Consistency, converse consistency, and aspirations in TU-games

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Received 1 January 2001; received in revised form 1 May 2002; accepted 1 October 2002

Abstract

In problems of choosing ‘aspirations’ for TU-games, we study two axioms, ‘MW-consistency’ and ‘converse MW-consistency.’ In particular, we study which subsolutions of the aspiration correspondence satisfy MW-consistency and/or converse MW-consistency. We also provide axiomatic characterizations of the aspiration kernel and the aspiration nucleolus.

Keywords: Aspirations; Consistency; Converse consistency; Reduced games; TU-games

JEL classification: C71

1. Introduction

A transferable utility coalitional game (TU-game, for short) associates with each coalition of agents a real number representing what the coalition can achieve on its own, its ‘worth’. Given a class of TU-games, a ‘solution’ associates with each game in the class a non-empty set of payoff vectors. The analysis of TU-games proposes solutions that answer two basic questions:

(i) Which coalitions form?
(ii) What is the payoff of each member of a coalition that forms?

Most studies on TU-games, however, assume that the grand coalition eventually
forms. Then, the analysis reduces to answering question (ii), that is, determining the payoff distribution for the grand coalition.

In this paper, we do not presuppose the formation of the grand coalition and study those solutions that, in a sense, simultaneously answer both questions (i) and (ii). An ‘aspiration’ for a given game is a payoff vector that summarizes predictions about which coalitions are likely to form and what the resulting payoffs of their members will be. For each agent $i$, let $x_i$ be a payoff that she demands in return for her cooperation. Then, it is natural to assume that a coalition $S$ forms only if the demands of its members are jointly compatible, namely, $(x_i)_{i \in S}$ is feasible for $S$. A payoff vector $x$ is an aspiration if no coalition can improve upon its component of $x$, and for each agent $i$, there exists at least one coalition for which agent $i$’s demand $x_i$ is jointly compatible with those of other members.

Studies of coalitional games have revealed that the set of aspirations is closely related to the outcomes obtained from two alternative approaches: the ‘multi-coalitional bargaining approach’ and the ‘noncooperative approach’.¹

In the multi-coalitional bargaining approach, each coalition is assigned a set of attainable payoffs and a ‘bargaining solution’, which is interpreted as summarizing the bargaining process among the members of that coalition. The disagreement point in each coalition’s bargaining problem is determined endogenously as the expectation of each member on what she can obtain from alternative coalitions. A payoff vector $x$ is a ‘multi-coalitional bargaining outcome’ if, for each coalition $S$, $(x_i)_{i \in S}$ is chosen by the assigned bargaining solution for the bargaining problem with the disagreement point associated with $x$. It so happens that every multi-coalitional bargaining outcome is an aspiration. Conversely, each aspiration can be obtained as a multi-coalitional bargaining outcome for some initial specification of bargaining solutions.

In the non-cooperative approach, versions of the following coalition formation game are analyzed: a randomly chosen agent proposes a coalition to be formed and a feasible payoff distribution for its members. The proposal is accepted if every member of the coalition agrees upon it. Otherwise, in the next period the first agent who rejected the proposal makes a new proposal. The game ends when a proposal is accepted. It turns out that the set of aspirations of the original coalitional game is equal to the set of stationary subgame perfect equilibrium proposals of this non-cooperative game.²

Its relation to the bargaining and non-cooperative approaches strongly suggests that the set of aspirations is an appropriate object to focus on if one wants to analyze coalitional games without imposing the assumption that the grand coalition eventually forms. Let us refer to the solution that assigns to each game its set of aspirations as the ‘aspiration correspondence’. In the literature, several subsolutions of the aspiration correspondence have been studied. We are interested in their properties. In particular, we focus on two properties: ‘consistency’ and ‘converse consistency’.³

¹See Bennett (1983) for more detail.
²The particular non-cooperative game described here is due to Selten (1981). The literature following the original paper includes Chatterjee et al. (1993), Perry and Reny (1994), and Moldovanu and Winter (1994b, 1995).
³See Thomson (1996) for an extensive survey of studies on these properties applied to various models of game theory and economics.
Consistency deduces from the desirability of a payoff vector in a game the desirability of its restrictions to all subgroups of agents in the associated ‘reduced games’. Suppose that a set of agents \( N \) is facing a game and a payoff vector \( x \) is agreed upon. Suppose then that some agents leave. Then let us reevaluate the situation from the viewpoint of the remaining agents \( N' \). Namely, for each coalition \( S \subseteq N' \), let us identify what \( S \) can obtain without any help from other members of \( N' \). In this context, since any agent \( i \) in \( N \setminus N' \) has agreed upon \( x \), it seems natural to assume that she will be willing to cooperate with \( S \) if offered \( x_i \). Additionally, suppose that \( S \) can choose such ‘partners’ from \( N \setminus N' \). Then the revised worth of \( S \) would be the maximal (total) payoff that \( S \) can obtain in this manner. This operation defines a game in which the set of agents is \( N' \). We refer to this game as an ‘MW-reduced game’ since it was introduced by Moldovanu and Winter (1994a) and Winter (1989). ‘MW-consistency’ states that, in this reduced game, the original agreement should be confirmed, namely, \( (x_i)_{i \in N} \) should be agreed upon.

Converse consistency deduces the desirability of a payoff vector in a game from the desirability of its restrictions to all pairs of agents in the associated two-agent reduced games. Consider a game for \( N \) and a payoff vector \( x \) under evaluation. Suppose that for each pair of agents \( \{i,j\} \) in \( N \), \( (x_i, x_j) \) is chosen for the MW-reduced game associated with \( x \) and \( \{i,j\} \). ‘Converse MW-consistency’ states that, in such a situation, \( x \) should be chosen for the original game.

On the domain of all TU-games, the aspiration correspondence and the partnered aspiration solution satisfy both MW-consistency and converse MW-consistency (Moldovanu and Winter, 1994a; Winter, 1989). In this paper, we analyze which other subsolutions of the aspiration correspondence satisfy MW-consistency and/or converse MW-consistency. Moreover, we obtain axiomatic characterizations of two solutions: the aspiration kernel and the aspiration nucleolus.

The paper is organized as follows. In Section 2, we introduce basic concepts and notations used later on. In Section 3, we study which subsolutions of the aspiration correspondence satisfy MW-consistency and/or converse MW-consistency. In Section 4, we provide axiomatic characterizations of the aspiration kernel and the aspiration nucleolus. In Section 5, we give some remarks on the non-transferable utility case.

2. Preliminary

There is an infinite set of ‘potential’ agents, indexed by the members of the set \( \mathbb{N} \) of natural numbers. Let \( \mathcal{N} \) denote the set of non-empty and finite subsets of \( \mathbb{N} \). Given a countable set \( A \), let \( \mathbb{R}^A \) denote the Cartesian product of \( |A| \) copies of the set \( \mathbb{R} \) of real numbers, indexed by the members of \( A \). We use \( \subset \) for strict set inclusion and \( \subseteq \) for weak set inclusion. To simplify the notation, given \( N \in \mathcal{N} \), \( x \in \mathbb{R}^N \), and \( S \subseteq N \), we often write \( x_S = (x_i)_{i \in S} \) and \( x(S) = \sum_{i \in S} x_i \).

Given \( N \in \mathcal{N} \), a transferable utility coalitional game for \( N \) (TU-game for \( N \), for short)
is a function $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$. For each $S \subseteq N$, the number $v(S)$ represents what coalition $S$ can obtain on its own, its ‘worth’. Let $\mathcal{V}_N^M$ denote the class of all TU-games for $N$, and $\mathcal{V}_N^M = \bigcup_{M \in \mathcal{V}_N^M} \mathcal{V}_N^M$.

Given $N \in \mathcal{N}$ and $v \in \mathcal{V}_N^M$, a payoff vector $x \in \mathbb{R}^N$ is \textit{individually feasible in} $v$ if for each $i \in N$, there exists $S \subseteq N$ such that $i \in S$ and $x(S) \leq v(S)$. It is \textit{coalitionally rational in} $v$ if for each $S \subseteq N$, $x(S) = v(S)$. An \textit{aspiration} for $v$ is a payoff vector in $\mathbb{R}^N$ satisfying individual feasibility and coalitional rationality.

Given $N \in \mathcal{N}$, $v \in \mathcal{V}_N^M$, and $x \in \mathbb{R}^N$, the \textit{set of generating coalitions} for $v$ and $x$, denoted $\mathcal{G}(v,x)$, contains those coalitions whose members’ promised payoffs in $x$ are jointly compatible:

$$\mathcal{G}(v,x) = \{S \subseteq N \mid x(S) \leq v(S)\}.$$ 

Thus, if each agent demands her component of the payoff vector, the generating coalitions are the only coalitions that are likely to form. In an aspiration for the game, each agent is a member of at least one generating coalition and each generating coalition distributes payoffs efficiently among its members.

A \textit{solution} on $\mathcal{V}_N^M$ is a correspondence from $\mathcal{V}_N^M$ to $\bigcup_{N \in \mathcal{N}} \mathbb{R}^N$ that associates with each $N \in \mathcal{N}$ and each $v \in \mathcal{V}_N^M$ a non-empty set of payoff vectors satisfying individual feasibility. We use $\varphi$ to denote a generic solution.

### 3. Consistency and converse consistency

Given $N \in \mathcal{N}$, $v \in \mathcal{V}_N^M$, $x \in \mathbb{R}^N$, and $N' \subseteq N$, the \textit{MW-reduced game} of $v$ relative to $x$ and $N'$, denoted $r_{N'}(v)$, is defined by setting for each $S \subseteq N'$

$$r_{N'}(v)(S) = \begin{cases} \max_{T \subseteq N \setminus S} [v(S \cup T) - x(T)], & \text{if } S \neq \emptyset, \\ 0, & \text{if } S = \emptyset. \end{cases}$$

‘MW-consistency’ (Moldovanu and Winter, 1994a; Winter, 1989) states that if a payoff vector is chosen for a game, then the restriction of it to any subgroup should be chosen for the associated MW-reduced game (see Fig. 1).

\textbf{MW-consistency}. For each $N \in \mathcal{N}$, each $v \in \mathcal{V}_N^M$, each $x \in \varphi(v)$, and each $N' \subseteq N$, we have $x_{N'} \in \varphi(r_{N'}(v))$.

‘Converse MW-consistency’ (Moldovanu and Winter, 1994a; Winter, 1989) states that if a payoff vector for a game is such that its restriction to any pair of agents is chosen for the associated two-agent MW-reduced game, then it should be chosen for the original game.

\textbf{Converse MW-consistency}. For each $N \in \mathcal{N}$, each $v \in \mathcal{V}_N^M$, and each $x \in \mathbb{R}^N$, if for each $N' \subseteq N$ with $|N'| = 2$, we have $x_{N'} \in \varphi(r_{N'}(v))$, then $x \in \varphi(v)$.

On $\mathcal{V}_N^M$, the ‘aspiration correspondence’ (Albers, 1979; Bennett, 1983; Cross, 1967)
Fig. 1. Aspirations in three-agent TU-games. Let $N = \{1,2,3\}$, $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$, $v(\{1,2\}) = 6$, $v(\{1,3\}) = 5$, $v(\{2,3\}) = 7$, $v(N) = 0$. The set of aspirations for $v$ is the union of the three thick line segments in (b).

and the 'partnered aspiration solution' (Albers, 1979; Bennett, 1983) are MW-consistent and conversely MW-consistent (Moldovanu and Winter, 1994a; Winter, 1989). It follows from this result that each of these two solutions is the unique MW-consistent and conversely MW-consistent extension of its two-agent version to the $n$-agent case.

In the following subsections, we introduce other subsolutions of the aspiration correspondence that have been studied in the literature, and check which of them satisfies MW-consistency and/or converse MW-consistency.

### 3.1. Balanced aspiration solution

Given $N \in \mathcal{N}$, a collection $\mathcal{B} \subseteq 2^N$ of coalitions is weakly balanced on $N$ if there exists a list of non-negative weights $(\delta_s)_{s \in \mathcal{B}}$ such that for each $i \in N$

$$\sum_{s \subseteq \mathcal{B}, s \ni i} \delta_s = 1.$$ 

It is strictly balanced on $N$ if, in addition, all weights are positive.

Cross (1967) and Bennett (1983) argue that the competition among the coalitions for 'scarce' agents leads to a balanced structure of the generating coalitions, by driving up the payoff demands of these agents and driving down the payoff demands of others. The following solution is based on this idea:

**Balanced aspiration solution, $\text{BalAsp}$**. For each $N \in \mathcal{N}$ and each $v \in \mathcal{V}_N^N$, $\text{BalAsp}(v)$ is the collection of aspirations $x$ for $v$ such that $\mathcal{F}(v,x)$ is weakly balanced on $N$.

**Proposition 3.1.** On $\mathcal{V}_N^N$, the balanced aspiration solution is MW-consistent.

**Proof.** Let $N \in \mathcal{N}$, $v \in \mathcal{V}_N^N$, $x \in \text{BalAsp}(v)$, and $\mathcal{B} = \mathcal{F}(v,x)$. Then there exists $(\delta_s)_{s \in \mathcal{B}} \in \mathbb{R}_{\geq 0}^{\mathcal{B}}$ such that for each $i \in N$
\[
\sum_{S \cap i} \delta_S = 1.
\]

Let \( N' \subset N \) and \( B' = \mathcal{G}(r^v_N(v), x_{N'}) \). Since the aspiration correspondence is MW-consistent, \( x_{N'} \) is an aspiration for \( r^v_N(v) \). For each \( S \in B' \), let

\[
\lambda_S = \sum_{T \subset N', \, s.t. \, \exists \, S \cup T \in B} \delta_S.
\]

Then \( (\lambda_S)_{S \in B'} \in \mathbb{R}^{B'}_{+} \) and for each \( i \in N' \)

\[
\sum_{S \cap i} \lambda_S = \sum_{S \cap i} \sum_{T \subset N', \, s.t. \, \exists \, S \cup T \in B} \delta_S = \sum_{R \in B} \delta_R = 1.
\]

Thus, \( B' \) is weakly balanced on \( N' \) and, hence, \( x_{N'} \in BalAsp(r^v_N(v)) \). \( \square \)

The following example shows that the balanced aspiration solution violates converse MW-consistency.

**Example 3.1.** Let \( N = \{1,2,3\} \). Consider the following TU-game: for each \( S \subseteq N \)

\[
v(S) = \begin{cases} 
1, & \text{if } |S| = 2, \\
0, & \text{otherwise.}
\end{cases}
\]

Let \( x = (1,1,0) \). Then \( x \) is an aspiration for \( v \) and \( \mathcal{G}(v,x) = \{(1,3), (2,3), (3)\} \). Note that the aspiration correspondence and the balanced aspiration solution coincide in the two-agent case (Fig. 2). Therefore, as the aspiration correspondence is MW-consistent, for each \( N' \subset N \) with \( |N'| = 2 \), we have \( x_{N'} \in BalAsp(r^v_{N'}(v)) \). However, since \( \mathcal{G}(v,x) \) is not weakly balanced, we have \( x \notin BalAsp(v) \).

### 3.2. Aspiration kernel

Next, we turn to a solution that is similar to the prekernel (Maschler et al., 1972). Given \( N \in \mathcal{N} \), \( v \in \mathcal{V}^N_{\text{all}} \), \( x \in \mathbb{R}^N \), and \( i,j \in N \) with \( i \neq j \), let

\[
s_{ij}(v,x) = \max_{S \subseteq N \setminus \{i,j\}} [v(S) - x(S)].
\]

The number \( s_{ij}(v,x) \) represents the maximum "surplus" that agent \( i \) can obtain without the cooperation of agent \( j \), supposing that other agents agree upon \( x \). For each game \( v \), the aspiration kernel (Bennett, 1981) chooses those aspirations that equalize these surpluses for each pair of agents.

**Aspiration kernel, AspKer.** For each \( N \in \mathcal{N} \) and each \( v \in \mathcal{V}^N_{\text{all}} \), \( \text{AspKer}(v) \) is the collection of aspirations \( x \) for \( v \) such that for each \( S \in \mathcal{G}(v,x) \) and each pair \( i,j \in S \), we have \( s_{ij}(v,x) = \).
Fig. 2. The aspiration kernel and the aspiration nucleolus in two-agent and three-agent TU-games. In these cases, the aspiration kernel coincides with the aspiration nucleolus. (a) If \( v(\{1,2\}) \approx v(\{1\}) + v(\{2\}) \), they select the ‘standard solution’ payoff vector: each agent is given first her individual worth, and then what remains is divided equally. (b) Otherwise, they select \( (v(\{1\}), v(\{2\})) \). In panels (c) and (d), \( v(\{1,2,3\}) = 0 \) and the payoff vector chosen by them is indicated as \( x^* \).

The next lemma is due to Peleg (1986), and it essentially implies that the aspiration kernel is MW-consistent and conversely MW-consistent.

**Lemma 3.1.** For each \( N \in \mathcal{N} \), each \( v \in \mathcal{V}^N \), each \( x \in \mathbb{R}^N \), each \( N' \subset N \), and each pair \( i,j \in N' \), we have \( s_{ij}(r_{N'}^N(v), x_{N'} \) \( = s_{ij}(v, x) \).

**Proposition 3.2.** On \( \mathcal{V}^N \), the aspiration kernel is MW-consistent and conversely MW-consistent.

**Proof.** (MW-consistency) Let \( N \in \mathcal{N} \), \( v \in \mathcal{V}^N \), \( x \in Asp(Ker(v)) \), and \( N' \subset N \). Since the aspiration correspondence is MW-consistent, \( x_{N'} \) is an aspiration for \( r_{N'}^N(v) \). Let \( S \in \mathcal{G}(r_{N'}^N(v), x_{N'}) \) and \( i,j \in S \) with \( i \neq j \). Then, by the definition of \( r_{N'}^N(v)(S) \), there exists \( T \subseteq NN' \), where \( T \) may be the empty set, such that \( x(S) \leq r_{N'}^N(v)(S) = v(S \cup T) - x(T) \). Since \( x \in Asp(Ker(v)) \), \( S \cup T \in \mathcal{G}(v, x) \), and \( i,j \in S \cup T \), we have \( s_{ij}(v, x) = s_{ij}(v, x) \). By Lemma 3.1, \( s_{ij}(r_{N'}^N(v), x_{N'}) = s_{ij}(r_{N'}^N(v), x_{N'}) \).

(Converse MW-consistency) Let \( N \in \mathcal{N} \), \( v \in \mathcal{V}^N \), and \( x \in \mathbb{R}^N \) be such that for each \( N' \subset N \) with \( |N'| = 2 \), \( x_{N'} \in Asp(Ker(r_{N'}^N(v))) \). Since the aspiration correspondence is conversely MW-consistent and the aspiration kernel is its subsolution, \( x \) is an aspiration for \( v \). Let \( S \in \mathcal{G}(v, x) \) and \( i,j \in S \) with \( i \neq j \). Then \( x_i + x_j \leq v(S) - x(S(\{i,j\})) \leq v(S) - x(S(\{i,j\})) \leq v(S) - x(S(\{i,j\})) \).
3.3. Aspiration nucleolus

Next, we analyze a solution which is closely related to the prenucleolus (Schmeidler, 1969).

For each \( x \in \mathbb{R}^N \), let \( e(v,x) \) be the vector in \( \mathbb{R}^{2^N} \) defined by setting for each \( S \in \mathcal{P}(\emptyset) \), \( e_S(v,x) = v(S) - x(S) \). The number \( e_S(v,x) \) represents the dissatisfaction of \( S \) in \( v \) at \( x \). Also, let \( \theta(e(v,x)) \in \mathbb{R}^{2^{|N|} - 1} \) be obtained by rearranging the coordinates of \( e(v,x) \) in non-increasing order.

Given \( m \in \mathbb{N} \) and \( z, z' \in \mathbb{R}^m \), \( z \) is lexicographically smaller than \( z' \) if either (i) \( z_i < z'_i \) or (ii) there exists \( k > 1 \) such that \( z_k < z'_k \) and for each \( \ell < k \), \( z_{\ell} = z'_{\ell} \).

For each game, the aspiration nucleolus (Bennett, 1981) selects a payoff vector that for each other aspiration \( y \) for \( v \), \( \theta(e(v,y)) \) is lexicographically smaller than \( \theta(e(v,y')) \).

The aspiration nucleolus is a subsolution of both the aspiration kernel and the balanced aspiration solution (Sharkey, 1993).

Given \( N \in \mathcal{N} \), \( v \in \mathcal{V}_a^N \), \( x \in \mathbb{R}^N \), and \( \alpha \in \mathbb{R} \), let

\[
\mathcal{S}_\alpha(v,x) = \{ S \subseteq N \mid e_S(v,x) \geq \alpha \}.
\]

The following lemma is a characterization of the aspiration nucleolus.

Lemma 3.2. (Sharkey, 1993) For each \( N \in \mathcal{N} \), each \( v \in \mathcal{V}_a^N \), and each aspiration \( x \) for \( v \), \( x = \text{AspNuc}(v) \) if and only if for each \( \alpha \in \mathbb{R} \) with \( \mathcal{S}_\alpha(v,x) \neq \emptyset \), \( \mathcal{S}_\alpha(v,x) \) is strictly balanced on \( N \).

We use Lemma 3.2 to prove the following result:

**Proposition 3.3.** On \( \mathcal{V}_a^N \), the aspiration nucleolus is MW-consistent.

**Proof.** Let \( N \in \mathcal{N} \), \( v \in \mathcal{V}_a^N \), \( x = \text{AspNuc}(v) \), and \( N' \subseteq N \). Since the aspiration correspondence is MW-consistent, \( x_{N'} \) is an aspiration for \( r_{N'}^v(v) \). Let \( \alpha \in \mathbb{R} \) be such that \( \mathcal{S}_\alpha(r_{N'}^v(v),x_{N'}) \neq \emptyset \). By the definition of \( r_{N'}^v(v) \), for each \( S \in \mathcal{S}_\alpha(r_{N'}^v(v),x_{N'}) \), there exists \( T \subseteq \mathcal{N} \setminus N' \), where \( T \) may be the empty set, such that \( r_{N'}^v(v)(S) = v(S \cup T) - x(T) \). Since \( v(S \cup T) - x(T) - x(S) \geq \alpha \), \( \mathcal{S}_\alpha(v,x) \neq \emptyset \). To simplify the notation, let \( \mathcal{R} = \mathcal{S}_\alpha(v,x) \) and

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1 A similar characterization of the (pre)nucleolus is due to Kohlberg (1971).
By Lemma 3.2, \( \mathcal{B} \) is strictly balanced on \( N \). Thus, there exists a list \((\delta_S)_{S \in \mathcal{B}}\) of positive weights such that for each \( i \in N \)
\[
\sum_{S \ni i} \delta_S = 1.
\]

For each \( S \in \mathcal{B}' \), let
\[
\lambda_S = \sum_{T \subseteq N, \, S \cup T \in \mathcal{B}} \delta_{S \cup T}.
\]

As shown above, for each \( S \in \mathcal{B}' \), there exists \( T \subseteq N \) such that \( S \cup T \in \mathcal{B} \). Thus, for each \( S \in \mathcal{B}' \), \( \lambda_S > 0 \). Note also that, for each \( i \in N' \)
\[
\sum_{S \ni i} \lambda_S = \sum_{S \ni i} \sum_{T \subseteq N, \, S \cup T \in \mathcal{B}} \delta_{S \cup T} = \sum_{R \ni i} \delta_R = 1.
\]

Thus, \( \mathcal{B}' \) is strictly balanced on \( N' \). Since this is true for each \( \alpha \in \mathbb{R} \) with \( \mathcal{F}_\alpha(r_\alpha^i(v),x_N) \neq \emptyset \), by Lemma 3.2 we have \( x_N = \text{AspNuc}(r_\alpha^i(v)) \). \( \square \)

As the following example shows, the aspiration nucleolus violates \textit{converse MW-consistency}.

**Example 3.2.** Let \( N = \{1,2,3,4\} \). Consider the following TU-game for \( N \): for each \( S \subseteq N \)
\[
v(S) = \begin{cases} 
6, & \text{if } S \in \{\{1,2,3\},\{1,2,4\}\}, \\
0, & \text{otherwise}.
\end{cases}
\]

Let \( x = (3,3,0,0) \) and \( y = (2,2,2,2) \). It can be shown that \( x \in \text{AspKer}(v) \), \( y \) is an aspiration for \( v \), and \( e(v,y) \) is lexicographically smaller than \( e(v,x) \). Thus, \( x \neq \text{AspNuc}(v) \). Since the aspiration kernel is \textit{MW-consistent} and it coincides with the aspiration nucleolus in the two-agent case, for each pair \( i,j \in N \)
\[
(x_i,x_j) \in \text{AspKer}(r_{i,j}^i(v)) = \{\text{AspNuc}(r_{i,j}^i(v))\}.
\]

Thus, \( x \) satisfies the hypothesis of \textit{converse MW-consistency} for the aspiration nucleolus. However, we have \( x \neq \text{AspNuc}(v) \). \( \square \)

### 3.4. Equal gains aspiration solution

The next subsolution of the aspiration correspondence is based on the premise that agents when bargaining tend to share the gains equally. In our context, by forming a coalition, the agents forego the payoffs that they could have attained by forming alternative coalitions. Therefore, each agent’s largest payoff from alternative coalitions serves as an ‘outside option’. Formally, given \( N \in \mathcal{N} \), \( v \in \mathcal{V}_N \), \( x \in \mathbb{R}^N \), \( S \in \mathcal{G}(v,x) \), and \( i \in S \), the \textit{outside option for} \( i \) \textit{relative to} \( v, x \), \textit{and} \( S \) is defined by
$$d_i^x(v, x) = \max_{T \ni x \cap S \neq S} [v(T) - x(T)\{i\}]$$.

**Equal gains aspiration solution, EqAsp.** For each $N \subseteq N$ and each $v \in \mathcal{V}_N$, $\text{EqAsp}(v)$ is the collection of aspirations for $v$ such that for each $S \subseteq \mathcal{G}(v, x)$ and each pair $i,j \in S$, we have $x_i - d_i^x(v, x) = x_j - d_j^x(v, x)$.

Note that the aspiration kernel is a subsolution of the equal gains aspiration solution. As the following example shows the equal gains aspiration solution violates MW-consistency.

**Example 3.3.** Let $N = \{1, 2, 3\}$. Consider the following TU-game: for each $S \subseteq N$

$$v(S) = \begin{cases} 6, & \text{if } S \in \{\{1, 2, 3\}, \{1\}\}, \\ 0, & \text{otherwise.} \end{cases}$$

Let $x = (4, 2, 0)$. Then $x$ is an aspiration for $v$ and $\mathcal{G}(v, x) = \{(3), \{1, 2\}, \{1, 2, 3\}\}$.

Note that

$$x_1 - d_1^x(v, x) = - \max_{S \ni 1 \atop S \neq \{1\}} [v(S) - x(S)] = - \max\{0, -0, 0, -4\} = 0,$$

$$x_2 - d_2^x(v, x) = - \max_{S \ni 2 \atop S \neq \{1\}} [v(S) - x(S)] = - \max\{0, 0, -2, 6\} = 0,$$

$$x_1 - d_1^{1, 2, 3}(v, x) = - \max_{S \ni 1 \atop S \neq \{1, 2, 3\}} [v(S) - x(S)] = - \max\{0, -0, 0, -4\} = 0,$$

$$x_2 - d_2^{1, 2, 3}(v, x) = - \max_{S \ni 2 \atop S \neq \{1, 2, 3\}} [v(S) - x(S)] = - \max\{0, -4, -6, 4\} = 0,$$

$$x_3 - d_3^{1, 2, 3}(v, x) = - \max_{S \ni 3 \atop S \neq \{1, 2, 3\}} [v(S) - x(S)] = - \max\{0, 0, -3\} = 0.$$

Thus, $x \in \text{EqAsp}(v)$. Note that

$$r^1_{1, 2}(v)(\{1\}) = \max\{v(\{1\}), v(\{1, 3\}) - x_3\} = \max\{0, 0, 0\} = 0,$$

$$r^1_{1, 2}(v)(\{2\}) = \max\{v(\{2\}), v(\{2, 3\}) - x_3\} = \max\{0, 0, 0\} = 0,$$

$$r^1_{1, 2}(v)(\{1, 2\}) = \max\{v(\{1, 2\}), v(\{1, 2, 3\}) - x_3\} = \max\{6, 6\} = 6.$$

Since $\text{EqAsp}(r^1_{1, 2}(v)) = \{(3, 3)\}$, we have $(x_1, x_2) \not\in \text{EqAsp}(r^1_{1, 2}(v))$.

**Proposition 3.4.** On $\mathcal{V}_{all}$, the equal gains aspiration solution is conversely MW-consistent.
**Proof.** Let $N \subseteq \mathcal{N}$, $v \in \mathcal{V}_{\text{all}}^N$, and $x \in \mathbb{R}^N$ be such that for each $N' \subseteq N$ with $|N'| = 2$, $x_{N'} \in \text{EqAsp}(r_{N'}^v(v))$. Note that the aspiration kernel and the equal gains aspiration solution coincide in the two-agent case. Since the aspiration kernel is conversely MW-consistent, $x \in \text{AspKer}(v)$. Since $\text{AspKer}(v) \subseteq \text{EqAsp}(v)$, we have $x \in \text{EqAsp}(v)$. □

4. Two axiomatic characterizations

In this section, we study the implications of MW-consistency, converse MW-consistency, and the following three basic axioms:

**Equal treatment of equals.** For each $N \subseteq \mathcal{N}$, each $v \in \mathcal{V}_{\text{all}}^N$, and each pair $i,j \in N$, if for each $S \subseteq N \setminus \{i,j\}$, $v(S \cup \{i\}) = v(S \cup \{j\})$, then for each $x \in \varphi(v)$, $x_i = x_j$.

**Anonymity.** For each $N,M \subseteq \mathcal{N}$ with $|N| = |M|$, each $v \in \mathcal{V}_{\text{all}}^N$, each $w \in \mathcal{V}_{\text{all}}^M$, and each bijection $b : N \rightarrow M$, if for each $S \subseteq N$, $w(b(i) | i \in S) = v(S)$, then

$$\varphi(w) = \{ x \in \mathbb{R}^M | \text{there exists } y \in \varphi(v) \text{ such that for each } i \in N, x_i = y_{b(i)} \}.$$

**Zero-independence.** For each $N \subseteq \mathcal{N}$, each pair $v,w \in \mathcal{V}_{\text{all}}^N$, and each $y \in \mathbb{R}^N$, if for each $S \subseteq N$, $w(S) = v(S) + \sum_{i \in S} y_i$, then for each $x \in \varphi(v)$, $x + y \in \varphi(w)$.

Table 1 summarizes which solutions satisfy which properties.

As mentioned in the Introduction, most studies on coalesional games assume that the grand coalition eventually forms. In this 'standard approach’, a notion of reduced games was first introduced by Davis and Maschler (1965). Given $N \subseteq \mathcal{N}$, $v \in \mathcal{V}_{\text{all}}^N$, $x \in \mathbb{R}^N$, and $N' \subseteq N$, the DM-reduced game of $v$ relative to $x$ and $N'$ is defined by setting for each $S \subseteq N'$

<table>
<thead>
<tr>
<th>Domain $\mathcal{V}_{\text{all}}^N$</th>
<th>Asp</th>
<th>ParAsp</th>
<th>BalAsp</th>
<th>AspNuc</th>
<th>AspKer</th>
<th>EqAsp</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equal treatment of equals</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
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<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Zero-independence</td>
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<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Single-valuedness</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>MW-consistency</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>Converse MW-consistency</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>(Example 3.1)</td>
<td>(Example 3.2)</td>
<td>(Example 3.3)</td>
</tr>
<tr>
<td>MW-consistency</td>
<td>(Example 3.1)</td>
<td>(Example 3.2)</td>
<td>(Example 3.3)</td>
<td>(Proposition 3.2)</td>
<td>(Proposition 3.4)</td>
<td></td>
</tr>
</tbody>
</table>
satisfying equal treatment of equals

Proof. Clearly, the aspiration kernel is a subsolution of the aspiration correspondence,

\[ f^*_N(v)(S) = \begin{cases} 
\max_{\tau \subseteq N} [v(S \cup \tau) - x(\tau)], & \text{if } S \subseteq \{N', \emptyset\}, \\
v(N) - x(N\setminus N'), & \text{if } S = N', \\
0, & \text{if } S = \emptyset.
\]

Associated axioms of consistency and its converse are defined as follows.

**DM-consistency.** For each \( N \in \mathcal{N} \), each \( v \in \mathcal{V}^N \), each \( x \in \varphi(v) \), and each \( N' \subset N \), we have \( x_{N'} \in \varphi(f^*_N(v)) \).

**Converse DM-consistency.** For each \( N \in \mathcal{N} \), each \( v \in \mathcal{V}^N \), and each \( x \in \mathbb{R}^N \) with \( x(N) = v(N) \), if for each \( N' \subset N \) with \( |N'| = 2 \) we have \( x_{N'} \in \varphi(f^*_N(v)) \), then \( x \in \varphi(v) \).

Given a game \( v \) for \( N \), a preimputation for \( v \) is a payoff vector \( x \in \mathbb{R}^N \) with \( x(N) = v(N) \). For each game \( v \), the prekernel (Maschler et al., 1972) chooses those preimputations that equalize the surpluses for each pair of agents. On \( \mathcal{V}_a \), the prekernel is the only subsolution of the preimputation correspondence satisfying equal treatment of equals, zero-independence, DM-consistency, and converse DM-consistency (Peleg, 1986). It so happens that a similar result holds for the aspiration kernel.

**Theorem 4.1.** On \( \mathcal{V}_a \), the aspiration kernel is the only subsolution of the aspiration correspondence satisfying equal treatment of equals, zero-independence, MW-consistency, and converse MW-consistency.

**Proof.** Clearly, the aspiration kernel is a subsolution of the aspiration correspondence satisfying equal treatment of equals and zero-independence. By Proposition 3.2, it also satisfies MW-consistency and converse MW-consistency.

Conversely, let \( \varphi \) be a subsolution of the aspiration correspondence satisfying the four axioms. Clearly, \( \varphi \) coincides with the aspiration kernel in the two-agent case. Let \( N, v, x \) be a pair of agents with \( |N| \geq 3 \), and \( v \in \mathcal{V}_a \). First, we show that \( \varphi(v) \subseteq \text{AspKer}(v) \). Let \( x \in \varphi(v) \). By MW-consistency of \( \varphi \), for each \( N' \subset N \) with \( |N'| = 2 \), we have \( x_{N'} \in \varphi(r^*_N(v)) = \text{AspKer}(r^*_N(v)) \). Since the aspiration kernel is conversely MW-consistent, \( x \in \text{AspKer}(v) \).

Next, we show that \( \text{AspKer}(v) \subseteq \varphi(v) \). Let \( y \in \text{AspKer}(v) \). Since the aspiration kernel is MW-consistent, for each \( N' \subset N \) with \( |N'| = 2 \)

\[ y_{N'} \in \text{AspKer}(r^*_N(v)) = \varphi(r^*_N(v)). \]

Since \( \varphi \) is conversely MW-consistent, \( y \in \varphi(v) \).

Altogether, \( \varphi(v) = \text{AspKer}(v) \). ∎

For each game, the prenucleolus (Schmeidler, 1969) selects a payoff vector that lexicographically minimizes the dissatisfaction of the coalitions over the set of preimputations. On \( \mathcal{V}_a \), the prenucleolus is the only subsolution of the preimputation correspondence satisfying single-valuedness, anonymity, zero-independence, and DM-consistency (Sobolev, 1975).

It turns out that, by using Lemma 3.2 and by following the argument in Sobolev
(1975), one can obtain a similar axiomatic characterization of the aspiration nucleolus. (Since the proof is very long, we provide it in Appendix A.)

**Theorem 4.2.** On \( \mathcal{V}^{m}_{u} \), the aspiration nucleolus is the only subsolution of the aspiration correspondence satisfying single-valuedness, anonymity, zero-independence, and MW-consistency.

5. Remarks on the NTU-case

The definitions of the aspiration correspondence, the partnered aspiration solution, the balanced aspiration solution, and the equal gains aspiration solution have been generalized to define corresponding solutions for non-transferable utility coalitional games (NTU-games, for short).

Moldovanu and Winter (1994a) study MW-consistency and converse MW-consistency on the domain of all NTU-games. They show that, on this domain, both the aspiration correspondence and the partnered aspiration solution satisfy these two properties. Here, we report, without proofs, two additional results:

- On the domain of all NTU-games, the balanced aspiration solution satisfies MW-consistency.
- On the domain of all NTU-games, the equal gains aspiration solution violates converse MW-consistency.

Acknowledgements

We thank William Thomson, François Maniquet, and an anonymous referee for helpful comments. We are responsible for any remaining errors. Financial support from the Murata Science Foundation is gratefully acknowledged.

Appendix A

In this appendix, we provide the proof of Theorem 4.2. As mentioned before, the proof is similar to that of a theorem in Sobolev (1975), which is written in Russian. The proof of Sobolev’s theorem (in English) can be found in Peleg (1988). Essential parts of Peleg’s proof are reproduced in Snijders (1995).

**Proof of Theorem 4.2.** Clearly, the aspiration nucleolus is a subsolution of the
aspiration correspondence satisfying single-valuedness, anonymity, and zero-independence. By Proposition 3.3, it is also MW-consistent.

Conversely, let \( \varphi \) be a subsolution of the aspiration correspondence satisfying the four axioms. Let \( N \subseteq \mathcal{N} \), \( v \in \mathcal{V}_{\mathcal{N}} \), and \( x = \text{AspNuc}(v) \). We show, in seven steps, that \( x = \varphi(v) \).

Let

\[
A = \{ \alpha \in \mathbb{R} \mid \text{there exists } S \subseteq N \text{ such that } \alpha = e_s(v, x) \},
\]

and \((\alpha_1, \alpha_2, \ldots, \alpha_{|A|}) \in \mathbb{R}^{|A|}\) be the enumeration of \( A \) with \( \alpha_1 > \alpha_2 > \cdots > \alpha_{|A|} \). To simplify the notation, for each \( k \in \{1, 2, \ldots, |A|\} \), we write \( \mathcal{I}_k = \mathcal{I}_{\alpha_k}(v, x) \). Also, for each \( k \in \{1, 2, \ldots, |A|\} \) and each \( i \in N \), let \( \mathcal{I}_{ki} = \{ S \in \mathcal{I}_k \mid i \in S \} \). Given \( k \in \{1, 2, \ldots, |A|\} \), by Lemma 3.2, \( \mathcal{I}_k \) is strictly balanced on \( N \). Moreover, the associated weights can be chosen to be rational. Thus, there exist a natural number \( \mu_k \) and a list of natural numbers \((\mu_b)_{b \in \mathcal{B}_k}\) such that for each \( i \in N \), \( \sum_{b \in \mathcal{B}_k} \mu_b = \mu_k \). Let \( \mathcal{B}_k \) be the partition of \( N \) such that for each pair \( i, j \in N \), there exists \( B \in \mathcal{B}_k \) with \( i, j \in B \) if and only if \( \mathcal{I}_k = \mathcal{I}_{ki} \). Let \( \beta_k = \max_{B \in \mathcal{B}_k} |B| \), \( \gamma_k = \sum_{b \in \mathcal{B}_k} \mu_b \), and

\[
\lambda_k = \left( \frac{\gamma_k}{\mu_k} \right).
\]

Step 1. Given \( k \in \{1, 2, \ldots, |A|\} \), we construct \( M_k \in \mathcal{N} \) and \( \mathcal{I}_k \subseteq 2^M \setminus \{0\} \) that satisfy the following conditions:

(i) \( N \subseteq M_k \);
(ii) \( |M_k| = \beta_k \cdot \lambda_k \);
(iii) for each \( S \subseteq \mathcal{I}_k \), there exists \( T \in \mathcal{I}_k \) such that \( S \subseteq T \);
(iv) for each \( S \subseteq N \) and each \( T \subseteq \mathcal{I}_k \), if \( S \subseteq T \), then \( T \cap N = S \) and \( S \in \mathcal{I}_k \);
(v) for each \( i \in M_k \), we have \( |\mathcal{I}_{ki}| = \mu_k \) and \(|\{ j \in M_k \mid \mathcal{I}_{ki} = \mathcal{I}_{kj} \}| = \beta_k \), where \( \mathcal{I}_k = \{ T \in \mathcal{I}_k \mid i \in T \} \) and \( \mathcal{I}_{ki} = \{ T \in \mathcal{I}_k \mid j \in T \} \).

Let \((B_1, B_2, \ldots, B_{|A|})\) be an enumeration of \( \mathcal{B}_k \). For each \( h \in \{1, 2, \ldots, A\} \), we construct a set \( D_h \) of agents as follows:

- if \( h \leq |\mathcal{B}_k| \) and \( |B_h| = \beta_k \), then let \( D_h = B_h \);
- if \( h \leq |\mathcal{B}_k| \) and \( |B_h| < \beta_k \), then let \( D_h \) be the union of \( B_h \) and \((\beta_k - |B_h|)\) agents chosen from \( N \setminus N \);
- if \( h > |\mathcal{B}_k| \), then let \( D_h \) be a set of \( \beta_k \) agents chosen from \( N \setminus N \).

Since \( \mathcal{B}_k \) is a partition of \( N \) and the set of potential agents is countably infinite, it is clear that, in the above construction of \( D_h \)'s, we can make them mutually exclusive. Then, let

\[
M_k = D_1 \cup D_2 \cup \cdots \cup D_{|A|}.
\]

Note that \((D_1, D_2, \ldots, D_{|A|})\) is a partition of \( M_k \). By construction, \( M_k \) satisfies conditions (i) and (ii).

Next, imagine that there are \( \gamma_k \) empty ‘rooms.’ We will fill these rooms with
(appropriately replicated) groups in \( \{D_1, D_2, \ldots, D_j\} \), and each room will correspond to an element of \( \mathcal{S}_k \). For each \( S \in \mathcal{S}_k \), create \( \mu_k \) copies of the set

\[
\bigcup_{h \in \{1, \ldots, |\beta_h|\} \text{ s.t. } D_h \cap S \neq \emptyset} D_h.
\]

Since the total number of these copies is \( \chi = \sum_{S \in \mathcal{S}_k} \mu_k \), we can put them into different rooms. Recall that for each \( i \in N \), \( \sum_{S \in \mathcal{S}_k} \mu_k = \mu_k \). This implies that for each \( h \in \{1, \ldots, |\beta_h|\} \), group \( D_h \) belongs to exactly \( \mu_k \) rooms. Next, for each \( h \in \{ |\beta_h| + 1, \ldots, \lambda_k \} \), create \( \mu_k \) copies of \( D_h \). Since

\[
\lambda_k = \left( \frac{\chi}{\mu_k} \right).
\]

we can place these copies of \( D_{|\beta_h|+1}, \ldots, D_{\lambda_k} \) into the rooms so that all \( \chi \) rooms contain the same number of groups and for each \( h \in \{ |\beta_h| + 1, \ldots, \lambda_k \} \), group \( D_h \) belongs to exactly \( \mu_k \) rooms. It is easy to see that \( \mathcal{S}_k \) thus constructed satisfies conditions (iii), (iv), and (v). (The above construction of \( M_k \) and \( \mathcal{S}_k \) is illustrated in Fig. 3 for a simple case.)

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 9 & \cdots & 67 & 69 \\
B_1 & B_2 & B_3 & D_1 & D_2 & D_3 & D_4 & D_5 & \cdots & D_{34} & D_{35}
\end{array}
\]

<table>
<thead>
<tr>
<th>( S_k )</th>
<th>{1,2,3}</th>
<th>{1,2}</th>
<th>{3,4}</th>
<th>{3}</th>
<th>{4}</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu_{(1,2,3)} = 1 )</td>
<td>( \mu_{(1,2)} = 2 )</td>
<td>( \mu_{(3,4)} = 1 )</td>
<td>( \mu_{(3)} = 1 )</td>
<td>( \mu_{(4)} = 2 )</td>
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</tr>
<tr>
<td>1 2 11 15</td>
<td>1 7 9 11 13</td>
<td>5 6 7 18 23</td>
<td>5 9 17 27 29</td>
<td>6 11 19 23 27</td>
<td></td>
</tr>
<tr>
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<td>2 8 10 12 14</td>
<td>3 4 8 16 24</td>
<td>3 10 18 28 30</td>
<td>4 12 20 24 28</td>
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<td>60 62 66 68 70</td>
<td>56 60 64 66 69</td>
<td></td>
</tr>
</tbody>
</table>

Fig. 3. Step 1 of the proof of Theorem 4.2. In the above example, \( N = \{1,2,3,4\} \), \( \mathcal{S}_k = \{\{1,2,3\},\{1,2\},\{3,4\},\{3\},\{4\}\} \), \( \mu_k = 3 \), \( \mu_{(1,2,3)} = \mu_{(3,4)} = \mu_{(3)} = 1 \), \( \mu_{(1,2)} = \mu_{(4)} = 2 \). Thus, \( \beta_k = \max\{2,1\} = 2 \), \( \chi = 1 + 1 + 1 + 2 + 2 = 7 \), and

\[
\lambda_k = \left( \frac{\chi}{\mu_k} \right) = \left( \frac{7}{3} \right) = 35.
\]
Step 2. Given $k \in \{1, 2, \ldots, |A|\}$, let $D_k = \{D_1, D_2, \ldots, D_{\lambda_k}\}$, $M_k$, and $\mathcal{T}_k$ be constructed as in Step 1. We show that, for each pair $i, j \in M_k$, there exists a permutation $\pi_{M_k}$ on $M_k$ such that $\pi_{M_k}(i) = j$ and for each $T \in \mathcal{T}_k$, $\pi_{M_k}(T) \in \mathcal{T}_k$.

Let $i, j \in M_k$. By condition (v) of $T_k$ in Step 1, $|\mathcal{T}_k| = |\mathcal{T}_k'| = \mu_k$. Thus, there exists a permutation $\pi_{M_k}$ on $\mathcal{T}_k$ such that $\pi_{M_k}(\mathcal{T}_k') = \mathcal{T}_k'$. Note that $\mathcal{T}_k$ has a property that if $\mu_k$ distinct coalitions in $\mathcal{T}_k$ are chosen, then there exists exactly one group in $\mathcal{T}_k$ that is included in all of these $\mu_k$ coalitions. Thus, $\pi_{M_k}$ induces a permutation on $D_k$. Formally, this permutation, denoted $\pi_{M_k}$, is defined by setting for each $D \in D_k$

$$\pi_{M_k}(D) = \bigcap_{T \in \mathcal{T}_k \atop T \supseteq D} \pi_{M_k}(T).$$

Note that each coalition in $\mathcal{T}_k$ can be viewed as a coalition of groups. For each $T \in \mathcal{T}_k$, its image under $\pi_{M_k}$ is defined by

$$\pi_{M_k}(T) = \bigcup_{D \in \mathcal{T}_k} \pi_{M_k}(D).$$

Now, we show that for each $T \in \mathcal{T}_k$, $\pi_{M_k}(T) \in \mathcal{T}_k$. Let $T \in \mathcal{T}_k$. Then, by the definition of $\pi_{M_k}$, for each $D \in D_k$ with $D \subseteq T$, we have $\pi_{M_k}(D) \subseteq \pi_{M_k}(T)$. Thus,

$$\pi_{M_k}(T) = \bigcup_{D \in \mathcal{T}_k} \pi_{M_k}(D) \subseteq \pi_{M_k}(T).$$

Since

$$|T| = |\pi_{M_k}(T)| = |\pi_{M_k}(T)| = \left(1 - \frac{\lambda_k - 1}{\mu_k - 1}\right),$$

we have $\pi_{M_k}(T) = \pi_{M_k}(T)$. Thus, $\pi_{M_k}(T) \in \mathcal{T}_k$.

By construction, each group in $D_k$ contains exactly $\beta_k$ agents. For each $D \in D_k$ with $i \in D$, choose a bijection $\pi_{M_k} : D \rightarrow \pi_{M_k}(D)$ such that $\pi_{M_k}(i) = j$. For each $D \in D_k$ with $i \in D$, choose an arbitrary bijection $\pi_{M_k} : D \rightarrow \pi_{M_k}(D)$. Given the list $(\pi_{M_k})_{D \in D_k}$ of such bijections, define the permutation $\pi_{M_k}$ on $M_k$ by setting for each $D \in D_k$ and each $h \in D$, $\pi_{M_k}(h) = \pi_{M_k}(h)$. Clearly, $\pi_{M_k}(i) = j$. Let $T \in \mathcal{T}_k$. Then

$$\pi_{M_k}(T) = \bigcup_{h \in T} \pi_{M_k}(h) = \bigcup_{D \subseteq T} \bigcup_{h \in D} \pi_{M_k}(h) = \bigcup_{D \subseteq T} \pi_{M_k}(D) = \pi_{M_k}(T) \in \mathcal{T}_k.$$

Thus, $\pi_{M_k}$ is a desired permutation on $M_k$.

Step 3. For each $k \in \{1, 2, \ldots, |A|\}$, let $M_k$ and $\mathcal{T}_k$ be constructed as in Step 1. Here, we construct a set $M$ and a partition of $2^{|M| \setminus \{M, \emptyset\}}$.

Let

$$M = M_1 \times M_2 \times \cdots \times M_{|A|}.$$
In order to stress the fact that $M$ is a Cartesian product of the sets of agents, we write its subsets and its elements in bold face. Note that, since the set of potential agents is countably infinite, in the presence of anonymity, $M$ can be viewed as an element of $N$.

For each $k \in \{1, 2, \ldots, |A|\}$, let
\[
\mathcal{S}_{|A|+1} = \{S \subseteq M \mid \text{there exists } T \in \mathcal{T}_k \text{ such that } S = M_1 \times \cdots \times M_{k-1} \times T \times M_{k+1} \times \cdots \times M_{|A|}\}.
\]

Let
\[
\mathcal{S}_{|A|+1} = 2^{|A|}(\{M, \emptyset\} \cup \bigcup_{k=1}^{|A|} \mathcal{S}_k).
\]

**Step 4.** We show that for each pair $i, j \in M$, there exists a permutation $\pi_M$ on $M$ such that (i) $\pi_M(i) = j$ and (ii) for each $k \in \{1, 2, \ldots, |A|\}$, $\pi_M(S) \subseteq \mathcal{S}_k$.

Let $i = (i_1, i_2, \ldots, i_{|A|}) \in M$ and $j = (j_1, j_2, \ldots, j_{|A|}) \in M$. By Step 2, for each $k \in \{1, 2, \ldots, |A|\}$, there exists a permutation $\pi_M$ on $M_k$ such that $\pi_M(i_k) = j_k$ and for each $T \in \mathcal{T}_k$, $\pi_M(T) \in \mathcal{T}_k$. Define the permutation $\pi_M$ on $M$ by setting for each $h = (h_1, h_2, \ldots, h_{|A|}) \in M$
\[
\pi_M(h) = (\pi_M(h_1), \pi_M(h_2), \ldots, \pi_M(h_{|A|})).
\]

Clearly, $\pi_M(i) = j$ and for each $k \in \{1, 2, \ldots, |A|\}$, $\pi_M(S) \subseteq \mathcal{S}_k$. Note that $\pi_M$ induces a permutation on $\bigcup_{k=1}^{|A|} \mathcal{S}_k$. Thus, also for each $S \in \mathcal{S}_{|A|+1}$, $\pi_M(S) \subseteq \mathcal{S}_{|A|+1}$.

**Step 5.** Let $w \in \mathcal{T}_M$ be defined as follows: (i) $w(M) = v(N) - x(N)$; (ii) for each $k \in \{1, 2, \ldots, |A|\}$ and each $S \in \mathcal{S}_k$, $w(S) = \alpha_k$; and (iii) for each $S \in \mathcal{S}_{|A|+1}$, $w(S) = \min\{v_{|A|}(N) - x(N)\}$. We show that for each $i \in M$, $\varphi_i(w) = 0$.

Let $i, j \in M$. By Step 4, there exists a permutation $\pi_M$ on $M$ such that $\pi_M(i) = j$, and for each $k \in \{1, 2, \ldots, |A|\}$, $\pi_M(S) \subseteq \mathcal{S}_k$. Let $w' \in \mathcal{T}_M$ be defined by setting for each $S \subseteq M$, $w'(S) = w((\pi_M)^{-1}(S))$. By anonymity, $\varphi_i(w) = \varphi_j(w')$.

Let $S \subseteq M$. Since $\bigcup_{k=1}^{|A|+1} \mathcal{S}_k = 2^{|A|}(\{M, \emptyset\})$, there exists $k \in \{1, 2, \ldots, |A|\}$ such that $S \subseteq \mathcal{S}_k$. Since $\pi_M(S) \subseteq \mathcal{S}_k$, by the definitions of $w$ and $w'$, $w'(S) = w(S)$. Clearly, $w'(M) = w(M)$. Thus, $w' = w$. Therefore, $\varphi_i(w) = \varphi_j(w') = \varphi_j(w)$.

The above argument can be applied to all $i, j \in M$. Since $x$ is an aspiration for $v$, $x$ is individually feasible and coalitionally rational. By coalitional rationality, for each $k \in \{1, 2, \ldots, |A|\}$, $\alpha_k \leq 0$. Together with individual feasibility, we have either $\alpha_i = 0$ or $v(N) = x(N)$. Thus, we have (i) for each $S \subseteq M$, $w(S) \leq 0$, and (ii) there exists $T \subseteq M$ such that $T \neq \emptyset$ and $w(T) = 0$. Let $i \in T$. Then, since $\varphi_i(w)$ is coalitionally rational in $w$,
\[
0 = w(T) \leq \sum_{j \in T} \varphi_i(w) = |T| \cdot \varphi_i(w),
\]
so that $\varphi_i(w) \geq 0$. By individual feasibility of $\varphi_i(w)$, $\varphi_i(w) = 0$.

Thus, for each $i \in M$, $\varphi_i(w) = 0$.

**Step 6.** Let
Thus, $M' = \{(i, i, \ldots, i) \mid i \in N\}$.

Clearly, $|M'| = |N|$. Let $b : N \to M'$ be the bijection defined by setting for each $i \in N$, $b(i) = (i, i, \ldots, i)$. We show that for each $S \subseteq N$, $r_{M'}^{w}(w)(b(S)) = v(S) - x(S)$.

Let $S \subseteq N$. Then there exists $k \in \{1, 2, \ldots, |A|\}$ such that $v(S) - x(S) = \alpha_k$. By properties (iii) and (iv) of $\mathcal{F}_k$ in Step 1, there exists $T \in \mathcal{F}_k$ such that $T \cap N = S$. Let

$$R = M_1 \times \cdots \times M_{k-1} \times T \times M_{k+1} \times \cdots \times M_{|A|}.$$ 

Then $b(S) \subseteq R$. Moreover, since $T \cap N = S$, we have $R \cap M' \subseteq M' \cap M'$. Thus,

$$r_{M'}^{w}(w)(b(S)) = \max_{Q \subseteq M' \cap M'} w(b(S) \cup Q) = w(b(S) \cup (R \cap b(S))) = w(R).$$

Since $R \subseteq S$, $w(R) = \alpha_k$. Thus,

$$r_{M'}^{w}(w)(b(S)) = w(R) = \alpha_k = v(S) - x(S).$$

Now, we claim that the opposite (weak) inequality also holds. Let $Q \subseteq M' \cap M'$. If $b(S) \cup Q \subseteq S_{|A|+1}$, then $w(b(S) \cup Q) = \alpha_{|A|} = v(S) - x(S)$. If there exists $\ell \leq |A|$ such that $b(S) \cup Q \subseteq S_{|A|}$, then there exists $T' \in \mathcal{F}_\ell$ such that $b(S) \cup Q \subseteq M_1 \times \cdots \times M_{\ell-1} \times T' \times M_{\ell+1} \times \cdots \times M_{|A|}$.

Since $Q \subseteq M' \cap M'$ and $S \subseteq T'$, we have $T' \cap N = S$. Thus, by property (iv) of $\mathcal{F}_\ell$ in Step 1, we have $S \subseteq \mathcal{F}_\ell$, so that $w(b(S) \cup Q) = \alpha_{\ell} = v(S) - x(S)$.

Thus, for each $Q \subseteq M' \cap M'$, we have $w(b(S) \cup Q) = v(S) - x(S)$. This implies that

$$r_{M'}^{w}(w)(b(S)) = \max_{Q \subseteq M' \cap M'} w(b(S) \cup Q) = v(S) - x(S).$$

By the definition of $w$, $w(M) = v(N) - x(N)$. For each $k \in \{1, 2, \ldots, |A|\}$, by the definition of $\mathcal{F}_\ell$, we have $N \subseteq \mathcal{F}_\ell$. Thus, for each $k \in \{1, 2, \ldots, |A|\}$, by property (iv) of $\mathcal{F}_\ell$ in Step 1, there exists no $T \in \mathcal{F}_k$ such that $T \supseteq N$. This implies that, for each $S \supseteq M'$, we have $S \subseteq S_{|A|+1}$, so that

$$w(S) = \min\{\alpha_{|A|}, v(N) - x(N)\} \leq v(N) - x(N).$$

Thus,

$$r_{M'}^{w}(w)(b(N)) = r_{M'}^{w}(w)(M') = \max_{Q \subseteq M' \cap M'} w(M' \cup Q) = v(N) - x(N).$$

Therefore, for each $S \subseteq N$

$$r_{M'}^{w}(w)(b(S)) = w(S) - x(S).$$

**Step 7.** By max consistency, for each $i \in M'$, $\varphi(r_{M'}^{w}(w)) = 0$.

Finally, by anonymity and zero-independence of $\varphi$, we deduce that, for each $i \in N$,

$$\varphi_i(v) = \varphi_{w(i)}(r_{M'}^{w}(w)) + x_i = 0 + x_i = x_i.$$

Thus, $\varphi(v) = \text{AspNuc}(v)$. □
References


