

# Multilevel Threshold Secret and Function Sharing based on the Chinese Remainder Theorem

Oğuzhan Ersoy, Kamer Kaya and Kerem Kaşkaloğlu

**Abstract**—A recent work of Harn and Fuyou presents the first multilevel (disjunctive) threshold secret sharing scheme based on the Chinese Remainder Theorem. In this work, we first show that the proposed method is not secure and also fails to work with a certain natural setting of the threshold values on compartments. We then propose a secure scheme that works for all threshold settings. In this scheme, we employ a refined version of Asmuth-Bloom secret sharing with a special and generic Asmuth-Bloom sequence called the *anchor sequence*. Based on this idea, we also propose the first multilevel conjunctive threshold secret sharing scheme based on the Chinese Remainder Theorem. Lastly, we discuss how the proposed schemes can be used for multilevel threshold function sharing by employing it in a threshold RSA cryptosystem as an example.

**Index Terms**—Secret sharing, multilevel function sharing, multilevel threshold cryptography, Chinese Remainder Theorem.

## I. INTRODUCTION

The concept of secret sharing is being used in many cryptographic protocols. As independently proposed by Shamir [1] and Blakley [2], a *secret-sharing scheme* (SSS) involves a *dealer* who has a *secret*  $s$ , a set of *participants*  $\mathcal{U}$  that the secret is shared amongst, and a collection  $\mathcal{A}$  of the authorized subsets of the  $\mathcal{U}$  which is called the *access structure*. In a SSS, the dealer distributes the shares to the participants such that only the subsets in  $\mathcal{A}$  can reconstruct the secret from the corresponding shares. Furthermore, a SSS is called *perfect* if all the subsets not in  $\mathcal{A}$  will have the same probability of guessing the secret as if they had no shares. We refer the reader to a comprehensive survey [3] for practical applications of secret sharing such as building authentication protocols which stay secure even under the leakage of a number of servers' data.

In *threshold secret sharing*, the access structure is defined by a threshold on the cardinality of authorized subsets: a  $(t, n)$ -SSS refers to a scheme in which any  $t$  out of  $n$  participants can recover the secret. Apart from Shamir's Lagrange interpolation-based scheme [1] and Blakley's scheme utilizing the idea that any  $n$  nonparallel  $(n - 1)$ -dimensional hyperplanes intersect at a specific point [2], Chinese Remainder Theorem (CRT)-based threshold schemes by Mignotte [4] and Asmuth and Bloom [5] also exist. While Mignotte's  $(t, n)$  scheme is not perfect in the sense that less than  $t$  shares reveal information about the secret, Asmuth-Bloom's scheme attains a better security level with a careful choice of parameters. We refer the reader to [6] for an extensive study on the security of CRT based SSSs.

Given the universal participant set  $\mathcal{U}$ , a partition of  $\mathcal{U}$  into disjoint subsets, i.e., *compartments*, is used to define a

*multipartite access structure* on  $\mathcal{U}$ . Unlike traditional threshold secret sharing that has only one threshold, different thresholds and conditions may be imposed for different compartments. On the other hand, multipartite schemes do not distinguish the members of the same compartment.

Although there exist methods for general access structures, e.g., [7], [8], [9], the schemes designed for specific access structures are almost always more efficient and hence, more practical. Such an access structure which has applications in practice is the *multilevel/hierarchical access structure*, a special form of the multipartite case, that employs a hierarchy between the compartments where the members of a superior compartment are more powerful and can replace the participants of an inferior one following the hierarchy definition of Simmons [10] that is further studied in [11]. Simmons gave the following example: assume that a bank transfer requires authorization and any two vice presidents or any three senior tellers are authorized to approve. In this example, there are two compartments (vice presidents and senior tellers) where the members of the former can also replace members of the latter. That is, a vice president together with two senior tellers are able to approve the transfer as well. In a recent work of Harn and Fuyou [12], a multilevel CRT-based SSS is proposed for an access structure involving a hierarchy of compartments as in the definition of Simmons.

The above mentioned access structure is *disjunctive*; if a coalition satisfies any of the threshold conditions of the compartments (that is, the presence of either two vice presidents or three senior tellers), then it is in the access structure. A more restricted *conjunctive* form where all the compartments' thresholds need to be satisfied by a valid coalition can also be employed in practice. For example, suppose a bank transfer now requires the authorization of any two vice presidents **and** any three senior tellers for the example above. Note that different from the disjunctive scheme, a coalition needs to satisfy all the thresholds in the conjunctive form. Hence, with this requirement, a vice president and two senior tellers (or only three senior tellers) cannot authorize a transfer as they could in the disjunctive case. Although a CRT-based conjunctive threshold SSS has been proposed by Iftene et al. [13], to the best of our knowledge, there is no hierarchical/multilevel conjunctive secret sharing scheme based on CRT in the literature.

There are multiple contributions of this paper: First, we show that the Harn-Fuyou scheme cannot be applied (i.e., is not well-defined) for all the access structures  $\mathcal{A}$  in the multilevel setting, and furthermore it is not secure, i.e., the secret can be reconstructed by an unauthorized coalition that is not in  $\mathcal{A}$ . Second, by using an *anchor Asmuth-Bloom sequence*,

we propose a more naive and novel CRT-based SSS for the multilevel access structures which does not suffer from these drawbacks. And third, based on similar techniques, we propose the first multilevel conjunctive threshold SSS based on CRT.

In addition to the main contributions, we also discuss how the proposed schemes can be employed for *multilevel function sharing*; a natural extension of secret sharing. A plain SSS is inappropriate for public key cryptography, since when a (shared) secret is reconstructed, it is known by all the participants and cannot be used again. A *function sharing scheme* (FSS) employs a SSS to share the keys/secrets so that operations such as decryption or signing can only be performed by a valid coalition (in  $\mathcal{A}$ ) and without revealing the secret. As usual, a coalition that is not in  $\mathcal{A}$  cannot perform such operations and cannot obtain any information on the secret. Function sharing schemes enable many applications in practice such as the fair sale of digital content in exchange for digital receipts, secure bidding, and secret election protocols. There are numerous studies on function sharing; the work of Shoup [14] can be considered as one of the milestones of the field, proposing a provably secure, non-interactive FSS joining RSA and Shamir's SSS. The first CRT based function sharing schemes for RSA, ElGamal and Paillier cryptosystems are given in [15].

After covering some preliminary definitions and schemes in Section II, we point out some shortcomings and the insecurity of the Harn-Fuyou scheme in Section III. We present our conjunctive and disjunctive multilevel secret sharing schemes in Section IV, and in Section V, we discuss and show how the proposed schemes can be adopted for function sharing by using RSA signature/decryption [16] as an example. Section VI concludes the paper.

## II. BACKGROUND AND PRELIMINARIES

Given the following system of congruences

$$\begin{aligned} x &= s_1 \bmod p_1 \\ x &= s_2 \bmod p_2 \\ &\vdots \\ x &= s_n \bmod p_n, \end{aligned}$$

the Chinese Remainder Theorem states that there is a unique solution  $x \in \mathbb{Z}_P$  such that

$$x = \sum_{i=1}^n \frac{P}{p_i} I_i s_i \bmod P,$$

where  $P = \text{lcm}(p_1, p_2, \dots, p_n)$  and  $I_i$  is the inverse of  $P/p_i$  in modulo  $p_i$ , i.e.,  $\frac{P}{p_i} I_i \bmod p_i = 1$ . Thus when the  $p_i$  values are chosen pairwise coprime (or all prime)  $P$  becomes  $p_1 p_2 \dots p_n$ .

### A. Mignotte's secret sharing

Mignotte's SSS is a direct application of CRT with one specification: With  $n$  participants and a threshold  $t \leq n$ , given the sequence of pairwise coprime positive integers (or primes)  $p_1 < p_2 < \dots < p_n$ , the secret  $s$  is chosen from

the interval  $(p_{n-t+2} p_{n-t+3} \dots p_n, p_1 p_2 \dots p_t)$ . The share of each participant  $u_i$  is  $s_i = s \bmod p_i$ . Since  $s$  is greater than the product of the greatest  $t-1$  primes, a set of  $t-1$  participants cannot (uniquely) reconstruct the secret. On the other hand,  $t$  or more participants can reconstruct  $s$  since it is smaller than the product of the smallest  $t$  primes. As all the parameters except the private shares  $s_i$  are public, the secret reconstruction is a straightforward application of CRT. It is important to notice that the Mignotte  $(t, n)$ -threshold secret-sharing scheme is not perfect in the sense that a set of less than  $t$  shares reveals some information about the secret.

### B. Asmuth-Bloom's secret sharing

Let  $p_0$  be a prime which defines the secret space and  $s \in \mathbb{Z}_{p_0}$  be the secret. Let  $M$  be  $\prod_{i=1}^t p_i$ , and  $p_0 < p_1 < p_2 < \dots < p_n$  be an increasing sequence of primes such that

$$p_0 \prod_{i=1}^{t-1} p_{n-i+1} < M. \quad (1)$$

To share the secret, the dealer first chooses a random positive integer  $\alpha$  such that  $0 \leq y = s + \alpha p_0 < M$ . The share of the participant  $u_i$  is equal to  $s_i = y \bmod p_i$ . Let  $A \in \mathcal{A}$  be a coalition of  $t$  participants and let  $M_A = \prod_{i \in A} p_i$ . Then the shared integer  $y$  can be uniquely reconstructed in  $\mathbb{Z}_{M_A}$  since  $y < M \leq M_A$ . Hence, the secret  $s$  can later be obtained by computing  $y \bmod p_0$ .

Asmuth Bloom's SSS has better security properties when compared to Mignotte's. When a coalition  $A'$  with  $t-1$  shares tries to reconstruct the secret, due to (1), there will be at least  $\frac{M}{M_{A'}} > p_0$  candidates for  $y$ . Furthermore, since  $p_0$  is relatively prime with  $M_{A'}$ , there will be at least one  $y$  candidate valid for each possible secret candidate in  $\mathbb{Z}_{p_0}$ . Thus,  $t-1$  or fewer participants cannot narrow down the secret space. However, since the number of  $y$  candidates for two secret candidates may differ (by one), the secret candidates are not equally probable, resulting in an imperfect distribution [15]. To solve this problem, Kaya et al. proposed to use the equation

$$p_0^2 \prod_{i=1}^{t-1} p_{n-i+1} < M \quad (2)$$

instead of (1), which forms a *statistical scheme* with respect to the definition given in [3]. We will follow the same idea in this work. For the rest of the paper, we will use the notation given in Table I.

### C. Multilevel threshold secret sharing

We employ Simmons' multilevel threshold secret sharing (MTSS) definition, which assumes a multipartite access structure and a hierarchy on it such that the members of the superior compartments (higher-level members) can replace the ones from inferior compartments (lower-level members). Throughout the paper, the terms *level* and *compartment* are used interchangeably for our context.

Let  $\mathcal{U}$  be a set of all participants composed of disjoint subsets called *levels*, i.e.,  $\mathcal{U} = \bigcup_{i=1}^m L_i$  where  $L_i \cap L_j = \emptyset$  for all  $1 \leq i, j \leq m$ . Here  $L_1$  is the highest level and

Notation	Explanation
$\mathcal{U}$	The set of participants.
$\mathcal{A}$	The collection of authorized subsets of $\mathcal{U}$ , the access structure.
$n$	The number of total participants.
$m$	The number of levels\compartments.
$u_k$	The $k^{th}$ participant.
$L_i$	The $i^{th}$ level\compartment.
$n_i$	The number of participants in $L_i$ .
$t_i$	The threshold, the minimum number of users required to construct the secret for level $L_i$ .
$U_i$	$\sum_{k=1}^i L_k$ .
$s$	The secret to be shared.
$s_k^j$	$y_j \bmod p_k$ , the share of user $u_k \in L_j$ .
$\Delta s_k^i$	$y_i - h_k(s_k^i, i) \bmod p_k$ , the public information of user $u_k$ for $L_i$ .
$M_i$	The modulus of smallest $t_i$ ones, $\prod_{j=1}^{t_i} p_j$ .
$M_A$	The modulus of coalition $A$ , $\prod_{u_i \in A} p_i$ .
$p_0$	A prime; specifies the domain of $s \in \mathbb{Z}_{p_0}$ .
$p_i$	The prime modulus for user $i$ .
$y_i$	$s_i + \alpha_i \cdot p_0$ , where $\alpha_i$ is the blinding factor.

TABLE I  
NOTATION

$L_m$  is the lowest one. Thus, a participant in  $L_1$  can take place of any other participant, and a participant in  $L_m$  can only take place of the participants in  $L_m$ . Let the integers  $0 < t_1 < t_2 < \dots < t_m$  be a sequence of threshold values such that  $t_j \leq |L_1| + |L_2| + \dots + |L_j|$  for all  $1 \leq j \leq m$ . When considered in the disjunctive setting, the access structure is defined by using the disjunction of the  $m$  conditions on  $m$  compartments as described below.

*Definition 1:* A  $(t, n)$  disjunctive multilevel threshold secret sharing scheme assigns each participant  $u \in \mathcal{U}$  a secret share such that the access structure is defined as  $\mathcal{A} = \{A \subset \mathcal{U} : \exists i \in \{1, 2, \dots, m\} \text{ s.t. } |A \cap (\bigcup_{j=1}^i L_j)| \geq t_i\}$ .

On the other hand, under the conjunctive setting, all the threshold conditions of the compartments need to be satisfied. We use the same access structure definition as of [17].

*Definition 2:* A  $(t, n)$  conjunctive multilevel threshold secret sharing scheme assigns each participant  $u \in \mathcal{U}$  a secret share such that the access structure is defined as  $\mathcal{A} = \{A \subset \mathcal{U} : \forall i \in \{1, 2, \dots, m\} \text{ s.t. } |A \cap (\bigcup_{j=1}^i L_j)| \geq t_i\}$ .

#### D. The Harn-Fuyou MTSS scheme

Assume that the participants are partitioned into  $m$  levels  $L_i$ ,  $i = 1, 2, \dots, m$ . Let  $|L_i| = n_i$  be the number of participants in  $L_i$  and let  $t_i < n_i$  define a threshold on it. The threshold of a higher-level is always smaller than the threshold of a lower-level (i.e.,  $t_j < t_i$  for  $j < i$ ) consistent with the above MTSS definition. The disjunctive MTSS of Harn and Fuyou has two phases:

- *Share generation:* The dealer first selects a prime  $p_0$ , defining the secret space as  $s \in \mathbb{Z}_{p_0}$ . For each subset  $L_i$  having  $n_i$  participants, she selects a sequence of pairwise coprime positive integers (or primes),  $p_1^i < p_2^i < \dots < p_{n_i}^i$ , such that

$$p_0 p_{n_i - t_i + 2}^i p_{n_i - t_i + 3}^i \dots p_{n_i}^i < p_1^i p_2^i \dots p_{t_i}^i,$$

and  $\gcd(p_0, p_k^i) = 1, k = 1, 2, \dots, n_i$ , where  $p_k^i$  is the public information associated with participant  $u_k^i$ , the  $k^{th}$

member of the subset  $L_i$ . For each such sequence, the dealer selects an integer  $\alpha_i$  such that the value  $s + \alpha_i p_0$  is in the  $t_i$ -threshold range [12]. That is,  $\alpha_i$  is chosen such that

$$p_{n_i - t_i + 2}^i p_{n_i - t_i + 3}^i \dots p_{n_i}^i < s + \alpha_i p_0 < p_1^i p_2^i \dots p_{t_i}^i$$

supposedly in order to prevent the recovery of the value  $s + \alpha_i p_0$  with fewer than  $t_i$  shares.<sup>1</sup>

For each participant  $u_k^i$ , the private share  $s_k^i$  that can directly be used for level  $L_i$  is generated as  $s_k^i = s + \alpha_i p_0 \bmod p_k^i$ . In order to enable the use of  $s_k^i$  in a compartment  $L_j$  ( $j > i$ ), the dealer first selects a prime  $p_{k,j}^i$  such that  $p_{t_j}^j < p_{k,j}^i < p_{n_j - t_j + 2}^j$ . She then computes

$$\Delta s_{k,j}^i = (s + \alpha_i p_0 - s_k^i) \bmod p_{k,j}^i$$

and broadcasts it with  $p_{k,j}^i$  as a public information.

All selected  $p_{k,j}^i$ s during this phase must be relatively coprime to all other moduli. At the end of the phase, each participant  $u_k^i \in L_i$  keeps a single private share  $s_k^i \in \mathbb{Z}_{p_k^i}$  accompanied with the public information  $(\Delta s_{k,j}^i, p_{k,j}^i)$  for  $j \in \{i+1, i+2, \dots, m\}$ .

- *Secret reconstruction:* The secret can be recovered by a coalition of participants if there are at least  $t_j$  participants in the coalition from levels  $L_i$  where  $1 \leq i \leq j$ . By using the corresponding shares, a system of equations regarding CRT can be established on the joined shares; if the participant  $u_k^i$  belongs to  $L_j$ , i.e.,  $i = j$ , she can use her share  $s_k^i$  and the modulus  $p_k^i$  directly. Otherwise, i.e., if  $i < j$ , her share needs to be modified as  $s_k^i + \Delta s_{k,j}^i$  to be used in the lower level  $L_j$  and the operations for this modified share need to be performed in modulo  $p_{k,j}^i$  while constructing the system of CRT equations. Using all these shares and a standard CRT construction, a unique solution  $y = s + \alpha_j p_0$  can be obtained. Then the secret can be reconstructed by computing  $s = y \bmod p_0$ .

### III. THE FALLACIES OF HARN-FUYOU MTSS SCHEME

Although the Harn-Fuyou scheme employs interesting and useful mini-mechanisms resulting in the first MTSS scheme employing CRT, there are some unresolved issues as will be discussed here. A minor problem is that their MTSS is based on the original Asmuth-Bloom scheme which is not perfect (i.e., the secret candidates are not statistically equally likely to be the secret for an invalid coalition with  $t - 1$  shares). Although, this can be neglected if the secret is shared only once, sharing the same secret multiple times with a non-perfect scheme in practice may cause significant probabilistic differences in the secret space. For that reason, we believe that instead of the original scheme, the modified version proposed in [15] is more appropriate for a MTSS scheme.

<sup>1</sup>In the Harn-Fuyou scheme, the lower-bound on  $y_i = s + \alpha_i p_0$  constitutes an extra restriction on the original Asmuth-Bloom scheme and this range is called **t-threshold range** therein. That is, while the upper bound  $M_i = \prod_{j=1}^{t_i} p_j^i$  remains the same, the lower bound that  $y_i = s + \alpha_i p_0$  can attain is restricted to be greater than  $p_{n_i - t_i + 2}^i \dots p_{n_i}^i$  rather than 0. Thus, Harn-Fuyou employs a slightly different version of the Asmuth-Bloom scheme. In our scheme, we will follow the original bounds.

The proposed scheme is not generic since there are practical cases for which it cannot be employed; as mentioned above, in the *share generation* phase, there are additional  $p_{k,j}^i$  values associated with each participant  $u_k^i$  for each level  $L_j$  lower than hers. These numbers need to fulfill the condition  $p_{t_j}^j < p_{k,j}^i < p_{n_j-t_j+2}^j$  and hence, the scheme implicitly compels the dealer to initially select the primes  $p_1^j < p_2^j < \dots < p_{n_j}^j$  with a gap allowing sufficient number of primes in between  $p_{t_j}^j$  and  $p_{n_j-t_j+2}^j$  so that  $p_{k,j}^i$ s can fill in. In addition to the gap,  $p_{t_j}^j < p_{k,j}^i < p_{n_j-t_j+2}^j$  explicitly states that  $t_j < n_j - t_j + 2$ . Therefore, the Harn-Fuyou scheme is not suitable for the cases where the compartment threshold composes at least one more than the majority of the participants as the following simple setting shows.

**Example 1:** Let there be two levels  $L_1$  and  $L_2$  involving  $n_1 = |L_1| = 2$  and  $n_2 = |L_2| = 3$  participants and let the thresholds be  $t_1 = 2$  and  $t_2 = 3$ . The dealer selects the primes  $p_0 < p_1^1 < p_2^1$  and  $p_0 < p_1^2 < p_2^2 < p_3^2$  which need to satisfy

$$p_0 p_2^1 < p_1^1 p_2^1$$

$$p_0 p_2^2 p_3^2 < p_1^2 p_2^2 p_3^2$$

to be secure. Recall that  $p_{k,j}^i$  is the prime distributed to  $k^{th}$  user in  $i^{th}$  level to be used for participation in a lower compartment  $j$ . Since  $p_{k,j}^i$  must be chosen such that  $p_{t_j}^j < p_{k,j}^i < p_{n_j-t_j+2}^j$ , we have  $p_3^2 < p_{1,2}^1 < p_2^2$  and  $p_3^2 < p_2^2$  contradicts with the initial choice of primes  $p_2^2 < p_3^2$ .

Hence, placing the primes  $p_{k,j}^i$  between  $p_{t_j}^j$  and  $p_{n_j-t_j+2}^j$  requires a condition which is not guaranteed to hold in a generic setting; it simply may be the case that  $p_{t_j}^j > p_{n_j-t_j+2}^j$ , i.e.,  $t_j > \lceil \frac{n_j}{2} \rceil + 1$ . That is, the existence of some interval in between the primes is not ensured since there is no order whatsoever among the primes of different compartments.

The most important problem of the Harn-Fuyou scheme is in fact its mismatch with the multilevel access structure of Simmons. In general, the range of the threshold values  $t_i$  are given such as  $1 \leq t_i \leq \sum_{j=1}^i |L_j|$  for  $i = 1, 2, \dots, m$ . Hence,  $t_i$  can be greater than  $n_i = |L_i|$  as  $\sum_{j=1}^i |L_j| > |L_i|$ . Nonetheless, in the Harn-Fuyou scheme, the specified primes  $p_1^i < p_2^i < \dots < p_{n_i}^i$  cease at the index  $n_i$ , resulting in the condition  $p_0 p_{n_i-t_i+2}^i p_{n_i-t_i+3}^i \dots p_{n_i}^i < p_1^i p_2^i \dots p_{t_i}^i$  being unclear for large enough  $t_i$  that exceeds  $n_i$ . For example, the scheme is not well-defined for a setting with two compartments  $L_1$  and  $L_2$ , where  $n_1 = 3$ ,  $n_2 = 3$ ,  $t_1 = 2$  and  $t_2 = 4$  since there are only 3 users in the second compartment. The threshold is 4 and a  $(t, n)$ -Asmuth-Bloom sequence with  $n = 3$  and  $t = 4$  does not exist.

#### A. A straightforward (yet insecure) modification of the Harn-Fuyou MTSS

One can make the Harn-Fuyou MTSS scheme suitable for any number of participants and threshold values by removing the necessity of the additional primes: In the share generation phase, instead of using a sequence with  $n_i$  primes  $p_1^i < p_2^i < \dots < p_{n_i}^i$  for compartment  $L_i$ , the dealer can use a sequence

with  $U_i$  primes  $p_1^i < p_2^i < \dots < p_{U_i}^i$  where  $U_i = \sum_{j=1}^i n_j$ . For security, the condition to be satisfied for this prime set is

$$p_0 p_{U_i-t_i+2}^i p_{U_i-t_i+3}^i \dots p_{U_i}^i < p_1^i p_2^i \dots p_{t_i}^i$$

that is well defined for any valid value of  $t_i$ . Here, the first  $n_i$  primes can be used for the participants in  $L_i$  and the extra primes  $p_\ell^i$  for  $\ell > n_i$  can be used for  $p_{k,j}^i$ s for the participants in higher compartments. The random integers  $\alpha_i$ ,  $1 \leq i \leq m$  are chosen such that  $0 \leq s + \alpha_i p_0 < p_1^i p_2^i \dots p_{t_i}^i$ . The share  $s_k^i$  for participant  $u_k^i$  is generated as  $s_k^i = s + \alpha_i p_0 \pmod{p_{k,j}^i}$  as before.

This approach indeed eliminates the need for  $p_{k,j}^i$  to fill in to a possibly non-existing gap in between  $p_{t_j}^j < p_{k,j}^i < p_{n_j-t_j+2}^j$ . As this is the only distinction we describe herein, the rest of the share generation phase and the secret reconstruction phase remains essentially intact, and can be performed in a similar fashion as described before.

Although we established a well-defined scheme for all possible threshold settings, this approach unfortunately does not provide security as the following example illustrates. The example below is given for the modified/fixed version without the *gap existence* problem. However, the weakness also exists in the original MTSS scheme of Harn-Fuyou since the public information with different prime modulus for a certain participant reveals extra information as we will show below.

**Example 2:** Consider the following setting emerging from the scheme with the basic fix above. Let  $p_0 = 5$  and  $s = 1 \in \mathbb{Z}_5$ . Suppose that we have two compartments  $L_1$  and  $L_2$  with  $n_1 = 4$ ,  $n_2 = 2$ ,  $t_1 = 2$  and  $t_2 = 3$ . Let

$$p_1^1 < p_2^1 < p_3^1 < p_4^1 \rightarrow 11 < 13 < 17 < 23$$

$$p_1^2 < p_2^2 < p_3^2 < p_4^2 < p_5^2 < p_6^2 \rightarrow 29 < 31 < 37 < 61 < 67 < 71$$

be the primes. The Asmuth-Bloom condition  $p_0 p_{U_i-t_i+2}^i p_{U_i-t_i+3}^i \dots p_{U_i}^i < p_1^i p_2^i \dots p_{t_i}^i$  is satisfied for both levels since

$$5 \times 23 = 115 < 143 = 11 \times 13,$$

$$5 \times 67 \times 71 = 23785 < 33263 = 29 \times 31 \times 37.$$

Let  $\alpha_1 = 5$  and  $\alpha_2 = 952$ . Hence,

$$y_1 = s + \alpha_1 p_0 = 1 + 5 \times 5 = 26,$$

$$y_2 = s + \alpha_2 p_0 = 1 + 952 \times 5 = 4761,$$

and these values are chosen from the  $t$ -threshold range since

$$23 < 26 < 143 = 11 \times 13,$$

$$67 \times 71 = 4757 < 4761 < 33263 = 29 \times 31 \times 37.$$

Similarly, let  $p_{1,2}^1 = p_6^2 = 71$ ,  $p_{1,2}^2 = p_5^2 = 67$ ,  $p_{1,2}^3 = p_4^2 = 61$ ,  $p_{1,2}^4 = p_3^2 = 37$  be the additional primes that will be used to enable the share of the participants in  $L_1$  for  $L_2$ . With these parameters, the shares are

$$s_1^1 = 4, s_2^1 = 0, s_3^1 = 9, \text{ and } s_4^1 = 3,$$

$$s_1^2 = 5 \text{ and } s_2^2 = 18$$

$$s_{1,2}^1 = 25, s_{2,2}^1 = 3, s_{3,2}^1 = 4, \text{ and } s_{4,2}^1 = 4,$$

and the public information is computed as

$$\Delta s_{1,2}^1 = 0, \Delta s_{2,1}^1 = 4, \Delta s_{3,1}^1 = 55, \Delta s_{4,1}^1 = 22.$$

To exemplify, the first participant in  $L_1$  is associated with the prime  $p_1^1 = 11$  as well as the prime  $p_{1,2}^1 = 71$  and the integer  $\Delta s_{1,2}^1 = (s + \alpha_2 p_0) - s_1^1 = 0 \pmod{p_{1,2}^1}$ .

Suppose that the adversary corrupted  $u_1^2$  and  $u_2^2$  hence obtained their shares. She knows that  $y_2$  is bounded by  $4757 < y_2 < 33263$  and she also can compute  $y_2 \pmod{p_1^2 p_2^2} = y_2 \pmod{899} = 266$  by using these shares. There are  $\lceil (33263 - 4757)/899 \rceil = 32$  candidates for  $y_2$  all in form  $266 + 899 \times K$  where  $5 \leq K \leq 36$ . Since, 899 is relatively prime with 5, each secret candidate in  $\mathbb{Z}_{p_0}$  must be valid for around 7 of these values, i.e., for  $266 + 899 \times 6$  the valid secret candidate is 0. Hence, without the public information, thanks to the perfectness of Asmuth-Bloom SSS, the adversary cannot have an information on the secret. Unfortunately, with public information that reveal extra information for the uncorrupted users, the adversary can gain some information on the secret and sometimes she even can find the secret as shown in the example below.

The participant  $u_1^1$  has a public information pair  $(\Delta s_{1,2}^1, p_{1,2}^1) = (0, 71)$  and her prime is  $p_1^1 = 11$ . Hence, the adversary knows that the value  $s_{1,2}^1$  is bounded by  $s_{1,2}^1 = s_1^1 + \Delta s_{1,2}^1 \in [0, 10]$  since  $s_1^1 \in \mathbb{Z}_{11}$ . Similarly, for  $u_2^2, u_3^3,$  and  $u_4^4$ , the adversary knows that

$$s_{2,2}^1 \in [4, 16], s_{3,2}^1 \in [55, 60] \cup [0, 10], s_{2,2}^2 \in [22, 36] \cup [0, 7].$$

As the Table II shows, there is only one  $y_2$  candidate in the form  $51 + 899K$ , which yields  $s_{\{1,2,3,4\},2}^1$  values within these ranges. Thus the adversary knows that  $y_2 = 4761$  and the secret  $s = 1$  is recovered in an unauthorized manner by corrupting only two participants from  $L_2$ .

#### IV. PROPOSED MULTILEVEL THRESHOLD SECRET SHARING SCHEMES

As described before, we are given a secret  $s \in \mathbb{Z}_{p_0}$  and a set of primes such that

$$p_0^2 \prod_{i=1}^{t-1} p_{n-i+1} < \prod_{i=1}^t p_i, \quad (3)$$

i.e., the *Asmuth-Bloom condition* holds. We will refer to the prime sequence  $p_0 < p_1 < p_2 < \dots < p_n$  satisfying the Asmuth-Bloom condition as a  $(t, n)$ -*Asmuth-Bloom sequence*. As the fallacies of the Harn-Fuyou scheme show, having the Asmuth-Bloom condition for all the compartments independently while keeping the level structure and being secure is not an easy task. We solve this problem by using a single *anchor Asmuth-Bloom sequence* as defined below so that each participant of the MTSS has only one prime modulus that can be used for all the levels she can contribute to.

**Definition 3:** An anchor Asmuth-Bloom sequence is a sequence of primes  $p_0 < p_1 < p_2 < \dots < p_n$  satisfying

$$p_0^2 \prod_{i=1}^{\lfloor n/2 \rfloor - 1} p_{n-i+1} < \prod_{i=1}^{\lfloor n/2 \rfloor} p_i. \quad (4)$$

As one can notice, an anchor sequence is a valid  $(\lfloor n/2 \rfloor, n)$ -Asmuth-Bloom sequence. Here, we will show that, an anchor sequence can be used not only for  $t = \lfloor n/2 \rfloor$  but also for other  $t$  values:

**Lemma 4:** An anchor Asmuth-Bloom sequence can be employed for any CRT-based  $(t, n)$  secret sharing scheme. That is an anchor prime sequence satisfies the Asmuth-Bloom condition for any  $1 \leq t \leq n$ .

**Proof:** We will investigate the sequence in two cases:

- 1) ( $t < \lfloor n/2 \rfloor$ ): To have (3) from (4) for a threshold value  $t < \lfloor n/2 \rfloor$ , one can remove  $\lfloor n/2 \rfloor - t$  primes from each side of (4). Note that for each prime  $p_i$  removed from the right side, one needs to remove  $p_{n-i+1}$  from the left. Since  $i \leq t < \lfloor n/2 \rfloor$  for all the primes removed,  $n - i + 1 > i$  and  $p_{n-i+1} > p_i$ . Thus, given the anchor inequality (4), the Asmuth-Bloom condition (3) is also satisfied for a threshold  $t < \lfloor n/2 \rfloor$  with the same set of primes.
- 2) ( $t \geq \lfloor n/2 \rfloor$ ): This case is similar to the former case except that to have (3) from (4), we need to add  $t - \lfloor n/2 \rfloor$  primes to each side of (4). For each prime pair  $(p_{n-i+1}, p_i)$  added to the left and right of the anchor inequality, respectively,  $p_{n-i+1} < p_i$  since  $t \leq i > \lfloor n/2 \rfloor$ . Thus given (4), (3) is also satisfied for a threshold value  $t \geq \lfloor n/2 \rfloor$  with the same prime sequence. ■

#### A. A novel CRT-based multilevel threshold (disjunctive) SSS

Let  $n = \sum_{i=1}^m n_i$  be the number of total participants. Let  $h_i : \mathbb{Z}_{p_i} \times \mathbb{Z}_m \rightarrow \mathbb{Z}_{p_i}$  for  $i \in \{1, \dots, n\}$  be a family of efficiently computable one-way hash functions. We employ an anchor sequence of  $n$  primes as follows:

- **Initialization:** The dealer first generates an anchor prime sequence  $p_0 < p_1 < p_2 < \dots < p_n$  satisfying (4) and assign each prime  $p_i$  to a participant  $u_i$ . Note that this will be the only prime modulus that will be used for the participant<sup>2</sup>.
- **Share generation:** Given a secret  $s \in \mathbb{Z}_{p_0}$ , the dealer chooses  $\alpha_i$ 's for all  $1 \leq i \leq m$  such that

$$0 \leq y_i = s + \alpha_i p_0 < M_i = p_1 p_2 \dots p_{t_i}.$$

For level  $L_i$ , the shares and the public information are generated as follows: Let  $u_k$  be a participant in  $L_i$ ; the original share  $s_k^i$  for  $u_k$  is generated as  $s_k^i = y_i \pmod{p_k}$ . If  $u_k$  is a participant in a higher compartment  $L_j$ , i.e.,  $j < i$ ; to enable the use of  $s_k^j$  in  $L_i$ , the dealer computes  $\Delta s_k^i = (y_i - h_k(s_k^j, i)) \pmod{p_k}$  and broadcasts it as the public information. This information will be used if  $u_k$  participates in the secret reconstruction within  $L_i$ .

- **Secret reconstruction:** Let  $A$  be a coalition gathered to reconstruct the secret.  $A$  is an authorized coalition if it has  $t_i$  or more participants from  $L_i$  or higher compartments for  $1 \leq i \leq m$ . If the participant is from  $L_i$ , her share  $s_k^i$  can be used as is. Any other share  $s_k^j$  of  $u_k$  from a higher

<sup>2</sup>While describing the proposed schemes, we will denote the primes and participants with a single subscript as opposed to the notation in Harn-Fuyou scheme. We believe this is more clear thanks to the compactness of the anchor sequence we employ.

candidate	$s_{1,2}^1$	$s_{2,2}^1$	$s_{3,2}^1$	$s_{4,2}^1$	candidate	$s_{1,2}^1$	$s_{2,2}^1$	$s_{3,2}^1$	$s_{4,2}^1$
4761	<b>4</b>	<b>4</b>	<b>3</b>	<b>25</b>	19145	46	<b>50</b>	52	16
5660	51	32	48	<b>36</b>	20044	22	<b>11</b>	36	<b>27</b>
6559	27	60	32	10	20943	69	39	20	<b>1</b>
7458	<b>3</b>	21	16	21	21842	45	0	<b>4</b>	12
8357	50	49	<b>0</b>	<b>32</b>	22741	21	28	49	<b>23</b>
9256	26	<b>10</b>	45	<b>6</b>	23640	68	56	33	<b>34</b>
10155	<b>2</b>	38	29	17	24539	44	17	17	8
11054	49	66	13	<b>28</b>	25438	20	45	<b>1</b>	19
11953	25	27	<b>58</b>	<b>2</b>	26337	67	<b>6</b>	46	<b>30</b>
12852	<b>1</b>	55	42	13	27236	43	34	30	<b>4</b>
13751	48	16	26	<b>24</b>	28135	19	62	14	15
14650	24	44	<b>10</b>	<b>35</b>	29034	66	23	<b>59</b>	<b>26</b>
15549	<b>0</b>	<b>5</b>	<b>55</b>	9	29933	42	51	43	<b>0</b>
16448	47	33	39	20	30832	18	<b>12</b>	27	11
17347	23	61	23	<b>31</b>	31731	65	40	11	<b>22</b>
18246	70	22	<b>7</b>	<b>5</b>	32630	41	1	<b>56</b>	<b>33</b>

TABLE II

SECRETS FOR EACH  $y_2$  CANDIDATE FROM ADVERSARY'S POINT OF VIEW FOR EXAMPLE 2.

THE VALUES CONSISTENT WITH THE RANGES OBTAINED BY PUBLIC INFORMATION ARE SHOWN IN BOLDFACE.

level needs to be modified as  $(s_k^j + \Delta s_k^i)$  and is used with the modulus  $p_k$  while constructing the system of congruences. Using the standard CRT, a unique solution  $y_i$  can be obtained. Then, the secret  $s$  is recovered by computing  $s = y_i \bmod p_0$ .

An authorized coalition can obtain the secret since with the help of public information, the coalition will have enough shares for a compartment  $L_i$ . Thanks to CRT, the corresponding  $y_i$  value and hence  $s = y_i \bmod p_0$  can be computed.

1) *Security analysis of the proposed MTSS*: The security of the proposed MTSS solely depends on the security of the Asmuth-Bloom scheme. We will argue that unlike the Harn-Fuyou scheme, the proposed MTSS scheme does not reveal any information on the secret with the public information used. Then, we will prove that an adversarial coalition cannot have any information on the secret.

To generate the public information, the proposed MTSS scheme employs a hash function for each participant. Let  $u_k$  be a participant in  $L_j$ . If the adversary corrupts  $u_k$  she will have  $s_k^j$  and she can compute the shares for all levels  $L_j$ ,  $j \leq i \leq m$ . If  $u_k$  remains uncorrupted, the adversary will only have the public information for  $u_k$ . Let  $L_i$  be a level lower than  $j$ ; the adversary will have

$$\Delta s_k^i = (s_k^i - h_k(s_k^j, i)) \bmod p_k \quad (5)$$

Hence, assuming the hash function  $h_k$  behaves like a random oracle,  $\Delta s_k^i$  will be random (which can be randomly generated in a zero-knowledge proof). Thus the adversary cannot learn anything on the shares of  $u_k$  for lower compartments. Furthermore, although the same hash function  $h_k$  is used to compute  $\Delta s_k^i$  and  $\Delta s_k^{i'}$  for two lower levels  $L_i$  and  $L_{i'}$ ,  $j \leq i, i' \leq m$ , these two values cannot be combined (as they could be without the hash function), since  $h_k$  takes  $i$  and  $i'$ , respectively, as an input.

*Theorem 5*: Given that the hash functions used in the MTSS scheme behave like random oracles, an unauthorized coalition cannot obtain any information about the secret.

**Proof**: Let  $A'$  be the adversarial coalition having  $t_i - 1$  participants from  $L_i$  and higher compartments. Let  $M_{A'}$  be the product of the prime modulus values assigned to these  $t_i - 1$  participants and  $y_i' = y_i \bmod M_{A'}$ . Since  $p_0^2 \prod_{j=1}^{t_i-1} p_{n-j+1} < \prod_{j=1}^{t_i} p_j < \prod_{j=1}^{t_i} p_j = M_i$ , we have

$M_i/M_{A'} > p_0^2$ . Hence  $y_i' + \beta M_{A'}$  is a valid candidate for  $y_i < M$  for all  $\beta < p_0^2$ . Since  $\gcd(p_0, M_{A'}) = 1$ , all  $(y_i' + \beta M_{A'}) \bmod p_0$  are distinct for  $\ell p_0 \leq \beta < (\ell + 1)p_0$ , for each  $0 \leq \ell < p_0$ . Thus  $s$  can be any integer from  $\mathbb{Z}_{p_0}$  and the secret space is not restricted from the adversary's point of view.

For each value  $s'$  in the secret space, from the adversary's point of view, there are either  $\lfloor M_i/(M_{A'} p_0) \rfloor$  or  $\lfloor M_i/(M_{A'} p_0) \rfloor + 1$  possible consistent  $y_i$  candidates consistent with  $s'$ . Considering  $M_i/M_{A'} > p_0^2$ , for two different integers  $s'$  and  $s''$  in  $\mathbb{Z}_{p_0}$ , the probabilities of  $s = s'$  or  $s = s''$  are almost equal and the difference between these two values reduces when  $p_0$  increases. More formally, thanks to the modified Asmuth-Bloom SSS we employed [15], the proposed MTSS scheme is *statistical*, i.e., the statistical distance between the probability distribution of the secret candidates being a secret and an uniform distribution is smaller than a given  $\epsilon$  with a carefully chosen  $p_0$ . ■

### B. A CRT-based multilevel threshold (conjunctive) SSS

The ideas presented above for the disjunctive scheme can also be employed to have a conjunctive SSS. Here, we present the first CRT-based conjunctive MTSS scheme which adopts Iftene's CRT-based compartmented SSS [13].

The setting is the same as that of the disjunctive MTSS scheme; compartment  $L_i$  with threshold  $t_i$  has  $n_i$  participants for  $1 \leq i \leq m$ . Hence, the total number of participants is  $n = \sum_{i=1}^m n_i$ . There is a hierarchy between the compartments; a member of  $L_j$  can act as a member of a lower compartment  $L_i$  if  $i > j$ . The proposed conjunctive scheme shares a given secret  $s \in \mathbb{Z}_{p_0}$  as follows:

- *Initialization*: The anchor prime sequence generation is the same. Let  $\sigma_1, \sigma_2, \dots, \sigma_{m-1}$  be random integers from  $\mathbb{Z}_{p_0}$  and  $\sigma_m \in \mathbb{Z}_{p_0}$  is chosen such that

$$s = (\sigma_1 + \sigma_2 + \dots + \sigma_m) \bmod p_0.$$

- *Share generation*: For all  $1 \leq i \leq m$ , a random  $\alpha_i$  is chosen such that  $0 \leq y_i = \sigma_i + \alpha_i p_0 < M_i = p_1 p_2 \dots p_{t_i}$ . The shares and public information are generated similar to the disjunctive case. Let  $u_k$  be a participant in  $L_i$ ; the original share  $s_k^i$  for  $u_k$  is generated as  $s_k^i = y_i \bmod p_k$ .

For all  $u_k$  who is from a higher level  $L_j$  to enable the use of  $s_k^j$  in  $L_i$ ,  $\Delta s_k^i = (y_i - h_k(s_k^j, i)) \bmod p_k$  is computed and broadcasted.

- *Secret reconstruction:* The secret  $s$  can be recovered if and only if all of the  $\sigma_i$  values for  $1 \leq i \leq m$  are recovered. A partial secret  $\sigma_i$  can be recovered if the number of shares from level  $L_i$  or from higher levels is greater than or equal to  $t_i$ . Let  $u_k$  be a coalition member participating in this task; if  $u_k \in L_i$ , her original share  $s_k^i$  can be used. Otherwise, if  $u_k \in L_j$  for  $j < i$ ,  $s_k^j + \Delta s_k^i$  is computed and used as  $s_k^i$ . After computing all  $\sigma_i$  values for  $1 \leq i \leq m$ , the secret  $s$  is constructed by  $s = (\sigma_1 + \sigma_2 + \dots + \sigma_m) \bmod p_0$ .

Since the scheme uses exactly the same set of public information and the underlying statistical SSS is the same, the security analysis for the disjunctive case can also be applied for the conjunctive MTSS scheme with minor modifications and is omitted here.

## V. MULTILEVEL THRESHOLD FUNCTION SHARING

In this section, we adapt our MTSS scheme to have a CRT-based multilevel function sharing scheme (FSS) which can be used for decrypting a ciphertext or signing a message in a way that no unqualified coalition of participants can perform this operation. Another important property of a FSS is that it does not disclose the secret and the shares; thus, it can be used several times without any rearrangement. Several protocols for function sharing [18], [19], [20] have been proposed in the literature where most of them are based on the Shamir SSS. Kaya and Selçuk [15] proposed the first CRT-based FSS for RSA signature [21] and the ElGamal [22] decryption functions. Here, we propose a secure CRT-based multilevel function sharing scheme (MFSS) for RSA signatures to demonstrate that the proposed MTSS scheme is applicable for function sharing.

*A Threshold (disjunctive) RSA Signature Scheme:* Let  $N = pq$  be product of two large primes. Choose public key  $e$  and private key  $d$  such that  $ed \equiv 1 \pmod{\phi(N)}$ . The signature of a message  $\text{msg}$  is  $\text{sgn} = \text{msg}^d \pmod{N}$  and the verification is done by checking  $\text{msg} \stackrel{?}{=} \text{sgn}^e \pmod{N}$ . The setup phase for the threshold multilevel RSA scheme is given in Figure 1 and the signature and verification steps can be found in Figure 2. Here we describe a disjunctive scheme but the scheme can be converted to the conjunctive case with minor modifications.

*Security Analysis:* Since the proposed MTSS scheme is as secure as the original Asmuth Bloom SSS by Theorem 5 and the adapted threshold signature scheme is proven to be secure with the Asmuth-Bloom structure [15], the proposed MFSS is also secure under the assumption of intractability of the RSA problem. Detailed explanations about random oracle proofs for the CRT-based threshold RSA can be found in [15].

## VI. CONCLUSION

The CRT-based multilevel threshold SSS of Harn-Fuyou in the literature cannot be used for all threshold settings. Furthermore, the scheme is not secure and an adversary can

extract the secret by using the private shares of the participants she corrupted and information revealed to the public during the secret sharing phase. We proposed novel, compact, and elegant disjunctive and conjunctive multilevel SSSs based on a special prime sequence called anchor sequence. We showed that the proposed schemes can easily be adopted for function sharing schemes which have numerous applications in applied cryptography.

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- Let  $N = pq$  be product of two strong primes, i.e.  $p = 2p' + 1$  and  $q = 2q' + 1$  where  $p'$  and  $q'$  are large primes.
- Choose  $e$  and  $d$  such that  $ed \equiv 1 \pmod{\phi(N)}$ , ( $\phi(N) = 4p'q'$ ).
- Use the proposed MTSS scheme in order to share the secret  $s = d$  with  $p_0 = 4p'q'$  to  $m$  levels where the  $i^{th}$  level  $L_i$  has  $n_i$  users with a threshold  $t_i$ .

Fig. 1. Setup of the proposed multilevel (disjunctive) threshold RSA signature scheme.

**Signing**

- Let  $\mathbf{msg} \in \mathbb{Z}_N^*$  be the message to be signed.
- Let  $A \in \mathcal{A}$  be a coalition in the access structure wants to sign  $\mathbf{msg}$ .
- Let  $i$  be an integer s.t.  $A_i = A \cap (\bigcup_{j=1}^i L_j)$  and  $|A_i| \geq t_i$ .
- Each user  $u_k \in A_i$  computes

$$M_{A_i} = \prod_{u_{k'} \in A_i} p_{k'} \text{ and } P_k = \frac{M_{A_i}}{p_k} \pmod{p_k}$$

and  $I_k$ , the inverse of  $P_k$  s.t.  $I_k P_k \equiv 1 \pmod{p_k}$ . She then computes the partial signature  $\mathbf{sgn}_k$  as

$$\overline{s}_k^i = \begin{cases} s_k^i & \text{if } u_k \in L_i \\ s_k^j & \text{if } u_k \in L_j \text{ and } j < i \end{cases}$$

$$\nu_k = \overline{s}_k^i P_k I_k \pmod{M_{A_i}},$$

$$\mathbf{sgn}_k = \mathbf{msg}^{\nu_k} \pmod{N}$$

and sends  $\mathbf{sgn}_k$  to the server.

- For each user  $u_k$ , Server computes public parts of the signature as

$$\overline{\Delta s}_k^i = \begin{cases} 0 & \text{if } u_k \in L_i \\ \Delta s_k^i & \text{if } u_k \in L_j \text{ and } j < i \end{cases}$$

$$\Delta \nu_k = \overline{\Delta s}_k^i P_k I_k \pmod{M_{A_i}},$$

$$\Delta \mathbf{sgn}_k = \mathbf{msg}^{\Delta \nu_k} \pmod{N}$$

Server combines all parts and computes incomplete signature  $\overline{\mathbf{sgn}}$

$$\overline{\mathbf{sgn}} = \prod_{u_{k'} \in A_i} (\mathbf{sgn}_k \times \Delta \mathbf{sgn}_k)$$

Server converts incomplete signature  $\overline{\mathbf{sgn}}$  to the signature  $\mathbf{sgn}$  by trying  $x$

$$(\overline{\mathbf{sgn}} \times \kappa^x)^e \stackrel{?}{\equiv} \mathbf{msg} \pmod{N} \quad (6)$$

for  $0 \leq x < 2|G|$  and let  $\delta$  denotes the value of  $x$  satisfying (6).  
Then the signature is computed as  $\mathbf{sgn} = \overline{\mathbf{sgn}} \times \kappa^\delta$

Fig. 2. Proposed multilevel (disjunctive) threshold RSA signature scheme: verification phase is not given since it is the same as the RSA verification

**Oğuzhan Ersoy** received his B.S. degrees in Electrical & Electronics Engineering and Mathematics from Boğaziçi University, İstanbul, Turkey, in 2012. He received his M.S. degree in 2015 in the Department of Electrical & Electronics Engineering at Boğaziçi University. He is working at TÜBİTAK BİLGEM, Kocaeli, Turkey since 2012.

**Kamer Kaya** is working as an Assistant Professor at the Faculty of Engineering and Natural Sciences at Sabancı University, Turkey. He got his PhD from Dept. of Computer Engineering at Bilkent University, Turkey. His current research interests are Parallel Programming, Cryptography, and High Performance Computing.

**Kerem Kaşkaloğlu** is working as an Assistant Professor at the department of Mathematics and Statistics at the American University of the Middle East, Kuwait. He received his PhD from the Middle East Technical University, Turkey. His current research interests are combinatorics and cryptography.