CONSTRAINTS FROM INFLATION AND BLACK-HOLES ON EMERGENT GRAVITY THEORIES

by
İLİM İRFAN ÇİMDİKER

Submitted to the Graduate School of Engineering and Natural Sciences
in partial fulfilment of
the requirements for the degree of Master of Science

Sabancı University
June 2021

CONSTRAINTS FROM INFLATION AND BLACK-HOLES ON EMERGENT GRAVITY THEORIES

Approved by:

Date of Approval: JUNE 28, 2021

ABSTRACT

CONSTRAINTS FROM INFLATION AND BLACK-HOLES ON EMERGENT GRAVITY THEORIES

İLİM İRFAN ÇİMDİKER

PHYSICS M.Sc. THESIS, JULY 2021

Thesis Supervisor: Prof. DURMUŞ ALİ DEMİR

Keywords: emergent gravity, induced gravity, inflation, black-hole, f(R)

In this work, we study the inflationary dynamics and the static spherically symmetric black hole solutions on induced gravity theories. Inflation is the central paradigm for explaining the isotropy and homogeneity of the observed universe, whereas induced gravity theories arise from the matter loops and can be qualified as natural. Proposed exact black hole solutions can be a testbed for these induced gravity theories with the help of future observations of Event Horizon Telescope (EHT) and Laser Interferometer Gravitational-wave Observatory (LIGO).

ÖZET

ENFLASYONUN VE KARA DELİK ÇÖZÜMLERİNİN KENDİLİĞİNDEN OLUŞAN KÜTLE ÇEKİM KURAMLARINDAKİ KISITLAMALARI

İLİM IRFAN ÇİMDİKER

FİZİK BÖLÜMÜ YÜKSEK LİSANS TEZİ, MAYIS 2021

Tez Danışmanı: Prof. Dr. DURMUŞ ALİ DEMİR

Anahtar Kelimeler: kütle çekim, zuhur, enflasyon, kara delik

Bu çalışmanın amacı kendiliğinden oluşan kütle çekim kuramlarında enflasyon dinamiğini ve zamanla değişmeyen küresel simetrik karadelik çözümlerini ortaya çıkarmaktır. Enflasyon, gözlemlenen evrenin eşyönlülüğünü ve tektürelliğini açıklayan ana örneklemdir. Aynı zamanda kendiliğinden meydana gelen kütle çekim kuramları, madde ilmeklerinden ortaya çıktığından dolayı doğal olarak kabul edilebilirler. Bu çalışmada ortaya koyduğumuz kara delik çözümleri, gelecekte Event Horizon Telescope (EHT) ve Laser Interferometer Gravitational-wave Observatory (LIGO) deneylerinin yardımı ile modifiye teoriler için test edilebilir bir alan sunmaktadır.

ACKNOWLEDGEMENTS

Foremost I would like to express my sincere gratitude to my mentor and advisor, Prof. Durmuş Ali Demir. He gave me the ropes on every possible situation, and I benefited from his immense knowledge in many ways. Also, I'm thankful for not giving me fish but teaching me how to fish. It was a pleasure.

I benefited much from the classical field theory lectures given by Prof. Cihan Saçlıoğlu, and I'm grateful for the many discussions we held. Also, I thank Assoc. Prof. Ali Övgün for the helpful conversations and reviews.

Special thanks to my beloved wife, Sinem Dilara Çimdiker, who always supported me during this challenging period. I couldn't achieve this without her encouragement. Lastly, I am indebted to my mother and recently passed father for their moral and financial support.

This project is supported in part by TÜBİTAK grant 118F387.

To my lovely wife, Sinem Dilara and my most beloved daughter, Leyla Hypatia.

LIST OF ABBREVIATIONS

GR: General Relativity

QFT: Quantum Field Theory

SM: Standart Model

IG: Induced Gravity

SG: Symmergent Gravity

BH: Black Hole

EH: Einstein Hilbert

CMB: Cosmic Microwave Background

EM: Electro-Magnetic

EW: Electro-Weak

SR: Special Relativity

UV: Ultra-Violet

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

1. INTRODUCTION

General Relativity (GR) is one of the most successful theories of modern physics (Will (2014)). The GR addresses gravity not as a force but a resulting act of curved space-time. Einstein came to the said theory through straightforward yet profound thought experiments via the help of the rich expressions of geometry.

While the GR may be one of the most understood and most robust theories in the literature, it is not without imperfections. The most prominent obstacle for the GR is that it is incompatible with Quantum Field Theory (QFT) (Macias & Camacho (2008)). Unification of GR and Standart Model (SM) is a stubborn problem that many have tried but remained futile (Dyson (2013); 't Hooft & Veltman (1974)).

The canonical idea was to quantize the gravity directly (Schulz (2014)); however, the inability to get a Fourier mod from a linearized gravity and the absence of a spin-two scalar graviton in the standard model makes this preposition unsatisfactory at best(Carlip (2001)). On the other hand, the current experimental methods cannot probe the energy levels that string theory proposes(Schwarz (2007)).

One can also accomplish this unification via Induced Gravity (or Emergent Gravity). Originally Induced Gravity was proposed by Sakharov (Sakharov (1967); Visser (2002)), which treats gravity not as a fundamental but rather as an emergent phenomenon induced by one-loop effects of QFT; hence it is called Induced Gravity (IG). There are also attempts to induce gravity from thermodynamics (Verlinde (2017)). In general, these theories have the gravitational scale emerge from the said matter loops or other physical aspects. Also, a novel idea called Symmergent Gravity(SG) takes this idea as a base and incorporates gravity into the SM via restoring the broken gauge symmetry(Demir (2021,1)). Just as in Sakharov's IG, space-time elasticities as geometry emerges from the action(Demir (2019)).

Both Sakharov's IG and SG contains quadratic terms of the scalar curvature. Contrary to the commonly used R in the Einstein-Hilbert action, IG and the SG contain linear R and quadratic R^2 terms. The quadratic scalar curvature is usually considered as a quantum correction to the Einstein-Hilbert(EH) action. Hence, actions of these theories can be considered an $f(R)$ theory at its core (Capozziello & De Laurentis (2011); De Felice & Tsujikawa (2010); Sotiriou & Faraoni (2010)).

It is also well known that $f(R)$ theories are equivalent to Brans-Dicke type Scalar-Tensor theories. It can be reduced to an EH action plus a canonical scalar action with the help of a conformal rescaling, i.e. passing from the Einstein frame to the Jordan frame. One can observe this in Starobinsky's inflationary model (Starobinsky (1980b)).

Inflation is the primary paradigm that solves the enigmatic horizon and flatness problems (Guth (1981); Linde (1982)) whose constraints are well defined by Planck Cosmic Microwave Background (CMB) results(Akrami & others (2020)). Starobinsky's model, an $R+R^2$ model, introduces a scalar mode called "inflaton", a single field that rolls slowly. The quadratic curvature term dominates the linear one in the early universe; thus, inflationary dynamics arise. While in the low curvature limit, linear R ends the inflation. Since both IG and SG have similar higher curvature forms, the inflation (Çimdiker (2020)) and Black-Hole(BH) solutions and results from Event Horizon Telescope (EHT) (Akiyama & others (2019)) should constrain these frameworks. Thus in this thesis, the aim is to designate these constraints via the help of Starobinsky's inflationary model and the BH solutions.

The plan of this thesis will be as follows;

In chapter two, we will briefly introduce the General Theory of Relativity. We will give background information on space-time geometry, and then we will delve into Action formalism.

Since both IG and SG actions are $f(R)$ functions, chapter three will be about the $f(R)$ theories and their applications. Again, we will use the Action formalism to obtain the equations of motions of a general function $f(R)$. Also, we will look upon the frameshift from the Einstein to Jordan frame and introduce the concept of conformal transformation to see the resemblance between the $f(R)$ theories and scalar-tensor theories. Moreover, at the end of the chapter, we will introduce the inflationary dynamics.

We also want to utilize the BH solutions of higher curvature theories to see the result in the underlying QFT. Thus we will reserve the fourth chapter for BH dynamics in Einstein Gravity and $f(R)$ theories. First, we will introduce the Spherically Symmetric solutions to Einstein gravity. Then we will briefly explain the thermodynamic properties of such solutions. In the end, we will generalize these to the $f(R)$ theories.

In chapter five, we will introduce the IG and SG to the reader. We will briefly derive their actions. We will see the relationship between gravity and the underlying QFT.

Furthermore, in the sixth chapter, we will use the dynamics obtained from chapters three and four and see their effects on the said emergent gravity theories. Finally, we conclude our thesis in chapter seven.

We will use the east coast $(-,+,+,+)$ metric through this thesis except Chapter five, where we will utilize the west coast one to match it with other particle physics manuscripts. Also we will take $\hbar$ and c to be 1 except otherwise stated.

2. Introduction to General Relativity

2.1 Gravity

Why do apples fall? It is maybe one of the first vital questions that dazzle many intellectuals for centuries. Aristotle thought that every element of nature had some inner affinity to be with its related. Hence, a falling rock would fall because stones tend to be with other stones, or air would rise since this was their natural tendency. On the other hand, a geocentric method of epicycles introduced by Ptolemy described the motion of the heavenly bodies accurately. After that, Copernicus paved a path to a heliocentric system. Measurements of Tycho Brahe and the analysis of Kepler further improved this idea by showing that the orbits of these bodies were ellipses rather than circles all along. However, none of these ideas was explaining the fundamental question of why apples fall. Only Newton came up with the brilliant idea that apples, Sun, and earth obey the same law.

Newton's ideas were relatively straightforward. All matter attracts each other via the help of an action at a distance. The force is proportional to the inverse square of the distance between these elements, and there exists a charge-like quantity called mass which creates this attractive force. There also exists a constant, namely Newton's constant which clears up the units. Force is always along the path between the masses.

$$F = G_N \frac{m_1 m_2}{r^2} \hat{r} \tag{2.1}$$

It is practical, elegant, and it makes sense. By the 19th century, field notation has taken its part in physics. The idea was that charged particles produce a field that influences all the surroundings and exists throughout the space. Via this notation, the particles could pull and push one another with the help of the field itself, thus creating a medium for a local transaction for forces. Nobody thought of these fields as literal beings; they were bookkeeping devices for the physicists. However, the ingenious physicist James Clark Maxwell changed the trajectory of physics by taking these fields as literal. Maxwell constructed his ideas by incorporating the electric and magnetic fields into two sides of a single coin. Also, it turned out that the propagation of the interaction between two body moves with a particular limiting speed was a recently measured value of the speed of light. However, the cosmic speed limit shown by Maxwell was nonexistent in the old Galilean relativity. In 1905 Einstein introduced the idea of Special Relativity(SR), where he assumed the followings;

1.1 The laws of physics are the same in all Galilean frames.

1.2 There is a cosmic speed limit for a causal connection in all Galilean frames independent of the observer's motion, which happens to be the speed of light.

Here Galilean frames are non-rotating, non-accelerating so-called inertial frames. It is important to note that the speed limit of special relativity is the limit of causality which means that two events that happen at specific space and time coordinates can not affect each other immediately. Event $\mathcal{A}$ can cause Event $\mathcal{B}$ via a propagating mediator, which must obey this speed limit. Since Electro-Magnetic(EM) quanta photon is a massless particle, it can and should travel at the boundary of this limit. Later publishing the idea of SR, Einstein was mainly concerned with incorporating gravity into his novel idea of relativity. After eight years of search for a particular relativistic gravitation theory, in 1915, Einstein achieved this by relating the curvature of a 4-dimensional merged space-time to the energy and matter content.

In flat geometry, two bodies that are comoving parallel to each other stay parallel. However, in curved geometry, things are different. Imagine a 3-dimensional sphere S^2 and two point particles with a velocity vector parallel to each other. The vectors start parallel but, with some time, they will lose this parallelity. The distance between these two objects will change too. The main observation is that in a curved geometry, two objects can be seen as attracted to each other even though there is no force to be seen anywhere. Hence we can conclude from this thought experiment that "Gravitation" may not be a force but could be a manifestation of 4-dimensional curved geometry.

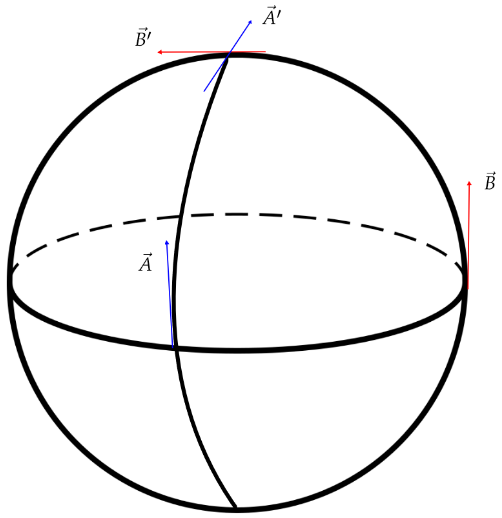

Figure 2.1 Vectors on S^2.

2.2 Fundamentals

Since we will try to describe gravity as elasticization of a four-dimensional space-time by a matter content, an object which contains some information about the matter should somehow be correlated to an object that shows us the geometry of space-time. Let us review what we know so far.

From SR, we know that space and time are a composite body that can be called space-time. Let us start with a formal definition of space-time and see how it behaves in a more general term.

" Space-time is a four-dimensional differentiable manifold that is equipped with a semi-Riemannian metric that is nondegenerate everywhere. "

The concept of a manifold is a continuous space of points that may be flat or curved (or even can have holes in it) globally. However, locally it can be considered as flat. These small flat patches construct a global may be curved manifold. A **differentiable manifold** is a space that is differentiable continuously; hence one can do calculus on it.

In this construct, there exists an infinitesimal interval that is invariant from an observer to observer, which is called space-time interval or line element that can be

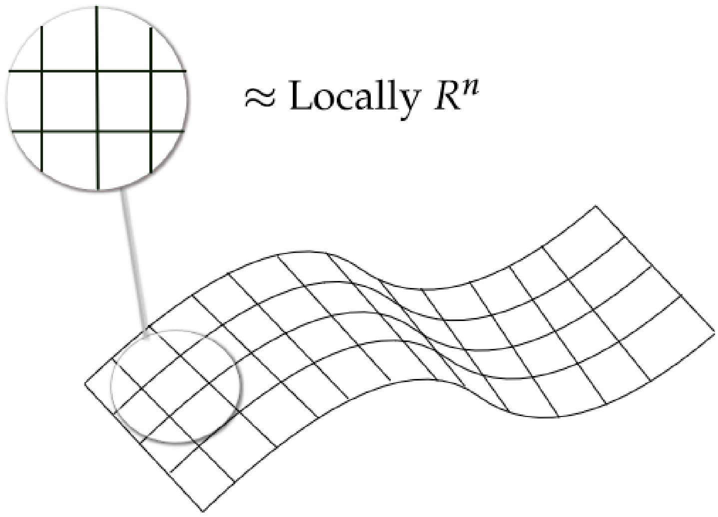

Figure 2.2 Representation of a curved manifold

written as

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 \tag{2.2}$$

One can see that space and time are on the same footing with a relative minus sign on the space part. Let us generalize this relation as

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} \tag{2.3}$$

We will use the Einstein summation convention throughout this work; thus, there will be a hidden sum for all indices, in this case, μ and ν. The second rank tensor $g_{\mu\nu}$ is called the metric tensor or loosely metric where this object defines the inner product. In flat space, it is called Minkowski metric, $\eta_{\mu\nu}$ which can be shown as

$$\eta_{\mu\nu} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} \tag{2.4}$$

The metric imposes geometry in GR. Space-time should be flat in empty space, but also it should be curved when there exists an energy content nearby. Hence in such cases, the metric will have dynamic entries. In 3-dimensions, we expect the inner product or norm of a vector to be positive or, formally, we expect them to be positive definite. The relative minus sign for the space part imposes a relaxation on this constraint. So that the norm of a vector in this particular manifold (space-time) can have either positive or negative values, such manifolds can be called **semi-Riemannian** pointing to this relaxation. Also, the metric g that is equipped to the manifold should be smooth, symmetric, and **nondegenerate** that is any map $g \to g'$ where g is a bilinear form, and g' is the dual of it, is an isomorphism.

In a flat space, parallel lines stay parallel. These straight lines are also the shortest distances between the selected two points on the line. The shortest distance between the two points is called geodesic, which may or may not be a straight line. For example, in a curved space embedded in a flat space, geodesics do not need to stay parallel. Lines both converge or diverge depending on the curvature of the selected area. Thus even though no force acts on a selected two body that moves in geodesics, the bodies can cross path at some point. Thus we can say that gravity is not a force; it is a manifestation of the curvature of space-time. (Landau & Lifschits (1975); Wald (1984)) However, to see this effect, we need to know how derivative works on a construct such as space-time. Consider a vector A which can be written explicitly with its basis as $A^\mu e_\mu$. To take the derivative, we need to use the Leibniz rule as,

$$\frac{\partial A}{\partial x^\nu} = \frac{\partial A^\mu}{\partial x^\nu} e_\mu + A^\mu \frac{\partial e_\mu}{\partial x^\nu} \tag{2.5}$$

For an ordinary flat space, the basis vector does not need to change. Thus the derivative of the basis yields null. However, in curved space, this is not the case. The derivative of the basis vector is another vector that can be written as a linear combination of the basis vectors with a weighted coefficient as

$$\frac{\partial e_\mu}{\partial x^\nu} = \Gamma^\lambda_{\mu\nu} e_\lambda \tag{2.6}$$

Coefficients Γ are called Christoffel symbols or connections (components of the connection concerning local coordinates.). Physically it can be considered as a correction term that makes the derivative operator covariant. Rewriting (2.5) using connection we obtain

$$\frac{\partial A}{\partial x^\mu} = \left(\frac{\partial A^\lambda}{\partial x^\mu} + \Gamma^\lambda_{\mu\nu} A^\nu \right) e_\lambda \tag{2.7}$$

The expression without the basis vector is what we define as a covariant derivative of a vector, in this case, vector A, which can be explicitly written as

$$\nabla_\mu A^\lambda = \frac{\partial A^\lambda}{\partial x^\mu} + \Gamma^\lambda_{\mu\nu} A^\nu \tag{2.8}$$

The connection is not a tensor, and it can exist without defining a metric on a manifold. In general, these connections without a metric definition are called affine connections. However, to do GR, we need a Riemannian manifold. Thus we need to introduce a metric to the manifold, and we can also demand some properties from the connection itself. The two things we apriori define can be shown as follows;

2.1 Torsion freeness: the connection is symmetric in the lower two indices, that is $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$.

2.2 Metric compatibility: Covariant derivative of the metric zero on the manifold everywhere, that is $\nabla_\rho g_{\mu\nu} = 0$.

These definitions allow us to define a unique connection that is called the Levi-Civita connection, which can be written as

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} (\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) \tag{2.9}$$

This object helps us to transfer some data along a curve on a manifold. It connects two points on a patch to do some iterations or predictions about the relationship between the two points. As said before, the connection does not need to be a function of a metric. One can easily define a connection without equipping the manifold with a metric; hence do some calculus on the said manifold. This Levi-Civita connection is a particular case. To construct a general theory of relativity, we need a Levi-Civita connection.

Next, we need to define an object to measure how curved a manifold is. This curvature, however, should be calculable on the manifold itself. Think of inventing

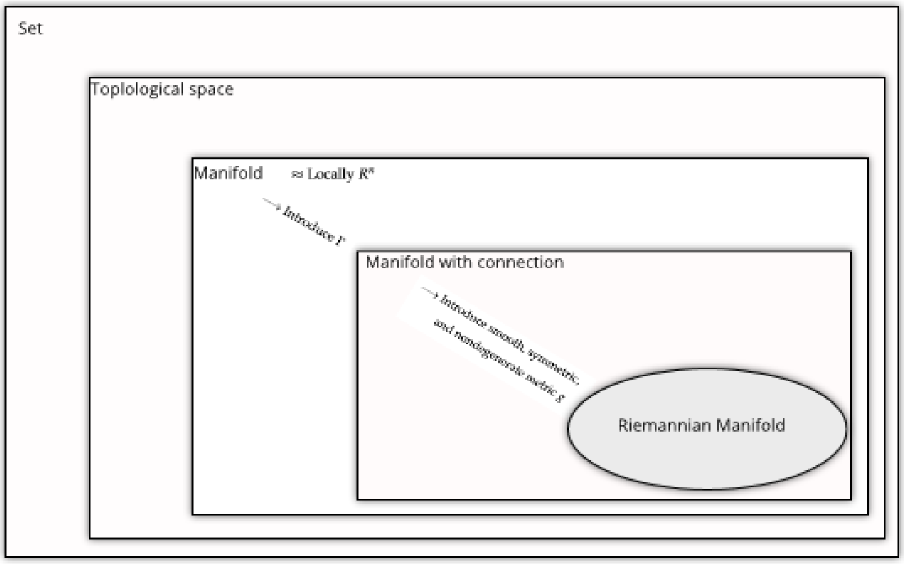

Figure 2.3 From open sets to Riemannian manifolds

a measurement to show that the earth is curved and try to prove it by only doing some calculations. No satellite images or space travels are allowed. To do this, we can introduce an operation called vector transportation. On flat surfaces, when we transport a vector pointing out to a specific direction along a close loop, it would not change its direction. However, in curved space, that is not the case. A vector in the equator pointing to the north pole can be parallel transported along a path to change its direction by full $\pi/2$ degrees. Hence just by transporting vectors on it, we can tell something about the curvature of the manifold.(Schutz (1985)) We can do this operation by introducing the following commutation relation,

$$[\nabla_\mu \nabla_\nu] A_\rho = \nabla_\mu \nabla_\nu A_\rho - \nabla_\nu \nabla_\mu A_\rho \tag{2.10}$$

Here we are taking a vector A_ρ and take the derivative of it in two distinct directions in sequence on a loop. If this object yields null, then we can conclude that the vector did not change its direction along the path. However, if this quantity changed along the path, we can relate this quantity to the manifold's curvature. Further unwrapping this quantity gives

$$
[\nabla_\mu \nabla_\nu] A_\rho = A_\lambda \partial_\mu \Gamma^\lambda_{\nu\rho} - \Gamma^\lambda_{\nu\rho} \partial_\mu A_\lambda - \Gamma^\sigma_{\mu\rho}(\partial_\nu A_\sigma - \Gamma^\lambda_{\nu\sigma} A_\lambda)
$$
$$
A_\lambda \partial_\nu \Gamma^\lambda_{\mu\rho} + \Gamma^\lambda_{\mu\rho} \partial_\nu A_\lambda + \Gamma^\sigma_{\nu\rho}(\partial_\mu A_\sigma - \Gamma^\lambda_{\mu\sigma} A_\lambda) \tag{2.11}
$$

After renaming indices and collecting similar terms, we get

$$
[\nabla_\mu \nabla_\beta] A_\nu = (\partial_\beta \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\beta} + \Gamma^\alpha_{\lambda\beta} \Gamma^\lambda_{\mu\nu} - \Gamma^\alpha_{\lambda\nu} \Gamma^\lambda_{\mu\beta}) V_\alpha \tag{2.12}
$$

The only quantity that can raise the null result is the tensorial quantity in the parenthesis. It can be seen that it has four indices that can be compactly written in the form,

$$
[\nabla_\mu \nabla_\beta] A_\nu = R^\alpha_{\mu\beta\nu} V_\alpha \tag{2.13}
$$

where our new tensorial quantity reads

$$
R^\alpha_{\mu\beta\nu} = \partial_\beta \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\beta} + \Gamma^\alpha_{\lambda\beta} \Gamma^\lambda_{\mu\nu} - \Gamma^\alpha_{\lambda\nu} \Gamma^\lambda_{\mu\beta} \tag{2.14}
$$

This object is called the famous Riemann tensor. It is a tensor field that is used to express the curvature of a Riemannian Manifold. It is the central piece of general relativity. Further contracting the α and β indices yields the Ricci Tensor $R_{\mu\nu}$ which is the trace of Riemannian tensor and beyond contracting Ricci Tensor with a dynamic metric gives Ricci Scalar or Scalar curvature R.

These objects can be differentiated between them by following interpretations. Riemann tensor $R^\alpha_{\mu\beta\nu}$ is a rank four tensor that holds all of the information about an object in free fall. It gives information about the size and shape change of an object that moves in a geodesic. Where Ricci Tensor $R_{\mu\nu}$ is a second rank tensor, has fewer components; thus, it only gives a rate about the volume change of an object. Lastly, Ricci scalar R is a scalar (thus rank zero). It can only give a comparative number about the difference of the volume of an object concerning a reference object in flat space.

2.3 Action Formalism in Metric Theory

Like a classical field theory, Einstein's field equations can be derived through the principle of least action. All systems in physics try to minimize a function that is called action, that is

$$\delta S = \delta \int d^4x \mathcal{L} = 0 \tag{2.15}$$

$\mathcal{L}$ is called the Lagrangian density or lagrangian that, when minimized, could give the equation of motions of a given system.(Carroll (2019)) For gravity, one can select any lagrangian in 4-dimensional space to produce a theoretical model as long as it is covariant. The most convenient and easy one would be the following.

$$\mathcal{L} = \sqrt{-g}R \tag{2.16}$$

Where R is the scalar curvature and g is the determinant of the metric $g_{\mu\nu}$. Scalar curvature can be obtained by contracting Ricci tensor with the metric as

$$R = g^{\mu\nu} R_{\mu\nu} = g^{\mu\nu} R^{\alpha}_{\mu\alpha\nu} \tag{2.17}$$

By selecting the lagrangian as in Eq.(2.16) and taking the dynamic metric $g_{\mu\nu}$ as an independent field, in the absence of matter, we can write the so-called EH action as follows;

$$S = \frac{M_{pl}^2}{2} \int d^4x \sqrt{-g}R \tag{2.18}$$

M_{pl} is Planck mass which can be taken as $1/\sqrt{8\pi G_N}$ in literature and is around 2.4×10^{18} GeV in mass-scale where G_N is Newton's constant. The dynamics of this action can be obtained by considering the variation of the action. In metric formalism, the dynamic metric is the only independent variable, hence taking the variation with respect to metric will yield the equations of motion. Let us do so by varying the action in Eq.(2.18) with respect to metric $g^{\mu\nu}$

$$\delta S = \frac{M_{pl}^2}{2}\int d^4x\{R\delta\sqrt{-g} + \sqrt{-g}(g^{\mu\nu}\delta R_{\mu\nu} + R_{\mu\nu}\delta g^{\mu\nu})\} \tag{2.19}$$

The last term, $\delta g^{\mu\nu}$, is the expected and the required version of the variation. Variation should be taken from the volume element $\sqrt{-g}$ and the Ricci Tensor $R_{\mu\nu}$ to the metric. The first thing to do is to vary the volume element. We can use the well-known matrix identity for it as

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu} \tag{2.20}$$

Also, the variation of the Ricci tensor $R_{\mu\nu}$ yields the well known Palatini identity.

$$\delta R_{\mu\nu} = \nabla_\lambda(\delta\Gamma^\lambda_{\mu\nu}) - \nabla_\nu(\delta\Gamma^\lambda_{\lambda\mu}) \tag{2.21}$$

where the variation of the connection can be calculated as

$$\delta\Gamma^\sigma_{\mu\nu} = -\frac{1}{2}\{g_{\lambda\mu}\nabla_\nu(\delta g^{\lambda\sigma}) + g_{\lambda\nu}\nabla_\mu(\delta g^{\lambda\sigma}) - g_{\mu\alpha}g_{\nu\beta}\nabla^\sigma(\delta g^{\alpha\beta})\}. \tag{2.22}$$

Using the variation of the Levi-Civita connection, we can easily shape the variation of the Ricci tensor as desired. Using Eq.(2.22) one can write;

$$\int d^4x\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} = \int d^4x\sqrt{-g}\nabla_\sigma[(g_{\mu\nu}\nabla^\sigma(\delta g^{\mu\nu}) - \nabla_\lambda(\delta g^{\sigma\lambda})] \tag{2.23}$$

The right-hand side of the Eq.(2.23) is a four-dimensional volume integral of divergence, particularly a surface term. Stoke's theorem states that at infinity, this term will vanish. Thus there will be no contribution from the variation of the Ricci tensor. In the end, variation of the gravitational action yields;

$$\delta S = \frac{M_{pl}^2}{2}\int d^4x\sqrt{-g}\{R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\}\delta g^{\mu\nu} \tag{2.24}$$

Now that the variation of the gravitational action has been written out, we can also define a matter action that connects matter fields to geometry. Let us define the following variational identity.

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} \tag{2.25}$$

$T^{\mu\nu}$ is the energy-momentum tensor that describes the energy, pressure, and shear properties of a given object. By defining such identity, the variation of the total action would take the form.

$$\delta S_T = \delta S + \delta S_M = 0 \tag{2.26}$$

thus the equation of motion for the total action, i.e. the Einstein-Hilbert action, can be written as;

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{M_{pl}^2}T_{\mu\nu} \tag{2.27}$$

Eqs.(2.27) is generally known as Einstein's field equations which are sets of ten differential equations, and $G_{\mu\nu}$ is known as Einstein tensor. The left-hand side of the equation describes the dynamics of the system's geometry, whereas the right-hand side delivers the distribution of matter.

3. Introduction to f(R) Theories

3.1 Action formalism in f(R) theories

Even though general relativity makes perfectly sensible and provable predictions on many occasions, corrections can provide substantial benefits. It helps build up a toy theory that can probe riddles that are yet to be unrevealed. Such correction terms can be added directly to the EH action to produce a new model. Selecting a function of R instead of a linear one would result in a higher-order action. Such theories are called $f(R)$ theories.(Sotiriou & Faraoni (2010)) $f(R)$ theories can be used to produce models for Dark Matter, Dark energy, Inflation(De Felice & Tsujikawa (2010)). This section will provide sufficient background for $f(R)$ gravity needed to work with inflationary dynamics. These theories come from the generalization of the EH action (2.18) that is scalar curvature R becomes a linear analytic function of R as

$$S = \frac{M_{pl}^2}{2}\int d^4x\sqrt{-g}f(R) \tag{3.1}$$

Here scalar curvature R is a function of the connection that may be or not related to a metric. In a standard way, the connection is generally taken to be a Levi-Civita one that is a function of a metric $g_{\mu\nu}$. Therefore, taking variation with respect to metric leads to the metric formalism of the $f(R)$ gravity. Moreover, Palatini (Bastero-Gil, Borunda & Janssen (2009)) and Metric-Affine (Shimada, Aoki & Maeda (2019); Vitagliano, Sotiriou & Liberati (2011))formalisms approach the $f(R)$ action by distinguishing the connection and metric, taking both constructs as separate things. In this way, one can obtain two equations of motions obtained via varying for both connection and the metric. Doing this generally yields the same results in bare cases. However, in borderline situations, this may differ the outcome.

In this section, we will use the general metric formalism. As done in the action formalism of the EH action, we can vary Eq.(3.1) with respect to metric to obtain the equation of motions. In the absence of matter, variation of the mentioned action will be in the form.

$$\delta S=\int d^4x\left(\delta\sqrt{-g}\,f(R)+\sqrt{-g}\,\delta f(R)\right) \tag{3.2}$$

which can be unwrapped as

$$\delta S=\int d^4x\,\delta\sqrt{-g}\,f(R)+\int d^4x f'(R)(g^{\mu\nu}\delta R_{\mu\nu}+R_{\mu\nu}\delta g^{\mu\nu}) \tag{3.3}$$

where the prime denotes the derivative with respect to the argument, in this case, R. We can see this action as the sum of three actions as

$$S_1=\int d^4x f(R)\delta\sqrt{-g} \tag{3.4}$$

$$S_2=\int d^4x f'(R)g^{\mu\nu}\delta R_{\mu\nu} \tag{3.5}$$

$$S_3=\int d^4x R_{\mu\nu}\delta g^{\mu\nu} \tag{3.6}$$

$$S=S_1+S_2+S_3 \tag{3.7}$$

We need to take the variations of the quantities, which can be taken directly from the variation of the Einstein-Hilbert action. Via using (2.20, 2.21, 2.22) we can take the variation of the Riemann tensor as

$$\begin{aligned}\delta R_{\mu\nu}=&\nabla_\rho[\frac{1}{2}g^{\lambda\rho}(\nabla_\mu\delta g_{\lambda\nu}+\nabla_\nu\delta g_{\lambda\mu}-\nabla_\lambda\delta g_{\mu\nu})]\\ &-\nabla_\nu[\frac{1}{2}g^{\rho\lambda}(\nabla_\mu\delta g_{\lambda\rho}+\nabla_\rho\delta g_{\mu\lambda}-\nabla_\lambda\delta g_{\mu\rho})]\end{aligned} \tag{3.8}$$

which is equivalent to

$$(3.9)\qquad \delta R_{\mu\nu} = \frac{1}{2}(\nabla^{\lambda}\nabla_{\mu}\delta g_{\lambda\nu} + \nabla^{\rho}\nabla_{\nu}\delta f_{\mu\rho} - g^{\lambda\rho}\nabla_{\nu}\nabla_{\mu}\delta g_{\lambda\rho} - \nabla^{\lambda}\nabla_{\lambda}\delta g_{\mu\nu})$$

Notice that in the EQ.(3.3), derivatives that come from Palatini identity would not create a divergence for a 4-dimensional volume. We generally have a good idea about healing the surface terms in the EH case where $f(R) = R$ (Chakraborty (2017)). However, the $f(R)$ version have terms that do not gather into a total divergence. Hence the variation of the Riemann tensor can not be integrated by parts in this method. Thus let us contract this variation with the metric while noting that.

$$(3.10)\qquad \delta g_{\mu\nu} = -g_{\mu\lambda}g_{\nu\rho}\delta g^{\lambda\rho}$$

results in

$$(3.11)\qquad g^{\mu\nu}\delta R_{\mu\nu} = g_{\kappa\sigma}\Box\delta g^{\kappa\sigma} - \nabla_{\kappa}\nabla_{\sigma}\delta g^{\kappa\sigma}$$

where the $\Box$ operator is the 4-dimensional Laplacian or D'Alembertian operator. The variation of the action S_2 becomes

$$(3.12)\qquad \delta S_2 = \int d^4x[g_{\mu\nu}\Box f'(R) - \nabla_{\mu}\nabla_{\nu}f'(R)]\delta g^{\mu\nu}$$

where the variation of the total action S would yield

$$(3.13)\qquad \delta S = \int d^4x[f'(R)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(R) + (g_{\mu\nu}\Box - \nabla_{\mu}\nabla_{\nu})f'(R)]\delta g^{\mu\nu}$$

In the absence of matter, this results in the following equation of motion.

$$(3.14)\qquad f'(R)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(R) + (g_{\mu\nu}\Box - \nabla_{\mu}\nabla_{\nu})f'(R) = 0$$

Eqs.(3.14) are fourth-order partial differential equations since R already contains the second derivative of the metric. Also, we can write the trace of the (3.14) in the presence of matter as

$$f'(R)R - 2f(R) + 3\Box f'(R) = \frac{2}{M_{pl}^2}T \tag{3.15}$$

T is the trace of the energy-momentum tensor; it is easy to see that for the $f(R) = R$ case Eqs. (3.14) reduces to Einstein's field equations. The extra term $g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu$ vanishes while $f'(R) = 1$. However, other than that case, this new term creates an energy surge to the system, and the equation of motion can be manipulated to a more convenient form that is

$$G_{\mu\nu} = \frac{2}{M_{pl}^2 f'(R)}(T_{\mu\nu}^{(M)} + T_{\mu\nu}^{(E)}) \tag{3.16}$$

where we compactified the extra terms to an effective energy-momentum tensor which can be explicitly written as

$$\frac{2}{M_{pl}^2}T_{\mu\nu}^{(E)} = g_{\mu\nu}\frac{(f(R) - f'(R)R)}{2} + \nabla_\mu\nabla_\nu f'(R) - g_{\mu\nu}\Box f'(R) \tag{3.17}$$

Notice that $T_{\mu\nu}^{(E)}$ is conserved under covariant derivative i.e. $\nabla^\mu T_{\mu\nu}^{(E)} = 0$. Hence continuity equation holds for both matter and effective stress-energy tensor. $\frac{2}{M_{pl}^2}f'(R)$ becomes the new gravitational scale for the effective theory.

Also, on this basis, we can use Friedman-Robertson-Walker(FRW) metric, which is

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2(t)dx^2 \tag{3.18}$$

where the scalar curvature R for this metric becomes

$$R = 12H^2 + 6\dot{H} \tag{3.19}$$

H is the hubble parameter which is $H = \dot{a}/a$ where dot represents a derivative with respect to time. $H = 2.13 \times 10^{-42}$ GeV is the value for our universe(Abbott & others (2018); Akramiet al. (2020)).

In the presence of matter, we can add a term to action (3.1) to yield an energy-momentum tensor as in done in (2.26). For a dust model $T^{\mu\nu}$ can be written as

$$T_{\mu\nu} = \begin{vmatrix} -\rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{vmatrix} \tag{3.20}$$

ρ is the energy density, and P are the pressure. With the FRW metric in hand, we can write down the field equations (3.14) as

$$3f'(R)H^2 = \frac{1}{2}(f'(R)R - f(R)) - 3H\frac{\partial}{\partial t}f'(R) + \frac{2}{M_{pl}^2}\rho \tag{3.21}$$

$$-2f'(R)\dot{H} = \frac{\partial^2}{\partial t^2}f'(R) - H\frac{\partial}{\partial t}f'(R) + \frac{2}{M_{pl}^2}(\rho + P) \tag{3.22}$$

3.2 Equivalence of $f(R)$ Theories with Scalar Tensor Theories

In metric formalism, an $f(R)$ action can be transformed into a scalar-tensor theory (Flanagan (2003)) with an effective scalar potential. To do this, let us introduce a new field κ and, in the absence of matter, and write the following action.

$$S = \frac{M_{pl}^2}{2}\int d^4x\sqrt{-g}[f(\kappa) + f'(\kappa)(R - \kappa)] \tag{3.23}$$

If we vary this action with respect to κ, we get the following eom

$$f''(\kappa)(R - \kappa) = 0 \tag{3.24}$$

Now observe the following. Provided that $f''(\kappa)$ is not null, then κ reduces to R dynamically. Hence action (3.23) is equialent to $f(R)$ action (3.1). Now define a scalar degree of freedom Λ as

$$\Lambda = f'(\kappa) \tag{3.25}$$

Action (3.23) becomes

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{pl}^2}{2}\Lambda R - U(\Lambda)\right] \tag{3.26}$$

where potential $U(\Lambda)$ can be defined as

$$U(\phi) = \frac{M_{pl}^2}{2}\left(\kappa(\Lambda)\Lambda - f(\kappa(\Lambda)\right) \tag{3.27}$$

Hence theory became a scalar-tensor theory. This theory is equivalent to Brans-Dicke type theories(Lu, Wu, Yang, Liu & Zhao (2018)), where Brans-Dicke action can be written as

$$S = \int d^4x\sqrt{-g}\left[\frac{\Lambda}{2}R - \frac{\lambda}{2\Lambda}\nabla^\rho\Lambda\nabla_\rho\Lambda - U(\Lambda)\right] \tag{3.28}$$

λ is the Brans-Dicke parameter; in general, $f(R)$ theories in metric formalism can be considered equivalent to Brans-Dicke type Scalar Tensor theory with parameter $\lambda = 0$ (Brans & Dicke (1961); O'Hanlon (1972)).

3.3 Transformation of a Connection

Suppose we have two manifolds $(\mathcal{M}, Q)$ and $(\mathcal{M}, R)$ that is equipped with Lorentzian metrics Q and R respectively. Since the metric Q is Lorentzian, and it is conserved with respect to covariant derivative, we can write the connection on $(\mathcal{M}, Q)$ as

$$ {}^{Q}\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}(Q^{-1})^{\lambda\rho}(\partial_{\mu}Q_{\nu\rho} + \partial_{\nu}Q_{\rho\lambda} - \partial_{\rho}Q_{\mu\nu}) \tag{3.29} $$

We can add and subtract the following quantities as

$$ \begin{aligned} {}^{Q}\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}(Q^{-1})^{\lambda\rho}(\partial_{\mu}Q_{\nu\rho} + \partial_{\nu}Q_{\rho\lambda} - \partial_{\rho}Q_{\mu\nu}) - 2{}^{R}\Gamma^{\lambda}_{\mu\nu}Q_{\lambda\rho} + 2{}^{R}\Gamma^{\lambda}_{\mu\nu}Q_{\lambda\rho} - \\ {}^{R}\Gamma^{\lambda}_{\mu\rho}Q_{\lambda\nu} + {}^{R}\Gamma^{\lambda}_{\mu\rho}Q_{\lambda\nu} - {}^{R}\Gamma^{\lambda}_{\nu\rho}Q_{\lambda\mu} + {}^{R}\Gamma^{\lambda}_{\nu\rho}Q_{\lambda\mu} \end{aligned} \tag{3.30} $$

Which can than compactified into

$$ \begin{aligned} {}^{Q}\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}(Q^{-1})^{\lambda\rho}\{(\partial_{\mu}Q_{\nu\rho} - {}^{R}\Gamma^{\lambda}_{\mu\nu}Q_{\lambda\rho} - {}^{R}\Gamma^{\lambda}_{\mu\rho}Q_{\lambda\nu}) \\ +(\partial_{\nu}Q_{\mu\rho} - {}^{R}\Gamma^{\lambda}_{\rho\nu}Q_{\lambda\mu} - {}^{R}\Gamma^{\lambda}_{\mu\nu}Q_{\lambda\rho}) \\ -(\partial_{\rho}Q_{\mu\nu} - {}^{R}\Gamma^{\lambda}_{\rho\mu}Q_{\lambda\nu} - {}^{R}\Gamma^{\lambda}_{\rho\nu}Q_{\lambda\mu}) + 2{}^{R}\Gamma^{\lambda}_{\nu\mu}Q_{\lambda\rho}\} \end{aligned} \tag{3.31} $$

Now notice that the impressions inside the parentheses are nothing but the covariant derivatives of the connections with respect to the metric R. Thus we can write.

$$ {}^{Q}\Gamma^{\lambda}_{\mu\nu} = {}^{R}\Gamma^{\lambda}_{\mu\nu} + \Delta^{\lambda}_{\mu\nu} \tag{3.32} $$

Where the quantity Δ becomes

$$ \Delta^{\lambda}_{\mu\nu} = \frac{1}{2}(Q^{-1})^{\lambda\rho}({}^{R}\nabla_{\mu}Q_{\nu\rho} + {}^{R}\nabla_{\nu}Q_{\rho\mu} - {}^{R}\nabla_{\rho}Q_{\mu\nu}) \tag{3.33} $$

3.4 Conformal Transformation and Frame Shift

A scalar-tensor theory Lagrangian can be expressed in two different conformal frames: Einstein and Jordan frames (Faraoni, Gunzig & Nardone (1999); Maeda (1989); Magnano & Sokolowski (1994)). Basically, in the Jordan frame, we have a scalar field that couples to the scalar curvature. This form can be changed to a scalar curvature added to a scalar field. We can use a conformal transformation to shift from Jordan's frame to Einstein frame to get to this form.(Odintsov & Oikonomou (2015); Postma & Volponi (2014)) Consider the following metric transformation.

$$\hat{g}_{\mu\nu} = e^{2\omega} g_{\mu\nu} \tag{3.34}$$

Where $e^{2\omega}$ is the conformal factor and hat represents the objects in the Einstein frame. The quantities we are looking forward to transforming are Riemann tensor, Ricci tensor and scalar curvature. To do this, first, we need to transform the Riemann Tensor. We can use Eqs. (3.32,3.33) to transform Riemann tensor as

$$\begin{aligned} R^{\alpha}_{\beta\mu\nu} = \partial_{\mu}(\hat{\Gamma}^{\alpha}_{\beta\nu} + \Delta^{\alpha}_{\beta\nu}) - \partial_{\nu}(\hat{\Gamma}^{\alpha}_{\beta\mu} + \Delta^{\alpha}_{\beta\mu}) + \\ (\hat{\Gamma}^{\gamma}_{\beta\nu} + \Delta^{\gamma}_{\beta\nu})(\hat{\Gamma}^{\alpha}_{\gamma\mu} + \Delta^{\alpha}_{\gamma\mu}) - (\hat{\Gamma}^{\gamma}_{\beta\mu} + \Delta^{\gamma}_{\beta\mu})(\hat{\Gamma}^{\alpha}_{\gamma\nu} + \Delta^{\alpha}_{\gamma\nu}) \end{aligned} \tag{3.35}$$

Here $\hat{\Gamma}$ is the new connection in the Einstein frame. Notice that partial derivatives already hitting the transformed coefficients, and from the multiplications, we will get the standard Riemann Tensor in the new frame beside some extra terms. Eq. (3.35) can be unwrapped to the following form

$$\begin{aligned} R^{\alpha}_{\beta\mu\nu} = \hat{R}^{\alpha}_{\beta\mu\nu} + \partial_{\mu}\Delta^{\alpha}_{\beta\nu} - \partial_{\nu}\Delta^{\alpha}_{\beta\mu} + \Delta^{\alpha}_{\gamma\nu}\Delta^{\gamma}_{\beta\mu} - \Delta^{\gamma}_{\beta\nu}\Delta^{\alpha}_{\gamma\mu} + \\ \hat{\Gamma}^{\gamma}_{\beta\nu}\Delta^{\alpha}_{\gamma\mu} + \hat{\Gamma}^{\alpha}_{\gamma\mu}\Delta^{\gamma}_{\beta\nu} - \hat{\Gamma}^{\gamma}_{\beta\mu}\Delta^{\alpha}_{\gamma\nu} - \hat{\Gamma}^{\alpha}_{\gamma\nu}\Delta^{\gamma}_{\beta\mu} \end{aligned} \tag{3.36}$$

Again we get terms that can be compactly written as covariant derivatives in the Einstein frame that is $\hat{\nabla}$ as

$$\hat{\nabla}_{\mu}\Delta^{\alpha}_{\beta\nu} = \partial_{\mu}\Delta^{\alpha}_{\beta\nu} + \hat{\Gamma}^{\alpha}_{\mu\gamma}\Delta^{\gamma}_{\beta\nu} - \hat{\Gamma}^{\gamma}_{\beta\mu}\Delta^{\alpha}_{\gamma\nu} - \hat{\Gamma}^{\gamma}_{\mu\nu}\Delta^{\alpha}_{\beta\gamma} \tag{3.37}$$

$$\hat{\nabla}_{\nu}\Delta^{\alpha}_{\beta\mu} = \partial_{\nu}\Delta^{\alpha}_{\beta\mu} + \hat{\Gamma}^{\alpha}_{\nu\gamma}\Delta^{\gamma}_{\beta\mu} - \hat{\Gamma}^{\gamma}_{\beta\nu}\Delta^{\alpha}_{\gamma\mu} - \hat{\Gamma}^{\gamma}_{\mu\nu}\Delta^{\alpha}_{\beta\gamma} \tag{3.38}$$

where Riemann tensor (3.36) becomes

$$R^{\alpha}_{\beta\mu\nu} = \hat{R}^{\alpha}_{\beta\mu\nu} + \hat{\nabla}_{\mu}\Delta^{\alpha}_{\beta\nu} - \hat{\nabla}_{\nu}\Delta^{\alpha}_{\beta\mu} + \Delta^{\alpha}_{\gamma\mu}\Delta^{\gamma}_{\beta\nu} - \Delta^{\gamma}_{\beta\mu}\Delta^{\alpha}_{\gamma\nu} \tag{3.39}$$

For clarity, we can list up the tensorial quantity Δ's as

$$\Delta^{\alpha}_{\gamma\mu} = \delta^{\alpha}_{\gamma}\hat{\nabla}_{\mu}\omega + \delta^{\alpha}_{\mu}\hat{\nabla}_{\gamma}\omega - \hat{g}_{\gamma\mu}\nabla^{\alpha}\omega \tag{3.40}$$

$$\Delta^{\gamma}_{\beta\nu} = \delta^{\gamma}_{\beta}\hat{\nabla}_{\nu}\omega + \delta^{\gamma}_{\nu}\hat{\nabla}_{\beta}\omega - \hat{g}_{\beta\nu}\nabla^{\gamma}\omega \tag{3.41}$$

$$\Delta^{\lambda}_{\alpha\beta} = \delta^{\lambda}_{\alpha}\hat{\nabla}_{\beta}\omega + \delta^{\lambda}_{\beta}\hat{\nabla}_{\alpha}\omega - \hat{g}_{\alpha\beta}\nabla^{\lambda}\omega \tag{3.42}$$

$$\Delta^{\gamma}_{\beta\mu} = \delta^{\gamma}_{\beta}\hat{\nabla}_{\mu}\omega + \delta^{\gamma}_{\mu}\hat{\nabla}_{\beta}\omega - \hat{g}_{\beta\mu}\nabla^{\gamma}\omega \tag{3.43}$$

$$\Delta^{\alpha}_{\gamma\nu} = \delta^{\alpha}_{\gamma}\hat{\nabla}_{\nu}\omega + \delta^{\alpha}_{\nu}\hat{\nabla}_{\gamma}\omega - \hat{g}_{\gamma\nu}\nabla^{\alpha}\omega \tag{3.44}$$

Where we can write the Riemann tensor explicitly as

$$\begin{aligned}
R^{\alpha}_{\beta\mu\nu} = \hat{R}^{\alpha}_{\beta\mu\nu} &+ \delta^{\alpha}_{\beta}\hat{\nabla}_{\mu}\hat{\nabla}_{\nu}\omega + \delta^{\alpha}_{\nu}\hat{\nabla}_{\mu}\hat{\nabla}_{\beta}\omega - \hat{g}_{\beta\nu}\hat{\nabla}_{\mu}\hat{\nabla}^{\alpha}\omega \\
&- \delta^{\alpha}_{\beta}\hat{\nabla}_{\mu}\hat{\nabla}_{\nu}\omega - \delta^{\alpha}_{\mu}\hat{\nabla}_{\nu}\hat{\nabla}_{\beta}\omega + \hat{g}_{\beta\mu}\hat{\nabla}_{\nu}\hat{\nabla}^{\alpha}\omega \\
&+ \delta^{\alpha}_{\beta}\hat{\nabla}_{\mu}\omega\hat{\nabla}_{\nu}\omega + \delta^{\alpha}_{\nu}\hat{\nabla}_{\mu}\omega\hat{\nabla}_{\beta}\omega - \hat{g}_{\beta\nu}\hat{\nabla}_{\mu}\omega\hat{\nabla}^{\alpha}\omega \\
&+ \delta^{\alpha}_{\mu}\hat{\nabla}_{\beta}\omega\hat{\nabla}_{\nu}\omega + \delta^{\alpha}_{\mu}\hat{\nabla}_{\nu}\omega\hat{\nabla}_{\beta}\omega - \delta^{\alpha}_{\mu}\hat{g}_{\beta\nu}\hat{\nabla}_{\gamma}\omega\hat{\nabla}^{\gamma}\omega \\
&- \hat{g}_{\beta\mu}\hat{\nabla}_{\nu}\omega\hat{\nabla}^{\alpha}\omega - \hat{g}_{\mu\nu}\hat{\nabla}_{\beta}\omega\hat{\nabla}^{\alpha}\omega + \hat{g}_{\beta\nu}\hat{\nabla}_{\mu}\omega\hat{\nabla}^{\alpha}\omega \\
&- \delta^{\alpha}_{\beta}\hat{\nabla}_{\mu}\omega\hat{\nabla}_{\nu}\omega - \delta^{\alpha}_{\nu}\hat{\nabla}_{\mu}\omega\hat{\nabla}_{\beta}\omega + \hat{g}_{\beta\nu}\hat{\nabla}_{\mu}\omega\hat{\nabla}^{\alpha}\omega \\
&- \delta^{\alpha}_{\mu}\hat{\nabla}_{\beta}\omega\hat{\nabla}_{\nu}\omega - \delta^{\alpha}_{\nu}\hat{\nabla}_{\beta}\omega\hat{\nabla}_{\mu}\omega + \hat{g}_{\mu\nu}\hat{\nabla}_{\beta}\omega\hat{\nabla}^{\alpha}\omega \\
&+ \hat{g}_{\beta\mu}\hat{\nabla}_{\nu}\omega\hat{\nabla}^{\alpha}\omega + \delta^{\alpha}_{\nu}g_{\beta\mu}\hat{\nabla}_{\rho}\omega\hat{\nabla}^{\rho}\omega - \hat{g}_{\beta\mu}\hat{\nabla}_{\nu}\omega\hat{\nabla}^{\alpha}\omega
\end{aligned} \tag{3.45}$$

After dealing with the cancelling terms, we can tidy up and write the Riemann tensor compactly.

$$\begin{aligned}
R^{\alpha}_{\beta\mu\nu} = \hat{R}^{\alpha}_{\beta\mu\nu} &+ (\delta^{\alpha}_{\nu}\hat{\nabla}_{\mu}\hat{\nabla}_{\beta}\omega - \delta^{\alpha}_{\mu}\hat{\nabla}_{\nu}\hat{\nabla}_{\beta}\omega) - (\hat{g}_{\beta\nu}\hat{\nabla}_{\mu}\hat{\nabla}^{\alpha}\omega - \hat{g}_{\beta\mu}\hat{\nabla}_{\nu}\hat{\nabla}^{\alpha}\omega) \\
&+ (\delta^{\alpha}_{\mu}\hat{\nabla}_{\beta}\omega\hat{\nabla}_{\nu}\omega - \delta^{\alpha}_{\nu}\hat{\nabla}_{\beta}\omega\hat{\nabla}_{\mu}\omega) - (\hat{g}_{\mu\beta}\hat{\nabla}_{\nu}\omega\hat{\nabla}^{\alpha}\omega - \hat{g}_{\nu\beta}\hat{\nabla}_{\mu}\omega\hat{\nabla}^{\alpha}\omega) \\
&- (\delta^{\alpha}_{\mu}\hat{g}_{\beta\nu}\hat{\nabla}_{\rho}\omega\hat{\nabla}^{\rho}\omega - \delta^{\alpha}_{\nu}\hat{g}_{\beta\mu}\hat{\nabla}_{\rho}\omega\hat{\nabla}^{\rho}\omega)
\end{aligned} \tag{3.46}$$

We can take the trace of the (3.46) to obtain the Ricci tensor.

$$(3.47)\quad R_{\mu\nu} = \hat{R}_{\mu\nu} - 2\hat{\nabla}_\mu\hat{\nabla}_\nu\omega - \hat{g}_{\mu\nu}\hat{\nabla}_\rho\hat{\nabla}^\rho\omega + 2\hat{\nabla}_\mu\omega\hat{\nabla}_\nu\omega - 2\hat{g}_{\mu\nu}\hat{\nabla}_\rho\omega\hat{\nabla}^\rho\omega$$

Lastly, contracting both sides with the metric in Jordan frame that is $g^{\mu\nu}$, keeping in mind that the right-hand side needs the metric $\hat{g}^{\mu\nu}$ we can get the scalar curvature.

$$(3.48)\qquad g^{\mu\nu}R_{\mu\nu} = R = e^{2\omega}\{\hat{R} - 6\hat{\nabla}_\rho\hat{\nabla}^\rho\omega - 6\hat{\nabla}_\rho\omega\hat{\nabla}^\rho\omega\}$$

Now let us rewrite the Scalar tensor action (3.28) in the Einstein frame noting that volume element changes as

$$(3.49)\qquad \sqrt{-g} = \sqrt{-\hat{g}}e^{-4\omega}$$

The action becomes

$$(3.50)\quad S = \int d^4x\sqrt{-g}\left[\frac{M_{pl}^2}{2}\Lambda e^{-2\omega}(\hat{R} - 6\hat{\nabla}_\rho\hat{\nabla}^\rho\omega - 6\hat{\nabla}_\rho\omega\hat{\nabla}^\rho\omega) - e^{-4\omega}U\right]$$

Now to model the action as a function that is linear in R we need the following Λ choice, which can be considered valid for $\Lambda > 0$

$$(3.51)\qquad \Lambda \approx e^{2\omega}$$

to make the action (3.50) canonical let us introduce a new scalar definition (i.e. scalaron (Starobinsky (1980a,8))) that is

$$(3.52)\qquad \sqrt{\frac{3}{2}}\log\Lambda = \frac{\Phi}{M_{pl}}$$

which changes omega to the form

$$\omega = \frac{\Phi}{\sqrt{6}M_{pl}} \tag{3.53}$$

and our action reduces to its canonical form

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{pl}^2}{2}\hat{R} - \frac{1}{2}\partial_\rho\Phi\partial^\rho\Phi - V(\Phi)\right] \tag{3.54}$$

where the potential for the new scalar field Φ can be written as

$$V(\Phi) = \frac{U(\Phi)}{\Lambda^2} = \frac{M_{pl}^2(\kappa(\Lambda)\Lambda - f(\kappa(\Lambda))}{2\Lambda} \longrightarrow \frac{M_{pl}^2 f'(R)R - f(R)}{2(f'(R))^2} \tag{3.55}$$

Hence, we obtain an action with a scalar Lagrangian with kinetic and potential terms added to the scalar curvature, EH action.

3.5 Friedmann Equations

In a broader sense, the universe can be considered isotropic and homogeneous. Every direction in the universe looks the same, and every point in the universe can be considered identical on a large scale. We can see this in the cosmic microwave background, ancient background radiation around 2.7K (Akramiet al. (2020)). We can construct an isotropic and homogeneous metric using this presumption to solve the Einsteins Field equations. Such a metric can be constructed as follows,

$$ds^2 = -dt^2 + a(t)^2dx^2 \tag{3.56}$$

Here $a(t)$ is a time-dependent function called the scale factor of the universe, and dx is the spatial part of the metric. Since we hold no apriori assumptions about the universe's expansion, this should be a valid proposition.

The connections for this metric can be calculated as follows;

$$\Gamma^t_{rr} = a\dot{a} \tag{3.57}$$

$$\Gamma^t_{\theta\theta} = a\dot{a}r^2 \tag{3.58}$$

$$\Gamma^t_{\phi\phi} = a\dot{a}r^2\sin^2\theta \tag{3.59}$$

$$\Gamma^r_{tr} = \frac{\dot{a}}{a} \tag{3.60}$$

$$\Gamma^r_{\theta\theta} = -r \tag{3.61}$$

$$\Gamma^r_{\phi\phi} = -r\sin^2\theta \tag{3.62}$$

$$\Gamma^\theta_{t\theta} = \frac{\dot{a}}{a} \tag{3.63}$$

$$\Gamma^\theta_{r\theta} = \frac{1}{r} \tag{3.64}$$

$$\Gamma^\theta_{\phi\phi} = \sin\theta\cos\theta \tag{3.65}$$

$$\Gamma^\phi_{\theta\phi} = \frac{\dot{a}}{a} \tag{3.66}$$

$$\Gamma^\phi_{r\phi} = \frac{1}{r} \tag{3.67}$$

$$\Gamma^\phi_{\theta\phi} = \cot\theta \tag{3.68}$$

Furthermore we can calculate the ricci scalar as

$$R = 6(2H^2 + \dot{H}) \tag{3.69}$$

where $H = \dot{a}/a$ is the Hubble parameter where its current value is around $2.13 \times 10^{-42} GeV$.

From the field equations (3.21,3.22) we can calculate the Friedmann equations for any given $f(R)$ as

$$3f'(R)H^2 = \frac{(f'(R)R - f(R))}{2} - 3H\dot{f'(R)} + K^2\rho_M \tag{3.70}$$

$$-2f'(R)\dot{H} = \ddot{f'(R)} - H\dot{f(R)} + K^2(\rho_M + P_M) \tag{3.71}$$

.

3.6 Inflation

In cosmology, Inflation is the theory of cosmic exponential acceleration in the early universe(Kalara, Kaloper & Olive (1990); Vázquez, Padilla & Matos (2018)). The conjectured inflationary epoch existed from the 10^{-36} to the 10^{-32} seconds. In theory, in this epoch, the universe expanded to 10^{26} of its Planck period size; hence it stayed homogeneous among its bulk (Hwang & Noh (2001); Mukhanov & Chibisov (1981); Starobinsky (1979)).

Let us consider the form of function $f(R)$ as

$$f(R) = R + \xi R^n \tag{3.72}$$

where both ξ and n are positive definite. From field equations (3.21,3.22) we get

$$3(1+n\xi R^{n-1})H^2 = \frac{(n-1)}{2}\xi R^n - 3n(n-1)\xi H R^{n-2}\dot{R} \tag{3.73}$$

In the inflationary epoch, the quantum corrections will be much more significant from the linear EH term (Starobinsky (1981); Vilenkin (1985)). Thus higher-order curvature terms will dominate the action. Thus in this epoch, we can make the following assumption.

$$f'(R) = 1 + n\xi R^{n-1} \gg 1 \rightarrow f'(R) \approx n\xi R^{n-1} \tag{3.74}$$

Thus (3.73) yields

$$H \approx \sqrt{\frac{n-1}{6n}(R - 6nH\frac{\dot{R}}{R})} \tag{3.75}$$

In the early epoch, Hubble evolves slowly; thus following assumptions are valid.

$$\frac{\dot{H}}{H^2} \ll 1 \text{ and } \frac{\ddot{H}}{H\dot{H}} \ll 1 \tag{3.76}$$

These furthermore reduce the scalar curvature and its time derivative to the followings.

$$R = 6(2H^2 + \dot{H}) \tag{3.77}$$

$$\dot{R} = 6(4H\dot{H} + \ddot{H}) \tag{3.78}$$

Thus (3.75) reduces to

$$\frac{\dot{H}}{H^2} \approx \frac{n-2}{2n^2 - 3n + 1} = -\epsilon \tag{3.79}$$

where ϵ is constant. We can see that for $n = 2$ ϵ reduces to 0. Thus for the most trivial correction to the EH term, in the early epoch where $f'(R) \gg 1$ the H stays constant. The inflation continues until this parameter becomes around the order of unity.

Let us elaborate on such a form of $f(R)$. Starobinsky's model of $f(R)$ is the most simplistic correction to the EH action, which is a quadratic one. It can be written as

$$f(R) = R + \frac{1}{6m_\phi^2} R^2 \tag{3.80}$$

Where m_ϕ is a constant on the dimension of mass. The field equations for such a function can be calculated as

$$\ddot{H} - \frac{\dot{H}^2}{2H} + \frac{1}{2} m_\phi^2 H = -3\dot{H}H \tag{3.81}$$

$$\ddot{R} + 3H\dot{R} + m_\phi^2 R = 0 \tag{3.82}$$

In the first era again, $\ddot{H}$ and $\dot{H}$ can be neglected, which yields.

$$H \approx H_{in} - \frac{m_\phi^2}{6}(t - t_{in}) \tag{3.83}$$

Expansion continues until the slow roll parameter ϵ reaches unity where in this case

$$\epsilon \approx \frac{m_\phi^2}{6H^2} \tag{3.84}$$

Also since the inflation is an exponential increase in Hubble we can define the number of e-foldings as

$$N = \int_{t_i}^{t_f} H dt \tag{3.85}$$

at the end of the inflation where (3.84) iterates $t_f \approx t_i + 6H_i/m_\phi^2$. N can be aproximated as

$$N \approx \frac{3H_i^2}{m_\phi^2} \tag{3.86}$$

From the CMB temperature anisotropies, we constrain the constant m_ϕ to be around 10^{13} GeV, and in order to solve the horizon and flatness problems, we need to have the $N \geqslant 70$.

Also, this model corresponds to a scalar-tensor theory in Einstein frame via a redefinition of curvature. By using (3.52), we can define the scalar field of Starobinsky's as

$$\Phi = \sqrt{\frac{3}{2}} M_{pl} \log f'(R) = \sqrt{\frac{3}{2}} M_{pl} \log\left(1 + \frac{R}{3m_\phi^2}\right) \tag{3.87}$$

which have a field potential as

$$U(\Phi) = \frac{3m_\phi^2 M_{pl}^2}{4}\left(1 - \exp - \sqrt{\frac{2}{3}}\frac{\Phi}{M_{pl}}\right)^2 \tag{3.88}$$

Inflation can be observed in this frame where $\Phi/M_{pl} \gg 1$ where the field potential stays constant and rolls slowly (Barrow & Cotsakis (1988)). In the late universe, however, the field oscillates around zero.

4. Spherical Symmetry and Black Holes

4.1 Spherically Symmetric Solutions of Einstein Gravity

To obtain a spherically symmetric solution to EH action, we can use the Schwarzchild metric that is

$$ds^2 = -B(r)dt^2 + \frac{dr^2}{B(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{4.1}$$

r is the parametrised radial coordinate, θ and ϕ represents the angular coordinates and $A(r)$ is the radial function that will be identified later on. In order to find the lapse function $B(r)$, one can solve the field equations in the vacuum that is $G_{\mu\nu} = 0$. The non zero Levi Civita connections can be calculated as

$$\Gamma^{t}_{rt} = \Gamma^{t}_{tr} = \frac{B'}{2B} \tag{4.2}$$

$$\Gamma^{r}_{tt} = \frac{BB'}{2} \tag{4.3}$$

$$\Gamma^{r}_{rr} = \frac{B'}{2B} \tag{4.4}$$

$$\Gamma^{r}_{\theta\theta} = rB \tag{4.5}$$

$$\Gamma^{r}_{\phi\phi} = rB\sin^2\theta \tag{4.6}$$

$$\Gamma^{\theta}_{r\theta} = \Gamma^{\theta}_{\theta r} = \frac{1}{r} \tag{4.7}$$

$$\Gamma^{\theta}_{\phi\phi} = -\cos\theta\sin\theta \tag{4.8}$$

$$\Gamma^{\phi}_{r\phi} = \Gamma^{\phi}_{\phi r} = \frac{1}{r} \tag{4.9}$$

$$\Gamma^{\phi}_{\phi\theta} = \Gamma^{\phi}_{\theta\phi} = \frac{\cos\theta}{\sin\theta} \tag{4.10}$$

Furthermore, the scalar curvature can be calculated by contracting the Ricci scalar with the metric as

$$R_{\mu\nu}g^{\mu\nu} = R = \frac{A''(r)r^2 + 4A'r + 2A - 2}{r^2} \tag{4.11}$$

The first two equations of the field equations, the time and radial component yield the same results. These equations emerges a constant in length dimension as a result.

$$G_{00} = 0 \rightarrow \frac{A^2}{r^2}\left(\frac{1}{A}\right)' + \frac{1}{r^2}(1-A) = 0 \tag{4.12}$$

$$G_{11} = 0 \rightarrow -\frac{A'}{r} + \frac{1}{r^2}(1-A) \tag{4.13}$$

$$A(r) = 1 - \frac{C}{r} \tag{4.14}$$

The constant C can be found out by taking the Newtonian limit of the Einstein equations that are taking the small metric perturbation about a flat metric as

$g_{\mu\nu} = \eta_{\mu\nu} + h\mu\nu$ where η is the Minkowski metric and the h is the non-relativistic perturbation that yields $C = 2G_N M$, where M is the mass of the spherical object. We can see that the metric blows up at two radial coordinates. One is the centre, and the other one is $r = 2G_N M$. These are two singularities of this metric. We can also see that the metric turns in to a flat one at infinity. That is, the metric is asymptotically flat. The singularities are taking the attention here since, generally speaking, we expect the "infinities" to be absent from any physically meaningful theory. When we take a trace of a curvature tensor, we also know that we lose some information about its physical implications. The tensor that holds the most information about the curvature of the system is the Riemann tensor; however, to calculate field equations, we already have taken its trace. So maybe we can think that some information about the identity of these singularities has gone away. To counter this, we can calculate the Kretschmann scalar that is the contraction of two Riemann tensors, which yields.

$$R^{\alpha\beta\gamma\lambda} R_{\alpha\beta\gamma\lambda} = \frac{48 G_N^2 M^2}{r^6} \tag{4.15}$$

Here we can see that the only singularity in the Kretschmann scalar at the centre that is $r = 0$ is the only essential singularity. But what about $r = 2G_N M$? It is called the horizon, which is not an essential singularity but a coordinate singularity. We can project the metric to a different one to this singularity to vanish. However, one can look at the killing vectors of this system to identify this radius as a horizon. Since the fastest information carrier on this manifold is a photon, one can find the killing vectors to see this radius is the last line of a photon escape limit. On the horizon, a piece of existing information cannot reach an outside observer. Thus we can make the following definition.

A spherically symmetric object containing an essential singularity at its centre contained by a coordinate singularity is called a black hole.

Physically this means that after a particle that travels to this object passes its horizon, its only direction will be the essential singularity on the centre. It cannot send any signal to the outside; however, it will travel as it is. From an outside observer, however, the object will be frozen on the surface of this object since the only information that remains from it is on the entry point.

4.2 Photon Sphere

To see the effects of a BH on a particle, we need to see the solution of its geodesic equation. As said, every object moves in a geodesic since gravity itself is not a force in GR. To do that, let us look for a stable orbit for a photon around a BH.

A BH has spherical symmetry. Thus for a photon, all axis for a circular photon orbit will be equivalent. This also means that the photon will travel without changing its radial coordinate, i.e. $dr = 0$. Also, we can take the initial rotation plane as a reference that makes $d\theta = 0$. Since dS is also zero for a photon, we can write the regular Schwarzschild metric as

$$\left(1 - \frac{2G_N M}{r}\right) dt^2 = r^2 \sin^2\theta d\phi^2 \tag{4.16}$$

Rearranging gives

$$\frac{d\phi}{dt} = \frac{1}{r\sin\theta}\sqrt{1 - \frac{2G_N M}{r}} \tag{4.17}$$

Also, we can solve the geodesic equation for the radial part that is

$$\frac{d^2 r}{d\tau^2} + \Gamma^r_{\mu\nu}\upsilon^\mu \upsilon^\nu = 0 \tag{4.18}$$

Since r and θ is constant derivatives with respect the proper time will also yield null. Thus we can easily solve the (4.18) with the given connections (4.2-4.10) as

$$\left(\frac{d\phi}{dt}\right)^2 = \frac{2G_N M}{r^3 \sin^2\theta} \tag{4.19}$$

which gives the following for the $\theta = \pi/2$ equatorial plane

$$r = 3G_N M \tag{4.20}$$

On this radial distance from the singularity, photons tend to create a circular orbit around the BH. We can see that it is larger than the horizon as $r_{ps} = 3/2r_h$. A free-falling object that crosses the photon sphere from the outside must spiral into the BH. From the inside, however, the maximum an object can do is to flow into infinity or fall into the BH again (Mishra, Chakraborty & Sarkar (2019)).

4.3 Shadow

To see the effects of a BH on a particle, we need to see the solution of its geodesic equation. As said, every object moves in a geodesic since gravity itself is not a force in GR. To do that, let us look for a stable orbit for a photon around a BH.

A BH has spherical symmetry. Thus for a photon, all axis for a circular photon orbit will be equivalent. This also means that the photon will travel without changing its radial coordinate, i.e. $dr = 0$. Also, we can take the initial rotation plane as a reference that makes $d\theta = 0$. Since dS is also zero for a photon, we can write the regular spherically-symmetric metric as

$$g_{\mu\nu}dx^{\mu}dx^{\nu} = -A(r)dt^2 + B(r)dr^2 + D(r)(d\theta^2 + \sin^2\theta d\phi^2 \tag{4.21}$$

And we can also write the Lagrangian as

$$\mathcal{L}(x, \dot{x}) = \frac{1}{2}g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} \tag{4.22}$$

and it takes the form

$$\mathcal{L}(x, \dot{x}) = -A(r)\dot{t}^2 + B(r)\dot{r}^2 + D(r)(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) \tag{4.23}$$

Now let us look to the BH directly from the equatorial plane. That is let us take $\theta = \pi/2$. From t and ϕ components of the Euler-Lagrange equation, we will get two equations of motions or, in this case, constants of motions as

$$C_1 = A(r)\dot{t} \tag{4.24}$$

$$C_2 = D(r)\dot{\phi} \tag{4.25}$$

Now if we consider massles particles where $g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = 0$ and

$$-A(r)\dot{t}^2 + B(r)\dot{r}^2 + D(r)\dot{\phi}^2 = 0 \tag{4.26}$$

we can write down the orbit equation for lightlikke geodesics with the newly obtined constants of motions., that is

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{D(r)}{B(r)}\left(\frac{D(r)C_1^2}{A(r)C_2^2} - 1\right) \tag{4.27}$$

We see that constant C_2/C_1 acts as an impact parameter b of the orbit equation. So at the turning point of the light ray trajectory, we expect to have $dr/d\phi = 0$, let us call this turning point r_t, and at r_t limit, we get the following relations.

$$\frac{C_2^2}{C_1^2} = \frac{D(r_t)}{A(r_t)} = \frac{1}{b^2} \tag{4.28}$$

To conventionalise, we can introduce a weighted function $h(r)$ as

$$h^2(r) = \frac{D(r)}{A(r)} \rightarrow b = h(R) \tag{4.29}$$

Thus our orbit equation becomes

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{D(r)}{B(r)}\left(\frac{h^2(r)}{h^2(R)} - 1\right) \tag{4.30}$$

Now to obtain an angular radius of shadow, we will assume a far away static observer at the coordinate r_o where we will obtain the trajectory of the light rays as

$$\cot\alpha = \frac{\sqrt{g_{rr}}}{\sqrt{g_{\phi\phi}}}\frac{dr}{d\phi}\bigg|_{r=r_o} = \frac{\sqrt{B(r)}}{\sqrt{D(r)}}\frac{dr}{d\phi}\bigg|_{r=r_o} \rightarrow \sin^2\alpha = \frac{h^2(r_t)}{h^2(r_o)} \tag{4.31}$$

At the limit where the turning point is the photon sphere, we naturally get the angular radius of the BH as

$$\sin\alpha = \frac{h(r_{ps})}{h(r_o)} \tag{4.32}$$

Therefore for considerable distances, we can calculate the shadow of a BH as

$$R_s = r_o \sin\alpha = r_o \frac{h(r_{ps})}{h(r_o)} \tag{4.33}$$

R_s is called the shadow radius of the Blackhole. A faraway observer will catch the outgoing photons from the photon sphere with this impact parameter. We saw that photons forming a circular orbit around a BH would be far from the actual horizon. A photon approach from infinity either fall into the BH or slingshot back to the infinity from the photon sphere. These deflected photons will have their trajectories bent. A faraway observer thus will see an illuminated ring that is bigger than its horizon and photon radius. The inner black region is called the shadow of the BH. This darkness in the celestial sphere can be observed via any methods from far. Thus, it is interesting to calculate the exact radius of a shadow (Luminet (1979); Perlick & Tsupko (2021); Synge (1966)).

for example, in a Schwarzchild space-time, we have

$$A(r) = \frac{1}{B(r)} = 1 - \frac{2M}{r}\ ,\ D(r) = r^2 \tag{4.34}$$

Thus

$$h^2(r) = \frac{r^2}{1 - \frac{2M}{r}} \tag{4.35}$$

$$r_{ph} = 3M \tag{4.36}$$

$$R_s = 3\sqrt{3}M \tag{4.37}$$

Shadow Radius is larger than the photon sphere radius as well as the BH itself.

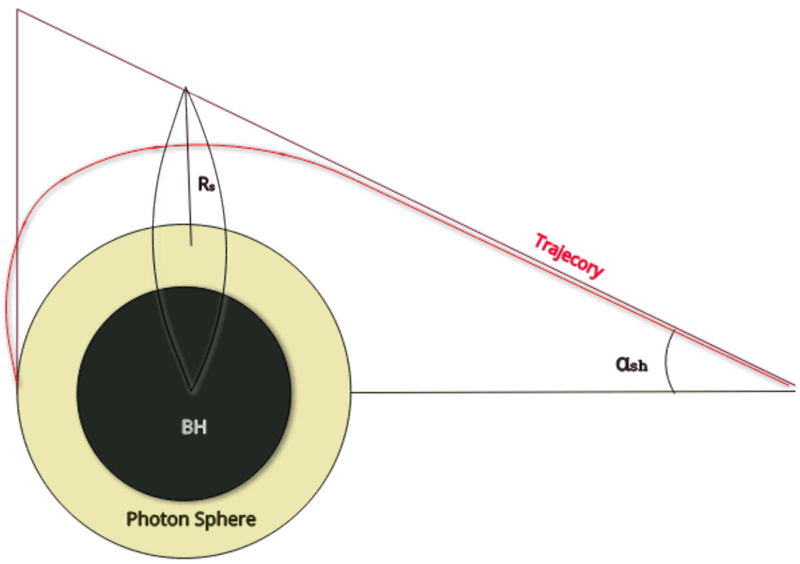

Figure 4.1 Shadow of a BH. It is the observed trajectory of photons that originates from the photon sphere to a faraway observer. It is bigger than both the photon sphere and the BH itself.

4.4 Entropy and Temprature

Entropy can be thought as randomness of energy dispersal in a system. If we throw an apple on a BH, it will fall obviously, but after it passes the horizon, one can not perceive any information about the apple other than its mass, its charge and its spin. The outer region can only influence these three properties of the BH; other information should be not reachable. This is famously known as Bekenstein's No Hair theorem (Bekenstein (1973)). The information loss is a bit problematic for physics, especially quantum mechanics. As well known from a given wave function, one can extract the future and pass of the given object. There exist a solution to this problem. We know that the horizon $2G_NM$ is only dependent on M, and since nothing can escape a BH, its mass and, therefore, its surface area can only increase just as entropy. So one can relate the entropy to the surface area of the BH. As we know, the surface area of the BH is nothing but the surface of a spherical shell with the Schwarzschild radius that is

$$A = 4\pi r_h^2 = 16\pi G_N^2 M^2 \tag{4.38}$$

Let us look at the differential of this area. One can get

$$dA = 32\pi G_N^2 M dM \tag{4.39}$$

We also know that entropy is related to the heat change divided by temperature as

$$dS = \frac{\delta Q}{T} = \frac{dE - \delta W}{T} \rightarrow dS = \frac{dE}{T}\,,\, \delta W = 0 \tag{4.40}$$

We can take the BH at rest and take its energy as $E = M$. Thus we expect to have the form as $dM = T \times dS$ on the area function. We can see that this is unit wise consistent with it as

$$dM = \frac{1}{8\pi G_N M} \times \frac{1}{4G_N} dA \tag{4.41}$$

Thus we can heuristically define a temperature and an Entropy for a given BH. With natural units written, they can be defined as

$$T_H = \frac{\hbar c^3}{8\pi G_N k_B M} \tag{4.42}$$

$$S = \frac{k_B c^3 A}{4\hbar G_N} \tag{4.43}$$

T_H is called the Hawking radiation, and S is the Bekenstain entropy of the BH. k_B is the usual Boltzman constant, and $\hbar$ is the reduced Planck constant.

4.5 Physical Properties of the Black Holes in f(R) Gravity

For a given function $f(R)$, one can obtain the shadow radius, temperature and entropy in the same way. The lapse function $B(r)$ will depend on the solutions of the field equations. We can get the following equations for the equations of motion in (4.2-4.10) as

$$2FB'' + 2B'F' - 2BF'' - \frac{4}{r}BF' + \frac{4}{r^2}F(1-B) = 0, \tag{4.44}$$

$$2FB'' + 2B'F' + 6BF'' - \frac{4}{r}BF' + \frac{4}{r^2}F(1-B) = 0, \tag{4.45}$$

$$2FB'' + 2B'F' + 2BF'' - \frac{4}{r}BF' + \frac{4}{r^2}F(1-B) = 0 \tag{4.46}$$

Here, subtracting any two equations of (4.44), (4.45) and (4.46) from each other yields $BF'' = 0$. Thus it leads consistently to the linear solution

$$F[R(r)] = a + br \tag{4.47}$$

with a and b undetermined constants.

One can then solve the Einstein equations with the condition that function $f(R)$ satisfies the (4.44-4.46)(Addazi, Capozziello & Odintsov (2021)). Then one can calculate the shadow as

$$R_s^2 = \frac{r_p^2}{B(r_p)} \tag{4.48}$$

Where r_p is the photon sphere for a given theory. Also, the temperature can be generalised as

$$T = \frac{B'(r_+)}{4\pi}. \tag{4.49}$$

where r_+ is the outer horizon of any BH. Furthermore the Entropy yields

$$S(r_+) = \frac{1}{4G_N}\mathcal{A}_h f'[R(r_+)], \tag{4.50}$$

where the Area can be calculated as

$$\mathcal{A}_h = 4\pi r_+^2. \tag{4.51}$$

In general, there is no constraint on the function $f(R)$ that comes from the thermodynamics other than the first derivative needs to be positive (Peralta & Jorás (2020)) since otherwise, it will result in negative entropy values. It is also important to note that for any $f(R)$ theory, this condition needs to be satisfied to not bother with ghosts in theory.

5. Emergent Gravity Theories

In the introduction part, we saw that the theory of GR sits on two assumptions. One is that space-time is a pseudo-Riemannian manifold. The other is the Einsteins equivalence principle. From these apriori assumptions, we can write a general action that contains the curvature to obtain specific field equations. The idea to induce gravity is to make these field equations emerge from an underlying QFT. This section will look at two emergent theories: Sakharov's idea of inducing the geometry from one-loop effective action (Sakharov (1967); Visser (2002)), and the other is Symmergent Gravity, which is a novel idea that gravity emerges by restoring broken gauge symmetries(Demir (2017,1,2,1,1)). This chapter will exclusively use the west coast metric $(+,-,-,-)$ for convenience with other particle physics texts. However, in the next chapter, we will return to the east coast metric, which will weigh the scalar curvature with a relative minus sign.

5.1 Sakharov's Induced Gravity

Let us start with a Lorentzian manifold. On this manifold, the metric will be considered as a classical background. Then consider the one-loop corrections to the effective action of this theory for a minimally coupled scalar field.

$$S = -\frac{1}{2}\log\det[\nabla^2 + m^2 + cR] = -\frac{1}{2}\operatorname{Tr}\log[\nabla^2 + m^2 + cR] \tag{5.1}$$

If we consider a difference of the action with a dynamic field and the background metric, we will get

$$S_g - S_{\hat{g}} = -\frac{1}{2}\operatorname{Tr}\log\left[\frac{\nabla^2{}_g + m^2 + cR_g}{\nabla^2_{\hat{g}} + m^2 + cR_{\hat{g}}}\right] \tag{5.2}$$

Then we can use the identity

$$\log\left[\frac{\alpha}{\beta}\right] = \int_0^\infty \frac{dx}{x}[e^{-\beta x} - e^{-\alpha x}] \tag{5.3}$$

to get

$$S_g - S_{\hat{g}} = \frac{1}{2}\operatorname{Tr}\int_0^\infty \frac{ds}{s}[e^{-s(\nabla^2_g + m^2 + cR)} - e^{-s(\nabla^2_{\hat{g}} + m^2 + cR_{\hat{g}})}] \tag{5.4}$$

where we used the Schwinger's proper time formalism. Now since this is a Ultra-Violet(UV) divergent quantity let us use a UV cutoff such that

$$\frac{1}{2}\operatorname{Tr}\int_0^\infty \frac{ds}{s}... \to \frac{1}{2}\operatorname{Tr}\int_{\Lambda^{-2}}^\infty \frac{ds}{s}... \tag{5.5}$$

And then use the heat kernel expansion up to order of $\mathcal{O}(s^2)$

$$e^{-s(\nabla^2_g + m^2 + cR)} = \frac{\sqrt{-g}}{(4\pi s)^2}[\mathcal{C}_{[0g]} + \mathcal{C}_{[1g]}s + \mathcal{C}_{[2g]}s^2 + \mathcal{O}(s^4)] \tag{5.6}$$

where $\mathcal{C}_{[ig]}$'s are the i'th Hemidew coefficents on the metric g where up to second order of s can be written as

$$\mathcal{C}_{[0g]} = 1 \tag{5.7}$$

$$\mathcal{C}_{[1g]} = \kappa_1 R_g - m^2 \tag{5.8}$$

$$\mathcal{C}_{[2g]} = \kappa_2 C_{\alpha\beta\gamma\lambda}C^{\alpha\beta\gamma\lambda} + \kappa_3 R_{\alpha\beta}R^{\alpha\beta} + \kappa_4 R_g^2 + \kappa_5 \nabla^2 R_g - \kappa_1 m^2 R_g + \frac{1}{2}m^4 \tag{5.9}$$

Then by using(5.5) and (5.6) we can integrate (5.4) as

$$\frac{1}{2}\operatorname{Tr}\int_{\Lambda^{-2}}^{\infty}\frac{ds}{s}\frac{\sqrt{-g}-\sqrt{-\hat{g}}}{(4\pi s)^2}\left(\mathcal{C}_{[0g]}-\mathcal{C}_{[0\hat{g}]}\right)\rightarrow\frac{1}{32\pi^2}\operatorname{Tr}\left(\mathcal{C}_{[0g]}-\mathcal{C}_{[0\hat{g}]}\right)\frac{\Lambda^4}{2} \tag{5.10}$$

$$\begin{aligned}\frac{1}{2}\operatorname{Tr}\int_{\Lambda^{-2}}^{\infty}\frac{ds}{s}\frac{\sqrt{-g}-\sqrt{-\hat{g}}}{(4\pi s)^2}\,s\left(\mathcal{C}_{[1g]}-\mathcal{C}_{[1\hat{g}]}\right)\\ \rightarrow\frac{1}{32\pi^2}\operatorname{Tr}\left(\mathcal{C}_{[1g]}-\mathcal{C}_{[1\hat{g}]}\right)\Lambda^2\end{aligned} \tag{5.11}$$

$$\begin{aligned}\frac{1}{2}\operatorname{Tr}\int_{\Lambda^{-2}}^{\infty}\frac{ds}{s}\frac{\sqrt{-g}-\sqrt{-\hat{g}}}{(4\pi s)^2}\,s^2\left(\mathcal{C}_{[2g]}-\mathcal{C}_{[2\hat{g}]}\right)\\ \rightarrow\frac{1}{32\pi^2}\operatorname{Tr}\left(\mathcal{C}_{[2g]}-\mathcal{C}_{[2\hat{g}]}\right)\log\left(\frac{\Lambda^2}{m^2}\right)\end{aligned} \tag{5.12}$$

the orders of s^3 however, will reduce to the order of Λ^{-2} hence will yield a finite value which will be suppressed by Λ hence the total difference between the actions can be written as

$$\begin{aligned}S_g-S_{\hat{g}}=\frac{1}{32\pi^2}\operatorname{str}[[\mathcal{C}_{[0g]}-\mathcal{C}_{[0\hat{g}]}]\frac{\Lambda^4}{2}+[\mathcal{C}_{[1g]}-\mathcal{C}_{[1\hat{g}]}]\Lambda^2\\ +[\mathcal{C}_{[2g]}-\mathcal{C}_{[2\hat{g}]}]\log\left(\frac{\Lambda^2}{m^2}\right)]+\mathcal{O}(\Lambda^{-2})\end{aligned} \tag{5.13}$$

We can unwrap this by using the Hemidew coefficients and collect the geometric terms out as

$$\int d^4x\sqrt{-g}[-\mathcal{K}_1-\mathcal{K}_2R+\mathcal{K}_3C_{\alpha\beta\gamma\lambda}C^{\alpha\beta\gamma\lambda}+\mathcal{K}_4R^2] \tag{5.14}$$

where

$$\mathcal{K}_1=\mathcal{K}_{1cl}-\frac{1}{32\pi^2}\operatorname{str}[\frac{\Lambda^4}{2}-m^2\Lambda^2+\frac{m^4}{2}\log\left(\frac{\Lambda^2}{m^2}\right)]+\mathcal{O}(\Lambda^{-2}) \tag{5.15}$$

$$\mathcal{K}_2 = \mathcal{K}_{2cl} - \frac{1}{32\pi^2}\,\mathrm{str}[\kappa_1\Lambda^2 - m^2\log\left(\frac{\Lambda^2}{m^2}\right)] + \mathcal{O}(\Lambda^{-2}) \tag{5.16}$$

$$\mathcal{K}_3 = \mathcal{K}_{3cl} + \frac{1}{32\pi^2}\,\mathrm{str}[\bar{\kappa}_2\log\left(\frac{\Lambda^2}{m^2}\right)] + \mathcal{O}(\Lambda^{-2}) \tag{5.17}$$

$$\mathcal{K}_4 = \mathcal{K}_{4cl} + \frac{1}{32\pi^2}\,\mathrm{str}[\bar{\kappa}_4\log\left(\frac{\Lambda^2}{m^2}\right)] + \mathcal{O}(\Lambda^{-2}) \tag{5.18}$$

Here we see that $\mathcal{K}_1$ acts as a cosmological constant, and $\mathcal{K}_2$ acts as the Newtons gravitational constant. However, constants $\mathcal{K}_3$ and $\mathcal{K}_4$ brings probes a new gravitational physics sector.

The original idea of the Sakharov was to make one loop dominate the classical contributions and observing the dominant terms as

$$\mathcal{K}_1 \approx -\frac{1}{64\pi^2}\,\mathrm{str}[I]\Lambda^4\ ,\ str[I] \approx= 0 \rightarrow \mathcal{K} \approx 0 \tag{5.19}$$

$$\lim_{\Lambda\rightarrow M_{pl}} \mathcal{K}_2 \approx \lim_{\Lambda\rightarrow M_{pl}} -\frac{1}{32\pi^2}\,\mathrm{str}[\kappa_1]\Lambda^2\ ,\ \mathrm{str}[\kappa_1] = -1 \rightarrow \mathcal{K}_2 \approx \frac{M_{pl}^2}{32\pi^2} \tag{5.20}$$

which yields an approximate Gravitational constant and a very small cosmological constant. Thus if we ignore the contributions from $\mathcal{K}_3$ and $\mathcal{K}_4$ we would get the standard EH action.

5.2 Symmergent Gravity

Symmergent Gravity is a novel and another type of emergent gravity framework that starts from the standard model action and emerges curvature as an affine one. The standard model action can be written as

$$S = S_{tree}(\eta,\mathcal{F}) + S_{log}(\eta,\mathcal{F},\log(\Lambda_w/\Lambda_u))$$
$$+S_o(\eta\Delta^2) + +S_H(\eta\Delta^2) + S_V(\eta,\Delta^2) \tag{5.21}$$

where η is the flat metric, Λ_u is the UV scale, $\Delta^2 = \Lambda_u^2 - \Lambda_w^2$ is the UV-Electroweak(EW) gap. $\mathcal{F}$ represents all field with energies lower than EW scale. S_{tree} is the tree level contrubitions S_{log} represents logarithmic corrections, S_o and S_H stands for vacuum and Higgs sector respectively and S_V represents gauge sector below the EW scale. We can unwrap the SM action as follows.

$$S_o = -\int d^4x\sqrt{\eta}[(2\bar{c_o}\Lambda_w^2 + \sum(\bar{c_F}m_F^2))\Delta^2 + \bar{c_o}\Delta^4] \tag{5.22}$$

$$S_H = -\int d^4x\sqrt{\eta}\bar{c_H}\Delta^2 H^\dagger H \tag{5.23}$$

$$S_V = \int d^4x\sqrt{\eta}\bar{c_V}\Delta^2\,\mathrm{Tr}[V_\mu V^\mu] \tag{5.24}$$

where H, and V_μ are the slow fields of the associated sector and parameters $\bar{c_i}$ are so-called Wilson coefficients or loop factors.

Unlike Sakharov's idea, SG uses Eddington's solution (Azri (2015); Demir (2014)) which can be represented as

$$\int d^4x\sqrt{-\det\left[\frac{\mathcal{R}}{\Delta^2}\right]} \tag{5.25}$$

yields the following equation of motion dynamiccally,

$$\mathcal{R}_{\mu\nu} = \Delta^2 g_{\mu\nu} \tag{5.26}$$

The UV-EW gap (Peskin & Schroeder (1995)) can be mapped to an affine connection (Azri (2015)) if our volume element is represented by (5.25). Thus the space-time elasticises dynamically. Now let us do the followings

3.1 map flat metric $\eta_{\mu\nu}$ to dynamic one $g_{\mu\nu}$

Hence on a manifold $\mathcal{M}(g,\Gamma)$ that is equipped with the metric $g_{\mu\nu}$ and an affine connection $\Gamma^{\lambda}_{\mu\nu}$ the standart model action can be written as

$$S[g,\Gamma,\phi,\upsilon] = \int d^4x\sqrt{-g}\{-Q^{\mu\nu}\mathcal{R}_{\mu\nu}(\Gamma) + \tfrac{1}{16}c_o\left(g^{\mu\nu}\mathcal{R}_{\mu\nu}(\Gamma)\right)^2 \\ -c_\upsilon R_{\mu\nu}({}^g\Gamma)\upsilon^\mu\upsilon^\nu\} + S_m[g,{}^g\Gamma,\phi,\upsilon] \tag{5.27}$$

where the disformal metric $Q^{\mu\nu}$ can be defined as follows

$$Q^{\mu\nu} = \left(\frac{\bar{M}^2}{2} + \frac{1}{8}c_O g^{\alpha\beta}\mathcal{R}_{\alpha\beta}(\Gamma) + \frac{1}{4}c_\phi\phi^2\right)g^{\mu\nu} - c_\upsilon\upsilon^\mu\upsilon^\nu \tag{5.28}$$

All the loop factors and the fields extended to a new physics sector beyond the standard model to properly induce gravity. $R_{\mu\nu}({}^g\Gamma)$ is metric curvature that depends on Levi-Civita connection ${}^g\Gamma$. Matter action $S_m[g,{}^g\Gamma,\phi,\upsilon]$ is independent of both curvature and affine connection . M can be considered as the apparent gravitational scale where it emerges from the combinations of slow fields and EW scale as

$$\frac{c_o}{2}\Lambda_w + \sum\frac{1}{4}c_f m_f^2 \rightarrow \frac{\bar{M}^2}{2} \tag{5.29}$$

here $\mathcal{F}$ also extends to designated NP sector as $F \rightarrow f$. The well known metrical theory of gravitation is dependent on the metrical connection or Levi Civita connection. This connection depends on the metric, and we can define a Ricci tensor that depends on the Levi-Civita connection to describe the curvature. Here in the disformal metric $Q^{\mu\nu}$ however we define an affine curvature that depends on a symmetric affine connection Γ that does not have to resemble a metric connection ${}^g\Gamma$, i.e. it does not have to depend on a metric. However, variation of action (5.27) with respect to affine connection yields the metricity condition that is the disformal metric is conserved with respect to the covariant derivative.

$${}^g\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right) \tag{5.30}$$

This allows us to connect two different connection, affine and metric, via the following relation.

$$\Gamma^{\lambda}_{\mu\nu} = {}^{g}\Gamma^{\lambda}_{\mu\nu} + \Delta^{\lambda}_{\mu\nu} \tag{5.31}$$

where $\Delta^{\lambda}_{\mu\nu}$ is a symmetric tensor field and can be shown as

$$\Delta^{\lambda}_{\mu\nu} = \frac{1}{2}(Q^{-1})^{\lambda\rho}(\nabla_{\mu}Q_{\nu\rho} + \nabla_{\nu}Q_{\mu\rho} - \nabla_{\rho}Q_{\mu\nu}) \tag{5.32}$$

Also it is important to note that covariant derivatives ∇ of $\Delta^{\lambda}_{\mu\nu}$ correspond to the Levi-Civita connection ${}^{g}\Gamma^{\lambda}_{\mu\nu}$. We can furthermore use this to relate the affine curvature to the metrical one with following

$$\mathcal{R}_{\mu\nu}(\Gamma) = R_{\mu\nu}({}^{g}\Gamma) + \nabla_{\lambda}\Delta^{\lambda}_{\mu\nu} - \nabla_{\nu}\Delta^{\lambda}_{\mu\lambda} + \Delta^{\gamma}_{\mu\nu}\Delta^{\lambda}_{\gamma\lambda} - \Delta^{\gamma}_{\mu\lambda}\Delta^{\lambda}_{\gamma\nu} \tag{5.33}$$

or explicitly, we can write

$$\begin{aligned}\mathcal{R}_{\mu\nu}(\Gamma) = R_{\mu\nu}({}^{g}\Gamma) &+ \frac{1}{2}(Q^{-1})^{\alpha\lambda}[\nabla_{\alpha}\nabla_{\mu}Q_{\lambda\nu} + \nabla_{\alpha}\nabla_{\nu}Q_{\lambda\mu} - \nabla_{\alpha}\nabla_{\lambda}Q_{\mu\nu} - \nabla_{\nu}\nabla_{\mu}Q_{\lambda\alpha}]\\ &+\frac{1}{4}(Q^{-1})^{\alpha\rho}(Q^{-1})^{\lambda\kappa}[\nabla_{\mu}Q_{\alpha\lambda}\nabla_{\nu}Q_{\rho\kappa} - 2\nabla_{\alpha}Q_{\rho\kappa}(\nabla_{\mu}Q_{\lambda\nu} + \nabla_{\nu}Q_{\lambda\mu} - \nabla_{\lambda}Q_{\mu\nu})\\ &\quad+\nabla_{\kappa}Q_{\rho\alpha}(\nabla_{\mu}Q_{\lambda\nu} + \nabla_{\nu}Q_{\lambda\mu} - \nabla_{\lambda}Q_{\mu\nu}) + 2\nabla_{\kappa}Q_{\nu\rho}(\nabla_{\alpha}Q_{\mu\lambda} - \nabla_{\lambda}Q_{\mu\alpha})]\end{aligned} \tag{5.34}$$

The affine curvature $\mathcal{R}$ inside the disformal metric $Q^{\mu\nu}$ is suppressed by $\bar{M}^2$ in (5.28), where one can write the inverse of it by expanding it in powers of $\bar{M}^2$ as

$$(Q^{-1})^{\mu\nu} = \frac{2}{\bar{M}^2}\left[\left(1 - \frac{c_o}{4\bar{M}^2}g^{\alpha\beta}\mathcal{R}_{\alpha\beta}(\Gamma) - \frac{c_{\phi}}{2\bar{M}^2}\phi^2\right)g^{\mu\nu} + \frac{2c_{\upsilon}}{\bar{M}^2}\upsilon^{\mu}\upsilon^{\nu}\right] \tag{5.35}$$

Thus we can say that double inverses in (5.34) that contain Planck masses in $\mathcal{O}(M^{-4})$ suppress the contributions from the single derivatives of the disformal metric. Thus up to first order, we can write

$$\begin{aligned}\mathcal{R}_{\mu\nu}(\Gamma) = \quad & R_{\mu\nu}({}^{g}\Gamma) + \tfrac{1}{\bar{M}^2}[\nabla^{\lambda}\nabla_{\mu}Q_{\lambda\nu} + \nabla^{\lambda}\nabla_{\nu}Q_{\lambda\mu} - \nabla^{\lambda}\nabla_{\lambda}Q_{\mu\nu}\\ &-\nabla_{\nu}\nabla_{\mu}Q^{\lambda}_{\lambda}] + \mathcal{O}(M^{-4})\end{aligned} \tag{5.36}$$

We can also take the trace of the affine curvature as

$$g^{\mu\nu}\mathcal{R}_{\mu\nu}(\Gamma) = R({}^{g}\Gamma) + \frac{1}{\bar{M}^2}\left[2\nabla^{\alpha}\nabla^{\beta}Q_{\alpha\beta} - 2\nabla^{\alpha}\nabla_{\alpha}Q^{\beta}_{\beta}\right] + O(M^{-4}) \tag{5.37}$$

We need to integrate the affine curvature out of the action to see the final contribution from it. To do that, we calculate the following contraction.

$$\begin{aligned} Q^{\mu\nu}\mathcal{R}_{\mu\nu}(\Gamma) = \; & \tfrac{\bar{M}^2}{2}g^{\mu\nu}R_{\mu\nu}({}^{g}\Gamma) + \tfrac{c_o}{8}\left(g^{\alpha\beta}\mathcal{R}_{\alpha\beta}(\Gamma)\right)g^{\mu\nu}R_{\mu\nu}({}^{g}\Gamma) \\ & + \tfrac{c_\phi}{4}\phi^2 g^{\mu\nu}R_{\mu\nu}({}^{g}\Gamma) - c_\upsilon \upsilon^{\mu}\upsilon^{\nu}R_{\mu\nu}({}^{g}\Gamma) + O(M^{-2}) \end{aligned} \tag{5.38}$$

If we integrate by parts the total derivatives and thus remove the boundary terms, up to first order, we can write our action as

$$\begin{aligned} S[g,\phi] = \; & \int d^4x\sqrt{-g}\left\{-(\tfrac{\bar{M}^2}{2} + \tfrac{c_\phi}{4}\phi^2)R({}^{g}\Gamma) - \tfrac{c_o}{8}R^2({}^{g}\Gamma)\right\} \\ & + S_m[g, {}^{g}\Gamma, \phi, \upsilon] \end{aligned} \tag{5.39}$$

We can see that in order to correct EH action to emerge, the apparent mass scale should be mapped as follows

$$\frac{\bar{M}^2}{2} + \frac{c_\phi}{4}\langle\phi^2\rangle \rightarrow \frac{M_{pl}^2}{2} \tag{5.40}$$

Which reduces the action to

$$S[g] = \int d^4x\sqrt{-g}\left\{-\tfrac{M_{pl}^2}{2}R({}^{g}\Gamma) - \tfrac{c_o}{8}R^2({}^{g}\Gamma)\right\} + S_m[g, {}^{g}\Gamma, \phi, \upsilon] \tag{5.41}$$

Which is a $R + R^2$ $f(R)$ action. Standard EH action can be obtained when the coefficient $c_o = 0$ where c_o is a loop induced quantity (Demir (2021)) that can be written as

$$c_o = \frac{1}{64\pi^2}(n_f - n_b) \tag{5.42}$$

where n_f and n_b represents the numbers of degrees of freedom inside the underlying QFT. In a semi super symmetric case where $n_b = n_f$ we get the standart EH action from the SG.

6. Implications

This chapter will look at the implications of inflation and the spherically symmetric solutions in both induced gravity and symmergent gravity. Both theories have quadratic curvature terms that can be bounded with inflationary and spherically symmetric solutions. We will use the solutions that we obtained in Chapter 4 and 5 and look at the implications of these results.

6.1 Sakharov's Induced Gravity

Action (5.14) gives us an $R+R^2$ model that can create an inflationary field, as we discussed in chapter 4. In order to look at the inflationary aspect of this action, we can use an FRW metric that nullifies the Weyl contribution. Linear and quadratic scalar curvatures are the only contributions that remain. We get the following action after crossing out the Weyl and Cosmological constant terms (where the cosmological constant is already tiny).

$$\int d^4x\sqrt{-g}\left(\frac{M_{pl}^2}{32\pi^2}R+\mathcal{K}_4R^2\right) \tag{6.1}$$

Let us cast out the Planck mass out and obtain the functional $f(R)$

$$f(R)_S=\left(R+\frac{32\pi^2\mathcal{K}_4}{M_{pl}^2}R^2\right) \tag{6.2}$$

The action of the Starobinsky inflation is in the same form that is

(6.3) $$f(R) = R + \frac{1}{6m^2}R^2$$

Where constant m in mass-scale drives the inflation, which we called the scalaron or inflaton. The coefficient of the quadratic scalar curvature in Sakharov's action is also in the inverse mass square scale. Thus Sakharov's action also can create inflation. Thus we can map the two quadratic coefficients to each other. This constrains the loop-coefficient as

(6.4) $$\frac{32\pi^2\mathcal{K}_4}{M_{pl}^2} \to \frac{1}{6m^2}$$

CMB temperature anisotropies constrain the scalar field mass m to be around 10^{13} GeV. So we can see that the loop-coefficient is constrained by

(6.5) $$\mathcal{K}_4 = 5 \times 10^9 \text{ numerically}$$

If we write down the loop coefficient explicitly, we get

(6.6) $$\mathcal{K}_4 = \mathcal{K}_{4cl} + \frac{1}{32\pi^2}\mathrm{str}[\bar{\kappa}_4 \log\left(\frac{\Lambda^2}{m^2}\right)] + \mathcal{O}(\Lambda^{-2})$$

We can see that the main contributions to this number should come from classical one since the classical contribution is expected to dominate the logarithmic correction. Furthermore, To look at the implications of the spherically symmetric solutions, we should calculate the lapse function $B(r)$ from the equations of motions. We find the $B(r)$ as

(6.7) $$B(r) = 1 - \frac{2G_N M}{r} + \frac{r^2(1-a)}{6144 G_N \mathcal{K}_4 \pi^3}$$

We set the integration constant to $2G_N M$ as we expect it to behave like a Schwarzschild solution in the low curvature limit.

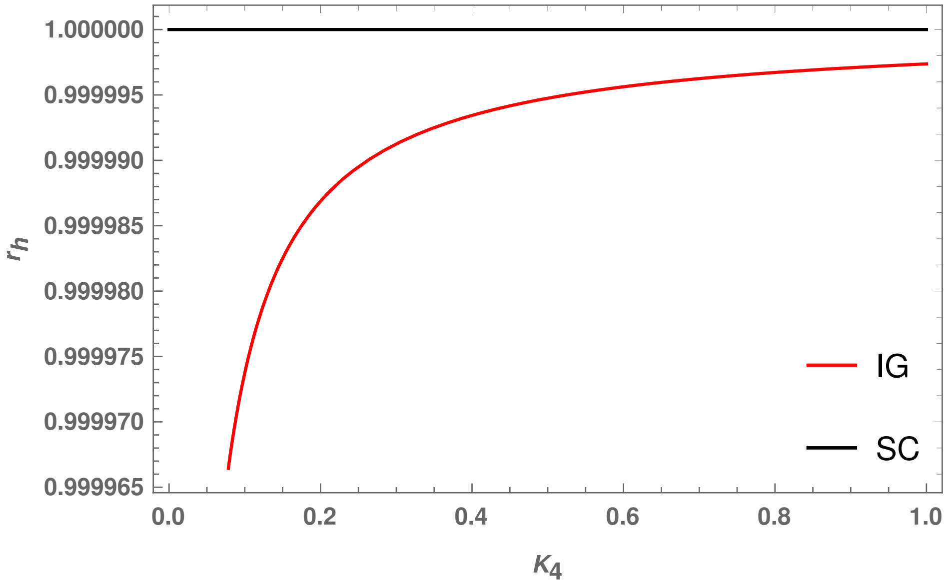

Figure 6.1 Dependence of the horizon r_h on the quadratic-curvature coefficient $\mathcal{K}_4$. The Induced Gravity solution (IG, the red curve) nears the Schwarzschild solution (SC, the black curve) for large values of $\mathcal{K}_4$, but deviates from it significantly for small values of $\mathcal{K}_4$

We can see that the lapse function $B(r)$ yields a not asymptotically flat metric. So we expect non-physical solutions away from the centre. Also, high values of $\mathcal{K}_4$ reduces the lapse function to Schwarzschild one. In Fig.6.1 as the value of $\mathcal{K}_4$ increases, it gradually and asymptotically approaches the value of the Schwarzschild horizon.

Also, we can see that the sign of the loop factor $\mathcal{K}_4$ significantly alters the horizon. Negative values give rise to a double horizon, whereas positive values give only one. In Fig.6.2 one can see that Sakharov's action have the horizon at Schwarzschild radii for small values of $\mathcal{K}_4$.

The singularities that are eminent on the metric are not needed to be real singularities. They can be either an absolute singularity or a coordinate singularity that the latter can vanish via a coordinate transformation. In order to see that whether the singularity is absolute or not, we should calculate the Kretschmann scalar for the action (6.1). The Kretschman scalar for Sakharov's action is found to be

$$K = R^{\alpha\beta\gamma\lambda}R_{\alpha\beta\gamma\lambda} = \frac{48G_N^2M^2}{r^6} + \frac{(-1+a)^2}{1572864\,\pi^2G_N^2\mathcal{K}_4^2} \tag{6.8}$$

As $\lim_{r\to\infty}$ we can see that only singularity is at the centre.

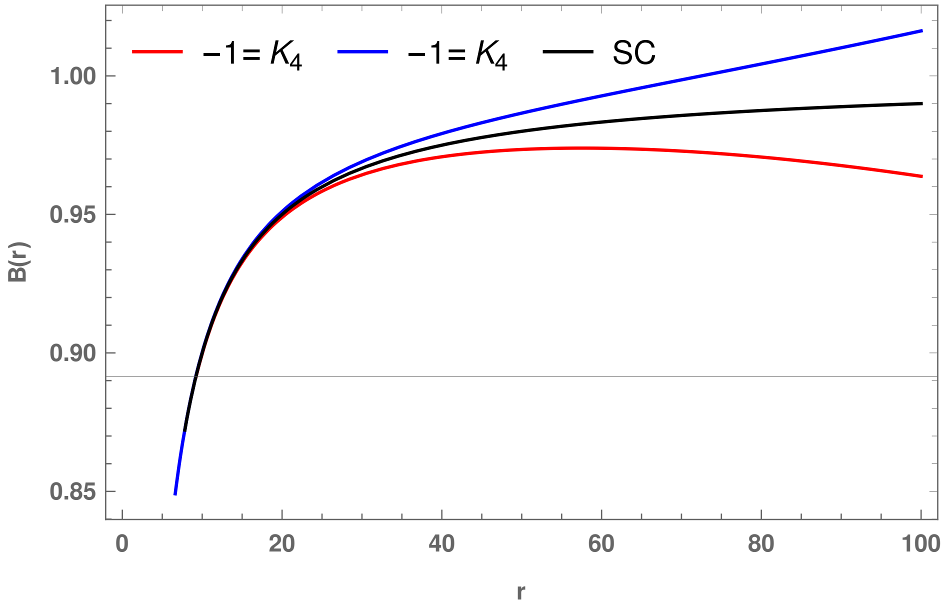

Figure 6.2 The lapse function $B(r)$ as a function of r for $M = a = 0.5$ (M in gravitational units) and for Schwarzschild solution (SC, the black curve) and for the induced gravity with $\mathcal{K}_4 = +1$ (blue curve) and $\mathcal{K}_4 = -1$ (red curve). The zeros of these curves give the event horizon of the corresponding BH.

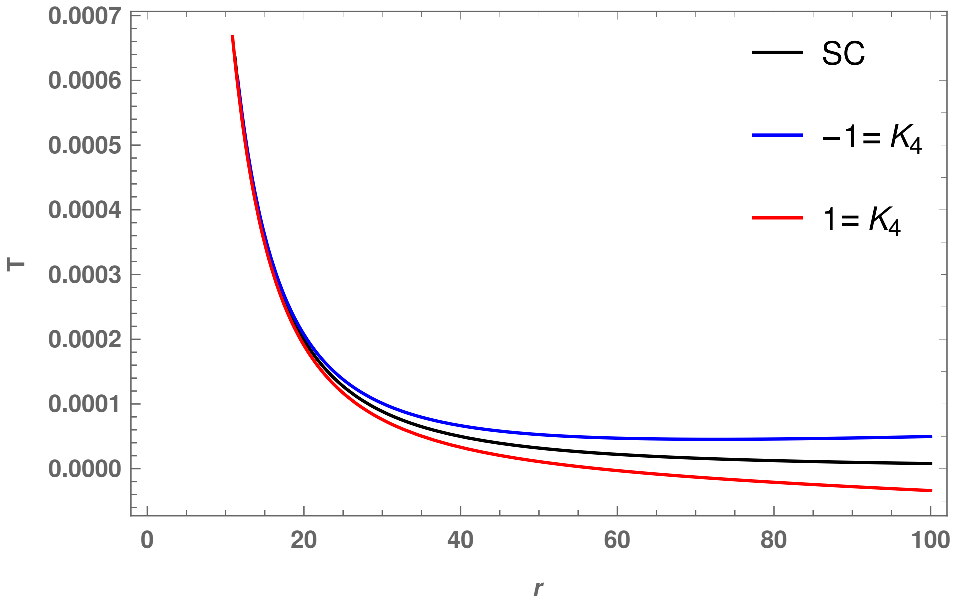

Figure 6.3 Hawking temperature T versus $r = r_h$ for the Schwarzschild (black curve), $\mathcal{K}_4 = 1$ (blue curve) and $\mathcal{K}_4 = -1$ (red curve) at $a = 0.5$ and $2G_N M = 1$.

Next, we can calculate the Hawking temperature of the Induced gravity BH via the following.

$$T = \frac{GM}{2\pi r^2} + \frac{r(1-a)}{1288 G\mathcal{K}_4 \pi^4} \tag{6.9}$$

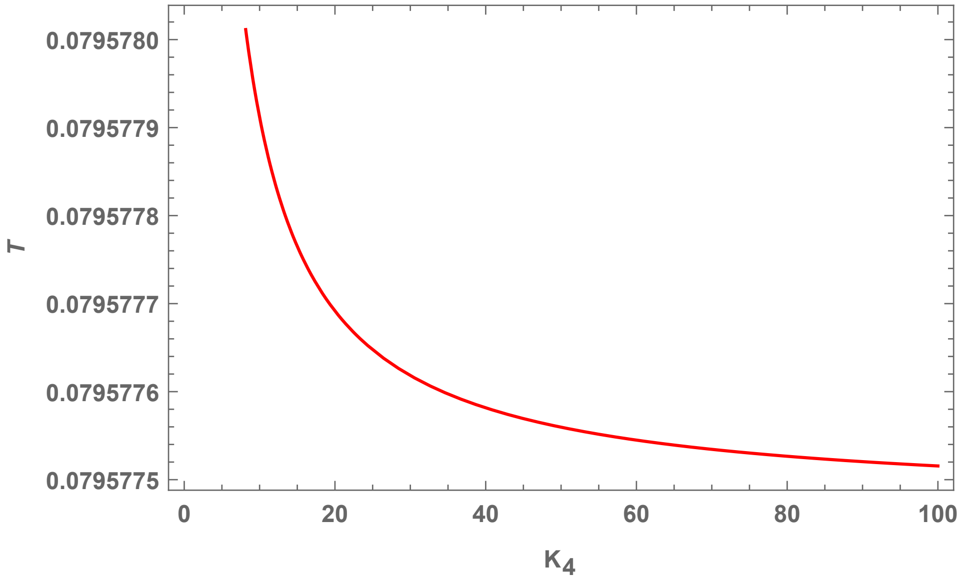

Figure 6.4 Hawking temperature T versus $\mathcal{K}_4$ for $a = 0.5$ and $2G_N M = 1$. Black line represents the Schwarzschild case.

We again see that the high values of the loop factor would yield temperature around Schwarzschild temperature. From Fig.6.3 We can see that hawking temperature increases with the horizon radius. Negative values of $\mathcal{K}_4$, however, reaches a negative temperature at high radius values. In Fig.6.4 we also can see that for large values of $\mathcal{K}_4$ IG reduces to the Schwarzschild temperature.

Entropy, on the other hand, depends on the horizon radius r_h. We can calculate the entropy for IG as

$$S = \frac{a\pi r_h^2}{G_N} \tag{6.10}$$

In Fig.6.5 one can see that as the horizon radius increases, IG have more surface entropy than a Schwarzschild BH. We can also calculate the shadow of the BH in induced gravity as

$$R = 96\sqrt{6}\pi^{3/2}\sqrt{-\frac{G_N^2\mathcal{K}_4 M^2}{9(a-1)G_N M^2 - 2048\mathcal{K}_4\pi^3}} \tag{6.11}$$

In Fig.6.6 as $\mathcal{K}_4$ increases shadow approaches to the Schwarzschild value asymptotically. Around an inflationary mass value, the IG still differs from SC by a small margin. This could be a testbed for both the IG and inflation.

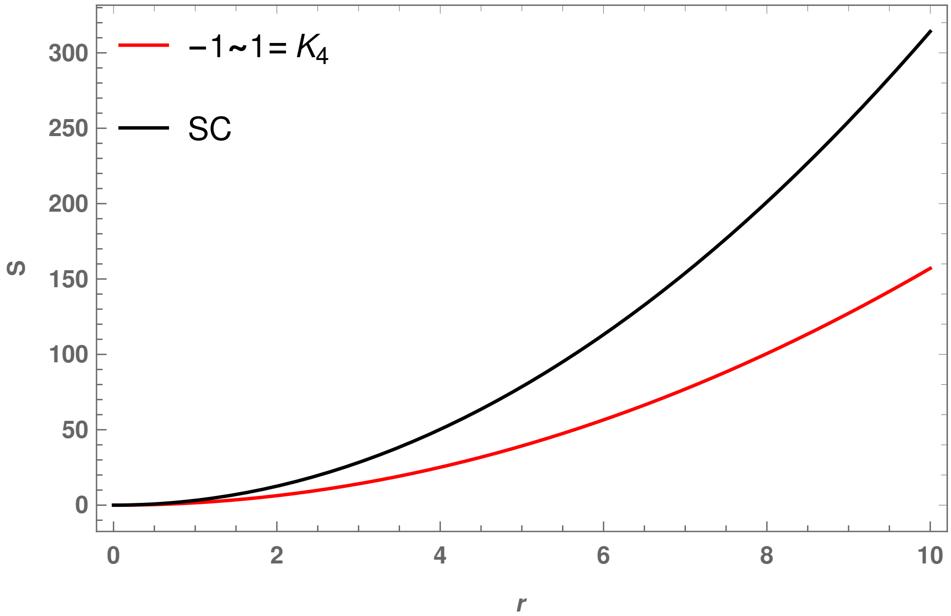

Figure 6.5 Bekenstein-Hawking entropy S versus $r = r_h$ for the Schwarzschild (black curve) and induced gravity (red curve) BH solutions for $\mathcal{K}_4 = 1$ and $2G_N M = 1$.

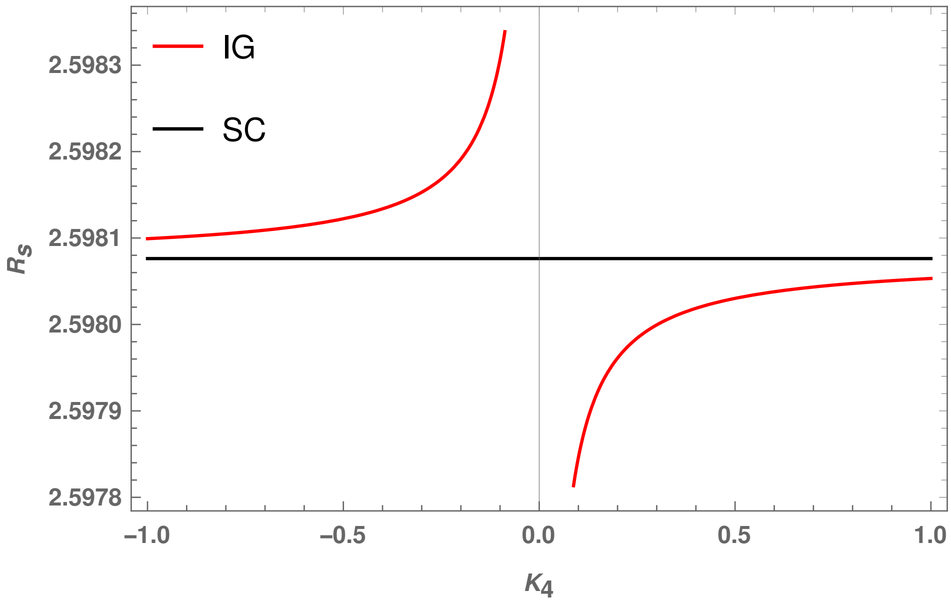

Figure 6.6 Shadow radius versus $\mathcal{K}_4$ at $a = 0.5$ and $2G_N M = 1$, with black line standing for the Schwarzschild solution.

.

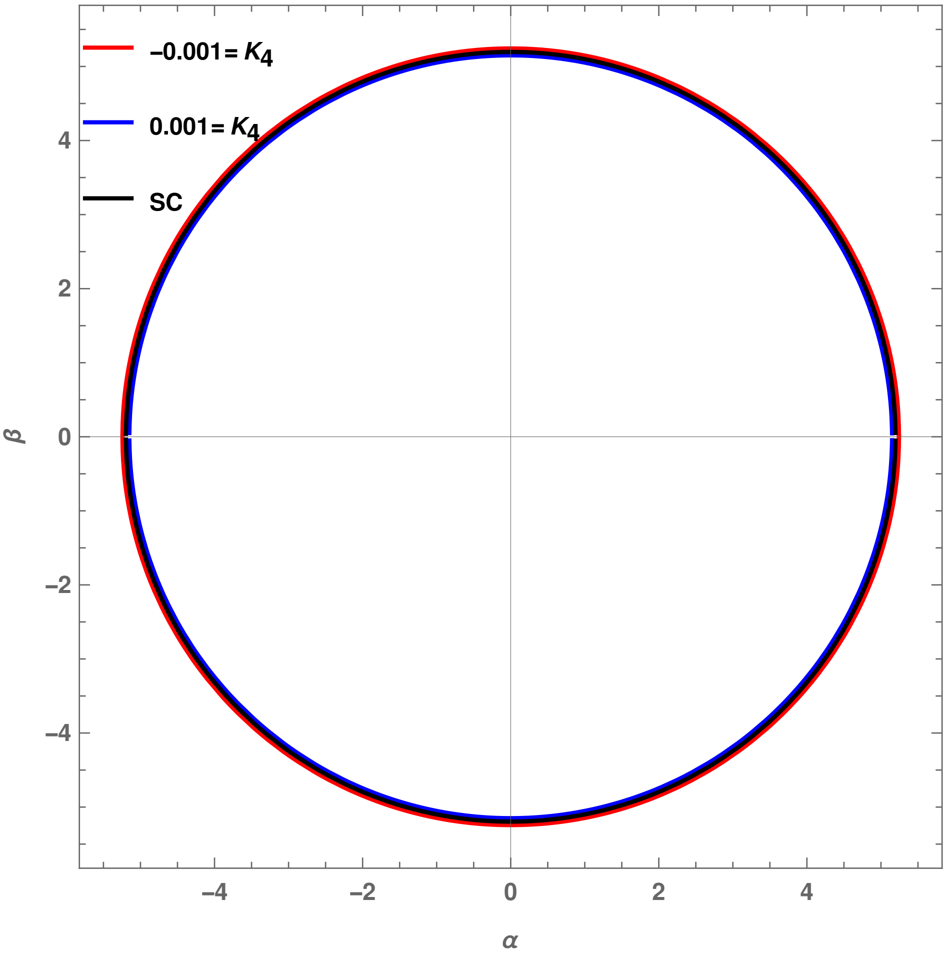

Figure 6.7 Shadow radius for Schwarzschild (black), $\mathcal{K}_4 = 1$ (blue), and $\mathcal{K}_4 = -1$ (red) BH solutions at $a = 0.5$ and $2G_N M = 1$.

6.2 Symmergent Gravity

SG also has a quadratic scalar curvature. We can write the action as

$$S = \int d^4x\sqrt{-g}\left\{\frac{M_{pl}^2}{2}R(\Gamma) - \frac{c_o}{8}R^2(\Gamma)\right\} \tag{6.12}$$

where we excluded the matter action from it. Furthermore we can identify the functional $f(R)$ of this action by casting out the placnk mass as follows

$$f(R) = \left(R - \frac{c_o}{4M_{pl}^2}R^2\right) \tag{6.13}$$

again we can map the quadratic coefficient of the SG to Starobinsky's model as

$$-\frac{c_o}{4M_{pl}^2} \to \frac{1}{6m^2} \tag{6.14}$$

c_o is directly bound to the underlying QFT via the following relation

$$c_o = \frac{1}{64\pi^2}(n_f - n_b) \tag{6.15}$$

Thus scalar mass m can constraint this number and degrees of freedom of the underlying QFT. We can calculate the required difference between the degrees of freedom of bosons of fermions via this relation. Thus the relation yields $n_b - n_f \approx 3 \times 10^{13}$. Already at one loop, we expect many new fields in the non identified NP sector. These new fields can drive the inflation if the new scalar fields do not behave like Higgs; however, if they behave like Higgs, we should not expect an inflationary scenario from these new fields. These new fields can also represent Dark matter since they do not need to interact with the SM. Next, we can calculate the lapse function for SG as

$$B(r) = 1 + \frac{C}{r} + \frac{r^2(a-1)}{24\pi G_N c_o} \tag{6.16}$$

with the integration constant C. again we can assume that $C = -2G_N M$ where M is the total mass within the spherically symmetric mass distribution around the origin. In general, the radius $r = r_h$ at which $B(r_h) = 0$ gives the event horizon:

$$r_h = \frac{h}{(18)^{1/3}} - \frac{(18)^{1/3} c_1}{3h} \tag{6.17}$$

where

$$h = ((12c_1^3 + 81c_2^2)^{1/2} - 9c_2)^{1/3} \tag{6.18}$$

with $c_1 = 1/A$, $c_2 = C/A$, $A = (a-1)/(24\pi G_N c_o)$.

Depicted in Fig. 6.8 we can see that negative values of c_o give the physical results.

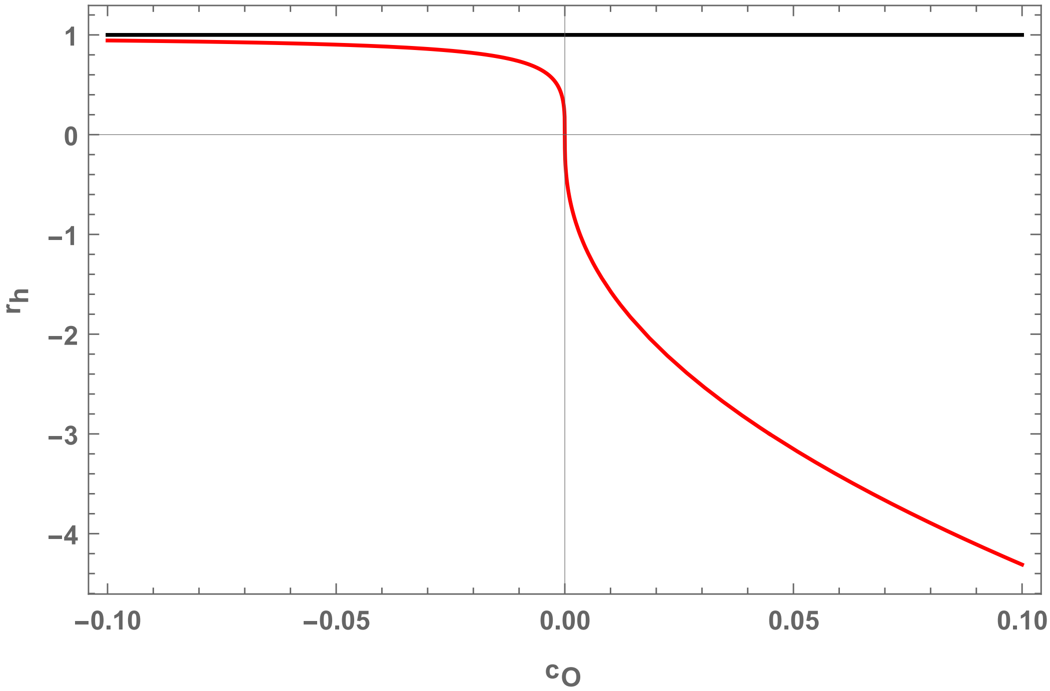

Figure 6.8 Dependence of the horizon r_h on the quadratic-curvature coefficient c_o. The symmergent gravity solution (SG, the red curve) nears the Schwarzschild solution (SC, the black curve) for negative c_o (when the underlying QFT has more bosons than fermions) but deviates from it significantly for positive c_o (when the underlying QFT has more fermions than bosons).

The SG solution approaches the Schwarzschild value as c_o decreases in negative value (when the underlying QFT has more fermions than bosons) but deviates significantly from it for positive values (when the underlying QFT has more bosons than fermions). Also shown in Fig.6.9, sign of c_o directly affects the lapse function $B(r)$. If the underlying QFT has more (less) bosons than fermions, then $B(r)$ increases (decreases) with r.

In Fig.6.9, each $B(r)$ intersects the horizontal axis at the radius giving its horizon. Here, positive value $c_o = 0.2$ (blue curve) generates two horizons: the inner horizon ($r = r_-$) and outer horizon ($r = r_+$). Negative value $c_o = -0.2$ (red curve) leads to one horizon. Black curve represents the Schwarzschild horizon for $r_h = 1$. Furthermore the Kretschman scalar for SG reads

$$(6.19)\quad R_{\alpha\beta\delta\gamma}R^{\alpha\beta\delta\gamma} = \frac{B''^2r^4 + 4B'^2r^2 + 4\,(B-1)^2}{r^4} = \frac{48G_N^2M^2}{r^6} + \frac{(a-1)^2}{24\pi G_N^2 c_o^2}$$

Again we see that the only absolute singularity at the centre. The Hawking temperature T of a BH at the horizon yields

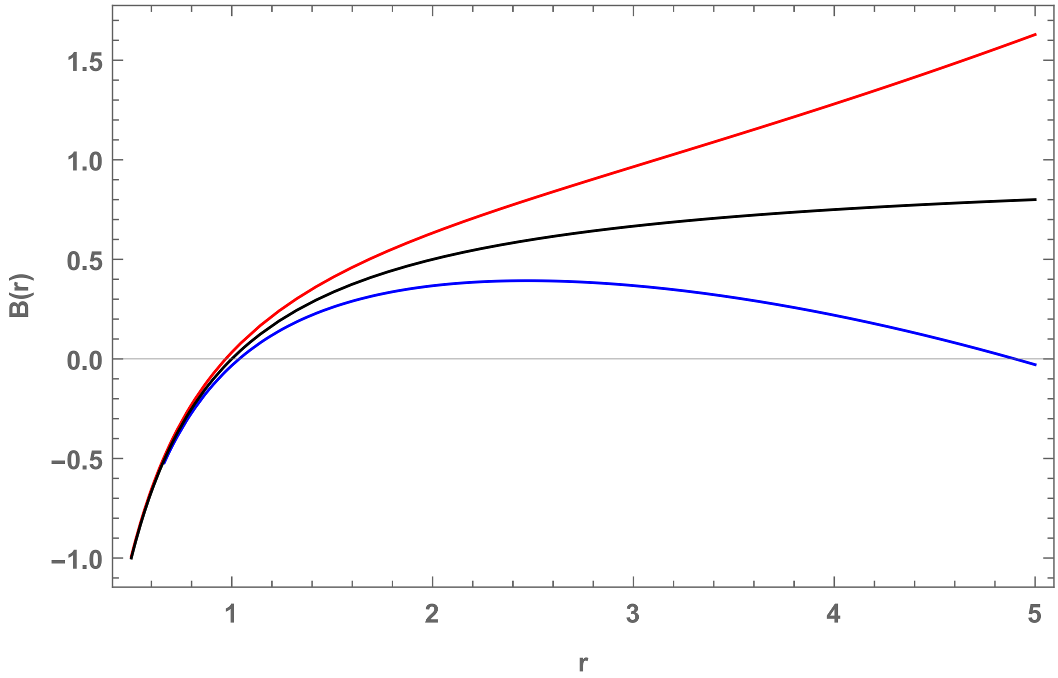

Figure 6.9 The lapse function $B(r)$ as a function of r for $M = a = 0.5$ (M in gravitational units) and for Schwarzschild solution (SC, the black curve) and for the SG with $c_o = +0.2$ (blue curve) and $c_o = -0.2$ (red curve). The zeros of these curves give the event horizon of the corresponding BH.

$$T = \frac{k_h}{2\pi} = \frac{G_N M}{8r_h^2} + \frac{(1-a)r_h}{192\pi c_o G_N} \tag{6.20}$$

whose variation with r_h is plotted in Fig. 6.10 and variation with c_o is plotted in Fig. 6.11. As seen from the plot, Hawking temperature increases (decreases) with r_+ for $c_o = -1$ ($c_o = 1$). Positive c_o values yields negative temprature values which is unphysical. As c_o reduces however it aproaches to schwarzchild temprature.

The Bekenstein-Hawking entropy is given by

$$S(r_h) = \frac{1}{4G_N}\mathcal{A}_h(r_h)F(r_h) \tag{6.21}$$

where $\mathcal{A}_h = 4\pi r_h^2$ is the horizon area, and $F(r_h) = a$ as found in (4.47) so that the entropy is

$$S = \frac{a\pi r_h^2}{G_N} \tag{6.22}$$

which depends on c_o through horizon radius. Its variation with r_h is depicted in Fig. 6.12, where it is seen that growth of the entropy with r_h is controlled by the

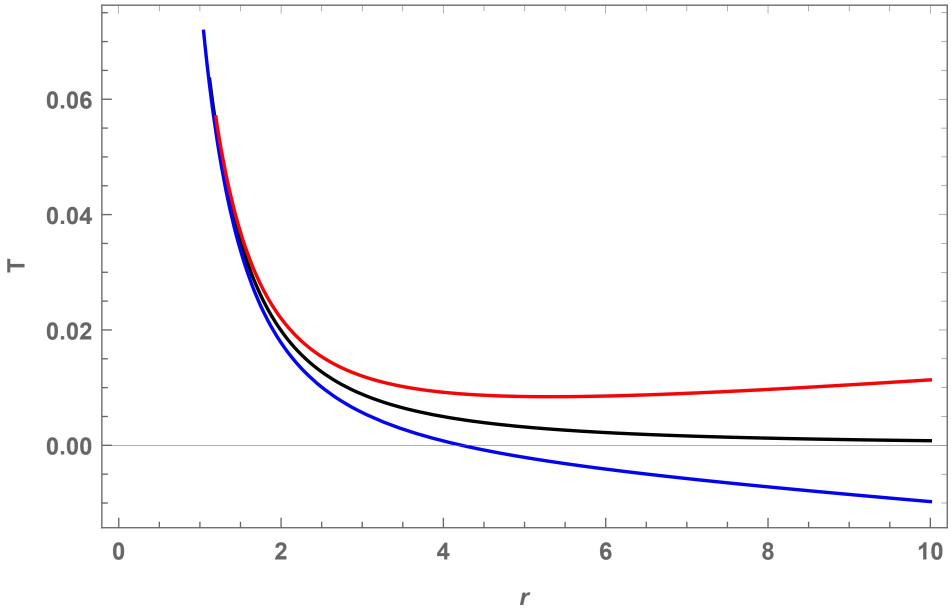

Figure 6.10 Hawking temperature T versus $r = r_h$ for the Schwarzschild (black curve), $c_o = 1$ (blue curve) and $c_o = -1$ (red curve) at $a = 0.5$ and $2G_NM = 1$.

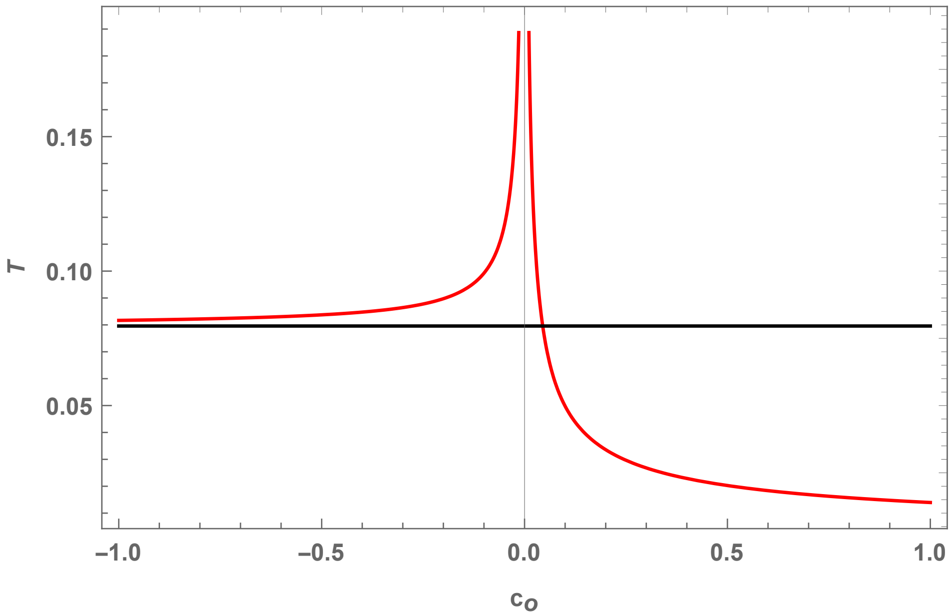

Figure 6.11 Hawking temperature T versus c_o for $a = 0.5$ and $2G_NM = 1$. Black line represents the Schwarzschild case.

parameter a.

we can also calculate the shadow of a BH on symmergent framework as

$$R = 6\sqrt{6\pi}\sqrt{\frac{c_oG_N^2M^2}{9(a-1)G_N^2M^2+8\pi c_o}} \tag{6.23}$$

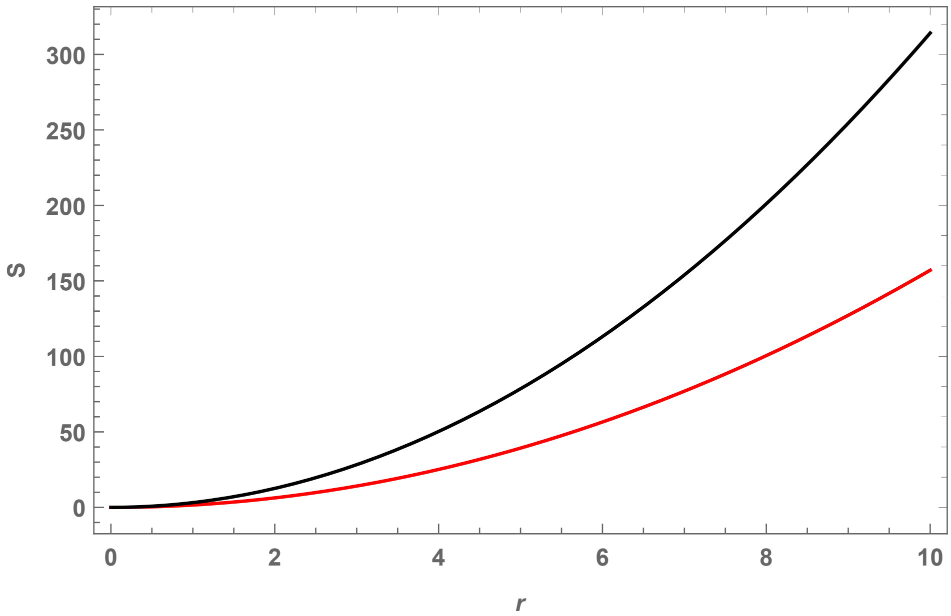

Figure 6.12 Bekenstein-Hawking entropy S versus $r = r_h$ for the Schwarzschild (black curve) and SG (red curve) BH solutions for $c_o = 1$ and $2G_N M = 1$.

In Fig. 6.14 we can see the shadow cast for a BH in SG for various values of c_o. Positive values of loop contributions cast larger shadows, whereas negative values decrease them. This can be a testbed for the SG framework.

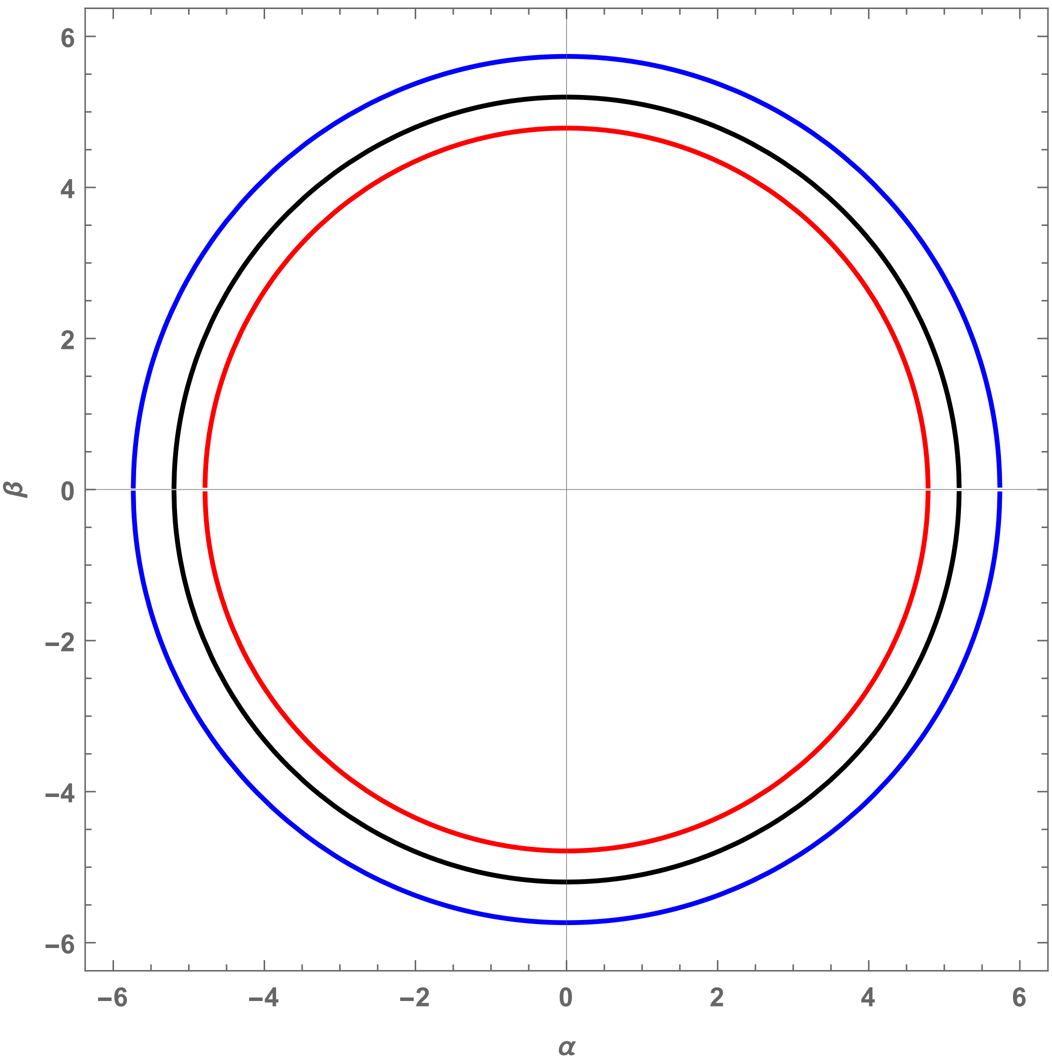

Figure 6.13 Shadow radius for Schwarzschild (black), $c_o = 1$ (blue), and $c_o = -1$ (red) BH solutions at $a = 0.5$ and $2G_NM = 1$.

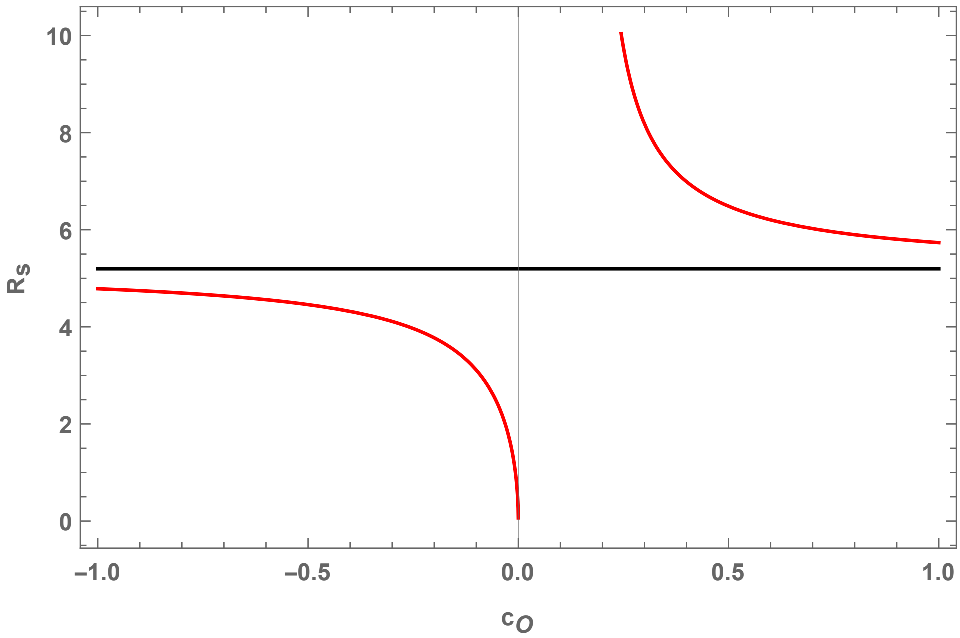

Figure 6.14 Shadow radius versus c_o at $a = 0.5$ and $2G_N M = 1$, with black line standing for the Schwarzschild solution.

7. Conclusion

In this modest work, we tried to describe the implications of inflation and the black hole solutions on the emergent gravity theories. Emergent gravity theories explain gravity not as a fundamental force but rather as an emergent phenomenon. They naturally contain quadratic scalar curvature terms besides linear ones. These kinds of actions are called $f(R)$ action for $f(R)$ theories, and these theories can be used to describe inflationary dynamics of the universe and many more. Generally, quadratic curvature describes the inflationary nature, whereas linear curvature ends the inflationary phase. The well defined Starobinsky inflationary model is also a quadratic curvature model with the same form as the emergent gravity theories. We described that to induce the inflation from Sakharov's induced gravity where gravity is generated from the effective action; the quadratic coefficient should be around 5×10^9 that needs to be induced from the classical contributions. On the other hand, a novel framework called symmergent gravity also induces gravity from loop contributions while restoring broken symmetries holds the same form. The quadratic coefficient of the symmergent gravity directly describes the underlying QFT's degrees of freedoms of bosons and fermions. To induce inflation, the underlying QFT should have more bosonic degrees of freedoms than fermions. That is, the difference between them needs to be around 3×10^{13}. This vast number of new fields does not need to interact with the standard model. Thus they can also represent dark matter.

Furthermore, we looked at the spherically symmetric solutions of these theories. Quadratic curvature actions yield non asymptotically flat lapse function, which can be described as a Schwarzschild part plus a suppressed correction terms to the physical terms such as the horizon, temperature and entropy.

The loop contribution in Sakharov's induced gravity $\mathcal{K}_4$ resides in the denominators of the contribution. As the loop contribution increases, results tend to behave like Schwarzschild solutions physical effects. The theory has higher temperature and lower entropy values. As contribution increases, the shadow of the black hole also increases significantly.

Theory	Function $f(R)$	$B(R)$ correction	T Correction
IG	$R+\frac{32\pi^2\mathcal{K}_4}{M_{pl}^2}R^2$	$\frac{r^2(1-a)}{6144G_N\mathcal{K}_4\pi^3}$	$\frac{r(1-a)}{1288G_N\mathcal{K}_4\pi^4}$
SG	$R-\frac{c_o}{M_{pl}^2 4}R^2$	$\frac{r^2(a-1)}{24\pi G_N c_O}$	$\frac{(1-a)r_h}{192\pi c_O G_N}$

Table 7.1 Lapse function and temprature corrections of the Sakharov's induced gravity and symmergent gravity to the schwarzchild solution.

Symmergent gravity's loop contribution also behave like Sakharov's. To avoid ghosts and unphysical results such as negative temperatures, c_o need to be negative. As it increases, the theory tends to behave as Schwarzschild. Such a black hole would have a higher temperature and lower horizon entropies. Positive values of the loop contribution give larger shadow cast values, whereas negative values have more minor. This can be a significant test bed for this theory as the current event horizon telescope would enlighten us about whether there exists a difference between the theoretical shadow and the observed one.

BIBLIOGRAPHY

Abbott, T. M. C. et al. (2018). Dark Energy Survey Year 1 Results: A Precise H0 Estimate from DES Y1, BAO, and D/H Data. *Mon. Not. Roy. Astron. Soc.*, *480*(3), 3879–3888.

Addazi, A., Capozziello, S., & Odintsov, S. (2021). Chaotic solutions and black hole shadow in $f(R)$ gravity. *Phys. Lett. B*, *816*, 136257.

Akiyama, K. et al. (2019). First M87 Event Horizon Telescope Results. I. The Shadow of the Supermassive Black Hole. *Astrophys. J. Lett.*, *875*, L1.

Akrami, Y. et al. (2020). Planck 2018 results. X. Constraints on inflation. *Astron. Astrophys.*, *641*, A10.

Azri, H. (2015). Eddington's gravity in immersed spacetime. *Class. Quant. Grav.*, *32*(6), 065009.

Barrow, J. D. & Cotsakis, S. (1988). Inflation and the Conformal Structure of Higher Order Gravity Theories. *Phys. Lett. B*, *214*, 515–518.

Bastero-Gil, M., Borunda, M., & Janssen, B. (2009). The Palatini formalism for higher-curvature gravity theories. *AIP Conf. Proc.*, *1122*(1), 189–192.

Bekenstein, J. D. (1973). Black holes and entropy. *Phys. Rev. D*, *7*, 2333–2346.

Brans, C. & Dicke, R. H. (1961). Mach's principle and a relativistic theory of gravitation. *Phys. Rev.*, *124*, 925–935.

Capozziello, S. & De Laurentis, M. (2011). Extended Theories of Gravity. *Phys. Rept.*, *509*, 167–321.

Carlip, S. (2001). Quantum gravity: A Progress report. *Rept. Prog. Phys.*, *64*, 885.

Carroll, S. M. (2019). *Spacetime and Geometry.* Cambridge University Press.

Çimdiker, I. I. (2020). Starobinsky inflation in emergent gravity. *Phys. Dark Univ.*, *30*, 100736.

Chakraborty, S. (2017). Boundary Terms of the Einstein–Hilbert Action. *Fundam. Theor. Phys.*, *187*, 43–59.

De Felice, A. & Tsujikawa, S. (2010). f(R) theories. *Living Rev. Rel.*, *13*, 3.

Demir, D. (2017). Naturalizing Gravity of the Quantum Fields, and the Hierarchy Problem.

Demir, D. (2019). Symmergent Gravity, Seesawic New Physics, and their Experimental Signatures. *Adv. High Energy Phys.*, *2019*, 4652048.

Demir, D. (2021). Emergent Gravity as the Eraser of Anomalous Gauge Boson Masses, and QFT-GR Concord. *Gen. Rel. Grav.*, *53*(2), 22.

Demir, D. A. (2014). Riemann-Eddington theory: Incorporating matter, degravitating the cosmological constant. *Phys. Rev. D*, *90*(6), 064017.

Demir, D. A. (2015). A Mechanism of Ultraviolet Naturalness.

Demir, D. A. (2016). Curvature-Restored Gauge Invariance and Ultraviolet Naturalness. *Adv. High Energy Phys.*, *2016*, 6727805.

Dyson, F. (2013). Is a graviton detectable? *Int. J. Mod. Phys. A*, *28*, 1330041.

Faraoni, V., Gunzig, E., & Nardone, P. (1999). Conformal transformations in classical gravitational theories and in cosmology. *Fund. Cosmic Phys.*, *20*, 121.

Flanagan, E. E. (2003). Higher order gravity theories and scalar tensor theories. *Class. Quant. Grav.*, *21*, 417–426.

Guth, A. H. (1981). The Inflationary Universe: A Possible Solution to the Horizon

and Flatness Problems. *Phys. Rev. D*, *23*, 347–356.
Hwang, J.-c. & Noh, H. (2001). f(R) gravity theory and CMBR constraints. *Phys. Lett. B*, *506*, 13–19.
Kalara, S., Kaloper, N., & Olive, K. A. (1990). Theories of Inflation and Conformal Transformations. *Nucl. Phys. B*, *341*, 252–272.
Landau, L. D. & Lifschits, E. M. (1975). *The Classical Theory of Fields*, volume Volume 2 of *Course of Theoretical Physics*. Oxford: Pergamon Press.
Linde, A. D. (1982). A New Inflationary Universe Scenario: A Possible Solution of the Horizon, Flatness, Homogeneity, Isotropy and Primordial Monopole Problems. *Phys. Lett. B*, *108*, 389–393.
Lu, J., Wu, Y., Yang, W., Liu, M., & Zhao, X. (2018). The solutions and the effects of Brans-Dicke field in the GBD theory.
Luminet, J. P. (1979). Image of a spherical black hole with thin accretion disk. *Astron. Astrophys.*, *75*, 228–235.
Macias, A. & Camacho, A. (2008). On the incompatibility between quantum theory and general relativity. *Phys. Lett. B*, *663*, 99–102.
Maeda, K.-i. (1989). Towards the Einstein-Hilbert Action via Conformal Transformation. *Phys. Rev. D*, *39*, 3159.
Magnano, G. & Sokolowski, L. M. (1994). On physical equivalence between nonlinear gravity theories and a general relativistic selfgravitating scalar field. *Phys. Rev. D*, *50*, 5039–5059.
Mishra, A. K., Chakraborty, S., & Sarkar, S. (2019). Understanding photon sphere and black hole shadow in dynamically evolving spacetimes. *Phys. Rev. D*, *99*(10), 104080.
Mukhanov, V. F. & Chibisov, G. V. (1981). Quantum Fluctuations and a Nonsingular Universe. *JETP Lett.*, *33*, 532–535.
Odintsov, S. D. & Oikonomou, V. K. (2015). Bouncing cosmology with future singularity from modified gravity. *Phys. Rev. D*, *92*(2), 024016.
O'Hanlon, J. (1972). Intermediate-range gravity - a generally covariant model. *Phys. Rev. Lett.*, *29*, 137–138.
Peralta, C. D. & Jorás, S. E. (2020). Thermodynamics of $f(R)$ Theories of Gravity. *JCAP*, *06*, 053.
Perlick, V. & Tsupko, O. Y. (2021). Calculating black hole shadows: review of analytical studies.
Peskin, M. E. & Schroeder, D. V. (1995). *An Introduction to quantum field theory*. Reading, USA: Addison-Wesley.
Postma, M. & Volponi, M. (2014). Equivalence of the Einstein and Jordan frames. *Phys. Rev. D*, *90*(10), 103516.
Sakharov, A. D. (1967). Vacuum quantum fluctuations in curved space and the theory of gravitation. *Dokl. Akad. Nauk Ser. Fiz.*, *177*, 70–71.
Schulz, B. (2014). Review on the quantization of gravity.
Schutz, B. F. (1985). *A FIRST COURSE IN GENERAL RELATIVITY*. Cambridge, UK: Cambridge Univ. Pr.
Schwarz, J. H. (2007). String theory: Progress and problems. *Prog. Theor. Phys. Suppl.*, *170*, 214–226.
Shimada, K., Aoki, K., & Maeda, K.-i. (2019). Metric-affine gravity and inflation. *Phys. Rev. D*, *99*, 104020.
Sotiriou, T. P. & Faraoni, V. (2010). f(R) Theories Of Gravity. *Rev. Mod. Phys.*,

82, 451–497.

Starobinsky, A. (1980a). A new type of isotropic cosmological models without singularity. *Physics Letters B*, *91*(1), 99–102.

Starobinsky, A. A. (1979). Spectrum of relict gravitational radiation and the early state of the universe. *JETP Lett.*, *30*, 682–685.

Starobinsky, A. A. (1980b). A New Type of Isotropic Cosmological Models Without Singularity. *Phys. Lett. B*, *91*, 99–102.

Starobinsky, A. A. (1981). NONSINGULAR MODEL OF THE UNIVERSE WITH THE QUANTUM GRAVITATIONAL DE SITTER STAGE AND ITS OBSERVATIONAL CONSEQUENCES. In *Second Seminar on Quantum Gravity*.

Synge, J. L. (1966). The Escape of Photons from Gravitationally Intense Stars. *Mon. Not. Roy. Astron. Soc.*, *131*(3), 463–466.

’t Hooft, G. & Veltman, M. J. G. (1974). One loop divergencies in the theory of gravitation. *Ann. Inst. H. Poincare Phys. Theor. A*, *20*, 69–94.

Vázquez, J. A., Padilla, L. E., & Matos, T. (2018). Inflationary Cosmology: From Theory to Observations.

Verlinde, E. P. (2017). Emergent Gravity and the Dark Universe. *SciPost Phys.*, *2*(3), 016.

Vilenkin, A. (1985). Classical and Quantum Cosmology of the Starobinsky Inflationary Model. *Phys. Rev. D*, *32*, 2511.

Visser, M. (2002). Sakharov’s induced gravity: A Modern perspective. *Mod. Phys. Lett. A*, *17*, 977–992.

Vitagliano, V., Sotiriou, T. P., & Liberati, S. (2011). The dynamics of metric-affine gravity. *Annals Phys.*, *326*, 1259–1273. [Erratum: Annals Phys. 329, 186–187 (2013)].

Wald, R. M. (1984). *General Relativity*. Chicago, USA: Chicago Univ. Pr.

Will, C. M. (2014). The Confrontation between General Relativity and Experiment. *Living Rev. Rel.*, *17*, 4.