CONSTRUCTION OF SERIES AS GENERATING FUNCTIONS AND VERIFICATION TYPE PROOFS FOR ROGERS-RAMANUJAN GENERALIZATION FOR PARTITIONS AND OVERPARTITIONS

by

MOHAMMAD ZADEH DABBAGH

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Mohammad Zadeh Dabbagh

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Abstract

During the last century, researchers studied integer partition theory extensively. We are more interested in exploring partition identities among many aspects of integer partitions. In this thesis, we study a constructive method developed by Kurşungöz to find new identities on Rogers-Ramanujan type integer partitions and overpartitions. For this aim, we give a reproof of two Rogers-Ramanujan identities using the constructive method.

Combining two types of partitions, we introduced 2-colored Rogers-Ramanujan partitions. By finding some functional equations and using the constructive method, some identities have been found. Our results coincide with some extreme cases of Rogers-Ramanujan-Gordon's identities. A correspondence between colored partitions and those overpartitions is provided.

Our second result is finding the missing cases of parity consideration on Rogers-Ramanujan-Gordon's identities due to Andrews's suggestion in his seminal paper about parity in partition identities. Four cases had proven by Sang, Shi, and Yee, we reproved them using the said constructive method and then found and proved the remaining cases by the same method.

PARÇALANIŞ VE ÜST-PARÇALANIŞLAR İÇİN ROGERS-RAMANUJAN GENELLEŞTİRMELERİNDE ÜRETEÇ FONKSİYON OLAN SERİLERİN İNŞASI VE BUNLARIN DOĞRULAMA TARZI KANITLARI

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Anahtar Kelimeler: Tamsayı Parçalanışları, Parçalanış özdeşlikleri, üst-parçalanışlar, Renkli Parçalanışlar, q-serileri, Rogers-Ramanujan Tarzı Parçalanışlar

Özet

Geçtiğimiz yüzyılda araştırmacılar tamsayı parçalanış teorisini kapsamlı bir biçimde çalışmışlardır. Biz tamsayı parçalanış özdeşlikleri ile daha çok ilgileniyoruz. Bu doktora tezinde parçalanışlar ve üst-parçalanışlar için Rogers-Ramanujan tarzı özdeşliklerin keşfi ve kanıtlanması için Kurşungöz tarafından geliştirilen inşalı bir metodu çalışıyoruz. Bu amaç doğrultusunda Rogers-Ramanujan özdeşliklerini yeniden kanıtlıyoruz.

İki tip parçalanışı birleştirerek iki renkli Rogers-Ramanujan parçalanışlarını tanımladık. Bazı fonksiyonel denklemleri bahsettiğimiz inşalı yöntem ile çözerek bazı özdeşlikler bulduk. Bulduklarımız üst-parçalanışlar için Rogers-Ramanujan-Gordon özdeşliklerinin uç durumları ile çakışmaktadır. Bu durumda renkli parçalanışlar ve bahsedilen üst-parçalanışlar arasında birebir bir eşleme verilmiştir.

İkinci sonucumuz ise Andrews'ün parçalanış özdeşliklerinde teklik ve çiftliği incelediği yeni ufuklar açan makalesinde önerdiği bir açık problemin kısmi bir çözümünün eksik durumlarının bulup kanıtlanarak tamamlanmasıdır. Bu problemin Sang, Shi ve Yee'nin bulduğu dört durumdaki özdeşlikleri bahsettiğimiz inşalı metotla yeniden kanıtlayıp kalan iki durumdaki özdeşlikleri de bularak kanıtladık.

To my Parents

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and

My Beloved

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Table of Contents

CHAPTER 1

Introduction

When we talk about a partition, it refers to breaking an object down into smaller parts. It is the same for integer partitions, we want to study positive integers when we decompose them into smaller positive integers. In fact, we are interested in classifying all the ways that a positive integer n can be broken into other positive integers such that the summation of them is n.

Definition 1.0.1 *[2] A partition of a positive integer n is a finite non-increasing sequence of positive integers $\lambda_1, \cdots \lambda_r$ such that $\sum_{i=1}^{r} \lambda_i = n$. The λ_i are called the parts of the partition.*

As an example, consider $n = 4$, then all the possible partitions are:

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1.$$

The number of all possible partitions of n is denoted by $p(n)$. In 1918, Hardy and Ramanujan found an asymptotic formula for $p(n)$.

Theorem 1.0.1 *(Hardy-Ramanujan) [2] For positive integer n*

$$p(n) \sim \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4n\sqrt{n}}.$$

Later in 1938, Rademacher found an exact formula for $p(n)$ [16]. We can add some restrictions or conditions on parts of all partitions of an integer. One may study those partition types for other asymptotic or exact formulas. Studying different partition types also leads us to find partition identities, which will help us classify integer partitions. In the majority of partition identities, there are two types of conditions, first, the multiplicity conditions such as all parts being distinct; second, the divisibility conditions, such as all parts being divisible by 2.

The first known partition identity was given by Euler [2].

Theorem 1.0.2 *The number of partitions of a positive integer n into distinct parts is equal to the number of partitions of n into odd parts.*

For $n = 6$, all four partitions of 6 into distinct parts are:

$$6,\ 5+1,\ 4+2,\ 3+2+1,$$

and partitions into odd parts are:

$$5+1,\ 3+3,\ 3+1+1+1,\ 1+1+1+1+1+1.$$

As we mentioned, partitions into distinct parts is a multiplicity condition, and it is clear that the other side, parts being odd, is a divisibility condition.

The Rogers-Ramanujan identities are a milestone in integer partitions. These identities were first discovered and proved by Rogers in 1894 [2], later, in 1913, Ramanujan rediscovered them without any proof [2], again, in 1919, Schur rediscovered and proved them independently [2]. The first identity is as follows:

Theorem 1.0.3 *(The first Rogers-Ramanujan identity) [2] The partitions of an integer n in which the difference between any two parts is at least 2 are equinumerous with the partitions of n into parts congruent to 1 or 4 modulo 5.*

For $n = 9$, there are 5 partitions in which the difference between parts is at least 2, as follows:

$$9,\ 8+1,\ 7+2,\ 6+3,\ 5+3+1,$$

and the number of partitions into parts congruent to 1 or 4 modulo 5 is again 5 and they are

$$9,\ 6+1+1+1,\ 4+4+1,\ 4+1+1+1+1+1,\ 1+\cdots+1.$$

The second one is:

Theorem 1.0.4 *(The second Rogers-Ramanujan identity) [2] The partitions of an integer n in which the difference between any two parts is at least 2 and parts are greater than 1 are equinumerous with the partitions of n into parts congruent to 2 or 3 modulo 5.*

Again, let $n = 9$. Then there are 3 partitions in which the difference between parts is at least 2 and parts greater than 1, as follows:

$$9,\ 7+2,\ 6+3,$$

and the number of partitions into parts congruent to 2 or 3 modulo 5 is again 3 and they are

$$7+2,\ 3+3+3,\ 3+2+2+2.$$

For our purposes, we write them in terms of generating functions. We can rewrite the first identity as follows:

$$R_1(q) = \sum_{n\geq 0} r_1(n)q^n = \frac{(q^2, q^3, q^5; q^5)_\infty}{(q;q)_\infty}$$

where $R_1(q)$ is the generating function for partitions of n in which the difference between any two parts is at least 2 and the second one as

$$R_2(q) = \sum_{n\geq 0} r_2(n)q^n = \frac{(q^1, q^4, q^5; q^5)_\infty}{(q;q)_\infty}$$

where $R_2(q)$ is the same as $R_1(q)$ with additional condition that parts are greater than 1, and

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}),$$

$$(a_1, a_2, \cdots, a_r; q)_n = (a_1;q)_n(a_2;q)_n\cdots(a_r;q)_n,$$

$$(a;q)_\infty = \lim_{n\to\infty}(a;q)_n,$$

$$(a;q)_0 = 1$$

are the q-Pochhammer symbols [2].

There are many generalization of the Rogers-Ramanujan identities, a very remarkable generalization is given by Gordon in 1961.

Theorem 1.0.5 *(The Rogers-Ramanujan-Gordon identities) [10] For* $1 \leq a \leq k$*, let* $A_{k,a}(n)$ *be the number of partitions of* n *into parts that are not congruent to* 0 *or* $\pm a$ *modulo* $2k+1$*. Let* $B_{k,a}(n)$ *be the number of partitions* π *of* n *of the form*

$$\pi_1 + \pi_2 + \cdots + \pi_j,$$

where $\pi_i \geq \pi_{i+1}$*,* $\pi_i - \pi_{i+k-1} \geq 2$*, and at most* $a-1$ *of the* π_i *are equal to* 1*. Then for all* $n \geq 0$*,*

$$A_{k,a}(n) = B_{k,a}(n).$$

In the study of integer partitions, we may go further and study other type of partitions such as overpartitions, in which an overlined part can occur among other parts [9]. Another one is colored partitions, in which parts may get different colors, we talk about it later [3].

Definition 1.0.2 *An overpartition of a positive integer* n *is a non-increasing sequence of positive integers such that the summation of them is* n *and the first occurrence (equivalently, the final occurrence) of any part can be overlined.*

As an example, all overpartitions of 4 are,

$$4,\ \bar{4},\ 3+1,\ \bar{3}+1,\ 3+\bar{1},\ \bar{3}+\bar{1},\ 2+2,\ \bar{2}+2,$$
$$2+1+1,\ \bar{2}+1+1,\ 2+\bar{1}+1,\ \bar{2}+\bar{1}+1,\ 1+1+1+1,\ \bar{1}+1+1+1$$

The number of all possible partitions of n is denoted by $\overline{p}(n)$.

Note that in terms of generating functions, for partitions and overpartitions, we have [2]

$$P(q) = \sum_{n\geq 0} p(n)q^n = \frac{1}{(q;q)_\infty},$$

and [7]

$$\overline{P}(q) = \sum_{n\geq 0} \overline{p}(n)q^n = \frac{(-q;q)_\infty}{(q;q)_\infty}.$$

In the following chapters, we will talk about the constructive method, developed by Kurşungöz [13] and [14], that we have used to find new identities of Rogers-Ramanujan type, then we will see colored partitions and some identities on them, after that, we will go through overpartitions, we will complete the parity condition consideration for

Rogers-Ramanujan-Gordon identities for overpartitions; and finally, we will give some open problems and ideas to generalize the results of this thesis.

The content of the thesis comes from parts of two articles, one of which is submitted [19], and the other one is going to be submitted [15].

CHAPTER 2

The Constructive Method

2.1 Jacobi's Triple Product Identity and the Constructive Method

In this chapter we will discuss the constructive method, developed by Kurşungöz in [13] and [14], to find new partition or overpartition identities of Rogers-Ramanujan type, starting from the definition (in particular the multiplicity side), then using the generating functions of them, we will find the divisibility side of the identities. For this purpose, one of the important tools is using a transformation such as Jacobi's triple product identity.

Theorem 2.1.1 *(Jacobi's triple product) [2] For $z \neq 0$, $|q| < 1$,*

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = \prod_{n=0}^{\infty} (1 - q^{2n+2})(1 + zq^{2n+1})(1 + z^{-1}q^{2n+1}).$$

By replacing q by $q^{t+\frac{1}{2}}$ and then setting $z = -q^{t+\frac{1}{2}-i}$, we can easily get the following form of Jacobi's triple product identity,

Corollary 2.1.2 *[2] For $|q| < 1$, integers t and i,*

$$\sum_{n=0}^{\infty}(-1)^n q^{(2t+1)n(n+1)/2-in}(1-q^{(2n+1)i}) = \prod_{n=0}^{\infty}(1-q^{(2t+1)(n+1)})(1-q^{(2t+1)n+i})(1-q^{(2t+1)(n+1)-i}).$$

In this method, first we enumerate a type of partitions with some special conditions on parts, this gives us a family of generating functions and then we apply the following steps to them:

step 1: We find functional equations relating the generating functions.

step 2: We guess the type of series we want as solutions. The inspiration is Andrews' H and J functions [2], which appear in the proof of many partition identities, in those proofs, the generating functions will be related to H and J functions. In the constructive method, we are trying to build those generating functions.

step 3: We use the functional equations and construct the series.

step 4: We apply $x = 1$, because we want the partitions to be independent from the length, and then we use the Jacobi's triple product theorem to find a q-series identity.

In the next section, we will provide a proof for Rogers-Ramanujan identities using the constructive method and we will see all the steps in more details. In fact, we will go over a proof of the Rogers-Ramanujan identities, and try to reverse engineer it.

2.2 Reproof of Rogers-Ramanujan Identities

Let $r_1(m,n)$ counts all Rogers-Ramanujan type partitions of n with m parts, and $r_2(m,n)$ count all Rogers-Ramanujan type partitions when all parts are more than 1 of n with m parts. Then $R_1(x)$ and $R_2(x)$ are the following double sums:

$$R_i(x) = \sum_{m\geq 0}\sum_{n\geq 0} r_i(m,n)x^m q^n \quad ; \quad i = 1,2$$

It is not hard to find the following relations between them.

$$R_2(x) - R_1(x) = xqR_1(xq), \tag{2.1}$$

$$R_1(x) = R_2(xq). \tag{2.2}$$

A proof to these functional equations is given in Andrews' book [2]. In fact, these functional equations and some initial conditions, which will be given shortly, uniquely determine the generating functions. We assume that each of them has the form

$$R_i(x) = \sum_{n\geq 0} \alpha_n(x)q^{nA_i} + \beta_n(x)x^{B_i}q^{C_i}q^{nD_i} \quad i = 1,2 \tag{2.3}$$

with the initial condition that $R_i(0) = 1$ (for the empty partition of 0).

Our first goal is to find A_i, B_i, C_i and D_i. For this end, we construct the functional equations using the equation (2.1) in the following form

$$\sum_{n\geq 0} \alpha_n(x)(q^{nA_2} - q^{nA_1}) + \beta_n(x)(x^{B_2}q^{C_2}q^{nD_2} - x^{B_1}q^{C_1}q^{nD_1})$$

$$= \sum_{n\geq 0} \alpha_n(xq)xqq^{nA_1} + \beta_n(xq)xqx^{B_1}q^{C_1}q^{nD_1}$$

in the last term, we substitute n by $n-1$, then we use another assumption that

$$\alpha_n(x)(q^{nA_2} - q^{nA_1}) = \beta_{n-1}(xq)xqx^{B_1}q^{C_1}q^{(n-1)D_1}$$

and

$$\beta_n(x)(x^{B_2}q^{C_2}q^{nD_2} - x^{B_1}q^{C_1}q^{nD_1}) = \alpha_n(xq)xqq^{nA_1}.$$

Note that these imply the functional equations (2.1) and (2.2), but not implied by them. After some calculation and simplification we can write $\alpha_n(x)$ in terms of $\alpha_0(xq^{2n})$ and $\beta_n(x)$ in terms of $\alpha_0(xq^{2n+1})$, which are

$$\alpha_n(x) = \alpha_0(xq^{2n})\frac{x^{2n}q^{n(2n+1)}q^{-nA_1}}{(q^E;q^E)_n(x^Fq^Gq^{F-G}q^{nH};q^{2F-H})_n},$$

and

$$\beta_n(x) = -\alpha_0(xq^{2n+1})\frac{x^{2n+1}q^{(n+1)(2n+1)}x^{-B_1}q^{-C_1}q^{-nD_1}}{(q^E;q^E)_n(x^Fq^Gq^{nH};q^{2F-H})_{n+1}}$$

where $E = A_2 - A_1$, $F = B_2 - B_1$, $G = C_2 - C_1$ and $H = D_2 - D_1$.

Now, we change the second finite product in both fractions into an infinite product, and rename part of the equation as $\tilde{\alpha_0}(x)$, then we have

$$\alpha_n(x) = \tilde{\alpha_0}(xq^{2n})\frac{x^{2n}q^{n(2n+1)}q^{-nA_1}}{(q^E;q^E)_n(x^Fq^Gq^{F-G}q^{nH};q^{2F-H})_\infty}$$

and

$$\beta_n(x) = -\tilde{\alpha_0}(xq^{2n+1})\frac{x^{2n+1}q^{(n+1)(2n+1)}x^{-B_1}q^{-C_1}q^{-nD_1}}{(q^E;q^E)_n(x^Fq^Gq^{nH};q^{2F-H})_\infty}$$

where

$$\tilde{\alpha_0}(xq^{2n}) = \alpha_0(xq^{2n})((xq^{2n})^Fq^Gq^{F-H};q^{2F-H})_\infty.$$

Next, we use these α_n and β_n in the equation (2.2) and again we shift n to $n-1$ in the last term on the right hand side, so we have

$$\begin{aligned}
&\sum_{n\geq 0}\tilde{\alpha_0}(xq^{2n})\frac{x^{2n}q^{n(2n+1)}q^{nE}}{(q^E;q^E)_n(x^Fq^Gq^{F-G}q^{nH};q^{2F-H})_\infty}\\
&\quad-\tilde{\alpha_0}(xq^{2n+1})\frac{x^{2n+1}q^{(n+1)(2n+1)}x^Fq^Gq^{nH}}{(q^E;q^E)_n(x^Fq^Gq^{nH};q^{2F-H})_\infty}\\
&\quad=\sum_{n\geq 0}\tilde{\alpha_0}(xq^{2n+1})\frac{x^{2n}q^{2n^2+3n}}{(q^E;q^E)_n(x^Fq^Gq^{2F-G}q^{nH};q^{2F-H})_\infty}\\
&\quad-\tilde{\alpha_0}(xq^{2n})\frac{x^{2n-1}q^{2n^2+n+1}}{(q^E;q^E)_n(x^Fq^Gq^{F-H}q^{nH};q^{2F-H})_\infty}
\end{aligned}$$

In this step, we want $\tilde{\alpha_0}$ to be independent of x and after some simplification we have

$$\begin{aligned}
&\frac{xq}{(x^Fq^Gq^{F-H}q^{nH};q^{2F-H})_\infty}-\frac{x^2q^{2n+2}}{(x^Fq^Gq^{nH};q^{2F-H})_\infty}\\
=&\frac{xq^{2n+1}q^{nE}(1-x^Fq^Gq^{nH})}{(x^Fq^Gq^{nH};q^{2F-H})_\infty}-\frac{x^Fq^{F+G}q^{(n-1)H}(1-q^{nE})}{(x^Fq^Gq^{F-H}q^{nH};q^{2F-H})_\infty}
\end{aligned}$$

Now, we need one more assumption to simplify the infinite products and make them into rational functions, then by cross-multiplying, we then obtain an identity between polynomials, and we can find E,F,G and H. This is also called similarity of two terms involving infinite products [13]. The assumption that we need here is $2F-H|F-H$, so $F=\frac{1-t}{1-2t}H$ for some integer t, in this part we choose the smallest or the simplest solutions among infinitely many ones, if they do not work, we choose other ones. Note that after rearranging monomials, there should be the same number of positive monomials on each side. Here, for $t=0$, we have $F=H$, then

$$xq-x^2q^2q^{2n}=xqq^{n(E+2)}-x^{F+1}q^{G+1}q^{n(E+F)}-x^Fq^Gq^{nF}+x^Fq^Gq^{n(E+F)}$$

we can rearrange it to

$$xq+x^{F+1}q^{G+1}q^{n(E+F)}+x^Fq^Gq^{nF}=xqq^{n(E+2)}+x^Fq^Gq^{n(E+F)}+x^2q^2q^{2n}.$$

So, there are three terms on each side, because every monomial on the left hand side must correspond to one monomial on the right hand side, this gives us $3!=6$ different linear systems of equations, such as

$$xq=xqq^{n(E+2)},$$

$$x^{F+1}q^{G+1}q^{n(E+F)}=x^Fq^Gq^{n(E+F)}$$

and

$$x^F q^G q^{nF} = x^2 q^2 q^{2n}.$$

The idea is to identify the exponents of x, q and q^n to find unknown parameters E, F, G and H. If our choice in the previous step works, one of them has a solution, in this case the solution is

$$F = G = H = 1 \ \ and \ \ E = -1.$$

We wanted $\tilde{\alpha_0}$ to be constant with respect to x. Note that in (2.3), if we put $x = 0$, in the right hand side, all terms will be eliminated except $\alpha_0(0)$ and in the left hand side, we have $R_i(0)$, the partitions of 0 which is 1 for the empty partition of 0, i.e. $\alpha_0(0) = 1$, so $\tilde{\alpha}_0(xq^{2n}) = 1$.

In this step, we apply $x = 1$, because x^m was the term for the length of partitions, but we want our generating function to be free of length, i.e. it works for any partition of any length. Then we have

$$R_1(1) = \frac{1}{(q)_\infty} \sum_{n \geq 0} (-1)^n q^{\frac{5n^2+3n}{2}} (1 - q^{2n+1})$$

and

$$R_2(1) = \frac{1}{(q)_\infty} \sum_{n \geq 0} (-1)^n q^{\frac{5n^2+n}{2}} (1 - q^{4n+2})$$

now, we use corollary (2.1.2), for $i = 1$ and $t = 2$, then we have

$$R_1(1) = \frac{\prod_{n=0}^{\infty} (1 - q^{5(n+1)})(1 - q^{5n+1})(1 - q^{5n+4})}{(q)_\infty}$$

which gives us the second Rogers-Ramanujan identity. Again, using corollary (2.1.2), for $i = 2$ and $t = 2$, we have

$$R_2(1) = \frac{\prod_{n=0}^{\infty} (1 - q^{5(n+1)})(1 - q^{5n+2})(1 - q^{5n+3})}{(q)_\infty}$$

which gives us the first one.

CHAPTER 3

2-Colored Rogers-Ramanujan Partition Identities

3.1 Introduction

There are different ways of coloring partitions, each introduced for different purposes. Andrews [3] introduced the two colored partitions, later, together with Agarwal [1], they defined partitions with N copies of N, another type is 4-colored partitions, also known as 4 parameters partitions introduced by Boulet [6] and developed by Uncu [5]. We will give a definition for arbitrary number of colors in a partition given by Chern, Fu and Tang [8],

Definition 3.1.1 *A k-colored partition of n is the one that each part can get any of k different colors.*

As an example, the fourteen 2-colored partitions of 3 are

$3, \textcolor{red}{3}, 2+1, \textcolor{red}{2}+1, 2+\textcolor{red}{1}, \textcolor{red}{2}+\textcolor{red}{1},$

$1+1+1, 1+1+\textcolor{red}{1}, 1+\textcolor{red}{1}+1, \textcolor{red}{1}+1+1, 1+\textcolor{red}{1}+\textcolor{red}{1}, \textcolor{red}{1}+1+\textcolor{red}{1}, \textcolor{red}{1}+\textcolor{red}{1}+1, \textcolor{red}{1}+\textcolor{red}{1}+\textcolor{red}{1}$

Combining Rogers-Ramanujan type partitions, defined in the second chapter, and t-colored partitions, we define the following partition type,

Definition 3.1.2 *A 2-colored Rogers-Ramanujan partition of n consists of two separate list of parts, each of the same color, and the difference between every two consecutive parts of the same color is at least two, moreover, parts in different colors do not overlap.*

As an example, the twelve 2-colored Rogers-Ramanujan partitions of 6 are

$$6, 6, 5+1, 5+1, 5+1, 5+1, 4+2, 4+2, 4+2, 4+2, 3+2+1, 3+2+1.$$

By this definition, we have the following identity.

Theorem 3.1.1 *Let $R_1(n)$ denote the number of 2-colored Rogers-Ramanujan partitions of n, then for $|q| < 1$,*

$$\sum_{n\geq 0} r_1(n)q^n = \frac{(-q)_\infty (q^2, q^2, q^4; q^4)_\infty}{(q)_\infty}.$$

Lovejoy [12] proved analogues of Gordon's theorem for overpartitions in the cases $i = 1$ and $i = k$. Later Chen et. al [7] found the missing cases as follows,

Theorem 3.1.2 *For $k \geq a \geq 1$, let $D_{k,a}(n)$ denote the number of overpartitions of n of the form $d_1 + d_2 + \cdots + d_s$, such that 1 can occur as a non-overlined part at most $a - 1$ times, and $d_j - d_{j+k-1} \geq 1$ if d_j is overlined and $d_j - d_{j+k-1} \geq 2$ otherwise. For $k > i \geq 1$, let $C_{k,i}(n)$ denote the number of overpartitions of n whose non-overlined parts are not congruent to $0, \pm i$ modulo $2k$ and let $C_{k,k}(n)$ denote the number of overpartitions of n with parts not divisible by k. Then $C_{k,i}(n) = D_{k,i}(n)$.*

In the following sections, we will go over the 2-colored Rogers-Ramanujan partition type, accordingly, we will find two functional equations, and then constructively, we will prove the theorem 3.1.1, we will also find two other partition identities. At the end, a correspondence between our identities and the ones for overpartitions is given.

3.2 Colored Rogers-Ramanujan Partitions and the Proof of Theorem 3.1.1

According to 2-colored Rogers-Ramanujan partitions, the following definition is given.

Definition 3.2.1 *For $1 \leq j \leq 2$, let $R_j(x)$ be the generating function of 2-colored Rogers-Ramanujan partitions with smallest part greater than or equal to j.*

With respect to these definitions, one can find the following functional equations relating $R_1(x)$ and $R_2(x)$. We use this equation as the construction equation in our method.

Theorem 3.2.1

$$R_1(x) - R_2(x) = xqR_1(xq) + xqR_2(xq). \tag{3.1}$$

Proof 3.2.2 *Let*

$$R_i(x) = \sum_{m\geq 0}\sum_{n\geq 0} r_i(m,n)x^m q^n \ \ ; \ \ i = 1, 2$$

be the generating function for the types that have been mentioned above, where m is referring to the number of parts in partitions.

Let π be a 2-Colored Rogers-Ramanujan partition of n with m parts. All 2-colored Rogers-Ramanujan partitions will be counted by $r_1(m,n)$, and if the smallest part is ≥ 2, then it will be counted by $r_2(m,n)$. So, $r_1(m,n) - r_2(m,n)$ will count the number of partitions with the smallest part 1. If we remove 1 from all partitions, then we have two cases:

(i) The smallest part is ≥ 2 with different color than 1, so one can subtract 1 from each part, the enumeration of these partitions is by $r_1(m-1, n-m)$.

(ii) The smallest part is ≥ 3 with the same color as 1, if 1 is subtracted from each part, the enumeration of these partitions is $r_2(m-1, m-n)$, note that a part 2 is not possible here.

So,

$$r_1(m,n) - r_2(m,n) = r_1(m-1, n-m) + r_2(m-1, n-m).$$

Multiplying all terms by $x^m q^n$ and taking the summation over m and n for all terms, $m, n \geq 0$ and both integers, we have

$$\sum_{m\geq 0}\sum_{n\geq 0} r_1(m,n)x^m q^n - \sum_{m\geq 0}\sum_{n\geq 0} r_2(m,n)x^m q^n =$$
$$\sum_{m\geq 0}\sum_{n\geq 0} r_1(m-1, n-m)x^m q^n + \sum_{m\geq 0}\sum_{n\geq 0} r_2(m-1, n-m)x^m q^n.$$

By changing $m-1$ to m and $n-m$ to $n-m+1$ on the right hand side of this equation, we have

$$\sum_{m\geq 0}\sum_{n\geq 0} r_1(m,n)x^m q^n - \sum_{m\geq 0}\sum_{n\geq 0} r_2(m,n)x^m q^n =$$
$$\sum_{m\geq 0}\sum_{n\geq 0} r_1(m,n)x^{m+1} q^{n+m+1} + \sum_{m\geq 0}\sum_{n\geq 0} r_2(m,n)x^{m+1} q^{n+m+1}.$$

This will get us the functional equation (3.1).

Theorem 3.2.3 *Another relation between $R_1(x)$ and $R_2(x)$ is as follows, we use this one as our check equation.*

$$R_2(x) = R_1(xq) \tag{3.2}$$

Proof 3.2.4 *Equation* (3.2) *is clear, as shifting every part of R_1 by 1 unit it will change it to R_2.*

Using steps described in the second chapter, with straightforward but long computations which we skipped here, by

$$R_i(x) = \sum_{n\geq 0} \alpha_n(x) q^{nA_i} + \beta_n(x) x^{B_i} q^{C_i} q^{nD_i}, \quad i = 1, 2$$

we can find α_n and β_n in terms of α_0,

$$\alpha_n(x) = \tilde{\alpha}_0(xq^{2n}) \frac{x^{2n} q^{n(2n+1)} q^{-nA_2} (-1; q^E)_n (-x^F q^G q^{F-H} q^{nH}; q^{2F-H})_\infty}{(q^E; q^E)_n (x^F q^G q^{F-H} q^{nH}; q^{2F-H})_\infty}$$

and

$$\beta_n(x) = -\tilde{\alpha}_0(xq^{2n+1}) \frac{x^{2n+1} q^{(n+1)(2n+1)} x^{-B_2} q^{-C_2} q^{-nD_2} (-1; q^E)_{n+1} (-x^F q^G q^{2F-H} q^{nH}; q^{2F-H})_\infty}{(q^E; q^E)_n (x^F q^G q^{nH}; q^{2F-H})_\infty}$$

where

$$\tilde{\alpha}_0(xq^{2n}) = \alpha_0(xq^{2n}) \frac{((xq^{2n})^F q^G q^{F-H}; q^{2F-H})_\infty}{(-(xq^{2n})^F q^G q^{F-H}; q^{2F-H})_\infty},$$

also, $E = A_2 - A_1$, $F = B_2 - B_1$, $G = C_2 - C_1$ and $H = D_2 - D_1$.

Then, by equation (3.2) and considering some assumptions for equations to be consistent, we can find E, F, G and H, in this case $F = G = H = 1$ and $E = -1$.

Putting them in the generating functions for $R_i(x)$, $\tilde{\alpha}_0$ being constant with respect to x, and applying $x = 1$, we have

$$R_1(1) = \sum_{n\geq 0} \frac{(-1)^n q^{n(2n+1)} (-1; q)_n (-q^{n+1}; q)_\infty}{(q)_n (q^{n+1}; q)_\infty} - \tag{3.3}$$

$$\sum_{n\geq 0} \frac{(-1)^n q^{(n+1)(2n+2)} (-1; q)_{n+1} (-q^{n+2}; q)_\infty}{(q)_n (q^{n+1}; q)_\infty} \tag{3.4}$$

and

$$R_2(1) = \sum_{n\geq 0} \frac{(-1)^n q^{n(2n+2)} (-1; q)_n (-q^{n+1}; q)_\infty}{(q)_n (q^{n+1}; q)_\infty} - \tag{3.5}$$

$$\sum_{n\geq 0} \frac{(-1)^n q^{(n+1)(2n+1)} (-1; q)_{n+1} (-q^{n+2}; q)_\infty}{(q)_n (q^{n+1}; q)_\infty}. \tag{3.6}$$

hey can be rewritten as follows,

$$R_1(1) = 2\frac{(-q)_\infty}{(q)_\infty}\sum_{n\geq 0}(-1)^n q^{n(2n+1)}(\frac{1}{1+q^n} - \frac{q^{3n+2}}{1+q^{n+1}})$$

and

$$R_2(1) = 2\frac{(-q)_\infty}{(q)_\infty}\sum_{n\geq 0}(-1)^n q^{n(2n+2)}(\frac{1}{1+q^n} - \frac{q^{n+1}}{1+q^{n+1}}).$$

So,

$$\begin{aligned} R_1(1) &= 2\frac{(-q)_\infty}{(q)_\infty}(\sum_{n\geq 0}\frac{(-1)^n q^{n(2n+1)}}{1+q^n} - \sum_{n\geq 0}\frac{(-1)^n q^{n(2n+1)}q^{3n+2}}{1+q^{n+1}}) \\ &= \frac{(-q)_\infty}{(q)_\infty}(1+2(\sum_{n\geq 1}\frac{(-1)^n q^{n(2n+1)}}{1+q^n} - \sum_{n\geq 1}\frac{(-1)^{n-1}q^{2n^2}}{1+q^n})) \\ &= \frac{(-q)_\infty}{(q)_\infty}(1+2(\sum_{n\geq 1}\frac{(-1)^n q^{2n^2}(q^n+1)}{1+q^n})) \\ &= \frac{(-q)_\infty}{(q)_\infty}(1+2\sum_{n\geq 1}(-1)^n q^{2n^2}) = \frac{(-q)_\infty}{(q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{2n^2}. \end{aligned}$$

By Theorem (2.1.1) for $z = -1$ and q^2, for 2-colored Rogers-Ramanujan type partitions defined in (3.2.1) the following identity holds

$$R_1(1) = \frac{(-q)_\infty(q^2, q^2, q^4; q^4)_\infty}{(q)_\infty}.$$

Moreover, the coefficients in the Taylor series of $R_2(1)$ coincides with the number of partitions for 2-colored Rogers-Ramanujan partitions with parts more than 1,

$$1 + 2q^2 + 2q^3 + 2q^4 + 4q^5 + 6q^6 + 8q^7 + 10q^8 + 14q^9 + 18q^{10} + \cdots.$$

We will come back to $R_2(n)$ at the end of this chapter.

3.3 Correspondence with Overpartitions

There is a one-to-one correspondence between 2-colored Rogers-Ramanujan type partitions and previously defined overpartitions $D_{k,a}(n)$ for $k = a = 2$.

Let $\pi = (y_1, \cdots, y_i, y_{i+1}, \cdots, y_m)$ be an arbitrary 2-colored partition of n, and $\overline{\pi} = (z_1, \cdots, z_i, z_{i+1}, \cdots, z_m)$ be an arbitrary overpartition of n, both into m parts. First of all, in both cases all parts are distinct. Secondly, for the case that there are t number of consecutive parts, for the colored case, there are only two possibilities, they should be alternatively red and black, e.g. for three consecutive parts i, $i+1$ and $i+2$, the cases are

$$j, i, {\color{red}i+1}, i+2, k \quad \text{and} \quad j, {\color{red}i}, i+1, {\color{red}i+2}, k$$

where $j < i-1$ and $k > i+3$. This means two consecutive parts can not be of the same color. For the overpartition case, the first $t-1$ parts should be overlined and there are two possibilities for the last one, e.g. for three consecutive parts we have

$$j, \bar{i}, \overline{i+1}, i+2, k \quad \text{and} \quad j, \bar{i}, \overline{i+1}, \overline{i+2}, k$$

where $j < i-1$ and $k > i+3$. This implies the first and the second part in the sequence should be overlined, so there are two possibilities for the last past in the sequence, it can be overlined or non-overlined.

If $y_{i+1} - y_i > 1$, and $z_{i+1} - z_i > 1$, then there are four cases for both colored cases and overpartition one, for colored partition, both can be of the same color or both may have different colors, and for the overpartition, it is possible for each part to be overlined or non-overlined, so in this case again, we have the same number of cases, and the correspondence in this case is also clear. So, there exists a one-to-one correspondence between them.

It is not hard to see another correspondence between $D_{2,1}(n)$ and the following partition type.

Definition 3.3.1 *Let $R_3(n)$ denote the number of 2-colored Rogers-Ramanujan partitions which do not allow to have a red 1 in the partition.*

With respect to this definition and the mentioned correspondence, we have the following identity.

Theorem 3.3.1 *For definition (3.3.1) and $|q| < 1$ the following identity holds*

$$\sum_{n\geq 0} r_3(n)q^n = \sum_{q\geq 0} d_{2,1}(n)q^n = \frac{(-q)_\infty (q^1, q^3, q^4; q^4)_\infty}{(q)_\infty}.$$

Here $D_{2,1}(n)$ is again as in [7] for $k = 2$ and $a = 1$.

Note that in the definition of $R_3(n)$, we can choose any of two colors. In fact, we have

$$R_2(x) = 2R_3(x) - R_1(x).$$

So, we have the following identity for $R_2(n)$.

Theorem 3.3.2 *For* $|q| < 1$,

$$\sum_{n\geq 0} r_2(n)q^n = 2\frac{(-q)_\infty(q^1, q^3, q^4; q^4)_\infty}{(q)_\infty} - \frac{(-q)_\infty(q^2, q^2, q^4; q^4)_\infty}{(q)_\infty}.$$

CHAPTER 4

Parity considerations in Rogers-Ramanujan-Gordon type overpartition, all cases

4.1 Definitions

In 2010, Andrews [4] applied parity conditions on Rogers-Ramanujan-Gordon identities. He asked the extension to overpartition as an open problem. In 2021, Sang, Shi and Yee [18] defined the following partition types, and discovered new identities with respect to these parity restrictions for overpartitions. In the two following definitions, if_i refers to the number of parts for integer i, if there is no confusion, we write f_i, note that $f_{\bar{i}} = 0$ or 1.

Definition 4.1.1 *[18] For $k \geq a \geq 1$, let $U_{k,a}(n)$ denote the number of overpartitions of n of the form $(\overline{1}f_{\overline{1}}, 1f_1, \overline{2}f_{\overline{2}}, 2f_2, \cdots)$ such that*

(i) $f_1 \leq a - 1 + f_{\overline{1}}$;

(ii) $f_{2l-1} \geq f_{\overline{2l-1}}$;

(iii) $f_{2l} + f_{\overline{2l}} \equiv 0 \pmod 2$;

(iv) $f_l + f_{\bar{l}} + f_{l+1} \leq k - 1 + f_{\overline{l+1}}$.

As an example, all 7 overpartitions of $U_{4,4}(6)$ are

$$5+1,\ 3+3,\ \overline{3}+3,\ 3+1+1+1,\ 3+\overline{1}+1+1,\ 2+2+\overline{1}+1,\ \overline{2}+2+\overline{1}+1.$$

Definition 4.1.2 *[18] For $k \geq a \geq 1$, let $\overline{U}_{k,a}(n)$ denote the number of overpartitions of n of the form $(\overline{1}f_{\overline{1}}, 1f_1, \overline{2}f_{\overline{2}}, 2f_2, \cdots)$ such that*

(i) $f_1 \leq a - 1 + f_{\overline{1}}$;

(ii) $f_{2l} \geq f_{\overline{2l}}$;

(iii) $f_{2l-1} + f_{\overline{2l-1}} \equiv 0 \pmod 2$;

(iv) $f_l + f_{\overline{l}} + f_{l+1} \leq k - 1 + f_{\overline{l+1}}$.

An example for this overpartition type will be all 10 overpartition of $\overline{U}_{4,4}(6)$,

$6,\ 4+2,\ 4+1+1,\ 4+\overline{1}+1,\ 3+3,\ \overline{3}+3,\ 2+2+2,\ \overline{2}+2+2,\ 2+2+\overline{1}+1,\ \overline{2}+2+\overline{1}+1.$

Note that with respect to the definition 4.1.2, we have the following lemma for the fixed first index and difference one for the second index in $\overline{U}$.

Lemma 4.1.1 *[18] For $k \geq a \geq 1$, if $a \equiv 0 \pmod 2$, then*

$$\overline{U}_{k,a}(n) = \overline{U}_{k,a-1}(n).$$

The proof is straightforward, from conditions (i) and (iii) in the definition (4.1.2) and the assumption $a \equiv 0 \ (mod\ 2)$, we see that f_1 never reaches the upper bound, $a - 1 + f_{\overline{1}}$, this means $f_1 \leq a - 2 + f_{\overline{1}}$, so both sides of the equality of this lemma count the same number of overpartitions. To find the new identities, we need to separate cases depending on the first index or the second one to be even or odd, so there will be 6 different identities. The first one is for the case that both indices are even of U.

Theorem 4.1.2 *For $k \geq a \geq 1$,*

$$\sum_{n \geq 0} u_{2k,2a}(n) q^n = \frac{(-q;q)_\infty (q^{2a}, q^{4k-2a}, q^{4k}; q^{4k})_\infty}{(q^2;q^2)_\infty}.$$

The other case when the first index is even and the second index is odd for U is the following.

Theorem 4.1.3 *For $k \geq a \geq 1$,*

$$\begin{aligned}\sum_{n \geq 0} u_{2k,2a+1}(n) q^n =& \frac{(-q^2;q)_\infty (q^{2a+2}, q^{4k-2a-2}, q^{4k}; q^{4k})_\infty}{(q^2;q^2)_\infty} \\ &+ \frac{q(-q^2;q)_\infty (q^{2a}, q^{4k-2a}, q^{4k}; q^{4k})_\infty}{(q^2;q^2)_\infty}.\end{aligned}$$

When both indices are odd, we have the following identity for U.

Theorem 4.1.4 *For $k \geq a \geq 1$,*

$$\sum_{n\geq 0} u_{2k+1,2a+1}(n)q^n = \frac{(-q;q)_\infty^2 (q^{2a+1}, q^{4k-2a+1}, q^{4k+2}; q^{4k+2})_\infty}{(q^2;q^2)_\infty}.$$

The last case for U will be as follows.

Theorem 4.1.5 *For $k \geq a \geq 1$,*

$$\begin{aligned}\sum_{n\geq 0} u_{2k+1,2a}(n)q^n =& \frac{(-q^2;q)_\infty (q^{2a+1}, q^{4k-2a+1}, q^{4k+2}; q^{4k+2})_\infty}{(q^2;q^2)_\infty} \\ &+ \frac{q(-q^2;q)_\infty (q^{2a-1}, q^{4k-2a+3}, q^{4k+2}; q^{4k+2})_\infty}{(q^2;q^2)_\infty}.\end{aligned}$$

For $\overline{U}$ with respect to (4.1.1), we have two following identities, then first one is for the case that both indices are even.

Theorem 4.1.6 *For $k \geq a \geq 1$,*

$$\sum_{n\geq 0} \overline{u}_{2k,2a}(n)q^n = \frac{(-q^2;q^2)_\infty^2 (q^{2a}, q^{4k-2a}, q^{4k}; q^{4k})_\infty}{(q^2;q^2)_\infty}.$$

And the next one is for the case that the first index is odd and the second index is even.

Theorem 4.1.7 *For $k \geq a \geq 1$,*

$$\sum_{n\geq 0} \overline{u}_{2k+1,2a}(n)q^n = \frac{(-q^2;q^2)_\infty^2 (q^{2a}, q^{4k+2-2a}, q^{4k+2}; q^{4k+2})_\infty}{(q^2;q^2)_\infty}.$$

Note that for both theorems 4.1.7 and 4.1.7, we can have $2a-1$ instead of $2a$ for the second index, with respect to the lemma 4.1.1.

In the following sections, we will give some functional equations that we will use for our constructive method, then we will prove those equations. At the end we will reprove the identities introduced by Chen, Shi and Yee, then the proofs of the remaining cases will be given.

4.2 Functional Equations

As described in the second chapter, we need some functional equations relating the generating functions of the overpartition types defined in the previous section to use for constructive method to find the divisibility part of our identities, to this purpose, some functional equation are given in this section. For all proofs, let π be an arbitrary overpartition on n with m parts, we denote the number of overpartitions of π in the first form by $r_{k,a}(m,n)$, then $U_{k,a}(n) = \sum_{m,n\geq 0} r_{k,a}(m,n)$, also $\overline{r}_{k,a}(m,n)$ denotes the number of overpartitions of π for the second definition, similarly, $\overline{U}_{k,a}(n) = \sum_{m,n\geq 0} \overline{r}_{k,a}(m,n)$.

Through all the proofs, note that $r_{k,a}(m,n) = 0$ whenever $m < 0$ or $n < 0$.

Theorem 4.2.1 *For $k \geq a \geq 1$, we have*

$$U_{2k,2a}(x) - U_{2k,2a-1}(x) = (xq)^{2a-1}\overline{U}_{2k,2k-2a+1}(xq) + (xq)^{2a+1}\overline{U}_{2k,2k-2a-1}(xq). \quad (4.1)$$

Proof 4.2.2 *Consider $r_{2k,2a}(m,n) - r_{2k,2a-1}(m,n)$. Then all overpartitions counted by $r_{2k,2a-1}(m,n)$ are also counted by $r_{2k,2a}(m,n)$, so with respect to the definition of $U_{2k,2a}(n)$, we have two cases,*

(i) there is no overlined 1 in the overpartition, i.e. $f_{\overline{1}} = 0$, so with respect to the first condition, there are exactly $2a-1$ of non-overlined 1's in the overpartition, now by the last condition, $f_2 \leq 2k-2a+f_{\overline{2}}$, removing all 1's and subtracting 1 from all remaining parts, $f_1 \leq (2k-2a+1)-1+f_{\overline{1}}$ and other conditions of the second definition also hold, so we have overpartitions of $n-m$ with $m-2a+1$ parts of the second type, $\overline{r}_{2k,2k-2a+1}(m-2a+1, n-m)$.

(ii) there is an overlined 1 in the overpartition, i.e. $f_{\overline{1}} = 1$, so we have exactly $2a$ non-overlined 1's and one overlined ones in the overpartition, note that by assumption $2a \geq 1$, so by the last condition $f_2 \leq 2k-2a-2+f_{\overline{2}}$, now by removing all 1's,and subtracting 1 from all remaining parts, $f_1 \leq (2k-2a-1)-1+f_{\overline{1}}$ and again other conditions of the second definition also hold, so we have overpartitions of $n-m$ with $m-2a-1$ parts of the second type, $\overline{r}_{2k,2k-2a-1}(m-2a-1, n-m)$.

So,

$$r_{2k,2a}(m,n) - r_{2k,2a-1}(m,n) = \overline{r}_{2k,2k-2a+1}(m-2a+1, n-m) + \overline{r}_{2k,2k-2a-1}(m-2a-1, n-m).$$

Multiplying all terms by $x^m q^n$ and taking the summation over m and n for all terms,

$m, n \geq 0$ and both integers, we have

$$\sum_{m,n\geq 0} r_{2k,2a}(m,n)x^m q^n - \sum_{m,n\geq 0} r_{2k,2a-1}(m,n)x^m q^n =$$

$$\sum_{m,n\geq 0} \overline{r}_{2k,2k-2a+1}(m-2a+1, n-m)x^m q^n + \sum_{m,n\geq 0} \overline{r}_{2k,2k-2a-1}(m-2a-1, n-m)x^m q^n.$$

By substituting n by $n+m$, then m by $m+2a-1$ for the first term and n by $n+m$, then m by $m+2a+1$ for the second term on the right hand side of this equation, we have

$$\sum_{m,n\geq 0} r_{2k,2a}(m,n)x^m q^n - \sum_{m,n\geq 0} r_{2k,2a-1}(m,n)x^m q^n =$$

$$\sum_{m,n\geq 0} \overline{r}_{2k,2k-2a+1}(m,n)x^{m+2a-1}q^{n+m+2a-1} + \sum_{m,n\geq 0} \overline{r}_{2k,2k-2a-1}(m,n)x^{m+2a-1}q^{n+m+2a-1} =$$

$$(xq)^{2a-1} \sum_{m,n\geq 0} \overline{r}_{2k,2k-2a+1}(m,n)(xq)^m q^n + (xq)^{2a+1} \sum_{m,n\geq 0} \overline{r}_{2k,2k-2a-1}(m,n)(xq)^m q^n.$$

This gives us the equation (4.1).

Next functional equation that we want to use in construction step will be as follows.

Theorem 4.2.3 *For $k \geq a \geq 1$, we have*

$$U_{2k,2a+1}(x) - U_{2k,2a}(x) = (xq)^{2a}\overline{U}_{2k,2k-2a}(xq) + (xq)^{2a+2}\overline{U}_{2k,2k-2a-2}(xq). \tag{4.2}$$

The proof for Theorem (4.2.3) is the same as proof of Theorem (4.2.1), we only need to replace $2a$ by $2a+1$, we will skip the rest. Now, we will introduce another functional equation from $\overline{U}$ to U.

Theorem 4.2.4 *For $k \geq a \geq 1$, we have*

$$\overline{U}_{2k,2a}(x) - \overline{U}_{2k,2a-2}(x) = (xq)^{2a}U_{2k,2k-2a}(xq) + (xq)^{2a-2}U_{2k,2k-2a+2}(xq). \tag{4.3}$$

Proof 4.2.5 *Consider $\overline{U}_{2k,2a}(x) - \overline{U}_{2k,2a-2}(x)$, when $1 \leq 2a \leq 2k$, in this case for $\overline{r}_{2k,2a}(m,n) - \overline{r}_{2k,2a-2}(m,n)$, there are two cases*

(i) $f_{\overline{1}} = 0$, then $f_1 = 2a$, so by the fourth condition $f_2 \leq k - 2a - 1 + f_{\overline{2}}$, removing all 1's and subtracting 1 from all remaining parts, we have $f_1 \leq (k-2a) - 1 + f_{\overline{1}}$ and other conditions of the definition of $U_{2k,2a}(n)$ also hold for the new partition, and they will be counted by $r_{2k,2k-2a}(m-2a, n-m)$.

(ii) $f_{\overline{1}} = 1$, *then* $f_1 = 2a+1$, *same as before we have that* $f_2 \le 2k-2a-3+f_{\overline{2}}$, *removing all* 1*'s and subtracting* 1 *from all other parts,* $f_1 \le (2k-2a-2)-1+f_{\overline{1}}$, *and other conditions hold for the first type partition hold, so they will counted by* $r_{2k,2k-2a-2}(m-2a+2,n)$.

So,

$$\overline{r}_{2k,2a}(m,n)-\overline{r}_{2k,2a-2}(m,n) = r_{2k,2k-2a}(m-2a,n-m)+r_{2k,2k-2a+2}(m-2a+2,n-m).$$

Multiplying all terms by $x^m q^n$ *and taking the summation over* m *and* n *for all terms,* $m,n \ge 0$ *and both integers, we have*

$$\sum_{m,n\ge 0} \overline{r}_{2k,2a}(m,n)x^m q^n - \sum_{m,n\ge 0} \overline{r}_{2k,2a-2}(m,n)x^m q^n =$$
$$\sum_{m,n\ge 0} r_{2k,2k-2a}(m-2a,n-m)x^m q^n + \sum_{m,n\ge 0} r_{2k,2k-2a-2}(m-2a+2,n-m)x^m q^n.$$

By substituting n *by* $n+m$, *then* m *by* $m+2a$ *for the first term and* n *by* $n+m$, *then* m *by* $m+2a-2$ *for the second term on the right hand side of this equation, we have*

$$\sum_{m,n\ge 0} \overline{r}_{2k,2a}(m,n)x^m q^n - \sum_{m,n\ge 0} \overline{r}_{2k,2a-2}(m,n)x^m q^n =$$
$$\sum_{m,n\ge 0} r_{2k,2k-2a}(m,n)x^{m+2a} q^{n+m+2a} + \sum_{m,n\ge 0} r_{2k,2k-2a-2}(m,n)x^{m+2a-2} q^{n+m+2a-2}$$
$$(xq)^{2a} \sum_{m,n\ge 0} r_{2k,2k-2a}(m,n)(xq)^m q^n + (xq)^{2a-2} \sum_{m,n\ge 0} r_{2k,2k-2a-2}(m,n)(xq)^m q^n.$$

This gives us the equation (4.3).

We will use these three functional equations to find identities for U and $\overline{U}$ when their first index is even. Next, we will prove three other functional equations using to find identities for the odd case of U and $\overline{U}$.

Theorem 4.2.6 *For* $k \ge a \ge 1$, *we have*

$$U_{2k+1,2a}(x) - U_{2k+1,2a-1}(x) = (xq)^{2a-1}\overline{U}_{2k+1,2k-2a+2}(xq) + (xq)^{2a+1}\overline{U}_{2k+1,2k-2a}(xq). \tag{4.4}$$

Proof 4.2.7 *Consider* $r_{2k+1,2a}(m,n) - r_{2k+1,2a-1}(m,n)$, *all overpartitions counted by* $r_{2k+1,a-1}(m,n)$ *are already in the* $r_{2k+1,2a}(m,n)$, *so with respect to the definition of* $U_{2k+1,2a}(n)$, *we can consider two cases,* *(i) there is no overlined* 1 *in the overpartition, i.e.* $f_{\overline{1}} = 0$*, so with respect to the first condition, there are exactly* $2a-1$ *of non-overlined* 1*'s in the overpartition, now by the last condition,* $f_2 \leq 2k-a+1+f_{\overline{2}}$*, removing all* 1*'s and subtracting* 1 *from all remaining parts,* $f_1 \leq (2k-2a+2)-1+f_{\overline{1}}$ *and other conditions of the second definition also hold, so we have overpartitions of* $n-m$ *with* $m-2a+1$ *parts of the second type,* $\overline{r}_{2k+1,2k-2a+2}(m-2a+1, n-m)$*.*

(ii) there is an overlined 1 *in the overpartition, i.e.* $f_{\overline{1}} = 1$*, so we have exactly* a *non-overlined 1's and one overlined ones in the overpartition, note that by assumption* $a \geq 1$*, so by the last condition* $f_2 \leq 2k-2a-1+f_{\overline{2}}$*, now by removing all* 1*'s,and subtracting* 1 *from all remaining parts,* $f_1 \leq (2k-2a)-1+f_{\overline{1}}$ *and again other conditions of the second definition also hold, so we have overpartitions of* $n-m$ *with* $m-2a-1$ *parts of the second type,* $\overline{r}_{2k+1,2k-2a}(m-2a-1, n-m)$*.*

So,

$$r_{2k+1,2a}(m,n) - r_{2k+1,2a-1}(m,n)$$
$$= \overline{r}_{2k+1,2k-2a+2}(m-2a+1, n-m) + \overline{r}_{2k+1,2k-2a}(m-2a-1, n-m).$$

Multiplying all terms by $x^m q^n$ *and taking the summation over* m *and* n *for all terms,* $m, n \geq 0$ *and both integers, we have*

$$\sum_{m,n\geq 0} r_{2k+1,2a}(m,n)x^m q^n - \sum_{m,n\geq 0} r_{2k+1,2a-1}(m,n)x^m q^n =$$
$$\sum_{m,n\geq 0} \overline{r}_{2k+1,2k-2a+2}(m-2a+1, n-m)x^m q^n +$$
$$\sum_{m,n\geq 0} \overline{r}_{2k+1,2k-2a}(m-2a-1, n-m)x^m q^n.$$

By substituting n *by* $n+m$*, then* m *by* $m+2a-1$ *for the first term and* n *by* $n+m$*, then* m *by* $m+2a+1$ *for the second term on the right hand side of this equation, we have*

$$\sum_{m,n\geq 0} r_{2k+1,2a}(m,n)x^m q^n - \sum_{m,n\geq 0} r_{2k+1,2a-1}(m,n)x^m q^n =$$
$$\sum_{m,n\geq 0} \overline{r}_{2k+1,2k-2a+2}(m,n)x^{m+2a-1}q^{n+m+2a-1} + \sum_{m,n\geq 0} \overline{r}_{2k+1,2k-2a}(m,n)x^{m+2a+1}q^{n+m+2a+1}$$
$$(xq)^{2a-1} \sum_{m,n\geq 0} \overline{r}_{2k+1,2k-2a+2}(m,n)(xq)^m q^n + (xq)^{2a+1} \sum_{m,n\geq 0} \overline{r}_{2k+1,2k-2a}(m,n)(xq)^m q^n.$$

This gives us the equation (4.4)*.*

The second one is similar.

Theorem 4.2.8 *For $k \geq a \geq 1$, we have*

$$U_{2k+1,2a+1}(x) - U_{2k+1,2a}(x) = (xq)^{2a}\overline{U}_{2k+1,2k-2a+2}(xq) + (xq)^{2a+2}\overline{U}_{2k+1,2k-2a}(xq). \quad (4.5)$$

The proof of Theorem (4.2.8) is the same as Theorem (4.2.6), we just need to replace $2a$ by $2a+1$, and at the end we need to use Lemma (4.1.1) as follows

$$\overline{U}_{2k+1,2k-2a+1}(xq) = \overline{U}_{2k+1,2k-2a+2}(xq)$$

and

$$\overline{U}_{2k+1,2k-2a-1}(xq) = \overline{U}_{2k+1,2k-2a}(xq).$$

Lastly, we have the following functional equation from $\overline{U}$ to U,

Theorem 4.2.9 *For $k \geq a \geq 1$, we have*

$$\overline{U}_{2k+1,2a}(x) - \overline{U}_{2k+1,2a-2}(x) = (xq)^{2a}U_{2k+1,2k-2a+1}(xq) + (xq)^{2a-2}U_{2k+1,2k-2a+3}(xq). \quad (4.6)$$

Proof 4.2.10 *Consider $\overline{U}_{2k+1,2a}(x) - \overline{U}_{2k+1,2a-2}(x)$, when $1 \leq 2a \leq 2k$, in this case for $\overline{r}_{2k+1,2a}(m,n) - \overline{r}_{2k+1,2a-2}(m,n)$, there are two cases*

(i) $f_{\bar{1}} = 1$, then $f_1 = 2a-1$, so according to the last condition in the definition, $f_2 \leq 2k-2a+f_{\bar{2}}$, removing all 1's and subtracting 1 from all remaining parts, the new partition satisfies in the conditions of the first definition for $2k$ and $2k-2a+1$, $r_{2k+1,2k-2a+1}(m-2a, n-m)$.

(ii) $f_{\bar{1}} = 0$, then $f_1 = 2a-2$, so according to the last condition in the definition, $f_2 \leq 2k-2a+2+f_{\bar{2}}$, removing all 1's and subtracting 1 from all remaining parts, the new partition satisfies in the conditions of the first definition for $2k$ and $2k-2a+3$, $r_{2k+1,2k-2a+3}(m-2a+2, n-m)$.

So,

$$\overline{r}_{2k+1,2a}(m,n) - \overline{r}_{2k+1,2a-2}(m,n) = r_{2k+1,2k-2a+1}(m-2a, n-m) + r_{2k+1,2k-2a+3}(m-2a+2, n-m).$$

Multiplying all terms by $x^m q^n$ and taking the summation over m and n for all terms, $m, n \geq 0$ and both integers, we have

$$\sum_{m,n\geq 0} \overline{r}_{2k+1,2a}(m,n)x^m q^n - \sum_{m,n\geq 0} \overline{r}_{2k+1,2a-2}(m,n)x^m q^n =$$
$$\sum_{m,n\geq 0} r_{2k+1,2k-2a+1}(m-2a,n-m)x^m q^n + \sum_{m,n\geq 0} r_{2k+1,2k-2a+3}(m-2a+2,n-m)x^m q^n.$$

By substituting n by $n+m$, then m by $m+2a$ for the first term and n by $n+m$, then m by $m+2a-2$ for the second term on the right hand side of this equation, we have

$$\sum_{m,n\geq 0} \overline{r}_{2k+1,2a}(m,n)x^m q^n - \sum_{m,n\geq 0} \overline{r}_{2k+1,2a-2}(m,n)x^m q^n =$$
$$\sum_{m,n\geq 0} r_{2k+1,2k-2a+1}(m,n)x^{m+2a}q^{n+m+2a} + \sum_{m,n\geq 0} r_{2k+1,2k-2a+3}(m,n)x^{m+2a-2}q^{n+m+2a-2}$$
$$(xq)^{2a}\sum_{m,n\geq 0} r_{2k+1,2k-2a+1}(m,n)(xq)^m q^n + (xq)^{2a-2}\sum_{m,n\geq 0} r_{2k+1,2k-2a+3}(m,n)(xq)^m q^n$$

This gives us the equation (4.6)*.*

4.3 Even cases

In this section, we will go through the results from Sang, Shi and Yee's paper, we will find identities for $U_{2k,2a}$, $U_{2k,2a+1}$ and $\overline{U}_{2k,2a} = \overline{U}_{2k,2a-1}$ using our constructive method. For this purpose, we will use the functional equations (4.1), (4.2) and (4.3).

Then, as it has been discussed in the chapter 2 and used for other Rogers-Ramanujan type partitions given in the chapters 2 and 3, we will use the following form of generating functions for U and $\overline{U}$,

$$U_{2k,2a}(x) = \sum_{n\geq 0} \alpha_n^e(x)q^{2anA_1} + \beta_n^e(x)x^{2aB_1}q^{2aC_1}q^{2anD_1}$$

for the even first index and the even second one,

$$U_{2k,2a+1}(x) = \sum_{n\geq 0} \alpha_n^o(x)q^{(2a+1)nA_1} + \beta_n^o(x)x^{(2a+1)B_1}q^{(2a+1)C_1}q^{(2a+1)nD_1}$$

for the even first index and the odd second one, and

$$\overline{U}_{2k,2a}(x) = \sum_{n\geq 0} \overline{\alpha}_n(x)q^{2anA_2} + \overline{\beta}_n(x)x^{2aB_2}q^{2aC_2}q^{2anD_2}.$$

Note that if the second index is odd, we can change it to even by Theorem (4.1.1). By functional equations (4.1) and (4.2), we have

$$\begin{aligned}\alpha_n^e(x)q^{2anA_1} - \alpha_n^o(x)q^{(2a-1)nA_1} &= \overline{\beta}_{n-1}(xq)(xq)^{2a-1}x^{(2k-2a+2)B_2} \\ \times q^{(2k-2a+2)(B_2+C_2)}q^{(2k-2a+2)(n-1)D_2}&(1+x^2q^2x^{-2B_2}q^{-2B_2-2C_2}q^{-2(n-1)D_2}),\end{aligned} \tag{4.7}$$

$$\begin{aligned}\alpha_n^o(x)q^{(2a+1)nA_1} - \alpha_n^e(x)q^{2anA_1} &= \overline{\beta}_{n-1}(xq)(xq)^{2a}x^{(2k-2a)B_2} \\ \times q^{(2k-2a)(B_2+C_2)}q^{(2k-2a)(n-1)D_2}&(1+x^2q^2x^{-2B_2}q^{-2B_2-2C_2}q^{-2(n-1)D_2}),\end{aligned} \tag{4.8}$$

$$\begin{aligned}\beta_n^e(x)x^{2aB_1}q^{2aC_1}q^{2anD_1} &- \beta_n^o(x)x^{(2a-1)B_1}q^{(2a-1)C_1}q^{(2a-1)nD_1} \\ &= \overline{\alpha}_n(xq)(xq)^{2a-1}q^{(2k-2a+2)nA_2}(1+x^2q^2q^{-2nA_2})\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}\beta_n^o(x)x^{(2a+1)B_1}q^{(2a+1)C_1}q^{(2a+1)nD_1} &- \beta_n^e(x)x^{2aB_1}q^{2aC_1}q^{2anD_1} \\ &= \overline{\alpha}_n(xq)(xq)^{2a}q^{(2k-2a)nA_2}(1+x^2q^2q^{-2nA_2}),\end{aligned} \tag{4.10}$$

also, by functional equation (4.3), we have

$$\begin{aligned}&\overline{\alpha}_n(x) = -\beta_{n-1}^e(xq) \\ &\times \frac{(xq)^{2a}x^{(2k-2a)B_1}q^{(2k-2a)(B_1+C_1)}q^{(2k-2a)(n-1)D_1}(1+x^{-2}q^{-2}x^{2B_1}q^{2B_1+2C_1}q^{2(n-1)D_1})}{q^{(2a-2)nA_2}(1-q^{2nA_2})}\end{aligned} \tag{4.11}$$

and

$$\overline{\beta}_n(x) = -\alpha_n^e(xq)\frac{(xq)^{2a}q^{(2k-2a)nA_1}(1+x^{-2}q^{-2}q^{2nA_1})}{x^{(2a-2)B_2}q^{(2a-2)C_2}q^{(2a-2)nD_2}(1-x^{2B_2}q^{2C_2}q^{2nD_2})}. \tag{4.12}$$

From the two first equations (4.7) and (4.8), we have

$$\begin{aligned}&\begin{pmatrix} q^{2anA_1} & -q^{(2a-1)nA_1} \\ -q^{2anA_1} & q^{(2a+1)nA_1}\end{pmatrix}\begin{pmatrix}\alpha_n^e(x) \\ \alpha_n^o(x)\end{pmatrix} = \overline{\beta}_{n-1}(xq)(xq)^{2a-1}x^{(2k-2a+2)B_2}q^{(2k-2a+2)(B_2+C_2)} \\ &\times q^{(2k-2a+2)(n-1)D_2}(1+x^2q^2x^{-2B_2}q^{-2B_2-2C_2}q^{-2(n-1)D_2})\begin{pmatrix} 1 \\ xqx^{-2B_2}q^{-2B_2-2C_2}q^{-2(n-1)D_2}\end{pmatrix}\end{aligned}$$

and similarly, for the second pair, from equations (4.9) and (4.10), we have

$$\begin{aligned}&\begin{pmatrix} x^{2aB_1}q^{2aC_1}q^{2anD_1} & -x^{(2a-1)B_1}q^{(2a-1)C_1}q^{(2a-1)nD_1} \\ -x^{2aB_1}q^{2aC_1}q^{2anD_1} & x^{(2a+1)B_1}q^{(2a+1)C_1}q^{(2a+1)nD_1}\end{pmatrix}\begin{pmatrix}\beta_n^e(x) \\ \beta_n^o(x)\end{pmatrix} \\ &= \overline{\alpha}_n(xq)(xq)^{2a-1}q^{(2k-2a+2)nA_2}(1+x^2q^2q^{-2nA_2})\begin{pmatrix} 1 \\ xqq^{-2nA_2}\end{pmatrix}.\end{aligned}$$

From this system, we need $\alpha_n^e(x)$ and $\beta_n^e(x)$, so

$$\overline{\alpha}_n(x) = -\beta_{n-1}^e(xq)\frac{(xq)^{2a}x^{(2k-2a)B_1}q^{(2k-2a)(B_1+C_1+(n-1)D_1)}}{q^{(2a-2)nA_2}} \times \frac{(1+x^{-2}q^{-2}x^{2B_1}q^{2B_1+2C_1+2(n-1)D_1})}{(1-q^{2nA_2})},$$

$$\overline{\beta}_n(x) = -\alpha_n^e(xq)\frac{(xq)^{2a}q^{(2k-2a)nA_1}(1+x^{-2}q^{-2}q^{2nA_1})}{x^{(2a-2)B_2}q^{(2a-2)C_2+(2a-2)nD_2}(1-x^{2B_2}q^{2C_2}q^{2nD_2})},$$

$$\alpha_n^e(x) = -\overline{\beta}_{n-1}(xq)\frac{(xq)^{2a-1}x^{(2k-2a+2)B_2}q^{(2k-2a+2)(B_2+C_2+(n-1)D_2)}}{q^{(2a-2)nA_1}} \times\frac{(1+x^2q^2x^{-2B_2}q^{-2B_2-2C_2-2(n-1)D_2})(1+xqx^{-2B_2}q^{-2nA_1-2B_2-2C_2-2(n-1)D_2})}{(1-q^{2nA_1})}$$

and

$$\beta_n^e(x) = -\overline{\alpha}_n(xq)\frac{(xq)^{2a-1}q^{(2k-2a+2)nA_2}(1+x^2q^2q^{-2nA_2})(1+xqx^{-2B_1}q^{-2nA_2-2C_1-2nD_1})}{x^{(2a-2)B_1}q^{(2a-2)(C_1+nD_1)}(1-x^{2B_1}q^{2C_1}q^{2nD_1})}.$$

By this recurrence, we can find α_n^e and $\overline{\beta}_n$ in terms of α_0^e, also $\overline{\alpha}_n$ and β_n^e in terms of $\overline{\alpha}_0$, then we can make some of finite products into infinite products, and we have

$$\begin{aligned}\alpha_n^e(x) =&\tilde{\alpha}_0^e(xq^{2n})f_a(x)\\ &\times\frac{(-x^2q^4q^{-2(n-1)A_1};q^{4+2A_1})_\infty(-x^2q^2x^{-2B_2}q^{-2B_2-2C_2-2(n-1)D_2};q^{4-4B_2+2D_2})_n}{(q^{-2A_1};q^{-2A_1})_n}\\ &\times\frac{(-x^{-1}q^{-1}x^{2B_2}q^{2nA_1+2B_2+2C_2+2(n-1)D_2};q^{-2-2A_1+4B_2-2D_2})_\infty}{(x^{2B_2}q^{2B_2+2C_2+2(n-1)D_2};q^{4B_2-2D_2})_\infty},\end{aligned}$$

$$\begin{aligned}\beta_n^e(x) =&\tilde{\overline{\alpha}}_0(xq^{2n+1})g_a(x)\\ &\times\frac{(-x^2q^2q^{-2nA_2};q^{4+2A_2})_\infty(-x^2q^4x^{-2B_1}q^{-4B_1-2C_1-2(n-1)D_1};q^{4-4B_1+2D_1})_n}{(q^{-2A_2};q^{-2A_2})_n}\\ &\times\frac{(-x^{-1}q^{-1}x^{2B_1}q^{2nA_2+2C_1+2nD_1};q^{-2-2A_2+4B_1-2D_1})_\infty}{(x^{2B_1}q^{2C_1+2nD_1};q^{4B_1-2D_1})_\infty},\end{aligned}$$

$$\begin{aligned}\overline{\alpha}_n(x) =&\tilde{\overline{\alpha}}_0(xq^{2n})f'_a(x)\\ &\times\frac{(-x^2q^4q^{-2(n-1)A_2};q^{4+2A_2})_\infty(-x^2q^2x^{-2B_1}q^{-2B_1-2C_1-2(n-1)D_1};q^{4-4B_1+2D_1})_n}{(q^{-2A_2};q^{-2A_2})_n}\\ &\times\frac{(-x^{-1}q^{-2}x^{2B_1}q^{2(n-1)A_2+2B_1+2C_1+2(n-1)D_1};q^{-2-2A_2+4B_1-2D_1})_\infty}{(x^{2B_1}q^{2B_1+2C_1+2(n-1)D_1};q^{4B_1-2D_1})_\infty}\end{aligned}$$

and

$$\begin{aligned}\overline{\beta}_n(x) =&\tilde{\alpha}_0^e(xq^{2n+1})g'_a(x)\\ &\times\frac{(-x^2q^2q^{-2nA_1};q^{4+2A_1})_\infty(-x^2q^4x^{-2B_2}q^{-4B_2-2C_2-2(n-1)D_2};q^{4-4B_2+2D_2})_n}{(q^{-2A_1};q^{-2A_1})_n}\\ &\times\frac{(-x^{-1}q^{-2}x^{2B_2}q^{2nA_1+4B_2+2C_2+2(n-1)D_2};q^{-2-2A_1+4B_2-2D_2})_\infty}{(x^{2B_2}q^{2C_2+2nD_2};q^{4B_2-2D_2})_\infty}\end{aligned}$$

where

$$\begin{aligned}f_a(x) =&(-1)^n x^{4an-2n+(2k-4a+2)nB_2}\\&\times q^{2n(2n+1)a-2n(n+1)+(n(n-1)(k-a+1)-n(n+1)(a+1))A_1+(k-2a+1)(2n^2B_2+2nC_2+n(n-1)D_2)},\end{aligned}$$

$$\begin{aligned}g_a(x) =&(-1)^n x^{2(2n+1)a-2n+(-4an-2a+2k+2n)B_1}\\&\times q^{(2n+1)(2n+2)a-2n(n+1)+(k-2a)n(n+1)A_2+(k-2a+1)(2n(n+1)B_1+2nC_1+n(n-1)D_1)},\end{aligned}$$

$$\begin{aligned}f'_a(x) =&(-1)^n x^{4an-2n+(2k-4a+2)nB_1}\\&\times q^{2n(2n+1)a-2n^2+(n(n-1)(k-a)-n(n+1)a)A_2+(k-2a+1)(2n^2B_1+2nC_1+n(n-1)D_1)},\end{aligned}$$

$$\begin{aligned}g'_a(x) =&(-1)^n x^{2(2n+1)a-2(n+1)+(2k-2a+2)nB_2-(2a-2)B_2} q^{(2n+1)(2n+2)a-2n^2-4n-2}\\&\times q^{(k-2a)n(n+1)A_1+(k-2a+1)(2n(n+1)B_2+2nC_2)-(2a-2)C_2+((k-a)n(n-1)-(a-1)n(n+1))D_2},\end{aligned}$$

$$\begin{aligned}\tilde{\alpha}_0^e(xq^{2n}) =&\alpha_0^e(xq^{2n})\frac{(x^{2B_2}q^{(4n+2)B_2+2C_2-2D_2};q^{4B_2-2D_2})_\infty}{(-x^2q^4q^{2A_1+4n};q^{4+2A_1})_\infty}\\&\times\frac{1}{(-x^{-1}q^{-1}x^{2B_2}q^{(4n+2)B_2+2C_2-2D_2-2n};q^{-2-2A_1+4B_2-2D_2})_\infty}\end{aligned}$$

and

$$\begin{aligned}\tilde{\overline{\alpha}}_0(xq^{2n}) =&\overline{\alpha}_0(xq^{2n})\frac{(x^{2B_1}q^{(4n+2)B_1+2C_1-2D_1};q^{4B_1-2D_1})_\infty}{(-x^2q^4q^{2A_2+4n};q^{4+2A_2})_\infty}\\&\times\frac{1}{(-x^{-1}q^{-2}x^{2B_1}q^{-2A_2+(4n+2)B_1+2C_1-2D_1-2n};q^{-2-2A_2+4B_1-2D_1})_\infty}.\end{aligned}$$

For the check step we will use the check equations

$$U_{2k,0}(x)=\overline{U}_{2k,0}(x)=0 \tag{4.13}$$

which is a boundary condition. We want the equations to be consistent, in fact, this check equation is a verification to the construction step, so we consider some assumptions, as $A_1=A_2$, $B_1=B_2$, $D_1=D_2$, $4B_1-2D_1 \mid 2B_1-2D_1$ and $4+2A_1 \mid 2$, so $B_1=\frac{1-s}{1-2s}D_1$ and $A_1=\frac{1-2s'}{s'}$ for some integers s and s', same as described in the second chapter, we choose the simplest ones, in this case $s=0$ and $s'=1$, then by putting in the check equation (4.13), we have $A_i=-1$ and $B_i=C_i=D_i=1$ for $i=1,2$. So,

$$\begin{aligned}U_{2k,2a}(x)=\sum_{n\geq 0}(-1)^n q^{n(n+1)(2k+1)}&\frac{(-x^2q^{2n+2};q^2)_\infty(-xq;q^2)_\infty(-q^{-2n};q^2)_n}{(q^2;q^2)_n(xq^{2n+2};q^2)_\infty}\\&\times(x^{2kn}q^{-2an}-x^{2a}q^{2a}q^{2an})\end{aligned}$$

and

$$\overline{U}_{2k,2a}(x) = \sum_{n\geq 0}(-1)^n x^{2kn} q^{n(n+1)(2k+1)} \frac{(-x^2q^{2n+2};q^2)_\infty(-xq^2;q^2)_\infty(-q^{-2n};q^2)_n}{(q^2;q^2)_n(xq^{2n+2};q^2)_\infty} \times (q^{-2an} - x^{(2n+3)a}q^{2a}q^{2an}).$$

Now, we apply $x=1$, so the generating functions is for any partition of any length.

$$U_{2k,2a}(1) = \frac{(-q;q)_\infty}{(q^2;q^2)_\infty}\sum_{n\geq 0}(-1)^n q^{2n(n+1)k-2an}(1-q^{(2n+1)2a})$$

and

$$\overline{U}_{2k,2a}(1) = \frac{(-q^2;q^2)^2_\infty}{(q^2;q^2)_\infty}\sum_{n\geq 0}(-1)^n q^{2n(n+1)k-2an}(1-q^{(2n+1)2a}).$$

Then by using (2.1.2), for $i=2a$ and $t=\frac{4k-1}{2}$, we have

$$U_{2k,2a}(1) = \frac{(-q;q)_\infty(q^{2a},q^{4k-2a},q^{4k};q^{4k})_\infty}{(q^2;q^2)_\infty}$$

and

$$\overline{U}_{2k,2a}(1) = \frac{(-q^2;q^2)^2_\infty(q^{2a},q^{4k-2a},q^{4k};q^{4k})_\infty}{(q^2;q^2)_\infty}.$$

These prove Theorems (4.1.2) and (4.1.6). For the remaining case, $U_{2k,2a+1}$, we use functional equations (4.2.1) and (4.2.3) to find $\alpha^o_n(x)$ in terms of $\overline{\beta}_{n-1}(xq)$ and $\beta^o_n(x)$ in terms of $\overline{\alpha}_n(xq)$, then having $\overline{\alpha}_n$ and $\overline{\beta}_{n-1}$ in terms of α_0, we can find our last identity in this case.

$$\alpha^o_n(x) = \tilde{\alpha}^e_0(xq^{2n+1}) \times \frac{(-1)^n x^{2kn}q^{2kn^2+2kn-n}(-x^2q^{2n+2};q^2)_\infty(-xq^3;q^2)_\infty(-q^2;q^2)_n(1+xq^{2n+1})}{(q^2;q^2)_n(x^2q^{2n+2};q^2)_\infty}$$

and

$$\beta^o_n(x) = -\tilde{\overline{\alpha}}_0(xq^{2n+2}) \times \frac{(-1)^n x^{2kn}q^{2kn^2+2kn-n}(-x^2q^{2n+2};q^2)_\infty(-xq^3;q^2)_\infty(-q^2;q^2)_n(1+xq^{2n+1})}{(q^2;q^2)_n(x^2q^{2n+2};q^2)_\infty}.$$

So, from

$$U_{2k,2a+1}(x) = \sum_{n\geq 0}\alpha^o_n(x)q^{-(2a+1)n} + \beta^o_n(x)x^{2a+1}q^{(2a+1)(n+1)}$$

we have

$$\begin{aligned}U_{2k,2a+1}(1) &= \frac{(-q^2;q)_\infty}{(q^2;q^2)_\infty}\sum_{n\geq 0}(-1)^n q^{2kn^2+2kn-n-(2a+1)n}(1+q^{2n+1})(1-q^{(2n+1)(2a+1)})\\ &= \frac{(-q^2;q)_\infty}{(q^2;q^2)_\infty}\sum_{n\geq 0}(-1)^n q^{2kn^2+2kn-(2a+2)n}(1-q^{(2n+1)(2a+2)})\\ &+ \frac{q(-q^2;q)_\infty}{(q^2;q^2)_\infty}\sum_{n\geq 0}(-1)^n q^{2kn^2+2kn-2an}(1-q^{(2n+1)2a}).\end{aligned}$$

Using (2.1.2) twice, once for $i = 2a+1$ and $t = \frac{4k-1}{2}$, and again for $i = 2a$ and $t = \frac{4k-1}{2}$, we have

$$U_{2k,2a+1}(1) = \frac{(-q^2;q)_\infty(q^{2a+2},q^{4k-2a-2},q^{4k};q^{4k})_\infty}{(q^2;q^2)_\infty} + \frac{q(-q^2;q)_\infty(q^{2a},q^{4k-2a},q^{4k};q^{4k})_\infty}{(q^2;q^2)_\infty}$$

This proves our last identity for even case, Theorem (4.1.3).

4.4 Odd cases

Same as the even case, we want to find identities for $U_{2k+1,2a}$, $U_{2k+1,2a+1}$ and $\overline{U}_{2k+1,2a} = \overline{U}_{2k+1,2a-1}$. We use the functional equations (4.2.6), (4.2.8) and (4.2.9).

Same as even case, by

$$U_{2k+1,2a}(x) = \sum_{n\geq 0}\alpha_n^e(x)q^{2anA_1} + \beta_n^e(x)x^{2aB_1}q^{2aC_1}q^{2anD_1}$$

for the odd first index and the even second one,

$$U_{2k+1,2a+1}(x) = \sum_{n\geq 0}\alpha_n^o(x)q^{(2a+1)nA_1} + \beta_n^o(x)x^{(2a+1)B_1}q^{(2a+1)C_1}q^{(2a+1)nD_1}$$

for the odd first index and the odd second one, and

$$\overline{U}_{2k+1,2a}(x) = \sum_{n\geq 0}\overline{\alpha}_n(x)q^{2anA_2} + \overline{\beta}_n(x)x^{2aB_2}q^{2aC_2}q^{2anD_2}.$$

Same as section (4.3), for odd a we use Lemma (4.1.1) to make it even. By functional equations (4.4) and (4.5), we have

$$\begin{aligned}\alpha_n^e(x)q^{2anA_1} - \alpha_n^o(x)q^{(2a-1)nA_1} &= \overline{\beta}_{n-1}(xq)(xq)^{2a-1}x^{(2k-2a+2)B_1}q^{(2k-2a+2)(B_2+C_2)}\\ &\quad q^{(2k-2a+2)(n-1)D_2}(1+x^2q^2x^{-2B_2}q^{-2B_2-2C_2}q^{-2(n-1)D_2}),\end{aligned} \tag{4.14}$$

$$\begin{aligned}\alpha_n^o(x)q^{(2a+1)nA_1} - \alpha_n^e(x)q^{2anA_1} &= \overline{\beta}_{n-1}(xq)(xq)^{2a}x^{(2k-2a+2)B_1}q^{(2k-2a+2)(B_2+C_2)}\\ &\quad q^{(2k-2a+2)(n-1)D_2}(1+x^2q^2x^{-2B_2}q^{-2B_2-2C_2}q^{-2(n-1)D_2}),\end{aligned} \tag{4.15}$$

$$\beta_n^e(x)x^{2aB_1}q^{2aC_1}q^{2anD_1} - \beta_n^o(x)x^{(2a-1)B_1}q^{(2a-1)B_1}q^{(2a-1)nD_1} = \overline{\alpha}_n(xq)(xq)^{2a-1}q^{(2k-2a+2)nA_2}(1+x^2q^2q^{-2nA_2}) \tag{4.16}$$

and

$$\beta_n^o(x)x^{(2a+1)B_1}q^{(2a+1)C_1}q^{(2a+1)nD_1} - \beta_n^e(x)x^{2aB_1}q^{2aB_1}q^{2anD_1} = \overline{\alpha}_n(xq)(xq)^{2a}q^{(2k-2a+2)nA_2} (1+x^2q^2q^{-2nA_2}), \tag{4.17}$$

also, by functional equation (4.6), we have

$$\overline{\alpha}_n(x)q^{(2a-2)nA_2}(1-q^{2nA_2}) = -\beta_{n-1}^o(xq)(xq)^{2a}x^{(2k-2a+1)B_1} q^{(2k-2a+1)(B_1+C_1)}q^{(2k-2a+)(n-1)D_1}(1+x^{-2}q^{-2}x^{2B_1}q^{2B_1+2C_1}q^{2(n-1)D_1}) \tag{4.18}$$

and

$$\overline{\beta}_n(x)x^{(2a-2)B_2}q^{(2a-2)C_2}q^{(2a-2)nD_2}(1-x^{2B_2}q^{2C_2}q^{2nD_2}) = -\alpha_n^o(xq)(xq)^{2a} q^{(2k-2a+1)nA_1}(1+x^{-2}q^{-2}q^{2nA_1}). \tag{4.19}$$

From the two first equations (4.14) and (4.15), we have

$$\begin{pmatrix} q^{2anA_1} & -q^{(2a-1)nA_1} \\ -q^{2anA_1} & q^{(2a+1)nA_1} \end{pmatrix} \begin{pmatrix} \alpha_n^e(x) \\ \alpha_n^o(x) \end{pmatrix} = \overline{\beta}_{n-1}(xq)(1+x^2q^2x^{-2B_2}q^{-2B_2-2C_2}q^{-2(n-1)D_2}) (xq)^{2a-1}x^{(2k-2a+2)B_2}q^{(2k-2a+2)(B_2+C_2+(n-1)D_2)} \begin{pmatrix} 1 \\ xq \end{pmatrix}$$

and from equations (4.16) and (4.17), we have

$$\begin{pmatrix} x^{2aB_1}q^{2a(C_+nD_1)} & -x^{(2a-1)B_1}q^{(2a-1)(C_1+nD_1)} \\ -x^{2aB_1}q^{2a(C_1+nD_1)} & x^{(2a+1)B_1}q^{(2a+1)(C_1+nD_1)} \end{pmatrix} \begin{pmatrix} \beta_n^e(x) \\ \beta_n^o(x) \end{pmatrix} = \overline{\alpha}_n(xq)(xq)^{2a-1} q^{(2k-2a+2)nA_2}(1+x^2q^2q^{-2nA_2}) \begin{pmatrix} 1 \\ xq \end{pmatrix}.$$

From these systems, we need $\alpha_n^o(x)$ and $\beta_n^o(x)$, in fact, with respect to equations (4.18) and (4.19), to make the recurrences work, we must take them $\alpha_n^o(x)$ and $\beta_n^o(x)$, so

$$\overline{\alpha}_n(x) = -\beta_{n-1}^o(xq)\frac{(xq)^{2a}x^{(2k-2a+1)B_1}q^{(2k-2a+1)(B_1+C_1+(n-1)D_1)}}{q^{(2a-2)nA_2}} \frac{(1+x^{-2}q^{-2}x^{2B_1}q^{2B_1+2C_1+2(n-1)D_1})}{(1-q^{2nA_2})},$$

$$\overline{\beta}_n(x) = -\alpha_n^o(xq)\frac{(xq)^{2a}q^{(2k-2a+1)nA_1}(1+x^{-2}q^{-2}q^{2nA_1})}{x^{(2a-2)B_2}q^{(2a-2)C_2+(2a-2)nD_2}(1-x^{2B_2}q^{2C_2}q^{2nD_2})},$$

$$\alpha_n^o(x) = -\overline{\beta}_{n-1}(xq)\frac{(xq)^{2a-1}x^{(2k-2a+2)B_2}q^{(2k-2a+2)(B_2+C_2+(n-1)D_2)}}{q^{(2a-1)nA_1}} \frac{(1+x^2q^2x^{-2B_2}q^{-2B_2-2C_2-2(n-1)D_2})(1+xq)}{(1-q^{2nA_1})}$$

and

$$\beta_n^o(x) = -\overline{\alpha}_n(xq)\frac{(xq)^{2a-1}q^{(2k-2a+2)nA_2}(1+x^2q^2q^{-2nA_2})(1+xq)}{x^{(2a-1)B_1}q^{(2a-1)(C_1+nD_1)}(1-x^{2B_1}q^{2C_1}q^{2nD_1})}.$$

Same as the even case, by this recurrence, we can find α_n^o and $\overline{\beta}_n$ in terms of α_0^o, also $\overline{\alpha}_n$ and β_n^o in terms of $\overline{\alpha}_0$, then we can make some of finite products into infinite products, and we have

$$\alpha_n^o(x) =\tilde{\alpha}_0^o(xq^{2n})f_a(x)\frac{(-x^2q^4q^{-2(n-1)A_1};q^{4+2A_1})_\infty}{(x^{2B_2}q^{2B_2+2C_2+2(n-1)D_2};q^{4B_2-2D_2})_\infty} \frac{(-xq;q^2)_\infty(-x^2q^2x^{-2B_2}q^{-2B_2-2C_2-2(n-1)D_2};q^{4-4B_2+2D_2})_n}{(q^{-2A_1};q^{-2A_1})_n},$$

$$\beta_n^o(x) =\tilde{\overline{\alpha}}_0(xq^{2n+1})g_a(x)\frac{(-x^2q^2q^{-2nA_2};q^{4+2A_2})_\infty}{(x^{2B_1}q^{2C_1+2nD_1};q^{4B_1-2D_1})_\infty} \frac{(-xq;q^2)_\infty(-x^2q^4x^{-2B_1}q^{-4B_1-2C_1-2(n-1)D_1};q^{4-4B_1+2D_1})_n}{(q^{-2A_2};q^{-2A_2})_n},$$

$$\overline{\alpha}_n(x) =\tilde{\overline{\alpha}}_0(xq^{2n})f_a'(x)\frac{(-x^2q^4q^{-2(n-1)A_2};q^{4+2A_2})_\infty}{(x^{2B_1}q^{2B_1+2C_1+2(n-1)D_1};q^{4B_1-2D_1})_\infty} \frac{(-xq^2;q^2)_\infty(-x^2q^2x^{-2B_1}q^{-2B_1-2C_1-2(n-1)D_1};q^{4-4B_1+2D_1})_n}{(q^{-2A_2};q^{-2A_2})_n}$$

and

$$\overline{\beta}_n(x) =\tilde{\alpha}_0^o(xq^{2n+1})g_a'(x)\frac{(-x^2q^2q^{-2nA_1};q^{4+2A_1})_\infty}{(x^{2B_2}q^{2C_2+2nD_2};q^{4B_2-2D_2})_\infty} \frac{(-xq^2;q^2)_\infty(-x^2q^4x^{-2B_2}q^{-4B_2-2C_2-2(n-1)D_2};q^{4-4B_2+2D_2})_n}{(q^{-2A_1};q^{-2A_1})_n}$$

where

$$f_a(x) =(-1)^n x^{4an-3n+(2k-4a+4)nB_2}q^{2n(2n+1)a-n(3n+2)+(n(n-1)(2k-2a+2)-\frac{n(n+1)}{2}(2a+1))A_1} q^{(2k-4a+4)(n^2B_2+nC_2+\frac{n(n-1}{2})D_2)},$$

$$g_a(x) = (-1)^n x^{2(2n+1)a-(3n+1)+(2k-4a+4)nB_1-(2a-1)B_1} q^{(2n+1)(2n+2)a-(n+1)(3n+1)}$$
$$q^{(k-2a+1)n(n+1)A_2+(2k-4a+4)(n(n+1)B_1+nC_1)-(2a-1)C_1+((2k-2a+3)\frac{n(n-1)}{2}-(2a-1)\frac{n(n+1)}{2})D_1)},$$

$$f'_a(x) = (-1)^n x^{4an-3n+(2k-4a+4)nB_1} q^{2n(2n+1)a-n(3n+1)+(n(n-1)(k-a+1)-n(n+1)a)A_2}$$
$$q^{(k-2a+2)(2n^2B_1+2nC_1+n(n-1)D_1)},$$

$$g'_a(x) = (-1)^n x^{2(2n+1)a-(3n+2)+(2k-4a+4)nB_2-(2a-2)B_2} q^{(2n+1)(2n+2)a-(n+1)(3n-2)}$$
$$q^{(k-2a+1)n(n+1)A_1+(2k-4a+4)(n(n+1)B_2+nC_2)-(2a-2)C_2+((k-2a+2)n(n-1)-(2a-2)n)D_2},$$

$$\tilde{\alpha}_0^o(xq^{2n}) = \alpha_0^o(xq^{2n}) \frac{(x^{2B_2}q^{2B_2+2C_2+2(n-1)D_2+n(4B_2-2D_2)}; q^{4B_2-2D_2})_\infty}{(-x^2q^2q^{-2(n-1)A_1+n(4+2A_1)}; q^{4+2A_1})_\infty(-xq^{2n+1}; q^2)_\infty}$$

and

$$\tilde{\overline{\alpha}}_0(xq^{2n}) = \overline{\alpha}_0(xq^{2n}) \frac{(x^{2B_1}q^{2B_1+2C_1+2(n-1)D_1+n(4B_1-2D_1)}; q^{4B_1-2D_1})_\infty}{(-x^2q^4q^{-2(n-1)A_2+n(4+2A_2)}; q^{4+2A_2})_\infty(-xq^{2n+2}; q^2)_\infty}.$$

The check equation $\overline{U}_{2k,0}(x) = 0$ is again a boundary condition, and with the same argument that we had for even case, we have $A_i = -1$ and $B_i = C_i = D_i = 1$ for $i = 1, 2$. So,

$$\overline{U}_{2k+1,2a}(x) = \sum_{n\geq 0} (-1)^n x^{(2k+1)n} q^{2n(n+1)(k+1)} (q^{-2an} - x^{2a}q^{2a}q^{2an})$$
$$\frac{(-x^2q^{2n+2}; q^2)_\infty(-xq^2; q^2)_\infty(-q^{-2n}; q^2)_n}{(q^2; q^2)_n(xq^{2n+2}; q^2)_\infty}$$

and

$$U_{2k+1,2a+1}(x) = \sum_{n\geq 0} (-1)^n x^{(2k+1)n} q^{2n(n+1)(k+1)} (q^{-(2a+1)n} - x^{2a+1}q^{(2a+1)(n+1)})$$
$$\frac{(-x^2q^{2n+2}; q^2)_\infty(-xq; q^2)_\infty(-q^{-2n}; q^2)_n}{(q^2; q^2)_n(xq^{2n+2}; q^2)_\infty}.$$

Applying $x = 1$, we have

$$\overline{U}_{2k+1,2a}(1) = \frac{(-q^2; q^2)_\infty^2}{(q^2; q^2)_\infty} \sum_{n\geq 0} (-1)^n q^{n(n+1)(2k+1)-2an} (1 - q^{(2n+1)2a})$$

and

$$U_{2k+1,2a+1}(1) = \frac{(-q; q)_\infty}{(q^2; q^2)_\infty} \sum_{n\geq 0} (-1)^n q^{n(n+1)(2k+1)-(2a+1)n} (1 - q^{(2n+1)(2a+1)}).$$

So, using Corollary (2.1.2), for $i = 2a+1$ and $t = \frac{4k+1}{2}$ for $\overline{U}$, and $i = 2a+1$ and $t = \frac{4k+1}{2}$ for U, we have

$$\overline{U}_{2k+1,2a}(1) = \frac{(-q^2;q^2)_\infty^2 (q^{2a}, q^{4k+2-2a}, q^{4k+2}; q^{4k+2})_\infty}{(q^2;q^2)_\infty}$$

and

$$U_{2k+1,2a+1}(1) = \frac{(-q;q)_\infty^2 (q^{2a+1}, q^{4k-2a+1}, q^{4k+2}; q^{4k+2})_\infty}{(q^2;q^2)_\infty}.$$

These prove Theorems (4.1.4) and (4.1.7). Finally, for the last identity of this case, $U_{2k+1,2a}$, same as even case, we have

$$\alpha_n^e(x) = \tilde{\alpha}_0^o(xq^{2n})(-1)^n x^{(2k+1)n} q^{2kn^2+2kn+n^2} \frac{(-x^2q^{2n+2};q^2)_\infty(-xq^3;q^2)_\infty(-q^2;q^2)_{n-1}(1+q^{2n})(1+xq^{2n+1})}{(q^2;q^2)_n(x^2q^{2n+2};q^2)_\infty}$$

and

$$\beta_n^e(x) = -\tilde{\tilde{\alpha}}_0(xq^{2n+1})(-1)^n x^{(2k+1)n} q^{2kn^2+2kn+n^2} \frac{(-x^2q^{2n+2};q^2)_\infty(-xq^3;q^2)_\infty(-q^2;q^2)_n(1+xq^{2n+1})}{(q^2;q^2)_n(x^2q^{2n+2};q^2)_\infty}.$$

So, from

$$U_{2k+1,2a}(x) = \sum_{n\geq 0} \alpha_n^o(x) q^{-2an} + \beta_n^o(x) x^{2a} q^{2a(n+1)}$$

we have

$$\begin{aligned} U_{2k+1,2a}(1) &= \frac{(-q^2;q)_\infty}{(q^2;q^2)_\infty} \sum_{n\geq 0} (-1)^n q^{2kn^2+2kn+n^2-2an}(1+q^{2n+1})(1-q^{(2n+1)2a}) \\ &= \frac{(-q^2;q)_\infty}{(q^2;q^2)_\infty} (\sum_{n\geq 0} (-1)^n q^{2kn^2+2kn+n^2+n-(2a+1)n}(1-q^{(2n+1)(2a+1)}) \\ &+ q\sum_{n\geq 0}(-1)^n q^{2kn^2+2kn+n^2+n-(2a-1)n}(1-q^{(2n+1)(2a-1)})) \end{aligned}$$

Using Corollary (2.1.2), twice, once for $i = 2a$ and $t = \frac{4k+1}{2}$, and again for $i = 2a+1$ and $t = \frac{4k+11}{2}$, we will have

$$\begin{aligned} U_{2k+1,2a}(1) =& \frac{(-q^2;q)_\infty (q^{2a+1}, q^{4k-2a+1}, q^{4k+2}; q^{4k+2})_\infty}{(q^2;q^2)_\infty} \\ &+ \frac{q(-q^2;q)_\infty (q^{2a-1}, q^{4k-2a+3}, q^{4k+2}; q^{4k+2})_\infty}{(q^2;q^2)_\infty}. \end{aligned}$$

This proves our last identity for odd case, Theorem (4.1.5).

CHAPTER 5

Conclusion and future work

The constructive method is effective for some Rogers-Ramanujan type partition identities other than the original identities, the ones that the partitions have multiplicity conditions on two consecutive parts, such as Rogers-Ramanujan-Gordon's identity for overpartitions, because the multiplicity condition is on i and $i+1$, but in Schur's identity [2], the multiplicity condition in on i, $i+1$ and $i+2$, so the exact method that we used does not work for that. There is a possibility that the constructive method works on other partition types by modifying the generating function, such as Schur's identity or Göllnitz-Gordon identities [11].

In this thesis, we construct identities for two other Rogers-Ramanujan types identities for overpartitions and colored partitions using this method. In fact, we finished the parity consideration on Rogers-Ramanujan-Gordon's identities for overpartitions with respect to the some restrictions on parts defined by Chen, Sang and Yee, the most important restriction they made was the second condition in the definitions (4.1.1) and (4.1.2), where the existence of an overlined part has a significant role in finding the new identities. Note that in the constructive method, we start with a definition for a partition type and if we can not reach to the other side of the identity, we may change the definition by adding new restrictions, then try the method for the new definition.

For future work, one idea is to remove some restrictions in the definitions (4.1.1) and (4.1.2), then find functional equations relating their generating functions and find identities in a more general case. Another problem to think of can be considering higher or even arbitrary congruence on parts instead of 2, e.g. three overpartition types, U_i's when $f_{3l+i} + f_{\overline{3l+i}} \equiv 0 \pmod 3$ for $i = 0, 1, 2$. In the third chapter, we realized that 2-colored Rogers-Ramanujan type partitions are related to some special case of overpartitions. We can generalize those colored partition identities for more colors, for generalized case of Rogers-Ramanujan type partitions, i.e parts of each color be of Rogers-Ramanujan-Gordon's type partitions or for both. Another idea can be considering the parity condition on the parts of each color.

In the process of our work, we used some some computer algebra software such as Maxima, Mathematica and Maple, first to find the partitions with those given conditions, and then to verify our final results. Other than those, we use them to verify some parts on construction step. For future, one of our plans is to write and develop programs in those software that generate partitions and overpartitions with respect to the condition that we put on parts, in this way, we may guess the relation between parts and then find new identities related to those partition or overpartition types. It may help us to find new identities.

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