreducible over $\mathbb{F}_q$. It remains to calculate the point-orbit distribution of S_{15}. The cubic surface $\Psi(S_{15})$ intersects the plane $\mathcal{Z}(Z)$ in a rational cubic curve consisting of $q+1$ points, and contains a further q^2-q points, parameterised as $(f(y,t),y,1,t)$ with $f(y,t)=(b+cy^2+y^3)/(t^2+y)$ and $t^2\neq y$. It meets $\mathcal{V}(\mathbb{F}_q)$ in a unique point, and the nucleus plane in a unique point. Hence, the point-orbit distribution of S_{15} is $[1,1,q^2-1,q^3+q]$. □

We now show that *every* solid with hyperplane-orbit distribution $[0,0,0,q+1]$ belongs to the K-orbit Ω_{15}. We need to know the sizes of the following unions of K-orbits, which are calculated via the orbit–stabiliser theorem using the relevant stabilisers (from Table 3.2) and the fact that $|K|=|\mathrm{PGL}(3,q)|=q^3(q^3-1)(q^2-1)$:

$$|\Omega_6\cup\Omega_7|=q^4(q^2+q+1),\ |\Omega_8\cup\Omega_{10}|=q^3(q^3-1)(q+1),\ |\Omega_9\cup\Omega_{13}|=\tfrac{1}{6}q^3(q^3-1)(q^2-1).$$

Note also that $|\mathcal{H}_1|=q^2+q+1$, $|\mathcal{H}_{2r}|=\frac{1}{2}q(q+1)(q^2+q+1)$, $|\mathcal{H}_{2i}|=\frac{1}{2}q(q-1)(q^2+q+1)$ and $|\mathcal{H}_3|=q^5-q^2$. Write $\mathcal{H}_2=\mathcal{H}_{2r}\cup\mathcal{H}_{2i}$ and note that $|\mathcal{H}_2|=q^2(q^2+q+1)$.

Lemma 3.23. *(Alnajjarine, Lavrauw & Popiel , 2022, Lemma 5.2)*
A hyperplane belonging to the K*-orbit* $\mathcal{H}_3$ *contains exactly* q^2 *solids that are contained in a hyperplane of type* $\mathcal{H}_1$ *and in a hyperplane of type* $\mathcal{H}_2$.

Proof. Since $\mathcal{H}_3$ is a K-orbit, each of its hyperplanes contains the same number of solids that are contained in a hyperplane of type $\mathcal{H}_j$ for both $j\in\{1,2\}$. Denote this number by k. Let $H\in\mathcal{H}_3$ and $H_1\in\mathcal{H}_1$. By Section 3.2.1, the solid $H\cap H_1$

belongs to one of the K-orbits Ω_5, Ω_6 or Ω_7, and accordingly has hyperplane orbit distribution $[1,0,0,q]$, $[1,1,0,q-1]$ or $[1,0,1,q-1]$ (by Lemma 3.11). If a solid $H \cap H_2$ with $H_2 \in \mathcal{H}_2$ belongs to a hyperplane of type $\mathcal{H}_1$, it therefore has type Ω_6 or Ω_7, and each such solid belongs to $q-1$ hyperplanes of type $\mathcal{H}_3$. Counting the flags (H,S) where $H \in \mathcal{H}_3$ and S is a solid contained in a hyperplane of type $\mathcal{H}_j$ for both $j \in \{1,2\}$ gives $|\mathcal{H}_3| \cdot k = |\Omega_6 \cup \Omega_7| \cdot (q-1)$, so $k = q^2$. □

Lemma 3.24. *(Alnajjarine, Lavrauw & Popiel , 2022, Lemma 5.3)*
There are exactly $\frac{1}{3}q^3(q^3-1)(q^2-1)$ solids with hyperplane-orbit distribution $[0,0,0,q+1]$.

Proof. Consider a hyperplane H of type $\mathcal{H}_3$. If a solid contained in H is contained in a hyperplane of type $\mathcal{H}_1$, then it is contained in exactly one such hyperplane, by the classification in Section 3.2, so there are $|\mathcal{H}_1| = q^2+q+1$ such solids in H. If a solid in H is not contained in a hyperplane of type $\mathcal{H}_1$, then it is contained in i hyperplanes of type $\mathcal{H}_2$ for some $i \in \{0,1,2,3\}$, by Lemma 3.10. Let n_i denote the number of solids contained in H in each case. The total number of solids in $\mathrm{PG}(5,q)$ with hyperplane-orbit distribution $[0,0,0,q+1]$ is then equal to

$$\frac{|\mathcal{H}_3| \cdot n_0}{q+1}, \tag{3.9}$$

so we must calculate n_0. The total number of solids in H is $N = (q^5-1)/(q-1)$, so $\sum_{i=0}^{3} n_i = N - |\mathcal{H}_1| = q(q^3+1)$. Now count the flags (S,H') where S is a solid in H that is not contained in a hyperplane of type $\mathcal{H}_1$ and H' is a hyperplane of type $\mathcal{H}_2$. By Lemma 3.23, we obtain $\sum_{i=1}^{3} i \cdot n_i = |\mathcal{H}_2| - q^2 = q(q^3+1)$. In particular, we have $\sum_{i=0}^{3} n_i = \sum_{i=1}^{3} i \cdot n_i$ and so $n_0 = n_2 + 2n_3$. Now, a solid contributing to n_2 belongs to $\Omega_8 \cup \Omega_{10}$, so $n_2 = (q-1)|\Omega_8 \cup \Omega_{10}|/|\mathcal{H}_3| = q(q^2-1)$. Similarly, a solid contributing to n_3 belongs to $\Omega_9 \cup \Omega_{13}$, giving $n_3 = (q-2)|\Omega_9 \cup \Omega_{13}|/|\mathcal{H}_3| = \frac{1}{6}q(q^2-1)(q-2)$. Therefore, $n_0 = n_2 + 2n_3 = \frac{1}{3}q(q+1)(q^2-1)$. Putting this into the expression in (3.9) completes the proof. □

Lemma 3.25. *(Alnajjarine, Lavrauw & Popiel , 2022, Lemma 5.4)*
If $S_{15} \in \Omega_{15}$ then $K_{S_{15}} \cong C_3$.

Proof. To prove this, consider the *cubic* extension $\overline{\mathcal{P}(S_{15})}$ of the pencil $\mathcal{P}(S_{15})$, namely its extension to $\mathrm{PG}(2,q^3)$. Since $\mathcal{P}(S_{15})$ contains no singular conics, $\overline{\mathcal{P}(S_{15})}$ contains exactly three singular conics (cf. Lemma 3.10), which must be conjugate under the Frobenius collineation σ of $\mathrm{PG}(2,q^3)$ induced by the automorphism $a \mapsto a^q$ of $\mathbb{F}_{q^3}$. In particular, these conics must all correspond to hyperplanes of $\mathrm{PG}(5,q^3)$ of the same type. According to the hyperplane-orbit distributions in Table 3.2, the only possibility is that S_{15} has type Ω_9 over $\mathbb{F}_{q^3}$. Hence, by Lemma 3.14, the stabiliser $\overline{G} \leqslant \mathrm{PGL}(3,q^3)$ of $\overline{\mathcal{P}(S_{15})}$ is isomorphic to the full permutation group of the four base points of $\overline{\mathcal{P}(S_{15})}$. Only one of these base points, call it Q, is $\mathbb{F}_q$-rational, since $\mathcal{P}(S_{15})$ has a unique base point; the other three are conjugate under σ, so we may label them as P, P^σ, P^{σ^2}. The stabiliser $G \leqslant \mathrm{PGL}(3,q)$ of $\mathcal{P}(S_{15})$ is therefore a subgroup of $\overline{G}_Q \cong \mathrm{Sym}_3$. We claim that G induces a group of order 3 on $\{P, P^\sigma, P^{\sigma^2}\}$. If $\alpha \in G$ fixes one of these points, say P, but is not the identity, then it swaps P^σ and P^{σ^2} (and fixes Q), so $P^{\alpha\sigma} = P^\sigma$ and $P^{\sigma\alpha} = P^{\sigma^2}$, contradicting the fact that α commutes with σ. Therefore, α is the identity, and so G induces no transpositions on $\{P, P^\sigma, P^{\sigma^2}\}$. Conversely, consider the element $\beta \in \mathrm{PGL}(3,q^3)$ in the stabiliser of $\overline{\mathcal{P}(S_{15})}$ corresponding to the 3-cycle $(P, P^\sigma, P^{\sigma^2})$. Then β commutes with σ and so belongs to $G \leqslant \mathrm{PGL}(3,q)$. Hence, G has order 3. □

Remark 3.6. *(Alnajjarine, Lavrauw & Popiel , 2022, Remark 5.5)*
For reference, we also record a matrix representative $g \in \mathrm{GL}(3,q)$ for a generator of $K_{S_{15}}$, where S_{15} is the solid given in (3.8). If $q = 2^n$ with n even then we may choose $c = 0$ and b a non-cube. In this case, $g = \mathrm{diag}(1, \zeta, \zeta^2)$ where $\zeta \in \mathbb{F}_q$ is a primitive third root of unity. If n is odd then all elements of $\mathbb{F}_q$ are cubes, so $c \neq 0$ and we can instead take $c = b$ after a change of variable $\lambda \to \sqrt{cb^{-1}}\lambda$. In this case,

$$g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta & b \\ 0 & b & \zeta^2 + b^2 \end{bmatrix}, \quad \text{where} \quad \zeta = b^{2^2} + b^{2^4} + \cdots + b^{2^{n-1}}.$$

Lemmas 3.24 and 3.25 together imply that there is a unique K-orbit of solids with hyperplane-orbit distribution $[0,0,0,q+1]$, as claimed (by the orbit–stabiliser theorem, since $|K| = q^3(q^3-1)(q^2-1)$).

3.4 Solids in PG(5,2)

Tables 3.1 and 3.2 are also correct for $q = 2$, but some of the arguments in Sections 3.1–3.3 do not apply in this case. For instance, the orbit Ω_1 can no longer be obtained by considering two pairs of real lines meeting in a point, because a pencil of conics $\mathcal{P}(S_1)$ corresponding to a solid $S_1 \in \Omega_1$ has a unique real line pair over $\mathbb{F}_2$. Similarly, if $S_9 \in \Omega_9$ then $\mathcal{P}(S_9)$ no longer contains any nonsingular conics, so the construction preceding Lemma 3.12 is not valid (but the generators given in

Table 3.1 are). Moreover, the point- and hyperplane-orbit distributions of S_9 now coincide with those of a solid $S_4 \in \Omega_4$, but the orbits of these solids can be distinguished either by their stabilisers, or by their line-orbit distributions: S_4 contains three lines of type o_6, while S_9 contains none (cf. Remark 3.5). In any case, it is straightforward to check the correctness of Tables 3.1 and 3.2 for $q=2$ either by hand or via the FinInG package in GAP (Bamberg, Betten, Cara, De Beule, Lavrauw & Neunhöffer , 2018; GAP, 2021). (Note that the descriptions of the stabilisers in Table 3.2 simplify in the obvious ways when $q=2$, i.e. C_{q-1} is the trivial group, $E_q \cong C_2$, and $\mathrm{GL}(2,q) \cong D_{2(q+1)} \cong \mathrm{Sym}_3$. Similarly, we necessarily have $\gamma = b = c = 1$ in Table 3.1.)

Remark 3.7. *By Remark 2.11, $K \cong \mathrm{PGL}(3,q)$ is not the full setwise stabiliser of the Veronese surface when $q=2$. The full stabiliser is Sym_7, and there are only 7 orbits of solids under this group, namely $\Omega_1 \cup \Omega_{10}$, $\Omega_2 \cup \Omega_8$, $\Omega_3 \cup \Omega_5 \cup \Omega_{15}$, $\Omega_4 \cup \Omega_9$, $\Omega_6 \cup \Omega_{11} \cup \Omega_{12}$, $\Omega_7 \cup \Omega_{14}$, and Ω_{13}. Finally, note that the point-orbit distribution of a subspace is not an invariant under Sym_7, since the nucleus plane is not preserved under the action.*

Theorem 3.2. *There are 7 J-orbits of solids, where $J \cong Sym_7$ is the group stabilising $\mathcal{V}(\mathbb{F}_2)$. In particular, these orbits split under the action of $\mathrm{PGL}(3,2)$ into 15 orbits as described in Remark 3.7.*

3.5 Comparison with Campbell's partial classification

Campbell provided a list of 17 "classes" and "sets of classes" of pencils of conics in $\mathrm{PG}(2,q)$, q even (Campbell, 1927). His analysis divided the classes of pencils into the following sets: pencils with at least one double line (set 1); pencils with no double lines and at least one real pair of lines (set 2); pencils with no double lines, no real pairs of lines, and at least one conjugate imaginary pair of lines (set 3); and pencils with no degenerate (singular) conics (set 4). The correspondence between our classification and Campbell's work (Campbell, 1927) is summarised in Table 3.4. We remark that in the study of his set 3, Campbell claimed that a pencil belonging to "set 15" has three imaginary pairs of lines and $q-2$ nonsingular conics. The non-existence of such a pencil was observed by Saniga (Saniga, 2000) (and also follows from Table 3.2). Moreover, the existence of the K-orbit Ω_{14}, whose elements have hyperplane-orbit distribution $[0,0,1,q]$, disproves Campbell's claim (Campbell,

Class/Set of pencils	Orbit(s) of solids
Class 1	Ω_3
Class 2	Ω_5
Class 3	Ω_1
Class 4	Ω_2
Class 5	Ω_7
Class 6	Ω_6
Class 7	Ω_9
Class 8	Ω_{12}
Class 9	Ω_8
Set 10	Ω_9, Ω_{12}, Ω_{13}
Class 11	Ω_{11}
Class 12	Ω_4
Class 13	Ω_{10}
Set 14	Ω_{14}
Set 15	Ω_{13}
Set 16	Ω_{15}
Set 17	Ω_{15}

Table 3.4 Correspondence between K-orbits of solids in $\mathrm{PG}(5,q)$ and Campbell's "classes" and "sets of classes" of pencils of conics in $\mathrm{PG}(2,q)$, q even.

1927, p. 405) that there exists no pencil with a unique pair of imaginary conjugate lines and q nonsingular conics.

Remark 3.8. *In Table 3.4, the blue colour indicates a completion of the discussion of Campbell's sets of classes of pencils, while the red colour indicates a completion and a correction of Campbell's sets of classes of pencils. In particular, we proved that the Set* 10 *splits into three orbits and each of the Sets* 14*,* 15*,* 16 *and* 17 *defines a unique orbit, we corrected as well the hyperplane-orbit distributions of the pencils in the Sets* 14 *and* 15 *as mentioned earlier.*

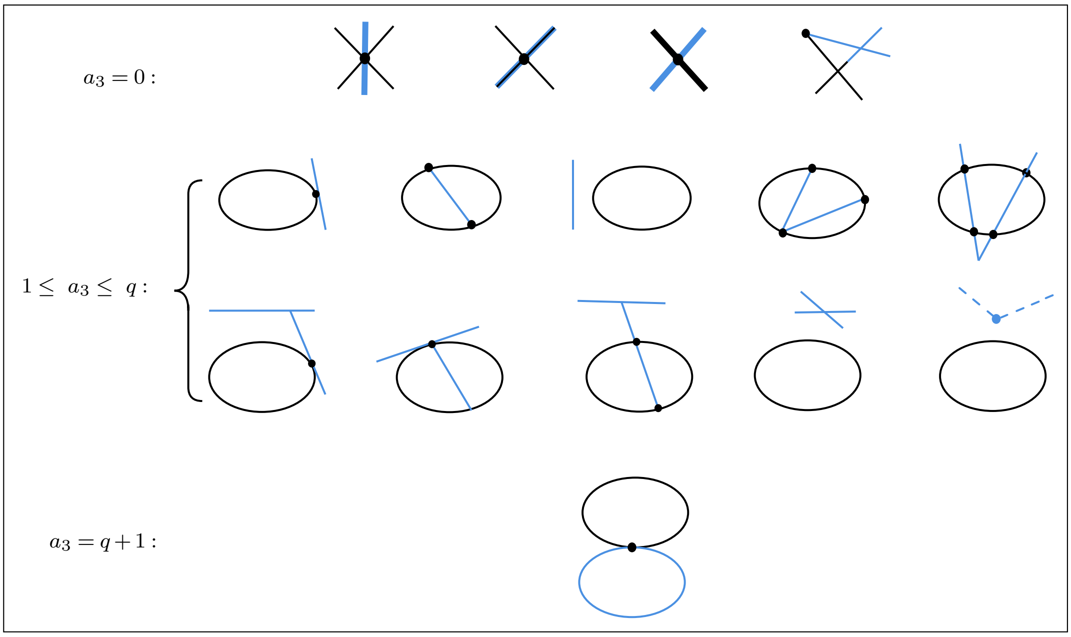

Figure 3.6 The 15 pencils of conics in $\mathrm{PG}(2,q)$, $q \neq 2$ even, up to projective equivalence.

4 PLANES INTERSECTING THE VERONESE SURFACE

NON-TRIVIALLY IN $\mathrm{PG}(5,q)$, q EVEN

In this chapter, we present our results from (Alnajjarine & Lavrauw, 2022). In particular, we classify planes intersecting the Veronese surface in at least one point in $\mathrm{PG}(5,q)$, q even, under the action of the subgroup K of $\mathrm{PGL}(6,q)$ stabilising the Veronese surface. We compute for each (type of) plane $\pi \subseteq \mathrm{PG}(5,q)$ its point-orbit distribution represented by the 4-tuple $[r_1, r_{2n}, r_{2s}, r_3]$, where r_i is the number of rank-i points in π for $i \in \{1,3\}$, r_{2n} is the number of rank-2 points in π meeting the nucleus plane and r_{2s} is the number of the remaining rank-2 points in π. In general, we distinguish between orbits using point-orbit distributions, line-orbit distributions and inflexion points defined in Chapter 2. Some of the arguments that we use here come from the classification of planes meeting the Veronese surface non-trivially over finite fields of odd characteristics (Lavrauw, Popiel & Sheekey, 2020). Note that, similar to solids' representations, planes in $\mathrm{PG}(5,q)$ can be seen as 3×3-matrices. For instance, the plane spanned by the first three points of the standard frame of $\mathrm{PG}(5,q)$ can be represented by:

$$\begin{bmatrix} x & y & z \\ y & \cdot & \cdot \\ z & \cdot & \cdot \end{bmatrix} := \left\{ \begin{bmatrix} x & y & z \\ y & 0 & 0 \\ z & 0 & 0 \end{bmatrix} : (x,y,z) \in \mathrm{PG}(2,q) \right\}. \tag{4.1}$$

In this chapter, the homogeneous coordinates in $\mathrm{PG}(2,q)$ and $\mathrm{PG}(5,q)$ are denoted by (X,Y,Z) and $(Y_0,\ldots,Y_5)$ respectively, and $\mathcal{Z}(f)$ denotes the zero locus of a form f.

Definition 4.1. *We define inflexion points of a plane π in $\mathrm{PG}(5,q)$ to be inflexion points of its associated cubic curve in $\mathrm{PG}(2,q)$ defined as the determinant of the matrix representation of π.*

Remark 4.1. *As we will see later, studying cubic curves associated with planes in $\mathrm{PG}(5,q)$ can be useful to differentiate between non-equivalent planes, but it is not sufficient to completely characterize each orbit. For instance, the representatives of the orbits Σ_8 and Σ_9 in Table 4.1 share the same cubic curve $\mathcal{Z}(XZ^2)$, however the two orbits are distinct by their intersection with the nucleus plane $\mathcal{N}$.*

This chapter is structured as follows. The proof of our main result, Theorem 4.1, is given in Sections 4.1–4.4. Note that, the case $q = 2$ requires special treatment, and is handled in Section 4.5. Finally, we give in Section 4.6 a comparison with the similar classification over finite fields of odd characteristic.

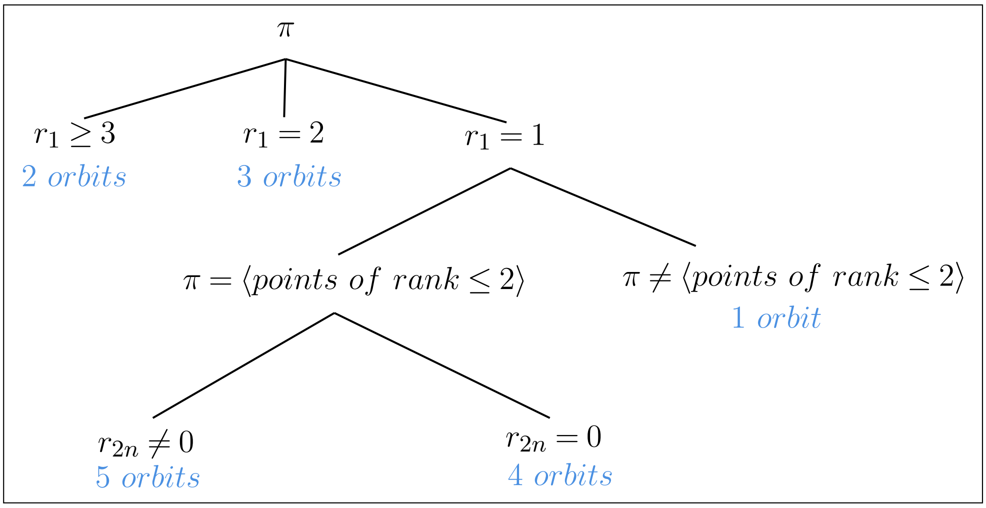

Figure 4.1 The discussion structure of Chapter 4.

Theorem 4.1. *(Alnajjarine & Lavrauw, 2022, Theorem 1.1)*
Let q be an even prime power. There are exactly 15 orbits of planes having at least one rank-1 point in $\mathrm{PG}(5,q)$ under the induced action of $\mathrm{PGL}(3,q) \leqslant \mathrm{PGL}(6,q)$ defined in Section 2.7.1. Representatives of these orbits are given in Table 4.1, the notation of which is also defined in Section 2.2.3.

Before we start recall the 15 K-orbits of lines in $\mathrm{PG}(5,q)$, q even, from (Lavrauw & Popiel, 2020), summarized in Table 2.2. The following two lemmas give bounds on the number of rank-2 points in planes of $\mathrm{PG}(5,q)$ meeting $\mathcal{V}(\mathbb{F}_q)$ in one or two points.

Lemma 4.1. *(Alnajjarine & Lavrauw, 2022, Lemma 2.8)*
There is no plane in $\mathrm{PG}(5,q)$ with rank distribution $[1,0,q^2+q]$.

Proof. Let Q_1 be the unique rank-1 point in a plane $\pi \subset \mathrm{PG}(5,q)$ having no rank-2 points. By inspecting point-orbit distributions of lines of $\mathrm{PG}(5,q)$ from Table 2.2, we conclude that all lines through Q_1 in π must be of type o_9. Therefore, we may assume without loss of generality that $\pi = \langle Q_1, Q_2, Q_3 \rangle$, where $\langle Q_1(1,0,0,0,0,0), Q_2(0,0,1,1,0,0) \rangle$ is the representative of the line orbit o_9 in (Lavrauw & Popiel, 2020, Table 2) and Q_3 is a point of rank 3 with homogeneous coordinates $(0,a,0,b,c,d)$; $a,b,c,d \in \mathbb{F}_q$. As Q_3 has rank three, it follows that $a,d \neq 0$. Thus, we may take Q_3 as the point $(0,1,0,a,b,c)$ for some $a,b,c \in \mathbb{F}_q$ with $c \neq 0$ and the representative of π becomes

$$\begin{bmatrix} x & y & z \\ y & ay+z & by \\ z & by & cy \end{bmatrix}.$$

The cubic curve associated with π has the form $XF(Y,Z) + G(Y,Z)$, where

$$F(Y,Z) = b^2Y^2 + acY^2 + cYZ, \;\; G(Y,Z) = aYZ^2 + cY^3 + Z^3.$$

Since F defines a quadric on $\mathrm{PG}(1,q)$ where each of its points satisfying $F(Y,Z) \neq 0$ corresponds to a point in π of rank 2, it follows that F must be identically zero. Therefore, $b = c = 0$, a contradiction. □

Lemma 4.2. *(Lavrauw, Popiel & Sheekey, 2020, Lemma 4.6)*
Every plane π in $\mathrm{PG}(5,q)$ with rank distribution $[2, r_2, r_3]$ has at least q rank-2 points, i.e., $r_2 \geq q$.

Proof. Let $Q_1, Q_2 \in \pi \cap \mathcal{V}(\mathbb{F}_q)$. Since points on $\langle Q_1, Q_2 \rangle$ have rank at most 2, it follows that π has at least $q-1$ rank-2 points. Assume by way of contradiction that $r_2 < q$. Then, $r_2 = q-1$, and thus all rank-2 points in π lie on the line $\langle Q_1, Q_2 \rangle$. Consequently, the cubic curve C defining points of rank at most 2 in π is the triple line $\langle Q_1, Q_2 \rangle$. Assume without loss of generality that $\pi = \langle Q_1, Q_2, Q_3 \rangle$ where $Q_1 = \nu(e_1)$, $Q_2 = \nu(e_2)$ and Q_3 is a point of rank 3. Then,

$$M_{Q_3} = \begin{bmatrix} 0 & a & b \\ a & 0 & c \\ b & c & d \end{bmatrix},$$

for some $a,b,c,d \in \mathbb{F}_q$. Hence, the cubic curve $C = \mathcal{Z}(dXYZ + c^2XZ^2 + a^2dZ^3 + b^2YZ^2)$ associated with π is a triple line. Therefore, $c = d = 0$, a contradiction with the rank of Q_3 being 3. □

4.1 Planes containing at least three rank-1 points

Let π be a plane in $\mathrm{PG}(5,q)$ with at least three rank-1 points. As $\mathcal{V}(\mathbb{F}_q)$ is a cap, it follows that no three rank-1 points in π are collinear. Thus, π can be viewed as $\pi = \langle Q_1, Q_2, Q_3\rangle$ where $Q_i = \nu(q_i)$ for $1 \le i \le 3$. We differentiate between the following two possibilities:

(i) If q_1, q_2 and q_3 are collinear in $\mathrm{PG}(2,q)$, then $Q_1, Q_2, Q_3 \in \mathcal{C}(Q_1, Q_2)$. As $\mathrm{PGL}(3,q)$ acts transitively on lines in $\mathrm{PG}(2,q)$, it follows that planes satisfying this configuration define a unique K-orbit Σ_1. In particular, by taking $\langle q_1, q_2\rangle$ as the line $\langle e_1, e_2\rangle$ we obtain the following representative

$$\Sigma_1 : \begin{bmatrix} x & y & \cdot \\ y & z & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}.$$

Lemma 4.3. *The point-orbit distribution of a plane in Σ_1 is $[q+1, 1, q^2-1, 0]$.*

Proof. Points of rank one in Σ_1 correspond to points on the quadric $\mathcal{Z}(XZ+Y^2)$. The remaining q^2 points in Σ_1 are of rank two, where only the point parametrized by $(x,y,z) = (0,1,0)$ is contained in the nucleus plane $\mathcal{N}$. Therefore, the point-orbit distribution of a plane in Σ_1 is $[q+1, 1, q^2-1, 0]$. □

(ii) If q_1, q_2 and q_3 are non-collinear in $\mathrm{PG}(2,q)$, then without loss of generality we may take $q_i = \langle e_i\rangle$ for $1 \le i \le 3$. This gives a new plane orbit Σ_2 whose representative is

$$\Sigma_2 : \begin{bmatrix} x & \cdot & \cdot \\ \cdot & y & \cdot \\ \cdot & \cdot & z \end{bmatrix}$$

and whose uniqueness is guaranteed by the 3-regular action of $\mathrm{PGL}(3,q)$ on points of $\mathrm{PG}(2,q)$.

Lemma 4.4. *The point-orbit distribution of a plane in Σ_2 is $[3, 0, 3q-3, q^2-2q+1]$ and $\Sigma_1 \neq \Sigma_2$.*

Proof. Points of rank at most two in Σ_2 correspond to points on the cubic curve $C_2 = \mathcal{Z}(XYZ)$. The rank-1 points are particularly those with parametrized coordinates $(x,y,z) = (1,0,0)$, $(0,1,0)$ and $(0,0,1)$. The remaining $3q-3$ points on

C_2 correspond to rank-2 points in Σ_2 where none of these is contained in the nucleus plane $\mathcal{Z}(Y_0, Y_3, Y_5)$. Therefore, the point-orbit distribution of a plane in Σ_2 is $[3, 0, 3q-3, q^2-2q+1]$ and $\Sigma_1 \neq \Sigma_2$ by their distinct point-orbit distributions. □

Remark 4.1. *Combining results of Lemma 4.1, Lemma 4.2 and Section 4.1 implies that every plane in* $\mathrm{PG}(5,q)$ *intersecting the Veronese surface in at least one point can be represented by* $\pi = \langle Q_1, Q_2, Q_3 \rangle$ *where the rank of* Q_1 *and* Q_2 *is at most* 2.

4.2 Planes containing two rank-1 points

We consider in this section planes of $\mathrm{PG}(5,q)$ intersecting the Veronese surface in exactly two points. Let π be such a plane containing the rank-1 points Q_1 and Q_2. By Lemma 4.2, there exists a rank-2 point in π not lying on the line Q_1Q_2. Hence, we may assume that $\pi = \langle Q_1, Q_2, Q_3 \rangle$ where $rank(Q_3) = 2$. Let $U = \mathcal{C}(Q_1, Q_2) \cap \mathcal{C}(Q_3)$ where $\mathcal{C}(Q_1, Q_2)$ and $\mathcal{C}(Q_3)$ are the two conics associated with $\{Q_1, Q_2\}$ and Q_3 respectively (see Section 2.7). We study separately the cases where $U \in \{Q_1, Q_2\}$ or $U \notin \{Q_1, Q_2\}$.

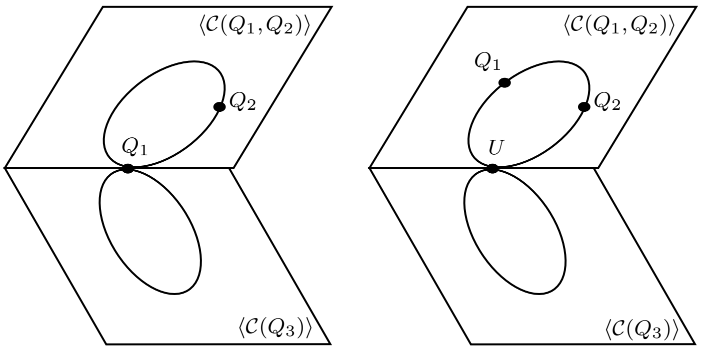

Figure 4.2 Configurations associated with cases (i) and (ii), respectively.

(i) If $U \in \{Q_1, Q_2\}$, then without loss of generality we may assume that $U = Q_1$. Let q_1, q_2 and l_3 be the preimages under ν of Q_1, Q_2 and $\mathcal{C}(Q_3)$ respectively. As the elation group $E(q_1, \langle q_1, q_2 \rangle)$, with centre q_1 and axis $\langle q_1, q_2 \rangle$, acts transitively on the affine points of $\mathrm{PG}(2,q) \setminus \langle q_1, q_2 \rangle$, it follows that we may fix $\langle q_1, q_2 \rangle$ and l_3 as $\langle e_1, e_2 \rangle$

and $\langle e_1, e_3\rangle$ respectively. Hence the points Q_1, Q_2 can be represented by

$$M_{Q_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } M_{Q_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

respectively. Since π contains the line $\langle Q_1, Q_2\rangle$, we have two possibilities: (*i-a*) $Q_3 \in T_{Q_1}(\mathcal{C}(Q_3))$, or (*i-b*) $Q_3 \in \langle \mathcal{C}(Q_3)\rangle \setminus (\mathcal{C}(Q_3) \cup T_{Q_1}(\mathcal{C}(Q_3)))$.

(i–a) If $Q_3 \in T_{Q_1}(\mathcal{C}(Q_3))$, then π is completely determined by $\langle Q_1, Q_2\rangle$ and $T_{Q_1}(\mathcal{C}(Q_3)) = \mathcal{Z}(Y_5)$, where $\mathcal{C}(Q_3) = \mathcal{Z}(Y_0Y_5 + Y_2^2) \cap \mathcal{Z}(Y_1, Y_3, Y_4)$, leading to a unique orbit represented by

$$\Sigma_3 : \begin{bmatrix} x & . & z \\ . & y & . \\ z & . & . \end{bmatrix}.$$

Lemma 4.5. *The point-orbit distribution of a plane in* Σ_3 *is* $[2, 1, 2q-2, q^2-q]$. *In particular,* $\Sigma_3 \notin \{\Sigma_1, \Sigma_2\}$.

Proof. Let π_3 be the above representative of Σ_3. Points of rank at most 2 in π_3 correspond to points on the cubic curve $C_3 = \mathcal{Z}(YZ^2)$. Among these $2q+1$ points, there are exactly two rank-1 points corresponding to points of $\mathcal{Z}(Z, XY)$ and a unique rank-2 point in $\mathcal{N} \cap \pi_3 = \mathcal{Z}(X, Y)$ with parametrized coordinates $(0, 0, 1)$. Therefore, the point-orbit distribution of a plane in Σ_3 is $[2, 1, 2q-2, q^2-q]$. In particular, $\Sigma_3 \notin \{\Sigma_1, \Sigma_2\}$. □

(i–b) Assume now that $Q_3 \in \langle \mathcal{C}(Q_3)\rangle \setminus (\mathcal{C}(Q_3) \cup T_{Q_1}(\mathcal{C}(Q_3)))$ and let $R_3 = \nu(r_3) = \langle Q_1, Q_3\rangle \cap \mathcal{C}(Q_3)$. The subgroup in $\mathrm{PGL}(3,q)$ stabilising $\{q_1, q_2\}$ and l_3 contains the elation group with center q_1 and axis $\langle q_1, q_2\rangle$, and thus it acts transitively on points of $l_3 \setminus \{q_1\}$. Hence, without loss of generality we may also fix r_3. Now , as $\pi = \langle Q_1, Q_2, Q_3\rangle = \langle Q_1, Q_2, R_3\rangle$, it follows that π intersects $\mathcal{V}(\mathbb{F}_q)$ in three points, returning us to the already obtained Σ_2.

(ii) If $U \notin \{Q_1, Q_2\}$, then the preimages of these points under ν must be collinear in $\mathrm{PG}(2,q)$. Without loss of generality, let $q_1 = \langle e_1\rangle$, $q_2 = \langle e_2\rangle$ and $u = \langle e_1 + e_2\rangle$. As $E(q_1, \langle q_1, q_2\rangle)$ acts transitively on the affine points of $\mathrm{PG}(2,q) \setminus \langle q_1, q_2\rangle$, it follows that we may fix $l_3 = \nu^{-1}(\mathcal{C}(Q_3))$ as $\langle e_1 + e_2, e_3\rangle$. We study separately the following possibilities of Q_3 in the conic plane $\langle \mathcal{C}(Q_3)\rangle$: (*ii-a*) $Q_3 = N(\mathcal{C}(Q_3))(0, 0, 1, 0, 1, 0)$

, *(ii-b)* $Q_3 \in T_U(\mathcal{C}(Q_3)) \setminus \{N(\mathcal{C}(Q_3)), U\}$ or *(ii-c)* $Q_3 \in \langle \mathcal{C}(Q_3)\rangle \setminus (\mathcal{C}(Q_3) \cup T_U(\mathcal{C}(Q_3)))$.

(ii–a) If Q_3 is the nucleus point $N(\mathcal{C}(Q_3))$, then we obtain the orbit represented by

$$\Sigma_4 : \begin{bmatrix} x & . & z \\ . & y & z \\ z & z & . \end{bmatrix}.$$

Lemma 4.6. *The point-orbit distribution of a plane in* Σ_4 *is* $[2,1,2q-2,q^2-q]$. *In particular,* $\Sigma_4 \notin \{\Sigma_1, \Sigma_2, \Sigma_3\}$.

Proof. Let π_4 be the above representative of Σ_4. Rank-1 points in π_4 correspond to points on $\mathcal{Z}(XY, Z)$. Namely, points with parametrized coordinates $(1,0,0)$ and $(0,1,0)$. The remaining points on the cubic curve $C_4 = \mathcal{Z}(Z^2(X+Y))$ correspond to points of rank 2, where only the point parametrized by $(0,0,1)$ lies in $\pi_4 \cap \mathcal{N} = \mathcal{Z}(X,Y)$. Therefore, the point-orbit distribution of a plane in Σ_4 is $[2,1,2q-2,q^2-q]$ and $\Sigma_4 \notin \{\Sigma_1, \Sigma_2\}$ by their different point-orbit distributions. Finally, by observing that C_3, the cubic curve associated with π_3, is the union of two lines of type o_5 and o_6, while C_4 is the union of two lines of type o_5 and $o_{12,2}$, we can deduce that Σ_3 and Σ_4, which share the same point-orbit distribution, are also distinct.

□

(ii–b) If $Q_3 \in T_U(\mathcal{C}(Q_3)) \setminus \{N(\mathcal{C}(Q_3)), U\}$, then without loss of generality we may assume that Q_3 is $(a,a,1,a,1,0)$ for some $a \in \mathbb{F}_q \setminus \{0\}$. It follows that π, represented by

$$\pi_a : \begin{bmatrix} x+az & az & z \\ az & y+az & z \\ z & z & . \end{bmatrix}$$

for some $a \in \mathbb{F}_q \setminus \{0\}$, intersects the nucleus plane in a unique point Q_3' with homogeneous coordinates $(0,a,1,0,1,0)$. By considering the two possibilities where $U' = \mathcal{C}(Q_1,Q_2) \cap \mathcal{C}_{Q_3'}$ belongs to $\{Q_1,Q_2\}$ or not, we end up in one of the orbits Σ_3 or Σ_4. Hence, this case will not define a new orbit.

(ii–c) Finally, if $Q_3 \in \langle \mathcal{C}(Q_3)\rangle \setminus (\mathcal{C}(Q_3) \cup T_U(\mathcal{C}(Q_3)))$, then let $R_3 = \nu(r_3) = \langle U, Q_3\rangle \cap \mathcal{C}(Q_3)$. The subgroup in $\mathrm{PGL}(3,q)$ stabilising $\{u, q_1, q_2\}$ and l_3 contains the elation group with center u and axis $\langle q_1, q_2\rangle$, and thus it acts transitively on points of $l_3 \setminus \{u\}$. Hence, without loss of generality we may also fix r_3. Now, as

PGL$(3,q)$ acts sharply transitively on frames in PG$(2,q)$, it follows that the subgroup stabilising $\{u,q_1,q_2,r_3\}$ pointwise acts transitively on points of $l_3\setminus\{u,r_3\}$. This shows that any choice of Q_3 as a point on the secant $\langle U,R_3\rangle$ defines the same orbit. More generally, any choice of a point on $\langle\mathcal{C}(Q_3)\rangle\setminus(\mathcal{C}(Q_3)\cup T_U(\mathcal{C}(Q_3)))$ defines a unique orbit which we denote by Σ_5 and has the representative

$$\Sigma_5:\begin{bmatrix} x & . & z \\ . & y & z \\ z & z & z \end{bmatrix},$$

for the choice $Q_3=(0,0,1,0,1,1)$.

Lemma 4.7. *The point-orbit distribution of a plane in* Σ_5 *is* $[2,0,2q-2,q^2-q+1]$. *In particular,* $\Sigma_5\notin\{\Sigma_1,\Sigma_2,\Sigma_3,\Sigma_4\}$.

Proof. Let π_5 be the above representative of Σ_5. Points of rank at most 2 in π correspond to points of the cubic curve $C_5=\mathcal{Z}(XYZ+XZ^2+YZ^2)$, which intersect the nucleus plane $\mathcal{N}$ trivially and the Veronese surface $\mathcal{V}(\mathbb{F}_q)$ in exactly two points. Namely, points with parametrized coordinates $(1,0,0)$ and $(0,1,0)$. Therefore, the point-orbit distribution of a plane in Σ_5 is $[2,0,2q-2,q^2-q+1]$ and $\Sigma_5\notin\{\Sigma_1,\Sigma_2,\Sigma_3,\Sigma_4\}$. □

4.3 Planes containing one rank-1 point and spanned by points of rank at most 2

We investigate in this section planes of PG$(5,q)$ spanned by points of rank at most 2 and which meet the Veronese surface in exactly one point. Let $\pi=\langle Q_1,Q_2,Q_3\rangle$ be such a plane where $rank(Q_1)=1$ and $rank(Q_2)=rank(Q_3)=2$, and consider the two conics $\mathcal{C}(Q_2)$ and $\mathcal{C}(Q_3)$ associated with Q_2 and Q_3 respectively. Denote by q_1, l_2 and l_3 the respective preimages of Q_1, $\mathcal{C}(Q_2)$ and $\mathcal{C}(Q_3)$ under the Veronese embedding. We discuss independently the following possibilities:

(a) $l_2=l_3$,

(b) $q_1=l_2\cap l_3$,

(c) $q_1\in l_2\setminus l_3$, and,

(d) $q_1 \notin l_2 \cup l_3$.

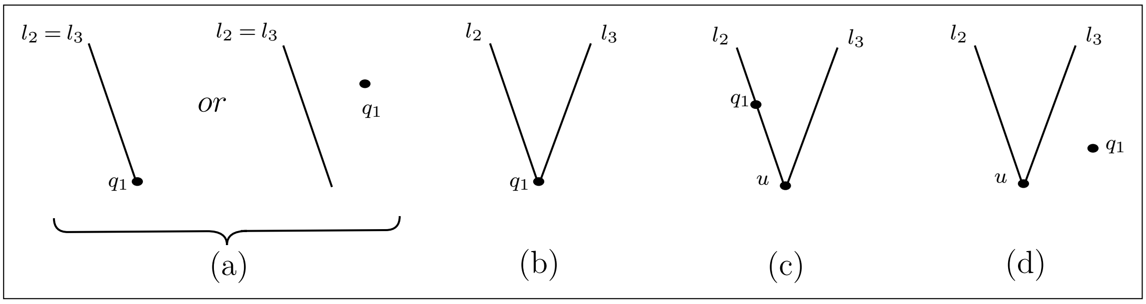

Figure 4.3 The configurations defined by cases (a), (b), (c) and (d) in Section 4.3.

4.3.1 (a) $l_2 = l_3$

If $l_2 = l_3$, then assume first that $q_1 \in l_2$. In this case, π becomes a conic plane and thus lies in Σ_1. Assume next that $q_1 \notin l_2$. As $\mathrm{PGL}(3,q)$ acts transitively on antiflags in $\mathrm{PG}(2,q)$ and π has a unique rank-1 point, it follows that we may fix Q_1 and $\mathcal{C}(Q_2)$ as $\nu(\langle e_1 \rangle)$ and $\nu(\langle e_2, e_3 \rangle)$ respectively, where the line $\langle Q_2, Q_3 \rangle$ must be external to $\mathcal{C}(Q_2)$. Now, as the group stabilising Q_1 and $\mathcal{C}(Q_2)$ acts transitively on external lines to $\mathcal{C}(Q_2)$, we obtain a unique orbit of such planes which we label as Σ_6. Indeed, we may fix Q_2Q_3 as the line $\mathcal{Z}(Y_3 + cY_4 + Y_5)$ where $Tr(c^{-1}) = 1$ to get the following representative

$$\Sigma_6 : \begin{bmatrix} x & \cdot & \cdot \\ \cdot & y+cz & z \\ \cdot & z & y \end{bmatrix}.$$

Lemma 4.8. *The point-orbit distribution of a plane in* Σ_6 *is* $[1, 0, q+1, q^2 - 1]$. *In particular,* $\Sigma_6 \notin \{\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4, \Sigma_5\}$.

Proof. Let π_6 be the above representative of Σ_6. Points of rank at most 2 in π_6 correspond to points on the cubic curve $C_6 = \mathcal{Z}(XY^2 + cXYZ + XZ^2)$. In particular, points of rank one in π_6 correspond to points on $\mathcal{Z}(XY, XZ, Y^2 + cYZ + Z^2)$. As $Tr(c^{-1}) = 1$, we obtain a unique rank-1 point parametrized by $(1,0,0)$. The remaining points on C_6 parametrize $q+1$ rank-2 points in π_6, where none of these is contained in the nucleus plane $\mathcal{N}$. Therefore, the point-orbit distribution of a plane in Σ_6 is $[1, 0, q+1, q^2 - 1]$, and thus $\Sigma_6 \notin \{\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4, \Sigma_5\}$. $\square$

Lemma 4.9. *A plane $\pi \in \Sigma_6$ has $q+1$ lines in $o_{8,1}$ and a unique line in o_{10}.*

Proof. By Lemma 4.8, π intersects the nucleus plane trivially and has $q+1$ rank-2 points lying on the line $\langle Q_2, Q_3\rangle$. Therefore, π has a unique line in o_{10} and each of the $q+1$ lines through the rank-1 point Q_1 must have q rank-3 points, and thus belongs to the line-orbit $o_{8,1}$. □

4.3.2 (b) $q_1 = l_2 \cap l_3$

If $q_1 = l_2 \cap l_3$, then as the group fixing q_1 in PGL$(3,q)$ acts transitively on lines passing through it, it follows that we may fix q_1, l_2 and l_3 as e_1, $\langle e_1, e_2\rangle$ and $\langle e_1, e_3\rangle$ respectively. Furthermore, as π contains a unique rank-1 point, it follows that $Q_2 \in T_{Q_1}(\mathcal{C}(Q_2))$ and $Q_3 \in T_{Q_1}(\mathcal{C}(Q_3))$. Therefore, π is completely determined by Q_1, $\mathcal{C}(Q_2)$ and $\mathcal{C}(Q_3)$. This yields to a unique K-orbit Σ_7 represented by

$$\Sigma_7 : \begin{bmatrix} x & y & z \\ y & . & . \\ z & . & . \end{bmatrix}.$$

Lemma 4.10. *The point-orbit distribution of a plane in Σ_7 is $[1, q+1, q^2-1, 0]$. In particular, $\Sigma_7 \notin \{\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4, \Sigma_5, \Sigma_6\}$.*

Proof. It follows from the above representative that points of Σ_7 have rank at most two. Particularly, Σ_7 has a unique rank-1 point obtained for $y = z = 0$ and $q+1$ points in the nucleus plane parametrized by $\{(0, y, z) : y, z \in \mathbb{F}_q;\ (y,z) \neq (0,0)\}$. Therefore, the point-orbit distribution of a plane in Σ_7 is $[1, q+1, q^2-1, 0]$. Moreover, by comparing this property with the previous orbits, we conclude that $\Sigma_7 \notin \{\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4, \Sigma_5, \Sigma_6\}$. □

4.3.3 (c) $q_1 \in l_2 \setminus l_3$

If $q_1 \in l_2 \setminus l_3$, then without loss of generality we may consider $U = \nu(u) = \mathcal{C}(Q_2) \cap \mathcal{C}(Q_3)$ and $Q_1 = \nu(q_1)$ as $\nu(\langle e_2\rangle)$ and $\nu(\langle e_1\rangle)$ respectively. The elation group $E(u, \langle u, q_1\rangle)$ acts transitively on the affine points of PG$(2,q) \setminus l_2$, and thus we may also fix l_3 as $\langle e_2, e_3\rangle$. Since π has a unique rank-1 point, it follows that Q_2 lies on the tangent line $T_{Q_1}(\mathcal{C}(Q_2))$. We next consider the following possibilities: *(c-i)* $Q_3 = N(\mathcal{C}(Q_3))$, *(c-ii)* $Q_3 \in T_U(\mathcal{C}(Q_3)) \setminus \{N(\mathcal{C}(Q_3)), U\}$ and *(c-iii)* $Q_3 \in \langle \mathcal{C}(Q_3)\rangle \setminus (\mathcal{C}(Q_3) \cup T_U(\mathcal{C}(Q_3)))$.

$\boxed{\text{(c−i)}}$ If Q_3 is the nucleus point $N(\mathcal{C}(Q_3))$, then $\pi = \langle T_{Q_1}(\mathcal{C}(Q_2)), Q_3\rangle$, which defines a new orbit represented by

$$\Sigma_8 : \begin{bmatrix} x & y & . \\ y & . & z \\ . & z & . \end{bmatrix}.$$

Lemma 4.11. *The point-orbit distribution of a plane in* Σ_8 *is* $[1, q+1, q-1, q^2-q]$. *In particular,* $\Sigma_8 \notin \{\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4, \Sigma_5, \Sigma_6, \Sigma_7\}$.

Proof. Points of rank at most 2 in Σ_8 correspond to points on the cubic curve $C_8 = \mathcal{Z}(XZ^2)$. Among these $2q+1$ points, there is a unique rank-1 point lying on $C_8 \cap \mathcal{Z}(Y, Z)$ and $q+1$ points in the nucleus plane lying on $C_8 \cap \mathcal{Z}(X)$. Therefore, the point-orbit distribution of a plane in Σ_8 is $[1, q+1, q-1, q^2-q]$ and $\Sigma_8 \notin \{\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4, \Sigma_5, \Sigma_6, \Sigma_7\}$ by their distinct point-orbit distributions. □

$\boxed{\text{(c−ii)}}$ Assume now that $Q_3 \in T_U(\mathcal{C}(Q_3)) \setminus \{N(\mathcal{C}(Q_3)), U\}$. The subgroup of $\mathrm{PGL}(3,q)$ fixing $\{q_1, u\}$ and l_3 contains the elation group $E(u, \langle u, q_1\rangle)$, and thus it acts transitively on points of $l_3 \setminus \{u\}$. Therefore, any different choice of Q_3' as a point on $T_U(\mathcal{C}(Q_3)) \setminus \{N(\mathcal{C}(Q_3)), U, Q_3\}$ defines the same orbit, Σ_9. Without loss of generality, we may choose Q_3 as $(0,0,0,1,1,0)$ to obtain the following representative

$$\Sigma_9 : \begin{bmatrix} x & y & . \\ y & z & z \\ . & z & . \end{bmatrix}.$$

Lemma 4.12. *The point-orbit distribution of a plane in* Σ_9 *is* $[1, 1, 2q-1, q^2-q]$. *In particular,* $\Sigma_9 \notin \{\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4, \Sigma_5, \Sigma_6, \Sigma_7, \Sigma_8\}$.

Proof. Similar to case Σ_8, points of rank at most 2 in Σ_9 correspond to points on the cubic curve $C_9 = \mathcal{Z}(XZ^2)$. In particular, Σ_9 has a unique rank-1 point parametrized by $(1,0,0)$ and a unique rank-2 point in $\mathcal{N}$ parametrized by $(0,1,0)$.

Therefore, the point-orbit distribution of a plane in Σ_9 is $[1,1,2q-1,q^2-q]$, and thus $\Sigma_9 \notin \{\Sigma_1,\Sigma_2,\Sigma_3,\Sigma_4,\Sigma_5,\Sigma_6,\Sigma_7,\Sigma_8\}$. □

Remark 4.2. *The two planes π_8 and π_9 define the same cubic curve $\mathcal{Z}(XZ^2)$, however they are not K-equivalent.*

$\boxed{\text{(c}-\text{iii)}}$ Finally, assume that $Q_3 \in \langle \mathcal{C}(Q_3)\rangle \setminus (\mathcal{C}(Q_3) \cup T_U(\mathcal{C}(Q_3)))$. The subgroup in $\mathrm{PGL}(3,q)$ stabilising $\{u,q_1\}$ and l_3 contains the elation group with center u and axis $\langle u,q_1\rangle$, and thus it acts transitively on points of $l_3 \setminus \{u\}$. Hence, without loss of generality we may fix $R_3 = \nu(r_3) = \langle U,Q_3\rangle \cap \mathcal{C}(Q_3)$ as the point $\nu(\langle e_3\rangle)$. Now, as $\mathrm{PGL}(3,q)$ acts sharply transitively on frames in $\mathrm{PG}(2,q)$, it follows that the subgroup stabilising $\{u,q_1,r_3\}$ pointwise acts transitively on points of $l_3 \setminus \{u,r_3\}$. This shows that any other choice of a point $Q_3' \neq Q_3$ on the secant $\langle U,R_3\rangle$ defines the same orbit. More generally, any choice of a point on $\langle \mathcal{C}(Q_3)\rangle \setminus (\mathcal{C}(Q_3) \cup T_U(\mathcal{C}(Q_3)))$ defines a unique K-orbit which we call Σ_{10} and represent by

$$\Sigma_{10} : \begin{bmatrix} x & y & . \\ y & z & . \\ . & . & z \end{bmatrix},$$

for the choice $Q_3 = (0,0,0,1,0,1)$.

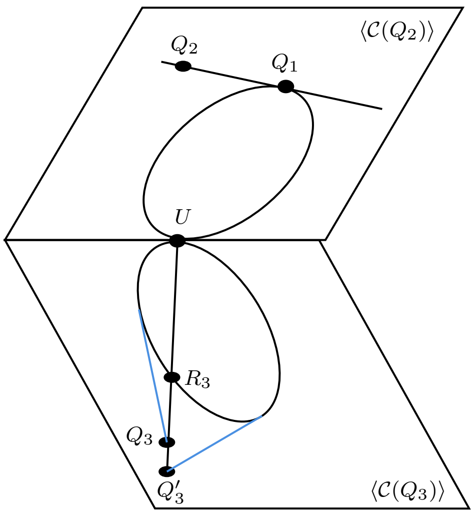

Figure 4.4 The configuration defined in case (*c-iii*), Section 4.3.3.

Lemma 4.13. *The point-orbit distribution of a plane in Σ_{10} is $[1,1,2q-1,q^2-q]$. In particular, $\Sigma_{10} \notin \{\Sigma_1,\Sigma_2,\Sigma_3,\Sigma_4,\Sigma_5,\Sigma_6,\Sigma_7,\Sigma_8,\Sigma_9\}$.*

Proof. Let π_{10} be the above representative of Σ_{10}. Points of rank at most 2 in π_{10} correspond to the $2q+1$ points on the cubic curve $C_{10} = \mathcal{Z}(XZ^2+Y^2Z)$. In particular, π_{10} has a unique rank-1 point parametrized by $(1,0,0)$ and a unique point lying on $\pi_{10} \cap \mathcal{N} = \mathcal{Z}(X,Z)$ parametrized by $(0,1,0)$. Therefore, the point-orbit distribution of a plane in Σ_{10} is $[1,1,2q-1,q^2-q]$ and $\Sigma_{10} \notin \{\Sigma_1,\Sigma_2,\Sigma_3,\Sigma_4,\Sigma_5,\Sigma_6,\Sigma_7,\Sigma_8\}$ by their distinct point-orbit distributions. It remains to show that $\Sigma_9 \neq \Sigma_{10}$. But this follows immediately by observing that C_9 is the union of two lines of type o_6 and $o_{12,2}$, while C_{10} is the union of a nonsingular conic and one of its tangent lines (which is a line of type o_6). □

4.3.4 (d) $q_1 \notin l_2 \cup l_3$

Finally, assume that $q_1 \notin l_2 \cup l_3$ and let $U = \nu(u) = \mathcal{C}(Q_2) \cap \mathcal{C}(Q_3)$. We study separately the following cases: *(d-i)* $\pi \cap \mathcal{N} \neq \emptyset$ and *(d-ii)* $\pi \cap \mathcal{N} = \emptyset$, where $\mathcal{N}$ is the nucleus plane.

4.3.4.1 *(d-i)* $\pi \cap \mathcal{N} \neq \emptyset$

As π intersects the nucleus plane non-trivially, we may assume that $Q_2 = N(\mathcal{C}(Q_2))$. The line joining Q_2 and the unique rank-1 point Q_1 is either of type o_6 or $o_{8,2}$ by Table 2.2. As lines of type o_6 in $\mathrm{PG}(5,q)$ are tangent lines to conics in $\mathcal{V}(\mathbb{F}_q)$, it follows that $\langle Q_1,Q_2\rangle \in o_{8,2}$. Hence, without loss of generality, we may start by fixing u, q_1 and r_2 as $\langle e_1\rangle$, $\langle e_2+e_3\rangle$, $\langle e_2\rangle$ respectively and consider l_2 as $\nu^{-1}(\mathcal{C}(Q_2)) = \langle e_1,e_2\rangle$. The group fixing $\{u,q_1,r_2\}$ acts transitively on points of $\mathrm{PG}(2,q)$ not lying on the triangle defined by $\{u,q_1,r_2\}$, and thus we may fix $l_3 = \nu^{-1}(\mathcal{C}(Q_3))$ as $\langle e_1,e_1+e_3\rangle$. Let $r_3 = \langle q_1,r_2\rangle \cap l_3$ and define R_i as $\nu(r_i)$ for $i = 2,3$. The subgroup of $\mathrm{PGL}(3,q)$ stabilising $\{u,q_1,r_2,l_3\}$ is induced by the elation group of centre u and axis $\langle u,q_1\rangle$, and acts on $\langle\mathcal{C}(Q_3)\rangle$ as the stabiliser of $\mathcal{C}(Q_3)$ and the two points U and R_3. If $Q_3 = N(\mathcal{C}(Q_3))$, then π contains the point $(0,1,1,0,0,0)$ which is the nucleus of the conic defined by $\nu(\langle q_1,u\rangle)$. Hence, $\pi = \langle Q_1,N(\mathcal{C}(U,Q_1)),Q_3\rangle$ and it is completely determined by $T_{Q_1}(\mathcal{C}(U,Q_1))$ and Q_3. Thus, this case returns us to the already obtained Σ_8. Assume next that $Q_3 \in T_U(\mathcal{C}(Q_3)) \setminus \{N(\mathcal{C}(Q_3)),U\}$. As the group stabilising $\{u,q_1,r_2,l_3\}$ acts on $\langle\mathcal{C}(Q_3)\rangle$ as the stabiliser of $\mathcal{C}(Q_3)$ and the two points U and R_3, it follows that any other choice of a point Q_3' on $T_U(\mathcal{C}(Q_3)) \setminus \{N(\mathcal{C}(Q_3)),U,Q_3\}$ defines the same orbit. Therefore, we may choose

Q_3 as the point $(1,0,1,0,0,0)$ to obtain the orbit represented by

$$\begin{bmatrix} x & y & x \\ y & z & z \\ x & z & z \end{bmatrix}.$$

This case will not define a new orbit as π intersects the Veronese surface in two points, namely $(1,0,0,0,0,0)$ and $(1,0,1,1,1,0)$, implying that $\pi \in \{\Sigma_3, \Sigma_4, \Sigma_5\}$. It remains to consider the case where $Q_3 \in \langle \mathcal{C}(Q_3) \rangle \setminus T_U(\mathcal{C}(Q_3))$. Similar to the previous argument, we may assume without loss of generality that Q_3 is $(1,0,0,0,0,1)$. This gives a unique new orbit Σ_{11} represented by

$$\Sigma_{11} : \begin{bmatrix} x & y & . \\ y & z & z \\ . & z & x+z \end{bmatrix}.$$

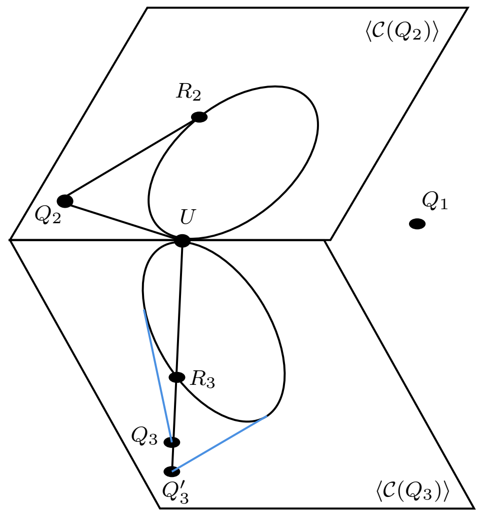

Figure 4.5 The configuration defining Σ_{11}.

Lemma 4.14. *The point-orbit distribution of a plane in* Σ_{11} *is* $[1,1,q-1,q^2]$. *In particular,* $\Sigma_{11} \notin \{\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4, \Sigma_5, \Sigma_6, \Sigma_7, \Sigma_8, \Sigma_9, \Sigma_{10}\}$.

Proof. Let π_{11} be the above representative of Σ_{11}. Points of rank at most 2 in π_{11} correspond to points on the cubic curve $C_{11} = \mathcal{Z}(X^2Z + XY^2 + Y^2Z)$. Particularly, π_{11} has a unique rank-1 point and a unique rank-2 point lying on $\mathcal{N} \cap \pi_{11} = \mathcal{Z}(X,Z)$ with parametrized coordinates $(0,0,1)$ and $(0,1,0)$ respectively. Therefore, the point-orbit distribution of a plane in Σ_{11} is $[1,1,q-1,q^2]$ and Σ_{11} is distinct from the previously defined orbits by their point-orbit distributions. □

4.3.4.2 (*d-ii*) $\pi \cap \mathcal{N} = \emptyset$

Assume now that π intersects the nucleus plane trivially, where the unique rank-1 point Q_1 is not lying on $\mathcal{C}(Q_2) \cup \mathcal{C}(Q_3)$. We begin with an essential lemma that gives a correspondence between types of lines spanned by two rank-2 points in $\mathrm{PG}(5,q)$ and their associated configurations defined by $\{\mathcal{C}(Q_2), \mathcal{C}(Q_3), U\}$, where $\mathcal{C}(Q_2) \neq \mathcal{C}(Q_3)$ and $U = \mathcal{C}(Q_2) \cap \mathcal{C}(Q_3)$.

Lemma 4.15. *Let L be a line in $\mathrm{PG}(5,q)$ intersecting the nucleus plane trivially and spanned by two rank-2 points R and S, where $\mathcal{C}(R) \neq \mathcal{C}(S)$. Then, $L \in \{o_{13,2}, o_{14}\}$. Furthermore, $L \in o_{13,2}$ if and only if $R \in T_V(\mathcal{C}(R))$ and $S \notin T_V(\mathcal{C}(S))$, and $L \in o_{14}$ if and only if $R \notin T_V(\mathcal{C}(R))$ and $S \notin T_V(\mathcal{C}(S))$, where $V = \mathcal{C}(R) \cap \mathcal{C}(S)$. In particular, if $L \in o_{14}$, then the preimage under the Veronese embedding of the three conics associated with rank-2 points on L define a triangle in $\mathrm{PG}(2,q)$.*

Proof. The hyperplane spanned by $\mathcal{C}(R)$ and $\mathcal{C}(S)$ intersects $\mathcal{V}(\mathbb{F}_q)$ in $\mathcal{C}(R) \cup \mathcal{C}(S)$, and thus L has no rank-1 points. It follows that $L \in \{o_{10}, o_{13,2}, o_{14}\}$ by Table 2.2. Since a line of type o_{10} lies in a conic plane and $\mathcal{C}(R) \neq \mathcal{C}(S)$, we conclude that $L \in \{o_{13,2}, o_{14}\}$. Let $L_{13,2}$ and L_{14} be the representatives of $o_{13,2}$ and o_{14} in (Lavrauw & Popiel, 2020, Table 2). The line $L_{13,2}$ has two rank-2 points with homogeneous coordinates $\{(0,1,0,1,0,0), (0,0,0,1,0,1)\}$ and the line L_{14} has three rank-2 points defined as $\{P_1(1,0,0,1,0,0), P_2(0,0,0,1,0,1), P_3(1,0,0,0,0,1)\}$. By a direct computation, we have $(0,1,0,1,0,0) \in T_V(\mathcal{C}((0,1,0,1,0,0)))$ and $(0,0,0,1,0,1) \notin T_V(\mathcal{C}((0,0,0,1,0,1)))$ where $V = \nu(e_2)$. A similar computation shows that the three conics associated with P_i, $1 \leq i \leq 3$;

$$\begin{aligned}
\mathcal{C}(P_1) &= \mathcal{Z}(Y_0Y_3 + Y_1^2, Y_2, Y_4, Y_5), \\
\mathcal{C}(P_2) &= \mathcal{Z}(Y_3Y_5 + Y_4^2, Y_0, Y_1, Y_2), \\
\mathcal{C}(P_3) &= \mathcal{Z}(Y_0Y_5 + Y_2^2, Y_1, Y_3, Y_4);
\end{aligned}$$

intersect pairwise in $V \in \{U_{12}(0,0,0,1,0,0), U_{13}(1,0,0,0,0,0), U_{23}(0,0,0,0,0,1)\}$, where each pair (P_i, P_j), $i < j$, has both of its points not lying on the tangent of their conics through U_{ij}. □

Remark 4.2. *By inspecting point-orbit distributions of lines in $\mathrm{PG}(5,q)$, we can see that $\langle Q_1, Q_i \rangle \in o_{8,1}$ for $i = 2,3$. Moreover, Lemma 4.15 implies that $\langle Q_2, Q_3 \rangle \in \{o_{13,2}, o_{14}\}$, where: $\langle Q_2, Q_3 \rangle \in o_{13,2}$ if and only if $Q_2 \in T_U(\mathcal{C}(Q_2))$ and $Q_3 \notin T_U(\mathcal{C}(Q_3))$, and $\langle Q_2, Q_3 \rangle \in o_{14}$ if and only if $Q_2 \notin T_U(\mathcal{C}(Q_2))$ and $Q_3 \notin T_U(\mathcal{C}(Q_3))$.*

Next, we consider the two possibilities where π can be represented by $\langle Q_1, Q_2, Q_3\rangle$ where the unique rank-1 point Q_1 is not lying on $\mathcal{C}(Q_2)\cup\mathcal{C}(Q_3)$ such that: (*d-ii-A*) $\langle Q_2, Q_3\rangle \in o_{13,2}$ or (*d-ii-B*) $\langle Q_2, Q_3\rangle \notin o_{13,2}$, i.e, π has no line of type $o_{13,2}$ and $\langle Q_2, Q_3\rangle \in o_{14}$ by Lemma 4.15.

(d−ii−A) Let $\pi = \langle Q_1, Q_2, Q_3\rangle$ where the unique rank-1 point Q_1 is not lying on $\mathcal{C}(Q_2)\cup\mathcal{C}(Q_3)$, $\pi\cap\mathcal{N} = \emptyset$ and $\langle Q_2, Q_3\rangle \in o_{13,2}$. Without loss of generality, take $\langle Q_2, Q_3\rangle$ as the representative of $o_{13,2}$ in (Lavrauw & Popiel, 2020, Table 2), and let Q_1 be a point with homogeneous coordinates $\nu(a,b,c)$. As $L_i = \langle Q_1, Q_i\rangle \in o_{8,1}$ for $i = 2,3$, it follows that L_i has a unique rank-1 point and a unique rank-2 point not contained in the nucleus plane, and thus $a, c \neq 0$. Therefore, π can be represented by

$$\pi_{b,c} : \begin{bmatrix} x & bx+y & cx \\ bx+y & b^2x+y+z & bcx \\ cx & bcx & c^2x+z \end{bmatrix},$$

which is K-equivalent to

$$\pi_c : \begin{bmatrix} x & y & cx \\ y & y+z & . \\ cx & . & c^2x+z \end{bmatrix}$$

for the choice of X as $\begin{bmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ with $X\pi_c X^T = \pi_{b,c}$.

Before proceeding with the study of planes of the form π_c, $c \neq 0$, recall the definition and the characterisation of inflexion points in Definition 2.2 and Lemma 2.1. Note that, for fields of characteristic different from two, inflexion points are points of the intersection of the cubic with the classical Hessian (the determinant of the 3×3 matrix of second derivatives), which is zero over fields of characteristic 2. For further details about inflexion points over characteristic two finite fields, we refer to (Glynn, 1998).

Lemma 4.16. *A plane π_c with $c \neq 0$ has*

- *three inflexion points if and only if $q \neq 4$, $Tr(c) = Tr(1)$ and c^{-1} is admissible.*
- *one inflexion point if and only if $Tr(c) \neq Tr(1)$.*
- *no inflexion points if and only if $Tr(c) = Tr(1)$ and c^{-1} is not admissible.*

Proof. Let $C = \mathcal{Z}(f)$ be the cubic curve associated with π_c defined by $f = X(Z^2 + YZ + c^2Y^2) + Y^2Z$. By Lemma 2.1, inflexion points of C correspond to nonsingular points of $C \cap C''$, where $C'' = \mathcal{Z}(f'')$ and $f'' = X(Z^2+YZ+c^2Y^2)+Z^3+(1+c^2)Y^2Z+c^2Y^3$. The points in $C \cap C''$ therefore satisfy the equation:

$$Z^3+c^2Y^2Z+c^2Y^3 = 0. \tag{4.2}$$

The affine points $(X,1,Z)$ in $\pi_c \setminus \mathcal{Z}(Y)$ satisfy

$$Z^3+c^2Z+c^2 = 0. \tag{4.3}$$

Let $\theta = c^{-1}Z$, then inflexion points of C correspond to solutions of

$$\theta^3+\theta+c^{-1} = 0, \tag{4.4}$$

where (4.4) has three solutions if and only if $q \neq 4$, $Tr(c) = Tr(1)$ and c^{-1} is admissible, a unique solution if and only if $Tr(c) \neq Tr(1)$, and no solution if and only if $Tr(c) = Tr(1)$ and c^{-1} is not admissible (see Lemma 2.19). □

Lemma 4.17. *Let $q = 2^h$, $h > 1$. There exist c_0 and c_1 in $\mathbb{F}_q \setminus \{0\}$ such that π_{c_0} has no inflexion points and π_{c_1} has a unique inflexion point. Moreover, if $h > 2$, then there exists $c_3 \in \mathbb{F}_q \setminus \{0\}$ such that π_{c_3} has three inflexion points.*

Proof. This is a consequence of having exactly $\lfloor \frac{q-2}{6} \rfloor$ admissible scalars in $\mathbb{F}_q \setminus \{0\}$, $q \neq 4$ (Berlekamp, Rumsey & Solomon, 1966, Lemma 1), and by noting that Tr is a $\frac{q}{2}$-to-1 map. □

Remark 4.3. *Let $q = 2^h$, $h > 1$. Lemma 4.17 implies the existence of at least three K-orbits of planes of the form π_c, when $h > 2$, and at least two K-orbits of planes of the form π_c, when $h = 2$. In particular, we denote by*

- *Σ_{12} the union of K-orbits of planes represented by π_c where $Tr(c) = 1$ and c^{-1} is not admissible if h is odd.*
- *Σ_{13} the union of K-orbits of planes represented by π_c where $Tr(c) = 0$ and c^{-1} is not admissible if h is even.*
- *Σ_{14} the union of K-orbits of planes represented by π_c where $h > 2$, $Tr(c) = Tr(1)$ and c^{-1} is admissible.*

Lemma 4.18. *For $q = 2^h > 4$, inflexion points of planes in Σ_{14} are collinear. Furthermore, there exists a one-to-one correspondence between planes in Σ_{14} and lines in o_{14} being their inflexion lines.*

Proof. Consider π_c as the plane defined in Section 4.3.4.2, where c is an admissible scalar in $\mathbb{F}_q \setminus \{0\}$. By Lemma 1 in (Berlekamp, Rumsey & Solomon, 1966), inflexion points of π_c are the points parametrized by $(\frac{z_i}{z_i^2+z_i+c^2}, 1, z_i)$ where

$$z_1 = (1+v+v^{-1})^2,\ z_2 = \frac{v(1+v+v^{-1})^2}{v+v^{-1}},\ z_3 = \frac{v^{-1}(1+v+v^{-1})^2}{v+v^{-1}},$$

and $v \in \mathbb{F}_q \setminus \mathbb{F}_4$. In particular, these points are collinear lying on the line L_v with parametrized dual coordinates

$$[(v+v^{-1})(1+v+v^{-1})^2, (v+v^{-1})(1+v+v^{-1})^2, \frac{(v+v^{-1})^2+(v+v^{-1})^4+(v^3+v^{-3})^2}{(v+v^{-1})^4}].$$

We call L_v an *inflexion line*. As rank-2 points in planes in Σ_{14} define distinct conic planes, it follows by Table 2.2 that $L_v \in o_{14}$. We prove next that no two planes in Σ_{14} have the same inflexion line L. Without loss of generality, we may start by fixing L as the representative of o_{14} in (Lavrauw & Popiel, 2020, Table 2). More precisely, let $E_1 = (1,0,0,1,0,0)$, $E_2 = (0,0,0,1,0,1)$, and $E_3 = (1,0,0,0,0,1)$ be the three inflexion points on L parametrised by $(0,1,0)$, $(0,0,1)$ and $(0,1,1)$ respectively, and consider $Q_{a,b,c} = \nu(a,b,c)$ as a point on $\mathcal{V}(\mathbb{F}_q)$. If $\pi_{a,b,c} = \langle L, Q_{a,b,c}\rangle$ is a plane of type Σ_{14}, then $\langle Q_{a,b,c}, E_i\rangle \in o_{8,1}$, $1 \le i \le 3$. This implies that $a,b,c \neq 0$. Therefore, we may assume without loss of generality that $a = 1$, $Q_{a,b,c} = Q_{b,c}$ and $\pi_{a,b,c} = \pi_{b,c}$, where

$$\pi_{b,c} = \begin{bmatrix} x+y & bx & cx \\ bx & b^2x+y+z & bcx \\ cx & bcx & c^2x+z \end{bmatrix}.$$

The cubic curve $C_{b,c}$ associated with $\pi_{b,c}$ is defined by

$$XZ^2 + c^2XY^2 + Y^2Z + YZ^2 + (1+b^2+c^2)XYZ = 0. \tag{4.5}$$

If $1+b+c = 0$, then $\pi_{b,c}$ intersects the nucleus plane $\mathcal{N}$ in a unique point parametrised by $(1,1,1+b^2)$, a contradiction as planes in Σ_{14} have no intersection with the nucleus plane. Therefore, we may assume that $1+b+c \neq 0$. By Lemma 2.1, inflexion points of $\pi_{b,c}$ are nonsingular points of $C_{b,c} \cap C''_{b,c}$, where $C''_{b,c} = \mathcal{Z}(h_{b,c})$, $\alpha = (1+b^2+c^2)$ and

$$\begin{aligned} h_{b,c} =& c^2\alpha^5XY^2 + \alpha^5XZ^2 + c^2(1+b^2)\alpha Y^3 + \alpha((1+b^2)+\alpha^3(b^2+c^2))YZ^2 + \\ & \alpha(c^2(b^2+c^2)+\alpha^3(1+b^2))Y^2Z + (b^2+c^2)\alpha Z^3. \end{aligned} \tag{4.6}$$

Imposing the conditions: $E_i \in C''_{b,c}$, $1 \le i \le 3$, implies that

$$\begin{aligned}(4.7)\quad & c^2(1+b^2)\alpha+\alpha((1+b^2)+\alpha^3(b^2+c^2))+\alpha(c^2(b^2+c^2)+\alpha^3(1+b^2))+(b^2+c^2)\alpha = \\ & c^2(1+b^2)\alpha = \\ & (b^2+c^2)\alpha = 0.\end{aligned}$$

As $\alpha, c \neq 0$, we get $b = c = 1$. Therefore, every line in o_{14} is the inflexion line of a unique plane in Σ_{14}, and thus we obtain a one-to-one correspondence between the set of planes in Σ_{14} and the set of lines in o_{14} being their inflexion lines. □

Lemma 4.19. *For $q = 2^h > 4$, planes in Σ_{14} define a unique K-orbit.*

Proof. Consider the plane

$$\pi_{1,1} = \begin{bmatrix} x+y & x & x \\ x & x+y+z & x \\ x & x & x+z \end{bmatrix},$$

defined in Lemma 4.18. The stabiliser of $\pi_{1,1}$ in K, denoted by $K_{\pi_{1,1}}$, is the intersection of the two subgroups of K stabilising the unique rank-1 point Q and the inflexion line L, i.e, $K_{\pi_{1,1}} = K_Q \cap K_L$. By (Lavrauw & Popiel, 2020), we have $K_L \cong \mathrm{Sym}_3$, being the group represented by the six 3×3 permutation matrices. Moreover, a matrix $g = (g_{ij}) \in \mathrm{GL}(3,q)$ stabilises Q if and only if $g_{11}+g_{12}+g_{13} = g_{21}+g_{22}+g_{23} = g_{31}+g_{32}+g_{33}$. Therefore, $K_Q \cong E_q^2 \rtimes \mathrm{GL}(2,q)$, and thus $K_{\pi_{1,1}} \cong \mathrm{Sym}_3$. Additionally, as the set Σ_{14} has $|K|/6$ planes by Lemma 4.18, it follows that Σ_{14} is equal to the K-orbit of $\pi_{1,1}$ in $\mathrm{PG}(5,q)$. Therefore, planes in Σ_{14} define a unique K-orbit represented by $\pi_{1,1}$. □

Remark 4.3. *In the next lemmas, the notations $(o_i)_{q^j}$ and $(\Sigma_i)_{q^j}$, $1 \le j \le 3$, are used to denote orbits of lines and planes considered over $\mathbb{F}_q$, $\mathbb{F}_{q^2}$ and $\mathbb{F}_{q^3}$ respectively. Furthermore, if L and π are a line and a plane in $\mathrm{PG}(5,q)$, then we denote by $L(\mathbb{F}_{q^s})$ and $\pi(\mathbb{F}_{q^s})$, $s \in \{2,3\}$, their extensions over $\mathbb{F}_{q^2}$ and $\mathbb{F}_{q^3}$ respectively.*

Lemma 4.20. *For $q = 2^h > 2$, where h is even, planes in Σ_{12} define a unique K-orbit with one inflexion point, and planes in Σ_{13} define a unique K-orbit with no inflexion points. Furthermore, there exists a one-to-one correspondence between planes in Σ_{12} (resp. Σ_{13}) and lines in o_{15} (resp. o_{17}).*

Proof. The uniqueness of $(\Sigma_{12})_q$ and $(\Sigma_{13})_q$ can be deduced from the uniqueness of $(\Sigma_{14})_q$ by expanding to the quadratic and the cubic extensions of $\mathbb{F}_q$. Recall that

$(\Sigma_{12})_q$ and $(\Sigma_{13})_q$ are the union of orbits represented by π_{c_1} and π_{c_0} respectively, where $c_1, c_0 \neq 0$, $Tr(c_1) = 1$ and $Tr(c_0) = 0$. If h is even, then $(\Sigma_{12})_q$ has a unique inflexion point while $(\Sigma_{13})_q$ has none. By expanding π_{c_1} to $\mathbb{F}_{q^2}$ and π_{c_0} to $\mathbb{F}_{q^3}$, we obtain two planes $\pi_{c_1}(\mathbb{F}_{q^2}) \subset \mathrm{PG}(5,q^2)$ and $\pi_{c_0}(\mathbb{F}_{q^3}) \subset \mathrm{PG}(5,q^3)$ of type $(\Sigma_{14})_{q^s}$, $s \in \{2,3\}$, where each plane is uniquely determined by an inflexion line of type $(o_{14})_{q^s}$, $s \in \{2,3\}$, say $L_1(\mathbb{F}_{q^2})$ and $L_0(\mathbb{F}_{q^3})$. Let σ_1 (resp. σ_0) be the Frobenius collineation of $\mathrm{PG}(5,q^2)$ (resp. $\mathrm{PG}(5,q^3)$) induced by the automorphism $x \mapsto x^q$ of $\mathbb{F}_{q^2}$ (resp. $\mathbb{F}_{q^3}$). As $\pi_{c_1}(\mathbb{F}_{q^2})$ has a unique $\mathbb{F}_q$-rational and two $\mathbb{F}_{q^2}$-conjugate inflexion points, while $\pi_{c_0}(\mathbb{F}_{q^3})$ has three $\mathbb{F}_{q^3}$-conjugate inflexion points, it follows that $L_1 = \pi_{c_1} \cap L_1(\mathbb{F}_{q^2}) \in (o_{15})_q$ and $L_0 = \pi_{c_0} \cap L_0(\mathbb{F}_{q^3}) \in (o_{17})_q$. Note that, L_1 cannot be of type $o_{16,2}$ as the representative of this orbit in (Lavrauw & Popiel, 2020, Table 2) is spanned by $(0,0,1,1,0,0)$ and $(0,0,0,0,1,1)$, which generate a line of type $o_{16,2}$ over $\mathbb{F}_{q^2}$. Therefore, L_1 and L_0 are uniquely determined in π_{c_1} and π_{c_0} respectively. Moreover, these lines uniquely determine the planes π_{c_1} and π_{c_0} as their extension define a unique inflexion line in $(o_{14})_{q^s}$, $s \in \{2,3\}$. Hence, there exists a one-to-one correspondence between planes in $(\Sigma_{12})_q$ (resp. $(\Sigma_{13})_q$) and lines in $(o_{15})_q$ (resp. $(o_{17})_q$). This yields to $|K|/2$ planes in $(\Sigma_{12})_q$ and $|K|/3$ planes in $(\Sigma_{13})_q$ by (Lavrauw & Popiel, 2020). On the other hand, let $K_{\pi_{c_i}}$ and K_{L_i} be the stabilisers in K of π_{c_i} and L_i respectively, $i \in \{0,1\}$, and consider K_Q as the stabiliser of the unique rank-1 point Q defined in Lemma 4.19. Then, $K_{\pi_{c_i}} = K_{L_i} \cap K_Q$, $i \in \{0,1\}$. Indeed, the description of stabilisers of lines of types $(o_{15})_q$ and $(o_{17})_q$ from (Lavrauw & Popiel, 2020), implies that $K_{\pi_{c_1}} \cong C_2$ and $K_{\pi_{c_0}} \cong C_3$. Therefore, each of Σ_{12} and Σ_{13} defines a unique K-orbit over $\mathbb{F}_{2^h}$, h is even. □

Lemma 4.21. *For $q = 2^h > 2$, where h is odd, planes in Σ_{12} define a unique K-orbit with no inflexion points, and planes in Σ_{13} define a unique K-orbit with one inflexion point. Furthermore, there exists a one-to-one correspondence between planes in Σ_{12} (resp. Σ_{13}) and lines in o_{17} (resp. o_{15}).*

Proof. Follows by a similar proof to that of Lemma 4.20. □

Lemma 4.22. *Point-orbit distributions of planes in Σ_{12}, Σ_{13} and Σ_{14} are given by $[1,0,q+1,q^2-1]$, $[1,0,q-1,q^2+1]$ and $[1,0,q\mp1,q^2\pm1]$, respectively. In particular, these orbits are distinct from each other and from the previously defined orbits Σ_i, $1 \leq i \leq 11$.*

Proof. Consider the cubic curve associated with Σ_i, $i \in \{12,13,14\}$, defined by

$$X(Z^2 + YZ + c^2Y^2) + Y^2Z = 0. \tag{4.8}$$

If $Tr(c) = 1$ (resp. $Tr(c) = 0$), then Σ_{12} (resp. Σ_{13}) has $q+1$ (resp. q) rank-2 points parametrized by

$$(\frac{y^2 z}{z^2 + yz + c^2 y^2}, y, z). \tag{4.9}$$

Furthermore, depending on the $Tr(c)$, being 0 if h is even or 1 if h is odd, the orbit Σ_{14} has either $q-1$ or $q+1$ rank-2 points. Therefore, point-orbit distributions of planes in Σ_{12}, Σ_{13} and Σ_{14} are given by $[1, 0, q+1, q^2-1]$, $[1, 0, q-1, q^2+1]$ and $[1, 0, q \mp 1, q^2 \pm 1]$ respectively. Note that, the orbits $\{\Sigma_{12}, \Sigma_{13}, \Sigma_{14}\}$ are distinct from each other by their inflexion points (see Lemma 4.17 and Remark 4.3), and from the previously defined orbits by their point-orbit distributions. Indeed, $\Sigma_6 \notin \{\Sigma_{12}, \Sigma_{14}\}$ as some rank-2 points of Σ_6 define the same conic plane, while all rank-2 points of Σ_{12} and Σ_{14} define distinct conic planes. $\square$

(d–ii–B) Finally, assume that $\pi = \langle Q_1, Q_2, Q_3 \rangle$, where $\pi \cap \mathcal{N} = \emptyset$, Q_1 is not lying on $\mathcal{C}(Q_2) \cup \mathcal{C}(Q_3)$ and Q_2, Q_3 are both not lying on the tangent of their conics through $U = \mathcal{C}(Q_2) \cap \mathcal{C}(Q_3)$. Indeed, provided that $q > 2$, we prove the existence of such planes if and only if $q = 4$. Without loss of generality, let $q_1 = \langle e_1 \rangle$, $u = \langle e_3 \rangle$, $l_2 = \mathcal{Z}(X_0)$ and $l_3 = \mathcal{Z}(X_0 + X_1)$, where (X_0, X_1, X_2) are the homogeneous coordinates in π, $Q_1 = \nu(q_1)$, $U = \nu(u)$, $\mathcal{C}(Q_2) = \nu(l_2)$ and $\mathcal{C}(Q_3) = \nu(l_3)$. Furthermore, let $R_2 = \nu(r_2)$ and $R_3 = \nu(r_3)$ denote $\mathcal{C}(Q_2) \cap \langle U, Q_2 \rangle$ and $\mathcal{C}(Q_3) \cap \langle U, Q_3 \rangle$ respectively. We have two possibilities, either $r_3 = \langle q_1, r_2 \rangle \cap l_3$ or $r_3 \neq \langle q_1, r_2 \rangle \cap l_3$. In the first case, we may fix r_3 as $e_1 + e_2$, and thus π can be represented by

$$\begin{bmatrix} x+z & z & \cdot \\ z & y+z & \cdot \\ \cdot & \cdot & by+cz \end{bmatrix},$$

for some $b, c \in \mathbb{F}_q$, where $Q_2 = (0,0,0,1,0,b)$ and $Q_3 = (1,1,0,1,0,c)$. This case will not define a new orbit, since for any point (x, y, z) on the line $\mathcal{Z}(bY + cZ)$, we obtain $Q \in \langle Q_2, Q_3 \rangle$ where $Q_1 \in \mathcal{C}(Q)$, returning us to Case 4.3.3. Therefore, we may assume without loss of generality that $r_3 \neq \langle q_1, r_2 \rangle \cap l_3$. Let $r_3 = \langle e_1 + e_2 + e_3 \rangle$. Then, $Q_2 = (0,0,0,1,0,b)$ and $Q_3 = (1,1,1,1,1,c)$ for some b, c in $\mathbb{F}_q$. It follows that π can be represented by

$$\pi_{b,c} = \begin{bmatrix} x+z & z & z \\ z & y+z & z \\ z & z & by+cz \end{bmatrix},$$

where $b(c-1) \neq 0$ as the rank of Q_2, Q_3 is 2.

Lemma 4.23. *If* $\pi_{b,c} \notin \Sigma_i$, $1 \le i \le 14$ *and* $b(c-1) \neq 0$, *then* $\pi_{b,c}$ *has* $q \pm 1$ *rank-2 points.*

Proof. The cubic curve $C_{b,c}$ associated with $\pi_{b,c}$ is defined by

$$Xf_{b,c}(Y,Z) + g_{b,c}(Y,Z) = 0, \tag{4.10}$$

where

$$f_{b,c}(Y,Z) = bY^2 + (b+c)YZ + (1+c)Z^2, \tag{4.11}$$

and

$$g_{b,c}(Y,Z) = bY^2Z + (1+c)YZ^2, \tag{4.12}$$

and thus $\pi_{b,c}$ has $q+1$, q or $q-1$ rank-2 points depending on points of $\mathcal{Z}(f_{b,c})$ on $\mathrm{PG}(1,q)$ being zero, one or two respectively. By Remark 4.2, any line in $\pi_{b,c}$ passing through two rank-2 points must belong to o_{14}. Therefore, fixing a rank-2 point $Q \in \pi_{b,c}$ and considering all lines spanned by Q and the remaining rank-2 points in $\pi_{b,c}$ gives a pair partition of the set of rank-2 points in $\pi_{b,c} \setminus \{Q\}$. Hence, the number of rank-2 points in $\pi_{b,c}$ is odd. More precisely, $\pi_{b,c}$ has $q+1$ rank-2 points if $\mathcal{Z}(f_{b,c})$ has no points in $\mathrm{PG}(1,q)$ and $q-1$ rank-2 points if $\mathcal{Z}(f_{b,c})$ has two points in $\mathrm{PG}(1,q)$. □

Lemma 4.24. *If* $\pi_{b,c} \notin \Sigma_i$, $1 \le i \le 14$ *and* $b(c-1) \neq 0$, *then* $q = 4$.

Proof. By Lemma 4.23, the cubic curve associated with $\pi_{b,c}$ is defined by

$$Xf_{b,c}(Y,Z) + g_{b,c}(Y,Z) = 0, \tag{4.13}$$

where $f_{b,c}$ and $g_{b,c}$ are as defined in (4.11) and (4.12) respectively. As $f_{b,c}$ has 0 or 2 points on $\mathrm{PG}(1,q)$, it follows that $b \neq c$. Consider the line L of $\pi_{b,c}$ parametrized by (x,y,z) where $x = z$. Since Q_1 is not lying on L, it follows that L is of type o_{14}, o_{15} or $o_{16,2}$, by Remark 4.2 and Table 2.2. More specifically, L has either 3 points of rank 2 or a unique point of rank 2. In particular, rank-2 points on L satisfy the equation

$$X^2((1+c)X + (1+b)b) = 0, \tag{4.14}$$

which has exactly two solutions unless $b = 1$. Similarly, we can consider the line L' parametrized by (x,y,z) where $y = z$. This line has no rank-1 points and has exactly two rank-2 points satisfying

$$X^2(X+cY)=0, \tag{4.15}$$

unless $c=0$. Therefore, $b=1$, $c=0$ and π reduces to $\pi_{1,0}$, which has $q+1$ rank-2 points if n is odd and $q-1$ rank-2 points if n is even. By Lemma 2.1, the Hessian of $C_{1,0}$ defined by

$$\mathcal{Z}(X(Y^2+YZ+Z^2)+Y^3+YZ^2+ZY^2), \tag{4.16}$$

intersects $C_{1,0}$ in three collinear points lying on the line L'' parametrized by (x,y,z); $x=y$. Since $Q_1 \notin L''$ and the configuration of $\pi_{1,0}$ coincides with the second configuration of Σ_{14} described in Lemma 4.18, it follows that for $q>4$, $\pi_{1,0} \in \Sigma_{14}$. Therefore, $\pi_{1,0}$ defines a new orbit if and only if $q=4$. □

We denote this orbit by Σ'_{14} which can be represented by

$$\Sigma'_{14} : \begin{bmatrix} x+z & z & z \\ z & y+z & z \\ z & z & y \end{bmatrix}.$$

Remark 4.4. *Lemma 4.24 shows that every plane $\pi=\langle Q_1,Q_2,Q_3\rangle$ in* $\mathrm{PG}(5,q)$*, $q=2^h>4$, containing a unique rank-1 point Q_1, where $Q_1 \notin \mathcal{C}(Q_2)\cup\mathcal{C}(Q_3)$ and $\pi\cap\mathcal{N}=\emptyset$, must belong to $\{\Sigma_{12},\Sigma_{13},\Sigma_{14}\}$.*

Lemma 4.25. *The point-orbit distribution of a plane in Σ'_{14} is* $[1,0,3,17]$*. In particular, $\Sigma'_{14} \notin \{\Sigma_1,\Sigma_2,\Sigma_3,\Sigma_4,\Sigma_5,\Sigma_6,\Sigma_7,\Sigma_8,\Sigma_9,\Sigma_{10},\Sigma_{12},\Sigma_{13}\}$.*

Proof. The first part is treated in the proof of Lemma 4.24. The second part follows from the difference of point-orbit distributions between Σ'_{14} and Σ_i; $1\le i\le 12$ and the property that Σ'_{14} has three inflexion points by Lemma 4.24 while Σ_{13} has none (see Lemma 4.17 and Remark 4.3). □

4.4 Planes containing one rank-1 point and not spanned by points of

rank at most 2

Let π be a plane containing a unique point Q_1 of $\mathcal{V}(\mathbb{F}_q)$ and not spanned by points of rank at most 2. Then, all rank-2 points in π through Q_1 must lie on a unique line (such points exist by Lemma 4.1) and each of the remaining q lines through Q_1 must have q rank-3 points, and thus belongs to the line-orbit o_9 by (Lavrauw & Popiel, 2020). Without loss of generality, let $\pi = \langle Q_1, Q_2, Q_3\rangle$ where $\langle Q_1, Q_3\rangle$ is the representative of o_9 in (Lavrauw & Popiel, 2020, Table 2). In particular, take $Q_1(1,0,0,0,0,0)$, $Q_3(0,0,1,1,0,0)$ and $Q_2(0,1,0,a,b,c)$ for some $a,b,c \in \mathbb{F}_q$, then π can be represented by

$$\begin{bmatrix} x & y & z \\ y & ay+z & by \\ z & by & cy \end{bmatrix}.$$

Since points of rank at most two in π lie on a line, it follows that the cubic curve $C = \mathcal{Z}(X(b^2Y^2 + acY^2 + cYZ) + aYZ^2 + cY^3 + Z^3)$ associated with π is a triple line. Hence, $a = b = c = 0$ and the equation of C reduces to $Z^3 = 0$. This gives a unique orbit of planes intersecting the Veronese surface in one point and not spanned by points of rank at most two. We denote this orbit by Σ_{15}, which can be represented by

$$\Sigma_{15} : \begin{bmatrix} x & y & z \\ y & z & \cdot \\ z & \cdot & \cdot \end{bmatrix}.$$

Lemma 4.26. *The point-orbit distribution of a plane in Σ_{15} is $[1,1,q-1,q^2]$. In particular, $\Sigma_{15} \neq \Sigma_i$ for $1 \leq i \leq 14$.*

Proof. Clearly, $\Sigma_{15} \neq \Sigma_i$, for all $1 \leq i \leq 14$, as Σ_{15} is not spanned by points of rank at most 2. Let π_{15} be the above representative of Σ_{15}. Points of rank at most 2 in π_{15} correspond to points on the line $\langle Q_1, Q_2\rangle$, where only the point with homogeneous coordinates $(0,1,0,0,0,0)$ is contained in the nucleus plane which intersects π_{15} in $\mathcal{Z}(X,Z)$. Therefore, the point-orbit distribution of a plane in Σ_{15} is $[1,1,q-1,q^2]$. □

4.5 Planes in PG(5, 2)

Table 4.1 is not completely correct under the action of $\mathrm{PGL}(3,2)$. In particular, the orbits Σ_1,...,Σ_{12} can be obtained analogously. However, the orbit Σ_{13} does not exist for $q=2$. Furthermore, Σ'_{14} can no be longer obtained by considering the span of a rank-1 point and a line of type o_{14} as described in Section 4.4 as no such line exists in this case. More interestingly, planes meeting the Veronese surface non-trivially and not spanned by points of rank at most 2 split under the action of $\mathrm{PGL}(3,2)$ into Σ_{15} and Σ'_{15} which is represented by

$$\Sigma'_{15}: \begin{bmatrix} x & y & z \\ y & z & . \\ z & . & y \end{bmatrix}.$$

Remark 4.4. *Over the field of two elements, the full setwise stabiliser of the Veronese surface is Sym*$_7$ *(see Remark 2.11) which strictly contains* $\mathrm{PGL}(3,2)$ *and does not preserve the nucleus plane. Therefore, under this action the number of orbits reduces to* 5*. Precisely, we have* $\Sigma_1=\Sigma_2$, $\Sigma_3=\Sigma_4=\Sigma_5$, $\Sigma_6=\Sigma_{10}$, $\Sigma_7=\Sigma_9=\Sigma_{12}$ *and* $\Sigma_8=\Sigma_{11}=\Sigma'_{14}=\Sigma_{15}=\Sigma'_{15}$, *which is easy to check by hand computations or by using the FinInG package in GAP (Bamberg, Betten, Cara, De Beule, Lavrauw & Neunhöffe, 2018; GAP, 2021).*

Theorem 4.2. *There are* 5 *J-orbits of planes meeting $\mathcal{V}(\mathbb{F}_2)$ is at least one point, where $J\cong$ Sym$_7$ is the group stabilising $\mathcal{V}(\mathbb{F}_2)$. In particular, these orbits split under the action of* $\mathrm{PGL}(3,2)$ *into* 15 *orbits as described in Remark 4.4.*

4.6 Comparison with the q odd case

Over finite fields of odd characteristic, there exists a polarity of $\mathrm{PG}(5,q)$ that maps the set of conic planes of $\mathcal{V}(\mathbb{F}_q)$ onto the set of tangent planes of $\mathcal{V}(\mathbb{F}_q)$. This is Theorem 4.25. in (Hirschfeld & Thas, 1991), which allows the correspondence between rank-1 nets of conics in $\mathrm{PG}(2,q)$, namely, nets with at least one double line, and planes in $\mathrm{PG}(5,q)$ meeting $\mathcal{V}(\mathbb{F}_q)$ in at least one point, q odd. This correspondence fails over finite fields of characteristic 2. For instance, let π_6 be the representative of Σ_6 defined in Table 4.1. Then, π_6 meets $\mathcal{V}(\mathbb{F}_q)$ in a unique point, however its associated net of conics $\mathcal{N}_6$ defined by

$$\alpha X_0X_1+\beta X_0X_2+\gamma(X_1^2+cX_1X_2+X_2^2)=0 \tag{4.17}$$

has by Lemma 2.2 $q+1$ pairs of real lines defined by the pencil of type Ω_4

$$\mathcal{Z}(X_0X_1, X_0X_2),$$

and a unique pair of conjugate imaginary lines given by

$$\mathcal{Z}(X_1^2 + cX_1X_2 + X_2^2).$$

Therefore, the hyperplane-orbit distribution of π_6 is $[0, q+1, 1, q^2-1]$, and Σ_6 has no double lines. In other words, $\mathcal{N}_6$ is not a rank-1 net of conics.

Corollary 4.1. *Rank-1 nets of conics in* $\mathrm{PG}(2,q)$ *do not correspond to planes having at least one rank-1 point in* $\mathrm{PG}(5,q)$ *for q even.*

K-orbits of planes	Representatives	Point-OD	Conditions
Σ_1	$\begin{bmatrix} x & y & \cdot \\ y & z & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$	$[q+1,1,q^2-1,0]$	
Σ_2	$\begin{bmatrix} x & \cdot & \cdot \\ \cdot & y & \cdot \\ \cdot & \cdot & z \end{bmatrix}$	$[3,0,3q-3,q^2-2q+1]$	
Σ_3	$\begin{bmatrix} x & \cdot & z \\ \cdot & y & \cdot \\ z & \cdot & \cdot \end{bmatrix}$	$[2,1,2q-2,q^2-q]$	
Σ_4	$\begin{bmatrix} x & \cdot & z \\ \cdot & y & z \\ z & z & \cdot \end{bmatrix}$	$[2,1,2q-2,q^2-q]$	
Σ_5	$\begin{bmatrix} x & \cdot & z \\ \cdot & y & z \\ z & z & z \end{bmatrix}$	$[2,0,2q-2,q^2-q+1]$	
Σ_6	$\begin{bmatrix} x & \cdot & \cdot \\ \cdot & y+cz & z \\ \cdot & z & y \end{bmatrix}$	$[1,0,q+1,q^2-1]$	$Tr(c^{-1})=1$
Σ_7	$\begin{bmatrix} x & y & z \\ y & \cdot & \cdot \\ z & \cdot & \cdot \end{bmatrix}$	$[1,q+1,q^2-1,0]$	
Σ_8	$\begin{bmatrix} x & y & \cdot \\ y & \cdot & z \\ \cdot & z & \cdot \end{bmatrix}$	$[1,q+1,q-1,q^2-q]$	
Σ_9	$\begin{bmatrix} x & y & \cdot \\ y & z & z \\ \cdot & z & \cdot \end{bmatrix}$	$[1,1,2q-1,q^2-q]$	
Σ_{10}	$\begin{bmatrix} x & y & \cdot \\ y & z & \cdot \\ \cdot & \cdot & z \end{bmatrix}$	$[1,1,2q-1,q^2-q]$	
Σ_{11}	$\begin{bmatrix} x & y & \cdot \\ y & z & z \\ \cdot & z & x+z \end{bmatrix}$	$[1,1,q-1,q^2]$	
Σ_{12}	$\begin{bmatrix} x & y & cx \\ y & y+z & \cdot \\ cx & \cdot & c^2x+z \end{bmatrix}$	$[1,0,q+1,q^2-1]$	$Tr(c)=1$, $(*)$
Σ_{13}	$\begin{bmatrix} x & y & cx \\ y & y+z & \cdot \\ cx & \cdot & c^2x+z \end{bmatrix}$	$[1,0,q-1,q^2+1]$	$Tr(c)=0$, $(**)$
Σ_{14}	$\begin{bmatrix} x & y & cx \\ y & y+z & \cdot \\ cx & \cdot & c^2x+z \end{bmatrix}$	$[1,0,q\mp 1,q^2\pm 1]$	$Tr(c)=Tr(1)$, $q\neq 4$, $(***)$
Σ'_{14}	$\begin{bmatrix} x+z & z & z \\ z & y+z & z \\ z & z & y \end{bmatrix}$	$[1,0,q-1,q^2+1]$	$q=4$
Σ_{15}	$\begin{bmatrix} x & y & z \\ y & z & \cdot \\ z & \cdot & \cdot \end{bmatrix}$	$[1,1,q-1,q^2]$	

Table 4.1 The K-orbits of planes in $\mathrm{PG}(5,q)$ meeting $\mathcal{V}(\mathbb{F}_q)$ in at least one point and their point-orbit distributions, where $q\neq 2$ and c is: $(*)$ not admissible if $q=2^{2m+1}$, $(**)$ not admissible if $q=2^{2m}$ and $(***)$ admissible if $q>4$. The point-orbit distribution in Σ_{14} is given with respect to $q=2^{2m}$ and $q=2^{2m+1}$ respectively.

5 TENSOR RANKS IN $\mathbb{F}_q^2 \otimes \mathbb{F}_q^3 \otimes \mathbb{F}_q^3$

In this chapter, we present our results from (Alnajjarine & Lavrauw, 2020). Particularly, we follow the classification of tensors in $V = \mathbb{F}_q^2 \otimes \mathbb{F}_q^3 \otimes \mathbb{F}_q^3$ under the action of the subgroup of $\mathrm{GL}(V)$ stabilising the set of fundamental tensors in V (Lavrauw & Sheekey, 2015), to define the GAP-package *T233* which determines ranks and orbits of points in $\mathrm{PG}(V) \cong \mathrm{PG}(17, q)$.

This chapter is structured as follows. We begin with an essential proposition that reflects the importance of studying contraction spaces while classifying tensors in $\mathbb{F}_q^m \otimes \mathbb{F}_q^n \otimes \mathbb{F}_q^n$; $m \neq n$. Then, we define in Section 5.1 the role of each function in *T233*. In Section 5.2, we explain the implementation of our main functions and give representatives of the orbits o_{17}, o_{10} and o_{15}. Finally, we end with Section 5.3 by an example illustrating the importance of *T233* while computing tensor ranks in $\mathrm{PG}(17, q)$, especially when q is large. For further details about the terminology used in this chapter we refer to Section 2.6.4. Note that, we consider the problem of determining tensors' orbits and ranks from a projective perspective, where nonzero rank-1 tensors in V correspond to points of the Segre variety $S_{1,2,2}(\mathbb{F}_q)$.

Proposition 5.1. *Let $B \neq C \in \mathbb{F}_q^m \otimes \mathbb{F}_q^n \otimes \mathbb{F}_q^n$; $m \neq n$, $G_1 = \mathrm{GL}(\mathbb{F}_q^n) \wr Sym(m)$ and $G_i = \mathrm{GL}(\mathbb{F}_q^m) \times \mathrm{GL}(\mathbb{F}_q^n)$; $i = 2, 3$, then the following are equivalent:*

- *B is G-equivalent to C.*
- *B_1 is G_1-equivalent to C_1.*
- *B_i is G_i-equivalent to one of $\{C_i, C_i^T\}$, where $i = 2, 3$ and T is the map sending*

$u \otimes v$ to $v \otimes u$ and expanded linearly.

Proof. Combine Lemma 2.1 and Corollary 2.2 in (Lavrauw & Sheekey, 2015). □

5.1 T233 package

T233 is a GAP-package (GAP, 2021) which uses some functionality from the FinInG package (Bamberg, Betten, Cara, De Beule, Lavrauw & Neunhöffe, 2018) to compute ranks and orbits of points in the projective space $\mathrm{PG}(V) \cong \mathrm{PG}(17, q)$. This package is formed of 2 main and 12 auxiliary codes which are described as follows:

- *OrbitOfTensor*: takes a point in $\mathrm{PG}(V)$ and returns its G-orbit and a canonical representative of the orbit.
- *RankOfTensor*: returns the rank of a point in $\mathrm{PG}(V)$ by computing its G-orbit.
- *MatrixOfPoint*: returns a matrix representation of a point in $\mathrm{PG}(V)$.
- *RankOfPoint*: returns the rank of the associated matrix representation of a point in $\mathrm{PG}(V)$.
- *RankDistribution*: computes the rank distribution of a projective subspace.
- *CubicalArrayFromPointInTensorProductSpace*: gives the horizontal slices of a point in $\mathrm{PG}(V)$.
- *ContractionOfPointInTensorProductSpace*: returns the projective contraction of a point in $\mathrm{PG}(V)$.
- *SubspaceOfContractions*: returns the contraction spaces of a point in $\mathrm{PG}(V)$.
- *Rank1PtsOftheContractionSubspace*: returns rank-1 points of the contraction subspaces associated with a point in $\mathrm{PG}(V)$.
- *RepO10odd*: returns a canonical representative of o_{10} when q is odd.
- *AlternativeRepresentationOfFiniteFieldElements*: gives an alternative representation of elements of $\mathbb{F}_q$.
- *RepO10even*: returns a canonical representative of o_{10} when q is even.
- *RepO15odd*: returns a canonical representative of o_{15} when q is odd.

- *RepO15even*: returns a canonical representative of o_{15} when q is even.

For more about the construction of these functions we refer to (Alnajjarine & Lavrauw, 2020).

5.2 Implementation

5.2.1 OrbitOfTensor

The *OrbitOfTensor* function uses information from Table A.1 to determine for an arbitrary tensor B in $\mathrm{PG}(V)$ its orbit and a representative of the orbit. It computes first the rank distribution R_1 and compares it with Table A.1 to specify the orbit containing B. However, sometimes R_1 is not sufficient to distinguish among orbits. For instance, o_6 and o_7 (resp. o_{10}, o_{11} and o_{12}) have the same first rank distribution R_1. In this case, we use properties of the second and third contraction spaces to differentiate among them. By (Lavrauw & Sheekey, 2015), o_4, o_7 and o_{11} are the only G-orbits of tensors which split under the action of $\mathrm{PGL}(2,q) \times \mathrm{PGL}(3,q)$ to o_i and o_i^T. Therefore, using properties of B_2 and B_3 directly from Table A.1 will not be sufficient to distinguish between o_6 and o_7 (resp. o_{10}, o_{11} and o_{12}). For this reason, we use algorithmically some extra possibilities of R_2 and R_3 to insure that if $B \in o_j$ then $B^T \in o_j$, for $j = 7, 11$ (see Proposition 5.1). Notice that, since o_4 is completely determined by R_1, there is no need for a similar work in this case.

Although in most cases the set $\{R_1, R_2, R_3\}$ is sufficient to specify tensors orbits, it is not helpful in distinguishing between o_{15} and o_{16} as they have same rank distributions. In this case, we use Lemma 5.1 to differentiate between them.

Lemma 5.1. *(Alnajjarine & Lavrauw, 2020, Lemma 1.1)*
Let $B \in \mathrm{PG}(V)$ such that $R_1 = [0,1,q]$ and $R_2 = R_3 = [1, q^2+q, 0]$, and denote by x_1 and x_2 a rank-3 point and the unique rank-2 point on the line B_1 respectively (see Table A.1). Then, there exists a unique solid S containing x_2 and intersecting $S_{2,2}(\mathbb{F}_q)$ in a subvariety $\mathcal{Q}(x_2)$ equivalent to the Segre variety $S_{1,1}(\mathbb{F}_q)$. Furthermore, for $U := \langle S, x_1 \rangle$, we have $B \in o_{15}$ if $U \setminus \mathcal{Q}(x_2)$ intersects $S_{2,2}(\mathbb{F}_q)$ nontrivially and $B \in o_{16}$ otherwise.

Proof. Let $x_2 \in \langle y, z\rangle$ where $y \neq z \in S_{2,2}(\mathbb{F}_q)$. If $y = \sigma_{2,2}(y_1 \times y_2)$ and $z = \sigma_{2,2}(z_1 \times z_2)$, then $x_2 \in \langle \mathcal{Q}_{y,z}\rangle$ where $\mathcal{Q}_{y,z} := \sigma_{2,2}(\langle y_1, z_1\rangle \times \langle y_2, z_2\rangle) \cong S_{1,1}(\mathbb{F}_q)$. We then identify $\mathcal{Q}(x_2)$ by $\mathcal{Q}_{y,z}$, whose uniqueness is guaranteed by Lemma 2.4 in (Lavrauw & Sheekey, 2015). Let $S = \langle \mathcal{Q}(x_2)\rangle$ and consider the two possibilities for B to have 2 points y'_i, $i = 1, 2$ of rank i such that x_1 is on the line $\langle y'_1, y'_2\rangle$ and $\mathcal{Q}(x_2) = \mathcal{Q}(y'_2)$ or no such points exist, to conclude that $B \in o_{15}$ or $B \in o_{16}$ respectively (see section 3.2 in (Lavrauw & Sheekey, 2015)). □

For the same reason, we deal with the case $q = 2$ separately. In particular, we can distinguish between o_{10} and $\{o_{12}, o_{14}\}$ by their second rank distribution R_2. But, as o_{12} and o_{14} share the same three rank distributions, we use the geometric description of the second contraction space to differentiate between them. More precisely, the difference between these orbits is that the second contraction space in o_{14} is a plane spanned by its three rank-1 points, however, this is not the case for o_{12} (see Table A.1).

Finally, except for o_{10}, o_{15} and o_{17}, representatives are obtained directly from Table A.1 and are defined by a set of two horizontal slices 2.6.5. For instance, a representative of o_{16} is $e_1 \otimes (e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3) + e_2 \otimes (e_1 \otimes e_2 + e_2 \otimes e_3)$ (see Table A.1) which can be represented by

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right\}.$$

5.2.2 Representative for o_{17}

A representative of o_{17} is given by: $e_1 \otimes (e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3) + e_2 \otimes (e_1 \otimes e_2 + e_2 \otimes e_3 + e_3 \otimes (\alpha e_1 + \beta e_2 + \gamma e_3))$ where $\lambda^3 + \gamma\lambda^2 - \beta\lambda + \alpha \neq 0$ for all λ in $\mathbb{F}_q$. Since determining α, β and γ is computationally infeasible for large q, we give an explicit construction that does not require any computations. First, notice that o_{17} is the unique orbit of lines in the space $\mathrm{PG}(\mathbb{F}_q^3 \otimes \mathbb{F}_q^3)$ consisting entirely of rank-3 points (Lavrauw & Popiel, 2020). Therefore, constructing a line of constant rank 3 is sufficient to obtain the desired representative. For this aim, consider the cubic extension of $\mathbb{F}_q$, $\mathbb{F}_{q^3}$, and define $U = \{M_\theta : \theta \in \mathbb{F}_{q^3}\}$ where M_θ is the matrix representative of the linear operator on $\mathbb{F}_{q^3}$ sending x to θx. Since U is a three dimensional $\mathbb{F}_q$-vector space containing $q^3 - 1$ matrices of rank three, it follows that any two dimensional

$\mathbb{F}_q$-subspace of U, W, can serve as a representative of o_{17}, where basis of W gives us the two horizontal slices. Particularly, let w be a primitive element of the extension $\mathbb{F}_{q^3}$ over $\mathbb{F}_q$ and consider the subspace generated by the identity matrix and the companion matrix of the minimal polynomial of w.

5.2.3 Representatives for o_{10} and o_{15}

The orbits o_{10} and o_{15} can be represented by $e_1 \otimes (e_1 \otimes e_1 + e_2 \otimes e_2 + ue_1 \otimes e_2) + e_2 \otimes (e_1 \otimes e_2 + ve_2 \otimes e_1)$ and $e_1 \otimes (e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3 + ue_1 \otimes e_2) + e_2 \otimes (e_1 \otimes e_2 + ve_2 \otimes e_1)$ respectively, where $u, v \in \mathbb{F}_q \setminus \{0\}$ and $v\lambda^2 + uv\lambda - 1 \neq 0$ for all λ in $\mathbb{F}_q$. Similar to the o_{17} case, we give an explicit construction of a representative of o_{10} which requires no computations. Observe first that o_{10} can be represented by a line of constant two-rank 2×2-matrices, which is external to a conic in $\mathcal{V}(\mathbb{F}_q)$ (Lavrauw & Popiel, 2020). Thus, constructing such a line will be sufficient to represent o_{10}. Before proceeding, recall that interior points of the conic $C = \mathcal{Z}(X_0X_2 - X_1^2)$ correspond to $\{(x, y, z) \in \mathrm{PG}(2, q) : xz - y^2 \notin \square\}$. Therefore, when q is odd, we can compute the image of a primitive root in $\mathbb{F}_q$ under the polarity associated with C to obtain an external line to C in $\mathrm{PG}(2, q)$. This line can be embedded in $\mathrm{PG}(8, q)$ by setting the third rows and columns of its points to zero. A similar argument works for q even. Particularly, we may start with the minimal polynomial of a generator of the group $\mathbb{F}_{q^2} \setminus \{0\}$ to obtain an irreducible quadratic polynomial over $\mathbb{F}_q$, whose coordinates can be viewed as the dual coordinates of a line in $\mathrm{PG}(2, q)$ external to the conic defined by $\{(a^2, ab, b^2) : a, b \in \mathbb{F}_q; (a, b) \neq (0, 0)\}$. We can then embed this line in $\mathrm{PG}(8, q)$ by setting the last columns and rows of its points to zero. Finally, by finding u and v from the obtained representative of o_{10}, we can obtain a representative of o_{15}.

5.3 Computations and summary

Example 5.1.

```
gap> q:=13441;
13441
gap> pg:=AmbientSpace(sv);
```

```
ProjectiveSpace(17, 13441)
gap> sv:=SegreVariety([PG(1,q),PG(2,q),PG(2,q)]);
Segre Variety in ProjectiveSpace(17, 13441)
gap> n:=Size(Points(pg));
152534889213444441555065109181873823540886905868308008700484628729939 38
gap> m:=Size(Points(sv)); # number of rank-1 points in PG(17,q)
43878809925060586561 8
gap> D:=VectorSpaceToElement(pg,[Z(q)^0,Z(q)^336,Z(q)^339,
> Z(q)^37,Z(q)^233,Z(q)^56,Z(q)^268,Z(q)^363,Z(q)^342,
> Z(q)^297,Z(q)^146,Z(q)^71,Z(q)^57,Z(q)^84,Z(q)^33,
> Z(q)^203,Z(q)^229,Z(q)^191]);
<a point in ProjectiveSpace(17, 13441)>
gap> OrbitOfTensor(D); # [orbit, representative]
[ 17,
  [ [ [ Z(13441)^0, 0*Z(13441), 0*Z(13441) ],
    [ 0*Z(13441), Z(13441)^0, 0*Z(13441) ],
    [ 0*Z(13441), 0*Z(13441), Z(13441)^0 ] ],
  [ [ 0*Z(13441), 0*Z(13441), Z(13441) ],
    [ Z(13441)^0, 0*Z(13441), Z(13441)^2008 ],
    [ 0*Z(13441), Z(13441)^0, 0*Z(13441) ] ] ] ]
gap> time; # in ms
2406
gap> RankOfTensor(D);
4
gap> time; # in ms
2438
```

T233 is an efficient tool to compute orbits and ranks of points in $\mathrm{PG}(17,q)$. Without this tool, it is computationally infeasible to find ranks of points in $\mathrm{PG}(17,q)$, especially when q is large. For instance, if we consider the point D in Example 5.1, we can see that its rank was computed within seconds. However, finding this manually requires to check an 82-digit number of possible 4-combinations of rank-1 points which might generate a solid containing D. This reflects how hard it would be to compute ranks of tensors in $\mathrm{PG}(17,q)$ without this algorithm.

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APPENDIX A

	Description and Representative	Tensor's Rank Rank Distributions
o_1	$e_1 \otimes e_1 \otimes e_1$	1
PG(A_1) :	Point on $S_{3,3}$	$[1,0,0]$
PG(A_2) :	Point on $S_{2,3}$	$[1,0,0]$
PG(A_3) :	Point on $S_{2,3}$	$[1,0,0]$
o_2	$e_1 \otimes (e_1 \otimes e_1 + e_2 \otimes e_2)$	2
PG(A_1) :	Point of rank 2	$[0,1,0]$
PG(A_2) :	Line on $S_{2,3}$	$[q+1,0,0]$
PG(A_3) :	Line on $S_{2,3}$	$[q+1,0,0]$
o_3	$e_1 \otimes (e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3)$	3
PG(A_1) :	Point of rank 3	$[0,0,1]$
PG(A_2) :	Plane on $S_{2,3}$	$[q^2+q+1,0,0]$
PG(A_3) :	Plane on $S_{2,3}$	$[q^2+q+1,0,0]$
o_4	$e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_1 \otimes e_2$	2
PG(A_1) :	Line on $S_{3,3}$	$[q+1,0,0]$
PG(A_2) :	Point of rank 2	$[0,1,0]$
PG(A_3) :	Line on $S_{2,3}$	$[q+1,0,0]$
o_5	$e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_2 \otimes e_2$	2
PG(A_1) :	Secant line	$[2,q-1,0]$
PG(A_2) :	Secant line	$[2,q-1,0]$
PG(A_3) :	Secant line	$[2,q-1,0]$
o_6	$e_1 \otimes e_1 \otimes e_1 + e_2 \otimes (e_1 \otimes e_2 + e_2 \otimes e_1)$	3
PG(A_1) :	Tangent line contained in an $< S_{2,2} >$	$[1,q,0]$
PG(A_2) :	Tangent line contained in an $< S_{2,2} >$	$[1,q,0]$
PG(A_3) :	Tangent line contained in an $< S_{2,2} >$	$[1,q,0]$
o_7	$e_1 \otimes e_1 \otimes e_3 + e_2 \otimes (e_1 \otimes e_1 + e_2 \otimes e_2)$	3
PG(A_1) :	Tangent line contained in an $< S_{2,3} >$, not contained in an $< S_{2,2} >$	$[1,q,0]$
PG(A_2) :	Tangent line, not contained in an $< S_{2,2} >$	$[1,q,0]$
PG(A_3) :	Plane containing 2 lines of an $S_{2,2}$	$[2q+1,q^2-q,0]$
o_8	$e_1 \otimes e_1 \otimes e_1 + e_2 \otimes (e_2 \otimes e_2 + e_3 \otimes e_3)$	3
PG(A_1) :	Tangent line not contained in an $< S_{2,3} >$, containing a point of rank 2	$[1,1,q-1]$
PG(A_2) :	Plane containing a line and a point of $S_{2,3}$ not contained in an $< S_{2,2} >$	$[q+2,q^2-1,0]$
PG(A_3) :	Plane containing a line and a point of $S_{2,3}$ not contained in an $< S_{2,2} >$	$[q+2,q^2-1,0]$
o_9	$e_1 \otimes e_3 \otimes e_1 + e_2 \otimes (e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3)$	4
PG(A_1) :	Tangent line not contained in an $< S_{2,3} >$,	$[1,0,q]$

	not containing a point of rank 2	
$\mathrm{PG}(A_2)$:	Plane containing a line of $S_{2,3}$,	$[q+1, q^2, 0]$
	not contained in an $< S_{2,2} >$	
$\mathrm{PG}(A_3)$:	Plane containing a line of $S_{2,3}$	$[q+1, q^2, 0]$
	not contained in an $< S_{2,2} >$	
o_{10}	$e_1 \otimes (e_1 \otimes e_1 + e_2 \otimes e_2 + u e_1 \otimes e_2) + e_2 \otimes (e_1 \otimes e_2 + v e_2 \otimes e_1)$	3
	$v\lambda^2 + uv\lambda - 1 \neq 0$ for all $\lambda \in \mathbb{F}_q$	
$\mathrm{PG}(A_1)$:	Line of constant rank 2, contained in an $< S_{2,2} >$,	$[0, q+1, 0]$
$\mathrm{PG}(A_2)$:	Line of constant rank 2, contained in an $< S_{2,2} >$,	$[0, q+1, 0]$
$\mathrm{PG}(A_3)$:	Line of constant rank 2, contained in an $< S_{2,2} >$,	$[0, q+1, 0]$
o_{11}	$e_1 \otimes (e_1 \otimes e_1 + e_2 \otimes e_2) + e_2 \otimes (e_1 \otimes e_2 + e_2 \otimes e_3)$	3
$\mathrm{PG}(A_1)$:	Line of constant rank 2, contained in an $< S_{2,3} >$,	$[0, q+1, 0]$
	but not in an $< S_{2,2} >$	
$\mathrm{PG}(A_2)$:	Line of constant rank 2, contained in an $< S_{2,3} >$,	$[0, q+1, 0]$
	but not in an $< S_{2,2} >$	
$\mathrm{PG}(A_3)$:	Plane in an $< S_{2,2} >$, meeting in a conic	$[q+1, q^2, 0]$
o_{12}	$e_1 \otimes (e_1 \otimes e_1 + e_2 \otimes e_2) + e_2 \otimes (e_1 \otimes e_3 + e_3 \otimes e_2)$	4
$\mathrm{PG}(A_1)$:	Line of constant rank 2, not contained in an $< S_{2,3} >$,	$[0, q+1, 0]$
$\mathrm{PG}(A_2)$:	Plane containing a line of $S_{2,3}$	$[q+1, q^2, 0]$
$\mathrm{PG}(A_3)$:	Plane containing a line of $S_{2,3}$	$[q+1, q^2, 0]$
o_{13}	$e_1 \otimes (e_1 \otimes e_1 + e_2 \otimes e_2) + e_2 \otimes (e_1 \otimes e_2 + e_3 \otimes e_3)$	4
$\mathrm{PG}(A_1)$:	Line with 2 points of rank 2	$[0, 2, q-1]$
$\mathrm{PG}(A_2)$:	Plane containing 2 points of $S_{2,3}$	$[2, q^2+q-1, 0]$
$\mathrm{PG}(A_3)$:	Plane containing 2 points of $S_{2,3}$	$[2, q^2+q-1, 0]$
o_{14}	$e_1 \otimes (e_1 \otimes e_1 + e_2 \otimes e_2) + e_2 \otimes (e_2 \otimes e_2 + e_3 \otimes e_3)$	3
$\mathrm{PG}(A_1)$:	Line with 3 points of rank 2	$[0, 2, q-1]$
$\mathrm{PG}(A_2)$:	Plane containing 3 points of $S_{2,3}$	$[0, 3, q-2]$
$\mathrm{PG}(A_3)$:	Plane containing 3 points of $S_{2,3}$	$[0, 3, q-2]$
o_{15}	$e_1 \otimes (e + u e_1 \otimes e_2) + e_2 \otimes (e_1 \otimes e_2 + v e_2 \otimes e_1)$;	4
	$v\lambda^2 + uv\lambda - 1 \neq 0$ for all $\lambda \in \mathbb{F}_q$	
	and $e = e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3$.	
$\mathrm{PG}(A_1)$:	Line having one point of rank 2	$[0, 1, q]$
$\mathrm{PG}(A_2)$:	Plane containing one point of $S_{2,3}$	$[1, q^2+q, 0]$
$\mathrm{PG}(A_3)$:	Plane containing one point of $S_{2,3}$	$[1, q^2+q, 0]$
o_{16}	$e_1 \otimes (e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3) + e_2 \otimes (e_1 \otimes e_2 + e_2 \otimes e_3)$	4
$\mathrm{PG}(A_1)$:	Line having one point of rank 2	$[0, 1, q]$
$\mathrm{PG}(A_2)$:	Plane containing one point of $S_{2,3}$	$[1, q^2+q, 0]$
$\mathrm{PG}(A_3)$:	Plane containing one point of $S_{2,3}$	$[1, q^2+q, 0]$
o_{17}	$e_1 \otimes (e) + e_2 \otimes (e_1 \otimes e_2 + e_2 \otimes e_3 + e_3 \otimes (\alpha e_1 + \beta e_2 + \gamma e_3))$;	4 if $q \geq 3$
	$\lambda^3 + \gamma\lambda^2 - \beta\lambda + \alpha \neq 0$ for all $\lambda \in \mathbb{F}_q$	5 if $q = 2$
	and $e = e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3$.	
$\mathrm{PG}(A_1)$:	Line of constant rank 3	$[0, 0, q+1]$
$\mathrm{PG}(A_2)$:	Plane disjoint from $S_{2,3}$	$[0, q^2+q+1, 0]$
$\mathrm{PG}(A_3)$:	Plane disjoint from $S_{2,3}$	$[0, q^2+q+1, 0]$

Table A.1 Projective description and properties of the G-orbits of tensors in V (Lavrauw & Sheekey, 2015).