CHARACTERIZATION OF POTENTIAL SMOOTHNESS AND RIESZ BASIS PROPERTY OF HILL-SCHRÖDINGER OPERATORS WITH SINGULAR PERIODIC POTENTIALS IN TERMS OF PERIODIC, ANTIPERIODIC AND NEUMANN SPECTRA

by

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Abstract

The Hill-Schrödinger operators, considered with singular complex valued periodic potentials, and subject to the periodic, anti-periodic or Neumann boundary conditions, have discrete spectra. For sufficiently large integer n, the disk with radius n and with center square of n, contains two periodic (if n is even) or anti-periodic (if n is odd) eigenvalues and one Neumann eigenvalue. We construct two spectral deviations by taking the difference of two periodic (or anti-periodic) eigenvalues and the difference of a periodic (or anti-periodic) eigenvalue and the Neumann eigenvalue. We show that asymptotic decay rates of these spectral deviations determine the smoothness of the potential of the operator, and there is a basis consisting of periodic (or anti-periodic) root functions if and only if the supremum of the absolute value of the ratio of these deviations over even (respectively, odd) n is finite. We also show that, if the potential is locally square integrable, then in the above results one can replace the Neumann eigenvalues with the eigenvalues coming from a special class of boundary conditions more general than the Neumann boundary conditions.

TEKİL VE PERİYODİK POTANSİYELE SAHİP HİLL-SCHRÖDİNGER OPERATÖRLERİNDE POTANSİYELİN TÜREVLENEBİLİRLİĞİNİN VE DE RİESZ BAZI ÖZELLİĞİNİN PERİYODİK, ANTİPERİYODİK VE NEUMANN SPEKTURUMU CİNSİNDEN KARAKTERİZASYONU

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Özet

Tekil ve periyodik potansiyele sahip Hill-Schrödinger operatörlerinin, periyodik, antiperiyodik ya da Neumann sınır koşulları altında ayrık spektrumları vardır. Yeterince büyük tamsayı n'ler için n yarıçaplı ve n kare merkezli diskler içinde eğer n çiftse periyodik, eğer n tekse antiperiyodik sınır koşullarından gelen iki özdeğer ve bir tane de Neumann sınır koşulundan gelen özdeğer bulunur. Bu iki periyodik (ya da antiperiyodik) özdeğerin farkını ve de bir periyodik özdeğerle (ya da antiperiyodik) Neumann özdeğerinin farkını alarak iki tane spektral sapma oluşturulmuş ve de potansiyelin "türevlenebilme" derecesinin bu spektral sapmaların asimtotik azalma hızlarıyla karakterize edilebileceği gösterilmiştir. Ayrıca periyodik (ya da antiperiyodik) kök fonksiyonlarının bir Riesz bazı oluşturmasının ancak ve ancak bu sapmaların oranlarının mutlak değerinin çift (ya da tek) n'ler üzerinden alınan supremumunun sonlu olmasıyla mümkün olduğu gösterilmiştir. Potansiyelin karesinin lokal integrallenebildiği durumlarda ise yukarıda ifade edilen sonuçlarda Neumann özdeğerlerinin, daha genel bir sınır koşulu sınıfından gelen özdeğerlerle değiştirilebileceği gösterilmiştir.

To my friends,

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CHAPTER 1

Introduction

We consider the Hill operator

$$Ly = -y'' + v(x)y, \quad x \in [0, \pi], \tag{1.1}$$

with the following boundary conditions (bc):

$$\begin{aligned}
&\text{Periodic } (bc = Per^+): && y(0) = y(\pi), \quad y'(0) = y'(\pi); \\
&\text{Antiperiodic } (bc = Per^-): && y(0) = -y(\pi), \ \ y'(0) = -y'(\pi); \\
&\text{Dirichlet } (bc = Dir): && y(0) = y(\pi) = 0; \\
&\text{Neumann } (bc = Neu): && y'(0) = y'(\pi) = 0.
\end{aligned}$$

For each of the above boundary conditions the spectrum of (1.1) is discrete. Moreover the spectrum is localized so that, for sufficiently large $n \in \mathbb{N}$, there exists a disc centered around n^2 consisting of two eigenvalues (counted with multiplicity) λ_n^- and λ_n^+ of periodic (if n is even) or antiperiodic (if n is odd) boundary conditions. It also consists one eigenvalue μ_n of Dirichlet and one eigenvalue ν_n of Neumann boundary conditions. There is a close relation between the eigenvalues λ_n^- and λ_n^+ and the spectrum of the same operator (1.1) but considered on the whole real line. (1.1) considered on $\mathbb{R}$ with a real-valued π-periodic potential $v \in L^2([0, \pi])$, is self-adjoint and its spectrum has a band-gap structure, i.e., it consists of intervals separated by spectral gaps (instability zones). The Floquet theory (e.g., see [1]) shows that the endpoints of these gaps are eigenvalues λ_n^-, λ_n^+ of (1.1) with periodic boundary conditions for even n and antiperiodic boundary conditions for odd n.

Hochstadt [2, 3] discovered that there is a close relation between the rate of decay of the *spectral gap* $\gamma_n = \lambda_n^+ - \lambda_n^-$ and the smoothness of the potential v. He proved that every finite zone potential is a C^∞-function, and moreover, *if v is infinitely differentiable then γ_n decays faster than any power of* $1/n$. Later several authors [4]- [6] studied this phenomenon and showed that *if γ_n decays faster than any power of* $1/n$, *then v is infinitely differentiable*. Moreover, Trubowitz [7] proved that *v is analytic if and only if γ_n decays exponentially fast*.

If v is a complex-valued function then the operator (1.1) considered on $\mathbb{R}$ is not self-adjoint and we cannot talk about spectral gaps. But $\lambda_n^{\pm}$ are still well defined for sufficiently large n as eigenvalues of (1.1) considered on the interval $[0, \pi]$ with periodic or antiperiodic boundary conditions, so we set again $\gamma_n = \lambda_n^+ - \lambda_n^-$ and call it the n-th spectral gap. Again the potential smoothness determines the decay rate of γ_n, but in general the opposite is not true. The decay rate of γ_n has no control on the smoothness of a complex valued potential v by itself as the Gasymov paper [8] shows.

Tkachenko [9]– [11] discovered that the smoothness of complex potentials could be controlled if one consider, together with the spectral gap γ_n, the deviation $\delta_n^{Dir} = \lambda_n^+ - \mu_n$. He characterized in these terms the C^∞-smoothness and analyticity of complex valued potentials v. Moreover, Sansuc and Tkachenko [12] showed that v is in the Sobolev space H^a, $a \in \mathbb{N}$ if and only if γ_n and δ_n^{Dir} are in the weighted sequence space $\ell_a^2 = \ell^2((1+n^2)^{a/2})$.

The above results have been obtained by using Inverse Spectral Theory. Kappeler and Mityagin [13] suggested another approach based on Fourier Analysis. To formulate their results, let us recall that the smoothness of functions could be characterized by weights $\Omega = (\Omega(k))_{k\in\mathbb{Z}}$, and the corresponding weighted spaces are defined by

$$H(\Omega) = \{v(x) = \sum_{k\in\mathbb{Z}} v_k e^{2ikx}, \quad \sum_{k\in\mathbb{Z}} |v_k|^2 (\Omega(k))^2 < \infty\}.$$

A weight Ω is called sub-multiplicative, if $\Omega(-k) = \Omega(k)$ and $\Omega(k+m) \le \Omega(k)\Omega(m)$ for $k, m \ge 0$. In these terms the main result in [13] says that if Ω is a sub-multiplicative weight, then

$$(A) \quad v \in H(\Omega) \quad \Longrightarrow \quad (B) \quad (\gamma_n), \left(\delta_n^{Dir}\right) \in \ell^2(\Omega). \tag{1.2}$$

Djakov and Mityagin [14–16] proved the inverse implication $(B) \Rightarrow (A)$ under some additional mild restrictions on the weight Ω. Similar results were obtained for 1D Dirac operators (see [16, 18, 19]).

The analysis in [13–16] is carried out under the assumption $v \in L^2([0, \pi])$. Using the quasi-derivative approach of Savchuk-Shkalikov [17], Djakov and Mityagin [20] developed a Fourier method for studying the spectra of L with periodic, antiperiodic, and Dirichlet boundary conditions in the case of periodic singular potentials and extended the above results. They proved that if $v \in H^{-1}_{per}(\mathbb{R})$ and Ω is a weight of the form $\Omega(m) = \omega(m)/|m|$ for $m \neq 0$, with ω being a sub-multiplicative weight, then $(A) \Rightarrow (B)$, and conversely, if in addition $(\log \omega(n))/n$ decreases to zero, then $(B) \Rightarrow (A)$ (see Theorem 37 in [21]).

A crucial step in proving the implications $(A) \Rightarrow (B)$ and $(B) \Rightarrow (A)$ is the following statement (which comes from Lyapunov-Schmidt projection method, e.g., see Lemma 21 in [16]): *For large enough n, there exists a matrix* $\begin{pmatrix} \alpha_n(z) & \beta_n^+(z) \\ \beta_n^-(z) & \alpha_n(z) \end{pmatrix}$ *such that a number $\lambda = n^2 + z$ with $|z| < n/4$ is a periodic or antiperiodic eigenvalue if and only if* *z is an eigenvalue of this matrix.* The entrees $\alpha_n(z) = \alpha_n(z; v)$ and $\beta_n^\pm(z) = \beta_n^\pm(z; v)$ are given by explicit expressions in terms of the Fourier coefficients of the potential v and depend analytically on z and v.

The functionals $\beta_n^\pm$ give lower and upper bounds for the gaps and deviations (e.g., see Theorem 29 in [21]): If $v \in H^{-1}_{per}(\mathbb{R})$ then, for sufficiently large n,

$$\frac{1}{72}(|\beta_n^+(z_n^*)| + |\beta_n^-(z_n^*)|) \leq |\gamma_n| + |\delta_n^{Dir}| \leq 58(|\beta_n^+(z_n^*)| + |\beta_n^-(z_n^*)|), \tag{1.3}$$

where $z_n^* = \frac{1}{2}(\lambda_n^+ + \lambda_n^-) - n^2$. Thus, the implications $(A) \Rightarrow (B)$ and $(B) \Rightarrow (A)$ are equivalent, respectively, to

$$(\tilde{A}): \quad v \in H(\Omega) \quad \Longrightarrow \quad (\tilde{B}): \quad (|\beta_n^+(z_n^*)| + |\beta_n^-(z_n^*)|) \in \ell^2(\Omega), \tag{1.4}$$

and $(\tilde{B}) \Rightarrow (\tilde{A})$. In this way the problem of analyzing the relationship between potential smoothness and decay rate of the sequence $(|\gamma_n| + |\delta_n^{Dir}|)$ is reduced to analysis of the functionals $\beta_n^\pm(z)$.

The asymptotic behavior of $\beta_n^\pm(z)$ (or γ_n and δ_n^{Dir}) plays also a crucial role in studying the Riesz basis property of the system of root functions of the operator L with periodic or antiperiodic boundary conditions. In [16, Section 5.2], it is shown (for potentials $v \in L^2([0, \pi])$) that if the ratio $\beta_n^+(z_n^*)/\beta_n^-(z_n^*)$ is not separated from 0 or ∞ then the system of periodic (or antiperiodic) root functions does not contain a Riesz basis (see Theorem 71 and its proof therein). Theorem 1 in [23] (or Theorem 2 in [22]) gives, for wide classes of L^2-potentials, a criterion for Riesz basis property in the same terms. In its most general form, for singular potentials, this criterion reads as follows (see Theorem 19 in [24]):

Criterion 1. Suppose $v \in H^{-1}_{per}(\mathbb{R})$; then the set of root functions of $L_{Per^\pm}(v)$ contains Riesz bases if and only if

$$0 < \inf_{\gamma_n \neq 0} |\beta_n^-(z_n^*)|/|\beta_n^+(z_n^*)|, \quad \sup_{\gamma_n \neq 0} |\beta_n^-(z_n^*)|/|\beta_n^+(z_n^*)| < \infty, \tag{1.5}$$

where n is even (respectively odd) in the case of periodic (antiperiodic) boundary conditions.

In [25] Gesztesy and Tkachenko obtained the following result.

Criterion 2. If $v \in L^2([0, \pi])$, then there is a Riesz basis consisting of root functions of the operator $L_{Per^\pm}(v)$ if and only if

$$\sup_{\gamma_n \neq 0} |\delta_n^{Dir}|/|\gamma_n| < \infty, \tag{1.6}$$

where n is even (respectively odd) in the case of periodic (antiperiodic) boundary conditions.

They also noted that a similar criterion holds if (1.6) is replaced by

$$\sup_{\gamma_n \neq 0} |\delta_n^{Neu}|/|\gamma_n| < \infty, \tag{1.7}$$

where $\delta_n^{Neu} = \lambda_n^+ - \nu_n$ (recall that ν_n is the n-th Neumann eigenvalue).

Djakov and Mityagin [24, Theorem 24] proved, for singular potentials $v \in H^{-1}_{per}(\mathbb{R})$, that the conditions (1.5) and (1.6) are equivalent, so (1.6) gives necessary and sufficient conditions for Riesz basis property for singular potentials as well.

On the other hand, the author has shown (see Theorems 1 and 2 in [26]), for potentials $v \in L^p([0,\pi])$, $p > 1$, that the Neumann version of Criterion 2 holds and the potential smoothness could be characterized by the rate of decay of $|\gamma_n| + |\delta_n^{Neu}|$. In this thesis we extend these results for singular periodic potentials $v \in H^{-1}_{per}(\mathbb{R})$. More precisely, the following theorems hold.

Theorem 1.1. *Suppose* $v \in H^{-1}_{per}(\mathbb{R})$ *and* Ω *is a weight of the form* $\Omega(m) = \omega(m)/m$ *for* $m \neq 0$, *where* ω *is a sub-multiplicative weight. Then*

$$v \in H(\Omega) \Longrightarrow (|\gamma_n|), (|\delta_n^{Neu}|) \in \ell^2(\Omega); \tag{1.8}$$

conversely, if in addition $(\log \omega(n))/n$ *eventually decreases to zero monotonically, then*

$$(|\gamma_n|), (|\delta_n^{Neu}|) \in \ell^2(\Omega) \Longrightarrow v \in H(\Omega). \tag{1.9}$$

If $\lim \frac{\log \omega(n)}{n} > 0$, *(i.e.* ω *is of exponential type), then*

$$(\gamma_n), (\delta_n^{Neu}) \in \ell^2(\Omega) \;\Rightarrow\; \exists \varepsilon > 0 : \; v \in H(e^{\varepsilon |n|}). \tag{1.10}$$

Theorem 1.2. *If* $v \in H^{-1}_{per}(\mathbb{R})$, *then there is a Riesz basis consisting of root functions of the operator* $L_{Per^{\pm}}(v)$ *if and only if*

$$\sup_{\gamma_n \neq 0} |\delta_n^{Neu}|/|\gamma_n| < \infty, \tag{1.11}$$

where n *is respectively even (odd) for periodic (antiperiodic) boundary conditions.*

We do not prove Theorem 1.1 and Theorem 1.2 directly, but show that they are valid by reducing their proofs to Theorem 37 in [21] and Theorem 19 in [24], respectively. For this end we prove the following theorem which generalizes Theorem 3 in [26].

Theorem 1.3. *If* $v \in H^{-1}_{per}(\mathbb{R})$, *then, for sufficiently large* n,

$$\frac{1}{80}(|\beta_n^+(z_n^*)| + |\beta_n^-(z_n^*)|) \le |\gamma_n| + |\delta_n^{Neu}| \le 19(|\beta_n^+(z_n^*)| + |\beta_n^-(z_n^*)|). \tag{1.12}$$

Next we show that Theorem 1.3 implies Theorem 1.1 and Theorem 1.2. By Theorem 29 in [21] and Theorem 1.3, (1.3) and (1.12) hold simultaneously, so the sequences $(|\gamma_n| + |\delta_n^{Dir}|)$ and $(|\gamma_n| + |\delta_n^{Neu}|)$ are asymptotically equivalent. Therefore, every claim in Theorem 1.1 follows from the corresponding assertion in [21, Theorem 37].

On the other hand the asymptotic equivalence of $|\gamma_n| + |\delta_n^{Dir}|$ and $|\gamma_n| + |\delta_n^{Neu}|$ implies that $\sup_{\gamma_n \neq 0} |\delta_n^{Dir}|/|\gamma_n| < \infty$ if and only if $\sup_{\gamma_n \neq 0} |\delta_n^{Neu}|/|\gamma_n| < \infty$, so (1.6) and (1.11) hold simultaneously if $v \in H^{-1}_{per}(\mathbb{R})$. By Theorem 24 in [24], (1.6) gives necessary and sufficient conditions for the Riesz basis property if $v \in H^{-1}_{per}(\mathbb{R})$. Hence, Theorem 1.2 is proved.

Theorem 1.3 is proved in Section 4, following the method developed in [15] in the case of Dirichlet boundary conditions.

Moreover in Section 5 we consider a special class of boundary conditions (3.1). These boundary conditions are first introduced by Kappeler and Mityagin in [13] and they noted that in (1.2) one can replace the Dirichlet deviations δ_n^{Dir} by the deviations δ_n coming from these boundary conditions , i.e. ; $v \in H(\Omega) \Longrightarrow (\gamma_n), (\delta_n) \in \ell^2(\Omega)$. In the last section (Theorem 3.4) we show that if $v \in L^2([0, \pi])$, then the sequences $|\beta_n^+(z_n^*)| + |\beta_n^-(z_n^*)|$ and $|\gamma_n| + |\delta_n|$ are asymptotically equivalent as well. Hence under the assumption $v \in L^2([0, \pi])$ Theorem 1.3 and therefore Theorem 1.1 and 1.2 are still valid if we replace δ_n^{Neu} by δ_n.

CHAPTER 2

Neumann Boundary Conditions

2.1 Preliminary Results

Let $\mathcal{D}$ be the space of test functions on $\mathbb{R}$, i.e., it consists of all infinitely differentiable functions with compact support. For each $T > 0$ let D_T be the space of test functions φ with $supp\varphi \subset [-T, T]$. We define $H^{-1}_{loc}(\mathbb{R})$ as the space of distributions v satisfying

$$\forall T > 0 \ \ \exists C_T : \quad |\langle v, \varphi \rangle| \leq C_T \|\varphi\|_T \quad \forall \varphi \in \mathcal{D}_T \tag{2.1}$$

where

$$\|\varphi\|_T^2 = \int_{-T}^{T} \left(|\varphi(x)|^2 + |\varphi'(x)|^2 \right) dx.$$

Since for each $\varphi \in \mathcal{D}_T$, $\varphi(x) = \int_{-T}^{x} \varphi'(t)dt$, one can easily see that

$$\int_{-T}^{T} |\varphi(x)|^2 dx \leq (2T)^2 \int_{-T}^{T} |\varphi'(x)|^2 dx.$$

Hence condition (2.1) can be rewritten as

$$\forall T > 0 \ \ \exists \widetilde{C}_T : \quad |\langle v, \varphi \rangle| \leq \widetilde{C}_T \|\varphi'\|_{L^2([-T,T])} \quad \forall \varphi \in \mathcal{D}_T. \tag{2.2}$$

A distribution v is called π-periodic if

$$\langle v, \varphi(x) \rangle = \langle v, \varphi(x - \pi) \rangle \quad \forall \varphi \in \mathcal{D}.$$

Further we denote the space of π-periodic distributions satisfying (2.1) by $H^{-1}_{per}(\mathbb{R})$. It is known (see [28], Remark 2.3) that the following proposition holds.

Proposition 2.1. *If $v \in H^{-1}_{per}(\mathbb{R})$ then it has the form*

$$v = C + Q', \tag{2.3}$$

where C is a constant and Q is a $\pi-$periodic $L^2_{loc}(\mathbb{R})$ function which is uniquely determined up to a constant.

Proof. We follow the proof of Proposition 1 in [20]. Let $\mathcal{D}' = \{\varphi' : \quad \varphi \in \mathcal{D}\}$ and $\mathcal{D}'_T = \{\varphi' : \quad \varphi \in \mathcal{D}_T\}$. If $v \in H^{-1}_{loc}(\mathbb{R})$, by (2.2) we see that for each $T > 0$ the functional q acting as

$$q(\varphi') = -\langle v, \varphi\rangle \quad \varphi' \in \mathcal{D}'$$

is a continuous linear functional in the space $\mathcal{D}'_T \subset L^2([T,T])$. Hence by the Riesz Representation Theorem there exists a function $Q_T(x) \in L^2([T,T])$ satisfying

$$q(\varphi') = \int_{-T}^{T} Q_T(x)\varphi'(x)dx \quad \forall \varphi \in \mathcal{D}'_T.$$

Since this is true for all $T > 0$ one can see that there is a function $Q(x) \in L^2_{loc}(\mathbb{R})$ such that

$$q(\varphi') = \int_{-\infty}^{\infty} Q(x)\varphi'(x)dx \quad \forall \varphi \in \mathcal{D}'. \tag{2.4}$$

The function Q is determined up to an additive constant since only constants are orthogonal to $\mathcal{D}'_T$ in $L^2([T,T])$. Therefore we obtain

$$\langle v, \varphi\rangle = -q(\varphi') = -\langle Q, \varphi'\rangle = \langle Q', \varphi\rangle,$$

i.e.,

$$v = Q'. \tag{2.5}$$

If v is π-periodic and $Q(x) \in L^2_{loc}(\mathbb{R})$ satisfies $v = Q'$ then by (2.4) we have

$$\int_{-\infty}^{\infty} Q(x+\pi)\varphi'(x)dx = \int_{-\infty}^{\infty} Q(x)\varphi'(x-\pi)dx = \int_{-\infty}^{\infty} Q(x)\varphi'(x)dx$$

i.e.,

$$\int_{-\infty}^{\infty} \left(Q(x+\pi) - Q(x)\right)\varphi'(x)dx = 0 \quad \forall \varphi \in \mathcal{D}.$$

Thus, there exists a constant c such that

$$Q(x+\pi) - Q(x) = c \quad a.e. \tag{2.6}$$

Now if we define the function $\widetilde{Q}(x) = Q(x) - \frac{c}{\pi}x$, by (2.5) and (2.6) we see that $\widetilde{Q}(x)$ is π-periodic and $v = \widetilde{Q}' - \frac{c}{\pi}$. □

Consider the Hill-Schrödinger operator on the interval $[0,\pi]$ generated by the differential expression

$$\ell(y) = -y'' + v \cdot y, \tag{2.7}$$

where $v \in H^{-1}_{per}(\mathbb{R})$. By Proposition 2.1, v has the form (2.3). Therefore for $\varphi \in \mathcal{D}$ we have

$$\langle -y'' + vy, \varphi\rangle = \langle y', \varphi'\rangle + \langle Q'y, \varphi\rangle + \langle Cy, \varphi\rangle.$$

The term $\langle Q'y, \varphi\rangle = \langle Q', y\varphi\rangle$ can be written as

$$\langle Q', y\varphi\rangle = -\langle Q, (y\varphi)'\rangle = -\langle Q, y'\varphi + y\varphi'\rangle = -\langle Qy', \varphi\rangle - \langle Qy, \varphi'\rangle.$$

Hence we get

$$\langle -y'' + vy, \varphi \rangle = \langle y' - Qy, \varphi' \rangle + \langle -Qy' + Cy, \varphi \rangle = -\langle (y' - Qy)' + Qy' - Cy, \varphi \rangle.$$

On the other hand, from now on we assume, without loss of generality, that $C = 0$ since a constant shift of the operator results in a shift of the spectra but the objects we analyze i.e., root functions, spectral gaps and deviations, do not change. Therefore the differential expression (2.7) can be written as

$$\ell(y) = -(y' - Qy)' - Qy'. \tag{2.8}$$

The expression $y' - Qy$ is called *quasi-derivative* of y. We define the appropriate boundary conditions and corresponding domains of the operator following the approach suggested and developed by A. Savchuk and A. Shkalikov [17,27] and R. Hryniv and Ya. Mykytyuk [28]. The classical periodic, antiperiodic, Dirichlet and Neumann boundary conditions (bc) are replaced by the following:

Periodic ($bc = Per^+$) : $\quad y(0) = y(\pi), \quad (y' - Qy)(0) = (y' - Qy)(\pi);$

Antiperiodic ($bc = Per^-$) : $\quad y(0) = -y(\pi), \quad (y' - Qy)(0) = -(y' - Qy)(\pi);$

Dirichlet ($bc = Dir$) : $\quad y(0) = y(\pi) = 0;$

Neumann ($bc = Neu$) : $\quad (y' - Qy)(0) = (y' - Qy)(\pi) = 0;$

Remark 2.2. Note that, for a given potential v, the function Q is determined up to a constant shift, i.e., Q can be replaced by $Q + t$ for any constant t. This freedom of choice of Q has no effect on how the operator acts, neither on the periodic, anti-periodic or Dirichlet bc's but it does change the Neumann bc we consider. So the above definition of Neumann bc describes a family of boundary conditions which depends on the choice of Q. In particular, if $v \in L^1([0, \pi])$, then Q is absolutely continuous and the Neumann bc we defined above can be rewritten as $y'(0) = ty(0)$ and $y'(\pi) = ty(\pi)$, where the parameter $t = Q(0) = Q(\pi)$ can be any complex number since we are free to shift Q. Hence any result we obtain about the Neumann bc as defined above applies to all members of this family of boundary conditions in the case of $v \in L^1([0, \pi])$ including the classical Neumann bc where $t = 0$.

For each of the above bc, we consider the closed operator L_{bc}, acting as $L_{bc}\, y = \ell(y)$ in the domain

$$Dom(L_{bc}) = \{ y \in W_2^1([0, \pi]) \ : \ y' - Qy \in W_1^1([0, \pi]),$$
$$\ell(y) \in L^2([0, \pi]), \text{ and } y \text{ satisfies } bc \}.$$

For each bc, $Dom(L_{bc})$ is dense in $L^2([0, \pi])$ and $L_{bc} = L_{bc}(v)$ satisfies

$$(L_{bc}(v))^* = L_{bc}(\overline{v}) \ \text{ for } \ bc = Per^{\pm}, \ Dir, \ Neu, \tag{2.9}$$

where $(L_{bc}(v))^*$ is the adjoint operator and $\overline{v}$ is the conjugate of v, i.e., $\langle \overline{v}, h\rangle = \overline{\langle v, \overline{h}\rangle}$ for all test functions h. In the classical case where $v \in L^2([0,\pi])$, (2.9) is a well known fact. In the case where $v \in H^{-1}_{per}(\mathbb{R})$ it is explicitly stated and proved for $bc = Per^{\pm}$, Dir in [20], see Theorem 6 and Theorem 13 there. Following the same argument as in [20] one can easily see that it holds for $bc = Neu$ as well.

If $v = 0$ we write L^0_{bc}, (or simply L^0). The spectra and eigenfunctions of L^0_{bc} are as follows:

(a) $Sp(L^0_{Per^+}) = \{n^2,\ n = 0, 2, 4, \dots\}$; its eigenspaces are $\mathcal{E}^0_n = Span\{e^{\pm inx}\}$ for $n > 0$ and $\mathcal{E}^0_0 = \mathbb{C}$, $\dim \mathcal{E}^0_n = 2$ for $n > 0$, and $\dim \mathcal{E}^0_0 = 1$.

(b) $Sp(L^0_{Per^-}) = \{n^2,\ n = 1, 3, 5, \dots\}$; its eigenspaces are $\mathcal{E}^0_n = Span\{e^{\pm inx}\}$, and $\dim \mathcal{E}^0_n = 2$.

(c) $Sp(L^0_{Dir}) = \{n^2,\ n \in \mathbb{N}\}$; each eigenvalue n^2 is simple; its eigenspaces are $\mathcal{S}^0_n = Span\{s_n(x)\}$, where $s_n(x)$ is the corresponding normalized eigenfunction $s_n(x) = \sqrt{2}\sin nx$.

(d) $Sp(L^0_{Neu}) = \{n^2,\ n \in \{0\} \cup \mathbb{N}\}$; each eigenvalue n^2 is simple; its eigenspaces are $\mathcal{C}^0_n = Span\{c_n(x)\}$, where $c_n(x)$ is the corresponding normalized eigenfunction $c_0(x) = 1$, $c_n(x) = \sqrt{2}\cos nx$ for $n > 0$.

The sets of indices $2\mathbb{Z}$, $2\mathbb{Z}+1$, $\mathbb{N}$, and $\{0\}\cup\mathbb{N}$ will be denoted by Γ_{Per^+}, Γ_{Per^-}, Γ_{Dir} and Γ_{Neu}, respectively. For each bc, we consider the corresponding canonical orthonormal basis consisting of eigenfunctions of L^0_{bc}, namely $\mathcal{B}_{Per^+} = \{e^{inx}\}_{n\in\Gamma_{Per^+}}$, $\mathcal{B}_{Per^-} = \{e^{inx}\}_{n\in\Gamma_{Per^-}}$, $\mathcal{B}_{Dir} = \{s_n(x)\}_{n\in\Gamma_{Dir}}$, $\mathcal{B}_{Neu} = \{c_n(x)\}_{n\in\Gamma_{Neu}}$.

In [20], Djakov and Mityagin developed a Fourier method for studying the operators L_{bc} for $bc = Per^{\pm}$, Dir in the case of $H^{-1}_{per}(\mathbb{R})$ potentials. To summarize their results let us denote by $\widehat{f}^{bc}_k$ the Fourier coefficients of a function $f \in L^1([0,\pi])$ with respect to the basis $\mathcal{B}_{bc}$, i.e.,

$$\widehat{f}^{bc}_k = \frac{1}{\pi}\int_0^{\pi} f(x)\overline{u^{bc}_k(x)}dx, \qquad k \in \Gamma_{bc} \quad u^{bc}_k(x) \in \mathcal{B}_{bc}. \tag{2.10}$$

Set also

$$V_+(k) = ik\widehat{Q}^{Per^+}_k, \quad \widetilde{V}(0) = 0, \quad \widetilde{V}(k) = k\widehat{Q}^{Dir}_k. \tag{2.11}$$

Let $\ell^2_1(\Gamma_{bc}) = \{a = (a_k)_{k\in\Gamma_{bc}} :\ \sum_{k\in\Gamma_{bc}}(1+k^2)|a_k|^2 < \infty\}$. Consider the unbounded operators $\mathcal{L}_{bc}$ acting in $\ell^2(\Gamma_{bc})$ as $\mathcal{L}_{bc}\, a = b = (b_k)_{k\in\Gamma_{bc}}$, where

$$b_k = k^2 a_k + \sum_{m\in\Gamma_{bc}} V_+(k-m)a_m \quad \text{ for } bc = Per^{\pm}, \tag{2.12}$$

$$b_k = k^2 a_k + \frac{1}{\sqrt{2}}\sum_{m\in\Gamma_{Dir}}\left(\widetilde{V}(|k-m|) - \widetilde{V}(k+m)\right)a_m \quad \text{ for } bc = Dir, \tag{2.13}$$

respectively in the domains

$$Dom(\mathcal{L}_{bc}) = \{a \in \ell^2_1(\Gamma_{bc}) :\ \mathcal{L}_{bc}\, a \in \ell^2(\Gamma_{bc})\}. \tag{2.14}$$

Then for $bc = Per^{\pm}$, Dir we have (Theorem 11 and 16 in [20])

$$Dom(L_{bc}) = \mathcal{F}_{bc}^{-1}(Dom(\mathcal{L}_{bc})) \quad and \quad L_{bc} = \mathcal{F}_{bc}^{-1} \circ \mathcal{L}_{bc} \circ \mathcal{F}_{bc}, \tag{2.15}$$

where $\mathcal{F}_{bc} : L^2([0,\pi]) \to \ell^2(\Gamma_{bc})$ is defined by $\mathcal{F}_{bc}(f) = (\widehat{f}_k^{bc})_{k\in\Gamma_{bc}}$. Similar facts hold in the case of Neumann boundary conditions as well. Indeed let us construct the unbounded operator $\mathcal{L}_{Neu}$ acting as $\mathcal{L}_{Neu}\, a = b$, where

$$b_k = k^2 a_k + \widetilde{V}(k)a_0 + \frac{1}{\sqrt{2}} \sum_{m=1}^{\infty} \left(\widetilde{V}(|k-m|) + \widetilde{V}(k+m) \right) a_m, \tag{2.16}$$

in the domain $Dom(\mathcal{L}_{Neu})$ given by (2.14) for $bc = Neu$. The following proposition implies that (2.15) holds in the case of Neumann bc as well.

Proposition 2.3. *In the above notations,*

$$y \in Dom(L_{Neu}) \quad and \quad L_{Neu}\, y = h \tag{2.17}$$

if and only if

$$\widehat{y} = (\widehat{y}_k^{Neu})_{k\in\Gamma_{Neu}} \in Dom(\mathcal{L}_{Neu}) \quad and \quad \mathcal{L}_{Neu}\, \widehat{y} = \widehat{h}, \tag{2.18}$$

where $\widehat{h} = (\widehat{h}_k^{Neu})_{k\in\Gamma_{Neu}}$.

Proof. The proof is similar to the proof of Proposition 15 in [20]. If (2.17) holds then $z = y' - Qy \in W_1^1([0,\pi])$ and $z(0) = z(\pi) = 0$. Hence by Lemma 14 in [20] we have

$$\widehat{z'}_0^{Neu} = 0 \;\text{ and }\; \widehat{z'}_k^{Neu} = k\widehat{z}_k^{Dir} \text{ for } k \in \mathbb{N} \tag{2.19}$$

and

$$\widehat{z}_k^{Dir} = -k\widehat{y}_k^{Neu} - (\widehat{Qy})_k^{Dir}. \tag{2.20}$$

On the other hand since $h = L_{Neu}y$, by (2.8) we have

$$h = -z' - Qy', \tag{2.21}$$

which together with (2.19) and (2.20) implies

$$\widehat{h}_k^{Neu} = k^2\widehat{y}_k^{Neu} + k(\widehat{Qy})_k^{Dir} - (\widehat{Qy'})_k^{Neu}. \tag{2.22}$$

Using trigonometric identities one can easily show that

$$(\widehat{Qy})_k^{Dir} = \frac{1}{\sqrt{2}} \sum_{m=1}^{\infty} \left(\widehat{Q}_{k+m}^{Dir} + sgn(k-m)\widehat{Q}_{|k-m|}^{Dir} \right) \widehat{y}_m^{Neu} + \widehat{Q}_k^{Dir}\widehat{y}_0^{Neu} \tag{2.23}$$

and

$$(\widehat{Qy'})_k^{Neu} = \frac{1}{\sqrt{2}} \sum_{m=1}^{\infty} \left(\widehat{Q}_{k+m}^{Dir} - sgn(k-m)\widehat{Q}_{|k-m|}^{Dir} \right) m\widehat{y}_m^{Neu}. \tag{2.24}$$

Combining (2.22), (2.23), (2.24) we get

$$\widehat{h}_k^{Neu} = k^2 \widehat{y}_k^{Neu} + k\widehat{Q}_k^{Dir}\widehat{y}_0^{Neu} + \frac{1}{\sqrt{2}}\sum_{m=1}^{\infty}\left((k+m)\widehat{Q}_{k+m}^{Dir} + |k-m|\widehat{Q}_{|k-m|}^{Dir}\right)\widehat{y}_m^{Neu}. \quad (2.25)$$

Comparing (2.25) with the definition of $\widetilde{V}(k)$ and the definition (2.16) of $\mathcal{L}_{Neu}$ we see that (2.18) holds.

Conversely, if (2.18) holds, then we can go from (2.25) back to (2.22) and see that $z = y' - Qy \in L_2([0,\pi])$ has the property that $k\widehat{z}_k^{Dir}$ are the cosine coefficients of an $L_1([0,\pi])$ function. Therefore, by Lemma 14 in [20], z is absolutely continuous, $z(0) = z(\pi) = 0$, and those numbers are the cosine coefficients of its derivative z'. Hence, $z = y' - Qy \in W_1^1([0,\pi])$ and $\ell(y) = h$, i.e., $y \in Dom(L_{Neu})$ and $L_{Neu}(y) = h$. □

In the sequel, for $bc = Per^{\pm}, Dir, Neu$, we identify the operator L_{bc} acting on the function space $L^2([0,\pi])$ with $\mathcal{L}_{bc}$ which acts on the corresponding sequence space $\ell^2(\Gamma_{bc})$ and use one and the same notation L_{bc} for both of them. Moreover, the matrix elements of an operator A acting on the sequence space $\ell^2(\Gamma_{bc})$ will be denoted by A_{nm}^{bc}, where $n, m \in \Gamma_{bc}$. The norm of a function $f \in L^a([0,\pi])$ and an operator A from $L^a([0,\pi])$ to $L^b([0,\pi])$ for $a, b \in [1,\infty]$ will be denoted by $\|f\|_a$ and $\|A\|_{a\to b}$, respectively. We may also write $\|f\|$ and $\|A\|$ instead of $\|f\|_2$ and $\|A\|_{2\to 2}$, respectively.

By (2.12), (2.13), and (2.16) we see that L_{bc} has the form $L_{bc} = L^0 + V$, where we define the operators L^0 and V, acting on the corresponding sequence space $\ell^2(\Gamma_{bc})$, by their matrix representations

$$L_{km}^0 = k^2\delta_{km} \qquad \text{for all } bc, \quad (2.26)$$

$$V_{km} = V_+(k-m) \qquad \text{for } bc = Per^{\pm}, \quad (2.27)$$

$$V_{km} = \frac{1}{\sqrt{2}}\left(\widetilde{V}(|k-m|) - \widetilde{V}(k+m)\right) \qquad \text{for } bc = Dir, \quad (2.28)$$

$$V_{km} = c_{k,m}\left(\widetilde{V}(|k-m|) + \widetilde{V}(k+m)\right) \qquad \text{for } bc = Neu, \quad (2.29)$$

where $c_{k,m} = 1/\sqrt{2}$ if $km \neq 0$ and $c_{k,m} = 1/2$ if $km = 0$. Note that in the notations of L^0 and V the dependence on the boundary conditions is suppressed.

Let $R_\lambda = (\lambda - L_{bc})^{-1}$ and $R_\lambda^0 = (\lambda - L_{bc}^0)^{-1}$. Since $\lambda - L_{bc} = \lambda - L_{bc}^0 - V = (1 - VR_\lambda^0)(\lambda - L_{bc}^0)$ we have $R_\lambda = R_\lambda^0(1 - VR_\lambda^0)^{-1}$. On the other hand $(1 - VR_\lambda^0)^{-1} = 1 + \sum_{s=1}^{\infty}(VR_\lambda^0)^s$ if the series on the right converges. Hence, assuming the series converge, we obtain

$$R_\lambda = R_\lambda^0 + \sum_{s=1}^{\infty} R_\lambda^0 (VR_\lambda^0)^s. \quad (2.30)$$

Moreover if there exists a square root K_λ of R_λ^0, i.e., $K_\lambda^2 = R_\lambda^0$, then (2.30) can be rewritten as

$$R_\lambda = R_\lambda^0 + \sum_{s=1}^{\infty} K_\lambda (K_\lambda V K_\lambda)^s K_\lambda. \quad (2.31)$$

Note that if

$$\|K_\lambda V K_\lambda\|_{2\to 2} < 1, \tag{2.32}$$

then series in (2.31) converges, hence R_λ exists.

By (2.26) we see that the matrix representation of R^0_λ is

$$(R^0_\lambda)^{bc}_{km} = \frac{1}{\lambda - m^2}\delta_{km}. \quad k, m \in \Gamma_{bc} \tag{2.33}$$

We define a square root $K = K_\lambda$ of R^0_λ by choosing its matrix representation as

$$(K_\lambda)^{bc}_{km} = \frac{1}{(\lambda - m^2)^{1/2}}\delta_{km}, \quad k, m \in \Gamma_{bc}, \tag{2.34}$$

where $z^{1/2} = |z|^{1/2}e^{i\theta/2}$ for $z = |z|e^{i\theta}$, $\theta \in [0, 2\pi)$.

Let

$$H^N = \{\lambda \in \mathbb{C} : Re\,\lambda \le N^2 + N\}, \tag{2.35}$$

$$R_N = \{\lambda \in \mathbb{C} : -N < Re\,\lambda < N^2 + N, \quad |Im\lambda| < N\}, \tag{2.36}$$

$$H_n = \{\lambda \in \mathbb{C} : (n-1)^2 \le Re\,\lambda \le (n+1)^2\}, \tag{2.37}$$

$$G_n = \{\lambda \in \mathbb{C} : n^2 - n \le Re\,\lambda \le n^2 + n\}, \tag{2.38}$$

$$D_n = \{\lambda \in \mathbb{C} : |\lambda - n^2| < r_n\}. \tag{2.39}$$

Assuming only $v \in H^{-1}_{per}(\mathbb{R})$, Djakov and Mityagin showed (see [20], Lemmas 19 and 20) that there exists $N > 0$, $N \in \Gamma_{bc}$ such that (2.32) holds for $\lambda \in H^N \backslash R_N$ and also for all $n > N$, $n \in \Gamma_{bc}$ (2.32) holds for $\lambda \in H_n \backslash D_n$ if $bc = Per^\pm$ and for $\lambda \in G_n \backslash D_n$ if $bc = Dir$ with $r_n = n$. Therefore, the following localization of the spectra holds:

$$Sp(L_{bc}) \subset R_N \cup \bigcup_{n>N, n\in\Gamma_{bc}} D_n, \quad bc = Per^\pm,\ Dir. \tag{2.40}$$

Moreover, using the method of continuous parametrization of the potential v, they showed that the spectrum is discrete for $bc = Per^\pm$, Dir and

$$\sharp(Sp(L_{Per^+}) \cap R_N) = 2N + 1, \quad \sharp(Sp(L_{Per^+}) \cap D_n) = 2, \quad n > N, n \in \Gamma_{Per^+},$$

$$\sharp(Sp(L_{Per^-}) \cap R_N) = 2N, \quad \sharp(Sp(L_{Per^-}) \cap D_n) = 2, \quad n > N, n \in \Gamma_{Per^-},$$

$$\sharp(Sp(L_{Dir}) \cap R_N) = N, \quad \sharp(Sp(L_{Dir}) \cap D_n) = 1, \quad n > N, n \in \Gamma_{Dir}.$$

Remark 2.4. Although in [20] Djakov and Mityagin formulated these lemmas for the discs D_n with $r_n = n$ they also pointed out (see the remark after Theorem 21) that the disks D_n can be chosen as $r_n = n\tilde{\varepsilon}_n$ where $\tilde{\varepsilon}_n \to 0$. Hence the localization of the spectra can be sharpen for all bc's we consider.

For Neumann bc the situation is similar. The Neumann eigenfunctions $c_k(x)$ of the free operator are uniformly bounded and form an orthonormal basis, so using the same argument as in [20] one can similarly localize the spectrum $Sp(L_{Neu})$ after showing that (2.32) holds for $\lambda \notin R_N \cup \left\{\bigcup_{n>N, n\in\Gamma_{Neu}} D_n\right\}$. To be more specific first note that the Hilbert-Schmidt norm

$$\|A\|_{HS} = \left(\sum_{k,m} |A_{km}|^2\right)^{1/2} \tag{2.41}$$

of an operator A majorizes its L^2 norm $\|A\|$. In [20] (inequality (5.22)) it is shown that

$$\|(K_\lambda V K_\lambda)^{Dir}\|_{HS}^2 \le \sum_{k,m\in\mathbb{Z}} \frac{(k-m)^2 |\widehat{Q}^{Dir}_{|k-m|}|^2}{|\lambda - k^2||\lambda - m^2|}, \tag{2.42}$$

($\widehat{Q}^{Dir}_0$ is defined to be zero for convenience). Then, using this estimate, it was shown that Lemma 19 and 20 in [20] hold for Dirichlet bc. In the case of Neumann bc, by (2.29), (2.34) and by definition of $\widetilde{V}$, the matrix representation of $(K_\lambda V K_\lambda)^{Neu}$ is

$$(K_\lambda V K_\lambda)^{Neu}_{km} = c_{k,m} \left(\frac{|k-m|\widehat{Q}^{Dir}_{|k-m|} + (k+m)\widehat{Q}^{Dir}_{k+m}}{(\lambda - j^2)^{1/2}(\lambda - m^2)^{1/2}}\right). \tag{2.43}$$

In view of (2.41) and (2.43), following the same argument as in [20], it is easy to see that inequality (2.42) still holds when we replace $(K_\lambda V K_\lambda)^{Dir}$ by $(K_\lambda V K_\lambda)^{Neu}$. Hence the proofs of Lemma 19, Lemma 20, and Theorem 21 in [20] apply to the case of Neumann bc as well. Therefore we have the following Propositions:

Proposition 2.5. *If $v \in H^{-1}_{per}(\mathbb{R})$, there are sequences $\varepsilon_n = \varepsilon_n(v)$ and $\tilde{\varepsilon}_n = \tilde{\varepsilon}_n(v)$ decreasing to zero and $N > 0$, $N \in \Gamma_{bc}$ such that*

$$\|K_\lambda V K_\lambda\| \le \varepsilon_N/2 < 1 \quad \textit{for} \quad \lambda \in H^N \setminus R_N, \tag{2.44}$$

and for $n > N$, $n \in \Gamma_{bc}$, with $r_n = n\tilde{\varepsilon}_n$,

$$\|K_\lambda V K_\lambda\| \le \varepsilon_n/2 \tag{2.45}$$

for $\lambda \in H_n \setminus D_n$ if $bc = Per^\pm$, and for $\lambda \in G_n \setminus D_n$ if $bc = Dir, Neu$.

Proposition 2.6. *For any potential $v \in H^{-1}_{per}(\mathbb{R})$, the spectrum of the operator $L_{Neu}(v)$ is discrete. Moreover there exists an integer N such that*

$$Sp(L_{Neu}) \subset R_N \cup \bigcup_{n>N, n\in\Gamma_{Neu}} D_n, \tag{2.46}$$

and

$$\sharp(Sp(L_{Neu}) \cap R_N) = N+1, \quad \sharp(Sp(L_{Neu}) \cap D_n) = 1, \quad n > N, n \in \Gamma_{Neu}.$$

2.2 Main Inequalities

For $bc = Per^\pm, Dir$ or Neu, we consider the Cauchy-Riesz projections

$$P_n = \frac{1}{2\pi i}\int_{\partial D_n} R_\lambda d\lambda, \quad P_n^0 = \frac{1}{2\pi i}\int_{\partial D_n} R_\lambda^0 d\lambda. \tag{2.47}$$

Proposition 2.7. *Let $D = \frac{d}{dx}$, and let P_n and P_n^0 be defined by (2.47). If $v \in H^{-1}_{per}(\mathbb{R})$ then we have, for large enough n,*

$$\|P_n - P_n^0\| \le \varepsilon_n \tag{2.48}$$

and

$$\|D(P_n - P_n^0)\| \le n\varepsilon_n. \tag{2.49}$$

Proof. In order to estimate $\|D(P_n - P_n^0)\|$, first we note that

$$D(P_n - P_n^0) = \frac{1}{2\pi i}\int_{\partial D_n} D(R_\lambda - R_\lambda^0)d\lambda. \tag{2.50}$$

Indeed, using integration by parts twice one can easily see that

$$\left\langle D\int_{\partial D_n}(R_\lambda - R_\lambda^0)f d\lambda, g\right\rangle = \left\langle \int_{\partial D_n} D(R_\lambda - R_\lambda^0)f d\lambda, g\right\rangle \tag{2.51}$$

for all $f \in L^2([0,\pi])$ and $g \in C_0^\infty([0,\pi])$. Since $C_0^\infty([0,\pi])$ is dense in $L^2([0,\pi])$, (2.51) implies (2.50). Hence

$$\|D(P_n - P_n^0)\| \le \frac{1}{2\pi}\int_{\partial D_n} \|D(R_\lambda - R_\lambda^0)\| d|\lambda| \le r_n \sup_{\lambda\in\partial D_n} \|D(R_\lambda - R_\lambda^0)\|. \tag{2.52}$$

By (2.31) we can write $D(R_\lambda - R_\lambda^0) = \sum_{s=1}^\infty DK_\lambda(K_\lambda V K_\lambda)^s K_\lambda$. It is easy to see that $\|DK_\lambda\| = \sup_{k\in\Gamma_{bc}} k/|\lambda - k^2|^{1/2} = n/|\lambda - n^2|^{1/2} = n/\sqrt{r_n}$ for $\lambda \in \partial D_n$, and similarly, $\|K_\lambda\| = \sup_{k\in\Gamma_{bc}} 1/|\lambda - k^2|^{1/2} = 1/|\lambda - n^2|^{1/2} = 1/\sqrt{r_n}$ for $\lambda \in \partial D_n$. Note also that, since $\lambda \in \partial D_n$, $\|K_\lambda V K_\lambda\| \le \varepsilon_n/2 \le 1/2$ for sufficiently large n's by Proposition 2.5. Hence we obtain $\|D(R_\lambda - R_\lambda^0)\| \le \sum_{s=1}^\infty \|DK_\lambda\|\|K_\lambda V K_\lambda\|^s\|K_\lambda\| \le 2\|DK_\lambda\|\|K_\lambda V K_\lambda\|\|K_\lambda\| \le n\varepsilon_n/r_n$. This together with (2.52) completes the proof of (2.49).

Following the same argument, we see that $\|P_n - P_n^0\| \le r_n \sup_{\lambda\in\partial D_n}\|R_\lambda - R_\lambda^0\|$ and $\|R_\lambda - R_\lambda^0\| \le 2\|K_\lambda\|^2\|K_\lambda V K_\lambda\| \le \varepsilon_n/r_n$ which imply (2.48). □

Let $L = L_{Per^\pm}$ and $L^0 = L^0_{Per^\pm}$, and let P_n and P_n^0 be the corresponding projections defined by (2.47). Then $\mathcal{E}_n = Ran\, P_n$ and $\mathcal{E}_n^0 = Ran\, P_n^0$ are invariant subspaces of L and L^0, respectively. By Lemma 30 in [21], $\mathcal{E}_n$ has an orthonormal basis $\{f_n, \varphi_n\}$ satisfying

$$Lf_n = \lambda_n^+ f_n \tag{2.53}$$

$$L\varphi_n = \lambda_n^+\varphi_n - \gamma_n\varphi_n + \xi_n f_n. \tag{2.54}$$

We denote the quasi-derivatives of f_n and φ_n by w_n and u_n, respectively. Then, in view of (2.8)and by definition of the quasi-derivative, we have

$$w_n = f_n' - Qf_n, \quad w_n' = -\lambda_n^+ f_n - Qf_n' \tag{2.55}$$

$$u_n = \varphi_n' - Q\varphi_n, \quad u_n' = -\lambda_n^+ \varphi_n - Q\varphi_n' + \gamma_n\varphi_n - \xi_n f_n. \tag{2.56}$$

Lemma 2.8. *In the above notations, for large enough n,*

$$\frac{1}{5}(|\beta_n^+(z_n^*)| + |\beta_n^-(z_n^*)|) \le |\xi_n| + |\gamma_n| \le 9(|\beta_n^+(z_n^*)| + |\beta_n^-(z_n^*)|) \tag{2.57}$$

Proof. Indeed, combining (7.13) and (7.18) and (7.31) in [21] one can easily see that $|\xi_n| \le 3(|\beta_n^+(z_n^*)| + |\beta_n^+(z_n^*)|) + 4|\gamma_n|$. This inequality, together with Lemma 20 in [21], implies that $|\xi_n| + |\gamma_n| \le 9(|\beta_n^+(z_n^*)| + |\beta_n^+(z_n^*)|)$ for sufficiently large n's. On the other hand by (7.31), (7.18), and (7.14) in [21] one gets $|\beta_n^+(z_n^*)| + |\beta_n^+(z_n^*)| \le 5(|\xi_n| + |\gamma_n|)$ for sufficiently large n's. □

Proposition 2.9. *Under the assumption $v \in H^{-1}_{per}(\mathbb{R})$, there exists a sequence κ_n converging to zero such that for large enough n*

$$|G(0) - G^0(0)| \le \kappa_n \|G\|, \qquad \text{for} \quad G \in \mathcal{E}_n, \tag{2.58}$$

$$|(G' - QG)(0) - (G^0)'(0)| \le n\kappa_n \|G\|, \qquad \text{for} \quad G \in \mathcal{E}_n, \tag{2.59}$$

where $G^0 = P_n^0 G$.

Proof. Since each $G \in \mathcal{E}_n$ is a linear combination of orthonormal functions f_n and φ_n, it is enough to show (2.58) and (2.59) for f_n and φ_n only. We provide a proof only for $G = \varphi_n$ because the same argument proves the claim for $G = f_n$. First we prove (2.59). Consider the function $\tilde{u}_n(x) = \cos mx\, u_n(x)$ where m is an integer chosen so that $m - n$ is odd. Then $\tilde{u}_n(x)$ satisfies $\tilde{u}_n(\pi) = -\tilde{u}_n(0)$, and therefore,

$$2u_n(0) = \tilde{u}_n(0) - \tilde{u}_n(\pi) = -\int_0^\pi \tilde{u}_n' dx = \int_0^\pi (m \sin mx\, u_n - \cos mx\, u_n')\, dx.$$

Using (2.56) and integrating by parts $\int_0^\pi m \sin mx\, \varphi_n' dx$, we obtain

$$\begin{aligned} 2u_n(0) = &- m \int_0^\pi \sin mx\, Q\varphi_n dx \\ &+ \int_0^\pi \cos mx \left(Q\varphi_n' + (\lambda_n^+ - m^2 - \gamma_n)\varphi_n + \xi_n f_n\right) dx \end{aligned} \tag{2.60}$$

Since $\varphi_n^0 = P^0\varphi_n$ is an eigenfunction of the free operator with eigenvalue n^2 we also have

$$2(\varphi_n^0)'(0) = (n^2 - m^2) \int_0^\pi \cos mx\, \varphi_n^0 dx. \tag{2.61}$$

Subtracting (2.61) from (2.60) we get

$$u_n(0) - (\varphi_n^0)'(0) = \frac{1}{2}\left(I_1 + I_2 + I_3 + I_4 + I_5\right), \tag{2.62}$$

where

$$I_1 = (n^2 - m^2) \int_0^\pi \cos mx \left(\varphi_n - \varphi_n^0\right) dx, \quad I_2 = -m \int_0^\pi \sin mx \; Q\varphi_n dx,$$
$$I_3 = \int_0^\pi \cos mx \; Q\varphi_n' dx, \quad I_4 = (\lambda_n^+ - n^2) \int_0^\pi \cos mx \; \varphi_n dx,$$
$$I_5 = \int_0^\pi \cos mx \left(-\gamma_n \varphi_n + \xi_n f_n\right) dx.$$

Next we estimate these integrals by choosing m appropriately. By Proposition 2.7, there is a positive sequence ε_n which dominates $\|P_n - P_n^0\|$ and converges to zero. We choose $m = m(n)$ so that $k_n = m - n$ is the largest odd number which is less than both n and $1/\sqrt{\varepsilon_n}$. Then

$$|I_1| \le \pi k_n(2n + k_n)\|\varphi_n - \varphi_n^0\|_1 \le \frac{3\pi n}{\sqrt{\varepsilon_n}}\|\varphi_n - \varphi_n^0\|_2. \tag{2.63}$$

Since

$$\|\varphi_n - \varphi_n^0\|_2 = \|(P_n - P_n^0)\varphi_n\| \le \|(P_n - P_n^0)\| \le \varepsilon_n \tag{2.64}$$

(by Proposition 2.7), it follows that

$$|I_1| \le 3\pi n\sqrt{\varepsilon_n}. \tag{2.65}$$

In order to estimate I_2, we first write it as $I_2 = I_{2a} + I_{2b}$ where

$$I_{2a} = -m \int_0^\pi \sin mx \; Q \left(\varphi_n - \varphi_n^0\right) dx, \qquad I_{2b} = -m \int_0^\pi \sin mx \; Q\varphi_n^0 dx.$$

Noting that $m = n + k_n \le 2n$, Cauchy-Schwartz inequality together with (2.64) implies that

$$|I_{2a}| \le 2\pi n\|Q\|_2 \|\varphi_n - \varphi_n^0\|_2 \le 2\pi n\|Q\|_2 \varepsilon_n. \tag{2.66}$$

For the second term I_{2b} note that $\mathcal{E}_n^0$ is spanned by orthonormal functions $\sqrt{2}\cos nx$ and $\sqrt{2}\sin nx$, so

$$\varphi_n^0 = \sqrt{2}\left(a_n \cos nx + b_n \sin nx\right), \tag{2.67}$$

where $|a_n|^2 + |b_n|^2 = \|\varphi_n^0\|^2 = \|P_n^0\varphi_n\|^2 \le \|\varphi_n\|^2 = 1$. Therefore, it follows that

$$\begin{aligned} I_{2b} = -\frac{n + k_n}{\sqrt{2}}\Bigg(a_n \int_0^\pi \big(\sin(2n + k_n)x + \sin k_n x\big)Q dx + \\ b_n \int_0^\pi \big(\cos k_n x - \cos(2n + k_n)x\big)Q dx\Bigg) \\ = -\frac{\pi(n + k_n)}{2}\Big(a_n\big(\widehat{Q}^{Dir}_{2n+k_n} + \widehat{Q}^{Dir}_{k_n}\big) + b_n\big(\widehat{Q}^{Neu}_{k_n} - \widehat{Q}^{Neu}_{2n+k_n}\big)\Big), \end{aligned}$$

Recalling $k_n \le n$ and $|a_n|, |b_n| \le 1$ we obtain

$$|I_{2b}| \le \pi n|\widehat{Q}|_n, \tag{2.68}$$

where we define $|\widehat{Q}|_n$ as $|\widehat{Q}|_n = |\widehat{Q}^{Dir}_{2n+k_n}| + |\widehat{Q}^{Dir}_{k_n}| + |\widehat{Q}^{Neu}_{k_n}| + |\widehat{Q}^{Neu}_{2n+k_n}|$. Note that k_n converges to infinity by the construction and Q is square integrable. Hence $|\widehat{Q}|_n$ tends to zero as n goes to infinity.

For I_3, we write it as $I_3 = I_{3a} + I_{3b}$, where

$$I_{3a} = \int_0^\pi \cos mx \, Q \left(\varphi_n - \varphi_n^0\right)' dx, \qquad I_{3b} = \int_0^\pi \cos mx \, Q \varphi_n^{0\prime} dx.$$

Applying the Cauchy-Schwartz inequality to I_{3a} we get

$$|I_{3a}| \le \pi \|Q\|_2 \| \left(\varphi_n - \varphi_n^0\right)' \|_2 \le \pi \|Q\|_2 n\varepsilon_n \tag{2.69}$$

since by Proposition 2.7

$$\| \left(\varphi_n - \varphi_n^0\right)' \|_2 \le \|D(P_n - P_n^0)\varphi_n\|_2 \le \|D(P_n - P_n^0)\| \le n\varepsilon_n. \tag{2.70}$$

I_{3b} can be treated similarly as I_{2b}. Inserting the derivative of (2.67) into I_{3b}, we obtain

$$\begin{aligned} I_{3b} = \frac{n}{\sqrt{2}} \Big(a_n \int_0^\pi \big(-\sin(2n+k_n)x + \sin k_n x \big) Q dx + \\ b_n \int_0^\pi \big(\cos k_n x + \cos(2n+k_n)x \big) Q dx \Big) \\ = \frac{\pi n}{2} \big(a_n \big(-\widehat{Q}^{Dir}_{2n+k_n} + \widehat{Q}^{Dir}_{k_n} \big) + b_n \big(\widehat{Q}^{Neu}_{k_n} + \widehat{Q}^{Neu}_{2n+k_n} \big) \big). \end{aligned}$$

Hence, as for I_{2b}, we obtain

$$|I_{3b}| \le \frac{\pi n}{2} |\widehat{Q}|_n. \tag{2.71}$$

For I_4 we have

$$|I_4| \le |\lambda_n^+ - n^2| \|\varphi_n\|_1 \le |\lambda_n^+ - n^2| \tag{2.72}$$

since $\|\varphi_n\|_1 \le \|\varphi_n\|_2 \le 1$. Recalling that each λ_n^+ lies in the disc $D_n = \{\lambda \, : \, |\lambda - n^2| < r_n\}$ where $r_n = n\tilde{\varepsilon}_n$ we get

$$|I_4| \le n\tilde{\varepsilon}_n. \tag{2.73}$$

Finally for I_5, in the view of Lemma 2.8, we have

$$|I_5| \le |\gamma_n| \|\varphi_n\| + |\xi_n| \|f_n\| \le |\gamma_n| + |\xi_n| \le 18(|\beta_n^+(z_n^*)| + |\beta_n^-(z_n^*)|).$$

Note that $|z_n^*| = |\frac{1}{2}(\lambda_n^+ - \lambda_n^-) - n^2|$ is in the disc D_n hence it is less that $n/2$ for sufficiently large n's. So by Proposition 15 in [21] there is a sequence $\hat{\varepsilon}_n$ converging to zero such that $|\beta_n^\pm(z_n^*) - V_+(\pm 2n)| \le n\hat{\varepsilon}_n$. Recall that $V_+(k) = ik\widehat{Q}^{Per^+}_k$. Hence

$$\begin{aligned} |I_5| &\le 18 \left(n\hat{\varepsilon}_n + |V_+(2n)| + |V_+(-2n)| \right) \\ &\le 36n \left(\hat{\varepsilon}_n + |\widehat{Q}^{Per^+}_{2n}| + |\widehat{Q}^{Per^+}_{-2n}| \right). \end{aligned} \tag{2.74}$$

Noting that $\widehat{Q}^{Per^+}_{\pm 2n}$ converges to zero, combining (2.62), (2.65), (2.66), (2.68), (2.69), (2.71), (2.73) and (2.74) we complete the proof of (2.59) for $G = \varphi_n$.

In order to prove (2.58) for $G = \varphi_n$, now we consider the function $\hat{u}_n(x) = \sin mx \, u_n(x)$, where $m - n$ is again odd. Then

$$0 = \hat{u}_n(\pi) - \hat{u}_n(0) = \int_0^\pi \hat{u}_n' dx = \int_0^\pi (m \cos mx \, u_n + \sin mx \, u_n') \, dx.$$

Using (2.56) and integrating by parts $\int_0^\pi m \cos mx \, \varphi_n' dx$ we obtain

$$\begin{aligned} 2m\varphi_n(0) = &- m \int_0^\pi \cos mx \, Q\varphi_n' dx \\ &- \int_0^\pi \sin mx \left(Q\varphi_n' + (\lambda_n^+ - m^2 - \gamma_n)\varphi_n + \xi_n f_n \right) dx. \end{aligned} \tag{2.75}$$

On the other hand

$$2m\varphi_n^0(0) = -(n^2 - m^2) \int_0^\pi \sin mx \, \varphi_n^0 dx. \tag{2.76}$$

Comparing (2.75) and (2.76) with (2.60) and (2.61) we see that following the same argument as in the proof of (2.59) one can prove (2.58). Note that now the multiplier n disappears since $\varphi_n(0)$ and $\varphi_n^0(0)$ in (2.75) and (2.76) are also multiplied by m which is greater than n by our choice. □

Corollary 2.10. *If $v \in L^1([0, \pi])$ then*

$$|G'(0) - (G^0)'(0)| \leq n\kappa_n \|G\|. \tag{2.77}$$

Proof. If $v \in L^1([0, \pi])$ then Q is absolutely continuous and can be chosen such that $Q(0) = Q(\pi) = 0$ (see Remark 2.2). Hence (2.59) implies (2.77). □

2.3 Proof of Theorem 1.3

In this section, we give a proof of Theorem 1.3, i.e., we show that the sequences $(|\gamma_n| + |\delta_n^{Neu}|)$ and $(|\beta_n^-(z_n^*)| + |\beta_n^+(z_n^*)|)$ are asymptotically equivalent. The proof is based on the methods developed in [13, 15, 16], but the technical details are different.

In the following, for simplicity, we suppress the dependence on n in all symbols containing n. From now on, P (P^0) denotes the Cauchy-Riesz projection associated with $L_{Per\pm}$ ($L^0_{Per\pm}$) only. We denote the projections associated with L_{Neu} and L^0_{Neu} by P_{Neu} and P^0_{Neu}, respectively, and $\mathcal{C} = \mathcal{C}(v)$ denotes the one dimensional invariant subspace of $L_{Neu} = L_{Neu}(v)$ corresponding to P_{Neu}.

Lemma 2.11. *Let f, φ be an orthonormal basis in $\mathcal{E}$ such that (2.53) and (2.54) hold. Then there is a unit vector $G = af + b\varphi$ in $\mathcal{E}$ satisfying*

$$(G' - QG)(0) = (G' - QG)(\pi) = 0, \tag{2.78}$$

and there is a unit vector $g \in \mathcal{C}$ satisfying

$$\langle G, \bar{g}\rangle \delta^{Neu} = b\langle \varphi, \bar{g}\rangle \gamma - b\langle f, \bar{g}\rangle \xi \tag{2.79}$$

such that $\langle G, \bar{g}\rangle \in \mathbb{R}$ and

$$\langle G, \bar{g}\rangle \geq 71/72 \tag{2.80}$$

for sufficiently large n.

(Remark. (2.78) means that G is in the domain of L_{Neu}.)

Proof. If $w(0) = 0$ then $w(\pi) = 0$ since f is either a periodic or antiperiodic eigenfunction (recall that we denote by w and u, the quasi-derivatives of f and φ, respectively). Hence we can set $G = f$. Otherwise we set $\tilde{G}(x) = u(0)f(x) - w(0)\varphi(x)$. Then $G = \tilde{G}/\|\tilde{G}\|$ satisfies (2.78) because the functions f and φ are simultaneously periodic or antiperiodic.

By (2.78), $G \in Dom(L) \cap Dom(L_{Neu})$, so we have $L_{Neu}G = LG$. Hence it follows

$$L_{Neu}G = aLf + bL\varphi = a\lambda^+ f + b(\lambda^+\varphi - \gamma\,\varphi + \xi f) \tag{2.81}$$

$$= \lambda^+(af + b\varphi) + b(\xi f - \gamma\varphi) = \lambda^+ G + b(\xi f - \gamma\varphi).$$

Fix a unit vector $g \in \mathcal{C}$ so that

$$\langle G, \bar{g}\rangle = |\langle G, \bar{g}\rangle|. \tag{2.82}$$

In view of (2.8), we have $-(g' - Qg) - Qg' = \nu g$. Passing to conjugates in the latter equation and in the Neumann boundary conditions for g, one can see that

$$L_{Neu}(\bar{v})\bar{g} = \bar{\nu}\bar{g}. \tag{2.83}$$

Taking inner product of both sides of (2.81) with $\bar{g}$ we get

$$\langle L_{Neu} G, \bar{g} \rangle = \lambda^+ \langle G, \bar{g} \rangle + b(\xi \langle f, \bar{g} \rangle - \gamma \langle \varphi, \bar{g} \rangle). \tag{2.84}$$

On the other hand, by (2.9) and (2.83), we have

$$\langle L_{Neu}(v) G, \bar{g} \rangle = \langle G, (L_{Neu}(v))^* \bar{g} \rangle = \langle G, L_{Neu}(\bar{v}) \bar{g} \rangle = \nu \langle G, \bar{g} \rangle. \tag{2.85}$$

Now (2.84) and (2.85) imply (2.79).

Let $G^0 = P^0 G$ and $\bar{g}^0 = P^0_{Neu} \bar{g}$; then $\|G^0\|, \|\bar{g}^0\| \le 1$ since P^0 and P^0_{Neu} are orthogonal projections and G and $\bar{g}$ are unit vectors. We have $\langle G, \bar{g} \rangle = \langle G^0, \bar{g}^0 \rangle + \langle G^0, \bar{g} - \bar{g}^0 \rangle + \langle G - G^0, \bar{g} \rangle$. So by the triangle and Cauchy inequalities it follows that $|\langle G, \bar{g} \rangle| \ge |\langle G^0, \bar{g}^0 \rangle| - \|\bar{g} - \bar{g}^0\| - \|G - G^0\|$. By Proposition 2.7 we have

$$\|G - G^0\| = \|(P - P^0) G\| \le \|P - P^0\| \le \varepsilon_n \tag{2.86}$$

and similarly

$$\|\bar{g} - \bar{g}^0\| = \|(P_{Neu}(\bar{v}) - P^0_{Neu}) \bar{g}\| \le \|P_{Neu}(\bar{v}) - P^0_{Neu}\| \le \varepsilon_n. \tag{2.87}$$

Hence, it follows that

$$|\langle G, \bar{g} \rangle| \ge |\langle G^0, \bar{g}^0 \rangle| - 2\varepsilon_n. \tag{2.88}$$

Next we estimate $|\langle G^0, \bar{g}^0 \rangle|$ from below in order to get a lower bound for $|\langle G, \bar{g} \rangle|$. Since $\mathcal{C}^0$ is spanned by $c_n(x) = \sqrt{2} \cos nx$, $\bar{g}^0$ is of the form

$$\bar{g}^0 = e^{i\theta} \|\bar{g}^0\| \left(\frac{1}{\sqrt{2}} e^{-inx} + \frac{1}{\sqrt{2}} e^{inx} \right) \tag{2.89}$$

for some $\theta \in [0, 2\pi)$. Now let G^0_1 and G^0_2 be the coefficients of G^0 in the basis $\{e^{inx}, e^{-inx}\}$, i.e.,

$$G^0(x) = G^0_1 e^{-inx} + G^0_2 e^{inx}. \tag{2.90}$$

Clearly $(G^0)'(0) = in(G^0_2 - G^0_1)$. Since $(G' - QG)(0) = 0$, by Proposition 2.9 we also have

$$|(G^0)'(0)| = |(G' - QG)(0) - (G^0)'(0)| \le n\kappa_n. \tag{2.91}$$

Hence we obtain

$$|G^0_1 - G^0_2| \le \kappa_n \tag{2.92}$$

and

$$|G^0_2| \le |G^0_1| + |G^0_1 - G^0_2| \le |G^0_1| + \kappa_n. \tag{2.93}$$

From (2.86) it follows that

$$\sqrt{|G^0_1|^2 + |G^0_2|^2} = \|G^0\| \ge \|G\| - \|G - G^0\| \ge 1 - \varepsilon_n,$$

so by (2.93) we get

$$|G^0_1| \ge \frac{1}{\sqrt{2}} - \sqrt{2}(\kappa_n + \varepsilon_n). \tag{2.94}$$

On the other hand (2.89) and (2.90) imply

$$|\langle G^0, \bar{g}^0\rangle| = \frac{1}{\sqrt{2}}\|\bar{g}^0\||G_1^0 + G_2^0| \geq \frac{1}{\sqrt{2}}\|\bar{g}^0\|\big(2|G_1^0| - |G_1^0 - G_2^0|\big). \tag{2.95}$$

Combining (2.92), (2.94), (2.95) and taking into account that

$$\|\bar{g}^0\| \geq \|\bar{g}\| - \|\bar{g} - \bar{g}^0\| \geq 1 - \varepsilon_n$$

due to (2.87), we obtain

$$|\langle G^0, \bar{g}^0\rangle| \geq 1 - 4\varepsilon_n - 2\kappa_n \tag{2.96}$$

which, together with (2.88) and (2.82), implies

$$\langle G, \bar{g}\rangle \geq 1 - 6\varepsilon_n - 2\kappa_n. \tag{2.97}$$

Hence, for a sufficiently large n, $\langle G, \bar{g}\rangle \geq 71/72$. □

Corollary 2.12. *For sufficiently large n, we have*

$$|\gamma_n| + |\delta_n^{Neu}| \leq 19\big(|\beta_n^+(z_n^*)| + |\beta_n^-(z_n^*)|\big). \tag{2.98}$$

Proof. Using (2.79), (2.80) and noting also that the absolute values of b and all inner products in the right-hand side of (2.79) do not exceed 1 we get $|\delta^{Neu}| \leq 72/71\big(|\xi|+|\gamma|\big)$. This inequality, together with Lemma 2.8, implies (2.98). □

Corollary 2.12 proves the second inequality in (1.12). In order to complete the proof of Theorem 1.3 it remains to prove the first inequality in (1.12).

By Proposition 34 in [21], if

$$\textit{Case 1}: \qquad \frac{1}{4}|\beta^-(z^+)| \leq |\beta^+(z^+)| \leq 4|\beta^-(z^+)|, \tag{2.99}$$

then we have

$$|\beta^+(z^*)| + |\beta^-(z^*)| \leq 2|\gamma|. \tag{2.100}$$

Next we consider the complementary cases

$$\textit{Case 2(a)}: \qquad 4|\beta^+(z^+)| < |\beta^-(z^+)|, \tag{2.101}$$

or

$$\textit{Case 2(b)}: \qquad 4|\beta^-(z^+)| < |\beta^+(z^+)|. \tag{2.102}$$

Lemma 2.13. *If* Case 2(a) *or* Case 2(b) *holds, then we have, for sufficiently large n,*

$$\frac{1}{4} \leq \frac{|w(0)|}{|u(0)|} \leq 4. \tag{2.103}$$

Proof. We consider only *Case 2(a)*, since the proof in *Case 2(b)* is similar. Let $f^0 = P^0 f$, $\varphi^0 = P^0\varphi$ and let $f^0 = f_1^0 e^{-inx} + f_2^0 e^{inx}$ and $\varphi^0 = \varphi_1^0 e^{-inx} + \varphi_2^0 e^{inx}$. In *Case 2(a)*, if $v \in L^2([0,\pi])$ it was shown in the proof of Lemma 64 in [16] that the following inequalities hold (inequalities (4.51), (4.52), (4.54), and (4.55) in [16]):

$$|f_1^0| \geq \frac{2}{\sqrt{5}} - 2\rho_n, \quad |f_2^0| \leq \frac{1}{\sqrt{5}}, \quad |\varphi_1^0| \leq \frac{1}{\sqrt{5}} + \rho_n, \quad |\varphi_2^0| \geq \frac{2}{\sqrt{5}} - 2\rho_n, \tag{2.104}$$

where ρ_n is a sequence converging to zero. These inequalities were derived using Lemma 21 and Proposition 11 in [16] which still hold in the case where $v \in H^{-1}([0,\pi])$, (see Lemma 6 and Proposition 44 in [21]) [1]. Hence we can safely use them.

Note that $(f^0)'(0) = in(f_2^0 - f_1^0)$. Using (2.104) we get

$$|(f^0)'(0)| \geq n(|f_1^0| - |f_2^0|) \geq n\left(\frac{1}{\sqrt{5}} - 2\rho_n\right) \geq \frac{n}{\sqrt{6}} \tag{2.105}$$

for sufficiently large n. On the other hand we have

$$|(f^0)'(0)| \leq n(|f_1^0| + |f_2^0|) \leq n\sqrt{2}\|f^0\| \leq \sqrt{2}n \tag{2.106}$$

Following the same argument for $(\varphi^0)'(0)$, we have both

$$n/\sqrt{6} \leq |(f^0)'(0)| \leq \sqrt{2}n \quad \text{and} \quad n/\sqrt{6} \leq |(\varphi^0)'(0)| \leq \sqrt{2}n. \tag{2.107}$$

On the other hand by Proposition 2.9 we have

$$|w(0) - (f^0)'(0)| \leq n\kappa_n \quad \text{and} \quad |u(0) - (\varphi^0)'(0)| \leq n\kappa_n. \tag{2.108}$$

Hence, for sufficiently large n's, we get

$$\frac{|w(0)|}{|u(0)|} \leq \frac{|f^{0\prime}(0)| + n\kappa_n}{|\varphi^{0\prime}(0)| - n\kappa_n} \leq \frac{n(\sqrt{2} + \kappa_n)}{n(1/\sqrt{6} - \kappa_n)} \leq 4 \tag{2.109}$$

and

$$\frac{|w(0)|}{|u(0)|} \geq \frac{|f^{0\prime}(0)| - n\kappa_n}{|\varphi^{0\prime}(0)| + n\kappa_n} \geq \frac{n(1/\sqrt{6} - \kappa_n)}{n(\sqrt{2} + \kappa_n)} \geq \frac{1}{4}. \tag{2.110}$$

□

Proposition 2.14. *For sufficiently large n, we have*

$$\left(|\beta_n^+(z_n^*)| + |\beta_n^-(z_n^*)|\right) \leq 80(|\gamma_n| + |\delta_n^{Neu}|) \tag{2.111}$$

Proof. In view of (2.100), it remains to prove (2.111) if *Case 2(a)* or *Case 2(b)* holds.

Now (2.79) implies that

$$|b||\langle f, \bar{g}\rangle||\xi| \leq |\delta^{Neu}| + |\gamma|. \tag{2.112}$$

[1] In the derivation of the inequalities (2.104), Proposition 44 in [21] is needed for its corollary (2.48). So one can directly use (2.48) instead of Proposition 44 in [21] to show (2.104) hold.

Thus, in order to estimate $|\xi|$ from above by $|\delta^{Neu}| + |\gamma|$ we need to find a lower bound to $|b||\langle f, \bar{g}\rangle|$. We have

$$|b||\langle f, \bar{g}\rangle| = |b|\big|\langle f, G\rangle + \langle f, \bar{g} - G\rangle\big| \geq |a||b| - \|\bar{g} - G\| \tag{2.113}$$

since $\|f\| = 1$, $|b| \leq 1$ and $\langle f, G\rangle = \bar{a}$. In view of (2.80)

$$\|\bar{g} - G\|^2 = \|\bar{g}\|^2 + \|G\|^2 - 2Re\,\langle \bar{g}, G\rangle = 2 - 2\langle \bar{g}, G\rangle \leq 1/36, \tag{2.114}$$

hence

$$\|\bar{g} - G\| \leq 1/6. \tag{2.115}$$

On the other hand, by the construction of G we know $|b/a| = |w(0)/u(0)|$, so Lemma 2.13 implies that $1/4 \leq |b/a| \leq 4$. Since $|a|^2 + |b|^2 = 1$, a standard calculus argument shows that

$$|a||b| \geq 4/17. \tag{2.116}$$

In view of (2.115) and (2.116), the right-hand side of (2.113) is not less than $4/17 - 1/6 > 1/15$, i.e., $|b||\langle f, \bar{g}\rangle| > 1/15$. Hence, by (2.112), it follows that

$$|\xi| \leq 15\big(|\delta^{Neu}| + |\gamma|\big). \tag{2.117}$$

Now we complete the proof combining (2.117) and Lemma 2.8. □

Corollary 2.12 and Proposition 2.14 show that (1.12) holds, so Theorem 1.3 is proved.

CHAPTER 3

More General Boundary Conditions

3.1 Spectrum and Root Functions of $H^0_{\sigma_0,\sigma_1}$

In Section 3.3 we show that, under the assumption $v \in L^2([0,\pi])$, Theorem 1.3 and therefore Theorem 1.1 and Theorem 1.2 still hold if we replace the Neumann deviations δ_n^{Neu} in those theorems by the deviations δ_n coming from a more general (σ_0,σ_1)-boundary conditions $((\sigma_0,\sigma_1)-bc)$:

$$\begin{aligned} y'(0)+y'(\pi)-\sigma_0(y(0)+y(\pi)) &= 0 \\ y'(0)-y'(\pi)-\sigma_1(y(0)-y(\pi)) &= 0. \end{aligned} \tag{3.1}$$

More precisely $\delta_n = \lambda_n^+ - \eta_n$ where η_n's are the eigenvalues of the operator H_{σ_0,σ_1} acting as

$$H_{\sigma_0,\sigma_1} y = -y'' + v(x)y \tag{3.2}$$

with a π-periodic potential $v \in L^2([0,\pi])$ and with the domain

$$Dom(H_{\sigma_0,\sigma_1}) = \{W_2^2([0,\pi]),\ f \text{ satisfies } (\sigma_0,\sigma_1)-bc\}. \tag{3.3}$$

The root functions of the free operator $H^0_{\sigma_0,\sigma_1}$ where $v=0$ are not the usual trigonometric functions such as exponentials or sines or cosines but combinations of them. Therefore verification of the Fourier method we use in our analysis becomes much harder under the assumption that v belongs to any space more general than $L^2([0,\pi])$ since in that case the root functions do not belong to $Dom(H_{\sigma_0,\sigma_1})$ in general. Recall that we show this verification in Section 2.1 in the case of Neumann boundary conditions using the convergence properties of the sine and cosine Fourier series. However in the case of (σ_0,σ_1)-boundary conditions, to show the existence of similar convergence properties of the Fourier series with respect to the basis consisting of the root functions of $H^0_{\sigma_0,\sigma_1}$ is not easy. Therefore we postpone the spectral analysis of H_{σ_0,σ_1} with more general potentials for now.

We have the following proposition.

Proposition 3.1. *The adjoint operator* $H^*_{\sigma_0,\sigma_1}$ *of* $H_{\sigma_0,\sigma_1} = H_{\sigma_0,\sigma_1}(v)$ *satisfies the identity*

$$H^*_{\sigma_0,\sigma_1}(v) = H_{\overline{\sigma}_1,\overline{\sigma}_0}(\overline{v}), \tag{3.4}$$

i.e., it is the operator acting as

$$H^*_{\sigma_0,\sigma_1} y = -y'' + \overline{v}(x)y$$

and having the boundary condition

$$\begin{aligned} y'(0) + y'(\pi) - \overline{\sigma}_1(y(0) + y(\pi)) = 0 \\ y'(0) - y'(\pi) - \overline{\sigma}_0(y(0) - y(\pi)) = 0. \end{aligned} \tag{3.5}$$

Proof. Let $y \in Dom(H^*_{\sigma_0,\sigma_1})$. So There exists an $e \in L^2([0,\pi])$ such that

$$\langle H_{\sigma_0,\sigma_1} f, y\rangle = \langle f, e\rangle, \tag{3.6}$$

which implies

$$-\int_0^\pi f''\bar{y}dx = \int_0^\pi f(\bar{e} - v\bar{y})dx \tag{3.7}$$

for all $f \in Dom(H_{\sigma_0,\sigma_1})$, in particular for all $f \in C_0^\infty([0,\pi])$. Let $E_1(x) = \int_0^x (e - \bar{v}h)dt$, and $E_2(x) = \int_0^x E_1 dt$. Assuming $f \in C_0^\infty([0,\pi])$ and integrating by parts the right side of (3.7) twice, we get $\int_0^\pi f''(\bar{E}_2 + \bar{y})dx = 0$, for all $f \in C_0^\infty([0,\pi])$. In other words $E_2 + y$ is in the orthogonal complement of the space $S = \{f''\ ,\ f \in C_0^\infty([0,\pi])\}$. On the other hand, using integration by parts one can easily see that $S = C_0^\infty([0,\pi]) \bigcap (Span\{1,\ x\})^\perp$. Since $C_0^\infty([0,\pi])$ is dense in $L^2([0,\pi])$ and $(Span\{1,\ x\})^\perp$ is closed, we obtain $cl(S) = (Span\{1,\ x\})^\perp$, which implies $E_2 + y \in S^\perp = cl(S)^\perp = ((Span\{1,\ x\})^\perp)^\perp = Span\{1,\ x\}$. Therefore $E_2 + y = mx + n$ for some constants m and n almost everywhere. Hence y is in $W_2^2([0,\pi])$ since E_2 is. Moreover $y'' = -E_2'' = \bar{v}y - e$, which, together with the definition of the adjoint operator, implies

$$H^*_{\sigma_0,\sigma_1} y = e = -y'' + \bar{v}y. \tag{3.8}$$

By (3.7) and (3.8) we see $\int_0^\pi f''\bar{y}dx = \int_0^\pi f\bar{y}''dx$ for all $f \in Dom(H_{\sigma_0,\sigma_1})$. Hence integration by parts gives us

$$f'(0)\bar{y}(0) - f'(\pi)\bar{y}(\pi) - f(0)\bar{y}'(0) + f(\pi)\bar{y}'(\pi) = 0. \tag{3.9}$$

Let F be the orthogonal complement of the space spanned by $(1,\ 1,\ -\overline{\sigma}_0,\ -\overline{\sigma}_0)$ and $(1,\ -1,\ -\overline{\sigma}_1,\ \overline{\sigma}_1)$. By a simple linear algebra argument we can see that

$$F = Span\{(\sigma_0,\ \sigma_0,\ 1,\ 1);\ (\sigma_1,\ -\sigma_1,\ 1,\ -1)\}. \tag{3.10}$$

Let us construct the vectors $\overrightarrow{f} = (f'(0),\ f'(\pi),\ f(0),\ f(\pi))$ and $\overrightarrow{y} = (y'(0),\ -y'(\pi),\ -y(0),\ y(\pi))$. Since f satisfies (3.1), $\overrightarrow{f} \in F$. Hence, by (3.9), $\overrightarrow{y}$ must be orthogonal to F, which, together with (3.10), implies (3.5). This completes the proof. □

Let u be an eigenfunction of the free operator with eigenvalue η. Then u satisfies

$$u'' + \eta u = 0. \tag{3.11}$$

Assume that u has the form

$$u = k_1 e^{i\sqrt{\eta}x} + k_2 e^{-i\sqrt{\eta}x}, \tag{3.12}$$

for some constants k_1 and k_2. Evaluating the above expression and its derivative at 0 and π and inserting them into (3.1) we obtain the following system of equations.

$$\left(1 + e^{i\sqrt{\eta}\pi}\right)\left(i\sqrt{\eta} - \sigma_0\right) k_1 - \left(1 + e^{-i\sqrt{\eta}\pi}\right)\left(i\sqrt{\eta} + \sigma_0\right) k_2 = 0, \tag{3.13}$$

$$\left(1 - e^{i\sqrt{\eta}\pi}\right)\left(i\sqrt{\eta} - \sigma_1\right) k_1 - \left(1 - e^{-i\sqrt{\eta}\pi}\right)\left(i\sqrt{\eta} + \sigma_1\right) k_2 = 0. \tag{3.14}$$

In order to get nontrivial solutions for k_1 and k_2, coefficients of (3.13) and (3.14) must satisfy

$$\begin{aligned} &\left(1 + e^{i\sqrt{\eta}\pi}\right)\left(i\sqrt{\eta} - \sigma_0\right)\left(1 - e^{-i\sqrt{\eta}\pi}\right)\left(i\sqrt{\eta} + \sigma_1\right) \\ &\quad - \left(1 + e^{-i\sqrt{\eta}\pi}\right)\left(i\sqrt{\eta} + \sigma_0\right)\left(1 - e^{i\sqrt{\eta}\pi}\right)\left(i\sqrt{\eta} - \sigma_1\right) = 0, \end{aligned}$$

which is equivalent to

$$\left(1 - e^{2i\sqrt{\eta}\pi}\right)\left(\eta + \sigma_0\sigma_1\right) = 0. \tag{3.15}$$

Hence either $\eta = -\sigma_0\sigma_1$ or $\eta = n^2$, where n is a positive integer. First assume that *Case I*: $-\sigma_0\sigma_1 \neq \tau^2$ where τ is a positive integer. Then, using (3.12), (3.13) and (3.14) one can compute that eigenfunction corresponding to the eigenvalue $\eta_0^0 = -\sigma_0\sigma_1$ is

$$u_0(x) = \begin{cases} \sigma_0\left(x - \frac{\pi}{2}\right) + 1 & \text{if } \sigma_1 = 0 \\ \frac{1}{2}\left(\left(1 + \sqrt{\frac{\sigma_0}{\sigma_1}}\right) e^{\sqrt{\sigma_0\sigma_1}x} + \left(1 - \sqrt{\frac{\sigma_0}{\sigma_1}}\right) e^{\sqrt{\sigma_0\sigma_1}(\pi - x)}\right) & \text{if } \sigma_1 \neq 0, \end{cases} \tag{3.16}$$

and eigenfunction corresponding to the eigenvalue $\eta_n^0 = n^2$ is

$$u_n(x) = \sqrt{2}\left(\cos(nx) + \frac{\sigma_n}{n}\sin(nx)\right), \tag{3.17}$$

where $\sigma_n = \sigma_0$ if n is even and $\sigma_n = \sigma_1$ if n is odd.

By (3.4), we see that the spectrum and the eigenfunctions of the adjoint of the free operator are as follows: $\overline{\eta}_0^0 = -\overline{\sigma}_0\overline{\sigma}_1$ with the corresponding eigenfunction

$$\tilde{u}_0(x) = \begin{cases} 1 & \text{if } \sigma_1 = 0 \\ \overline{\sigma}_1\left(x - \frac{\pi}{2}\right) + 1 & \text{if } \sigma_0 = 0 \\ A_0\left(\left(1 + \sqrt{\frac{\overline{\sigma}_1}{\overline{\sigma}_0}}\right) e^{\sqrt{\overline{\sigma}_1\overline{\sigma}_0}x} + \left(1 - \sqrt{\frac{\overline{\sigma}_1}{\overline{\sigma}_0}}\right) e^{\sqrt{\overline{\sigma}_1\overline{\sigma}_0}(\pi - x)}\right) & \text{if } \sigma_1, \sigma_0 \neq 0. \end{cases} \tag{3.18}$$

$\overline{\eta}_n^0 = n^2$ with the corresponding eigenfunction

$$\tilde{u}_n(x) = \sqrt{2}A_n\left(\cos(nx) + \frac{\overline{\sigma}_{n+1}}{n}\sin(nx)\right). \tag{3.19}$$

Here A_0 and A_n are the normalization constants which are chosen such that $A_0 = \pi\sqrt{\overline{\sigma}_0\overline{\sigma}_1}/(e^{2\pi\sqrt{\overline{\sigma}_0\overline{\sigma}_1}} - 1)$ and $A_n = n^2/(n^2 + \overline{\sigma}_0\overline{\sigma}_1)$. Indeed one can check that with this choice of A_0 and A_n the eigenfunctions $\{u_n\}_{n=0}^{\infty}$ and $\{\tilde{u}_n\}_{n=0}^{\infty}$ form a bi-orthogonal system, i.e., $\langle u_k, \tilde{u}_j\rangle = \delta_{kj}$.

It is easy to see that $\{u_n\}_{n=0}^{\infty}$ form a Riesz basis. First note that $\{u_n\}_{n=0}^{\infty}$ is minimal since $\{u_n\}_{n=0}^{\infty}$ and $\{\tilde{u}_n\}_{n=0}^{\infty}$ are bi-orthogonal. On the other hand

$$\sum_{n=1}^{\infty} \|u_n - \sqrt{2}\cos(nx)\|^2 \leq (|\sigma_0|^2 + |\sigma_1|^2) \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty \tag{3.20}$$

and $\{1\} \cup \{\sqrt{2}\cos(nx)\}_{n=1}^{\infty}$ is an orthonormal basis in $L^2([0,\pi])$. Then by Bari-Marcus theorem $\{u_n\}_{n=0}^{\infty}$ is a Riesz basis in $L^2([0,\pi])$.

However in *Case II*: $\sigma_0\sigma_1 = -\tau^2$ for some positive integer τ, eigenfunction system does not form a basis since in this case eigenfunctions u_0 and u_τ "coincide". Nevertheless one can show that there exists an associated function w satisfying $H^0_{\sigma_0,\sigma_1} w = \tau^2 w - u_0$ and eigenfunctions together with this associated function w form a Riesz basis. For convenience we will denote the eigenfunction corresponding to τ^2 by u_0 and the associated function w by u_τ. More precisely, in *Case II* we have the following eigenvalues and associated system:

$\eta^0_n = n^2$, $n \in \mathbb{N}\backslash\{\tau\}$, with the corresponding eigenfunctions u_n in (3.17). $\eta^0_0 = \eta^0_\tau = \tau^2$ with the corresponding eigenfunction

$$u_0(x) = \sqrt{2}\,(\tau\cos(\tau x) + \sigma_\tau \sin(\tau x)) \tag{3.21}$$

and associated function

$$u_\tau(x) = \frac{1}{\sqrt{2}}\left(-\frac{\sigma_\tau}{\tau}(x + \chi_1)\cos(\tau x) + (x + \chi_2)\sin(\tau x)\right) \tag{3.22}$$

where χ_1 and χ_2 are constants satisfying

$$\sigma_\tau \chi_1 - \sigma_{\tau+1}\chi_2 = 1 + \frac{\pi}{2}(\sigma_{\tau+1} - \sigma_\tau). \tag{3.23}$$

Indeed, one can check that with this choice of χ_1 and χ_2, u_τ satisfies the (σ_0, σ_1)-boundary conditions and

$$H^0_{\sigma_0,\sigma_1} u_\tau = \tau^2 u_\tau - u_0. \tag{3.24}$$

The root function system $\{u_n\}_{n=0}^{\infty}$ of $H^0_{\sigma_0,\sigma_1}$ form a bi-orthogonal system together with the root function system $\{\tilde{u}_n\}_{n=0}^{\infty}$ of $(H^0_{\sigma_0,\sigma_1})^*$. However while denoting the eigen and associated function of $(H^0_{\sigma_0,\sigma_1})^*$ corresponding to τ^2 it is better to exchange the roles of $\tilde{u}_0$ and $\tilde{u}_\tau$. Let us choose $\tilde{u}_n$, $n \in \mathbb{N}\backslash\{\tau\}$, as in (3.19) and take

$$\tilde{u}_\tau(x) = -\sqrt{2}\,(\tau\cos(\tau x) + \overline{\sigma}_{\tau+1}\sin(\tau x)) \tag{3.25}$$

and

$$\tilde{u}_0(x) = -\frac{1}{\sqrt{2}}\left(-\frac{\overline{\sigma}_{\tau+1}}{\tau}(x + \widetilde{\chi}_1)\cos(\tau x) + (x + \widetilde{\chi}_2)\sin(\tau x)\right) \tag{3.26}$$

where $\widetilde{\chi}_1$ and $\widetilde{\chi}_2$ satisfy

$$\overline{\sigma}_{\tau+1}\widetilde{\chi}_1 - \overline{\sigma}_\tau\widetilde{\chi}_2 = 1 + \frac{\pi}{2}(\overline{\sigma}_\tau - \overline{\sigma}_{\tau+1}). \tag{3.27}$$

Then one can check that $\{u_n\}_{n=0}^\infty$ and $\{\tilde{u}_n\}_{n=0}^\infty$ form a bi-orthogonal system with a proper choice of $\widetilde{\chi}_1$ and $\widetilde{\chi}_2$. Indeed if $\langle u_\tau, \tilde{u}_0\rangle \neq 0$ one can replace $\tilde{u}_0$ by $\tilde{u}_0 - \overline{\langle u_\tau, \tilde{u}_0\rangle}\tilde{u}_\tau$.

After having that $\{u_n\}_{n=0}^\infty$ and $\{\tilde{u}_n\}_{n=0}^\infty$ is a bi-orthogonal system, the same argument in *Case I* applies to *Case II* to show that $\{u_n\}_{n=0}^\infty$ forms a Riesz basis.

3.2 Localization of the Spectra of H_{σ_0,σ_1}

Let V denote the operator of multiplication by v, i.e., $(Vf)(x) = v(x)f(x)$. Then $H_{\sigma_0,\sigma_1} = H^0_{\sigma_0,\sigma_1} + V$ and we may use the perturbation formula (2.31). Note that the matrix representation of R^0_λ in *Case I* is

$$(R^0_\lambda)_{kj} = \frac{\delta_{kj}}{\lambda - \eta^0_k} \tag{3.28}$$

and in *Case II* it has the form

$$(R^0_\lambda)_{kj} = \frac{\delta_{kj}}{\lambda - \eta^0_k} - \frac{\delta_{0k}\delta_{\tau j}}{(\lambda - \eta^0_0)^2}. \tag{3.29}$$

Hence we can define a square root K_λ of R^0_λ by choosing its matrix representation as

$$(K_\lambda)_{kj} = \frac{\delta_{kj}}{(\lambda - \eta^0_k)^{1/2}} \tag{3.30}$$

in *Case I* and as

$$(K_\lambda)_{kj} = \frac{\delta_{kj}}{(\lambda - \eta^0_k)^{1/2}} - \frac{\delta_{0k}\delta_{\tau j}}{2(\lambda - \eta^0_0)^{3/2}} \tag{3.31}$$

in *Case II*, where $z^{1/2} = |z|^{1/2}e^{i\theta/2}$ for $z = |z|e^{i\theta}$, $\theta \in [0, 2\pi)$.

For (σ_0, σ_1)-boundary conditions, using the same argument as in [20] one can similarly localize and count the spectrum $Sp(H_{\sigma_0,\sigma_1})$ after showing that (2.32) holds for $\lambda \notin R_N \cup \left\{\bigcup_{n>N, n\in\mathbb{N}} D_n\right\}$ (Recall the definitions (2.35)-(2.39)). However in the case where $v \in L^1([0,\pi])$ we estimate $\|K_\lambda V K_\lambda\|$ explicitly for $(\sigma_0, \sigma_1) - bc$ since we need this estimate later.

Proposition 3.2. *If $v \in L^1([0,\pi])$, there exist $C = C(v)$ and $N > 0$, $N \in \mathbb{N}$ such that*

$$\|K_\lambda V K_\lambda\| \le C\frac{\log N}{\sqrt{N}} \quad \text{for} \quad \lambda \in H^N \backslash R_N. \tag{3.32}$$

Moreover for all $n > N$, $n \in \mathbb{N}$,

$$\|K_\lambda V K_\lambda\| \le C\frac{\log n}{n} \quad \text{for} \quad \lambda \in G_n \backslash D_n. \tag{3.33}$$

Proof. It is a well known fact that if we have Riesz basis then there exists a positive constant C_0 satisfying

$$\frac{1}{C_0}\|f\|^2 \le \sum_j |f_j|^2 \le C_0\|f\|^2, \tag{3.34}$$

for any $f \in L^2([0,\pi])$. Moreover for any operator A, $\|A\| \le C_0\|A\|_{HS}$ where

$$\|A\|_{HS} = \left(\sum_{k,j\in\in\mathbb{Z}^*} |A_{jk}|^2\right)^{1/2} \tag{3.35}$$

is the modified Hilbert-Schmidt norm of A. So instead of estimating $\|K_\lambda V K_\lambda\|$ directly, we will estimate the modified Hilbert-Schmidt norm of $K_\lambda V K_\lambda$. Note that the matrix representation of $K_\lambda V K_\lambda$ is

$$(K_\lambda V K_\lambda)_{kj} = \frac{V_{kj}}{(\lambda - \eta_k^0)^{1/2}(\lambda - \eta_j^0)^{1/2}} \tag{3.36}$$

in *Case I* and

$$(K_\lambda V K_\lambda)_{kj} = \frac{1}{(\lambda - \eta_k)^{1/2}}\left(\frac{V_{kj}}{(\lambda - \eta_j^0)^{1/2}} - \frac{V_{k0}\delta_{\tau j}}{2(\lambda - \eta_0^0)^{3/2}}\right) \tag{3.37}$$

$$-\frac{\delta_{0k}}{2(\lambda - \eta_0^0)^{3/2}}\left(\frac{V_{\tau j}}{(\lambda - \eta_j^0)^{1/2}} - \frac{V_{\tau 0}\delta_{\tau j}}{2(\lambda - \eta_0^0)^{3/2}}\right)$$

in *Case II*. Hence in both cases we have

$$\sum_{k,j\in\mathbb{Z}^*} |(K_\lambda V K_\lambda)_{kj}|^2 \le 8 \sup_{k,j\in\mathbb{Z}^*} |V_{kj}|^2 \left(\left(\sum_{k\in\mathbb{Z}^*} \frac{1}{|\lambda - \eta_k^0|}\right)^2 + \frac{1}{16|\lambda - \eta_0^0|^6}\right) \tag{3.38}$$

Note that

$$\sum_{k\in\mathbb{Z}^*} \frac{1}{|\lambda - \eta_k^0|} = \frac{1}{|\lambda + \sigma_0\sigma_1|} + \sum_{k\in\mathbb{N}} \frac{1}{|\lambda - k^2|}. \tag{3.39}$$

By (5.27) and (5.28) in [20] there exists an integer $N > 0$ and an absolute constant $C_1 > 0$ such that

$$\sum_{k\in\mathbb{N}} \frac{1}{|\lambda - k^2|} \le C_1 \frac{\log n}{n} \quad \text{for} \quad \lambda \in G_n \setminus D_n, \quad n > N. \tag{3.40}$$

On the other hand, if $\lambda = x + it \in H^N \backslash R_N$, with $n^2 - n \le x < n^2 + n$ in the case when $x \ge 0$, then one can see that

$$|\lambda - k^2| \ge \begin{cases} (k^2 + N)/\sqrt{2} & \text{if} \quad x \le 0, \\ (|n^2 - k^2| + 2N)/2\sqrt{2} & \text{if} \quad x > 0, \quad k \ne \pm n, \\ N & \text{if} \quad x > 0, \quad k = \pm n. \end{cases} \tag{3.41}$$

By Lemma 79 in [16], for large enough N we also have the inequality

$$\sum_k \frac{1}{|n^2 - k^2| + N} \le C_2 \frac{\log N}{\sqrt{N}}, \tag{3.42}$$

where C_2 is an absolute constant. Hence

$$\sum_{k\in\mathbb{N}} \frac{1}{|\lambda - k^2|} \le C_2 \frac{\log N}{\sqrt{N}} \quad \text{for} \quad \lambda \in H^N \backslash R_N. \tag{3.43}$$

Let us choose N large enough such that $N \geq 2|\sigma_0\sigma_1|$. Then it is easy to see that we also have

$$\frac{1}{|\lambda + \sigma_0\sigma_1|} \leq \begin{cases} \frac{2}{n^2} & \text{for} \quad \lambda \in G_n \setminus D_n, \quad n > N, \\ \frac{2}{N} & \text{for} \quad \lambda \in H^N \backslash R_N. \end{cases} \tag{3.44}$$

Hence we can write

$$\sum_{k \in \mathbb{Z}^*} \frac{1}{|\lambda - \eta_k|} \leq \begin{cases} C_3 \frac{\log n}{n} & \text{for} \quad \lambda \in G_n \setminus D_n, \quad n > N, \\ C_3 \frac{\log N}{\sqrt{N}} & \text{for} \quad \lambda \in H^N \backslash R_N. \end{cases} \tag{3.45}$$

Now (3.38), (3.44) and (3.45) imply that

$$\left(\sum_{k,j \in \mathbb{Z}^*} |(K_\lambda V K_\lambda)_{kj}|^2 \right)^{1/2} \leq 3C_3 \sup_{k,j \in \mathbb{Z}^+} |V_{kj}| \frac{\log n}{n} \quad \text{for} \quad \lambda \in G_n \setminus D_n, \quad n > N \tag{3.46}$$

and

$$\left(\sum_{k,j \in \mathbb{Z}^*} |(K_\lambda V K_\lambda)_{kj}|^2 \right)^{1/2} \leq 3C_3 \sup_{k,j \in \mathbb{Z}^+} |V_{kj}| \frac{\log N}{\sqrt{N}} \quad \text{for} \quad \lambda \in H^N \backslash R_N. \tag{3.47}$$

We complete the proof noting that since

$$V_{kj} = \frac{1}{\pi} \int_0^\pi v(x) u_j(x) \overline{\tilde{u}_k(x)} dx, \tag{3.48}$$

we have

$$\sup_{k,j \in \mathbb{Z}^*} |V_{kj}| \leq C_4 \|v\|_1 \tag{3.49}$$

where $C_4 = \sup_{k,j \in \mathbb{Z}^*} \|u_j\|_\infty \|\tilde{u}_k\|_\infty$ which is finite since u_n's and $\tilde{u}_n$'s are uniformly bounded in both cases by construction. □

Proposition 3.3. *For any potential $v \in L^2([0,\pi])$, the spectrum of the operator $H_{\sigma_0,\sigma_1}(v)$ is discrete. Moreover there exists an integer N such that*

$$Sp(H_{\sigma_0,\sigma_1}) \subset R_N \cup \bigcup_{n>N, n \in \mathbb{N}} D_n, \tag{3.50}$$

and

$$\sharp(Sp(H_{\sigma_0,\sigma_1}) \cap R_N) = N + 1, \quad \sharp(Sp(H_{\sigma_0,\sigma_1}) \cap D_n) = 1, \quad n > N, n \in \mathbb{N}. \tag{3.51}$$

Proof. Apply the proof of Theorem 21 in [20] but use Proposition 3.2 instead of Lemmas 19 and 20 in [20]. □

3.3 Proof of Theorem 3.4

In this section, we give a proof of the following theorem.

Theorem 3.4. *If $v \in L^2([0,\pi])$ then, for sufficiently large n,*

$$\frac{1}{80}(|\beta_n^+(z_n^*)| + |\beta_n^-(z_n^*)|) \leq |\gamma_n| + |\delta_n| \leq 19(|\beta_n^+(z_n^*)| + |\beta_n^-(z_n^*)|). \tag{3.52}$$

Proposition 3.5. *Let $(P_{\sigma_0,\sigma_1})_n$ and $(P^0_{\sigma_0,\sigma_1})_n$ be defined by (2.47) where R_λ and R^0_λ are the corresponding resolvents of H_{σ_0,σ_1} and $H^0_{\sigma_0,\sigma_1}$, respectively. If $v \in L^2([0,\pi])$ then we have, for large enough n,*

$$\|(P_{\sigma_0,\sigma_1})_n - (P^0_{\sigma_0,\sigma_1})_n\| \leq M\frac{\log n}{n} \tag{3.53}$$

where $M = M(v)$.

Proof. In view of (2.47),

$$\|(P_{\sigma_0,\sigma_1})_n - (P^0_{\sigma_0,\sigma_1})_n\| \leq \frac{1}{2\pi}\int_{\partial D_n} \|R_\lambda - R^0_\lambda\| d|\lambda| \leq n \sup_{\lambda\in\partial D_n} \|R_\lambda - R^0_\lambda\|. \tag{3.54}$$

By (2.31) we have

$$\|R_\lambda - R^0_\lambda\| \leq \|K_\lambda\|^2 \sum_{m=1}^{\infty} \|K_\lambda V K_\lambda\|^m \leq 2\|K_\lambda\|^2 \|K_\lambda V K_\lambda\| \tag{3.55}$$

since for $\lambda \in \partial D_n$, $\|K_\lambda V K_\lambda\| \leq 1/2$ for sufficiently large n's by Proposition 3.2.

To estimate $\|K_\lambda\|$ note that in *Case I*

$$K_\lambda f = \sum_{k\in\mathbb{Z}^*} \frac{f_k}{(\lambda - \eta_k^0)^{1/2}} u_k \tag{3.56}$$

and in *Case II*

$$K_\lambda f = \sum_{k\in\mathbb{Z}^*} \frac{f_k}{(\lambda - \eta_k^0)^{1/2}} u_k - \frac{f_\tau}{2(\lambda - \eta_0^0)^{3/2}} u_0. \tag{3.57}$$

So in both cases we have

$$\|K_\lambda f\|^2 \leq 2C_1^2 \left(\sup_{k\in\mathbb{Z}^*} \frac{1}{|\lambda - \eta_k^0|} + \frac{\|u_0\|^2}{4|\lambda - \eta_0^0|^3} \right) \|f\|^2 \tag{3.58}$$

where C_0 is a constant satisfying (3.34). Hence for $\lambda \in \partial D_n$, $n \geq N \geq 2|\sigma_0\sigma_1|$, we have

$$\|K_\lambda\| \leq C_5 \sup_{k\in\mathbb{Z}^*} \frac{1}{|\lambda - \eta_k^0|^{1/2}} = \frac{2C_5}{\sqrt{n}} \tag{3.59}$$

for some constant C_5. Combining (3.33), (3.55) and (3.59) we get

$$\|R_\lambda - R^0_\lambda\| \leq 8CC_5^2 \frac{\log n}{n^2} \tag{3.60}$$

for sufficiently large n's. Finally, (3.54) and (3.60) imply (3.53), which completes the proof. □

In the following, for simplicity, we suppress n in all symbols containing n. Let $\tilde{P}_{\sigma_0,\sigma_1}$ and $\tilde{P}^0_{\sigma_0,\sigma_1}$ denote the Cauchy-Riesz projection associated with $H^*_{\sigma_0,\sigma_1}$ and ${H^0_{\sigma_0,\sigma_1}}^*$, respectively, i.e., $\tilde{P}_{\sigma_0,\sigma_1} = P_{\bar{\sigma}_1,\bar{\sigma}_0}(\bar{v})$ and $\tilde{P}^0_{\sigma_0,\sigma_1} = P_{\bar{\sigma}_1,\bar{\sigma}_0}(0)$. Moreover let $\tilde{\mathcal{C}}_{\sigma_0,\sigma_1} = Ran\tilde{P}_{\sigma_0,\sigma_1}$ and $\tilde{\mathcal{C}}^0_{\sigma_0,\sigma_1} = Ran\tilde{P}^0_{\sigma_0,\sigma_1}$ be the corresponding one dimensional invariant subspace of $H^*_{\sigma_0,\sigma_1}$ and $(H^0_{\sigma_0,\sigma_1})^*$, respectively.

Lemma 3.6. *Let f, φ be an orthonormal basis in $\mathcal{E}$ such that (2.53) and (2.54) hold. Then there is a unit vector $G = af + b\varphi$ in $\mathcal{E}$ satisfying*

$$\begin{aligned} G'(0) + G'(\pi) - \sigma_0(G(0) + G(\pi)) = 0 \\ G'(0) - G'(\pi) - \sigma_1(G(0) - G(\pi)) = 0 \end{aligned} \tag{3.61}$$

and there is a unit vector $\tilde{g} \in \tilde{\mathcal{C}}_{\sigma_0,\sigma_1}$ satisfying

$$\langle G, \tilde{g}\rangle\delta = b\langle\varphi, \tilde{g}\rangle\gamma - b\langle f, \tilde{g}\rangle\xi \tag{3.62}$$

such that $\langle G, \tilde{g}\rangle \in \mathbb{R}$ and

$$\langle G, \tilde{g}\rangle \geq \frac{71}{72} \tag{3.63}$$

for sufficiently large n.

(Remark. (3.61) means that G is in the domain of H_{σ_0,σ_1}.)

Proof. Choose $\tilde{G}$ as $\tilde{G} = f$ if $f'(0) - \sigma f(0) = 0$ and as

$$\tilde{G}(x) = -(\varphi'(0) - \sigma\varphi(0))f(x) + (f'(0) - \sigma f(0))\varphi(x)$$

otherwise. Then one can check that $G = \tilde{G}/\|\tilde{G}\|$ satisfies (3.61).

By (3.61), $G \in Dom(L) \cap Dom(H_{\sigma_0,\sigma_1})$, so we have $H_{\sigma_0,\sigma_1}G = LG$. Hence it follows

$$\begin{aligned} H_{\sigma_0,\sigma_1}G &= aLf + bL\varphi = a\lambda^+ f + b(\lambda^+\varphi - \gamma\,\varphi + \xi f) \\ &= \lambda^+(af + b\varphi) + b(\xi f - \gamma\varphi) = \lambda^+ G + b(\xi f - \gamma\varphi). \end{aligned} \tag{3.64}$$

Fix a unit vector $\tilde{g} \in \tilde{\mathcal{C}}$ so that

$$\langle G, \tilde{g}\rangle = |\langle G, \tilde{g}\rangle|, \tag{3.65}$$

$$H^*_{\sigma_0,\sigma_1}\tilde{g} = \bar{\eta}\tilde{g}. \tag{3.66}$$

Taking inner product of both sides of (3.64) with $\tilde{g}$ we get

$$\langle H_{\sigma_0,\sigma_1}G, \tilde{g}\rangle = \lambda^+\langle G, \tilde{g}\rangle + b(\xi\langle f, \tilde{g}\rangle - \gamma\langle\varphi, \tilde{g}\rangle). \tag{3.67}$$

On the other hand, we have

$$\langle H_{\sigma_0,\sigma_1}G, \tilde{g}\rangle = \langle G, H^*_{\sigma_0,\sigma_1}\tilde{g}\rangle = \eta\langle G, \tilde{g}\rangle. \tag{3.68}$$

Now (3.67) and (3.68) imply (3.62).

Let $G^0 = P^0 G$ and $\tilde{g}^0 = \tilde{P}^0_{\sigma_0,\sigma_1}\tilde{g}$. We write

$$\langle G, \tilde{g}\rangle = \langle G^0, \tilde{g}^0\rangle + \langle G^0, \tilde{g} - \tilde{g}^0\rangle + \langle G - G^0, \tilde{g}\rangle,$$

Note that $\tilde{g}$ is a unit vector and also $\|G^0\| \le 1$ since P^0 is an orthogonal projection and G is a unit vector. Hence by the triangle and Cauchy inequalities we get

$$|\langle G, \tilde{g}\rangle| \ge |\langle G^0, \tilde{g}^0\rangle| - \|\tilde{g} - \tilde{g}^0\| - \|G - G^0\|. \tag{3.69}$$

By Proposition 2.7 we have

$$\|G - G^0\| = \|(P - P^0)G\| \le \|P - P^0\| \le \varepsilon_n \tag{3.70}$$

and by Proposition 3.5

$$\|\tilde{g} - \tilde{g}^0\| = \|(\tilde{P}_{\sigma_0,\sigma_1} - \tilde{P}^0_{\sigma_0,\sigma_1})\tilde{g}\| \le \|\tilde{P}_{\sigma_0,\sigma_1} - \tilde{P}^0_{\sigma_0,\sigma_1}\| \le M\frac{\log n}{n}. \tag{3.71}$$

Hence, (3.69) implies

$$|\langle G, \tilde{g}\rangle| \ge |\langle G^0, \tilde{g}^0\rangle| - \varepsilon_n - M\frac{\log n}{n}. \tag{3.72}$$

Since $\tilde{\mathcal{C}}^0$ is spanned by $\tilde{u}_n$, $\tilde{g}^0$ is in the form of

$$\tilde{g}^0 = e^{i\theta}\|\tilde{g}^0\|\frac{\sqrt{2}n}{\sqrt{n^2 + |\sigma_{n+1}|^2}}\left(\cos(nx) + \frac{\overline{\sigma}_{n+1}}{n}\sin(nx)\right). \tag{3.73}$$

On the other hand since $G \in \mathcal{E}^0$ it has the form

$$G^0(x) = G^0_1\sqrt{2}\cos(nx) + G^0_2\sqrt{2}\sin(nx) \tag{3.74}$$

Thus

$$\langle G^0, \tilde{g}^0\rangle = e^{-i\theta}\|\tilde{g}^0\|\frac{n}{\sqrt{n^2 + |\sigma_{n+1}|^2}}\cdot\left(G^0_1 + \frac{\sigma_{n+1}G^0_2}{n}\right) \tag{3.75}$$

So

$$|\langle G^0, \tilde{g}^0\rangle| \ge \frac{n}{\sqrt{n^2 + |\sigma_{n+1}|^2}}\|\tilde{g}^0\|\left(|G^0_1| - \frac{|\sigma_{n+1}||G^0_2|}{n}\right) \tag{3.76}$$

By (3.61) we see that $G'(0) = \sigma_0 G(0)$ if G is periodic (if n is even) and $G'(0) = \sigma_1 G(0)$ if G is anti-periodic (if n is odd). Since $\sigma_n = \sigma_0$ if n is even and $\sigma_n = \sigma_1$ if n is odd suppressing the index n we can write

$$G'(0) = \sigma G(0). \tag{3.77}$$

On the other hand by (3.74)

$$G^{0\prime}(0) = \sqrt{2}nG^0_2 \quad \text{and} \quad G^0(0) = \sqrt{2}G^0_1. \tag{3.78}$$

Combining Proposition 2.9, Corollary 2.10, (3.77), and (3.78) we see that

$$\begin{aligned} |nG_2^0 - \sigma G_1^0| &= \frac{1}{\sqrt{2}} |G^{0\prime}(0) - \sigma G^0(0)| \\ &\le \frac{1}{\sqrt{2}} \Big(|G'(0) - G^{0\prime}(0)| + |\sigma G(0) - \sigma G^0(0)| \Big) \\ &\le \frac{1}{\sqrt{2}} (n + |\sigma|)\, \kappa_n \le (n + |\sigma|)\, \kappa_n \end{aligned} \tag{3.79}$$

which implies

$$|G_2^0| \le \frac{1}{n}(|nG_2^0 - \sigma G_1^0| + |\sigma G_1^0|) \le \left(1 + \frac{|\sigma|}{n}\right) \kappa_n + \frac{|\sigma|}{n} \le 2\kappa_n + \frac{|\sigma|}{n}. \tag{3.80}$$

On the other hand, by (3.70),

$$\sqrt{|G_1^0|^2 + |G_2^0|^2} = \|G^0\| \ge \|G^0\| - \|G^0 - G\| \ge 1 - \varepsilon_n \tag{3.81}$$

which implies

$$|G_1^0|^2 + |G_2^0|^2 \ge 1 - 2\varepsilon_n. \tag{3.82}$$

Hence

$$|G_1^0|^2 = |G_1^0|^2 + |G_2^0|^2 - |G_2^0|^2 \ge 1 - 2\varepsilon_n - \left(2\kappa_n + \frac{|\sigma|}{n}\right)^2 \tag{3.83}$$

which also implies

$$|G_1^0| \ge 1 - 2\varepsilon_n - \left(2\kappa_n + \frac{|\sigma|}{n}\right)^2 \tag{3.84}$$

since $|G_1^0| \le 1$. Note also that by (3.71) and the fact that $\tilde{g}$ is unit we have

$$\|\tilde{g}^0\| \ge \|\tilde{g}\| - \|\tilde{g} - \tilde{g}^0\| \ge 1 - M\frac{\log n}{n}. \tag{3.85}$$

Now combining (3.76), (3.84), (3.85) and the fact that $|G_2^0| \le 1$ we obtain

$$\begin{aligned} |\langle G^0, \tilde{g}^0\rangle| \ge \frac{n}{\sqrt{n^2 + |\sigma_{n+1}|^2}} \left(1 - \frac{M \log n}{n}\right) \times \\ \left(1 - 2\varepsilon_n - \left(2\kappa_n + \frac{|\sigma_n|}{n}\right)^2 - \frac{|\sigma_{n+1}|}{n}\right). \end{aligned} \tag{3.86}$$

Since the right hand side tends to 1 as n goes to infinity, together with (3.65) and (3.72), we obtain

$$\langle G, \tilde{g}\rangle \ge \frac{71}{72} \tag{3.87}$$

for sufficiently large n. □

Corollary 3.7. *For sufficiently large n, we have*

$$|\gamma_n| + |\delta_n| \le 19\big(|\beta_n^+(z_n^*)| + |\beta_n^-(z_n^*)|\big). \tag{3.88}$$

Proof. Using (3.62), (3.63) and noting also that the absolute values of b and all inner products in the right-hand side of (3.62) do not exceed 1 we get $|\delta| \le 72/71\big(|\xi| + |\gamma|\big)$. This inequality, together with Lemma 2.8, implies (3.88). □

Corollary 3.7 proves the second inequality in (3.52). In order to complete the proof of Theorem 3.4 it remains to prove the first inequality in (3.52).

Lemma 3.8. *If* Case 2(a) *or* Case 2(b) *holds, then we have, for sufficiently large n,*

$$\frac{1}{4} \le \frac{|f'(0) - \sigma f(0)|}{|\varphi'(0) - \sigma\varphi(0)|} \le 4. \tag{3.89}$$

Proof. First recall the definitions (2.101) and (2.102) of *Case 2(a)* and *Case 2(b)* and note that in these cases (2.107) holds. Moreover

$$|f^0(0)| \le \sqrt{2} \quad \text{and} \quad |\varphi^0(0)| \le \sqrt{2} \tag{3.90}$$

Indeed $f^0(0) = f_1^0 + f_2^0$ and $|f_1^0| + |f_2^0| \le \sqrt{2}\|f^0\| \le \sqrt{2}$ since P^0 is an orthogonal projection and f is unit. Thus in view of Proposition 2.9, Corollary 2.10, (2.107) and (3.90) we get

$$\begin{aligned} |f'(0) - \sigma f(0)| &\le |f^{0\prime}(0)| + |f'(0) - f^{0\prime}(0)| + |\sigma| \left(|f^0(0)| + |f(0) - f^0(0)|\right) \\ &\le \sqrt{2}n + n\kappa_n + |\sigma| \left(\sqrt{2} + \kappa_n\right) = \sqrt{2}n \left(1 + O\left(\frac{1}{n}\right)\right) \end{aligned} \tag{3.91}$$

and

$$\begin{aligned} |\varphi'(0) - \sigma\varphi(0)| &\ge |\varphi^{0\prime}(0)| - \left(|\varphi'(0) - \varphi^{0\prime}(0)| + |\sigma| \left(|\varphi^0(0)| + |\varphi(0) - \varphi^0(0)|\right)\right) \\ &\ge \frac{n}{\sqrt{6}} - \left(n\kappa_n + |\sigma| \left(\sqrt{2} + \kappa_n\right)\right) = \frac{n}{\sqrt{6}} \left(1 + O\left(\frac{1}{n}\right)\right). \end{aligned} \tag{3.92}$$

Hence

$$\frac{|f'(0) - \sigma f(0)|}{|\varphi'(0) - \sigma\varphi(0)|} \le 4 \tag{3.93}$$

and similarly

$$\frac{|f'(0) - \sigma f(0)|}{|\varphi'(0) - \sigma\varphi(0)|} \ge \frac{1}{4}. \tag{3.94}$$

□

Proposition 3.9. *For sufficiently large n, we have*

$$\left(|\beta_n^+(z_n^*)| + |\beta_n^-(z_n^*)|\right) \le 80(|\gamma_n| + |\delta_n|) \tag{3.95}$$

Proof. The proof is the same as the proof of Proposition 2.14. The only difference is that in this case $|b/a| = |f'(0) - \sigma f(0)|/|\varphi'(0) - \sigma\varphi(0)|$, so one needs to use Lemma 3.8 instead of Lemma 2.13 in the corresponding place. □

Corollary 3.7 and Proposition 3.9 show that (3.52) holds, so Theorem 3.4 is proved.

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