ON M-RECTANGLE CHARACTERISTICS AND ISOMORPHISMS OF MIXED (F)-, (DF)- SPACES

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Abstract

In this thesis, we consider problems on the isomorphic classification and quasiequivalence properties of mixed (F)-, (DF)- power series spaces which, up to isomorphisms, consist of basis subspaces of the complete projective tensor products of power series spaces and (DF)- power series spaces.

Important linear topological invariants in this consideration are the m-rectangle characteristics, which compute the number of points of the defining sequences of the mixed (F)-, (DF)- power series spaces, that are inside the union of m rectangles. We show that the systems of m-rectangle characteristics give a complete characterization of the quasidiagonal isomorphisms between Montel spaces that are in certain classes of mixed (F)-, (DF)- power series spaces under proper definitions of equivalence. Using compound invariants, we also show that the m-rectangle characteristics are linear topological invariants on the class of mixed (F)-, (DF)- power series spaces that consist of basis subspaces of the complete projective tensor products of a power series space of finite type and a (DF)- power series space of infinite type. From these invariances, we obtain the quasiequivalence of absolute bases in the spaces of the same class that are Montel and quasidiagonally isomorphic to their Cartesian square.

M-DİKDÖRTGEN KARAKTERİSTİKLERİ VE KARIŞIK (F)-, (DF)- UZAYLARININ EŞDÖNÜŞÜMLERİ ÜZERİNE

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Anahtar Kelimeler: doğrusal topolojik invaryantlar, bileşik invaryantlar, m-dikdörtgen karakteristikleri, karışık (F)-, (DF)- uzayları, bazlarn sanki denklikleri.

Özet

Bu tezde, kuvvet serisi uzayları ve (DF)- kuvvet serisi uzaylarının tam projektif tensör çarpımlarının baz altuzaylarına eş yapılı olan karışık (F)-, (DF)- kuvvet serisi uzaylarının eş yapı sınıflandırmaları ve sanki denklik özelliklerine dair problemler incelenmiştir.

Bu incelemedeki önemli doğrusal topolojik invaryantlar, karışık (F)-, (DF)- kuvvet serisi uzaylarını tanımlayan dizilerin m adet dikdörtgen içinde kalan noktalarını hesaplayan m-dikdörtgen karakteristikleridir. İlgili denklik tanımları altında, m-dikdörtgen karakteristik sistemlerinin, bazı karışık (F)-, (DF)- kuvvet serisi uzayları sınıflarına ait Montel uzayları arasındaki sanki diyagonal eşdönüşümleri tamamen karakterize ettiği gösterilmiştir. Bileşik invaryantlar kullanılarak, m-dikdörtgen karakteristiklerinin sonlu tipli kuvvet serisi uzayları ve sonsuz tipli (DF)- kuvvet serisi uzaylarının tensör çarpımlarının baz altuzaylarına eş yapılı olan karışık (F)-, (DF)- kuvvet serisi uzayları sınıfı üzerinde doğrusal topolojik invaryantlar olduğu ispatlanmıştır. Bu invaryantlar aracılığıyla, aynı sınıfa ait, Montel ve kendisiyle Kartezyen çarpımlarına sanki diyagonal olarak eş yapılı olan uzaylarda mutlak bazların sanki denkliği elde edilmiştir.

To my family

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CHAPTER 1

Introduction

In this thesis, we aim to characterize isomorphisms between certain classes of locally convex spaces. Linear topological invariants are powerful tools in this regard, as they are a way to distinguish non-isomorphic spaces.

By an *isomorphism* between two locally convex spaces X and Y, we mean that there exists a continuous linear map from X into Y that is one-to-one, onto, and with a continuous inverse. The spaces X and Y are then called *isomorphic*, denoted by $X \simeq Y$. If $\mathcal{X}$ is a class of locally convex spaces and Γ is a set with an equivalence relation $\sim$, then $\gamma : \mathcal{X} \to \Gamma$ is called a *linear topological invariant* if $X \simeq Y$ implies $\gamma(X) \sim \gamma(Y)$ for all $X, Y \in \mathcal{X}$.

Results on isomorphic classification of non-normable locally convex spaces and related problems were initiated by the introduction of *the approximative dimensions* by Kolmogorov ([23]) and Pełczyński ([29]). Shortly after, variations of the approximative dimensions called *the diametral dimensions* were introduced by Bessaga, Pełczyński, Rolewicz ([1]) and Mityagin ([25]), and these invariants were proven to be more convenient for certain classes of locally convex spaces.

Definition 1.0.1 *Let U and V be absolutely convex sets in a locally convex space X such that $V \subset cU$ for some constant $c > 0$. Then, for every $n \in \mathbb{N}$, the n*th Kolmogorov diameter *of V with respect to U is defined by*

$$d_n(V, U) = \inf_{L \in L_n} \inf\{\rho > 0 : V \subset \rho U + L\},$$

where L_n denotes the collection of all subspaces of X with dimension less than or equal to n. Then, the diametral dimensions *of X are defined by*

$$\begin{aligned} \Gamma(X) &= \{(\xi_n) : \forall U \ \exists V \ \lim \xi_n d_n(V, U) = 0\}, \\ \Gamma'(X) &= \left\{(\xi_n) : \exists U \ \forall V \ \lim \frac{\xi_n}{d_n(V, U)} = 0\right\}. \end{aligned}$$

These invariants were especially useful for the classes of Köthe spaces with a regular basis, where the Köthe spaces are defined as follows.

Definition 1.0.2 *A matrix $A = (a_{i,p})_{i,p\in\mathbb{N}}$ of non-negative numbers satisfying*

(i) for each $i \in \mathbb{N}$ there exists $p = p(i)$ such that $a_{i,p} > 0$,

(ii) $a_{i,p} \leq a_{i,p+1}$ for all $i, p \in \mathbb{N}$,

is called a Köthe Matrix. *For a Köthe matrix A, the locally convex space $K(A)$ of all sequences $\xi = (\xi_i)_{i\in\mathbb{N}}$ with the locally convex topology generated by the system of seminorms $\{||.||_p : p \in \mathbb{N}\}$, where*

$$||\xi||_p = \sum_{i\in\mathbb{N}} |\xi_i| a_{i,p} < \infty,$$

is called the Köthe space *defined by A.*

For any Köthe matrix $A = (a_{i,p})_{i,p\in\mathbb{N}}$, $K(A)$ is a Fréchet space, that is, a complete metrizable locally convex space. Also, for a Köthe matrix $A = (a_{i,p})_{i,p\in\mathbb{N}}$ with non-zero terms, we have the isomorphism

$$K(A) \simeq \text{proj}_{\leftarrow p} l_1 \left((a_{i,p})_{i\in\mathbb{N}}\right).$$

A sequence (x_n) in a locally convex space X is called a *(Schauder) basis*, if for each x in X there is a unique sequence of scalars (t_n) such that $x = \sum t_n x_n$, where the sum converges in the topology of X. Moreover, (x_n) is called an *absolute basis* if for each continuous seminorm p on X there exists a continuous seminorm q on X and a constant $C > 0$ such that

$$\sum |t_n| p(x_n) \leq C q(x)$$

for every $x \in X$. Every Fréchet space with an absolute basis is isomorphic to a Köthe space. From Grothendieck-Pietsch theorem, $K(A)$ is nuclear if and only if for every $p \in \mathbb{N}$ there exists $q \in \mathbb{N}$, $q \geq p$ so that

$$\sum_{i=1}^{\infty} \frac{a_{i,p}}{a_{i,q}} < \infty.$$

Since any basis in a nuclear Fréchet space is an absolute basis by Dynin-Mityagin theorem, any nuclear Fréchet space with a basis is isomorphic to a nuclear Köthe space.

An important subclass of Köthe spaces are the power series spaces, which are defined as follows.

Definition 1.0.3 *For any positive sequence* $a = (a_i)_{i \in \mathbb{N}}$*,*

$$E_\alpha(a) = proj_{\lambda<\alpha} l_1(\exp(\lambda a))$$

where $-\infty < \alpha \leq \infty$*, is called a* power series space of finite type *if* $\alpha < \infty$*, or a* power series space of infinite type *if* $\alpha = \infty$*.*

If the sequence a increases to infinity, then $E_\alpha(a)$ is a Schwartz space. Without loss of generality, we only need to consider

$$E_0(a) = \text{proj}_{\leftarrow p}\, l_1(\exp(-\frac{1}{p}a)), \quad E_\infty(a) = \text{proj}_{\leftarrow p}\, l_1(\exp(pa))$$

for representing power series spaces, since any power series space of finite type is isomorphic to $E_0(a)$ and for every strictly increasing sequence $(\lambda_p)_{p \in \mathbb{N}}$ with $\lim \lambda_p = \alpha$ we have $E_\alpha(a) = K(A)$ where $A = (\exp(\lambda_p a_i))_{i,p \in \mathbb{N}}$.

Many concrete spaces in analysis are isomorphic to power series spaces. As important examples, let $A(\mathbb{D})$ denote the space of analytic functions in the unit disk on the complex plane and $A(\mathbb{C})$ denote the space of entire functions on the complex plane, both endowed with the topology of uniform convergence on compact subsets. Then, $A(\mathbb{D})$ is isomorphic to a power series space of finite type and $A(\mathbb{C})$ is isomorphic to a power series space of infinite type. Also, the space of infinitely differentiable functions on the interval $[0, 1]$, denoted by $C^\infty[0, 1]$, is isomorphic to *the space of rapidly decreasing sequences*, denoted by s, and defined by

$$s = E_\infty((\log i)_{i \in \mathbb{N}}).$$

The isomorphic classification of power series spaces were considered by Mityagin and, for Schwartz power series spaces, the following result was shown in [25] by using diametral dimensions and their computation in terms of their defining sequences.

Proposition 1.0.1 *For positive sequences* $a = (a_i)_{i \in \mathbb{N}}$ *and* $b = (b_i)_{i \in \mathbb{N}}$ *both monotonically increasing to infinity, the following statements are equivalent:*

(i) $E_0(a) \simeq E_0(b)$*.*

(ii) $E_\infty(a) \simeq E_\infty(b)$.

(iii) *There exists a constant* $C > 1$ *such that* $\frac{1}{C}a_i \leq b_i \leq Ca_i$ *for all* $i \in \mathbb{N}$.

Mityagin also investigated the isomorphic classification of non-Schwartz power series spaces in [26], [27], and later in [28], by analysing the counting functions

$$N_a(u, v) = |\{i \in \mathbb{N} : u \leq a_i \leq v\}|, \quad 0 \leq u \leq v < \infty,$$

where $|S|$ denotes the number of elements of a given set S if S is a finite set and equal to ∞ if S is an infinite set, and obtained the following criterion.

Proposition 1.0.2 *For positive sequences* $a = (a_i)_{i\in\mathbb{N}}$ *and* $b = (b_i)_{i\in\mathbb{N}}$*, the following conditions are equivalent:*

(i) $E_0(a) \simeq E_0(b)$.

(ii) $E_\infty(a) \simeq E_\infty(b)$.

(iii) *There exists a constant* $R > 0$ *such that for any* u, v, $0 \leq u \leq v < \infty$,

$$N_a(u, v) \leq N_b(Ru, \frac{v}{R}), \quad N_b(u, v) \leq N_a(Ru, \frac{v}{R}).$$

A related question in isomorphic classification of locally convex spaces is whether a locally convex space has the *quasiequivalence property*, that is, if any two bases in a locally convex space are quasiequivalent.

Definition 1.0.4 *Two bases* (e_n) *and* (f_n) *of a locally convex space* X *are called* quasiequivalent *if the operator* $T : X \to X$ *where* $Te_n = t_n f_{\sigma(n)}$ *for some sequence of scalars* (t_n) *and a bijection* $\sigma : \mathbb{N} \to \mathbb{N}$ *for every* $n \in \mathbb{N}$ *is an isomorphism.*

It was shown by Dragilev ([14], [15]) that $A(\mathbb{D})$ has the quasiequivalence property. Mityagin has shown in [26] that nuclear power series spaces have the quasiequivalence property. Zakharyuta has shown in [34] that Schwartz power series spaces have the quasiequivalence property. The quasiequivalence property for arbitrary power series spaces was then shown by Mityagin in [27].

Dragilev has also considered nuclear Fréchet spaces in the classes (d_1) and (d_2) with regular basis, where regular bases and the classes (d_1) and (d_2) are defined as follows.

Definition 1.0.5 *A basis $\{e_i : i \in \mathbb{N}\}$ in a Fréchet space E is called* regular *if there is a sequence of seminorms $\{||.||_p : p \in \mathbb{N}\}$ generating the topology of E such that*

$$\frac{||e_i||_p}{||e_i||_{p+1}} \geq \frac{||e_{i+1}||_p}{||e_{i+1}||_{p+1}}$$

for all $i, p \in \mathbb{N}$.

Definition 1.0.6 *Let X be a Fréchet space with an absolute basis $(e_n)_{n=1}^{\infty}$ and a system of seminorms $\{||.||_p : p \in \mathbb{N}\}$ defining the topology of X. Then, X said to belong in class (d_1) if there exists p such that for every q there exists r and n_0 such that*

$$||e_n||_q^2 \leq ||e_n||_p ||e_n||_r, \quad n \geq n_0.$$

X said to belong in class (d_2) if for every p there exists q such that for every r and n_0

$$||e_n||_q^2 \geq ||e_n||_p ||e_n||_r, \quad n \geq n_0.$$

As examples of spaces in these classes, any power series space of finite type belongs in class (d_2), and any power series space of infinite type belongs in class (d_1).

It was shown by Dragilev in [16], by using the diametral dimension $\Gamma(X)$, that nuclear Fréchet spaces in classes (d_1) and (d_2) with regular basis have the quasiequivalence property. Crone, Robinson ([9]), and Kondakov ([24]), has later shown that the diametral dimension $\Gamma'(X)$ distinguishes regular bases, hence any nuclear Fréchet space with a regular basis has the quasiequivalence property. Djakov has shown in [10] that equivalence of characteristics can be used instead of equality in the proof of Crone and Robinson, which provided a new method in the consideration of linear topological invariants.

In the case of distinguishing spaces without a regular basis, the diametral dimensions are not very efficient as the following example, due to Rolewicz ([30]), shows.

Example 1.0.1 *The cartesian product $A(\mathbb{D}) \times A(\mathbb{C})$ has no regular basis and $A(\mathbb{D})$ and $A(\mathbb{D}) \times A(\mathbb{C})$ are non-isomorphic. However, $\Gamma'(A(\mathbb{D})) = \Gamma'(A(\mathbb{D}) \times A(\mathbb{C}))$.*

To investigate Köthe spaces without a regular basis, more generalized linear topological invariants were constructed Zakharyuta in [35], [36] and [37]. Subsequently, new geometrical invariants named *compound invariants* were introduced by Zakharyuta in [38], [39] and [40], where the asymptotic behaviour of Kolmogorov n-diameters of certain absolutely convex sets that are geometrically constructed (by taking intersections, convex hulls, etc.) from given bases of neighborhoods of zero, called *synthetic* sets, were analysed and shown to be equivalent to the generalized invariants in [36] and [37]. Also, by considering characteristics other than Kolmogorov n-diameters, and using interpolational methods in geometric constructions, new linear topological invariants were introduced by Zakharyuta, and used in joint papers by Chalov, Djakov, Terzioğlu, Yurdakul and Zakharyuta ([3], [4], [6], [7], [11], [12], [33]) for the isomorphic classification of cartesian products and tensor products of power series spaces, and more generally, the *power Köthe spaces of first type*, that is, the class of spaces

$$E(\lambda, a) = K\left(\exp\left(\left(-\frac{1}{p} + p\lambda_i\right) a_i\right)\right),$$

where $\lambda = (\lambda_i)_{i\in\mathbb{N}}$ and $a = (a_i)_{i\in\mathbb{N}}$ are sequences of positive numbers, containing cartesian and projective tensor products of power series spaces. An important invariant in the consideration of power Köthe spaces of first type is the m-rectangle characteristics, introduced by Chalov in [2] for the isomorphic classification of certain classes of Hilbert spaces, which compute the number of the points (λ_i, a_i) that are inside the union of m-rectangles.

Definition 1.0.7 *let $\lambda = (\lambda_i)_{i\in\mathbb{N}}$ and $a = (a_i)_{i\in\mathbb{N}}$ be sequences of positive numbers and let $m \in \mathbb{N}$. Then, the function*

$$\mu_m^{(\lambda,a)}(\delta, \varepsilon; \tau, t) = \left|\bigcup_{k=1}^{m}\{i \ : \ \delta_k \le \lambda_i \le \varepsilon_k \ , \ \tau_k \le a_i \le t_k\}\right|$$

defined for $\delta = (\delta_k)$, $\varepsilon = (\varepsilon_k)$, $\tau = (\tau_k)$ and $t = (t_k)$ such that $0 \le \delta_k \le \varepsilon_k \le 2$, $0 < \tau_k \le t_k < \infty$, where $k = 1, 2, \cdots, m$, is called the m-rectangle characteristic *of the pair (λ, a).*

Compound invariants were also used in joint papers by Goncharov, Terzioğlu and Zakharyuta in [18], [19] and [20] for the isomorphic classification of complete projective tensor products of power series spaces with the (DF)- power series spaces, where the (DF)- power series spaces are defined as follows.

Definition 1.0.8 *For a sequence of positive numbers $a = (a_i)_{i\in\mathbb{N}}$,*

$$E_0'(a) = ind_{q\to}\ l_1(\exp(\frac{1}{q}a_i))$$

is called a (DF)- power series space of finite type*, and*

$$E'_\infty(a) = ind_{q\to}\ l_1(\exp(-qa))$$

is called a (DF)- power series of infinite type.

(DF)- power series spaces are ultrabornological (DF)-spaces since they are countable inductive limits of Banach spaces. Note that (DF)- power series spaces are not necessarily the duals of power series spaces, such an identification is true only in the case of nuclearity of the corresponding power series space.

Problems on isomorphic classification and quasiequivalence of bases of a wider class of spaces

$$G(\lambda, a) = \mathrm{proj}_{\leftarrow p}\big(\mathrm{ind}_{q\to}\ l_1\left(\omega(p,q)\right)\big), \tag{1.1}$$

where $\omega_i(p,q) = \exp\left(\left(p - q\lambda_i\right) a_i\right)$ for sequences of positive numbers $\lambda = (\lambda_i)_{i\in\mathbb{N}}$, $a = (a_i)_{i\in\mathbb{N}}$, which includes the basis subspaces of the tensor products

$$E_\infty(c)\hat{\otimes}_\pi E'_\infty(d),$$

were investigated by Chalov, Terzioğlu and Zakharyuta in [5], and it was shown that for each $m \in \mathbb{N}$, the corresponding m-rectangle characteristic is a linear topological invariant for this class under some equivalence.

In this thesis, we consider problems on isomorphic classification of the mixed (F)-, (DF)- spaces

$$G_{\alpha,\beta}(\lambda, a) = \mathrm{proj}_{\leftarrow p}\left(\mathrm{ind}_{q\to}\ l_1\left(\omega^{\alpha,\beta}(p,q)\right)\right) \tag{1.2}$$

for $\alpha, \beta \in \{0, \infty\}$ with $p, q \in \mathbb{N}$ and $\omega^{\alpha,\beta}(p,q) = (\omega_i^{\alpha,\beta}(p,q))_{i\in\mathbb{N}}$ when

(1) $\omega_i^{\infty,\infty}(p,q) = \exp\left(\left(p - q\lambda_i\right) a_i\right)$,

(2) $\omega_i^{0,\infty}(p,q) = \exp\left(\left(-\frac{1}{p} - q\lambda_i\right) a_i\right)$,

(3) $\omega_i^{0,0}(p,q) = \exp\left(\left(-\frac{1}{p}\lambda_i + \frac{1}{q}\right) a_i\right)$,

(4) $\omega_i^{\infty,0}(p,q) = \exp\left(\left(p\lambda_i + \frac{1}{q}\right) a_i\right)$,

where $\lambda = (\lambda_i)_{i\in\mathbb{N}}$, $a = (a_i)_{i\in\mathbb{N}}$ are sequences of positive numbers.

These classes, up to isomorphisms, consist of basis subspaces of projective tensor products $E_\infty(c)\hat{\otimes}_\pi E'_\infty(d)$, $E_0(c)\hat{\otimes}_\pi E'_\infty(d)$, $E_0(c)\hat{\otimes}_\pi E'_0(d)$, $E_\infty(c)\hat{\otimes}_\pi E'_0(d)$ respectively, where c and d are sequences of positive numbers.

In Chapter 2 we establish the notation and give preliminary results. In Chapter 3, we obtain criteria for quasidiagonal isomorphisms between the spaces in each of the four classes above. In Chapter 4, we present the m-rectangle characteristics and related equivalences, and show that the systems of m-rectangle characteristics completely characterize the quasidiagonal isomorphisms between the spaces in each of these four classes. In Chapter 5, by using compound invariants, we prove that the m-rectangle characteristics are linear topological invariants for each $m \in \mathbb{N}$ on the class of spaces (2) when $\omega_i^{0,\infty}(p,q) = \exp\left(\left(-\frac{1}{p} - q\lambda_i\right) a_i\right)$. In Chapter 6, we show the quasiequivalence of absolute bases for the spaces in the class (2) that are Montel and quasidiagonally isomorphic to their Cartesian square.

CHAPTER 2

Preliminaries

2.1 Mixed (F)-, (DF)- Spaces

We consider the classes of mixed (F)-, (DF)- spaces

$$G_{\alpha,\beta}(\lambda, a) = \mathrm{proj}_{\leftarrow p} \left(\mathrm{ind}_{q\rightarrow}\ l_1\left(\omega^{\alpha,\beta}(p,q)\right)\right) \tag{2.1}$$

for $\alpha, \beta \in \{0, \infty\}$, with $p, q \in \mathbb{N}$, and $\omega^{\alpha,\beta}(p,q) = (\omega_i^{\alpha,\beta}(p,q))_{i\in\mathbb{N}}$, when

(1) $\omega_i^{\infty,\infty}(p,q) = \exp\left((p - q\lambda_i)\, a_i\right)$,

(2) $\omega_i^{0,\infty}(p,q) = \exp\left(\left(-\frac{1}{p} - q\lambda_i\right) a_i\right)$,

(3) $\omega_i^{0,0}(p,q) = \exp\left(\left(-\frac{1}{p}\lambda_i + \frac{1}{q}\right) a_i\right)$,

(4) $\omega_i^{\infty,0}(p,q) = \exp\left(\left(p\lambda_i + \frac{1}{q}\right) a_i\right)$,

where $\lambda = (\lambda_i)_{i\in\mathbb{N}}$, $a = (a_i)_{i\in\mathbb{N}}$ are sequences of positive numbers.

Here, $l_1\left(\omega^{\alpha,\beta}(p,q)\right)$ denote the weighted l_1-spaces

$$l_1\left(\omega^{\alpha,\beta}(p,q)\right) = \left\{ x = (\xi_i)_{i\in\mathbb{N}} : ||x||_{p,q} = \sum_{i=1}^{\infty} |\xi_i| \omega_i^{\alpha,\beta}(p,q) < \infty \right\}.$$

For each $p \in \mathbb{N}$, we put $X_p := \bigcup_{q\in\mathbb{N}} l_1\left(\omega_i^{\alpha,\beta}(p,q)\right)$, equipped with the inductive topology, that is, the finest locally convex topology for which the inclusion maps

$$i_q : l_1\left(\omega^{\alpha,\beta}(p,q)\right) \rightarrow X_p$$

are continuous. Then, X_p is an inductive limit for each $p \in \mathbb{N}$. We have $X_{p+1} \subset X_p$ for every $p \in \mathbb{N}$, hence we define the projective limit

$$G_{\alpha,\beta}(\lambda, a) = \mathrm{proj}_{\leftarrow p}\ X_p$$

and endow it with the projective topology, that is, the coarsest topology for which the inclusion maps

$$\pi_p : G_{\alpha,\beta}(\lambda, a) \to X_p$$

are continuous.

$G_{\alpha,\beta}(\lambda, a)$ is a Montel space, that is, a quasibarrelled space in which every bounded set is relatively compact, if and only if $(a_i) \to \infty$.

For the spaces $G_{\alpha,\beta}(\lambda, a)$ in the classes $(1)-(4)$, the coordinate basis $\{e_n : n \in \mathbb{N}\}$, where e_n are the sequences which are zero at each coordinate except the nth coordinate and one at the nth coordinate, is an absolute basis. A subspace of $G(\lambda, a)$ that is generated by a subset of the coordinate basis is called a *basis subspace* (or *step* subspace as in [17]).

Lemma 2.1.1 *Any space in one of the classes* $(1)-(4)$ *is isomorphic to a space* $G_{\alpha,\beta}(\lambda, a)$*, where* λ *and* a *satisfy the conditions*

$$a_i \geq 1, \quad \frac{1}{a_i} \leq \lambda_i \leq 1. \tag{2.2}$$

Proof. For any space $G_{\alpha,\beta}(\tilde{\lambda}, \tilde{a})$, take

$$(\lambda_i, a_i) = \begin{cases} \left(\max\left\{\frac{1}{1+\tilde{a}_i}, \tilde{\lambda}_i\right\}, 1+\tilde{a}_i\right) & \text{if } \tilde{\lambda}_i \leq 1, \\ \left(1, 1+\tilde{\lambda}_i\tilde{a}_i\right) & \text{if } \tilde{\lambda}_i > 1. \end{cases}$$

For example, if we consider a space $G_{\infty,0}(\tilde{\lambda}, \tilde{a})$ in the class (4) where $\omega_i^{\infty,0}(p,q) = \exp((p\lambda_i + \frac{1}{q})a_i)$, then we have the inequalities

$$\left(p\tilde{\lambda}_i + \frac{1}{q}\right)\tilde{a}_i \leq \left(p\lambda i + \frac{1}{q}\right)a_i \leq \left(p\tilde{\lambda}_i + \frac{1}{q}\right)\tilde{a}_i + 2p$$

for every $p, q \in \mathbb{N}$, which imply that the identity map and its inverse map are continuous, hence the identity map is an isomorphism between $G_{\infty,0}(\tilde{\lambda}, \tilde{a})$ and $G_{\infty,0}(\lambda, a)$. The other cases can be obtained similarly. ■

2.2 Projective Spectra of (LB)-Spaces

Any space $G_{\alpha,\beta}(\lambda, a)$ of the form (2.1) can also be considered as a projective spectrum $\mathcal{X} = \left(X_p, \pi_p^r\right)$ where $X_p = \operatorname{ind}_{q\to} l_1(\omega^{\alpha,\beta}(p,q))$ and the connecting maps π_p^r are inclusions. In this case, $\mathcal{X}$ is a strongly reduced spectrum of complete Haussdorff

(LB)-spaces. Hence, the spaces $G_{\alpha,\beta}(\lambda, a)$ have the following property that is mentioned in [31] and stated in [32] (Proposition 3.3.8) as follows.

Proposition 2.2.1 *Let $\mathcal{X} = (X_n, \varrho^n_m)$ and $\mathcal{Y} = (Y_n, \sigma^n_m)$ be two strongly reduced spectra of complete Haussdorff (LB)-spaces, and T : Proj $\mathcal{X} \to$ Proj $\mathcal{Y}$ a continuous linear map. Then there is a morphism of locally convex spectra $\tilde{T} : \tilde{\mathcal{X}} \to \mathcal{Y}$, where $\tilde{\mathcal{X}}$ is a subsequence of $\mathcal{X}$, such that $T =$ Proj $\mathcal{X}$. In particular, Proj $\mathcal{X} \simeq$ Proj $\mathcal{Y}$ implies that $\mathcal{X}$ and $\mathcal{Y}$ are equivalent.*

By this proposition, if $T : G_{\alpha,\beta}(\lambda, a) \to G_{\alpha,\beta}(\tilde{\lambda}, \tilde{a})$ is a continuous linear operator, then for every $r \in \mathbb{N}$ there exists $p \geq r$ and a continuous linear map T_r such that we have the following commutative diagram:

$$\begin{array}{ccc} G_{\alpha,\beta}(\lambda, a) & \xrightarrow{\quad T \quad} & G_{\alpha,\beta}(\tilde{\lambda}, \tilde{a}) \\ \Big\downarrow \pi_p & & \Big\downarrow \tilde{\pi}_r \\ \text{ind}_{q\to}\ l_1(\omega^{\alpha,\beta}(p,q)) & \xrightarrow{T_r} & \text{ind}_{s\to}\ l_1(\tilde{\omega}^{\alpha,\beta}(r,s)) \end{array}$$

For each $r \in \mathbb{N}$, T_r is continuous if and only if $T_r \circ i_q$ is continuous for every $q \in \mathbb{N}$. So, for each $q \in \mathbb{N}$, by applying Grothendieck's factorization theorem, we get $s \in \mathbb{N}$ and a continuous linear operator $T_{r,q}$ so that the following diagram commutes:

$$\begin{array}{ccc} \text{ind}_{q\to}\ l_1(\omega^{\alpha,\beta}(p,q)) & \xrightarrow{T_r} & \text{ind}_{s\to}\ l_1(\tilde{\omega}^{\alpha,\beta}(r,s)) \\ \Big\uparrow i_q & \nearrow & \Big\uparrow \tilde{i}_s \\ l_1(\omega^{\alpha,\beta}(p,q)) & \xrightarrow{\quad T_{r,q} \quad} & l_1(\tilde{\omega}^{\alpha,\beta}(r,s)) \end{array}$$

2.3 Power Series Spaces and (DF)- Power Series Spaces

The spaces $G_{\alpha,\beta}(\lambda, a)$ with the corresponding weight sequences $\omega^{\alpha,\beta}(p,q)$ for the cases $(1) - (4)$ are isomorphic to power series spaces or (DF)- power series spaces under the following conditions.

Proposition 2.3.1 *Given the sequences of positive numbers $\lambda = (\lambda_i)_{i\in\mathbb{N}}$ and $a = (a_i)_{i\in\mathbb{N}}$, the following statements are equivalent:*

(i) $G_{\infty,\infty}(\lambda, a) \simeq E_\infty(a)$.

(ii) $G_{0,\infty}(\lambda, a) \simeq E_0(a)$.

(iii) $G_{0,0}(\lambda, a) \simeq E_0'(a)$.

(iv) $G_{\infty,0}(\lambda, a) \simeq E_0'(a)$.

(v) $\lim_{i\to\infty} \lambda_i = 0$.

Also, the following statements are equivalent:

(i) $G_{\infty,\infty}(\lambda, a) \simeq E_\infty'(a)$.

(ii) $G_{0,\infty}(\lambda, a) \simeq E_\infty'(a)$.

(iii) $G_{0,0}(\lambda, a) \simeq E_0(a)$.

(iv) $G_{\infty,0}(\lambda, a) \simeq E_\infty(a)$.

(v) $\inf\{\lambda_i : i \in \mathbb{N}\} > 0$.

In the case when a space $G_{\alpha,\beta}(\lambda, a)$ is not isomorphic to a power series space or a (DF)- power series space, $G_{\alpha,\beta}(\lambda, a)$ is said to be a *mixed* (F)-, (DF)- space.

Given two sequences of positive numbers $a = (a_i)_{i\in\mathbb{N}}$ and $\tilde{a} = (\tilde{a}_i)_{i\in\mathbb{N}}$, we denote by $a \asymp \tilde{a}$, if there exists a constant $\alpha > 1$ such that

$$\frac{1}{\alpha} a_i \leq \tilde{a}_i \leq \alpha a_i, \quad i \in \mathbb{N}.$$

For Schwartz power series spaces and (DF)- power series spaces, we have the following criteria for isomorphisms.

Proposition 2.3.2 *If $a = (a_i)_{i\in\mathbb{N}}$ and $\tilde{a} = (\tilde{a}_i)_{i\in\mathbb{N}}$ are sequences of positive numbers monotonically increasing to ∞, and $\theta, \vartheta \in \{0, \infty\}$, then*

(i) $E_\theta(a) \simeq E_\theta(\tilde{a}) \Leftrightarrow a \asymp \tilde{a}$,

(ii) $E_\vartheta'(a) \simeq E_\vartheta'(\tilde{a}) \Leftrightarrow a \asymp \tilde{a}$.

Also, $E_0(a)$ is never isomorphic to $E_\infty(\tilde{a})$, and $E_\theta(a)$ and $E_\vartheta'(\tilde{a})$ are not isomorphic if one of the sequences a or $\tilde{a}$ is not bounded.

The statements (i) and (ii) is due to Mityagin ([25]). The fact that a power series space of finite type cannot be isomorphic to a power series space of infinite type is a well known result which is shown by using diametral dimensions. To show that $E_\theta(a)$ and $E'_\vartheta(\tilde{a})$ are not isomorphic if one of the sequences a or $\tilde{a}$ is not bounded, assume contrarily that $E_\theta(a)$ and $E'_\vartheta(\tilde{a})$ are isomorphic, where one of the sequences a or $\tilde{a}$ is not bounded. Since $E_\theta(a)$ is a Fréchet space , it admits a fundamental sequence of bounded sets if and only if it is normable. (See [22], Corollary 12.4.4) As $E'_\vartheta(\tilde{a})$ is a (DF)-space, both spaces should admit a fundamental sequence of bounded sets. However, one of the sequences a or $\tilde{a}$ is not bounded, so one of the spaces is not normable, which is a contradiction. Therefore, $E_\theta(a)$ and $E'_\vartheta(\tilde{a})$ cannot be isomorphic if one of the sequences a or $\tilde{a}$ is not bounded.

2.4 Tensor Products of (F)- and (DF)- Spaces

Given two Hausdorff locally convex spaces E and F, we denote by $E\hat{\otimes}_\pi F$ the complete projective tensor product of E and F, that is, the completion of the finest locally convex topology on $E \otimes F$ for which the canonical bilinear map $\otimes : E \times F \to E \otimes F$ is continuous.

The tensor products $E_\infty(c)\hat{\otimes}E'_\infty(d)$, $E_0(c)\hat{\otimes}E'_\infty(d)$, $E_0(c)\hat{\otimes}E'_0(d)$ and $E_\infty(c)\hat{\otimes}E'_0(d)$ are isomorphic to spaces in classes $(1)-(4)$, respectively. For example, $E_\infty(c)\hat{\otimes}E'_\infty(d)$ can be considered as a space of the form (2.1) where

$$\omega_i(p,q) = \exp(pc_{k(i)} - qd_{l(i)})$$

for some bijection $\mathbb{N} \to \mathbb{N} \times \mathbb{N}$ that sends $i \in \mathbb{N}$ to $(k(i), l(i)) \in \mathbb{N}$. If we take

$$a_i = \max\{c_{k(i)}, d_{l(i)}\}, \quad \lambda_i = \frac{d_{k(i)}}{a_i},$$

then this space is isomorphic to a space in class (1) with $\omega_i^{\infty,\infty}(p,q) = \exp((p - q\lambda_i)a_i)$.

Actually, the spaces in the classes $(1)-(4)$, up to isomorphisms, consist of basis subspaces of the projective tensor products

(1) $E_\infty(c)\hat{\otimes}E'_\infty(d)$

(2) $E_0(c)\hat{\otimes}E'_\infty(d)$

(3) $E_0(c)\hat{\otimes}E_0'(d)$

(4) $E_\infty(c)\hat{\otimes}E_0'(d)$

respectively, where $c=(c_i)_{i\in\mathbb{N}}$ and $d=(d_i)_{i\in\mathbb{N}}$ are sequences of positive numbers.

Let us show the above claim for the spaces that are in class (1) where $\omega_i^{\infty,\infty}(p,q)=\exp\left((p-q\lambda_i)\,a_i\right)$. The claim for the spaces in the classes $(2)-(4)$ can be obtained analogously. For this purpose, we need the following proposition which can be found in [22] (Theorem 15.4.2, Corollary 15.5.4).

Proposition 2.4.1 *(a) If $E=proj_{i\in I}\ E_i$ and $F=proj_{j\in J}\ F_j$ are reduced projective limits of Haussdorff locally convex spaces, then*

$$E\hat{\otimes}_\pi F\simeq proj_{(i,j)\in I\times J}\ E_i\hat{\otimes}_\pi F_j.$$

(b) If E and F be Haussdorff locally convex spaces such that F is normable and $E=ind_{i\in I}\ E_i$ is an inductive limit of locally convex spaces, then

$$E\hat{\otimes}_\pi F\simeq ind_{i\in I}\ E_i\hat{\otimes}_\pi F.$$

Now, let $G_{\infty,\infty}(\lambda,a)$ be a space in the class (1) with $\omega_i^{\infty,\infty}(p,q)=\exp\left((p-q\lambda_i)\,a_i\right)$. Then, we have

$$G_{\infty,\infty}(\lambda,a)=\text{proj}_{\leftarrow p}\ \text{ind}_{q\rightarrow}\ l_1\left(\exp(pc_i-qd_i)\right),$$

where $c_i=a_i$ and $d_i=\lambda_i a_i$. Considering the cross norms for tensor products of l_1 spaces, we have the natural isomorphism

$$l_1(\exp(pc_i))\hat{\otimes}_\pi l_1(\exp(-qd_i))\simeq l_1(\exp(pc_j-qd_k)),$$

where $(j,k)\in\mathbb{N}\times\mathbb{N}$. Hence, $G_{\infty,\infty}(\lambda,a)$ is isomorphic to a basis subspace of

$$X:=\text{proj}_{\leftarrow p}\ \text{ind}_{q\rightarrow}\left(l_1(\exp(pc_i))\hat{\otimes}_\pi l_1(\exp(-qd_i))\right).$$

For each $p\in\mathbb{N}$, $l_1(\exp(pc_i))$ is a Banach space and $\text{ind}_{q\rightarrow}\ l_1(\exp(-qd_i))$ is an inductive limit, hence by Proposition 2.4.1 (b),

$$\text{ind}_{q\rightarrow}\ \left(l_1(\exp(pc_i))\hat{\otimes}_\pi l_1(\exp(-qd_i))\right)\simeq l_1(\exp(pc_i))\hat{\otimes}_\pi\text{ind}_{q\rightarrow}\ l_1(\exp(-qd_i)),$$

which implies that

$$X\simeq\text{proj}_{\leftarrow p}\left(l_1(\exp(pc_i))\hat{\otimes}_\pi\text{ind}_{q\rightarrow}\ l_1(\exp(-qd_i))\right).$$

Then, by Proposition 2.4.1 (a), we obtain

$$X \simeq \text{proj}_{\leftarrow p}\, l_1(\exp(pc_i))\hat{\otimes}_\pi \text{ind}_{q\rightarrow}\, l_1(\exp(-qd_i)) = E_\infty(c)\hat{\otimes}_\pi E'_\infty(d).$$

Therefore, $G_{\infty,\infty}(\lambda, a)$ is isomorphic to a basis subspace of $E_\infty(c)\hat{\otimes}_\pi E'_\infty(d)$.

2.5 Quasidiagonal Isomorphisms

Two locally convex topological vector spaces X, $\tilde{X}$, with respective absolute bases $\{x_i\}_{i\in\mathbb{N}}$ and $\{\tilde{x}_i\}_{i\in\mathbb{N}}$, are called *quasidiagonally isomorphic*, denoted by $X \overset{qd}{\simeq} \tilde{X}$, if there exists a locally convex space isomorphism $T : X \to \tilde{X}$ such that

$$Tx_i = t_i\tilde{x}_{\sigma(i)}$$

for a sequence of scalars (t_i), and a bijection $\sigma : \mathbb{N} \to \mathbb{N}$. If such a quasidiagonal isomorphism exists, then the bases $\{x_i\}_{i\in\mathbb{N}}$ and $\{\tilde{x}_i\}_{i\in\mathbb{N}}$ are called *quasiequivalent*. X is said to be *quasidiagonally embedded* in $\tilde{X}$ if X is quasidiagonally isomorphic onto its image in $\tilde{X}$.

If T is a quasidiagonal isomorphism such that $t_i = 1$ for all $i \in \mathbb{N}$, then X and $\tilde{X}$ are called *permutationally isomorphic*, denoted by $X \overset{p}{\simeq} \tilde{X}$. If T is a quasidiagonal isomorphism such that $\sigma(i) = i$ for all $i \in \mathbb{N}$, then X and $\tilde{X}$ are called *diagonally isomorphic*, denoted by $X \overset{d}{\simeq} \tilde{X}$.

The following proposition is a well known result ([36], [40]), which is shown by using Cantor-Bernstein-Schröder theorem.

Proposition 2.5.1 *Given the mixed (F)-, (DF)- spaces X and $\tilde{X}$ of the form (2.1), if X is quasidiagonally embedded in $\tilde{X}$, and $\tilde{X}$ is quasidiagonally embedded in X, then $X \overset{qd}{\simeq} \tilde{X}$.*

2.6 Hall-König Theorem

In order to construct quasidiagonal embeddings, we will need the following theorem from combinatorics, referred to as Hall-König Theorem, which can be found in [21].

Theorem 2.6.1 *Suppose that for each i of a system of indices I corresponds a finite subset S_i of a set S. Then, there exists an injection $\sigma : I \to S$ such that $\sigma(i) \in S_i$ if and only if*

$$|\bigcup_{j=1}^{m} S_{i_j}| \geq m$$

for any choice of m distinct indices $i_1, \ldots, i_m$.

CHAPTER 3

Criteria For Quasidiagonal Isomorphisms

In this section, we establish criteria for the quasidiagonal isomorphisms between Montel spaces $G_{\alpha,\beta}(\lambda, a)$ that are in the classes $(1) - (4)$ in terms of certain properties of their defining sequences λ and a. The following criteria for quasidiagonal isomorphisms between the spaces $G_{\infty,\infty}(\lambda, a)$ belonging to class (1), where

$$\omega_i^{\infty,\infty}(p, q) = \exp\left((p - q\lambda_i)\, a_i\right),$$

was given in [5].

Proposition 3.0.2 *For Montel spaces $G_{\infty,\infty}(\lambda, a)$ and $G_{\infty,\infty}(\tilde{\lambda}, \tilde{a})$, the following conditions are equivalent:*

(i) $G_{\infty,\infty}(\lambda, a) \overset{p}{\simeq} G_{\infty,\infty}(\tilde{\lambda}, \tilde{a})$

(ii) $G_{\infty,\infty}(\lambda, a) \overset{qd}{\simeq} G_{\infty,\infty}(\tilde{\lambda}, \tilde{a})$

(iii) there exists a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that

$$a_i \asymp \tilde{a}_{\sigma(i)},$$

and for any subsequence (i_k) of $\mathbb{N}$,

$$(\lambda_{i_k}) \to 0 \iff (\tilde{\lambda}_{\sigma(i_k)}) \to 0.$$

For the spaces $G_{0,\infty}(\lambda, a)$ that are in class (2), where

$$\omega_i^{0,\infty}(p, q) = \exp\left(\left(-\frac{1}{p} - q\lambda_i\right) a_i\right),$$

we show that analogous criteria hold for quasidiagonal isomorphisms. For this purpose, we need the following lemma.

Lemma 3.0.3 *For any subsequence* $\nu = (i_k)$ *of* $\mathbb{N}$,

(i) $(\lambda_{i_k}) \to 0 \Rightarrow X^{(\nu)} \simeq E_0(a^{(\nu)})$,

(ii) $\inf\{\lambda_{i_k} : i_k \in \nu\} > 0 \Rightarrow X^{(\nu)} \simeq E'_\infty(a^{(\nu)})$,

where $a^{(\nu)} = (a_{i_k})$ *and* $X^{(\nu)}$ *is the basis subspace of* $G_{0,\infty}(\lambda, a)$ *corresponding to*

$$\{e_{i_k} : i_k \in \nu\}.$$

Proof. Let $\nu = (i_k)$ be a subsequence of $\mathbb{N}$. If $(\lambda_{i_k}) \to 0$, then there exists $N \in \nu$ such that $\lambda_{i_k} \leq \frac{1}{pq}$ whenever $i_k \geq N$. Hence, we obtain the inequalities

$$-\frac{2}{p} \leq -\frac{1}{p} - q\lambda_{i_k} \leq -\frac{1}{p}, \quad i_k \geq N,$$

which imply that the identity map

$$I : G_{0,\infty}(\lambda, a) \to E_0(a)$$

is a homeomorphism. Therefore, we have

$$G_{0,\infty}(\lambda, a) \simeq E_0(a).$$

If we assume $\inf\{\lambda_{i_k} : i_k \in \nu\} > 0$, then there exists $\delta > 0$ such that $\lambda_{i_k} \geq \delta$ for every $i_k \in \nu$. Hence, we have the inequality

$$-\frac{1}{p} - q\lambda_{i_k} \leq -q\delta,$$

which implies that the identity map

$$I : E'_\infty(a) \to G_{0,\infty}(\lambda, a)$$

is continuous. Given $p, q \in \mathbb{N}$, if we choose $s \geq 2q$, then we have the inequality

$$-s \leq -\frac{1}{p} - q\lambda_{i_k},$$

which implies that the inverse map $I : G_{0,\infty}(\lambda, a) \to E'_\infty(a)$ is also continuous. Therefore, we have

$$G_{0,\infty}(\lambda, a) \simeq E'_\infty(a).$$

■

Proposition 3.0.4 *For Montel spaces $G_{0,\infty}(\lambda, a)$ and $G_{0,\infty}(\tilde{\lambda}, \tilde{a})$, the following are equivalent:*

(i) $G_{0,\infty}(\lambda, a) \overset{p}{\simeq} G_{0,\infty}(\tilde{\lambda}, \tilde{a})$.

(ii) $G_{0,\infty}(\lambda, a) \overset{qd}{\simeq} G_{0,\infty}(\tilde{\lambda}, \tilde{a})$.

(iii) There exists a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that

$$a_i \asymp \tilde{a}_{\sigma(i)}, \tag{3.1}$$

and for any subsequence (i_k) of $\mathbb{N}$,

$$(\lambda_{i_k}) \to 0 \iff (\tilde{\lambda}_{\sigma(i_k)}) \to 0. \tag{3.2}$$

Proof. The implication $(i) \Rightarrow (ii)$ follows trivially from the definitions of quasidiagonal and permutational isomorphisms.

In order to show $(ii) \Rightarrow (iii)$, let $T : G_{0,\infty}(\lambda, a) \to G_{0,\infty}(\tilde{\lambda}, \tilde{a})$ be a quasidiagonal isomorphism. Then, there exist scalars (t_i) and a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ so that

$$Te_i = t_i \tilde{e}_{\sigma(i)},$$

where $\{e_i : i \in \mathbb{N}\}$ and $\{\tilde{e}_i : i \in \mathbb{N}\}$ are the coordinate bases for $G_{0,\infty}(\lambda, a)$ and $G_{0,\infty}(\tilde{\lambda}, \tilde{a})$, respectively.

To show that $(\lambda_{i_k}) \to 0 \Leftrightarrow (\tilde{\lambda}_{\sigma(i_k)}) \to 0$ for any subsequence (i_k) of $\mathbb{N}$, assume contrarily that $(\lambda_{i_k}) \to 0 \Leftrightarrow (\tilde{\lambda}_{\sigma(i_k)}) \to 0$ does not hold for some subsequence (i_k). Then, we can find a subsequence ν of (i_k) such that either

$$\lambda^{(\nu)} \to 0 \text{ and } \inf\{\tilde{\lambda}^{(\sigma(\nu))}\} > 0, \text{ or, } \tilde{\lambda}^{(\sigma(\nu))} \to 0 \text{ and } \inf\{\lambda^{(\nu)}\} > 0.$$

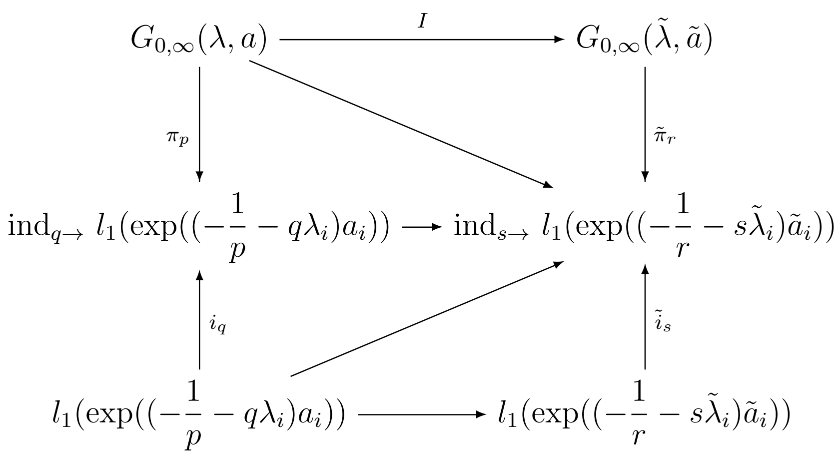
If $\lambda^{(\nu)} \to 0$ and $\inf\{\tilde{\lambda}^{(\sigma(\nu))}\} > 0$, then by Lemma 3.0.3,

$$X^{(\nu)} \simeq E_0(a^{(\nu)}) \text{ and } \tilde{X}^{(\sigma(\nu))} \simeq E'_\infty(\tilde{a}^{(\sigma(\nu))}).$$

However, by proposition 2.3.2, $E_0(a^{(\nu)})$ cannot be isomorphic to $E'_\infty(\tilde{a}^{(\nu)})$ since $a^{(\nu)}$ is not bounded. So, $X^{(\nu)}$ is not isomorphic to $\tilde{X}^{(\nu)}$, which contradicts the assumption that T is an isomorphism. Similarly, we obtain a contradiction in the case when $\tilde{\lambda}^{(\sigma(\nu))} \to 0$ and $\inf\{\lambda^{(\nu)}\} > 0$, hence $(\lambda_{i_k}) \to 0 \Leftrightarrow (\tilde{\lambda}_{\sigma(i_k)}) \to 0$ holds for any subsequence (i_k).

To show that $a_i \asymp \tilde{a}_{\sigma(i)}$, assume contrarily that $a_i \asymp \tilde{a}_{\sigma(i)}$ does not hold for the bijection σ coming from the quasidiagonal isomorphism. Then, there exists a subsequence (i_k) such that either $\frac{\tilde{a}_{\sigma(i_k)}}{a_{i_k}} \to \infty$ or $\frac{a_{i_k}}{\tilde{a}_{\sigma(i_k)}} \to \infty$. For the case when $\frac{\tilde{a}_{\sigma(i_k)}}{a_{i_k}} \to \infty$, we can find a subsequence $\nu = (i_{k_l})$ of (i_k) so that either $\left(\lambda_{i_{k_l}}\right) \to 0$, or $\inf\{\lambda_{i_{k_l}} : i_{k_l} \in \nu\} > 0$. If $\left(\lambda_{i_{k_l}}\right) \to 0$, then $\left(\tilde{\lambda}_{\sigma(i_{k_l})}\right) \to 0$, so by Lemma 3.0.3, $X^{(\nu)} \simeq E_0(a^{(\nu)})$ and $\tilde{X}^{(\sigma(\nu))} \simeq E_0(\tilde{a}^{(\sigma(\nu))})$. $X^{(\nu)} \simeq \tilde{X}^{(\sigma(\nu))}$ since T is an isomorphism, hence $E_0(a^{(\nu)}) \simeq E_0(a^{(\sigma(\nu))})$. This implies, by Proposition 2.3.2, that $a_\nu \asymp \tilde{a}_{\sigma(\nu)}$, which is a contradiction since $\frac{\tilde{a}_{\sigma(\nu)}}{a_\nu} \to \infty$. For $\inf\{\lambda_{i_{k_l}} : i_{k_l} \in \nu\} > 0$, we can find a subsequence η of ν such that $\inf\{\tilde{\lambda}_{\sigma(i_{k_l})} : i_{k_l} \in \eta\} > 0$. Then, by Lemma 3.0.3, $X^{(\eta)} \simeq E'_\infty(a^{(\eta)})$ and $\tilde{X}^{(\sigma(\eta))} \simeq E'_\infty(\tilde{a}^{(\sigma(\eta))})$. We have $X^{(\eta)} \simeq \tilde{X}^{(\sigma(\eta))}$ since T is an isomorphism, so $E'_\infty(a^{(\eta)}) \simeq E'_\infty(\tilde{a}^{(\sigma(\eta))})$. This implies, by Proposition 2.3.2, that $a_\eta \asymp \tilde{a}_{\sigma(\eta)}$, which contradicts the assumption that $\frac{\tilde{a}_{\sigma(\eta)}}{a_\eta} \to \infty$. We can similarly obtain a contradiction for the case when $\frac{a_{i_k}}{\tilde{a}_{\sigma(i_k)}} \to \infty$. Therefore, $a_i \asymp \tilde{a}_{\sigma(i)}$.

$(iii) \Rightarrow (i)$ We can assume, without loss of generality, that $\sigma(i) = i$ and $a_i = \tilde{a}_i$ since $G_{0,\infty}(\lambda, a) \overset{p}{\simeq} G_{0,\infty}(\lambda, \tilde{a})$ if $a \asymp \tilde{a}$. So, in order to show that the identity map $I : G_{0,\infty}(\lambda, a) \to G_{0,\infty}(\tilde{\lambda}, \tilde{a})$ is continuous, we have the following commutative diagram:



Considering the properties of projective and inductive topologies, we can observe from the diagram that I is continuous if

$$\forall r \exists p \forall q \exists s \quad I : l_1(\exp((-\frac{1}{p} - q\lambda_i)a_i)) \to l_1(\exp((-\frac{1}{r} - s\tilde{\lambda}_i)\tilde{a}_i))$$

is continuous. Hence, we need to show

$$\forall r \exists p \forall q \exists s \exists C \quad \exp((-\frac{1}{r} - s\tilde{\lambda}_i)\tilde{a}_i) \le C \exp((-\frac{1}{p} - q\lambda_i)a_i).$$

Since $(\lambda_{i_k}) \to 0 \Leftrightarrow (\tilde{\lambda}_{\sigma(i_k)}) \to 0$ for any subsequence (i_k) of $\mathbb{N}$, we can find a nondecreasing function $\varphi : (0,1] \to (0,1]$ such that $\lim_{t\to 0^+} = 0$, and $\lambda_i \geq \delta$ implies $\tilde{\lambda}_i \geq \varphi(\delta)$ for every $\delta \in (0,1]$.

Taking arbitrary p, q, r with $r < p < q$, choose $\delta \in (0, \frac{p-r}{rpq})$ and let $s > \frac{q}{\varphi(\delta)}$. For $\lambda_i \geq \delta$, we have $\tilde{\lambda}_i \geq \varphi(\delta)$, and hence

$$-\frac{1}{r} - s\tilde{\lambda}_i < -\frac{1}{p} - s\tilde{\lambda}_i \leq -\frac{1}{p} - s\varphi(\delta) < -\frac{1}{p} - q \leq -\frac{1}{p} - q\lambda_i.$$

For $\lambda_i < \delta$,

$$-\frac{1}{r} - s\tilde{\lambda}_i < -\frac{1}{r} < -\frac{1}{p} - q\delta < -\frac{1}{p} - q\lambda_i.$$

Thus, we have the inequality

$$\exp((-\frac{1}{r} - s\tilde{\lambda}_i)a_i) \leq \exp((-\frac{1}{p} - q\lambda_i)a_i),$$

which implies that I is continuous. Similarly, one can show that I^{-1} is also continuous. Therefore, I is an isomorphism and $G_{0,\infty}(\lambda, a) \overset{p}{\simeq} G_{0,\infty}(\tilde{\lambda}, \tilde{a})$. ■

The criteria for quasidiagonal isomorphisms between spaces $G_{0,0}(\lambda, a)$ belonging to class (3), where $\omega_i^{0,0}(p,q) = \exp\left((-\frac{1}{p}\lambda_i + \frac{1}{q})a_i\right)$, is given in [8] as follows.

Proposition 3.0.5 *For Montel spaces $G_{0,0}(\lambda, a)$ and $G_{0,0}(\tilde{\lambda}, \tilde{a})$, the following conditions are equivalent:*

(i) $G_{0,0}(\lambda, a) \overset{p}{\simeq} G_{0,0}(\tilde{\lambda}, \tilde{a})$

(ii) $G_{0,0}(\lambda, a) \overset{qd}{\simeq} G_{0,0}(\tilde{\lambda}, \tilde{a})$

(iii) there exists a bijection $\sigma : \mathbb{N} \to \mathbb{N}$, a constant $\Delta > 1$, and a strictly decreasing function $\Psi : [1, \infty) \to \mathbb{R}^+$, $\Psi(t) \to 0$ as $t \to \infty$ such that

$$a_i \asymp \tilde{a}_{\sigma(i)},$$

for any subsequence (i_k) of $\mathbb{N}$

$$(\lambda_{i_k}) \to 0 \iff (\tilde{\lambda}_{\sigma(i_k)}) \to 0,$$

and

$$\frac{1}{\Delta}\lambda_i \leq \tilde{\lambda}_{\sigma(i)} \leq \Delta\lambda_i \text{ for } \lambda_i \geq \Psi(a_i).$$

We obtain analogous criteria for quasidiagonal isomorphisms between spaces $G_{\infty,0}(\lambda, a)$ belonging to class (4), where $\omega_i^{\infty,0}(p,q) = \exp\Big((p\lambda_i + \frac{1}{q})a_i\Big)$. For this purpose, we need the following lemmas.

Lemma 3.0.6 *For any subsequence $\nu = (i_k)$ of $\mathbb{N}$,*

(i) $\inf\{\lambda_{i_k} : i_k \in \nu\} > 0 \Rightarrow X^{(\nu)} \simeq E_\infty(a^{(\nu)})$,

(ii) $(\lambda_{i_k}) \to 0 \Rightarrow X^{(\nu)} \simeq E_0'(a^{(\nu)})$,

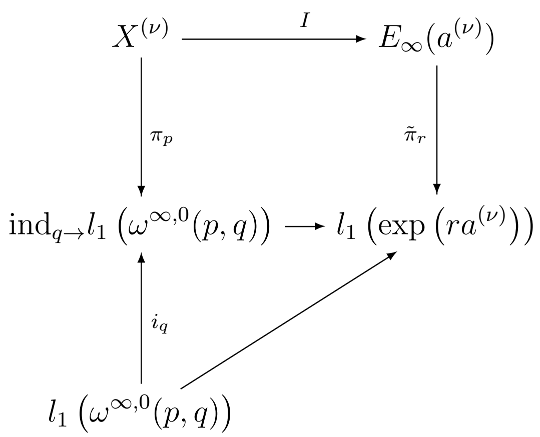
where $a^{(\nu)} = (a_{i_k})$, and $X^{(\nu)}$ is the basis subspace of $G_{\infty,0}(\lambda, a)$ corresponding to

$$\{e_{i_k} : i_k \in \nu\}.$$

Proof. Let $\nu = (i_k)$ be a subsequence of $\mathbb{N}$ such that $\inf\{\lambda_{i_k} : i_k \in \nu\} = \delta > 0$. Consider the identity map

$$I : X^{(\nu)} \to E_\infty(a^{(\nu)}).$$

Then, we have the following diagram:



For any $r \in \mathbb{N}$, choose p so that $r \le p\delta$. Then, for any $q \in \mathbb{N}$, we obtain

$$r \le p\delta \le p\lambda_{i_k} < p\lambda_{i_k} + \frac{1}{q}, \quad i_k \in \nu.$$

Hence, we have

$$\forall r \exists p \forall q \exists C \quad \exp(ra_{i_k}) \le C \exp\left(\left(p\lambda_{i_k} + \frac{1}{q}\right) a_{i_k}\right), \quad i_k \in \nu,$$

which implies that I is continuous. Similarly, one can obtain

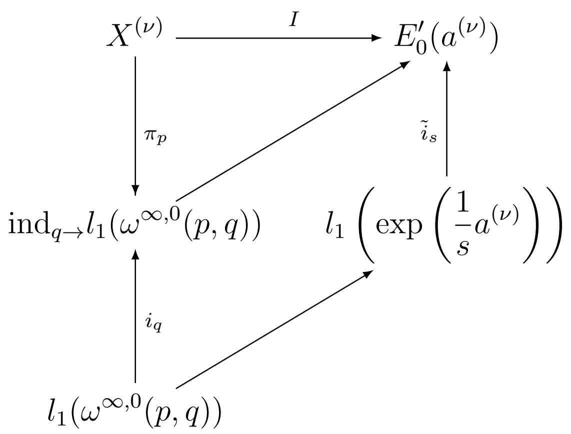
$$\forall p \exists r \exists q \exists C \quad \exp\left(\left(p\lambda_{i_k} + \frac{1}{q}\right) a_{i_k}\right) \le C \exp(ra_{i_k}),$$

hence the inverse of I is also continuous. Therefore, I is an isomorphism.

If $\nu = (i_k)$ is a subsequence of $\mathbb{N}$ such that $\lambda_{i_k} \to 0$, then for the identity map

$$I : X^{(\nu)} \to E_0'(a^{(\nu)})$$

we have the following diagram



Hence, I is continuous if

$$\exists p \forall q \exists s \exists C \quad \exp(\frac{1}{s}a_{i_k}) \leq C \exp(p\lambda_{i_k} + \frac{1}{q})a_{i_k}), \quad i_k \in \nu,$$

which is true, since for any $p, q \in \mathbb{N}$ if we choose $s > q$, then we have

$$\frac{1}{s} < \frac{1}{q} < p\lambda_{i_k} + \frac{1}{q}.$$

Also, the inverse of I is continuous if

$$\forall p \forall s \exists q \exists C \quad \exp(p\lambda_{i_k} + \frac{1}{q})a_{i_k}) \leq C \exp(\frac{1}{s}a_{i_k}), \quad i_k \in \nu.$$

For any $p, s \in \mathbb{N}$ there exists $i_0 \in \mathbb{N}$ such that $p\lambda_{i_k} \leq \frac{1}{2s}$ for $i_k \geq i_0$, since $\lim \lambda_{i_k} = 0$. By choosing $q < 2s$ we obtain

$$p\lambda_{i_k} + \frac{1}{q} < \frac{1}{2s} + \frac{1}{2s} = \frac{1}{s}, \quad i_k \geq i_0,$$

which implies that the inverse of I is continuous. Therefore I is an isomorphism. ■

Lemma 3.0.7 *Let the Montel spaces $G_{\infty,0}(\lambda, a)$ and $G_{\infty,0}(\tilde{\lambda}, \tilde{a})$ be quasidiagonally isomorphic. Then, there exists a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ and a positive constant β such that*

(i) $a_i \asymp \tilde{a}_{\sigma(i)}$,

(ii) for any subsequence (i_k) of $\mathbb{N}$,

$$(\lambda_{i_k}) \to 0 \Leftrightarrow (\tilde{\lambda}_{\sigma(i_k)}) \to 0,$$

(iii) for any subsequence (i_k) of $\mathbb{N}$, where $\lim \lambda_{i_k}$ and $\lim \tilde{\lambda}_{\sigma(i_k)}$ exist and are positive,

$$\frac{1}{\beta} \leq \lim \frac{\tilde{\lambda}_{\sigma(i_k)}}{\lambda_{i_k}} \leq \beta.$$

Proof. Let $T : G_{\infty,0}(\lambda, a) \to G_{\infty,0}(\tilde{\lambda}, \tilde{a})$ be a quasidiagonal isomorphism. Then, there exist scalars (t_i), and a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ so that $Te_i = t_i \tilde{e}_{\sigma(i)}$, where $\{e_i : i \in \mathbb{N}\}$ and $\{\tilde{e}_i : i \in \mathbb{N}\}$ are coordinate bases for $G_{\infty,0}(\lambda, a)$ and $G_{\infty,0}(\tilde{\lambda}, \tilde{a})$, respectively.

In order to show (ii), assume contrarily that $(\lambda_{i_k}) \to 0 \Leftrightarrow (\tilde{\lambda}_{\sigma(i_k)}) \to 0$ does not hold for some subsequence (i_k). Then, we can find a subsequence ν of (i_k) such that either $(\lambda_i)_{i\in\nu} \to 0$ and $\inf\{\tilde{\lambda}_i : i \in \sigma(\nu)\} > 0$, or $(\tilde{\lambda}_i)_{i\in\sigma(\nu)} \to 0$ and $\inf\{\lambda_i : i \in \nu\} > 0$. If $(\lambda_i)_{i\in\nu} \to 0$ and $\inf\{\tilde{\lambda}_i : i \in \nu\} > 0$, then $X^{(\nu)} \simeq E_0'(a^{(\nu)})$ and $\tilde{X}^{(\sigma(\nu))} \simeq E_\infty(\tilde{a}^{(\sigma(\nu))})$ by Lemma 3.0.6. By Proposition 2.3.2, $E_0'(a^{(\nu)})$ is not isomorphic to $E_\infty(\tilde{a}^{(\nu)})$ since $a^{(\nu)}$ is not bounded, hence $X^{(\nu)}$ is not isomorphic to $\tilde{X}^{(\nu)}$, which contradicts the assumption that T is an isomorphism. Similarly, we obtain a contradiction in the case when $(\tilde{\lambda}_i)_{i\in\sigma(\nu)} \to 0$ and $\inf\{\lambda_i : i \in \nu\} > 0$, hence $(\lambda_{i_k}) \to 0 \Leftrightarrow (\tilde{\lambda}_{\sigma(i_k)}) \to 0$ holds for any subsequence (i_k).

In order to show (i), assume contrarily that $a_i \asymp \tilde{a}_{\sigma(i)}$ does not hold for the bijection σ coming from the quasidiagonal isomorphism. Then, there exists a subsequence (i_k) such that either $\frac{\tilde{a}_{\sigma(i_k)}}{a_{i_k}} \to \infty$ or $\frac{a_{i_k}}{\tilde{a}_{\sigma(i_k)}} \to \infty$. For the case when $\frac{\tilde{a}_{\sigma(i_k)}}{a_{i_k}} \to \infty$, we can find a subsequence $\nu = (i_{k_l})$ of (i_k) so that either $\lambda_{i_{k_l}} \to 0$, or $\inf\{\lambda_{i_{k_l}} : i_{k_l} \in \nu\} > 0$. If $\lambda_{i_{k_l}} \to 0$, then $\tilde{\lambda}_{\sigma(i_{k_l})} \to 0$, so by Lemma 3.0.3,

$$X^{(\nu)} \simeq E_0'(a^{(\nu)}) \text{ and } \tilde{X}^{(\sigma(\nu))} \simeq E_0'(\tilde{a}^{(\sigma(\nu))}).$$

We have $X^{(\nu)} \simeq \tilde{X}^{(\sigma(\nu))}$ since T is an isomorphism, hence $E_0'(a^{(\nu)}) \simeq E_0'(a^{(\sigma(\nu))})$. This implies, by Proposition 2.3.2, that $a_\nu \asymp \tilde{a}_{\sigma(\nu)}$, which contradicts the assumption that $\frac{\tilde{a}_{\sigma(\nu)}}{a_\nu} \to \infty$. If $\inf\{\lambda_{i_{k_l}} : i_{k_l} \in \nu\} > 0$, then we can find a subsequence η of ν such that $\inf\{\tilde{\lambda}_{\sigma(i_{k_l})} : i_{k_l} \in \eta\} > 0$. Then, by Lemma 3.0.3, $X^{(\eta)} \simeq E_\infty(a^{(\eta)})$ and $\tilde{X}^{(\sigma(\eta))} \simeq E_\infty(\tilde{a}^{(\sigma(\eta))})$. We also have $X^{(\eta)} \simeq \tilde{X}^{(\sigma(\eta))}$ since T is an isomorphism, so $E_\infty(a^{(\eta)}) \simeq E_\infty(\tilde{a}^{(\sigma(\eta))})$. This implies, by Proposition 2.3.2, that $a_\eta \asymp \tilde{a}_{\sigma(\eta)}$, which contradicts the assumption that $\frac{\tilde{a}_{\sigma(\eta)}}{a_\eta} \to \infty$. We can similarly obtain a contradiction in the case when $\frac{a_{i_k}}{\tilde{a}_{\sigma(i_k)}} \to \infty$. Therefore, $a_i \asymp \tilde{a}_{\sigma(i)}$.

In order to show (iii), let (i_k) be a subsequence of $\mathbb{N}$ such that $(\lambda_{i_k}) \to \Lambda$ and $(\tilde{\lambda}_{\sigma(i_k)}) \to \tilde{\Lambda}$, where $\Lambda, \tilde{\Lambda}$ are positive numbers. Since T is continuous, we have

$$\forall r \exists p \forall q \exists s \exists C \quad |t_i| \exp\left(\left(r\tilde{\lambda}_{\sigma(i)} + \frac{1}{s}\right)\tilde{a}_{\sigma(i)}\right) \leq C \exp\left(\left(p\lambda_i + \frac{1}{q}\right)a_i\right), \quad i \in \mathbb{N}.$$

Taking the logaritms of both sides, and dividing by a_i, we obtain

$$\frac{\ln|t_i|}{a_i} + \left(r\tilde{\lambda}_{\sigma(i)} + \frac{1}{s}\right)\frac{\tilde{a}_{\sigma(i)}}{a_i} \leq \frac{\ln C}{a_i} + \left(p\lambda_i + \frac{1}{q}\right).$$

Using (i) and rearranging terms, we get

$$\frac{\ln|t_i|}{a_i} \leq \frac{\ln C}{a_i} + p\lambda_i - \frac{r}{\alpha}\tilde{\lambda}_{\sigma(i)} + \frac{1}{q} - \frac{1}{\alpha s}. \tag{3.3}$$

Also, since T^{-1} is continuous, we have

$$\forall r' \exists p' \forall q' \exists s' \exists C' \quad \exp\left(\left(r'\lambda_i + \frac{1}{s'}\right)a_i\right) \leq C'|t_i| \exp\left(\left(p'\tilde{\lambda}_{\sigma(i)} + \frac{1}{q'}\right)\tilde{a}_{\sigma(i)}\right).$$

Taking the logarithms of both sides, dividing both sides by a_i, using (i) and rearranging terms, we get

$$-\frac{\ln C'}{a_i} + r'\lambda_i - p'\alpha\tilde{\lambda}_{\sigma(i)} + \frac{1}{s'} - \frac{\alpha}{q'} \leq \frac{\ln|t_i|}{a_i}. \tag{3.4}$$

From the inequalities (3.3) and (3.4) we obtain

$$0 \leq \frac{\ln C}{a_i} + \frac{\ln C'}{a_i} + (p - r')\lambda_i + \left(p'\alpha - \frac{r}{\alpha}\right)\tilde{\lambda}_{\sigma(i)} + \left(\frac{1}{q} + \frac{\alpha}{q'}\right) - \left(\frac{1}{s'} + \frac{1}{\alpha s}\right).$$

This inequality holds for all i_k, hence if we take $i_k \to \infty$, then $a_{i_k} \to \infty$, $(\lambda_{i_k}) \to \Lambda$ and $(\tilde{\lambda}_{\sigma(i_k)}) \to \tilde{\Lambda}$, then we have the following inequality

$$0 \leq (p - r')\Lambda + \left(p'\alpha - \frac{r}{\alpha}\right)\tilde{\Lambda} + \left(\frac{1}{q} + \frac{\alpha}{q'}\right) - \left(\frac{1}{s'} + \frac{1}{\alpha s}\right).$$

By fixing the quantifiers so that they satisfy the inequalities

$$r' < p' < q' < s' < r < p < q < s, \ r > 2p'\alpha^2, \ q' > \frac{1}{p'\tilde{\Lambda}},$$

the above inequality gives

$$\frac{\tilde{\Lambda}}{\Lambda} \leq \frac{p - r'}{\frac{r}{\alpha} - 2p'\alpha},$$

where the right hand side is a positive constant. By using the continuity of T^{-1} again to fix r'', p'', q'', s'' such that

$$r < p < q < s < r'' < p'' < q'' < s'', \ q > \frac{1}{p\Lambda}, \ r'' > 2p,$$

we similarly obtain

$$\frac{\tilde{\Lambda}}{\Lambda} \geq \frac{r'' - 2p}{p''\alpha - \frac{r}{\alpha}},$$

where the right hand side is again a positive constant. Therefore, if we take a positive constant β, so that $\beta > \max\{\frac{p-r'}{\frac{r}{\alpha}-2p'\alpha}, \frac{p''\alpha-\frac{r}{\alpha}}{r''-2p}\}$, then

$$\frac{1}{\beta} \leq \lim \frac{\tilde{\lambda}_{\sigma(i_k)}}{\lambda_{i_k}} \leq \beta.$$

■

Proposition 3.0.8 *For Montel spaces $G_{\infty,0}(\lambda, a)$ and $G_{\infty,0}(\tilde{\lambda}, \tilde{a})$, the following are equivalent:*

(i) $G_{\infty,0}(\lambda, a) \overset{p}{\simeq} G_{\infty,0}(\tilde{\lambda}, \tilde{a})$,

(ii) $G_{\infty,0}(\lambda, a) \overset{qd}{\simeq} G_{\infty,0}(\tilde{\lambda}, \tilde{a})$,

(iii) there exists a bijection $\sigma : \mathbb{N} \to \mathbb{N}$, a constant $\Delta > 1$, and a strictly decreasing function $\Psi : [1, \infty) \to \mathbb{R}^+$, $\Psi(t) \to 0$ as $t \to \infty$ such that

$$a_i \asymp \tilde{a}_{\sigma(i)},$$

for any subsequence (i_k) of $\mathbb{N}$

$$(\lambda_{i_k}) \to 0 \iff (\tilde{\lambda}_{\sigma(i_k)}) \to 0,$$

and

$$\frac{1}{\Delta}\lambda_i \leq \tilde{\lambda}_{\sigma(i)} \leq \Delta\lambda_i \text{ for } \lambda_i \geq \Psi(a_i).$$

Proof. $(i) \Rightarrow (ii)$ is trivial since any permutational isomorphism is a quasidiagonal isomorphism by definition.

In order to show $(ii) \Rightarrow (iii)$, let $T : G_{\infty,0}(\lambda, a) \to G_{\infty,0}(\tilde{\lambda}, \tilde{a})$ be a quasidiagonal isomorphism. Then, there exist scalars (t_i) and a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ so that

$$Te_i = t_i\tilde{e}_{\sigma(i)},$$

where $\{e_i : i \in \mathbb{N}\}$ and $\{\tilde{e}_i : i \in \mathbb{N}\}$ are the coordinate bases for $G_{\infty,0}(\lambda, a)$ and $G_{\infty,0}(\tilde{\lambda}, \tilde{a})$, respectively. Then, by Lemma 3.0.7, $a_i \asymp \tilde{a}_{\sigma(i)}$ and for any subsequence (i_k) of $\mathbb{N}$, $(\lambda_{i_k}) \to 0 \Leftrightarrow (\tilde{\lambda}_{\sigma(i_k)}) \to 0$. To show the existence of a constant Δ and a function Ψ, by Lemma 3.0.7, take a positive constant β so that for any subsequence (i_k) of $\mathbb{N}$ where $\lim \lambda_{i_k}$ and $\lim \tilde{\lambda}_{\sigma(i_k)}$ exist and are positive,

$$\frac{1}{\beta} \leq \lim \frac{\tilde{\lambda}_{\sigma(i_k)}}{\lambda_{i_k}} \leq \beta.$$

Let us define the set

$$S = \{i \in \mathbb{N} : \frac{1}{\beta^2}\lambda_i < \tilde{\lambda}_{\sigma(i)} < \beta^2 \lambda_i\},$$

and the function $\psi : [1, \infty) \to (0, \infty)$, where

$$\psi(t) = \sup\{\lambda_j : j \in \mathbb{N}\backslash S, a_j \geq t\},$$

which is monotonically decreasing by definition. If $\mathbb{N}\backslash S$ is a finite set, then ψ can be extended so that $\psi(t) \to 0$ as $t \to \infty$. For the case when $\mathbb{N}\backslash S$ is infinite, assume that $(\lambda_i)_{i\in\mathbb{N}\backslash S}$ does not go to 0. Then, there exists a subsequence ν of $\mathbb{N}\backslash S$ so that $(\lambda_i)_{i\in\nu} \to \Lambda$ for some $\Lambda > 0$, which implies that there is a subsequence η of ν where $\inf\{\tilde{\lambda}_{\sigma(i)} : i \in \eta\} > 0$, hence we can find a subsequence (i_k) of η such that $\tilde{\lambda}_{\sigma(i_k)} \to \tilde{\Lambda}$ for some $\tilde{\Lambda} > 0$. As (i_k) is a sequence in $\mathbb{N}\backslash S$, $\frac{\tilde{\lambda}_{\sigma(i_k)}}{\lambda_{i_k}} \notin \left(\frac{1}{\beta^2}, \beta^2\right)$ for any i_k, which implies that $\lim \frac{\tilde{\lambda}_{\sigma(i_k)}}{\lambda_{i_k}} \notin [\frac{1}{\beta}, \beta]$, contradicting our initial assumption. So, $(\lambda_i)_{i\in\mathbb{N}\backslash S} \to 0$, which implies, by definition of the function ψ, that $\psi(t) \to 0$ as $t \to \infty$.

Since ψ is a monotonically decreasing function where $\psi(t) \to 0$ as $t \to \infty$, we can take a strictly decreasing function $\Psi : [1, \infty) \to (0, \infty)$ so that $\Psi(t) > \psi(t)$ for all $t \in [1, \infty)$ and $\Psi(t) \to 0$ as $t \to \infty$. So, whenever $\lambda_i \geq \Psi(a_i)$, we have $\lambda_i > \sup\{\lambda_j : j \in \mathbb{N}\backslash S, a_j \geq a_i\}$, which implies that $i \in S$. Therefore, by taking Δ such that $\Delta \geq \beta^2$, we obtain

$$\frac{1}{\Delta}\lambda_i \leq \tilde{\lambda}_{\sigma(i)} \leq \Delta\lambda_i \text{ for } \lambda_i \geq \Psi(a_i).$$

To show $(iii) \Rightarrow (i)$, assume that there exists a bijection $\sigma : \mathbb{N} \to \mathbb{N}$, a strictly decreasing function $\Psi : [1, \infty) \to (0, \infty)$ where $\Psi(t) \to 0$ as $t \to \infty$, and there exist constants $\alpha, \Delta > 1$ so that $\frac{1}{\alpha}a_i \leq \tilde{a}_{\sigma(i)} \leq \alpha a_i$ for all $i \in \mathbb{N}$, $(\lambda_{i_k}) \to 0 \Leftrightarrow (\tilde{\lambda}_{\sigma(i_k)}) \to 0$ for any subsequence (i_k) of $\mathbb{N}$, and $\frac{1}{\Delta}\lambda_i \leq \tilde{\lambda}_{\sigma(i)} \leq \Delta\lambda_i$ for $\lambda_i \geq \Psi(a_i)$.

Consider the operator $P : G_{\infty,0}(\lambda, a) \to G_{\infty,0}(\tilde{\lambda}, \tilde{a})$ defined as $Pe_i = \tilde{e}_{\sigma(i)}$. To show the continuity of P, for any r choose p so that $p > \alpha\Delta r$, and for any q choose s so that $s > 2\alpha q$. Then, since $a_i \asymp \tilde{a}_{\sigma(i)}$,

$$\left(r\tilde{\lambda}_{\sigma(i)} + \frac{1}{s}\right)\tilde{a}_{\sigma(i)} \leq \left(r\tilde{\lambda}_{\sigma(i)} + \frac{1}{s}\right)\alpha a_i.$$

For $\lambda_i \geq \Psi(a_i)$,

$$\begin{aligned}\left(r\tilde{\lambda}_{\sigma(i)} + \frac{1}{s}\right)\alpha a_i &\leq \left(r\Delta\lambda_i + \frac{1}{s}\right)\alpha a_i \\ &= \left(\alpha\Delta r\lambda_i + \frac{\alpha}{s}\right)a_i \\ &< \left(p\lambda_i + \frac{1}{q}\right)a_i.\end{aligned}$$

For $\lambda_i < \Psi(a_i)$, $a_i \to \infty$ as $i \to \infty$, which implies that $\Psi(a_i) \to 0$ as $i \to \infty$ since Ψ is decreasing. Hence $(\lambda_i) \to 0$, which implies $\left(\tilde{\lambda}_{\sigma(i)}\right) \to 0$, that is, there exists N such that $\tilde{\lambda}_{\sigma(i)} < \frac{1}{rs}$ whenever $i \geq N$. So, for $i \geq N$ where $\lambda_i < \Psi(a_i)$, we have the estimates

$$\begin{aligned}\left(r\tilde{\lambda}_{\sigma(i)} + \frac{1}{s}\right)\alpha a_i &\leq \left(r\left(\frac{1}{rs}\right) + \frac{1}{s}\right)\alpha a_i \\ &= \frac{2\alpha}{s}a_i \\ &< p\lambda_i a_i + \frac{2\alpha}{s}a_i \\ &< \left(p\lambda_i + \frac{1}{q}\right)a_i.\end{aligned}$$

By these estimates, one can show that P is continuous. Continuity of P^{-1} can be shown similarly. Therefore, P is a permutational isorphism. ■

CHAPTER 4

m-rectangle Characteristics and Quasidiagonal Isomorphisms

In this section, we give the definitions of m-rectangle characteristics and necessary equivalences, and we show that under these equivalences, the systems of m-rectangle characteristics give a complete characterization of quasidiagonal isomorphisms between the spaces that are in classes $(1)-(4)$.

Let $\lambda=(\lambda_i)_{i\in\mathbb{N}}$ and $a=(a_i)_{i\in\mathbb{N}}$ be sequences of positive numbers and let $m\in\mathbb{N}$. Then, the function

$$\mu_m^{(\lambda,a)}(\delta,\varepsilon;\tau,t) \;=\; \left|\bigcup_{k=1}^{m}\{i\;:\;\delta_k\le\lambda_i\le\varepsilon_k\;,\;\tau_k\le a_i\le t_k\}\right|$$

defined for $\delta=(\delta_k)_{k=1}^m$, $\varepsilon=(\varepsilon_k)_{k=1}^m$, $\tau=(\tau_k)_{k=1}^m$ and $t=(t_k)_{k=1}^m$ such that $0\le\delta_k\le\varepsilon_k\le 2$, $0<\tau_k\le t_k<\infty$, where $k=1,2,\cdots,m$, is called the *m-rectangle characteristic* of the pair (λ,a).

Given another couple of positive sequences $\tilde\lambda=(\tilde\lambda_i)$ and $\tilde a=(\tilde a_i)$, and a fixed $m\in\mathbb{N}$, the functions $\mu_m^{(\lambda,a)}$ and $\mu_m^{(\tilde\lambda,\tilde a)}$ are said to be *equivalent*, denoted by $\mu_m^{(\lambda,a)}\sim\mu_m^{(\tilde\lambda,\tilde a)}$, if there exists a strictly increasing function $\varphi:[0,2]\to[0,1]$ with $\varphi(0)=0$ and $\varphi(2)=1$, and a positive constant α such that the inequalities

$$\begin{aligned}\mu_m^{(\lambda,a)}(\delta,\varepsilon;\tau,t) &\le \mu_m^{(\tilde\lambda,\tilde a)}\left(\varphi(\delta),\varphi^{-1}(\varepsilon);\frac{\tau}{\alpha},\alpha t\right)\\ \mu_m^{(\tilde\lambda,\tilde a)}(\delta,\varepsilon;\tau,t) &\le \mu_m^{(\lambda,a)}\left(\varphi(\delta),\varphi^{-1}(\varepsilon);\frac{\tau}{\alpha},\alpha t\right)\end{aligned}$$

hold where $\varphi(\delta)=(\varphi(\delta_k))_{k=1}^m$, $\varphi^{-1}(\varepsilon)=(\varphi^{-1}(\varepsilon_k))_{k=1}^m$, $\frac{\tau}{\alpha}=(\frac{\tau_k}{\alpha})_{k=1}^m$, $\alpha t=(\alpha t_k)_{k=1}^m$ for all collections of parameters $\delta,\varepsilon,\tau,t$.

The systems of characteristics $\left(\mu_m^{(\lambda,a)}\right)_{m\in\mathbb{N}}$ and $\left(\mu_m^{(\tilde{\lambda},\tilde{a})}\right)_{m\in\mathbb{N}}$ are said to be *equivalent*, denoted by $\left(\mu_m^{(\lambda,a)}\right) \sim \left(\mu_m^{(\tilde{\lambda},\tilde{a})}\right)$, if the function φ and the constant α can be chosen so that the inequalities above hold for all $m \in \mathbb{N}$.

It is shown in [5] that with this definition of equivalence, the systems of m-rectangle characteristics gives a complete characterization of quasidiagonal isomorphisms between Montel spaces $G_{\infty,\infty}(\lambda, a)$ belonging to class (1), where

$$\omega_i^{\infty,\infty}(p, q) = \exp\left((p - q\lambda_i)a_i\right),$$

as given in the following theorem.

Theorem 4.0.9 *For Montel spaces $G_{\infty,\infty}(\lambda, a)$ and $G_{\infty,\infty}(\tilde{\lambda}, \tilde{a})$,*

$$G_{\infty,\infty}(\lambda, a) \overset{qd}{\simeq} G_{\infty,\infty}(\tilde{\lambda}, \tilde{a}) \iff \left(\mu_m^{(\lambda,a)}\right) \sim \left(\mu_m^{(\tilde{\lambda},\tilde{a})}\right).$$

We obtain ana analogous result for Montel spaces $G_{0,\infty}(\lambda, a)$ belonging to class (2), where $\omega_i^{0,\infty}(p, q) = \exp\left((-\frac{1}{p} - q\lambda_i)a_i\right)$.

Theorem 4.0.10 *For Montel spaces $G_{0,\infty}(\lambda, a)$ and $G_{0,\infty}(\tilde{\lambda}, \tilde{a})$,*

$$G_{0,\infty}(\lambda, a) \overset{qd}{\simeq} G_{0,\infty}(\tilde{\lambda}, \tilde{a}) \iff \left(\mu_m^{(\lambda,a)}\right) \sim \left(\mu_m^{(\tilde{\lambda},\tilde{a})}\right).$$

Proof. Let $G_{0,\infty}(\lambda, a)$ and $G_{0,\infty}(\tilde{\lambda}, \tilde{a})$ be quasidiagonally isomorphic Montel spaces. Then, by proposition 3.0.4, there exists a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ so that

$$(\lambda_{i_k}) \to 0 \Leftrightarrow (\tilde{\lambda}_{\sigma(i_k)}) \to 0$$

for any subsequence (i_k) of $\mathbb{N}$, which implies that for every $\delta \in (0, 1]$, $\tilde{\lambda}_{\sigma(i)} \geq \varepsilon_1 > 0$ for some $\varepsilon_1 > 0$ if $\lambda_i \geq \delta$, and $\lambda_i \geq \varepsilon_2 > 0$ for some $\varepsilon_2 > 0$ if $\tilde{\lambda}_{\sigma(i)} \geq \delta$. If we define the functions $\phi_1 : (0, 1] \to (0, 1]$ and $\phi_2 : (0, 1] \to (0, 1]$ by $\phi_1(\delta) = \inf_{i\in S_\delta} \tilde{\lambda}_{\sigma(i)}$ where $S_\delta = \{i : \lambda_i \geq \delta\}$, and $\phi_2(\delta) = \inf_{i\in\tilde{S}_\delta} \lambda_i$ where $\tilde{S}_\delta = \{i : \tilde{\lambda}_{\sigma(i)} \geq \delta\}$, then ϕ_1 and ϕ_2 are monotonically increasing functions such that $\phi_1(\delta) \to 0$ and $\phi_2(\delta) \to 0$ as $\delta \to 0$. So, we can take a function $\phi(\delta) : (0, 1] \to (0, 1]$ with

$$\phi(\delta) < \min\{\phi_1(\delta), \phi_2(\delta)\},$$

that is strictly increasing, and $\phi(\delta) \to 0$ as $\delta \to 0$. Taking $\phi(1) < 1$, one can extend ϕ to a function $\varphi : [0, 2] \to [0, 1]$ which is strictly increasing, $\varphi(0) = 0$, and $\varphi(2) = 1$.

As φ is strictly increasing, its inverse $\varphi^{-1} : [0,1] \to [0,2]$ also exists. So, from the construction of the function φ, we have the inclusions $S_\delta \subset \tilde{S}_{\varphi(\delta)}$ and $\tilde{S}_\delta \subset S_{\varphi(\delta)}$, which imply

$$\{i : \delta \leq \lambda_i \leq \varepsilon\} \subset \{i : \varphi(\delta) \leq \tilde{\lambda}_{\sigma(i)} \leq \varphi^{-1}(\varepsilon)\} \tag{4.1}$$

$$\{i : \delta \leq \tilde{\lambda}_{\sigma(i)} \leq \varepsilon\} \subset \{i : \varphi(\delta) \leq \lambda_i \leq \varphi^{-1}(\varepsilon)\} \tag{4.2}$$

for any δ, ε.

Also, $a_i \asymp \tilde{a}_{\sigma(i)}$ by Proposition 3.0.4, so there exists $\alpha > 1$ such that the inclusions

$$\{i : \tau \leq a_i \leq t\} \subset \{i : \frac{\tau}{\alpha} \leq \tilde{a}_{\sigma(i)} \leq \alpha t\} \tag{4.3}$$

$$\{i : \tau \leq \tilde{a}_{\sigma(i)} \leq t\} \subset \{i : \frac{\tau}{\alpha} \leq a_i \leq \alpha t\} \tag{4.4}$$

hold for any τ, t.

As σ is a bijection, the inclusions (4.1)-(4.4) imply the inequalities

$$\begin{aligned} \mu_m^{(\lambda,a)}(\delta, \varepsilon; \tau, t) &\leq \mu_m^{(\tilde{\lambda},\tilde{a})}(\varphi(\delta), \varphi^{-1}(\varepsilon); \frac{\tau}{\alpha}, \alpha t), \\ \mu_m^{(\tilde{\lambda},\tilde{a})}(\delta, \varepsilon; \tau, t) &\leq \mu_m^{(\lambda,a)}(\varphi(\delta), \varphi^{-1}(\varepsilon); \frac{\tau}{\alpha}, \alpha t), \end{aligned}$$

for any $m \in \mathbb{N}$. Therefore, $\left(\mu_m^{(\lambda,a)}\right) \sim \left(\mu_m^{(\tilde{\lambda},\tilde{a})}\right)$.

Now assume that $\left(\mu_m^{(\lambda,a)}\right) \sim \left(\mu_m^{(\tilde{\lambda},\tilde{a})}\right)$. For every $i \in \mathbb{N}$, consider the sets

$$S_i = \{j : \varphi(\lambda_i) \leq \tilde{\lambda}_j \leq \varphi^{-1}(\lambda_i), \frac{a_i}{\alpha} \leq \tilde{\alpha}_j \leq \alpha a_i\}.$$

If we take distinct indices $i_1, \ldots, i_m$, then

$$\begin{aligned} \left|\bigcup_{k=1}^m S_{i_k}\right| &= \left|\bigcup_{k=1}^m \{j : \varphi(\lambda_{i_k}) \leq \tilde{\lambda}_j \leq \varphi^{-1}(\lambda_{i_k}), \frac{a_{i_k}}{\alpha} \leq \tilde{\alpha}_j \leq \alpha a_{i_k}\}\right| \\ &= \mu_m^{(\tilde{\lambda},\tilde{a})}(\varphi(\delta), \varphi^{-1}(\varepsilon); \frac{\tau}{\alpha}, \alpha t) \\ &\geq \mu_m^{(\lambda,a)}(\delta, \varepsilon; \tau, t) \end{aligned}$$

for $\delta = (\lambda_{i_k})$, $\varepsilon = (\lambda_{i_k})$, $\tau = (a_{i_k})$ and $t = (a_{i_k})$, where the last inequality holds since the m-rectangle characteristics are equivalent. Also,

$$\mu_m^{(\lambda,a)}(\delta, \varepsilon; \tau, t) = \left|\bigcup_{k=1}^m \{j : \lambda_{i_k} \leq \lambda_j \leq \lambda_{i_k}, a_{i_k} \leq \alpha_j \leq a_{i_k}\}\right| \geq m,$$

since

$$\{i_1, \cdots, i_m\} \subset \bigcup_{k=1}^m \{j : \lambda_{i_k} \leq \lambda_j \leq \lambda_{i_k}, a_{i_k} \leq \alpha_j \leq a_{i_k}\}.$$

Hence, $|\bigcup_{k=1}^{m} S_{i_k}| \geq m$ for any distinct indices $i_1, \cdots, i_m$, and we can apply Theorem 2.6.1 to obtain an injection $\sigma : \mathbb{N} \to \mathbb{N}$, where $\sigma(i) \in S_i$ for every $i \in \mathbb{N}$. Thus, $T : G_{0,\infty}(\lambda, a) \to G_{0,\infty}(\tilde{\lambda}, \tilde{a})$ defined as $Te_i = \tilde{e}_{\sigma(i)}$, for respective coordinate bases $\{e_i\}$ and $\{\tilde{e}_i\}$, is a quasidiagonal embedding by Proposition 3.0.4.

As the inequality $\mu_m^{(\tilde{\lambda},\tilde{a})}(\delta, \varepsilon; \tau, t) \leq \mu_m^{(\lambda,a)}(\varphi(\delta), \varphi^{-1}(\varepsilon); \frac{\tau}{\alpha}, \alpha t)$ also holds since the systems of m-rectangle characteristics are equivalent, we can repeat the same argument to obtain a quasidiagonal embedding $S : G_{0,\infty}(\tilde{\lambda}, \tilde{a}) \to G_{0,\infty}(\lambda, a)$. Therefore, by Proposition 2.5.1, we have $G_{0,\infty}(\lambda, a) \overset{qd}{\simeq} G_{0,\infty}(\tilde{\lambda}, \tilde{a})$. ■

In order to show similar characterizations of quasidiagonal isomorphisms between the spaces in the classes (3) and (4) in terms m-rectangle characteristics, we need to have a slightly different definition of equivalence as follows.

For any $m \in \mathbb{N}$, we again call the m-rectangle characteristics equivalent, and denote in this case by $\mu_m^{(\lambda,a)} \approx \mu_m^{(\tilde{\lambda},\tilde{a})}$, if there exists a constant $c > 1$, a strictly decreasing function $\Psi : [1, \infty) \to (0, \infty)$ where $\Psi(\xi) \to 0$ as $\xi \to \infty$, and a strictly increasing function $\varphi : [0, 2] \to [0, 1]$ where $\varphi(0) = 0$, $\varphi(2) = 1$, and $\varphi(\xi) < \frac{\xi}{c}$ for all $\xi \in [0, 2]$, such that the inequalities

$$\mu_m^{(\lambda,a)}(\delta, \varepsilon; \tau, t) \leq \mu_m^{(\tilde{\lambda},\tilde{a})}(\Phi_1(\delta, \tau), \Phi_2(\delta, \varepsilon, \tau); \frac{\tau}{c}, ct) \tag{4.5}$$

$$\mu_m^{(\tilde{\lambda},\tilde{a})}(\delta, \varepsilon; \tau, t) \leq \mu_m^{(\lambda,a)}(\Phi_1(\delta, \tau), \Phi_2(\delta, \varepsilon, \tau); \frac{\tau}{c}, ct) \tag{4.6}$$

hold for all collections of parameters $\delta, \varepsilon, \tau, t$, where $\varphi(\delta) = (\varphi(\delta_k))$, $\varphi^{-1}(\varepsilon) = (\varphi^{-1}(\varepsilon_k))$, $\frac{\tau}{\alpha} = (\frac{\tau_k}{\alpha})$, $\alpha t = (\alpha t_k)$, and the functions

$$\Phi_1(\delta, \tau) = (\Phi_1(\delta_k, \tau_k)) \text{ and } \Phi_2(\delta, \varepsilon, \tau) = (\Phi_2(\delta_k, \varepsilon_k, \tau_k)),$$

defined as

$$\Phi_1(\delta_k, \tau_k) = \begin{cases} \frac{\delta_k}{c} & \text{if } \delta_k \geq \Psi(\tau_k), \\ \varphi(\delta_k) & \text{if } \delta_k < \Psi(\tau_k), \end{cases}$$

$$\Phi_2(\delta_k, \varepsilon_k, \tau_k) = \begin{cases} c\varepsilon_k & \text{if } \delta_k \geq \Psi(\tau_k), \\ \varphi^{-1}(\varepsilon_k) & \text{if } \delta_k < \Psi(\tau_k). \end{cases}$$

The systems of characteristics $\left(\mu_m^{(\lambda,a)}\right)_{m\in\mathbb{N}}$ and $\left(\mu_m^{(\tilde{\lambda},\tilde{a})}\right)_{m\in\mathbb{N}}$ are then called *equivalent*, denoted by $\left(\mu_m^{(\lambda,a)}\right) \approx \left(\mu_m^{(\tilde{\lambda},\tilde{a})}\right)$, if the constant c and the functions Ψ, φ can be chosen so that the inequalities above hold for all $m \in \mathbb{N}$.

It is shown in [8] that with this equivalence, systems of m-rectangle characteristics completely characterise the quasidiagonal isomorphisms between Montel spaces $G_{0,0}(\lambda, a)$ belonging to class (3), where $\omega_i(p,q) = \exp\left((-\frac{1}{p}\lambda_i + \frac{1}{q})a_i\right)$, as given in the following theorem.

Theorem 4.0.11 *For Montel spaces $G_{0,0}(\lambda, a)$ and $G_{0,0}(\tilde{\lambda}, \tilde{a})$,*

$$G_{0,0}(\lambda, a) \overset{qd}{\simeq} G_{0,0}(\tilde{\lambda}, \tilde{a}) \iff \left(\mu_m^{(\lambda,a)}\right) \approx \left(\mu_m^{(\tilde{\lambda},\tilde{a})}\right).$$

For Montel spaces $G_{\infty,0}(\lambda, a)$ belonging to class (4), where

$$\omega_i(p,q) = \exp\left((p\lambda_i + \frac{1}{q})a_i\right),$$

we obtain an analogous result as follows.

Theorem 4.0.12 *For Montel spaces $G_{\infty,0}(\lambda, a)$ and $G_{\infty,0}(\tilde{\lambda}, \tilde{a})$,*

$$G_{\infty,0}(\lambda, a) \overset{qd}{\simeq} G_{\infty,0}(\tilde{\lambda}, \tilde{a}) \iff \left(\mu_m^{(\lambda,a)}\right) \approx \left(\mu_m^{(\tilde{\lambda},\tilde{a})}\right).$$

Proof. Suppose that $G_{\infty,0}(\lambda, a) \overset{qd}{\simeq} G_{\infty,0}(\tilde{\lambda}, \tilde{a})$. Then, by Proposition 3.0.8, there exists a bijection $\sigma : \mathbb{N} \to \mathbb{N}$, a strictly decreasing function $\Psi : [1,\infty) \to (0,\infty)$ where $\Psi(\xi) \to 0$ as $\xi \to \infty$, and there exist constants $\alpha, \Delta > 1$ so that $\frac{1}{\alpha}a_i \le \tilde{a}_{\sigma(i)} \le \alpha a_i$ for all $i \in \mathbb{N}$, $(\lambda_{i_k}) \to 0 \Leftrightarrow (\tilde{\lambda}_{\sigma(i_k)}) \to 0$ for any subsequence (i_k) of $\mathbb{N}$, and $\frac{1}{\Delta}\lambda_i \le \tilde{\lambda}_{\sigma(i)} \le \Delta\lambda_i$ for $\lambda_i \ge \Psi(a_i)$.

Since $(\lambda_{i_k}) \to 0 \Leftrightarrow (\tilde{\lambda}_{\sigma(i_k)}) \to 0$ for any subsequence (i_k) of $\mathbb{N}$, as in the proof of Theorem 4.0.10, we can find a function $\varphi : [0,2] \to [0,1]$ which is strictly increasing, $\varphi(0) = 0$, $\varphi(2) = 1$, and we have the inclusions

$$\begin{aligned} \{i : \delta_k \le \lambda_i \le \varepsilon_k\} &\subset \{i : \varphi(\delta_k) \le \tilde{\lambda}_{\sigma(i)} \le \varphi^{-1}(\varepsilon_k)\}, \\ \{i : \delta_k \le \tilde{\lambda}_{\sigma(i)} \le \varepsilon_k\} &\subset \{i : \varphi(\delta_k) \le \lambda_i \le \varphi^{-1}(\varepsilon_k)\}, \end{aligned}$$

for any δ_k, ε_k. Choosing a constant c such that $c > \max\{\alpha, \Delta\}$, the construction of the function φ allows us to take $\varphi(\xi) \le \frac{\xi}{c}$ for all $\xi \in [0,2]$.

So, let us fix δ_k, ε_k, τ_k, t_k and take any $i \in \mathbb{N}$ so that $\delta_k \le \lambda_i \le \varepsilon_k$ and $\tau_k \le a_i \le t_k$. Then, since $\frac{1}{\alpha}a_i \le \tilde{a}_{\sigma(i)} \le \alpha a_i$ and $c > \alpha$, we have $\frac{\tau_k}{c} \le \tilde{a}_{\sigma(i)} \le ct_k$. Also, if $\delta_k \ge \Psi(\tau_k)$, we have $\lambda_i \ge \Psi(a_i)$, which implies that $\frac{1}{\Delta}\lambda_i \le \tilde{\lambda}_{\sigma(i)} \le \Delta\lambda_i$. Since $c > \Delta$, by the definitions of functions Φ_1 and Φ_2, $\Phi_1(\delta_k, \tau_k) \le \tilde{\lambda}_{\sigma(i)} \le \Phi_2(\delta_k, \varepsilon_k, \tau_k)$. If $\delta_k < \Psi(\tau_k)$, then $\delta_k \le \lambda_i \le \varepsilon_k$ implies $\varphi(\delta_k) \le \tilde{\lambda}_{\sigma(i)} \le \varphi^{-1}(\varepsilon_k)$. So, by the definitions of functions Φ_1 and Φ_2, $\Phi_1(\delta_k, \tau_k) \le \tilde{\lambda}_{\sigma(i)} \le \Phi_2(\delta_k, \varepsilon_k, \tau_k)$. From these inequalities, we obtain the inclusions

$$\{i \in \mathbb{N} : \delta_k \le \lambda_i \le \varepsilon_k, \tau_k \le a_i \le t_k\} \subset$$
$$\{i \in \mathbb{N} : \Phi_1(\delta_k, \tau_k) \le \tilde{\lambda}_{\sigma(i)} \le \Phi_2(\delta_k, \varepsilon_k, \tau_k), \frac{\tau_k}{c} \le \tilde{a}_{\sigma(i)} \le ct_k\}.$$

Thus, for every $m \in \mathbb{N}$, $\delta = (\delta_k)$, $\varepsilon = (\varepsilon_k)$, $\tau = (\tau_k)$ and $t = (t_k)$, we have the inequality

$$\mu_m^{(\lambda,a)}(\delta, \varepsilon; \tau, t) \le \mu_m^{(\tilde{\lambda},\tilde{a})}(\Phi_1(\delta, \tau), \Phi_2(\delta, \varepsilon, \tau); \frac{\tau}{c}, ct).$$

By a similar argument, one can also show

$$\mu_m^{(\tilde{\lambda},\tilde{a})}(\delta, \varepsilon; \tau, t) \le \mu_m^{(\lambda,a)}(\Phi_1(\delta, \tau), \Phi_2(\delta, \epsilon, \tau); \frac{\tau}{c}, ct).$$

For this purpose, we may need to choose a different constant Δ and a different function Ψ which exist from the quasidiagonal isomorphism, and then take the maximum of the corresponding constants and functions.

As these inequalities hold for all $m \in \mathbb{N}$, the system of characteristics is equivalent, that is, $(\mu_m^{(\lambda,a)}) \approx (\mu_m^{(\tilde{\lambda},\tilde{a})})$.

Now suppose that $\left(\mu_m^{(\lambda,a)}\right) \approx \left(\mu_m^{(\tilde{\lambda},\tilde{a})}\right)$. Then, there exists a constant $c > 1$, a strictly decreasing function $\Psi : [1, \infty) \to (0, \infty)$ where $\Psi(\xi) \to 0$ as $\xi \to \infty$, and a strictly increasing function $\varphi : [0, 2] \to [0, 1]$ where $\varphi(0) = 0$, $\varphi(2) = 1$, and $\varphi(\xi) < \frac{\xi}{c}$ for all $\xi \in [0, 2]$, such that the inequalities (4.5) and (4.6) hold for any $m \in \mathbb{N}$.

As $c > 1$ and Ψ is a strictly decreasing function where $\Psi(\xi) \to 0$ as $\xi \to \infty$, for any $k \in \mathbb{N}$, there exists $\nu_k \in \mathbb{N}_0$ such that

$$\frac{1}{c^{\nu_k+1}} < \Psi(c^{k-1}) \le \frac{1}{c^{\nu_k}}.$$

So, let us define the following sets

$$\begin{aligned}
N_{l,k} &= \left\{i \in \mathbb{N} : \frac{1}{c^{l+1}} \le \lambda_i \le \frac{1}{c^l},\ c^{k-1} \le a_i \le c^k\right\},\ k \in \mathbb{N},\ l = 0, 1, \cdots, \nu_k - 1, \\
N_{\nu_k,k} &= \left\{i \in \mathbb{N} : \lambda_i \le \frac{1}{c^{\nu_k}},\ c^{k-1} \le a_i \le c^k\right\},\ k \in \mathbb{N} \\
\tilde{N}_{l,k} &= \left\{i \in \mathbb{N} : \frac{1}{c^{l+2}} \le \tilde{\lambda}_i \le \frac{1}{c^{l-1}},\ c^{k-2} \le \tilde{a}_i \le c^{k+1}\right\},\ k \in \mathbb{N},\ l = 0, 1, \cdots, \nu_k - 1, \\
\tilde{N}_{\nu_k,k} &= \left\{i \in \mathbb{N} : \tilde{\lambda}_i \le \varphi^{-1}\left(\frac{1}{c^{\nu_k}}\right),\ c^{k-2} \le \tilde{a}_i \le c^{k+1}\right\},\ k \in \mathbb{N},
\end{aligned}$$

which are finite subsets of $\mathbb{N}$ since a and $\tilde{a}$ tend to infinity.

Hence, for each $i \in \mathbb{N}$, there is a finite subset $S_i = \tilde{N}_{l,k}$ if $i \in N_{l,k}$ for $k \in \mathbb{N}$, $l = 0, 1, \cdots, \nu_k$. For distinct indices $i_1, \cdots, i_m$, the inequality (4.5) implies that

$$\left| \bigcup_{j=1}^{m} S_{i_j} \right| \geq m,$$

so we can apply Theorem 2.6.1 to obtain an injection $\sigma : \mathbb{N} \to \mathbb{N}$ such that $\sigma(i) \in S_i$ for every $i \in \mathbb{N}$. Then, the operator $P : G_{\infty,0}(\lambda, a) \to G_{\infty,0}(\tilde{\lambda}, \tilde{a})$ defined as $Pe_i = \tilde{e}_{\sigma(i)}$ is a quasidiagonal embedding by Proposition 3.0.8. Similarly, one can find a quasidiagonal embedding $\tilde{P} : G_{\infty,0}(\tilde{\lambda}, \tilde{a}) \to G_{\infty,0}(\lambda, a)$. Therefore, by Proposition 2.5.1,

$$G_{\infty,0}(\lambda, a) \overset{qd}{\simeq} G_{\infty,0}(\tilde{\lambda}, \tilde{a}).$$

■

CHAPTER 5

Invariance of m-rectangle Characteristics

In this section, we prove that the m-rectangle characteristics are linear topological invariants for the spaces $G_{0,\infty}(\lambda, a)$ in class (2) where $\omega_i^{0,\infty}(p,q) = \exp\Big((-\frac{1}{p} - q\lambda_i)a_i\Big)$, that is, $G_{0,\infty}(\lambda, a) \simeq G_{0,\infty}(\tilde{\lambda}, \tilde{a})$ implies $\mu_m^{(\lambda,a)} \sim \mu_m^{(\tilde{\lambda},\tilde{a})}$ for every $m \in \mathbb{N}$. For this purpose, we will need the following characteristic that was first used in the construction of compound invariants in [13].

Let X be a locally convex space and U, V be absolutely convex sets in X. Then, the β-characteristics of V and U, denoted by $\beta(V,U)$, is defined by

$$\beta(V,U) = \sup\{\dim L : L \text{ is a finite dimensional subspace of } \overline{\text{span} V}, U \cap L \subset V\}.$$

The β-characteristics have the following useful properties.

Remark 5.0.1 *For absolutely convex sets U, V, $\tilde{U}$, $\tilde{V}$ of X, and $\alpha > 0$,*

(a) $\beta(\alpha V, U) = \beta(V, \frac{1}{\alpha}U)$,

(b) if $V \subset \tilde{V}$ and $\tilde{U} \subset U$, then $\beta(V,U) \leq \beta(\tilde{V}, \tilde{U})$.

For a locally convex space X with an absolute basis $e = \{e_i : i \in \mathbb{N}\}$ and a sequence $a = (a_i)_{i\in\mathbb{N}}$ of positive numbers, the set

$$B^e(a) = \left\{ x = \sum_{i=1}^{\infty} \xi_i e_i \in X : \sum_{i=1}^{\infty} |\xi_i| a_i \leq 1 \right\}$$

is called the weighted l_1-ball with the weight sequence a with respect to the basis e. Weighted l_1-balls have the following geometrical properties.

Proposition 5.0.13 *Let X be a locally convex space with an absolute basis $e = \{e_i\}_{i\in\mathbb{N}}$ and $a^{(j)} = (a_i^{(j)})$ be sequences of positive numbers for $j = 1, \cdots, m$. Then,*

$$B^e(c) \subset \bigcap_{j=1}^{m} B^e(a^{(j)}) \subset mB^e(c), \quad B^e(d) = conv\left(\bigcup_{j=1}^{m} B^e(a^{(j)})\right),$$

where $c = (c_i)_{i\in\mathbb{N}}$ and $d = (d_i)_{i\in\mathbb{N}}$ are sequences such that $c_i = \max\{a_i^{(j)} : j = 1, \ldots, m\}$, $d_i = \min\{a_i^{(j)} : j = 1, \ldots, m\}$.

The following proposition, which can be found in [13], provides a method for computing the β-characteristics of weighted l_1-balls in terms of their weight sequences.

Proposition 5.0.14 *For sequences of positive numbers $a = (a_i)_{i\in\mathbb{N}}$ and $b = (b_i)_{i\in\mathbb{N}}$,*

$$\beta(B^e(a), B^e(b)) = |\{i \in \mathbb{N} : a_i \le b_i\}|.$$

In order to estimate the m-rectangle characteristics with β-characteristics of certain weighted l_1-balls, we will use the following sets. Given an isomorphism

$$T : G_{\alpha,\beta}(\tilde{\lambda}, \tilde{a}) \to G_{\alpha,\beta}(\lambda, a),$$

where $\alpha, \beta \in \{0, \infty\}$, consider the coordinate basis $e = (e_i)_{i\in\mathbb{N}}$ of $G_{\alpha,\beta}(\lambda, a)$, and the image of the coordinate basis $\tilde{e} = (\tilde{e}_i)_{i\in\mathbb{N}}$ of $G_{\alpha,\beta}(\tilde{\lambda}, \tilde{a})$ under the isomorphism T, and denote it by $f = (f_i)$, where $f_i = T\tilde{e}_i$, $i \in \mathbb{N}$. Then, e and f are absolute bases in $G_{\alpha,\beta}(\lambda, a)$, so we define, for all $p, q \in \mathbb{N}$, the sets

$$\begin{aligned} A_{p,q} &= \left\{ x = \sum_{i=1}^{\infty} \xi_i e_i \in G_{0,\infty}(\lambda, a) : \sum_{i=1}^{\infty} |\xi_i| \omega_i^{\alpha,\beta}(p,q) \le 1 \right\}, \\ \tilde{A}_{p,q} &= \left\{ x = \sum_{i=1}^{\infty} \eta_i f_i \in G_{0,\infty}(\lambda, a) : \sum_{i=1}^{\infty} |\eta_i| \tilde{\omega}_i^{\alpha,\beta}(p,q) \le 1 \right\}, \end{aligned}$$

which are weighted l_1-balls in $G_{\alpha,\beta}(\lambda, a)$, that is,

$$A_{p,q} = B^e(\omega^{\alpha,\beta}(p,q)) \text{ and } \tilde{A}_{p,q} = B^f(\tilde{\omega}^{\alpha,\beta}(p,q))$$

for every $p, q \in \mathbb{N}$.

Lemma 5.0.15 *If $G_{\alpha,\beta}(\lambda, a) \simeq G_{\alpha,\beta}(\tilde{\lambda}, \tilde{a})$, then for every $r \in \mathbb{N}$ there exists $p \ge r$ such that for every $q \in \mathbb{N}$ there exists $s \ge q$ and a constant $C > 1$ so that the following inclusions hold:*

$$A_{p,q} \subset C\tilde{A}_{r,s}, \quad \tilde{A}_{p,q} \subset CA_{r,s}.$$

Proof. Let $T : G_{\alpha,\beta}(\tilde{\lambda}, \tilde{a}) \to G_{\alpha,\beta}(\lambda, a)$ be an isomorphism. Then, their projective spectra are equivalent by Proposition 2.2.1, hence for every $r \in \mathbb{N}$, there exists $p \in \mathbb{N}$ and a continuous linear operator T_r so that the following diagram commutes:

$$\begin{array}{ccc} G_{\alpha,\beta}(\tilde{\lambda}, \tilde{a}) & \xrightarrow{\quad T \quad} & G_{\alpha,\beta}(\lambda, a) \\ \Big\downarrow {\scriptstyle \tilde{\pi}_p} & & \Big\downarrow {\scriptstyle \pi_r} \\ \operatorname{ind}_{q\to}\, l_1(\tilde{\omega}^{\alpha,\beta}(p,q)) & \xrightarrow{T_r} & \operatorname{ind}_{s\to}\, l_1(\omega^{\alpha,\beta}(r,s)) \end{array}$$

For each $r \in \mathbb{N}$, T_r is continuous if and only if $T_r \circ \tilde{i}_q$ is continuous for every $q \in \mathbb{N}$. So, for each $q \in \mathbb{N}$, applying Grothendieck's factorization theorem, we get $s \in \mathbb{N}$ and a continuous linear operator $T_{r,q}$ such that the following diagram commutes:

$$\begin{array}{ccc} \operatorname{ind}_{q\to}\, l_1(\tilde{\omega}^{\alpha,\beta}(p,q)) & \xrightarrow{T_r} & \operatorname{ind}_{s\to}\, l_1(\omega^{\alpha,\beta}(r,s)) \\ \Big\uparrow {\scriptstyle \tilde{i}_q} & \nearrow & \Big\uparrow {\scriptstyle i_s} \\ l_1(\tilde{\omega}^{\alpha,\beta}(p,q)) & \xrightarrow{\quad T_{r,q} \quad} & l_1(\omega^{\alpha,\beta}(r,s)) \end{array}$$

Therefore, for every $r \in \mathbb{N}$ there exists $p \in \mathbb{N}$, and for every $q \in \mathbb{N}$ there exists $s \in \mathbb{N}$, so that the operator $T_{r,q} : l_1(\tilde{\omega}^{\alpha,\beta}(p,q)) \to l_1(\omega^{\alpha,\beta}(r,s))$ is continuous, that is,

$$\forall r \exists p \forall q \exists s \exists C \quad ||Tx||_{r,s} \leq C||x||_{p,q}, \quad x \in l_1(\tilde{\omega}^{\alpha,\beta}(p,q)). \tag{5.1}$$

Since T^{-1} is continuous, by (5.1), $\forall r \exists p \forall q \exists s \exists C$ $||T^{-1}x||_{r,s} \leq C||x||_{p,q}$. So, if we take $x \in A_{p,q}$, then $\sum_{i=1}^{\infty} |\xi_i|\omega_i^{\alpha,\beta}(p,q) \leq 1$ where $x = \sum \xi_i e_i$. Hence, $||x||_{p,q} \leq 1$, which implies $||T^{-1}x||_{r,s} \leq C$. Also,

$$T^{-1}x = T^{-1}(\sum \eta_i f_i) = T^{-1}(\sum \eta_i T\tilde{e}_i) = \sum \eta_i \tilde{e}_i.$$

Hence $||T^{-1}x||_{r,s} = \sum |\eta_i|\tilde{\omega}_i^{\alpha,\beta}(r,s) \leq C$, which implies that $x \in C\tilde{A}_{r,s}$. Therefore,

$$\forall r \exists p \forall q \exists s \exists C \; A_{p,q} \subset C\tilde{A}_{r,s}. \tag{5.2}$$

Since T is continuous, by (5.1), $\forall r' \exists p' \forall q' \exists s' \exists C'$ $||Ty||_{r',s'} \leq C'||y||_{p',q'}$. So, if we take $x \in \tilde{A}_{p,q}$, then $\sum_{i=1}^{\infty} |\eta_i|\tilde{\omega}_i^{\alpha,\beta}(p',q') \leq 1$ where $x = \sum \eta_i f_i$. Define $y = \sum \eta_i \tilde{e}_i$. Then, $||y||_{p',q'} = \sum_{i=1}^{\infty} |\eta_i|\tilde{\omega}_i^{\alpha,\beta}(p,q) \leq 1$. Hence, $||Ty||_{r',s'} \leq C'||y||_{p',q'} \leq C'$. Also,

$$Ty = T(\sum \eta_i \tilde{e}_i) = \sum \eta_i T\tilde{e}_i = \sum \eta_i f_i = x.$$

So, the seminorms satisfy

$$||Ty||_{r',s'} = ||x||_{r',s'} = \sum |\xi_i|\omega_i(r',s') \le C',$$

where $x = \sum \xi_i e_i$, which implies that $x \in C'A_{r',s'}$. Therefore,

$$\forall r' \exists p' \forall q' \exists s' \exists C' \ \tilde{A}_{p',q'} \subset C' A_{r',s'}. \tag{5.3}$$

■

In order to show that $G_{0,\infty}(\lambda, a) \simeq G_{0,\infty}(\tilde{\lambda}, \tilde{a})$ implies $\mu_m^{(\lambda,a)} \sim \mu_m^{(\tilde{\lambda},\tilde{a})}$ for every $m \in \mathbb{N}$, we need the following main lemma.

Lemma 5.0.16 *Let $G_{0,\infty}(\lambda, a) \simeq G_{0,\infty}(\tilde{\lambda}, \tilde{a})$, and $m \in \mathbb{N}$. Then, there exists a strictly increasing function $\gamma : [0,2] \to [0,1]$ where $\gamma(0) = 0$ and $\gamma(2) = 1$, a decreasing function $M : (0,1] \to (0,\infty)$, and a constant $\alpha > 1$ such that*

$$\mu_m^{(\lambda,a)}(\delta, \varepsilon; \tau, t) \le \mu_m^{(\tilde{\lambda},\tilde{a})}\left(\gamma(\delta) - \frac{M(\delta)}{\tau}, \gamma^{-1}(\varepsilon) + \frac{M(\varepsilon)}{\tau}; \frac{\tau}{\alpha}, \alpha t\right) \tag{5.4}$$

for all $\delta = (\delta_k)$, $\varepsilon = (\varepsilon_k)$, $\tau = (\tau_k)$ and $t = (t_k)$ where $0 < \delta_k \le \varepsilon_k \le 1$ and $0 < \tau_k \le t_k < \infty$, $k = 1, \cdots, m$.

Proof. Assume that $G_{0,\infty}(\lambda, a) \simeq G_{0,\infty}(\tilde{\lambda}, \tilde{a})$, and take $m \in \mathbb{N}$. In order to show inequality (5.4), we estimate the m-rectangle characteristics with the β-characteristics of some specific weighted l_1-balls that are constructed as follows.

By using Lemma 5.0.15 repeatedly, we can choose a chain of positive integers

$$\begin{aligned} r_{m+1} < p_{m+1} < r'_{m+1} < \cdots < r_k < p_k < r'_k < \cdots < r_0 < p_0 < r'_0 \\ < s'_0 < q_0 < s_0 < \cdots < s'_k < q_k < s_k < \cdots < s'_{m+1} < q_{m+1} < s_{m+1} \\ < n_1 < \cdots < n_j < \cdots \end{aligned} \tag{5.5}$$

with the additional conditions that each integer in the chain is at least two times greater than the previous one, and $2r'_0 n_j < n_{j+1}$ for each $j \in \mathbb{N}$, so that we have the inclusions

$$\begin{aligned} A_{p_k,q_k} \subset C\tilde{A}_{r_k,s_k}, \ \tilde{A}_{r'_k,s'_k} \subset CA_{p_k,q_k}, \ k = 0, \cdots, m+1, \\ A_{p_0,n_j} \subset C_j\tilde{A}_{r_0,n_{j+1}}, \ \tilde{A}_{r'_0,n_j} \subset C_jA_{p_0,n_{j+1}}, \ j \in \mathbb{N}, \end{aligned} \tag{5.6}$$

for some constants C, depending on m, and C_j, $j \in \mathbb{N}$.

Let $\delta = (\delta_k)$, $\varepsilon = (\varepsilon_k)$, $\tau = (\tau_k)$ and $t = (t_k)$ where $0 < \delta_k \le \varepsilon_k \le 1$ and $0 < \tau_k \le t_k < \infty$, $k = 1, \cdots, m$. If we reorder δ_k so that $\delta_1 \le \cdots \le \delta_m$, and define the sequence $(\zeta_j)_{j\in\mathbb{N}_0}$ by $\zeta_0 = 1$, $\zeta_j = \frac{1}{n_j}$ for $j \in \mathbb{N}$, then there exist finite subsequences (ν_k) and (j_k) of $\mathbb{N}_0$ such that

$$\zeta_{\nu_k} \le \delta_k < \zeta_{\nu_k - 1}, \ \zeta_{j_k+1} < \varepsilon_k \le \zeta_{j_k}, \ k = 1, \cdots, m. \tag{5.7}$$

For each $k = 1, \cdots, m$, we define the following sets

$$\begin{aligned}
U_1^{(k)} &= \begin{cases} A_{p_0, n_{j_k-1}}, \ j_k > 2 \\ A_{p_0,q_0}, \ j_k = 1,2 \end{cases}, \\
U_2^{(k)} &= \exp\left(\frac{\tau_k}{2p_k}\right) A_{p_0,q_0}, \\
U_3^{(k)} &= \exp(-2q_{m+1}t_k) A_{p_{m+1},q_{m+1}}, \\
U_4^{(k)} &= A_{p_k,q_k}, \\
V_1^{(k)} &= A_{p_0,n_{\nu_k}}, \\
V_2^{(k)} &= \exp\left(\frac{-\tau_k}{2p_{m+1}}\right) A_{p_{m+1},q_{m+1}}, \\
V_3^{(k)} &= \exp(2q_k t_k) A_{p_0,q_0}, \\
V_4^{(k)} &= A_{p_k,q_k}, \\
\tilde{U}_1^{(k)} &= \begin{cases} \frac{1}{C_{j_k-2}} \tilde{A}_{r_0', n_{j_k-2}}, \ j_k > 2 \\ \frac{1}{C} \tilde{A}_{r_0',s_0'}, \ j_k = 1,2 \end{cases}, \\
\tilde{U}_2^{(k)} &= \frac{1}{C} \exp\left(\frac{\tau_k}{2p_k}\right) \tilde{A}_{r_0',s_0'}, \\
\tilde{U}_3^{(k)} &= \frac{1}{C} \exp(-2q_{m+1}t_k) \tilde{A}_{r_{m+1}',s_{m+1}'}, \\
\tilde{U}_4^{(k)} &= \frac{1}{C} \tilde{A}_{r_k',s_k'}, \\
\tilde{V}_1^{(k)} &= C_{\nu_k} \tilde{A}_{r_0, n_{\nu_k+1}}, \\
\tilde{V}_2^{(k)} &= C \exp\left(\frac{-\tau_k}{2p_{m+1}}\right) \tilde{A}_{r_{m+1},s_{m+1}}, \\
\tilde{V}_3^{(k)} &= C \exp(2q_k t_k) \tilde{A}_{r_0,s_0}, \\
\tilde{V}_4^{(k)} &= C \tilde{A}_{r_k,s_k}.
\end{aligned}$$

These sets are, by definition, weighted l_1-balls, that is,

$$U_\theta^{(k)} = B^e\left(u_\theta^{(k)}\right), \ v_\theta^{(k)} = B^e\left(v_\theta^{(k)}\right), \ \tilde{U}_\theta^{(k)} = B^f\left(\tilde{u}_\theta^{(k)}\right), \ \tilde{V}_\theta^{(k)} = B^f\left(\tilde{v}_\theta^{(k)}\right),$$

where we denote by $u_\theta^{(k)} = \left(u_{\theta,i}^{(k)}\right)_{i\in\mathbb{N}}$, $v_\theta^{(k)} = \left(v_{\theta,i}^{(k)}\right)_{i\in\mathbb{N}}$, $\tilde{u}_\theta^{(k)} = \left(\tilde{u}_{\theta,i}^{(k)}\right)_{i\in\mathbb{N}}$, and $\tilde{v}_\theta^{(k)} = \left(\tilde{v}_{\theta,i}^{(k)}\right)_{i\in\mathbb{N}}$, their respective weight sequences, for every $k = 1, \cdots, m$ and $\theta = 1, \cdots, 4$.

The inclusions (5.6) imply that $\tilde{U}_\theta^{(k)} \subset U_\theta^{(k)}$ and $V_\theta^{(k)} \subset \tilde{V}_\theta^{(k)}$ for all $k = 1, \cdots, m$, $\theta = 1, \cdots, 4$. If we define the following sets

$$\begin{aligned} U &= \bigcap_{k=1}^{m} \operatorname{conv}\left(\bigcup_{\theta=1}^{4} U_\theta^{(k)}\right), V = \operatorname{conv}\left(\bigcup_{k=1}^{m}\left(\bigcap_{\theta=1}^{4} V_\theta^{(k)}\right)\right), \\ \tilde{U} &= \bigcap_{k=1}^{m} \operatorname{conv}\left(\bigcup_{\theta=1}^{4} \tilde{U}_\theta^{(k)}\right), \ \tilde{V} = \operatorname{conv}\left(\bigcup_{k=1}^{m}\left(\bigcap_{\theta=1}^{4} \tilde{V}_\theta^{(k)}\right)\right), \end{aligned}$$

then, we have the inclusions

$$\tilde{U} \subset U, \quad V \subset \tilde{V}.$$

So, by remark 5.0.1, we have

$$\beta(V, U) \leq \beta(\tilde{V}, \tilde{U}). \tag{5.8}$$

However, these sets are not necessarily weighted l_1-balls, hence it may not be possible to compute their β-characteristics with some corresponding weight sequences. In order to overcome this, we apply Proposition 5.0.13 to obtain the inclusions

$$B^e(c) \subset V, \ U \subset mB^e(d), \ \tilde{V} \subset 4B^f(\tilde{c}), \ B^f(\tilde{d}) \subset \tilde{U}, \tag{5.9}$$

for the sequences $c = (c_i)_{i\in\mathbb{N}}$, $d = (d_i)_{i\in\mathbb{N}}$, $\tilde{c} = (\tilde{c}_i)_{i\in\mathbb{N}}$, $\tilde{d} = (\tilde{d}_i)_{i\in\mathbb{N}}$, where

$$\begin{aligned} c_i &= \min_{k=1,\ldots,m}\left\{\max_{\theta=1,\ldots,4}\left\{v_{\theta,i}^{(k)}\right\}\right\}, \ d_i = \max_{k=1,\ldots,m}\left\{\min_{\theta=1,\ldots,4}\left\{u_{\theta,i}^{(k)}\right\}\right\}, \\ \tilde{c}_i &= \min_{k=1,\ldots,m}\left\{\max_{\theta=1,\ldots,4}\left\{\tilde{v}_{\theta,i}^{(k)}\right\}\right\}, \ \tilde{d}_i = \max_{k=1,\ldots,m}\left\{\min_{\theta=1,\ldots,4}\left\{\tilde{u}_{\theta,i}^{(k)}\right\}\right\}. \end{aligned}$$

Then, by Remark 5.0.1, the inclusions (5.9) imply that

$$\beta(B^e(c), B^e(d)) \leq \beta(V, \frac{1}{m}U) \text{ and } \beta(\tilde{V}, \tilde{U}) \leq \beta(4B^f(\tilde{c}), B^f(\tilde{d}).$$

Hence, by (5.8) and Remark 5.0.1, we have

$$\beta(B^e(c), B^e(d)) \leq \beta(4mB^f(\tilde{c}), B^f(\tilde{d})). \tag{5.10}$$

Now we show that, with this construction of weighted l_1-balls, we have the desired estimates. First, we claim that $\mu_m^{(\lambda,a)}(\delta, \varepsilon; \tau, t) \leq \beta(B^e(c), B^e(d))$.

Using Proposition 5.0.14, and considering the definitions of the weight sequences c and d, we get

$$\begin{aligned}
\beta(B^e(c), B^e(d)) &= |\{i \in \mathbb{N} : c_i \le d_i\}| \\
&= \left|\left\{i \in \mathbb{N} : \min_{k=1,\dots,m}\left\{\max_{\theta=1,\dots,4}\left\{v_{\theta,i}^{(k)}\right\}\right\} \le \max_{k=1,\dots,m}\left\{\min_{\theta=1,\dots,4}\left\{u_{\theta,i}^{(k)}\right\}\right\}\right\}\right| \\
&= \left|\bigcup_{k=1}^{m}\bigcup_{l=1}^{m}\left\{i \in \mathbb{N} : \max_{\theta=1,\dots,4}\left\{v_{\theta,i}^{(k)}\right\} \le \min_{\theta=1,\dots,4}\left\{u_{\theta,i}^{(l)}\right\}\right\}\right| \\
&\ge \left|\bigcup_{k=1}^{m}\left\{i \in \mathbb{N} : \max_{\theta=1,\dots,4}\left\{v_{\theta,i}^{(k)}\right\} \le \min_{\theta=1,\dots,4}\left\{u_{\theta,i}^{(k)}\right\}\right\}\right|
\end{aligned}$$

Note that $u_4^{(k)} = v_4^{(k)}$, so we have

$$\left\{i \in \mathbb{N} : \max_{\theta=1,\dots,4}\left\{v_{\theta,i}^{(k)}\right\} \le \min_{\theta=1,\dots,4}\left\{u_{\theta,i}^{(k)}\right\}\right\} = \bigcap_{\theta=1}^{3}\left\{i \in \mathbb{N} : v_{\theta,i}^{(k)} \le u_{4,i}^{(k)},\ v_{4,i}^{(k)} \le u_{\theta,i}^{(k)}\right\}.$$

Hence,

$$\beta(B^e(c), B^e(d)) \ge \left|\bigcup_{k=1}^{m}\bigcap_{\theta=1}^{3}\left\{i \in \mathbb{N} : v_{\theta,i}^{(k)} \le u_{4,i}^{(k)},\ v_{4,i}^{(k)} \le u_{\theta,i}^{(k)}\right\}\right|.$$

Since $\mu_m^{(\lambda,a)}(\delta, \varepsilon; \tau, t) = |\bigcup_{k=1}^{m}\{i \in \mathbb{N} : \delta_k \le \lambda_i \le \varepsilon_k, \tau_k \le a_i \le t_k\}|$, the claim is true if the inclusions

$$\{i \in \mathbb{N} : \delta_k \le \lambda_i \le \varepsilon_k, \tau_k \le a_i \le t_k\} \subset \bigcap_{\theta=1}^{3}\left\{i \in \mathbb{N} : v_{\theta,i}^{(k)} \le u_{4,i}^{(k)},\ v_{4,i}^{(k)} \le u_{\theta,i}^{(k)}\right\}$$

hold for all $k = 1, \cdots, m$, which we show as follows.

Given $k \in \{1, \cdots, m\}$ and $i \in \mathbb{N}$, if $\lambda_i \ge \delta_k$, then, since $p_k < p_0$, $p_k > 2$, $n_{\nu_k} > 2q_k$ from (5.5) and $\zeta_{\nu_k} \le \delta_k$ by (5.7), we have the estimates

$$0 < \frac{1}{p_k} - \frac{1}{p_0} < \frac{1}{p_k} < \frac{1}{2}, \quad n_{\nu_k} - q_k > \frac{n_{\nu_k}}{2} = \frac{1}{2\zeta_{\nu_k}} \ge \frac{1}{2\delta_k} \ge \frac{1}{2\lambda_i},$$

which imply that

$$\begin{aligned}
& \frac{1}{p_k} - \frac{1}{p_0} \le \lambda_i\,(n_{\nu_k} - q_k) \\
\Rightarrow\ & -\frac{1}{p_0} - n_{\nu_k}\lambda_i \le -\frac{1}{p_k} - q_k\lambda_i \\
\Rightarrow\ & \exp\left(\left(-\frac{1}{p_0} - n_{\nu_k}\lambda_i\right)a_i\right) \le \exp\left(\left(-\frac{1}{p_k} - q_k\lambda_i\right)a_i\right) \\
\Rightarrow\ & v_{1,i}^{(k)} \le u_{4,i}^{(k)}.
\end{aligned}$$

Hence,

$$\{i \in \mathbb{N} : \lambda_i \ge \delta_k\} \subset \left\{i \in \mathbb{N} : v_{1,i}^{(k)} \le u_{4,i}^{(k)}\right\}. \tag{5.11}$$

If $\lambda_i \le \varepsilon_k$, then, since $p_0 > 2p_k$, $n_{j_k} > 2r_0' n_{j_k-1} > 2p_k n_{j_k-1}$ from (5.5) and $\zeta_{j_k} \ge \varepsilon_k$ by (5.7), we have the estimates

$$\frac{1}{p_k} - \frac{1}{p_0} > \frac{1}{2p_k}, \quad n_{j_k-1} - q_k < n_{j_k-1} < \frac{n_{j_k}}{2p_k} = \frac{1}{2p_k\zeta_{j_k}} \le \frac{1}{2p_k\varepsilon_k} \le \frac{1}{2p_k\lambda_i},$$

which imply for $j_k > 2$ that

$$\begin{aligned}
&\lambda_i \left(n_{j_k-1} - q_k\right) \le \frac{1}{p_k} - \frac{1}{p_0} \\
\Rightarrow\ & -\frac{1}{p_k} - q_k\lambda_i \le -\frac{1}{p_0} - n_{j_k-1}\lambda_i \\
\Rightarrow\ & \exp\left(\left(-\frac{1}{p_k} - q_k\lambda_i\right) a_i\right) \le \exp\left(\left(-\frac{1}{p_0} - n_{j_k-1}\lambda_i\right) a_i\right) \\
\Rightarrow\ & v_{4,i}^{(k)} \le u_{1,i}^{(k)}.
\end{aligned}$$

For $j_k \le 2$, we have $p_0 > p_k$ and $q_k > q_0$ from (5.5), so the inequality

$$-\frac{1}{p_k} - q_k\lambda_i \le -\frac{1}{p_0} - q_0\lambda_i$$

is satisfied for all λ_i. From these inequalities, we obtain

$$\{i \in \mathbb{N} : \lambda_i \le \varepsilon_k\} \subset \left\{i \in \mathbb{N} : v_{4,i}^{(k)} \le u_{1,i}^{(k)}\right\}. \tag{5.12}$$

If $a_i \ge \tau_k$, we have the inequalities

$$\frac{1}{2p_{m+1}} < \frac{1}{p_{m+1}} - \frac{1}{p_k} < \frac{1}{p_{m+1}} - \frac{1}{p_k} + \lambda_i \left(q_{m+1} - q_k\right)$$

since $p_k > 2p_{m+1}$ and $q_{m+1} > q_k$ from (5.5), so we obtain

$$\frac{\tau_k}{2p_{m+1}} < \left(\frac{1}{p_{m+1}} - \frac{1}{p_k} + \lambda_i \left(q_{m+1} - q_k\right)\right) a_i.$$

By rearranging terms and taking the exponential function of both sides, we get

$$\exp\left(\frac{\tau_k}{2p_{m+1}}\right) \exp\left(\left(-\frac{1}{p_{m+1}} - q_{m+1}\lambda_i\right) a_i\right) < \exp\left(\left(-\frac{1}{p_k} - q_k\lambda_i\right) a_i\right).$$

Then, by the definitions of the weight sequences, we have $v_{2,i}^{(k)} \le u_{4,i}^{(k)}$. Hence,

$$\{i \in \mathbb{N} : \tau_k \le a_i\} \subset \left\{i \in \mathbb{N} : v_{2,i}^{(k)} \le u_{4,i}^{(k)}\right\}. \tag{5.13}$$

If $a_i \ge \tau_k$, we also have the inequalities

$$\frac{1}{2p_k} < \frac{1}{p_k} - \frac{1}{p_0} < \frac{1}{p_k} - \frac{1}{p_0} + \lambda_i \left(q_k - q_0\right)$$

since $p_0 > 2p_k$ and $q_k > q_0$ from (5.5), and we obtain

$$\frac{\tau_k}{2p_k} < \left(\frac{1}{p_k} - \frac{1}{p_0} + \lambda_i (q_k - q_0)\right) a_i.$$

By rearranging terms and taking the exponential function of both sides, we get

$$\exp\left(\left(-\frac{1}{p_k} - q_k\lambda_i\right) a_i\right) < \exp\left(-\frac{\tau_k}{2p_k}\right) \exp\left(\left(-\frac{1}{p_0} - q_0\lambda_i\right) a_i\right).$$

So, by the definitions of the weight sequences, we have $v_{4,i}^{(k)} \leq u_{2,i}^{(k)}$. Hence,

$$\{i \in \mathbb{N} : \tau_k \leq a_i\} \subset \left\{i \in \mathbb{N} : v_{4,i}^{(k)} \leq u_{2,i}^{(k)}\right\}. \tag{5.14}$$

If $a_i \leq t_k$, since $p_k < p_0$, $q_k > q_0$ from (5.5) and $\lambda_i \leq 1$ by lemma 2.1.1, we have the inequalities

$$0 < \frac{1}{p_k} - \frac{1}{p_0} < q_k, \quad 0 < (q_k - q_0)\,\lambda_i \leq q_k - q_0 < q_k,$$

which imply that

$$\frac{1}{p_k} - \frac{1}{p_0} + (q_k - q_0)\,\lambda_i < 2q_k,$$

and we obtain

$$\left(\frac{1}{p_k} - \frac{1}{p_0} + (q_k - q_0)\,\lambda_i\right) a_i < 2q_k t_k.$$

By rearranging terms and taking the exponential function of both sides, we get

$$\exp(-2q_k t_k) \exp\left(\left(-\frac{1}{p_0} - q_0\lambda_i\right) a_i\right) < \exp\left(\left(-\frac{1}{p_k} - q_k\lambda_i\right) a_i\right).$$

By the definitions of the weight sequences, we have $v_{3,i}^{(k)} \leq u_{4,i}^{(k)}$. Hence,

$$\{i \in \mathbb{N} : a_i \leq t_k\} \subset \left\{i \in \mathbb{N} : v_{3,i}^{(k)} \leq u_{4,i}^{(k)}\right\}. \tag{5.15}$$

For $a_i \leq t_k$, we also have the inequalities

$$0 < \frac{1}{p_{m+1}} - \frac{1}{p_k} < q_{m+1}, \quad 0 < (q_{m+1} - q_k)\,\lambda_i \leq q_{m+1} - q_k < q_{m+1}$$

since $p_{m+1} < p_k$, $q_{m+1} > q_k$ from (5.5) and $\lambda_i \leq 1$ by lemma 2.1.1, and we get

$$\left(\frac{1}{p_{m+1}} - \frac{1}{p_k} + (q_{m+1} - q_k)\,\lambda_i\right) a_i < 2q_{m+1} t_k.$$

By rearranging terms and taking the exponential function of both sides, we get

$$\exp\left(\left(-\frac{1}{p_k} - q_k\lambda_i\right) a_i\right) < \exp(2q_{m+1} t_k) \exp\left(\left(-\frac{1}{p_{m+1}} - q_{m+1}\lambda_i\right) a_i\right).$$

By the definition of the weight sequences, we have $v_{4,i}^{(k)} \le u_{3,i}^{(k)}$. Hence,

$$\{i \in \mathbb{N} : a_i \le t_k\} \subset \left\{i \in \mathbb{N} : v_{4,i}^{(k)} \le u_{3,i}^{(k)}\right\}. \tag{5.16}$$

From the inclusions (5.11)-(5.16), we obtain

$$\{i \in \mathbb{N} : \delta_k \le \lambda_i \le \varepsilon_k, \tau_k \le a_i \le t_k\} \subset \bigcap_{\theta=1}^{3} \left\{i \in \mathbb{N} : v_{\theta,i}^{(k)} \le u_{4,i}^{(k)},\ v_{4,i}^{(k)} \le u_{\theta,i}^{(k)}\right\}.$$

Therefore,

$$\mu_m^{(\lambda,a)}(\delta, \varepsilon; \tau, t) \le \beta(B^e(c), B^e(d)). \tag{5.17}$$

Now, we claim that

$$\beta(4mB^f(\tilde{c}), B^f(\tilde{d})) \le \mu_m^{(\tilde{\lambda},\tilde{a})}\left(\gamma(\delta) - \frac{M(\delta)}{\tau}, \gamma^{-1}(\varepsilon) + \frac{M(\varepsilon)}{\tau}; \frac{\tau}{\alpha}, \alpha t\right)$$

for some strictly increasing function $\gamma : [0,2] \to [0,1]$ where $\gamma(0) = 0$ and $\gamma(2) = 1$, decreasing function $M : (0,1] \to (0,\infty)$, and constant $\alpha > 1$.

Using Proposition 5.0.14 and considering the definitions of the weight sequences $\tilde{c}$ and $\tilde{d}$, we obtain

$$\begin{aligned}
\beta(4mB^f(\tilde{c}), B^f(\tilde{d})) &= \left|\{i \in \mathbb{N} : \tilde{c}_i \le 4m\tilde{d}_i\}\right| \\
&= \left|\left\{i \in \mathbb{N} : \min_{k=1,\ldots,m}\left\{\max_{\theta=1,\ldots,4}\left\{\tilde{v}_{\theta,i}^{(k)}\right\}\right\} \le 4m \max_{k=1,\ldots,m}\left\{\min_{\theta=1,\ldots,4}\left\{\tilde{u}_{\theta,i}^{(k)}\right\}\right\}\right\}\right| \\
&= \left|\bigcup_{k=1}^{m}\bigcup_{l=1}^{m}\left\{i \in \mathbb{N} : \max_{\theta=1,\ldots,4}\left\{\tilde{v}_{\theta,i}^{(k)}\right\} \le 4m \min_{\theta=1,\ldots,4}\left\{\tilde{u}_{\theta,i}^{(l)}\right\}\right\}\right|
\end{aligned}$$

Also, for any $k, l = 1, \ldots, m$, we have

$$\begin{aligned}
&\left\{i \in \mathbb{N} : \max_{\theta=1,\ldots,4}\left\{\tilde{v}_{\theta,i}^{(k)}\right\} \le 4m \min_{\theta=1,\ldots,4}\left\{\tilde{u}_{\theta,i}^{(l)}\right\}\right\} = \bigcap_{\theta=1}^{4}\bigcap_{\rho=1}^{4}\left\{i \in \mathbb{N} : \tilde{v}_{\theta,i}^{(k)} \le 4m\tilde{u}_{\rho,i}^{(l)}\right\} \\
&\subset \bigcap_{\theta=1}^{3}\left\{i \in \mathbb{N} : \tilde{v}_{\theta,i}^{(k)} \le 4m\tilde{u}_{4,i}^{(l)},\ \tilde{v}_{4,i}^{(k)} \le 4m\tilde{u}_{\theta,i}^{(l)}\right\} \bigcap \left\{i \in \mathbb{N} : \tilde{v}_{4,i}^{(k)} \le 4m\tilde{u}_{4,i}^{(l)}\right\}
\end{aligned} \tag{5.18}$$

Now, using the sets on the right hand side of the inclusion (5.18), we construct the corresponding m-rectangles.

If $\tilde{v}_{1,i}^{(k)} \le 4m\tilde{u}_{4,i}^{(l)}$, then

$$\frac{1}{C_{\nu_k}}\exp\left(\left(-\frac{1}{r_0} - n_{\nu_k+1}\tilde{\lambda}_i\right)\tilde{a}_i\right) \le 4mC\exp\left(\left(-\frac{1}{r'_l} - s'_l\tilde{\lambda}_i\right)\tilde{a}_i\right).$$

Taking the logarithm of both sides and rearranging terms, we get

$$\tilde{\lambda}_i \ge \frac{\frac{1}{r'_l} - \frac{1}{r_0}}{n_{\nu_k+1} - s'_l} - \frac{\ln(4mCC_{\nu_k})}{\tilde{a}_i(n_{\nu_k+1} - s'_l)}.$$

Since $r_0 > 2r'_l$, $n_{\nu_k+1} > 2s'_l$, $n_{\nu_k+1} > n_{\nu_k}$, and $n_{\nu_k+2} > 2r'_l n_{\nu_k+1}$ by the choice of the chain (5.5), we obtain the inclusions

$$\frac{\frac{1}{r'_l} - \frac{1}{r_0}}{n_{\nu_k+1} - s'_l} > \frac{1}{2r'_l(n_{\nu_k+1} - s'_l)} > \frac{1}{2r'_l n_{\nu_k+1}} > \frac{1}{n_{\nu_k+2}} = \zeta_{\nu_k+2},$$

$$\frac{\ln(4mCC_{\nu_k})}{\tilde{a}_i(n_{\nu_k+1} - s'_l)} < \frac{2\ln(4mCC_{\nu_k})}{n_{\nu_k+1}\tilde{a}_i} = \frac{2\zeta_{\nu_k+1}\ln(4mCC_{\nu_k})}{\tilde{a}_i} < \frac{2\zeta_{\nu_k}\ln(4mCC_{\nu_k})}{\tilde{a}_i},$$

which imply that

$$\tilde{\lambda}_i \geq \zeta_{\nu_k+2} - \frac{2\zeta_{\nu_k}\ln(4mCC_{\nu_k})}{\tilde{a}_i}. \tag{5.19}$$

Let $\tilde{v}^{(k)}_{4,i} \leq 4m\tilde{u}^{(l)}_{1,i}$. We need to consider the cases $j_l > 2$ and $j_l \leq 2$ separately due to the definition of $\tilde{u}^{(l)}_{1,i}$. For $j_l > 2$, we have

$$\frac{1}{C}\exp\left(\left(-\frac{1}{r_k} - s_k\tilde{\lambda}_i\right)\tilde{a}_i\right) \leq 4mC_{j_l-2}\exp\left(\left(-\frac{1}{r'_0} - n_{j_l-2}\tilde{\lambda}_i\right)\tilde{a}_i\right).$$

Taking the logarithm of both sides and rearranging terms, we get

$$\tilde{\lambda}_i \leq \frac{\frac{1}{r_k} - \frac{1}{r'_0}}{n_{j_l-2} - s_k} + \frac{\ln(4mCC_{j_l-2})}{\tilde{a}_i(n_{j_l-2} - s_k)}.$$

Since $n_{j_l-2} > 2s_k$ and $\frac{1}{r_k} - \frac{1}{r'_0} < \frac{1}{r_k} < \frac{1}{2}$ by the choice of the chain (5.5), the above inequality implies

$$\begin{aligned}\tilde{\lambda}_i &< \frac{2\left(\frac{1}{r_k} - \frac{1}{r'_0}\right)}{n_{j_l-2}} + \frac{2\ln(4mCC_{j_l-2})}{\tilde{a}_i n_{j_l-2}} \\ &< \frac{1}{n_{j_l-2}} + \frac{2\ln(4mCC_{j_l-2})}{\tilde{a}_i n_{j_l-2}} \\ &= \zeta_{j_l-2} + \frac{2\zeta_{j_l-2}\ln(4mCC_{j_l-2})}{\tilde{a}_i}.\end{aligned} \tag{5.20}$$

For $j_l \leq 2$, the inequality $\tilde{v}^{(k)}_{4,i} \leq 4m\tilde{u}^{(l)}_{1,i}$ implies

$$\frac{1}{C}\exp\left(\left(-\frac{1}{r_k} - s_k\tilde{\lambda}_i\right)\tilde{a}_i\right) \leq 4mC\exp\left(\left(-\frac{1}{r'_0} - s'_0\tilde{\lambda}_i\right)\tilde{a}_i\right).$$

Taking the logarithm of both sides and rearranging terms, we get

$$\left[\left(\frac{1}{r'_0} - \frac{1}{r_k}\right) + (s'_0 - s_k)\,\tilde{\lambda}_i\right]\tilde{a}_i \leq \ln(4mC^2).$$

Since $r_k < r'_0$ and $s'_0 < s_k$, the left hand side of the above inequality is negative for all $i \in \mathbb{N}$, hence the inequality holds for all $i \in \mathbb{N}$. As $\tilde{\lambda}_i \leq 1$ for all $i \in \mathbb{N}$ by Lemma 2.1.1, and $\zeta_0 = 1$, for $j_l \leq 2$, we have

$$\left\{i \in \mathbb{N} : \tilde{v}^{(k)}_{4,i} \leq 4m\tilde{u}^{(l)}_{1,i}\right\} = \left\{i \in \mathbb{N} : \tilde{\lambda}_i \leq \zeta_0\right\} = \mathbb{N}.$$

Take a strictly increasing function $\gamma : [0,2] \to [0,1]$ so that $\gamma(0) = 0$, $\gamma(2) = 1$, and $\gamma(\zeta_j) = \zeta_{j+3}$ for all $j \in \mathbb{N}_0$. Since γ is strictly increasing, γ^{-1} exists, it is strictly increasing, $\gamma^{-1}(0) = 0$, $\gamma^{-1}(1) = 2$, and $\gamma^{-1}(\zeta_j) = \zeta_{j-3}$ for all $j \geq 3$. Also, take a decreasing function $M : (0,1] \to (0,\infty)$ so that

$$M(\zeta_j) \geq 2\alpha \max\{\zeta_j \ln(4mCC_{j+1}), \zeta_{j-2}\ln(4mCC_{j-2})\}$$

for all $j \in \mathbb{N}_0$, where the constant α will be chosen explicitly later in the proof to simultaneously satisfy other inclusions. Then, $\tilde{v}_{1,i}^{(k)} \leq 4m\tilde{u}_{4,i}^{(l)}$ implies

$$\begin{aligned}
\tilde{\lambda}_i &\geq \zeta_{\nu_k+2} - \frac{2\zeta_{\nu_k}\ln(4mCC_{\nu_k})}{\tilde{a}_i} \quad \text{by (5.19)},\\
&= \gamma(\zeta_{\nu_k-1}) - \frac{2\zeta_{\nu_k}\ln(4mCC_{\nu_k})}{\tilde{a}_i} \quad \text{since } \gamma(\zeta_j) = \zeta_{j+3},\\
&\geq \gamma(\zeta_{\nu_k-1}) - \frac{M(\zeta_{\nu_k-1})}{\alpha\tilde{a}_i} \quad \text{since } M(\zeta_{\nu_k-1}) \geq 2\alpha\zeta_{\nu_k}\ln(4mCC_{\nu_k}),\\
&> \gamma(\delta_k) - \frac{M(\delta_k)}{\alpha\tilde{a}_i} \quad \text{since } \delta_k < \zeta_{\nu_k-1} \text{ by (5.7)}.
\end{aligned}$$

Therefore, for any $k, l = 1, \ldots, m$, we have

$$\left\{i \in \mathbb{N} : \tilde{v}_{1,i}^{(k)} \leq 4m\tilde{u}_{4,i}^{(l)}\right\} \subset \left\{i \in \mathbb{N} : \gamma(\delta_k) - \frac{M(\delta_k)}{\alpha\tilde{a}_i} \leq \tilde{\lambda}_i\right\}. \tag{5.21}$$

If we consider the inequality $\tilde{v}_{4,i}^{(k)} \leq 4m\tilde{u}_{1,i}^{(l)}$, then for $j_l > 2$,

$$\begin{aligned}
\tilde{\lambda}_i &\leq \zeta_{j_l-2} + \frac{2\zeta_{j_l-2}\ln(4mCC_{j_l-2})}{\tilde{a}_i} \quad \text{by (5.20)},\\
&= \gamma^{-1}(\zeta_{j_l+1}) + \frac{2\zeta_{j_l-2}\ln(4mCC_{j_l-2})}{\tilde{a}_i} \quad \text{since } \zeta_{j_l-2} = \gamma^{-1}(\zeta_{j_l+1}),\\
&\leq \gamma^{-1}(\zeta_{j_l+1}) + \frac{M(\zeta_{j_l})}{\alpha\tilde{a}_i} \quad \text{since } M(\zeta_{j_l}) \geq 2\alpha\zeta_{j_l-2}\ln(4mCC_{j_l-2}),\\
&< \gamma^{-1}(\varepsilon_l) + \frac{M(\varepsilon_l)}{\alpha\tilde{a}_i} \quad \text{since } \zeta_{j_l+1} < \varepsilon_l \leq \zeta_{j_l} \text{ by (5.7)}.
\end{aligned}$$

For $j_l \leq 2$, $\tilde{v}_{1,i}^{(k)} \leq 4m\tilde{u}_{4,i}^{(l)}$ implies $\tilde{\lambda}_i \leq \zeta_0 = 1$. Since $\zeta_3 \leq \zeta_{j_l+1}$ for $j_l \leq 2$, γ^{-1} is increasing and $\zeta_{j_l+1} < \varepsilon_l$, we have

$$\gamma^{-1}(\zeta_3) = \zeta_0 \leq \gamma^{-1}(\zeta_{j_l+1}) < \gamma^{-1}(\varepsilon_l).$$

Hence, $\tilde{\lambda}_i \leq \gamma^{-1}(\varepsilon_l) + \frac{M(\varepsilon_l)}{\alpha\tilde{a}_i}$. Therefore, for any $k, l = 1, \ldots, m$, we have

$$\left\{i \in \mathbb{N} : \tilde{v}_{4,i}^{(k)} \leq 4m\tilde{u}_{1,i}^{(l)}\right\} \subset \left\{i \in \mathbb{N} : \tilde{\lambda}_i \leq \gamma^{-1}(\varepsilon_l) + \frac{M(\varepsilon_l)}{\alpha\tilde{a}_i}\right\}. \tag{5.22}$$

Let us choose the constant α so that

$$\alpha > 4r_0' \max\left\{\ln(4mC^2), 2s_{m+1}\right\}. \tag{5.23}$$

If $\tilde{v}_{2,i}^{(k)} \leq 4m\tilde{u}_{4,i}^{(l)}$, then

$$\frac{1}{C}\exp\left(\frac{\tau_k}{2p_{m+1}}\right)\exp\left(\left(-\frac{1}{r_{m+1}} - s_{m+1}\tilde{\lambda}_i\right)\tilde{a}_i\right) \leq 4mC\exp\left(\left(-\frac{1}{r_l'} - s_l'\tilde{\lambda}_i\right)\tilde{a}_i\right).$$

Taking the logarithm of both sides and rearranging terms, we get

$$\frac{\tau_k}{2p_{m+1}} \leq \ln(4mC^2) + \left[\left(\frac{1}{r_{m+1}} - \frac{1}{r_l'}\right) + (s_{m+1} - s_l')\,\tilde{\lambda}_i\right]\tilde{a}_i.$$

Since $r_l' > r_{m+1}$, $s_{m+1} > s_l'$ by (5.5) and $\tilde{\lambda}_i \leq 1$ by Lemma 2.1.1, we have the inequalities

$$0 < \frac{1}{r_{m+1}} - \frac{1}{r_l'} < s_{m+1}, \quad 0 < (s_{m+1} - s_l')\tilde{\lambda}_i \leq s_{m+1} - s_l' < s_{m+1},$$

which imply that

$$\frac{\tau_k}{2p_{m+1}} \leq \ln(4mC^2) + 2s_{m+1}\tilde{a}_i.$$

As $\tilde{a}_i \geq 1$ by Lemma 2.1.1, we get

$$\tau_k \leq 2p_{m+1}\left(\ln(4mC^2) + 2s_{m+1}\right)\tilde{a}_i.$$

From (5.23), we have $\tau_k \leq \alpha\tilde{a}_i$. Thus, for any $k, l = 1, \ldots, m$ we have

$$\left\{i \in \mathbb{N} : \tilde{v}_{2,i}^{(k)} \leq 4m\tilde{u}_{4,i}^{(l)}\right\} \subset \left\{i \in \mathbb{N} : \frac{\tau_k}{\alpha} \leq \tilde{a}_i\right\}. \tag{5.24}$$

If $\tilde{v}_{4,i}^{(k)} \leq \tilde{u}_{2,i}^{(l)}$, then

$$\frac{1}{C}\exp\left(\left(-\frac{1}{r_k} - s_k\tilde{\lambda}_i\right)\tilde{a}_i\right) \leq 4mC\exp\left(-\frac{\tau_l}{2p_l}\right)\exp\left(\left(-\frac{1}{r_0'} - s_0'\tilde{\lambda}_i\right)\tilde{a}_i\right).$$

Taking the logarithm of both sides and rearranging terms, we get

$$\frac{\tau_l}{2p_l} \leq \ln(4mC^2) + \left[\left(\frac{1}{r_k} - \frac{1}{r_0'}\right) + (s_k - s_0')\,\tilde{\lambda}_i\tilde{a}_i\right].$$

Since $r_0' > r_k$, $s_k > s_0'$ by (5.5) and $\tilde{\lambda}_i \leq 1$ by Lemma 2.1.1, we have the inequalities

$$0 < \frac{1}{r_k} - \frac{1}{r_0'} < s_k, \quad 0 < (s_k - s_0')\tilde{\lambda}_i \leq s_k - s_0' < s_k.$$

which imply that

$$\tau_l \leq 2p_l\ln(4mC^2) + 2s_k\tilde{a}_i.$$

As $\tilde{a}_i \geq 1$ by Lemma 2.1.1, we get

$$\tau_l \leq 2p_l \left(\ln(4mC^2) + 2s_k\right) \tilde{a}_i.$$

From (5.23), we have $\tau_l \leq \alpha \tilde{a}_i$. Thus, for any $k, l = 1, \ldots, m$, we have

$$\left\{i \in \mathbb{N} : \tilde{v}_{4,i}^{(k)} \leq 4m\tilde{u}_{2,i}^{(l)}\right\} \subset \left\{i \in \mathbb{N} : \frac{\tau_l}{\alpha} \leq \tilde{a}_i\right\}. \tag{5.25}$$

Let $\tilde{v}_{3,i}^{(k)} \leq 4m\tilde{u}_{4,i}^{(l)}$. Then

$$\frac{1}{C} \exp\left(-2q_k t_k\right) \exp\left(\left(-\frac{1}{r_0} - s_0 \tilde{\lambda}_i\right) \tilde{a}_i\right) \leq 4mC \exp\left(\left(-\frac{1}{r_l'} - \frac{1}{s_l'} \tilde{\lambda}_i\right) \tilde{a}_i\right).$$

Taking the logarithm of both sides and rearranging terms, we get

$$\left[\left(\frac{1}{r_l'} - \frac{1}{r_0}\right) + \left(s_l' - s_0\right) \tilde{\lambda}_i\right] \tilde{a}_i \leq \ln(4mC^2) + 2q_k t_k.$$

Since $r_0 > 2r_l'$ and $s_l' > s_0$ by the choice of the chain, we have

$$\left(\frac{1}{r_l'} - \frac{1}{r_0}\right) + \left(s_l' - s_0\right) \tilde{\lambda}_i > \frac{1}{r_l'} - \frac{1}{r_0} > \frac{1}{2r_l'}.$$

Without loss of generality, we can take $t_k \geq 1$ since $a_i \geq 1$ by Lemma 2.1.1. We also have $r_0' > r_l'$, $s_{m+1} > q_k$ by the choice of the chain (5.5), so we obtain

$$\begin{aligned}
\tilde{a}_i &\leq 2r_l' \left(\ln(4mC^2) + 2q_k t_k\right) \\
&\leq 2r_l' \left(\ln(4mC^2) + 2q_k\right) t_k \\
&\leq \left(2r_0' \ln(4mC^2) + 4r_0' s_{m+1}\right) t_k \\
&\leq \left(\frac{\alpha}{2} + \frac{\alpha}{2}\right) t_k = \alpha t_k
\end{aligned}$$

Thus, for any $k, l = 1, \ldots, m$, we have

$$\left\{i \in \mathbb{N} : \tilde{v}_{3,i}^{(k)} \leq 4m\tilde{u}_{4,i}^{(l)}\right\} \subset \left\{i \in \mathbb{N} : \tilde{a}_i \leq \alpha t_k\right\}. \tag{5.26}$$

Let $\tilde{v}_{4,i}^{(k)} \leq 4m\tilde{u}_{3,i}^{(l)}$. Then

$$\frac{1}{C} \exp\left(\left(-\frac{1}{r_k} - s_k \tilde{\lambda}_i\right) \tilde{a}_i\right) \leq 4mC \exp\left(2q_{m+1} t_l\right) \exp\left(\left(-\frac{1}{r_{m+1}'} - \frac{1}{s_{m+1}'} \tilde{\lambda}_i\right) \tilde{a}_i\right).$$

Taking the logarithm of both sides and rearranging terms, we get

$$\left[\left(\frac{1}{r_{m+1}'} - \frac{1}{r_k}\right) + \left(s_{m+1}' - s_k\right) \tilde{\lambda}_i\right] \tilde{a}_i \leq \ln(4mC^2) + 2q_{m+1} t_l.$$

Since $r_k > 2r'_{m+1}$ and $s'_{m+1} > s_k$ by the choice of the chain (5.5), we have

$$\left(\frac{1}{r'_{m+1}} - \frac{1}{r_k}\right) + \left(s'_{m+1} - s_k\right)\tilde{\lambda}_i > \frac{1}{r'_{m+1}} - \frac{1}{r_k} > \frac{1}{2r'_{m+1}}.$$

Without loss of generality, we can take $t_l \geq 1$. We also have $r'_0 > r'_{m+1}$, $s_{m+1} > q_{m+1}$ by the choice of the chain, hence we get

$$\begin{aligned}\tilde{a}_i &\leq 2r'_{m+1}\left(\ln(4mC^2) + 2q_{m+1}t_l\right)\\ &\leq 2r'_{m+1}\left(\ln(4mC^2) + 2q_{m+1}\right)t_l\\ &\leq \left(2r'_0\ln(4mC^2) + 4r'_0 s_{m+1}\right)t_l\\ &\leq \left(\frac{\alpha}{2} + \frac{\alpha}{2}\right)t_l = \alpha t_l\end{aligned}$$

Thus, for any $k, l = 1, \ldots, m$, we have

$$\left\{i \in \mathbb{N} : \tilde{v}^{(k)}_{4,i} \leq 4m\tilde{u}^{(l)}_{3,i}\right\} \subset \left\{i \in \mathbb{N} : \tilde{a}_i \leq \alpha t_l\right\}. \tag{5.27}$$

From the inclusions (5.21)-(5.27), we obtain

$$\bigcap_{\theta=1}^{3}\left\{i \in \mathbb{N} : \tilde{v}^{(k)}_{\theta,i} \leq 4m\tilde{u}^{(l)}_{4,i},\ \tilde{v}^{(k)}_{4,i} \leq 4m\tilde{u}^{(l)}_{\theta,i}\right\} \subset R_{k,l} \tag{5.28}$$

for any $k, l = 1, \ldots, m$, where

$$\begin{aligned}R_{k,l} = \Bigg\{i \in \mathbb{N} : \gamma(\delta_k) - \frac{M(\delta_k)}{\max\{\tau_k, \tau_l\}} \leq \tilde{\lambda}_i \leq \gamma^{-1}(\varepsilon_l) + \frac{M(\varepsilon_l)}{\max\{\tau_k, \tau_l\}},\\ \frac{\max\{\tau_k, \tau_l\}}{\alpha} \leq \tilde{a}_i \leq \alpha\min\{t_k, t_l\}\Bigg\}.\end{aligned}$$

If $\tilde{v}^{(k)}_{4,i} \leq 4m\tilde{u}^{(l)}_{4,i}$, then

$$\frac{1}{C}\exp\left(\left(-\frac{1}{r_k} - s_k\tilde{\lambda}_i\right)\tilde{a}_i\right) \leq 4mC\exp\left(\left(-\frac{1}{r'_l} - s'_l\tilde{\lambda}_i\right)\tilde{a}_i\right).$$

Taking the logarithm of both sides and rearranging terms, we get

$$\left[\left(\frac{1}{r'_l} - \frac{1}{r_k}\right) + \left(s'_l - s_k\right)\tilde{\lambda}_i\right]\tilde{a}_i \leq \ln(4mC^2) < \frac{\alpha}{2r'_0}.$$

If $k < l$, then $r_k > 2r'_l$ and $s'_l > s_k$ by the choice of the chain (5.5). Hence,

$$\left(\frac{1}{r'_l} - \frac{1}{r_k}\right) + \left(s'_l - s_k\right)\tilde{\lambda}_i > \frac{1}{r'_l} - \frac{1}{r_k} > \frac{1}{2r'_l},$$

which implies that $\tilde{a}_i < 2r'_l\frac{\alpha}{2r'_0} < \alpha$ since $r'_l < r'_0$. So,

$$\left\{i \in \mathbb{N} : \tilde{v}^{(k)}_{4,i} \leq 4m\tilde{u}^{(l)}_{4,i}\right\} \subset \left\{i \in \mathbb{N} : \tilde{a}_i \leq \alpha\right\}.$$

If $k \geq l$, then $r_k < r'_l$ and $s'_l < s_k$ by (5.5), hence

$$\left(\frac{1}{r'_l} - \frac{1}{r_k}\right) + (s'_l - s_k)\,\tilde{\lambda}_i < 0.$$

So the inequality $\tilde{v}_{4,i}^{(k)} \leq 4m\tilde{u}_{4,i}^{(l)}$ is satisfied for all $i \in \mathbb{N}$. Therefore,

$$\left\{i \in \mathbb{N} : \tilde{v}_{4,i}^{(k)} \leq 4m\tilde{u}_{4,i}^{(l)}\right\} \subset S_{k,l} \tag{5.29}$$

for all $k, l = 1, \ldots, m$, where

$$S_{k,l} = \begin{cases} \{i \in \mathbb{N} : \tilde{a}_i \leq \alpha\}, & k < l, \\ \mathbb{N}, & k \geq l. \end{cases}$$

Now, we claim that $R_{k,l} \cap S_{k,l} \subset R_{l,l}$. Since the claim is trivially true if $k = l$ or $R_{k,l} \cap S_{k,l}$ is empty, take k, l such that $k \neq l$ and the intersections $R_{k,l} \cap S_{k,l}$ are nonempty.

If $k > l$, then $\delta_k \geq \delta_l$ by the reordering of δ. Since γ is increasing and M is decreasing, we have $\gamma(\delta_k) \geq \gamma(\delta_l)$ and $M(\delta_k) \leq M(\delta_l)$, which imply the following inequalities

$$\gamma(\delta_k) - \frac{M(\delta_k)}{\max\{\tau_k, \tau_l\}} \geq \gamma(\delta_l) - \frac{M(\delta_l)}{\max\{\tau_k, \tau_l\}} \geq \gamma(\delta_l) - \frac{M(\delta_l)}{\tau_l},$$

$$\gamma^{-1}(\varepsilon_l) + \frac{M(\varepsilon_l)}{\max\{\tau_k, \tau_l\}} \leq \gamma^{-1}(\varepsilon_l) + \frac{M(\varepsilon_l)}{\tau_l}.$$

From these inequalities, we obtain $R_{k,l} \subset R_{l,l}$. Hence,

$$R_{k,l} \cap S_{k,l} = R_{k,l} \cap \mathbb{N} = R_{k,l} \subset R_{l,l}.$$

For $k < l$, take $i \in R_{k,l} \cap S_{k,l}$. Then, $\tilde{a}_i \leq \alpha$ since $i \in S_{k,l}$. As $\frac{1}{\tilde{a}_i} \leq \tilde{\lambda}_i$ by lemma 2.1.1, we have $\frac{1}{\alpha} \leq \tilde{\lambda}_i$. Also, since γ is increasing,

$$\gamma(\delta_l) - \frac{M(\delta_l)}{\tau_l} < \gamma(\delta_l) < \gamma(1) = \gamma(\zeta_0) = \zeta_3 = \frac{1}{n_3}.$$

So, if we take $n_3 > \alpha$, which is possible since α depends only on m, we get

$$\gamma(\delta_l) - \frac{M(\delta_l)}{\tau_l} < \frac{1}{n_3} < \frac{1}{\alpha} \leq \tilde{\lambda}_i.$$

With this estimate for $\tilde{\lambda}_i$ from below, and the estimates for $\tilde{\lambda}_i$ and $\tilde{a}_i$ that come from the fact that $i \in R_{k,l}$, we obtain that $i \in R_{l,l}$. Therefore, $R_{k,l} \cap S_{k,l} \subset R_{l,l}$ for all $k, l = 1, \ldots, m$.

From the inclusions (5.18), (5.29) and (5.28), we have

$$\left\{i \in \mathbb{N} : \max_{\theta=1,\ldots,4} \left\{\tilde{v}_{\theta,i}^{(k)}\right\} \leq 4m \min_{\theta=1,\ldots,4} \left\{\tilde{u}_{\theta,i}^{(l)}\right\}\right\} \subset R_{k,l} \cap S_{k,l} \subset R_{l,l}$$

for all $k, l = 1, \ldots, m$. Hence,

$$\begin{aligned} \beta(4mB^f(c), B^f(d)) &\leq \left|\bigcup_{k=1}^{m}\bigcup_{l=1}^{m}\left\{i \in \mathbb{N} : \max_{\theta=1,\ldots,4} \left\{\tilde{v}_{\theta,i}^{(k)}\right\} \leq 4m \min_{\theta=1,\ldots,4} \left\{\tilde{u}_{\theta,i}^{(l)}\right\}\right\}\right| \\ &\leq \left|\bigcup_{l=1}^{m} R_{l,l}\right| \\ &= \mu_m^{(\tilde{\lambda},\tilde{a})}\left(\gamma(\delta) - \frac{M(\delta)}{\tau}, \gamma^{-1}(\varepsilon) + \frac{M(\varepsilon)}{\tau}; \frac{\tau}{\alpha}, \alpha t\right). \end{aligned} \tag{5.30}$$

From (5.10), (5.17) and (5.30), we obtain

$$\mu_m^{(\lambda,a)}(\delta, \varepsilon; \tau, t) \leq \mu_m^{(\tilde{\lambda},\tilde{a})}\left(\gamma(\delta) - \frac{M(\delta)}{\tau}, \gamma^{-1}(\varepsilon) + \frac{M(\varepsilon)}{\tau}; \frac{\tau}{\alpha}, \alpha t\right).$$

■

Remark 5.0.2 *The function γ in the proof of Lemma 5.0.16 satisfies*

$$\gamma(\xi) \leq \xi \leq \gamma^{-1}(\xi), \quad \xi \in [0,1].$$

For $\xi = 0$, we have $\gamma(0) = 0 = \gamma^{-1}(0)$. For $\xi \in (0,1]$, there exist $\nu_k \in \mathbb{N}_0$ such that $\zeta_{\nu_k+1} < \xi \leq \zeta_{\nu_k}$. As γ is increasing, we have $\gamma(\zeta_{\nu_k+1}) < \gamma(\xi) \leq \gamma(\zeta_{\nu_k})$. Since $\gamma(\zeta_j) = \zeta_{j+3}$, we get $\gamma(\xi) \leq \zeta_{\nu_k+3} < \zeta_{\nu_k+1} < \xi$. Since γ^{-1} is also increasing, we obtain $\xi < \gamma^{-1}(\xi)$.

Theorem 5.0.17 *If $G_{0,\infty}(\lambda, a) \simeq G_{0,\infty}(\tilde{\lambda}, \tilde{a})$, then $\mu_m^{(\lambda,a)} \sim \mu_m^{(\tilde{\lambda},\tilde{a})}$ for every $m \in \mathbb{N}$.*

Proof. Let $G_{0,\infty}(\lambda, a) \simeq G_{0,\infty}(\tilde{\lambda}, \tilde{a})$ and $m \in \mathbb{N}$. Then, by Lemma 5.0.16, there exists a strictly increasing function $\gamma : [0,2] \to [0,1]$ where $\gamma(0) = 0$ and $\gamma(2) = 1$, a decreasing function $M : (0,1] \to (0,\infty)$, and a constant $\alpha > 1$ such that

$$\mu_{2m}^{(\lambda,a)}(\delta, \varepsilon; \tau, t) \leq \mu_{2m}^{(\tilde{\lambda},\tilde{a})}\left(\gamma(\delta) - \frac{M(\delta)}{\tau}, \gamma^{-1}(\varepsilon) + \frac{M(\varepsilon)}{\tau}; \frac{\tau}{\alpha}, \alpha t\right)$$

for all $\delta = (\delta_k)$, $\varepsilon = (\varepsilon_k)$, $\tau = (\tau_k)$ and $t = (t_k)$ where $0 < \delta_k \leq \varepsilon_k \leq 1$ and $0 < \tau_k \leq t_k < \infty$, $k = 1, \cdots, m$. So, we can take a strictly decreasing function $\Psi : (0,1] \to (0,\infty)$ such that

$$\Psi(\xi) > \frac{2M(\xi)}{\gamma(\xi)}, \quad \xi \in (0,1].$$

Given $\delta = (\delta_k)$, $\varepsilon = (\varepsilon_k)$, $\tau = (\tau_k)$ and $t = (t_k)$, $k = 1, \dots, m$,

$$\mu_m^{(\lambda,a)}(\delta, \varepsilon; \tau, t) = \left| \bigcup_{k=1}^{m} \{ i \in \mathbb{N} : (\lambda_i, a_i) \in P_k \} \right|,$$

where $P_k = [\delta_k, \varepsilon_k] \times [\tau_k, t_k]$. If we define the following sets

$$P'_k = \begin{cases} [\delta_k, \varepsilon_k] \times [\tau_k, t_k], & \tau_k > \Psi(\delta_k), \\ [\delta_k, \varepsilon_k] \times [\Psi(\delta_k), t_k], & \tau_k \le \Psi(\delta_k) \le t_k, \\ \emptyset & t_k < \Psi(\delta_k), \end{cases}$$

$$P''_k = \begin{cases} \emptyset & \tau_k > \Psi(\delta_k), \\ [\delta_k, \varepsilon'_k] \times [\tau_k, \Psi(\delta_k)], & \tau_k \le \Psi(\delta_k) \le t_k, \\ [\delta_k, \varepsilon'_k] \times [\tau_k, t_k], & t_k \le \Psi(\delta_k), \end{cases}$$

where

$$\varepsilon'_k = \begin{cases} \max\{\varepsilon_k, \Psi^{-1}(\tau_k)\}, & \tau_k \ge \Psi(1), \\ 1 & \tau_k < \Psi(1), \end{cases}$$

then, $P_k \subset P'_k \cup P''_k$ for every $k = 1, \dots, m$. Hence,

$$\left| \bigcup_{k=1}^{m} \{ i \in \mathbb{N} : (\lambda_i, a_i) \in P_k \} \right| \le \left| \bigcup_{k=1}^{m} \{ i \in \mathbb{N} : (\lambda_i, a_i) \in P'_k \cup P''_k \} \right|.$$

By applying Lemma 5.0.16, we obtain

$$\left| \bigcup_{k=1}^{m} \{ i \in \mathbb{N} : (\lambda_i, a_i) \in P'_k \cup P''_k \} \right| \le \left| \bigcup_{k=1}^{m} \left\{ i \in \mathbb{N} : \left(\tilde{\lambda}_i, \tilde{a}_i\right) \in \tilde{P}'_k \cup \tilde{P}''_k \right\} \right|,$$

where

$$\tilde{P}'_k = \begin{cases} \left[\gamma(\delta_k) - \frac{M(\delta_k)}{\tau_k}, \gamma^{-1}(\varepsilon_k) + \frac{M(\varepsilon_k)}{\tau_k}\right] \times \left[\frac{\tau_k}{\alpha}, \alpha t_k\right], & \tau_k > \Psi(\delta_k), \\ \left[\gamma(\delta_k) - \frac{M(\delta_k)}{\Psi(\delta_k)}, \gamma^{-1}(\varepsilon_k) + \frac{M(\varepsilon_k)}{\Psi(\delta_k)}\right] \times \left[\frac{\Psi(\delta_k)}{\alpha}, \alpha t_k\right], & \tau_k \le \Psi(\delta_k) \le t_k, \\ \emptyset & t_k < \Psi(\delta_k), \end{cases}$$

$$\tilde{P}''_k = \begin{cases} \emptyset & \tau_k > \Psi(\delta_k), \\ \left[\gamma(\delta_k) - \frac{M(\delta_k)}{\tau_k}, \gamma^{-1}(\varepsilon'_k) + \frac{M(\varepsilon'_k)}{\tau_k}\right] \times \left[\frac{\tau_k}{\alpha}, \alpha \Psi(\delta_k)\right], & \tau_k \le \Psi(\delta_k) \le t_k, \\ \left[\gamma(\delta_k) - \frac{M(\delta_k)}{\tau_k}, \gamma^{-1}(\varepsilon'_k) + \frac{M(\varepsilon'_k)}{\tau_k}\right] \times \left[\frac{\tau_k}{\alpha}, \alpha t_k\right], & t_k \le \Psi(\delta_k). \end{cases}$$

Take a strictly increasing function $\varphi : [0, 2] \to [0, 1]$ so that $\varphi(0) = 0$, $\varphi(2) = 1$, and

$$\varphi(\xi) \le \min\left\{ \frac{\gamma(\xi)}{2}, \gamma\left(\frac{2}{3}\xi\right), \frac{1}{\alpha\Psi(\xi)}, \frac{1}{\Psi\left(\gamma\left(\frac{2}{3}\xi\right)\right)} \right\}, \quad \xi \in (0, 1]. \tag{5.31}$$

If $\tau_k > \Psi(\delta_k)$, then since $\Psi(\delta_k) > \frac{2M(\delta_k)}{\gamma(\delta_k)}$ and $\varphi(\delta_k) \leq \frac{\gamma(\delta_k)}{2}$ by (5.31), we have

$$\gamma(\delta_k) - \frac{M(\delta_k)}{\tau_k} > \gamma(\delta_k) - \frac{\gamma(\delta_k)}{2} = \frac{\gamma(\delta_k)}{2} \geq \varphi(\delta_k).$$

Also, since $\delta_k \leq \varepsilon_k$ and Ψ is decreasing,

$$\tau_k > \Psi(\delta_k) \geq \Psi(\varepsilon_k) > 2\frac{M(\varepsilon_k)}{\gamma(\varepsilon_k)},$$

which implies that

$$\gamma^{-1}(\varepsilon_k) + \frac{M(\varepsilon_k)}{\tau_k} < \gamma^{-1}(\varepsilon_k) + \frac{\gamma(\varepsilon_k)}{2}.$$

As $\gamma(\varepsilon_k) \leq \gamma^{-1}(\varepsilon_k)$ by Remark 5.0.2 and $\varphi(\varepsilon_k) < \gamma\left(\frac{2}{3}\varepsilon_k\right)$ by (5.31), we have

$$\gamma^{-1}(\varepsilon_k) + \frac{M(\varepsilon_k)}{\tau_k} < \gamma^{-1}(\varepsilon_k) + \frac{\gamma^{-1}(\varepsilon_k)}{2} = \frac{3\gamma^{-1}(\varepsilon_k)}{2} \leq \varphi^{-1}(\varepsilon_k).$$

Hence, for $\tau_k > \Psi(\delta_k)$,

$$\tilde{P}_k' \subset \left[\varphi(\delta_k), \varphi^{-1}(\varepsilon_k)\right] \times \left[\frac{\tau_k}{\alpha}, \alpha t_k\right].$$

One can show similarly that the same inclusion holds if $\tau_k \leq \Psi(\delta_k) \leq t_k$. Therefore, for any $k =, 1 \ldots, m$, we have

$$\left\{i \in \mathbb{N} : \left(\tilde{\lambda}_i, \tilde{a}_i\right) \in \tilde{P}_k'\right\} \subset \left\{i \in \mathbb{N} : \varphi(\delta_k) \leq \tilde{\lambda}_i \leq \varphi^{-1}(\varepsilon_k), \frac{\tau_k}{\alpha} \leq \tilde{a}_i \leq \alpha t_k\right\}. \quad (5.32)$$

Let $\left(\tilde{\lambda}_i, \tilde{a}_i\right) \in \tilde{P}_k''$. Then, $\tilde{a}_i \leq \alpha\Psi(\delta_k)$ by the definition of $\tilde{P}_k''$. As $\tilde{\lambda}_i \geq \frac{1}{\tilde{a}_i}$ by Lemma 2.1.1 and $\varphi(\delta_k) \leq \frac{1}{\alpha\Psi(\delta_k)}$ by (5.31), we have

$$\tilde{\lambda}_i \geq \frac{1}{\alpha\Psi(\delta_k)} \geq \varphi(\delta_k).$$

Also, by the definition of $\tilde{P}_k''$,

$$\tilde{\lambda}_i \leq \gamma^{-1}(\varepsilon_k') + \frac{M(\varepsilon_k')}{\tau_k}.$$

We have the cases when $\varepsilon_k' = \varepsilon_k$, $\varepsilon_k' = \Psi^{-1}(\tau_k)$ or $\varepsilon_k' = 1$.

If $\varepsilon_k' = \varepsilon_k$, then $\varepsilon_k \geq \Psi^{-1}(\tau_k)$, so $\Psi(\varepsilon_k) \leq \tau_k$ since Ψ is decreasing. Hence,

$$\frac{1}{\tau_k} \leq \frac{1}{\Psi(\varepsilon_k')} < \frac{\gamma(\varepsilon_k')}{2M(\varepsilon_k')},$$

which implies that

$$\gamma^{-1}(\varepsilon_k') + \frac{M(\varepsilon_k')}{\tau_k} < \gamma^{-1}(\varepsilon_k') + \frac{\gamma(\varepsilon_k')}{2}.$$

As $\gamma(\varepsilon_k') \le \gamma^{-1}(\varepsilon_k')$ by Remark 5.0.2 and $\varphi(\varepsilon_k) < \gamma\left(\frac{2}{3}\varepsilon_k\right)$ by (5.31), we have

$$\gamma^{-1}(\varepsilon_k') + \frac{M(\varepsilon_k')}{\tau_k} < \frac{3}{2}\gamma^{-1}(\varepsilon_k') = \frac{3}{2}\gamma^{-1}(\varepsilon_k) \le \varphi^{-1}(\varepsilon_k).$$

If $\varepsilon_k' = \Psi^{-1}(\tau_k)$, then $\tau_k = \Psi(\varepsilon_k')$, so

$$\frac{1}{\tau_k} = \frac{1}{\Psi(\varepsilon_k')} < \frac{\gamma(\varepsilon_k')}{2M(\varepsilon_k')}.$$

Hence,

$$\gamma^{-1}(\varepsilon_k') + \frac{M(\varepsilon_k')}{\tau_k} < \gamma^{-1}(\varepsilon_k') + \frac{\gamma(\varepsilon_k')}{2} \le \frac{3}{2}\gamma^{-1}(\varepsilon_k') = \frac{3}{2}\gamma^{-1}\left(\Psi^{-1}(\tau_k)\right).$$

Without loss of generality, we can take $\tau_k \ge \frac{1}{\varepsilon_k}$ since $\mu_m^{(\lambda,a)}(\delta,\varepsilon;\tau,t) = \mu_m^{(\lambda,a)}(\delta,\varepsilon;\tau',t)$ for $\tau' = (\tau_k')$ where

$$\tau_k' = \begin{cases} \frac{1}{\varepsilon_k}, & \tau_k < \frac{1}{\varepsilon_k}, \\ \tau_k, & \tau_k \ge \frac{1}{\varepsilon_k}. \end{cases}$$

So, since Ψ^{-1} is decreasing and γ^{-1} is increasing, we have

$$\gamma^{-1}\left(\Psi^{-1}(\tau_k)\right) \le \gamma^{-1}\left(\Psi^{-1}\left(\frac{1}{\varepsilon_k}\right)\right),$$

which implies that

$$\gamma^{-1}(\varepsilon_k') + \frac{M(\varepsilon_k')}{\tau_k} < \frac{3}{2}\gamma^{-1}\left(\Psi^{-1}\left(\frac{1}{\varepsilon_k}\right)\right).$$

Since $\varphi(\varepsilon_k) \le \frac{1}{\Psi\left(\gamma\left(\frac{2}{3}\xi\right)\right)}$ by (5.31), we get

$$\gamma^{-1}(\varepsilon_k') + \frac{M(\varepsilon_k')}{\tau_k} < \varphi^{-1}(\varepsilon_k).$$

If $\varepsilon_k' = 1$, then

$$\gamma^{-1}(\varepsilon_k') + \frac{M(\varepsilon_k')}{\tau_k} = \gamma^{-1}(1) + \frac{M(1)}{\tau_k} > 1$$

since $\gamma^{-1}(1) = 2$ and $\frac{M(1)}{\tau_k}$ is positive. As $\alpha > 1$, we have $\varphi(1) \le \frac{1}{\alpha\Psi(1)}$. Also, without loss of generality, we can take $\tau_k \ge \frac{1}{\varepsilon_k}$. Hence, we have

$$\varepsilon_k \ge \frac{1}{\tau_k} > \frac{1}{\Psi(1)} > \frac{1}{\alpha\Psi(1)} \ge \varphi(1),$$

which implies that $\varphi^{-1}(\varepsilon_k) > 1$. As $\tilde{\lambda}_i \le 1$ by Lemma 2.1.1, we have $\tilde{\lambda}_i \le \gamma^{-1}(\varepsilon_k') + \frac{M(\varepsilon_k')}{\tau_k}$, which implies that $\tilde{\lambda}_i \le \varphi^{-1}(\varepsilon_k)$.

Thus, for any $k =, 1 \ldots, m$, we have

$$\left\{i \in \mathbb{N} : \left(\tilde{\lambda}_i, \tilde{a}_i\right) \in \tilde{P}_k''\right\} \subset \left\{i \in \mathbb{N} : \varphi(\delta_k) \le \tilde{\lambda}_i \le \varphi^{-1}(\varepsilon_k), \frac{\tau_k}{\alpha} \le \tilde{a}_i \le \alpha t_k\right\}. \quad (5.33)$$

From (5.32) and (5.33), we obtain

$$\bigcup_{k=1}^{m}\left\{i\in\mathbb{N}:\left(\tilde{\lambda}_i,\tilde{a}_i\right)\in\tilde{P}'_k\cup\tilde{P}''_k\right\}\subset\bigcup_{k=1}^{m}\left\{i\in\mathbb{N}:\varphi(\delta_k)\leq\tilde{\lambda}_i\leq\varphi^{-1}(\varepsilon_k),\frac{\tau_k}{\alpha}\leq\tilde{a}_i\leq\alpha t_k\right\},$$

which implies that

$$\mu_m^{(\lambda,a)}(\delta,\varepsilon;\tau,t)\leq\mu_m^{(\tilde{\lambda},\tilde{a})}\left(\varphi(\delta),\varphi^{-1}(\varepsilon);\frac{\tau}{\alpha},\alpha t\right).$$

By interchanging (λ,a) and $(\tilde{\lambda},\tilde{a})$, we similarly get a strictly increasing function $\tilde{\varphi}$, and a constant $\tilde{\alpha}$ so that

$$\mu_m^{(\tilde{\lambda},\tilde{a})}(\delta,\varepsilon;\tau,t)\leq\mu_m^{(\lambda,a)}\left(\tilde{\varphi}(\delta),\tilde{\varphi}^{-1}(\varepsilon);\frac{\tau}{\tilde{\alpha}},\tilde{\alpha}t\right).$$

Taking the minimum of the functions φ, $\tilde{\varphi}$, and the maximum of the constants α, $\tilde{\alpha}$, we obtain

$$\mu_m^{(\lambda,a)}\sim\mu_m^{(\tilde{\lambda},\tilde{a})}.$$

■

CHAPTER 6

Quasiequivalence of Bases

As an application of m-rectangle characteristics, we obtain the quasiequivalence of absolute bases in Montel spaces $G_{0,\infty}(\lambda, a)$ that are in class (2), where $\omega_i^{0,\infty}(p,q) = \exp\left((-\frac{1}{p} - q\lambda_i)a_i\right)$, such that $G_{0,\infty}(\lambda, a) \overset{qd}{\simeq} G_{0,\infty}(\lambda, a) \times G_{0,\infty}(\lambda, a)$.

Proposition 6.0.18 *Let $X = G_{0,\infty}(\lambda, a)$ and $\tilde{X} = G_{0,\infty}(\tilde{\lambda}, \tilde{a})$ be Montel spaces such that $X \overset{qd}{\simeq} X^2$ and $\tilde{X} \overset{qd}{\simeq} \tilde{X}^2$. Then, $X \simeq \tilde{X}$ implies $X \overset{qd}{\simeq} \tilde{X}$.*

Proof. Assume that $X = G_{0,\infty}(\lambda, a)$ and $\tilde{X} = G_{0,\infty}(\tilde{\lambda}, \tilde{a})$ are Montel spaces such that $X \overset{qd}{\simeq} X^2$, $\tilde{X} \overset{qd}{\simeq} \tilde{X}^2$ and $X \simeq \tilde{X}$.

First, we show that X can be quasidiagonally embedded in $\tilde{X}^3$. By Theorem 5.0.17, $X \simeq \tilde{X}$ implies $\mu_1^{(\lambda,a)} \sim \mu_1^{(\tilde{\lambda},\tilde{a})}$. Hence, there exists a strictly increasing function $\varphi : [0,2] \to [0,1]$ with $\varphi(0) = 0$ and $\varphi(2) = 1$, and a positive constant α so that

$$\mu_1^{(\lambda,a)}(\delta, \varepsilon; \tau, t) \leq \mu_1^{(\tilde{\lambda},\tilde{a})}\left(\varphi(\delta), \varphi^{-1}(\varepsilon); \frac{\tau}{\alpha}, \alpha t\right), \tag{6.1}$$

for all parameters $\delta, \varepsilon, \tau, t$. Taking $\varepsilon = \varphi(1)$, where $\varepsilon \in (0,1)$, we define the following sets

$$\begin{aligned} N_{j,k} &= \left\{i \in \mathbb{N} : \varphi^j(\varepsilon) \leq \lambda_i \leq \varphi^{j-1}(\varepsilon), \alpha^k \leq a_i \leq \alpha^{k+1}\right\}, \quad j,k \in \mathbb{N}_0, \\ \tilde{N}_{j,k} &= \left\{i \in \mathbb{N} : \varphi^{j+1}(\varepsilon) \leq \lambda_i \leq \varphi^{j-2}(\varepsilon), \alpha^{k-1} \leq a_i \leq \alpha^{k+2}\right\}, \quad j,k \in \mathbb{N}_0. \end{aligned}$$

Then, the inequality (6.1) implies that

$$|N_{j,k}| \leq |\tilde{N}_{j,k}|, \quad j,k \in \mathbb{N}_0. \tag{6.2}$$

So, if we define the following sets

$$N_k = \bigcup_{j=1}^{\infty} N_{j,k}, \quad \tilde{N}_k = \bigcup_{j=1}^{\infty} \tilde{N}_{j,k}, \quad k \in \mathbb{N}_0,$$

then, for every $k \in \mathbb{N}_0$, the multi-valued functions $S_k : N_k \to \tilde{N}_k$, defined by

$$S_k(i) = \tilde{N}_{j,k} \text{ if } i \in N_{j,k},$$

satisfy the conditions of Hall-König Theorem (Theorem 2.6.1). Hence, for each $k \in \mathbb{N}_0$, there exists an injection $\sigma_k : N_k \to \tilde{N}_k$ such that $\sigma_k(i) \in \tilde{N}_{j,k}$ whenever $i \in N_{j,k}$.

If we set $N^{(\theta)} = \bigcup_{k=0}^{\infty} N_{3k+\theta}$ for $\theta = 0, 1, 2$, then the maps $\sigma^{(\theta)} : N^{(\theta)} \to \mathbb{N}$ defined by

$$\sigma^{(\theta)}(i) = \sigma_k(i) \text{ if } i \in N_{3k+\theta},\ k \in \mathbb{N}_0,$$

are also injective. Hence, we can define an injective map $\sigma : \mathbb{N} \to \mathbb{N}^3$ by

$$\sigma(i) = \sigma^{(\theta)}(i) \text{ if } i \in N^{(\theta)},\ \theta = 0, 1, 2.$$

From the construction of the sets $N_{j,k}$ and the injection σ, we obtain $a_i \asymp \tilde{a}_{\sigma(i)}$, and $(\lambda_{i_k}) \to 0 \Leftrightarrow (\tilde{\lambda}_{\sigma(i_k)}) \to 0$ for any subsequence (i_k) of $\mathbb{N}$. Hence, by Proposition 3.0.4, X can be quasidiagonally embedded in $\tilde{X}^3$.

By the assumption $\tilde{X} \overset{qd}{\simeq} \tilde{X}^2$, we have $\tilde{X} \overset{qd}{\simeq} \tilde{X}^3$, hence X can be quasidiagonally embedded in $\tilde{X}$. By interchanging X and $\tilde{X}$, we can analogously show that $\tilde{X}$ can be quasidiagonally embedded in X. Therefore, by Proposition 2.5.1, we have $X \simeq \tilde{X}$. ∎

Corollary 6.0.19 *If $G_{0,\infty}(\lambda, a) \overset{qd}{\simeq} G_{0,\infty}(\lambda, a) \times G_{0,\infty}(\lambda, a)$, then the absolute bases in $G_{0,\infty}(\lambda, a)$ are pairwise quasiequivalent.*

Bibliography

[1] C. Bessaga, A. Pełczyński, S. Rolewicz, *On diametral approximative dimension and linear homogeneity of F-spaces*, Bull. Acad. Pol. Sci. 9 (1961), 677-683.

[2] P.A. Chalov, *On the quasiequivalence of bases in families of Hilbert spaces*, (in Russian), Ph.D. Thesis, Rostov-na-Donu, 1980.

[3] P.A. Chalov, P.B. Djakov, T. Terzioğlu, V.P. Zahariuta, *On cartesian products of locally convex spaces*, In: Linear Topological Spaces and Complex Analysis II, edited by A. Aytuna, METU-TUBITAK, 1995, 9-33.

[4] P.A. Chalov, P.B. Djakov, V.P. Zahariuta, *Compound invariants and embeddings of cartesian products*, Studia Math. 137 (1999), 33-47.

[5] P.A. Chalov, T. Terzioğlu, V.P. Zahariuta, *Compound invariants and mixed F-, DF- power spaces*, Can. J. Math. 50 (1998), 1138-1162.

[6] P.A. Chalov, T. Terzioğlu, V.P. Zahariuta, *First type power Köthe spaces and m-rectangular invariants*, In: Linear Topological Spaces and Complex Analysis III, edited by A. Aytuna, METU-TUBITAK, 1997, 30-44.

[7] P.A. Chalov, T. Terzioğlu, V.P. Zahariuta, *Multirectangular invariants for power Köthe spaces*, J. Math. Anal. Appl. 297 (2004), 673-695.

[8] P.A. Chalov, V.P. Zahariuta, *Mixed F-, DF- power spaces*, (preprint).

[9] L. Crone, W.B. Robinson, *Every nuclear Fréchet space with a regular basis has the quasi-equivalence property*, Studia Math. 52 (1975), 203-207.

[10] P.B. Djakov, *A short proof of the theorem of Crone and Robinson on quasi-equivalence of regular bases*, Studia Math. 53 (1975), 269-271.

[11] P.B. Djakov, M. Yurdakul, V.P. Zahariuta *On cartesian products of power series spaces*, Bull. Pol. Acad. Sci. Math 43 (1995), 113-117.

[12] P.B. Djakov, M. Yurdakul, V.P. Zahariuta *Isomorphic classification of cartesian products of power series spaces*, Michigan Math. J. 43 (1996), 221-229.

[13] P.B. Djakov, V.P. Zahariuta *On Dragilev type power Köthe spaces*, Studia Math. 120 (1996), 219-234.

[14] M.M. Dragilev, *On the stability of basis* $\{z^n\}$, (in Russian), Izv. Vyssh. Uchebn. Zaved. Mat. 3 (1958), 67-73.

[15] M.M. Dragilev, *Standard form of basis for the space of analytic functions*, (in Russian), Uspekhi Mat. Nauk 15 (1960), 181-188.

[16] M.M. Dragilev, *On regular bases in nuclear spaces*, (in Russian), Matem. Sbornik 68 (1965), 153-173.

[17] E. Dubinsky, *The Structure of Nuclear Fréchet Spaces*, Lecture Notes in Mathematics vol. 720, Springer-Verlag, 1979.

[18] A. Goncharov, T. Terzioğlu, V.P. Zahariuta, *On isomorphic classification of spaces* $s\hat{\otimes}E'_\infty(a)$, In: Linear Topological Spaces and Complex Analysis I, METU-TUBITAK, 1994, 14-24.

[19] A. Goncharov, T. Terzioğlu, V.P. Zahariuta, *On isomorphic classification of tensor products* $E_\infty(a)\hat{\otimes}E'_\infty(b)$ Dissertationes Math. 350 (1996), 1-27.

[20] A. Goncharov, V.P. Zahariuta, *Linear topological invariants for tensor products of power F- and DF- spaces*, Turkish J. Math. 19 (1995), 90-101.

[21] M. Hall, *Combinatorial Theory*, John Wiley & Sons, 1986.

[22] H. Jarchow, *Locally Convex Spaces*, Stuttgart Teubner, 1981.

[23] A.N. Kolmogorov, *On the linear dimension of topological vector spaces*, (in Russian), Dokl. Akad. Nauk SSSR 120 (1958), 210-213.

[24] V.P. Kondakov, *On quasi-equivalence of regular bases in Köthe spaces*, (in Russian), Mat. Anal. i Ego Pril. 5 (1974) 210-213.

[25] B.S. Mityagin, *Approximative dimension and bases in nuclear spaces*, Russian Math. Surveys 16 (1961), 59-127.

[26] B.S. Mityagin, *Nuclear Riesz scales*, (in Russian), Dokl. Akad. Nauk SSSR 137 (1961), 519-522.

[27] B.S. Mityagin, *Equivalence of bases in Hilbert scales*, (in Russian), Studia Math. 37 (1970), 111-137.

[28] B.S. Mityagin, *Non-Schwartzian power series spaces*, Math. Z. 182 (1983), 303-310.

[29] A. Pełczyński, *On the approximation of S-spaces by finite dimensional spaces*, Bull. Acad. Pol. Sci. 5 (1957), 879-881.

[30] S. Rolewicz, *On spaces of holomorphic functions*, Studia Math. 21 (1962), 135-160.

[31] D. Vogt, *Topics on projective spectra of (LB)-spaces*, T. Terzioğlu (ed.), Advances in the Theory of Fréchet Spaces, Kluwer Acad. Publ., Dordrecht, 1989, 11-27.

[32] J. Wengenroth, *Derived Functors in Functional Analysis*, Lecture Notes in Mathematics vol. 1810, Springer, 2003.

[33] M. Yurdakul, V.P. Zahariuta, *Linear topological invariants and isomorphic classification of cartesian products of locally convex spaces*, Turkish J. Math. 19 (1995), 37-47.

[34] V.P. Zahariuta, *Quasi-equivalence of bases in finite centers of Hilbert scales*, (in Russian), Dokl. Akad. Nauk SSSR 180 (1968), 786-788.

[35] V.P. Zahariuta, *On the isomorphism of cartesian products of locally convex spaces*, Studia Math. 46 (1973), 201-221.

[36] V.P. Zahariuta, *On isomorphisms and quasi-equivalence of bases of power Köthe spaces*, (in Russian), In: Proceedings of 7th Winter School in Drogobych, CEMI, Moscow, 1976, 101-126.

[37] V.P. Zahariuta, *Generalized Mityagin's invariants and continuum of pairwise nonisomorphic spaces of analytic functions*, (in Russian), Functional Anal. i Prilozhen 11 (1977), 24-30.

[38] V.P. Zahariuta, *Synthetic diameters and linear topological invariants*, (in Russian), In: School on Operator Theory in Function Spaces, Minsk, 1978, 51-52.

[39] V.P. Zahariuta, *Linear Topological Invariants and Their Applications to Generalized Power Spaces*, (in Russian), Rostov University, 1979.

[40] V.P. Zahariuta, *Linear topological invariants and their applications to isomorphic classification of generalized power spaces*, Turkish J. Math. 20 (1996), 237-289.