Improvements in Finite State Machine Based Testing

by Uraz Cengiz Türker

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Uraz Cengiz Türker

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Thesis Supervisor: Assistant Prof. Hüsnü Yenigün

Thesis Co-supervisor: Prof. Dr. Robert Hierons

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ABSTRACT

Finite State Machine (FSM) based testing methods have a history of over half a century, starting in 1956 with the works on machine identification. This was then followed by works checking the conformance of a given implementation to a given specification. When it is possible to identify the states of an FSM using an appropriate input sequence, it's been long known that it is possible to generate a *Fault Detection Experiment* with fault coverage with respect to a certain fault model in polynomial time. In this thesis, we investigate two notions of fault detection sequences; *Checking Sequence* (CS), *Checking Experiment* (CE). Since a fault detection sequence (either a CS or a CE) is constructed once but used many times, the importance of having short fault detection sequences is obvious and hence recent works in this field aim to generate shorter fault detection sequences.

In this thesis, we first investigate a strategy and related problems to reduce the length of a CS. A CS consists several components such as *Reset Sequences* and *State Identification Sequences*. All works assume that for a given FSM, a reset sequence and a state identification sequence are also given together with the specification FSM M. Using the given reset and state identification sequences, a CS is formed that gives full fault coverage under certain assumptions. In other words, any faulty implementation N can be identified by using this test sequence. In the literature, different methods for CS construction take different approaches to put these components together, with the aim of coming up with a shorter CS incorporating all of these components. One obvious way of keeping the CS short is to keep components short. As the reset sequence and the state identification sequence are the biggest components, having short reset and state identification sequences is very important as well.

It was shown in 1991 that for a given FSM M, shortest reset sequence cannot be computed in polynomial time if $\mathsf{P} \neq \mathsf{NP}$. Recently it was shown that when the FSM has particular type ("monotonic") of transition structure, constructing one of the shortest reset word is polynomial time solvable. However there has been no work on constructing one of the shortest reset word for a monotonic partially specified machines. In this thesis, we showed that this problem is NP-hard.

On the other hand, in 1994 it was shown that one can check if M has special type of state identification sequence (known as an adaptive distinguishing sequence) in polynomial time. The same work also suggests a polynomial time algorithm to construct a state identification sequence when one exists. However, this algorithm generates a state identification sequence without any particular emphasis on generating a short one. There has been no work on the generation of state identification sequences for complete or partial machines after this work. In this thesis, we showed that construction of short state identification sequences is NP-complete and NP-hard to approximate. We propose methods of generating short state identification sequences and experimentally validate that such state identification sequences can reduce the length of fault detection sequences by 29.2% on the average.

Another line of research, in this thesis, devoted for reducing the cost of checking experiments. A checking experiment consist of a set of input sequences each of which aim to test different properties of the implementation. As in the case of CSs, a large portion of these input sequences contain state identification sequences. There are several kinds of state identification sequences that are applicable in CEs. In this work, we propose a new kind of state identification sequence and show that construction of such sequences are PSPACE-complete. We propose a heuristic and we perform experiments on benchmark FSMs and experimentally show that the proposed notion of state identification sequence can reduce the cost of CEs by 65% in the extreme case.

Testing distributed architectures is another interesting field for FSM based fault detection sequence generation. The additional challenge when such distributed architectures are considered is to generate a fault detection sequence which does not pose controllability or observability problem. Although the existing methods again assume that a state identification sequence is given using which a fault detection sequence is constructed, there is no work on how to generate a state identification sequence which do not have controllability/observability problem itself. In this thesis we investigate the computational complexities to generate such state identification sequences and show that no polynomial time algorithm can construct a state identification sequence for a given distributed FSM.

Sonlu Durum Makinelerine Dayalı Sınama Dizilerinde Iyileştirmeler

Uraz Cengiz Türker

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Tez danışmanı: Yrd. Doç. Dr. Hüsnü Yenigün

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Özet

Sonlu durum makinelerine (SDM'e) dayalı sınama yöntemleri 1956 yılında makine tanıma üzerine yapılan çalışmalar ile başlamış ve elli yılı aşkın bir süredir üzerinde çalışılan bir konu olmuştur. Makine tanıma çalışmalarını takiben bir gerçekleştirmenin bir spesifikasyona uygun olup olmadığının sınanması üzerine çalışmalar başlamış ve verilen SDM'nin durumları tanımlandığı ve belli bir hata kümesi göz önüne anlındığı zaman verilen bir SDM için sınama dizilerinin üretilmesi için polinom zamana ihtiyaç duyulduğu bilinmektedir. Bu tezde iki farklı sınama dizisi ele alınmıştır: Sınama Dizisi (SDi) ve Sınama Deneyleri (SDe). Sınama dizileri ister SDi ister SDe olsun genelde belli bir prensipte çalışır: bir kez üret ve çok kez kullan. Bu yüzden sınama dizilerinin boylarının kısa olması sınama sırasında geçen yekün süreyi azaltacağı gerekçesi ile oldukça önemlidir. Bu yüzden literatürde bu alanda çalışmalar yapılmaya başlanmıştır.

Bu tezde ilk önce SDi'lerin boylarını kısaltmayı amaçlayan stratejiler gösterilmiştir. Bir SDi birden fazla, kendisinden ufak Sıralama Dizisi, Durum Tanıma Dizisi gibi dizilerinden oluşur. Bu konu üzerine yapılan hemen hemen tüm çalışmalar bu dizilerin SDM ile birlikte verildiğini tahmin etmişlerdir ve bu diziler ile oluşturulacak SDi'ler belli bir hata kümesi göz önünde bulundurularak üretildiğinde bir spesifikasyonun hatalı tüm gerçekleştirmelerini saptayacağı bilinmektedir. Bir başka değiş ile verilen hatalı bir gerçekleştirme üretilen bir SDi tarafından belirlenebilir. Farklı SDi oluşturma yöntemleri bu dizileri farklı şekilde bir araya getirerek SDi'leri daha kısa boyda oluşturmayı amaçlamışlardır. Ancak sıralama ve durum tanıma dizileri bir SDi'nin en büyük parçaları olduğu bilgisi ile hareket edersek bu dizilerin boylarının kısaltılması, oluşturulacak SDi'lerin boylarını'da kısaltacağı düşünülmelidir.

1991'de verilen bir SDM'nin en kısa sıralama dizinin üretilmesinin NP != P eşitsizliği var olduğu sürece polinom zamanda üretilemeyeceği ispat edilmiştir. Ancak yakın geçmişte bir SDM'nin durumlar arası geçişlerinin özel bir türde olması "monotonik" durumunda en kısa sıralama dizisinin polinom zamanda üretileceği gösterilmiştir. Ancak kısmi tanımlı bir monotonik SDM'nin en kısa sıralama dizisinin hesaplanma zorluğu açık bir problemdi. Bu tezde bu problemin NP-Zor olduğunu gösterdik.

Öteyandan, 1994 yılında özellikli bir durum tanıma dizisinin (uyarlamalı ayrıştırma dizisi (UAD)) polinom zamanda üretilebileceği gösterilmiştir. Aynı çalışmada yazarlar bir SDM için bu diziyi polinom zamanda üretebilen bir algoritma da göstermişlerdir. Ancak bu algoritma herhangi bir ayrıştırma dizisini büyüklüğüne bakmadan üretmektedir. Bu çalışmadan başka tam tanımlı yada kısmi tanımlı SDM'ler için uyarlamalı ayrıştırma dizisi üretebilen başka bir çalışma yoktur. Bu tezde kısa uyarlamalı ayrıştırma dizisi üretmenin NP-TAM ve en kısa UAD'ye yaklaşmanın da NP-Zor olduğunu gösterdik. Bunun yanında SDi'lerin boyunu ortalama %29.2 kadar kısaltabilmeye yarayan UAD'leri retebilen sezgisel yöntemler sunduk.

Bu tezde SDe'lerin boyunu kısaltmayı hedefleyen çalışmalar yaptık. SDe'ler SDi'lerin aksine birbiri ile birleşmeyen çok sayıda ufak sınama konuları içerir ve her bir sınama konusu gerçekleştirmenin farklı bir yönünü sınar. Ancak SDi'ler de olduğu gibi bu sınama

konularının büyük bir bölümü yine durum tanıma dizilerinden oluşur. SDe'ler için sınırlı sayıda durum tanıma dizisi mevcuttur, bu tezde yeni bir durum tanıma dizisi sunduk ve gösterdik ki bu yeni durum tanıma dizisinin oluşturulmasının zorluğu PSPACE-Tam. Bu sonucu takiben bu dizileri üretmek için sezgisel yöntem ürettik ve endüstriden alınmış SDM'ler üzerinde deneylar yaptık ve teklif edilen yöntem ile SDe'lerin boylarını %65'e varan oranlarda kısaltılabileceğini gösterdik.

Dağıtık SDM'lerin (DSDM'lerin) sınanması SDM tabanlı sınama çalışmalarının ilginç bir ayağı olmaktadır. Sınama dizilerinin üretiminde yaşanan zorluklara ek olarak dağıtık mimarilerin getirmiş olduğu kontrolledilebilirlik ve gözlemlenebilirlik problemleri karşımıza çıkmaktadır. Her ne kadar mevcut SDi üretme yöntemlerinde durum tanıma dizilerinin DSDM ile birlikte verildiği düşünülmüşsede kontroledilebir durum tanıma disizin üretlimesine değinen bir çalışma yoktur. Bu tezde bu dizilerin üretilmesinin zorluğunu araştırmış ve bu dizilerin polinom zamanda üretilemeyeceğini ispatlamş bulunmaktayz.

Contents

List of Figures

List of Tables

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1. Introduction

Although the concept of Finite State Machines (FSMs) had been existed for so long, its popularity today in the computer science and engineering fields can be attributed to the pioneering efforts of George H. Mealy [1] and Edward Forrest Moore [2] performed at Bell Labs and IBM around 1960s. After their efforts, finite state machines became popular in computer science and engineering disciplines, remarkably due to the ability of modelling systems such as sequential circuits [3], communication protocols [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14], object-oriented systems [15], and web services [4, 16, 17, 18, 19, 20].

The operation of an FSM can be described as follows: the system is always in one of the defined states. It reacts to an input by producing an output, and by changing its state. For a Mealy machine, the output is generated by a transition. For a Moore machine, an output is generated by a state. Due to this reactive behaviour, FMSs are also called *reactive systems*. An input to an FSM may be a message, or a simple event flag. Likewise, an output from an FSM may be a message interpreted by an observer, or setting an event flag. Multiple transitions are allowed from one state to other states. We refer [21, 22] for detailed information on FSMs. In this work we focus on Mealy machines. However, Mealy and Moore machines are equivalent and can be converted to each other [2].

When a system is modelled by an FSM, it is possible to generate a test from this model. Here, by *testing* we refer to the *Black Box Testing* where the tester is only allowed to observe outputs. The first paper in this field was given by Moore [2], where Moore suggested to generate a machine identification sequence: a special input sequence which is capable of distinguishing a copy of M from any other FSMs which have same

number of input/output symbols and states as M.

In principle, testing FSM refers to a *Fault Detection Experiment* [22] which consists of applying an experiment (derived from a specification FSM M) to an implementation N of M, observing the output sequence produced by N in response to the application of the experiment, and comparing the output sequence to the expected output sequence. In this thesis, we consider two notions of fault detection experiments: *Checking Sequences* (CSs) [23] and *Checking Experiments* (CEs)[4]. If the applied experiment contains a single input sequence then it is called a CS and if the applied experiment contains a set of input sequences then it is called a CE. These fault detection experiments determine whether *System Under Test* (SUT) N is a correct or faulty implementation of M [4, 21, 23]. After Moore, Arthur Gill [21] and Frederick C. Hennie [23, 24] present a line of research on testing FSMs. As fault detection experiments (CSs/CEs) are used to test an implementation, and the fact that a specification may have multiple implementations, reducing the size of fault detection experiments is important. In [23], Hennie considers the specification machine as *the master plan*, and he encodes the behaviour of this master plan as a CS. Then based on this sequence he tests if the implementation has the same behaviour. Due to this strategy; a CS refers to an input sequence that is constructed from M and is guaranteed to distinguish a correct implementation from any faulty implementation, which have the same input and output alphabets as M and no more states than M. Following him, Charles R. Kime enhanced the methods given by Hennie and lessen the lengths of the CS to some extend [25]. Following Kime and Hennie another influential scientist Güney Gönenç proposed an algorithm that shortens the length of such sequences considerably [26]. After this point researchers have been working on to shorten the lengths of the CSs by putting the pieces that need to exist in such a CS together in a better way [4, 17, 27, 28, 29, 30, 31, 32, 33].

In general, a CS consists of four different type of components. *Reset Sequence* is a component in which the machine N is brought to the initial state regardless of the current state of N and the output produced by N. *State Verification* component is carried out by bringing N to a certain state s of M, checking if N is at state s by

applying a *state identification sequence* for s and repeating this procedure until all the states of M are recognised in N. The *transition verification* component is performed for each transition of M in N. To verify a transition, one brings N to the state from which the transition starts, applies the input that labels the transition (to check correct implementation of the output of the transition) and then verifies the ending state by using a state identification sequence. The final component is *transfer sequences*. Transfer sequences are used to combine all the components to form the final CS.

When examining the structure of a CS, the motivation to study reset sequences becomes natural i.e. shorter reset sequences lead to shorter CSs. However for a given FSM computing the shortest reset sequence is known to be NP-complete in general [34]. Therefore, we investigated open problems and raise several problems related to constructing reset sequences and try to draw the computational complexities for these problems.

State identification sequences are used many times in a CS and there are different type of state identification sequences: *Unique Input Output (UIO) sequences*, or *Separating Family* (also known as the *Characterizing Set*), or *Distinguishing sequences (DSs)*. A UIOs is a set of input sequences that verifies the states of an FSM. Since it is PSPACE-complete to construct UIOs for an FSM [35], it may be impractical to use UIOs for large FSMs [7, 13, 36, 37, 38, 39, 40, 41]. Separating family can also be used to verify states and transitions of an FSM [4]. Although this method is strong in the sense that every minimal FSM posseses a characterizing set and it is polynomial time computable, it requires a reliable reset feature in the implementation or otherwise results in exponentially long CSs [4, 22, 21]. DSs are used to identify the current state of N. Thanks to the efficient state identification capabilities, distinguishing sequences simplify the problem of generating CSs. They do not require reliable reset, and by using a distinguishing sequence, one can construct a CS of a length polynomial in the size of the FSM and the distinguishing sequence[1] [23, 29, 35, 42, 43, 44]. Therefore many techniques for constructing CSs use DSs to resolve the state identification problem.

[1] That is, the FSM and its distinguishing sequence are considered as the inputs for such CS generation algorithms.

There are two types of distinguishing sequences, *Preset Distinguishing Sequences*, and *Adaptive Distinguishing Sequences* (also known as *Distinguishing Sets*). As it was noted before [35, 42], the use of ADS instead of PDS is also possible for these methods and shown to yield polynomial length CSs [43]. There are numerous advantages of using ADSs over PDSs. Lee and Yannakakis have reported that checking the existence and computing a PDS is a PSPACE-complete problem whereas it is polynomial time solvable in case of ADS [35]. They have also shown that an FSM which posses an ADS may not have a PDS and not the other way around [35, 42]. Moreover, it is also known that the shortest ADS for an FSM can not be longer than the shortest PDS of the same FSM [35, 42, 24]. Furthermore, because during the distinguishing experiment the next input is chosen according to the previous response of FSM, ADS based testing methods is accepted as more powerful means of testing than the PDS based methods [45, chp.2]. Hierons et al.[43] reported that CSs are relatively shorter when designed by ADS.

All ADS based CS generation methods start with the assumption that an ADS is given. The given ADS is repeatedly applied in state verification and transition verification components of the CS. Thus, these ADS applications form a considerably large part of the CS and hence, reducing the size of ADSs is a reasonable way to reduce the length of the CSs.

Earlier ADS construction algorithms [21, 22, 23] are exhaustive and require exponential space and time. The only polynomial time algorithm was proposed by Lee and Yannakakis (*LY Algorithm*). Let us assume that p, n refers to the number of inputs and number of states respectively then the LY algorithm can check if M has an ADS in $O(pn \log n)$ time [35], and if one exists, we can construct an ADS in $O(pn^2)$ time [35]. Alur et al. show that checking the existence of an ADS for non-deterministic FSMs is EXPTIME-complete [46]. Recently, Kushik et al. present an algorithm (KEY algorithm) for constructing ADSs for non-deterministic observable FSMs [47]. We believe that the KEY algorithm can also construct ADSs for deterministic FSMs, since the class of deterministic FSMs is a sub-class of non-deterministic FSMs.

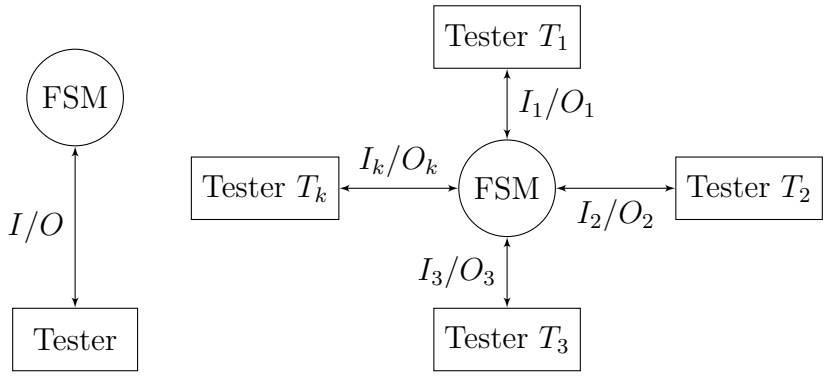
These ADS construction algorithms are not guaranteed to compute the minimum cost

ADS for a given FSM. Moreover, to our knowledge, there is no work that analyses the computational complexity of constructing minimum cost ADSs. In this thesis, we also analyse the computational complexity of constructing minimum cost ADSs and devise methods for computing such ADSs.

Although the existence of ADSs and PDSs are very useful, not all FSMs possess an ADS or PDS. For such cases instead of CSs, another fault detection sequence *Checking Experiments* (CEs) are constructed. The key difference between CSs and CEs is that a CE can contain multiple test sequences (or *test cases*). A test sequence is simply an input sequence that, when applied, the machine N has to produce expected output. Most of the approaches use separating family, or an enhanced version called *Harmonized State Identifiers* to identify the states [4, 21, 48, 49, 50, 51, 52]. We refer [53] for comparison of such methods. In this thesis we to try to broaden the use of ADSs and PDSs on FSMs that do not have one, by introducing *Incomplete* ADSs/PDSs and use these sequences for constructing CEs.

As a matter of fact, most CS generation approaches[2] assume that the SUT interacts with a single tester (Figure 1.1a). However, many systems interact with their environment at multiple physically distributed interfaces, called ports (Figure 1.1b). Examples include communications protocols, cloud systems, web services, and wireless sensor networks. In testing such a system, we might place a separate independent (local) tester at each port. The ISO standardised *distributed test architecture* dictates that while testing there is no global clock and testers do not synchronize during testing [55]. However, sometimes, rather than using the distributed test architecture, we allow the testers to exchange coordination messages through a network in order to synchronise their actions (see, for example, [56, 57, 58]). However, this can make testing more expensive, since it requires us to establish a network to connect the local testers, and may not be feasible where there are timing constraints. In addition, the message exchange may use the same network as the SUT and so change the behaviour of the SUT. As a result, there

[2]Such as **HEN** method given in [23], **UWZ** method given in [30], **HIU** method given in [29], **SP** method given in [33], and **DY** method given in [54].



(a) A Localized Architecture

(b) A Distributed Architecture

Figure 1.1.: Localized and Distributed Architectures

has been much interest in testing in the distributed test architecture (see, for example, [59, 60, 61, 62, 63, 64, 65, 66, 67, 68]).

Early work, regarding the distributed test architecture, was motivated by protocol conformance testing [62, 63, 68]. This work identified two problems introduced by distributed testing. First, there might be a *controllability problem* in which a local tester, at a port p, cannot determine when to supply an input. Controllability problems lead to non-determinism in testing and so there has been interest in the problem of generating test sequences that do not cause controllability problems [59, 61, 64, 69, 70, 71, 72]. *Observability problems* refer to the fact that, since we only make local observations, we may not be able to distinguish between two different behaviours (global traces). Observability problems can reduce the effectiveness of a test sequence and so there has been interest in producing test sequences that do not suffer from such observability problems [60, 63, 73, 74, 75].

British scientist Robert Hierons has shown that if we are testing from multi–port FSM (MPFSM) M then it is undecidable whether there is a test case that is guaranteed to move M to a particular state or to distinguish two states and these results hold even if

M is deterministic [66]. In contrast, these problems can be solved in low order polynomial time if we have a deterministic FSM M. If we restrict attention to controllable test sequences[3] then there are low-order polynomial time algorithms to decide whether there is a separating sequence for two states [65] and to decide whether there is a controllable sequence that forces M into a particular state [76]. However, as noted above, if we use separating sequences then we require many test sequences to test a single transition. This motivates the final leg of this thesis: investigate computational complexity of constructing PDSs and ADSs for distributed testing.

1.1. Contributions

The contributions of this thesis are manifold. However, we believe that all these contributions aim to enhance FSM based testing by presenting new problems and investigating their computational complexities, providing algorithms for the proposed problems and introducing new problems.

The major contributions of our work can be summarized as follows:

1. We introduce several problems related to reset sequences: We investigate their computational complexities.

2. We provide a rather unique way of reducing the length of checking sequences: We propose several objective functions to minimize adaptive distinguishing sequences and we show that constructing a minimum cost ADS is computationally hard and hard to approximate. We provide two modifications on the existing ADS construction algorithm that aim to construct minimum cost ADSs and provide a new lookahead based algorithm to construct minimum cost ADSs. Finally, we experimentally show that minimum cost ADSs can reduce the length of the checking sequence by 29.20% on the average.

3. We show how the state identification capabilities of DSs can be utilized on FSMs

[3] Controllable test sequences are formally defined in Section 6.2.

that do not have a DS: We introduce the notion of *Incomplete DSs*. We investigate the computational complexity of constructing such incomplete DSs and we provide a heuristic to compute incomplete DSs. We experimentally show that the use of incomplete DSs reduce the cost of checking experiments.

4. We investigate computational complexities of constructing preset and adaptive distinguishing sequences for distributed testing: We show that constructing adaptive and preset distinguishing sequences are computationally hard. We left the bounds of ADSs and PDSs as open problems. We consider DSs with limited size and provide computational complexities of constructing such DSs. We also provide a sub–class of multi–port FSMs where the PDS construction is decidable.

1.2. Outline of the Thesis

The organization of this thesis is as follows: Chapter 2, introduces some basic notation that are going to be used throughout the thesis. In Chapter 3, we examine the problems related to reset sequences, focusing mainly on computational complexities of open and introduced problems. In Chapter 4, we describe the computational complexity of constructing minimum cost ADSs provide methods to construct minimum cost ADSs and experimentally show what we can earn by using minimum cost ADSs while constructing CSs. In Chapter 5, we introduce the notion of incomplete ADSs/PDSs, give computational cost of constructing them, and experimentally show the effect of using such ADSs/ PDSs while constructing CEs. The Chapter 6 is devoted for the contributions related to the distributed testing and in Chapter 7 we conclude the thesis.

All the proofs for Lemmas, Propositions, and Theorems of Chapter 3, Chapter 4, Chapter 5 and Chapter 6 are given in Appendix A, Appendix B, Appendix C and Appendix D, respectively.

2. Preliminaries

2.1. Finite State Machines

An FSM is formally defined as a 5-tuple $M = (S, s_0, X, Y, \delta, \lambda)$ where:

- S is the finite set of states.
- X is the finite set of input symbols
- Y is the set of output symbols
- s_0 is the initial state[1]
- δ is the transition function $\delta : S \times X \to S$
- λ is the output function $\lambda : S \times X \to Y$

At any given time, M is at one of its states. If an input $x \in X$ is applied when M is at state s, M changes its state to $\delta(s, x)$ and during this transition, the output symbol $\lambda(s, x)$ is produced. It is assumed that only one input is applied at a time and similarly only one output is produced at a time.

When δ and λ are described as functions as above, the FSM is called *deterministic*. For an FSM which is not deterministic (in which case it is called *non-deterministic*), δ and λ are defined as relations. In this thesis we will only be interested in deterministic FSMs. To denote a transition from a state s to a state s' with an input x and an output y, we write $(s, s', x/y)$, where $s' = \delta(s, x)$ and $y = \lambda(s, x)$. We call x/y an input/output

[1] In this thesis we mostly omit the initial states from definitions of FSMs.

pair. For a transition $\tau = (s, s', x/y)$, we use $start(\tau)$, $end(\tau)$, $input(\tau)$, $output(\tau)$, and $label(\tau)$ to refer to state s (the starting state of τ), state s' (the ending state of τ), input x (the input of τ), output y (the output of τ), and input/output pair x/y (the input/output label of τ), respectively.

An FSM M can be by a directed graph with a set of vertices and a set of edges. Each vertex represents one state and each edge represents one transition between the states of the machine M.

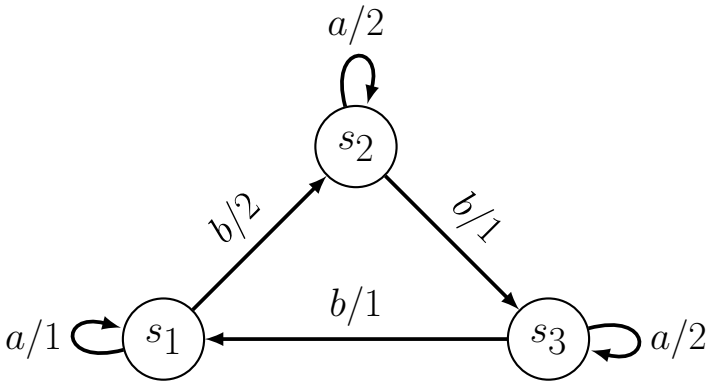


Figure 2.1.: An example FSM M_1

Figure 2.1 is an example of a FSM. Where $S = \{s_1, s_2, s_3\}$, $X = \{a, b\}$ and $Y = \{1, 2\}$.

Throughout this thesis we will use juxtaposition to denote sequences (e.g. *abba* is an input sequence where a and b are input symbols) and variables with bars to denote variables with sequence values (e.g. $\bar{x} \in X^*$ to denote an input sequence). We use ε to denote the empty sequence. We define extensions of transition and output functions over sequences of inputs as follows:

- $\bar{\delta}(s, \varepsilon) = s$
- $\bar{\delta}(s, x\bar{x}) = \bar{\delta}(\delta(s, x), \bar{x})$ where $x \in X, \bar{x} \in X^*$
- $\bar{\lambda}(s, \varepsilon) = \varepsilon$
- $\bar{\lambda}(s, x\bar{x}) = \lambda(s, x)\bar{\lambda}(\delta(s, x), \bar{x})$ where $x \in X, \bar{x} \in X^*$

By abusing the notation, we will again use δ and λ instead of $\bar{\delta}$ and $\bar{\lambda}$.

A *walk* in M is a sequence $(s_1, s_2, x_1/y_1), \ldots, (s_m, s_{m+1}, x_m/y_m)$ of consecutive transitions. This walk has *starting state* s_1, *ending state* s_{m+1}, and *label* $x_1/y_1, x_2/y_2, \ldots, x_m/y_m$. Here $x_1/y_1, x_2/y_2, \ldots, x_m/y_m$ is an *input/output sequence*, also called a *global trace*, and $x_1, x_2, \ldots, x_m$ is the *input portion* and $y_1, y_2, \ldots, y_m$ is the *output portion* of this global trace. An example walk in M_2 of Figure 2.1 is $\bar{\tau} = (s_1, s_2, b/2)(s_2, s_3, b/1)$, its starting state is s_1 and ending state is s_3 its label is $b/2$ $b/1$, which has input portion bb and output portion $2, 1$.

An FSM M defines the language $L(M)$ of labels of walks with starting state s_0. Likewise, $L_M(s)$ denotes the set of labels of walks of M with starting state s. For example, $L(M_1)$ contains the global trace[2] $b/2, a/2$ and $L_{M_1}(s_3)$ contains the global trace $b/1, b/2$. Given $S' \subseteq S$ we let $L_M(S')$ denote the set of labels of walks of M with starting state in S' and so $L_M(S') = \cup_{s \in S'} L_M(s)$. Two states s, s' are *indistinguishable* or *equivalent* if $L_M(s) = L_M(s')$. Similarly, FSMs M and N are equivalent if $L(M) = L(N)$. An FSM M is said to be *minimal* if there is no equivalent FSM that has fewer states. Assuming every state s of M is reachable we have that M is minimal if and only if $L_M(s) \neq L_M(s')$ for all $s, s' \in S$ with $s \neq s'$. We write *pre* to denote a function that takes a set of sequences and returns the set of prefixes of these, similarly we write *post* to denote a function that returns the set of postfixes of these. Note that if $x_1/y_1, x_2/y_2, \ldots, x_m/y_m$ is an input/output sequence then its prefixes are of the form $x_1/y_1, x_2/y_2, \ldots, x_n/y_n$ for $n \leq m$. Formal definitions for PDSs and ADSs (DSs) are given respectively.

We use barred symbols to denote sequences and ε for the empty sequence. Suppose that we are given a rooted tree $\mathcal{K}$ where the nodes and the edges are labeled. The term *internal node* is used to refer to a node which is not a leaf. For two nodes p and q in $\mathcal{K}$, we say p *is under* q, if p is a node in the subtree rooted at node q. A node is by definition under itself. Consider a node p in $\mathcal{K}$. We use the notation $\bar{p}_v$ (v for vertices) to denote the sequence obtained by concatenating the node labels on the path from the

[2]Assume s_1 is the initial state.

root of $\mathcal{K}$ to p *excluding* the label of p. The notation p_v is used to denote the label of p itself. Similarly, $\bar{p}_e$ (e for edges) denotes the sequence obtained by concatenating the edge labels on the path from the root of $\mathcal{K}$ to p. If p is the root, $\bar{p}_v$ and $\bar{p}_e$ are both considered ε. For a child p' of p, if the label of the edge from p to p' is l, then we call p' the *l–successor of p*. In this thesis, we will always have distinct labels for the edges emanating from an internal node, hence l–successor of a node will always be unique.

Definition 1 *Given FSM $M = (S, X, Y, \delta, \lambda)$ and S, input sequence $\bar{x}$ is a* Preset Distinguishing Sequence (PDS) *for S if for all $s, s' \in S$ with $s \neq s'$ we have that $\lambda(s, \bar{x}) \neq \lambda(s', \bar{x})$.*

On the other hand, an ADS can be thought as a decision tree. The nodes of the tree are labeled by input symbols, edges are labeled by output symbols providing that edges emanating from a common node have different labels and leaves are labeled by state ids.

Definition 2 *An* Adaptive Distinguishing Sequence *of an FSM $M = (S, X, Y, \delta, \lambda)$ with n states is a rooted tree $\mathcal{A}$ with n leaves such that:*

1. *Each leaf of $\mathcal{A}$ is labeled by a distinct state $s \in S$.*
2. *Each internal node of $\mathcal{A}$ is labeled by an input symbol $x \in X$.*
3. *Each edge is labeled by an output symbol $y \in Y$.*
4. *If an internal node has two or more outgoing edges, these edges are labeled by distinct output symbols.*
5. *For a leaf node p, $\lambda(p_v, \bar{p}_v) = \bar{p}_e$ (i.e. the state labeling a leaf node p produces the output sequence labeling the path from the root to p to the input sequence labeling the path from the root to p).*

The use of ADS is straightforward: to identify the current state of the FSM apply the input symbol that labels the current node of the tree, and select the outgoing edge of the current node that is labeled by the output symbol produced by the FSM and read

the label of the new node. If the label is a state id then the initial state of the FSM is identified, otherwise repeat the procedure. Figure 2.2 is an example ADS for FSM M_1 given in Figure 2.1. If an FSM M has an ADS, then M is minimal. However, a minimal FSM may or may not have an ADS. An FSM may also have more than one ADS, e.g. Figure 2.3 is another ADS for M_1 of Figure 2.1. In this work, we write (DS) to refer to PDSs and ADSs.

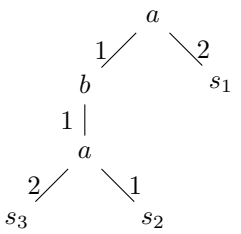


Figure 2.2.: An ADS for M_1 of Figure 2.1

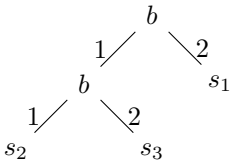


Figure 2.3.: Another ADS for M_1 of Figure 2.1

For a set of states S', an input sequence $\bar{x}$ and an output sequence $\bar{y}$, let $S'_{\bar{x}/\bar{y}}$ be $\{s \in S' | \lambda(s, \bar{x}) = \bar{y}\}$. In other words, $S'_{\bar{x}/\bar{y}}$ is the set of states in S' which produce the output sequence $\bar{y}$ to the input sequence $\bar{x}$. Followings are easy to see consequences of definitions. Let $\mathcal{A}$ be an ADS for an FSM $M = (S, X, Y, \delta, \lambda)$.

Lemma 1 *Let p be a leaf node in $\mathcal{A}$ and q be an internal node in $\mathcal{A}$ on the path from the root to p. If p is under the y–successor of q, then $\lambda(\delta(p_v, \bar{q}_v), q_v) = y$.*

Lemma 2 *Let p be a leaf node in $\mathcal{A}$. For any state $s \neq p_v$, $\lambda(s, \bar{p}_v) \neq \lambda(p_v, \bar{p}_v)$.*

Lemma 3 *For a node p in $\mathcal{A}$, (i) if p is a leaf node, then $|\delta(S_{\bar{p}_v/\bar{p}_e}, \bar{p}_v)| = 1$, and (ii) if p is an internal node, then $|\delta(S_{\bar{p}_v/\bar{p}_e}, \bar{p}_v)| > 1$.*

Lemma 4 *For an internal node p in $\mathcal{A}$, p_v is a valid input for the set of states $\delta(S_{\bar{p}_v/\bar{p}_e}, \bar{p}_v)$.*

A *Partial* FSM (PSFSM) M is defined by tuple $M = (S, X, Y, \delta, \lambda, D)$ where S, X, Y are finite sets of states, inputs and outputs respectively. $D \subset S \times X$ is the domain,

$\delta : D \to S$ is the transition function, and $\lambda : D \to Y$ is the output function. If $(s, x) \in D$ then x is *defined* at s. Given input sequence $\bar{x} = x_1 x_2 \dots x_k$ and $s \in S$, $\bar{x}$ *is defined at* s if there exist $s_1, s_2, \dots s_k \in S$ such that $s = s_1$ and for all $1 \leq i \leq k$, x_i is defined at s_i and $\delta(s_i, x_i) = s_{i+1}$. The transition and output functions can be extended to input sequences as described above. An example PSFSM M_1 is given in Figure 2.4 where $X = \{a, b\}$, $Y = \{0, 1\}$ $S = \{s_1, s_2, s_3\}$. Note that input a is not defined at state s_3.

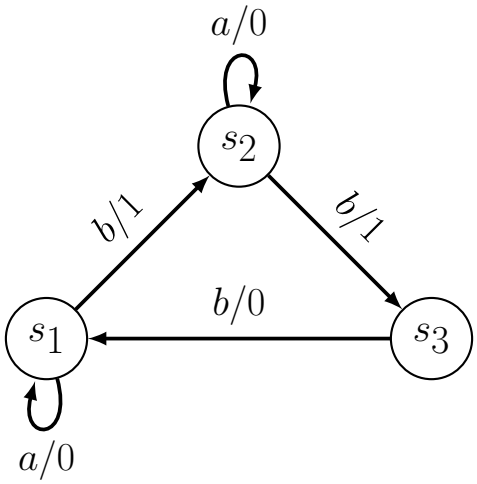


Figure 2.4.: PSFSM M_1

Although DSs are important and useful on their own right, they are important for another reason: they have been useful to solve *fault detection problem.*

Fault detection problem is referred to also as the *machine verification* or *conformance testing* problem depending on the subject (i.e. it is refereed as conformance testing in communication protocol spectra). Let us assume that we are given an FSM M with n number of states, and a finite set $\phi(M)$ of all faulty FSMs such that each of which has at most n number of states. Also let us assume that we are given an FSM N which is known to be an implementation of M, the *Fault Detection Problem* is to decide if $N \notin \phi(M)$. The *Fault Detection Experiment* is an experiment that solves the fault

detection problem. The underlying input sequence can be a CS or CE. A CS of M is an input sequence starting at the initial state s_0 of M that distinguishes M from any fault implementation of M that is not isomorphic to M. (i.e., the output sequence produced by any such N of $\phi(M)$ is different from the output sequence produced by M). Formally;

Definition 3 *An input sequence $\bar{x}$ is a checking sequence if and only if $\lambda(s_M, \bar{x}) \neq \lambda(s_N, \bar{x})$ where $N \in \phi(M)$, and s_M, s_N are initial states of FSMs M and N respectively.*

On the other hand, a CE contains a set of input sequences called *test sequences*. A test sequence is simply an input sequence. In testing we will apply the inputs from a test sequence in the order specified and compare the outputs produced with those specified.

Definition 4 *Given FSM $M = (S, X, Y, \delta, \lambda, s_0)$ and integer m, a test suite $\mathcal{T} \subseteq X^*$ is a checking experiment if, for every FSM $N = (S', X, Y, \delta', \lambda', s'_0)$ that has the same input alphabet as M and no more than m states, N produces expected output for $\mathcal{T}$ if and only if $\forall \bar{x} \in \mathcal{T}$ we have that $\lambda(s_0, \bar{x}) = \lambda(s'_0, \bar{x})$.*

2.1.1. Multi–port Finite State Machines

A multi-port finite state machine MPFSM is an FSM with a set $\mathcal{P}$ of ports at which it interacts with its environment. The ports are physically distributed and each has its own input/output alphabet. An input can only be applied at a specific port, and an output can only be observed at a specific port. Therefore, for each port $p \in \mathcal{P}$ there is a separate local tester that applies the inputs to p and observes the outputs produced at p.

A deterministic MPFSM is defined by a tuple $M = (\mathcal{P}, S, s_0, X, Y, \delta, \lambda)$ where:

- $\mathcal{P} = \{1, 2, \ldots, k\}$ is the set of ports.
- S is the finite set of states and $s_0 \in S$ is the initial state.
- X is the finite set of inputs and $X = X_1 \cup X_2 \cup \cdots \cup X_k$ where X_p $(1 \leq p \leq k)$ is the input alphabet for port p. We assume that the input alphabets of the ports

are disjoint: for all $p, p' \in \mathcal{P}$, such that $p \neq p'$, we have $X_p \cap X_{p'} = \emptyset$. For an input $x \in X$, we use $inport(x)$ to denote the port to which x belongs and so $inport(x) = p$ if $x \in X_p$. We consider the projection of an input onto a port and defined it as $\pi_p(x) = x$ if $x \in X_p$, and $\pi_p(x) = \varepsilon$ if $x \notin X_p$. The symbol "ε" will be used to denote an empty/null input or output and also the empty sequence.

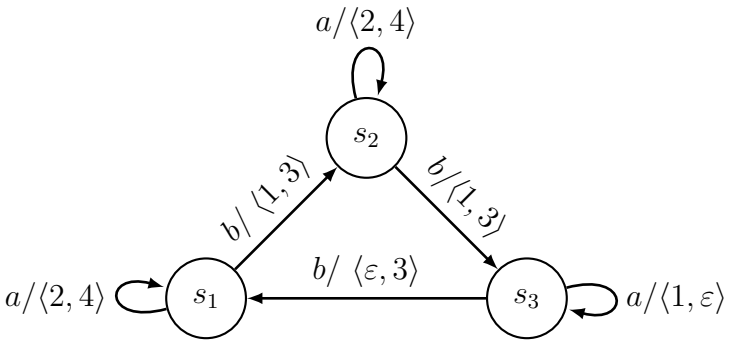
- $Y = \prod_{p=1}^{k}(Y_p \cup \{\varepsilon\})$ is the set of outputs where Y_p is the output alphabet for port p. We assume that the output alphabets of the ports are disjoint: for two ports $p, p' \in \mathcal{P}$, such that $p \neq p'$, we have $Y_p \cap Y_{p'} = \emptyset$. An output $y \in Y$ is a vector $\langle o_1, o_2, \ldots, o_k \rangle$ where $o_p \in Y_p \cup \{\varepsilon\}$ for all $1 \leq p \leq k$. We also assume that X is disjoint from $\cup_{1 \leq i \leq k} Y_k$. The notation $\pi_p(y)$ is used to denote the projection of y onto port p, which is simply the p^{th} component of the output vector y. We define $outport(y) = \{p \in \mathcal{P} \mid \pi_p(y) \neq \varepsilon\}$, which is the set of ports at which an output is produced.

- δ is the state transfer function of type $S \times X \to S$.

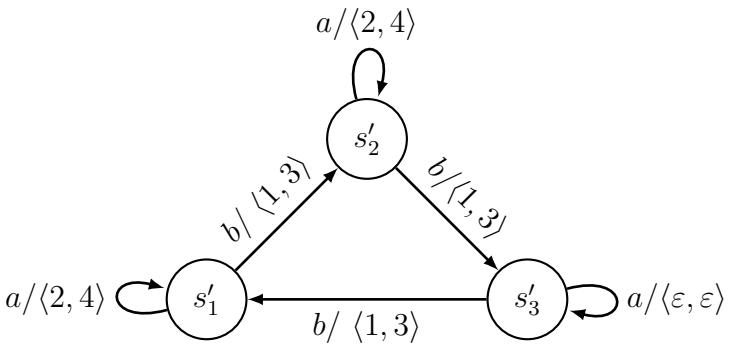
- During a state transition M also produces an output vector. The output function $\lambda : S \times X \to Y$ gives the output vector produced in response to an input.

Let $(s, s', x/y)$ be a transition of M then we define $inport(\tau) = inport(x/y) = inport(x)$ and we also define $outport(\tau) = outport(x/y) = outport(y)$ and finally we define $ports(\tau) = ports(x/y) = \{inport(x)\} \cup outport(y)$ to denote the ports used in the transition. Figure 2.5a gives an example of a 2-port MPFSM. The output and state transfer functions can be extended to input sequences as usual.

Since we assume that the ports are physically distributed, no tester observes a global trace: the tester connected to port p will observe only the inputs and outputs at p. We use Σ to denote the set of global observations (inputs and outputs) that a hypothetical global tester can observe and Σ_p to denote the set of observations that can be made at port p. Thus, $\Sigma = X \cup Y$ contains inputs and vectors of outputs while $\Sigma_p = X_p \cup Y_p$ contains only inputs and outputs at p. Let $\sigma \in \Sigma^*$ be a global trace, then $\pi_p(\sigma)$ is the



(a) Example MPFSMM_1



(b) Faulty implementation of machine M_1

Figure 2.5.: Example MPFSMM_1 and its faulty implementation M'_1

local trace at p: a sequence of inputs and outputs at port p that is the projection of σ at p. Here π_p is defined by the following in which ε denotes the empty sequence.

$$\pi_p(\varepsilon) = \varepsilon$$
$$\pi_p((x/\langle o_1, o_2, \ldots, o_m\rangle)\sigma) = \pi_p(\sigma) \text{ if } x \notin X_p \wedge o_p = \varepsilon$$
$$\pi_p((x/\langle o_1, o_2, \ldots, o_m\rangle)\sigma) = x\pi_p(\sigma) \text{ if } x \in X_p \wedge o_p = \varepsilon$$
$$\pi_p((x/\langle o_1, o_2, \ldots, o_m\rangle)\sigma) = o_p\pi_p(\sigma) \text{ if } x \notin X_p \wedge o_p \neq \varepsilon$$
$$\pi_p((x/\langle o_1, o_2, \ldots, o_m\rangle)\sigma) = xo_p\pi_p(\sigma) \text{ if } x \in X_p \wedge o_p \neq \varepsilon$$

Since the local testers observe only the local projections of global traces, these testers can only distinguish two global traces if one or more of their local projections differ. Thus, two global traces σ_1, σ_2 are *indistinguishable*, written $\sigma_1 \sim \sigma_2$, if for all $p \in \mathcal{P}$ we have that $\pi_p(\sigma_1) = \pi_p(\sigma_2)$. For instance let us consider MPFSM M_1 given in Figure 2.5,

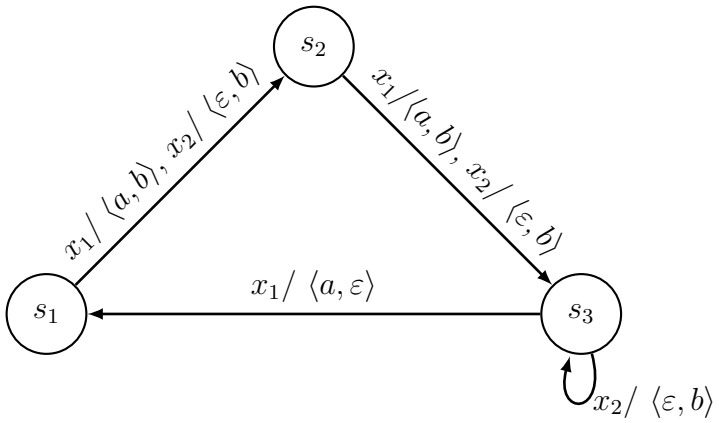


Figure 2.6.: Example MPFSM M_2

and global traces $\sigma_1 = b/\langle 1,3\rangle, b/\langle \varepsilon, 3\rangle$, $\sigma_2 = b/\langle \varepsilon, 3\rangle, b/\langle 1,3\rangle$, then $\pi_2(\sigma_1) = 1$, $\pi_1(\sigma_1) = b3b3$, $\pi_2(\sigma_2) = 1$ and $\pi_1(\sigma_2) = b3b3$ and so $\sigma_1 \sim \sigma_2$.

Recall that in distributed testing, the testers are physically distributed and they are not capable of communicating between other testers. This reduced observational power can lead to situations in which a traditional PDS or ADS fails to distinguish certain states.

Example 1 *Consider the FSM given in Figure 2.6. We have that x_1x_1 is a traditional* PDS *since it leads to different global traces from the states: from s_1 we have $x_1/\langle a,b\rangle, x_1/\langle a,b\rangle$; from s_2 we have $x_1/\langle a,b\rangle, x_1/\langle a,\varepsilon\rangle$; and from s_3 we have $x_1/\langle a,\varepsilon\rangle, x_1/\langle a,b\rangle$. However, if we consider the local traces we find that x_1x_1 does not distinguish states s_2 and s_3 in distributed testing since in each case the project at port 1 is x_1ax_1a and the projection at port 2 is b.*

We can formalise this observation as follows.

Proposition 1 *Given FSM M, a traditional* PDS $\bar{x}$ *of M might fail to distinguish some states of M when local observations are made.*

Since PDS defines an ADS the result immediately follows to ADSs.

Proposition 2 *Given FSM M, a traditional* ADS $\bar{x}$ *of M might fail to distinguish some states of M when local observations are made.*

Therefore the definitions supplied for PDSs and ADSs are slightly different when we consider distributed architectures. We will present formal definitions for such sequences in Chapter 6.

2.1.2. Finite Automata

A *Deterministic Finite Automaton* (or simply an *automaton*) is defined by a triple $A = (Q, \Sigma, \delta)$ where,

- Q is a finite set of states.
- Σ is a finite set of input alphabet.
- $\delta : Q \times \Sigma \to Q$ is a transition function.

If δ is a partial function, A is called a *partially specified automaton (PSA)*. Otherwise, when δ is a total function, A is called a *completely specified automaton (CSA)*. The transition function can be extended for a sequence of input symbols in the usual way. Moreover, for a $\bar{Q} \subseteq Q$, we use $\delta(\bar{Q}, \bar{x})$ to denote the set $\cup_{q \in \bar{Q}} \delta(q, \bar{x})$. For a PSA, a word $\bar{x} \in \Sigma^\star$ is said to be *defined at a state* $q \in Q$, if $\forall \bar{x}', \bar{x}'' \in \Sigma^\star, \forall x \in \Sigma$ such that $\bar{x} = \bar{x}' x \bar{x}''$, $\delta(\delta(q, \bar{x}'), x)$ is defined. Throughout this thesis, we use the term *automaton* to refer to general automata (both PSA and CSA). We will specifically use PSA or CSA to refer to the respective classes of automata.

A CSA $A = (Q, \Sigma, \delta)$ is *synchronizable* if there exists a word $\bar{x} \in \Sigma^\star$ such that $|\delta(Q, \bar{x})| = 1$. A synchronizable CSA has a reset functionality, i.e. it can be reset to a single state by reading a special word. In this case $\bar{x}$ is called a *reset word* (or a *synchronizing sequence*). Similarly, a PSA $A = (Q, \Sigma, \delta)$ is *synchronizable* if there exists a word $\bar{x} \in \Sigma^\star$ such that $\bar{x}$ is defined at all states and $|\delta(Q, \bar{x})| = 1$. Throughout this thesis, we use terms reset word and synchronizing sequence interchangeably. It is known that not all automata are synchronizing. We call such automata *non–synchronizable automata (NSA)*. A CSA is a *monotonic CSA* when states preserve a linear order $<$ under the transition function. In other words, a CSA $A = (Q, \Sigma, \delta)$ is monotonic if

for all $q, q' \in Q$ where $q < q'$ then we have that $\delta(q, a) < \delta(q', a)$ or $\delta(q, a) = \delta(q', a)$. Similarly, a PSA is a *monotonic PSA*[3] when states preserve a linear order $<$ under the transition function when they are defined. Formally, a PSA $A = (Q, \Sigma, \delta)$ is monotonic if for all $q, q' \in Q$ where $q < q'$ such that both $\delta(q, a)$ and $\delta(q', a)$ are defined, then we have $\delta(q, a) < \delta(q', a)$ or $\delta(q, a) = \delta(q', a)$.

[3]It is called *Partially Monotonic* in [77]

3. Complexities of Some Problems Related to Synchronizing, Non-synchronizing and Monotonic Automata

3.1. Introduction

A *Reset Sequence / Reset Word*, or a *Synchronizing Sequence / Synchronizing Word* of an FSM M takes M to a specified state, regardless of the initial state of the M and the output sequence produced by the M. As output sequence produced by M is not important, the problem of constructing synchronizing sequence usually studied on finite automata. Therefore in the rest of this chapter, we are going to consider finite automata.

As the need for reset operation is natural, synchronizing sequences are used in various fields including automata theory, robotics, bio–computing, set theory, propositional calculus and many more [4, 34, 38, 42, 48, 49, 78, 79, 80, 81, 82].

For instance, consider an automaton $A = (Q, \Sigma, \delta)$. The transition function introduces functions on the set of states of the form $f_x : Q \to Q$ for all $x \in \Sigma$, where $f_x(q) = q'$ iff $\delta(q, x) = q'$. Finding a synchronizing sequence can then be seen as the problem of finding a composition g of the functions f_x in the form $g(q) = f_{x_1}(f_{x_2}(\dots f_{x_k}(q)))$ such that $x_1, x_2, \dots, x_k \in \Sigma$ and g is a constant function.

An other interesting example arises in bio–computing. In [80, 83] researchers use a set

of automata (in work [83] authors mentioned the number of automata is $3 * 10^{12}/\mu\ell$, and can perform a total of $6.6 * 10^{13}$ transitions per second) made of synthetic molecules and the task is to construct reset sequences, which is a synthetic DNA made of synthetic nucleotides, in order to be able to re-use the automata. Moreover, in [84] authors propose an automaton called MAYA, a molecular automaton that plays TIC-TAC-TOE against human opponent. Such an automaton, after a game ends, requires a reset word to bring the automaton to the "new game state". For a survey of automata based bio-computing, we direct the reader to [85]. In model based testing, the checking experiment construction requires a synchronizing sequence to bring the implementation to the specific state at which the designed test sequence is to be applied (e.g. see [26, 30, 31]).

On the other hand, for NSAs instead of resetting all the states in Q into a single state, one may consider restricted type of reset operations, such as resetting into a given set of states $F \subset Q$, or resetting a certain number $\mathcal{K}$ of states into F. A word $\bar{x} \in \Sigma^\star$ is called $\mathcal{K}/F$*–reducing word* for automata $A = (Q, \Sigma, \delta)$ if there exists a subset $\bar{Q}$ of states such that $\delta(\bar{Q}, \bar{x}) \subseteq F$ and $|\bar{Q}| = \mathcal{K}$. A word $\bar{x}$ is called Max/F*–reducing word* for automata $A = (Q, \Sigma, \delta)$ if $\bar{x}$ is a $\mathcal{K}/F$–reducing work for A and there does not exist $\bar{x}'$ and $\mathcal{K}' > \mathcal{K}$ such that $\bar{x}'$ is a $\mathcal{K}'/F$–reducing word for A. These problems are introduced in [86] and solved negatively.

3.1.1. Problems

Consider an FSM M such that $W : X \to \mathbb{Z}^+$ be a function assigning a cost to each input symbol of machine M and we have a budget $K \in \mathbb{Z}_{>0}$. Our aim is to extract subset of these inputs such that the total implementation of costs of these inputs are not higher than the budget and we can still construct a synchronizing sequence for the FSM M.

Suprisingly this problem is also find practical application in robotics. In the seminal work [79], Natarajan studied a practical problem of automated part orienting on an assembly line. He, having some assumptions, converted the parts orienting problem to the problem of constructing synchronizing sequences for deterministic finite automata as follows: He considered an assembly line on which parts to be process are dropped in

a random fashion. Therefore, the initial orientation of the parts are not known. The next station that will process the parts, however, requires that parts have a particular orientation. One can change the orientation of the parts on the assembly line by putting some obstacles, or by using tilted surfaces. The task is to find a sequence of such orienting steps such that no matter which orientation a part has at the beginning, it ends up in a certain orientation after it passes through these orienting steps. Natarajan modelled this problem as an automaton A as follows: he considers each orientation as a state and orienting functions as input alphabet such that the reset word of A corresponds to a sequence of orienting operations that brings these parts to unique orientation no matter which orientation it started at. Following Natarajans analogy, we considered an assembly line, a description of a part, and a set of tilt functions with implementation costs. Our aim is to extract a subset of these tilt functions such that the total implementation costs of these tilt functions are minimum and we can still rotate the part to a single orientation.

A similar problem might appear in bio–computing. As discussed in [80, 83, 84] in order to re-use automata one has to supply reset words (reset DNA's) which made of DNAs. As these DNA's made of commercially obtained synthetic deoxyoligonucleotides, it is sometimes possible, due to the lack of some nucleotides or due to the cost, for one to construct reset DNA's by the use of only a subset of nucleotides. That is, we want to find the cheapest set of synthetic deoxyoligonucleotides to construct a synchronizable subautomaton, knowing that we can construct reset DNA's using the cheapest (or available) synthetic deoxyoligonucleotides.

Now consider the automaton $A = (Q, \Sigma, \delta)$. Sub-automaton $A|_{\bar{\Sigma}}$ with respect to $\bar{\Sigma}$ is defined in the following way: $A|_{\bar{\Sigma}} = (Q, \bar{\Sigma}, \delta')$ where for two states $s, s' \in S$ and an input $x \in \bar{\Sigma}$, $\delta'(s, x) = s'$ if $\delta(s, x) = s'$. In other words, we simply keep the transitions with the inputs in $\bar{\Sigma}$ and delete the other transitions from A. If A is a CSA, then so is $A|_{\bar{\Sigma}}$. However, for if A is a PSA, we may have $A|_{\bar{\Sigma}}$ as a PSA or a CSA.

We first formalize the problem for CSAs as follows:

Definition 5 Minimum Synchronizable Sub-Automaton Problem *(MSS–Problem):*

Let $A = (Q, \Sigma, \delta)$ be a synchronizable CSA, $W : \Sigma \to \mathbb{Z}^+$ be a function assigning a cost to each input symbol, and $K \in \mathbb{Z}^+$. Find a sub-automaton $A|_{\bar{\Sigma}}$ such that $\bar{\Sigma} \subseteq \Sigma$ and $\sum_{x \in \bar{\Sigma}} W(x) \leq K$ and $A|_{\bar{\Sigma}}$ is synchronizable.

We show that the MSS–Problem is an NP-complete problem, implying that the minimization version of the MSS–Problem is NP-hard. We also show that the minimization version is hard to approximate.

Having determined the complexity of the MSS–Problem for CSAs, we consider the computational complexity of the MSS–Problem for PSAs. The primary motivation behind to study PSAs is obvious; finite automata with partial transition function is a generalization of completely specified finite automata; that is, partially specified automata can model a wider range of problems. The decision version of the MSS–Problem for PSA is defined as follows:

Definition 6 Minimum Synchronizable Sub-Automaton Problem for PSA*: Let $A = (Q, \Sigma, \delta)$ be a synchronizable PSA, $W : \Sigma \to \mathbb{Z}^+$ be a function assigning a cost to each input symbol, and $K \in \mathbb{Z}^+$. Find a sub-automaton $A|_{\bar{\Sigma}}$ such that $\bar{\Sigma} \subseteq \Sigma$ and $\sum_{x \in \bar{\Sigma}} W(x) \leq K$ and $A|_{\bar{\Sigma}}$ is synchronizable.*

We show that finding such partially specified sub automaton is PSPACE-complete.

Consider an FSM M such that taking M to a specified state is very expensive from a subset of state and we want to construct a synchronizing sequence that takes FSM to a specified state if and only if the current state of the FSM is not in this set. That is let $M = (S_1 \cup S_2, X, Y, \delta, \lambda)$ is given and our aim is to construct a synchronizing sequence $\bar{x}$ such that $\delta(s, \bar{x}) \in \bar{S}$ if and only if $s \in S_1$, where $\bar{S} \in S$.

This problem might also appear in robotics, consider the Natarjans analogy again. We are given an assembly line with a set of orienting functions and a set of parts. These parts have identical shapes but they are made of different materials. The set of initial positions of these parts are disjoint. Our aim is to find a sequence of tilt operations such that we can orient a given part to predefined position where parts with different types are guaranteed to be oriented at different positions. The problem is formally defined as follows:

Definition 7 Exclusive Synchronizing-Word *Problem for Synchronizable Automata (*ESW–SA*): Given a synchronizable automaton* $A = (Q, \Sigma, \delta)$ *and subsets of states* $\bar{Q} \subseteq Q$ *and* $F \subset Q$*. Is there a word* $\bar{x}$ *such that* $\delta(q, w) \subseteq F$ *if and only if* $q \in \bar{Q}$*?*

We show that although the underlying automaton is synchronizable this problem is PSPACE-complete and there exist a constant $\varepsilon > 0$ such that approximating the maximization version of the problem within ratio n^{ε} is PSPACE-hard.

In the second part of this work, we investigate the computational complexities of problems related to monotonic automata. In particular we consider Partially Specified Monotonic Automata (PSMA) and Non-Synchronizing Monotonic Automata (NSMA). In [87], Martyugin showed that constructing a reset word for a PSA is PSPACE-complete. Recall that there exist a complexity reduction for computing shortest synchronizing sequences when monotonic automata are considered [78, 34]. Hence it is natural to ask if we have a similar complexity reduction for computing a synchronizing sequence when we consider a *monotonic* PSA. However, until now no work revealed the complexity of computing a synchronizing sequence for a given PSMA.

Definition 8 Synchronizability Problem for PSMA*: Given a monotonic PSA* $A = (Q, \Sigma, \delta)$*, is* A *synchronizable ?*

Definition 9 Synchronizing Word Problem for PSMA*: Given a monotonic PSA* $A = (Q, \Sigma, \delta)$*, find a synchronizing sequence for* A*.*

Definition 10 Minimum Synchronizing Word Problem for PSMA*: Given a monotonic PSA* $A = (Q, \Sigma, \delta)$*, find a shortest synchronizing word for* A*.*

Unfortunately we show that these problems are at least as hard as NP-complete problems.

In [86] $\mathcal{K}/F$–reducing problem is introduced as follows: "Given a non-synchronizable automata A, is there a reset word that can reset K states into a set of states F?" and they proved that it is PSPACE-complete for the general automata. Again we investigate if monotonicity reduces the complexity of the original problem. The formal definition of the problem is given as follows:

Definition 11 $\mathcal{K}/F-$REDUCING-WORD *Problem for Non-Synchronizable Monotonic Automata (*KFW–NSMA*): Given a non-synchronizable monotonic automaton* $A = (Q, \Sigma, \delta)$*, a constant* $K \in \mathbb{Z}^+$*, and a subset of states* $F \subset Q$*, find a* $\mathcal{K}/F-$*reducing word for automaton* A*.*

We also study the maximization version of the problem.

Definition 12 MAX/F REDUCING-WORD *Problem for Non-Synchronizable Monotonic Automata (*MFW–NSMA*): Given a non-synchronizable monotonic automaton* $A = (Q, \Sigma, \delta)$ *and a subset of states* $F \subseteq Q$*, find a* Max/F*–reducing word for automaton* A*.*

Although the underlying automata is monotonic, we report that they are all NP-hard problems.

The rest of the chapter is organized as follows: In the next three sections we discuss and present our results related to MSS–Problem, ESW–SA problem and problems related to monotonic automata, respectively. In the last section we summarize the key results of this study and present some future directions.

3.2. Minimum Synchronizable Sub-Automaton Problem

We show that the MSS–Problem is computationally hard by reducing the SET COVER problem to the MSS–Problem.

In SET COVER problem, we are given a finite set of items $U = \{u_1, u_2, \ldots, u_m\}$ called the *Universal Set* and a finite set of set of items $C = \{c_1, c_2, \ldots, c_n\}$ where $\forall c \in C, c \subset U$. A subset C' of C is called a *cover* if $\cup_{c \in C'} = U$. The problem is to find a cover C' where $|C'|$ is minimized. The decision version of the SET COVER problem is NP-complete and its optimization version is NP-hard [88, 89].

From a given instance (U, C) of SET COVER problem we construct an automaton $\mathbb{F}(U, C) = (Q, \Sigma, \delta)$ as follows: for each item u in the universal set U we introduce a state q_u and we introduce another state $Sink$. For each set of items $c_i \in C$ we introduce an input symbol x_i. We construct the transition function of the automaton $\mathbb{F}(U, C)$ as follows:

- $\forall q_u \in Q \setminus \{Sink\}, \forall x_i \in \Sigma$

$$\delta(q_u, x_i) = \begin{cases} Sink, & u \in c_i \\ q_u, & \text{otherwise} \end{cases}$$

- $\forall x_i \in \Sigma$, $\delta(Sink, x_i) = Sink$

Lemma 5 *Let (U, C) be an instance of a* Set Cover *problem and $C' = \{c_1, c_2, \ldots, c_m\}$ be a cover. Then the sub-automaton $\mathbb{F}(U, C)|_{\bar{\Sigma}}$ is synchronizable, where $\bar{\Sigma} = \{x_i | c_i \in C'\}$.*

Lemma 6 *Let $\bar{\Sigma} = \{x_1, x_2, \ldots, x_m\}$ be a subset of alphabet of $\mathbb{F}(U, C)$ such that $\mathbb{F}(U, C)|_{\bar{\Sigma}}$ is synchronizable. Then $C' = \{c_1, c_2, \ldots, c_m\}$ is a cover.*

Hence we reach to the following result.

Theorem 1 *Given a synchronizable CSA $A = (Q, \Sigma, \delta)$ and a constant $K \in \mathbb{Z}^+$, it is* NP-complete *to decide if there exists a set $\bar{\Sigma} \subseteq \Sigma$ such that $|\bar{\Sigma}| < K$ and $A|_{\bar{\Sigma}}$ is synchronizable.*

Theorem 2 *MSS–Problem is* NP-complete.

In [90, 91] authors reported that the minimization version of the Set Cover problem cannot be approximated within a factor in $o(\log n)$ unless NP has quasipolynomial time algorithms. Moreover, it was also shown that Set Cover problem does not admit an $o(\log n)$ approximation under the weaker assumption that P $\neq$ NP [92, 93]. Therefore relying on the construction of the automaton $\mathbb{F}(U, C)$, it is also possible for us to deduce such inapproximability results apply to the MSS–Problem.

Lemma 7 *Let OPT_{sc} is the size of minimum cover for the* Set Cover *problem instance (U, C), and let $OPT_{\bar{\Sigma}}$ is the size of minimum cardinality input alphabet such that $\mathbb{F}(U, C)|_{\bar{\Sigma}}$ is synchronizable. Then $OPT_{sc} = OPT_{\bar{\Sigma}}$.*

Theorem 3 *MSS–Problem does not admit an $o(\log n)$ approximation algorithm unless* P = NP.

Although checking the existence and constructing one synchronizing sequence for a CSA are polynomial time solvable problems, we seen that the MSS–Problem is NP-complete for CSAs. That is, under the assumption that P $\neq$ NP there is a complexity jump. Recall that in [87] Pavel Martyugin showed that it is PSPACE-complete to construct a reset word for a PSA. Having these observations, it is natural to ask if there is a complexity jump when we consider the MSS–Problem for PSAs.

Before going any further please note that for a synchronizable PSA A, the sub-automaton $A|_{\bar{\Sigma}}$ can be a CSA. To see that this is the case, consider a synchronizable PSA with input alphabet $\{a, b, c\}$ such that only input b is not defined at some states. When we drop transitions that are labeled with input b, we obtain a completely specified automaton. Therefore since we showed, for CSAs, that the problem is NP-hard, we assume that, from now on, for any non-empty subset $\bar{\Sigma} \subset \Sigma$ the sub-automaton $A|_{\bar{\Sigma}}$ is a PSA if and only if A is a PSA.

Lemma 8 *MSS–Problem for PSA is in* PSPACE.

Now we are going to show that MSS–Problem for PSA is PSPACE-hard.

Lemma 9 *MSS–Problem for PSA is* PSPACE-hard.

Finally using Lemma 8 and Lemma 9 we reach to the following conclusion.

Theorem 4 *MSS–Problem for PSA is* PSPACE-complete.

3.3. Exclusive Synchronizing Word Problems

In [86], Igor K. Rystsov considers the problem of constructing a word that synchronizes a given set of states $\bar{Q} \subset Q$ at a given set of states F where the underlying automaton is non–synchronizable. The problem is named as Inclusion Problem for the Weakly Synchronizing Automaton. Rystsov showed that this problem is PSPACE-complete. However the problem we consider here is different for two reasons: (1) We consider synchronizable automata and (2) We prohibit resetting a state $q \in \hat{Q}$ at F.

We show that Exclusive Synchronizing Word (ESW–SA) problem is PSPACE-complete by a reduction from Finite Automata Intersection Problem(FA-INT), which was introduced by Dexter Kozen [94].

Definition 13 *Let $\mathbb{A} = \{A_1, A_2, \ldots, A_z\}$ be a set of deterministic finite automata with a common alphabet Σ. The FA-INT problem is to determine whether the given set of automata accept a common word in $\Sigma^\star$, i.e. whether there is a word $\bar{x}$ such that $w \in L(A_i)$ for all $1 \leq i \leq z$.*

In the same work Kozen proved that the FA-INT problem is PSPACE-complete. Given an instance of an FA-INT problem, with a set of automaton $\mathbb{A}$ having a common alphabet Σ, we construct a synchronizable automaton $\mathcal{A}$ such that we can find a solution $\bar{x}$ for the ESW–SA problem for automaton $\mathcal{A}$ if and only if we can find a word that is accepted by every automata in $\mathbb{A}$.

From each automaton, we pick initial state and form set $\bar{Q}$. We add $|F|$ number of new states and form set F. We introduce another state called $Sink$ state. We introduce transitions from accepting states of each automaton to a state $q \in F$ labeled by S and we introduce transitions from all the states of the automaton (except states of $\bar{Q}$) to $Sink$ state labeled by input R. Moreover, we introduce transitions from all the states of the automaton (except the accepting states of $\mathbb{A}$ and the states in F) to $Sink$ state labeled by input S. For each $q_i \in \bar{Q}$, we introduce self loop transitions labelled with input R. $Sink$ state loops with all inputs. We represent the reduction in Figure 3.1.

The intuition of the construction is as follows: note that the input sequence RS can reset the automaton at the $Sink$ state at any time. However, in order to reset initial states exclusively into F, we must avoid applying this input sequence. On the other hand, we must apply input S to reach to the set F. However, since S takes all non-accepting states to $Sink$ state all states in set $\bar{Q}$ must reach to accepting states at the time of application of S. Plus, in order to avoid resetting the set $\bar{Q} \cup F$ into F, we have to apply input R to reset F into the $Sink$ state. Now we show that the construction works.

Lemma 10 *ESW–SA problem is* PSPACE-hard.

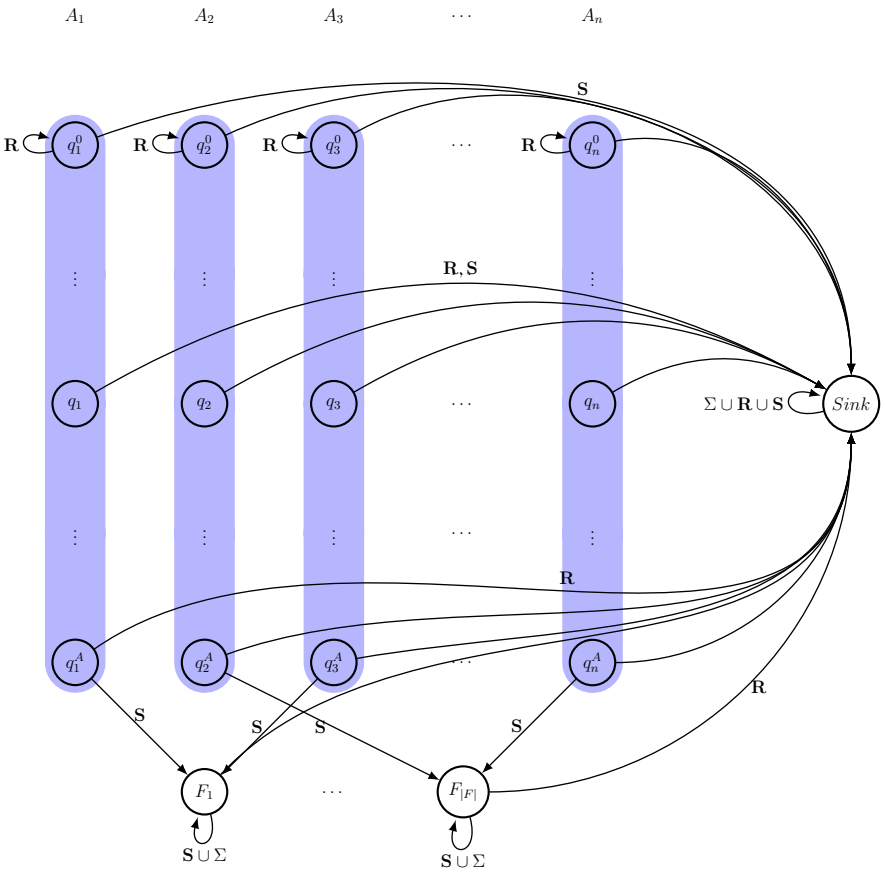


Figure 3.1.: Synchronizable Automaton $\mathcal{A}$ constructed from an FA-INT problem. States $q_1^0, q_2^0, q_3^0, \ldots, q_n^0$ form $\bar{Q}$

We will show that ESW–SA problem is in PSPACE after we make the following observations. Let us consider an arbitrary CSA $A = (Q, \Sigma, \delta)$ and sets $\bar{Q}, F \subset Q$. Let $\hat{Q} = Q \setminus \bar{Q}$, and $|\bar{Q}| = m, |F| = k, |Q| = n$. For an exclusive synchronizing word $\bar{x}$, let us denote the set of states reached by a prefix w' of $\bar{x}$ from $\bar{Q}$ and $\hat{Q}$ by using a pair $\pi = (\delta(\bar{Q}, w'), \delta(\hat{Q}, w'))$, which we call a *state configuration pair*. We must have $\delta(\bar{Q}, w') \cap \delta(\hat{Q}, w') = \emptyset$, since otherwise we can reset a state in $\hat{Q}$ at F. Note that there are $(2^m - 1) \times (2^{n-m} - 1) < 2^n$ different state configurations possible that can be reached by prefixes of exclusive synchronizing sequences. This indicates that $L = 2^n$ is an upper bound for a shortest exclusive synchronizing sequence.

Now we can propose a PSPACE algorithm for ESW–SA problem.

Lemma 11 *ESW–SA problem is in* PSPACE.

Consequently we have the following result.

Theorem 5 *ESW–SA problem is* PSPACE-complete.

We now consider the inapproximability of the ESW–SA problem. Condon et al introduce the maximization version of the FA–INT problem [95].

Definition 14 MAXIMIZATION OF FA-INT PROBLEM*: Let* $\mathbb{A} = \{A_1, A_2, \ldots, A_m\}$ *be a set of deterministic finite automata with input alphabet* Σ*.* MAXIMUM FINITE AUTOMATON INTERSECTION PROBLEM*(MAX FA-INT) problem is to find a subset* $\mathbb{A}'$ *of* $\mathbb{A}$ *such that automata in* $\mathbb{A}'$ *accept a common word* $w \in \Sigma^\star$*, and* $|\mathbb{A}'|$ *is maximized.*

However they proved that it is PSPACE-hard to approximate MAX FA-INT problem within a factor n^ε for $\varepsilon > 0$. As we reduce from the FA-INT problem, we can give a similar result for the maximization version of the ESW–SA problem.

Definition 15 *Max* EXCLUSIVE SYNCHRONIZING WORD FOR SYNCHRONIZABLE AUTOMATA *(Max* ESW–SA*): Given a synchronizable automaton* $A = (Q, \Sigma, \delta)$*, subsets* $\bar{Q}$ *and* F *of states* Q*, find a subset* $\bar{Q}' \subseteq \bar{Q}$ *where there exists an exclusive synchronizing word for* $\bar{Q}'$ *to* F*, such that* $|\bar{Q}'|$ *is maximized.*

Lemma 12 *Let* $\mathbb{A}'$ *be a subset of* $\mathbb{A}$ *where automata in* $\mathbb{A}'$ *has a common word and* $|\mathbb{A}'|$ *is maximized. Also let* $\bar{Q}'$ *be a subset of states* $\bar{Q}$ *of* $\mathcal{A}$*, where there exists an exclusive synchronizing word for* $\bar{Q}'$ *to* F *and* $|\bar{Q}'|$ *is maximized. Then* $|\bar{Q}'| = |\mathbb{A}'|$*.*

Therefore we reach to the following result.

Theorem 6 *There exists a constant* $\varepsilon > 0$ *such that approximating Max ESW–SA problem within ratio* n^ε *is* PSPACE-hard.

3.4. Synchronizing Monotonic Automata

In this section we consider problems related to monotonic automata. We first show that Synchronizability Problem for PSMA is an NP-hard problem by reducing the Exact Cover problem which is defined as follows:

Definition 16 Exact Cover Problem*: Let $U = \{1, 2, \ldots, m\}$ be the universal set and $C = \{c_1, c_2, \ldots, c_n\}$ is a finite set of set of items such that $\forall c \in C, c \subset U$. Is there a subset C' of C such that $\cup_{c \in C'} = U$ and for all $c, c' \in C'$ we have $c \cap c' = \emptyset$.*

Let us assume that we are given an instance of an Exact Cover problem (U, C). We now introduce a reduction $\mathbb{F}$ that maps a given Exact Cover instance (U, C) into a partially specified monotonic automaton $\mathbb{F}(U, C) = (Q, \Sigma, \delta)$.

We form the set Q as follows: for each $u \in U$ we introduce two states q_u^0 and q_u^1. We refer to states with superscript 1 (i.e. q_u^1) as the *satellite state* of u and we refer to states with superscript 0 (i.e. q_u^0) as the *base state* of u. The pair (q_u^0, q_u^1) is called as the *pair set* of u and denoted as Q_u. Finally we introduce a state S such that the state set Q is given as $Q = \{q_i^0 | i \in U\} \cup \{q_i^1 | i \in U\} \cup \{S\}$.

The input alphabet and the transition function of the automaton $\mathbb{F}(U, C)$ are given by:

- $\Sigma = \{x_1, x_2, \ldots, x_{|C|}\} \cup \{X, Y\}$
- $\forall q \in Q, \forall x \in \Sigma$

$$\delta(q, x) = \begin{cases} q_u^0, & \text{if } q = q_u^0 \text{ and } x = x_i \text{ where } u \notin c_i \\ q_u^1, & \text{if } q = q_u^0 \text{ and } x = x_i \text{ where } u \in c_i \\ q_u^1, & \text{if } q = q_u^1 \text{ and } x = x_i \text{ where } u \notin c_i \\ q_u^0, & \text{if } q \in \{q_u^1, q_u^0\} \text{ where } x = X \\ S, & \text{if } q = q_u^1 \text{ where } x = Y \\ q_1^0, & \text{if } q = S \text{ and } x = X \end{cases}$$

Clearly, any input sequence $w \in \Sigma^*$ must begin with input symbol X such that for any $q \in Q$, $\delta(q, X)$ is a base state. Moreover, the transition structure of the automaton

$\mathbb{F}(U, C)$ allows us to reset Q into state S by input symbol Y only. However the input Y is only defined at the satellite states. Therefore after the application of input symbol X and before the application of input symbol Y, we must apply some input sequence $w \in \Sigma^*$ such that for all states $q \in Q$ of $\mathbb{F}(U, C)$ we have that $\delta(q, w)$ is a satellite state. Moreover in order to take a base state q_u^0 to a satellite state q_u^1, we must apply an input symbol x_c if and only if $u \in c$. Besides since the input $x_{c'}$ is defined at a satellite state q_u^1 if and only if $u \notin c'$, input x_c can appear in $\bar{x}$ at most once.

We demonstrate the reduction with an example. Let (U_1, C_1) be given as $U_1 = \{1, 2, 3, 4, 5, 6\}$ and $C_1 = \{(1, 2, 5), (3, 4, 6), (1, 4, 2)\}$. The automaton $\mathbb{F}(U_1, C_1)$ obtained by the reduction is given in Figure 3.2:

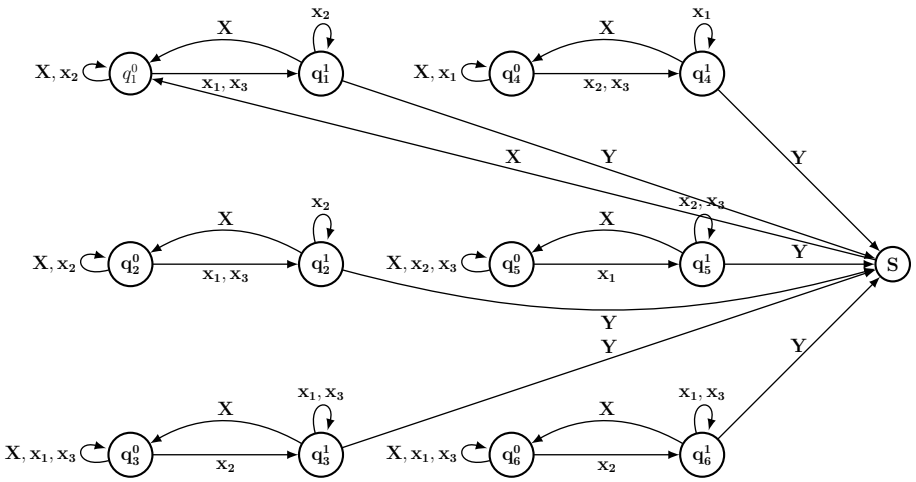


Figure 3.2.: Monotonic Partially Specified Automaton $\mathbb{F}(U_1, C_1)$ constructed from the Exact Cover instance $U_1 = \{1, 2, 3, 4, 5, 6\}$ and $C_1 = \{(1, 2, 5), (3, 4, 6), (1, 4, 2)\}$

The automaton is synchronizable (with word $w = X12Y$), which suggests that the corresponding Exact Cover problem instance has a solution ($C' = \{c_1, c_2\}$).

Now we show that the transition function of the automaton $\mathbb{F}(U, C)$ preserves some linear ordering $<$ of the states and so it is monotonic.

Lemma 13 *Let (U, C) be an arbitrary* Exact Cover *problem instance, then the states of the automaton $\mathbb{F}(U, C)$ admits a linear order for all input symbols in set Σ.*

We demonstrate the monotonicity of the automaton in Figure 3.3.

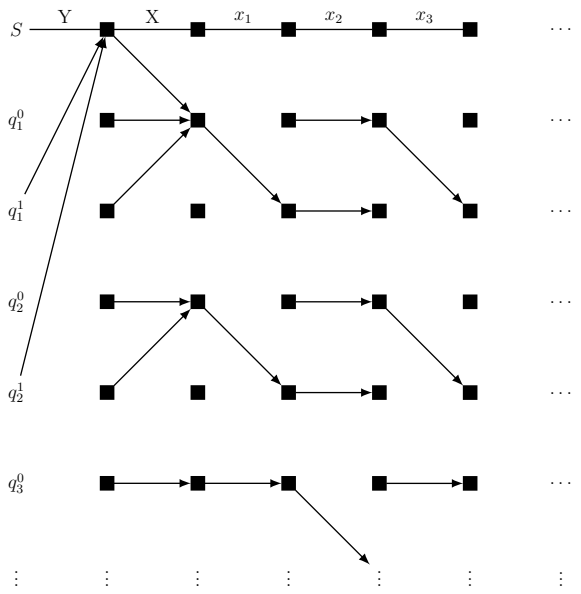


Figure 3.3.: Monotonicity of the automaton $\mathbb{F}(U_1, C_1)$ constructed from the Exact Cover problem instance (U_1, C_1).

Theorem 7 Synchronizability Problem for PSMA *is* NP-hard.

Since the existence check for a synchronizing sequence of a given PSMA is NP-hard, this implies the following results.

Theorem 8 Synchronizing Word Problem *and* Minimum Synchronizing Word Problem *for a PSMA are* NP-hard *problems.*

We now consider the problems related to NSMA. We will show the hardness of these problems using a reduction from the N–Queens puzzle. The N–Queens puzzle has been an important and interesting subject for many aspects of Mathematics and Computer Science. In particular, the N–Queens puzzle is often used as a benchmark for algorithms designed for AI research and combinatorial optimization. The N–Queens puzzle is known to be an NP-complete problem [96], since it is a slight generalization of the Exact Cover problem [97, 98].

Definition 17 N Queens Puzzle*: Given an integer N the* N–Queens *puzzle is to produce a placements of N number of Queens on an $N \times N$ chessboard such that no queen is under attack.*

In the N–Queens puzzle, we have an $N \times N$ chessboard, each cell being represented by a pair (i, j) of integers with $1 \leq i, j \leq N$. A cell is said to be *occupied* if there is a queen at that cell. A queen can move an unlimited distance up and down, left and right, and diagonally. Thus, a queen in cell (i, j) attacks another cell (n, m) (or a queen in that cell) if $i = n$, or $j = m$, or $|n - i| = |m - j|$.

A cell that is attacked by a queen is said to be a *dead cell*; otherwise it is called a *live cell*. A simple case is demonstrated in Figure 3.4.

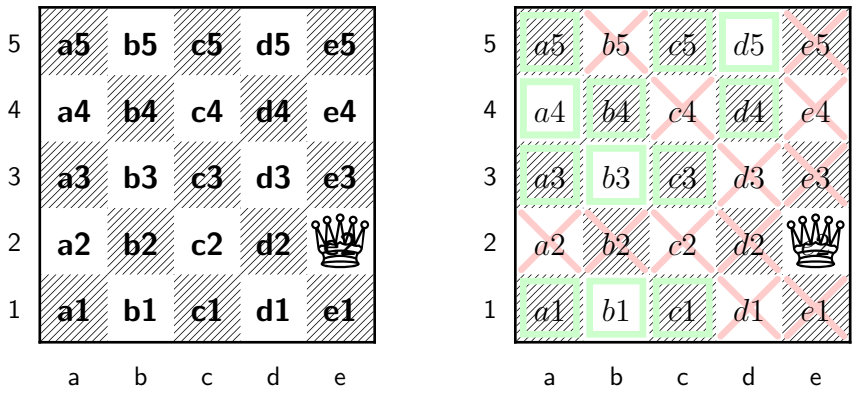


Figure 3.4.: A 5x5 Chessboard in which a queen is placed at board position $(e, 2)$ (left image). Chessboard places with red crosses are dead cells and chessboard places with green squares are live cells (right image).

For a given an instance B of an N–Queens puzzle, we construct a monotonic automaton $\mathbb{F}(B) = (Q, \Sigma, \delta)$ such that a solution to the N–Queens puzzle instance B constitutes a solution to the KFW–NSMA problem for the automaton $\mathbb{F}(B)$.

For each board position (i, j), we introduce three states $\{q^0_{i,j}, q^+_{i,j}, q^-_{i,j}\}$. We group states with $(0), (-)$ and $(+)$ superscripts as Q^0, Q^- and Q^+ respectively i.e. $Q = Q^0 \cup Q^- \cup Q^+$ where $Q^0 = \{q^0_{i,j} | 1 \leq i, j \leq N\}$, $Q^- = \{q^-_{i,j} | 1 \leq i, j \leq N\}$ and $Q^+ = \{q^+_{i,j} | 1 \leq i, j \leq N\}$.

From now on states in $Q^+ \cup Q^-$ are called the *satellite states* and states in set Q^0 are called the *board states*.

The input alphabet and the transition function of $\mathbb{F}(B)$ are given as follows:

- $\Sigma = \{x_{i,j} | 1 \leq i, j \leq N\}$
- $\forall q^0_{i,j} \in Q^0, \forall x_{k,l} \in \Sigma$

$$\delta(q^0_{i,j}, x_{k,l}) = \begin{cases} q^+_{i,j}, & i = k, j = l \\ q^-_{i,j}, & (i,j) \ attacks \ (k,l) \\ q^0_{i,j}, & else \end{cases}$$

  For a board state $q^0_{i,j}$, an input symbol $x_{k,l}$ is called an *attacking input* if (i, j) attacks (k, l).

- $\forall q \in Q^+ \cup Q^-, \forall x \in \Sigma$ we have $\delta(q, x) = q$

As a final, step we set K and the set F as follows: $K = N + N^2$, $F = Q^+$.

Before going any further, we first show that for any N–Queens puzzle instance B, automaton $\mathbb{F}(B)$ is monotonic.

Lemma 14 *Let B be an arbitrary* N–Queens *puzzle instance, the states Q of the automaton $\mathbb{F}(B)$ admits a linear order.*

In Figure 3.5, we show that the states of the automaton $\mathbb{F}(B)$ admits a linear order. Due to space limitations we present a small portion, however the reader can easily verify that the linear order is preserved for all input symbols at all the states of the automaton.

Let $w \in \Sigma^\star$ be a word that resets $N + N^2$ states of the automaton $\mathbb{F}(B)$ at F. For a decomposition $w = w' x_{i,j} \bar{x}''$ of $\bar{x}$, where $w', \bar{x}'' \in \Sigma^\star$, $x_{i,j}$ is called an *effective input* if $\delta(q^0_{i,j}, w') = q^0_{i,j}$.

Lemma 15 *Let $w \in \Sigma^\star$ be a reset word that resets $N + N^2$ states of the automaton $\mathbb{F}(B)$ at F. Then $\bar{x}$ defines a solution for the* N–Queens *puzzle instance B.*

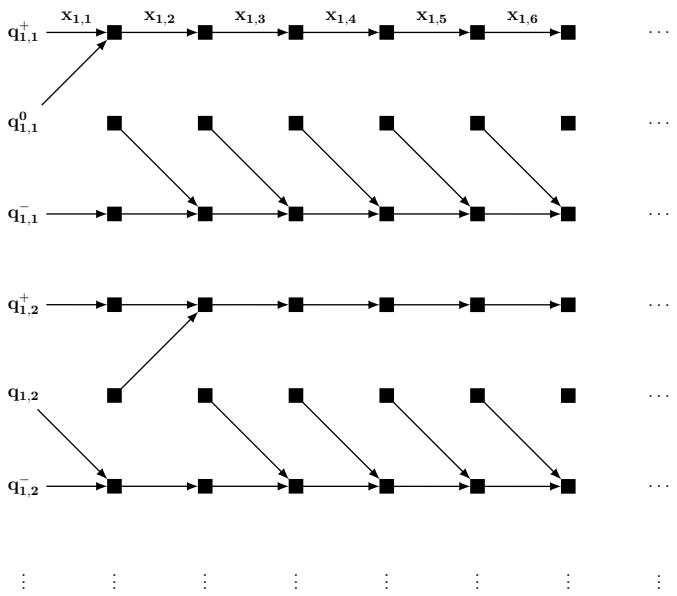


Figure 3.5.: Monotonicity of the automaton $\mathbb{F}(B)$ constructed from the N–Queens instance B.

Lemma 16 *Let* $\{(i_1, j_1), (i_2, j_2) \ , \ldots, (i_N, j_N)\}$ *be a solution to the* N–Queens *puzzle. Then set of positions* $(i_1, j_1), (i_2, j_2) \ , \ldots, (i_N, j_N)$ *defines a solution for the* KFW–NSMA *Problem for automaton* $\mathbb{F}(B)$.

Theorem 9 KFW–NSMA *Problem is* NP-hard.

Note that given an N–Queens problem instance B, for automaton $\mathbb{F}(B)$ and let $F = Q^{+}$. In this case, there is a word $\bar{x}$ that resets at most $N + N^2$ states at F, however finding such a word is NP-hard. Therefore MFW–NSMA is NP-hard.

Theorem 10 MFW–NSMA *Problem is* NP-hard.

3.5. Chapter Summary and Future Directions

In this chapter we introduced several problems. In the first problem (MSS problem), we assume that the input symbols of the underlying FSM (Automaton) have costs and our aim is to extract a subset of input alphabet such that one can construct a reset

word using transitions that are labeled by symbols from the reduced input alphabet and the cost of the reduced input alphabet is less than K. We show that if the transition function of the underlying automata is a total function, then the problem is NP-complete. Otherwise, if the transition function is partial then the problem is PSPACE-complete.

In the second problem we consider constructing an exclusive reset word (ESW-SA). By exclusive we mean that the reset word resets a set of states and is guaranteed not to reset states out of this set. We show that even if the underlying automaton is synchronizable the problem is PSPACE-complete and there exists a constant $\varepsilon > 0$ such that approximating the maximization version of the problem within ratio n^{ε} is PSPACE-hard.

Later we consider problems related to monotonic automata. We first consider monotonic automata with partial transition functions. Monotonicity is a feature that simplifies the complexity of the synchronizability problems. On the contrary, having a partial transition function makes the related synchronizability problems harder. We investigated the case when we have both of these features. We showed that checking the existence, computing one and computing a shortest synchronizing sequence are NP-hard problems.

Later we consider non-snychronizing monotonic automata. We showed that resetting $\mathcal{K}$ number of states (KFW–NSMA problem), or maximum number of states (MFW-NSMA problem) of a monotonic non-synchronizing automaton are NP-hard problems.

In line with these results we can propose some future works. Hardness results indicate that instead of exact algorithms we must design greedy algorithms for the problems introduced in this work. Consequently, designing and implementing such greedy algorithms for the MSS and the ESW–SA problems is one possible research direction. For monotonic partially specified machines, the problems investigated in this Chapter are shown to be NP-hard. It is also known that these problems are in PSPACE, since these are partially specified. Hence, for these problems we currently have a gap and it remains open to show whether these problems are in NP or PSPACE-hard. The upper bounds for the KFW–NSMA and MFW–NSMA problems remain open. Thus it would also be an interesting research direction to find upper bounds for these problems.

4. Hardness and Inaproximability of Minimizing Adaptive Distinguishing Sequences

4.1. Introduction

Usually, for a given specification M, a checking sequence is constructed and afterwards the checking sequence is applied to all implementations of M. Due to this "construct once and use many times" nature, one clearly desire a short checking sequence.

Until now, most of the efforts spent on to construct short checking sequences aim to overlap the contents of the checking sequences to reduce their lengths. In this Chapter, we consider a rather unique way of reducing the length of checking sequences. All ADS based checking sequence generation methods start with the assumption that an ADS is given. The given ADS is repeatedly applied within the checking sequence to identify the states and to verify the transitions. These ADS applications form a considerably large part of the checking sequence. Therefore we believe that reducing the size of ADSs is a reasonable way to reduce the length of the checking sequences.

Earlier ADS construction algorithms [22, 21, 23] are exhaustive and require exponential space and time. The only polynomial time algorithm was proposed by Lee and Yannakakis (*LY Algorithm*). It can check if M has an ADS in $O(pn \log n)$ time [35], and if one exists, we can construct an ADS in $O(pn^2)$ time [35]. Alur et al. show that checking the existence of an ADS for non-deterministic FSMs is EXPTIME-complete [46].

Recently, Kushik et al. present an algorithm (KEY algorithm) for constructing ADSs for non-deterministic observable FSMs [47]. We believe that the KEY algorithm can also construct ADSs for deterministic FSMs, since the class of deterministic FSMs is a sub-class of non-deterministic FSMs. However, since the KEY algorithm tends to construct all subset of states for the state set, it may require exponential time and space.

In summary, these ADS construction algorithms are not guaranteed to compute the minimum cost ADS for a given FSM. Moreover, to our knowledge, there is no work that analyses the computational complexity of constructing minimum cost ADSs.

In this Chapter, we investigate the complexity of constructing minimum sized ADS. We consider a number of different size definitions for an ADS, such as the height of the ADS (MinHeightADS problem), the total root–to–leaf path length over all the leaves of the ADS tree (MinADS problem), and the depth of a particular leaf node in the ADS tree (MinSDS problem). We show that for each one of these definitions, the minimization problem is hard to decide, and hard to approximate.

When the height of the ADS tree is considered as the size of an ADS, it was proven by Sokolovskii that if an FSM M with n states has an ADS, the shortest ADS for M cannot be longer than $\pi^2 n^2/12$ [99]. A known lower bound is $n(n-1)/2$ [35], i.e. there exist FSMs with a shortest ADS of length $n(n-1)/2$.

The LY algorithm can compute ADSs having the upper bound of $n(n-1)/2$ for the height of ADS and therefore matches this known lower bound. Thus, LY algorithm can actually generate the minimum ADS for those FSMs with shortest ADS of height $n(n-1)/2$. However, LY algorithm is not guaranteed to produce a minimum ADS in general. Although the use of a reduced ADS can be useful in several contexts (e.g. for constructing test sequences), except the exhaustive algorithms [22, 21, 23], there is no work in the literature on constructing reduced size ADS. As we emphasize, to our knowledge, there is no work that analyses the computational complexity of constructing minimum cost ADSs. As a matter of fact, in this work we show that this is a hard problem and polynomial time algorithms for constructing an ADS, may not generate minimum ADSs.

In order to validate our motivation that using minimized ADSs can yield shorter checking sequences, we first introduce two modifications on the LY algorithm to construct reduced size ADSs. Then we propose a Lookahead base algorithm to construct reduced size ADSs. We perform experiments on both randomly generated FSMs, and FSM specifications of sequential circuits obtained from industry. Experiments show that modifications on LY algorithm can construct better ADSs in terms of the size of the generated ADSs but requires extra computation time. Experiments also suggest that using reduced ADSs in fact gives rise to shorter checking sequences.

4.1.1. A Motivating Example

In the followings we try to demonstrate what we could gain by using a reduced size ADS in checking sequence construction on a concrete example. Several size definitions will be given for ADSs later in the Chapter. We will consider the total root–to–leaf path length over all leaves as the size of the ADS in this section. Let us consider the FSM M_1 given in Figure 4.1.

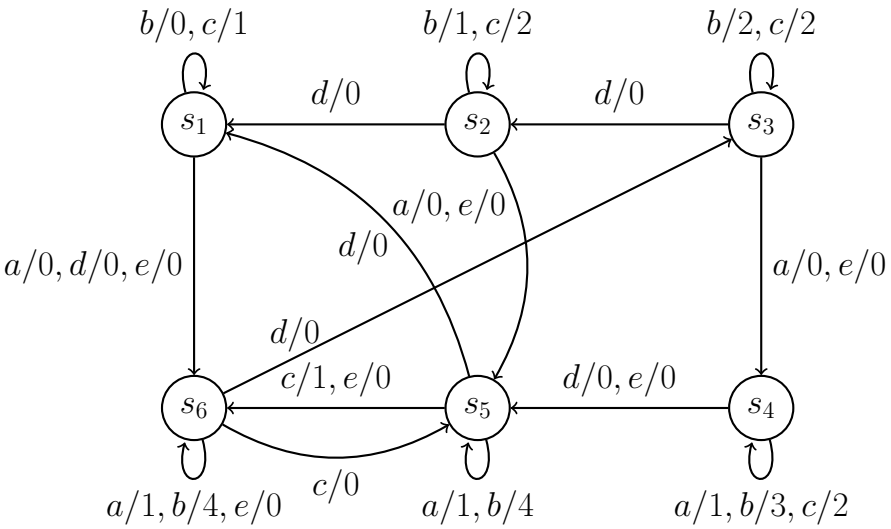


Figure 4.1.: An example FSM M_2

LY algorithm generates the ADS $\mathcal{A}_1$ given in Figure 4.2. The size of $\mathcal{A}_1$ is 16. However, by manual inspection, one can see that the ADS $\mathcal{A}_2$ of size 8, given in Figure 4.3 is also

an ADS for M_1. $\mathcal{A}_2$ is a minimum size ADS for M_1, since it is not possible to distinguish all the states of M_1 by using 1 input only.

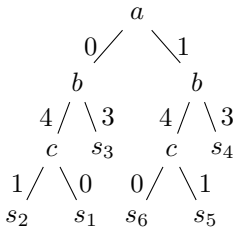


Figure 4.2.: An ADS $\mathcal{A}_1$ for M_2 of Figure 4.1 generated by LY algorithm

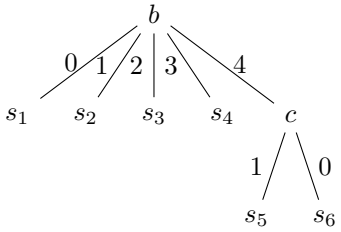


Figure 4.3.: A manually designed minimum size ADS $\mathcal{A}_2$ for M_1 of Figure 4.1

In order to see the effect of using $\mathcal{A}_1$ or $\mathcal{A}_2$ in checking sequence construction, we consider the following checking sequence construction methods: **HEN** method given in [23], **UWZ** method given in [30], **HIU** method given in [29], **SP** method given in [33], and **DY** method given in [54]. These methods span the history of distinguishing sequence based checking sequence generation methods, starting from the first (**HEN**) to the most recent ones (**SP** and **DY**), and also with some important improvements on the early versions (**UWZ** and **HIU**).

For an FSM M with n states, and an implementation N to be verified, a checking sequence is a test sequence that would identify any faulty implementation N of M. Although the methods constructing checking sequences differ in the way they form the final checking sequence, the components that need to exists in the sequence are the same.

In the checking sequence, there are repeated applications of distinguishing sequences to identify the current state of N. First, in order to see that each state s in M also exists in N, N is brought to a state that is supposed to correspond to s and a distinguishing sequence is applied. The application of the distinguishing sequence for this purpose is called a *state verification*.

Second, in order to see that every transition in M from a state s to state s' with input symbol x and output symbol y, N is brought to the state that corresponds to s and

the input symbol x is applied (with the hope that y will be observed). In order to see that N made a transition into the state that corresponds to s', another application of distinguishing sequence takes place. The application of the input symbol x followed by the application of the distinguishing sequence is called *transition verification*.

Therefore, for a completely specified FSM M with n states and input alphabet X, there will be n state verification and $n|X|$ transition verification components, each one having an application of the distinguishing sequence.

The state verification and transition verification components can be performed in any order, and they are combined by using appropriate *transfer sequences*, which is an input sequence that transfers M from one state to another state. The methods **HEN**, **UWZ**, **HIU**, **SP**, and **DY** mainly differ in what order they consider state verification and transition verification components, and how they select the transfer sequences to put these components together. The earliest method we consider, **HEN**, imposes a prior ordering of these components, and then finds the necessary transfer sequences to combine the components. The later improvements on **HEN** (namely **UWZ** and **HIU**) let the components be combined in any order (without fixing a prior ordering), hence they are able to use shorter transfer sequences, using empty transfer sequences whenever possible. Therefore, the total length of transfer sequences used in **UWZ** and **HIU** is shorter compared to **HEN**. The most recent methods (**SP** and **DY**) even consider overlapping these components, hence the transfer sequence lengths become even shorter. For more details on these methods, we direct reader to the references [23, 30, 29, 33, 54].

Table 4.1 gives the length of checking sequences when $\mathcal{A}_1$ and $\mathcal{A}_2$ are used with these methods.

The results on this single example are promising and show that reducing the height of ADS does provide an opportunity to reduce the length of checking sequences. Of course, an extensive set of experiments would establish a more convincing evidence in this direction. We present the results of such experiments later in Section 4.5. We will first focus on the question of whether it is possible to find minimum ADSs efficiently.

In order to show the hardness of constructing a minimum ADS, we will use reduc-

	$\mathcal{A}_1$	$\mathcal{A}_2$	Reduction (%)
HEN	299	224	25%
UWZ	176	96	45%
HIU	158	92	41%
SP	149	111	25%
DY	105	83	20%

Table 4.1.: Comparison of checking sequence lengths for FSM M_1

tions from known NP-complete problems. These problems are related to the binary identification problem [100]. In next section we introduce this problem and review existing hardness results related to binary decision trees used in the context of the binary identification problem.

4.2. Binary Decision Trees

Let $Z = \{z_1, z_2, \ldots, z_n\}$ be a finite set of distinct objects and $T = \{t_1, t_2, \ldots, t_m\}$ be a finite set of tests where each test $t \in T$ is a function $t : Z \to \{0, 1\}$. Intuitively, when a test t is applied to an object z, the object z produces the response $t(z)$, i.e. either a 0 or a 1 is obtained as an answer.

The set of objects $Z = \{z_1, z_2, \ldots, z_n\}$ and the set of tests $T = \{t_1, t_2, \ldots, t_m\}$ can also be presented as a table $D[T, Z]$ (which we will call a *decision table*) with m rows and n columns where the rows are indexed by the tests and the columns are indexed by the objects. An element $D[t, z]$ is set to the value $t(z)$. Table 4.2 is an example of such a decision table where there are 4 objects and 3 tests. A row corresponds to a test t and it gives the vector of responses of the objects to t. Similarly, a column corresponds to an object z and it gives the vector of responses of z to the tests. For a test t and an object z, we will use the notation $D[t, .]$ and $D[., z]$ to refer to the row of $D[T, Z]$ corresponding to the test t and the column of $D[T, Z]$ corresponding to the object z, respectively.

D	z_1	z_2	z_3	z_4
t_1	0	1	1	0
t_2	1	0	1	0
t_3	1	0	1	1

Table 4.2.: An example decision table

Suppose that we are given a decision table $D[T, Z]$ and an unknown object from Z, and we are asked to identify this object. One can apply all the tests in T and the results of the tests will be corresponding to a unique column of the table identifying the unknown object, provided that for all objects $z \in Z$, $D[., z]$ is unique, which we will assume throughout the work (as otherwise such an identification is not possible). Note that if for a test t, $D[t, .]$ is a row where every elements is 0 (or every element is 1), this means t does not distinguish between any objects, hence the test t is useless for the identification of the unknown object. Therefore, we assume that there is no such useless test t in T. We can in fact find such useless tests and eliminate them in polynomial time, by performing a single pass over the table $D[T, Z]$. Also note that if there are two different tests t and t' such that $D[t, .]$ and $D[t', .]$ are the same, then the information provided by t and t' are the same, hence they are duplicate tests. We will also assume that no such duplicate tests exist, as otherwise we can find and eliminate them in polynomial time as well, by checking the equality of every pair of rows of $D[T, Z]$.

Identifying an unknown object of Z by using tests in T can also be performed adaptively. In this case the procedure to be applied can be described in the form of a (binary) decision tree $\mathcal{T}$ having the following properties.

Definition 18 *A* decision tree *for a decision table $D[T, Z]$ where $Z = \{z_1, z_2, \ldots, z_n\}$ and $T = \{t_1, t_2, \ldots, t_m\}$ is a rooted tree $\mathcal{T}$ with n leaves such that:*

1. *Each leaf of $\mathcal{T}$ is labeled by a distinct object $z \in Z$.*

2. *Each internal node of $\mathcal{T}$ is labeled by a test $t \in T$.*

3. *Each internal node has two outgoing edges, one with label 0 and the other with label 1.*

4. *Consider a path from the root to a leaf node p labeled by an object z. Let q be an internal node on this path and t be the test labeling the node q. If p is under the 0–successor of q then $t(z) = 0$, and if p is under the 1–successor of q then $t(z) = 1$.*

Figure 4.4 and Figure 4.5 present two different decision trees for the decision table given in Table 4.2. The identification procedure based on a given decision tree $\mathcal{T}$ proceeds as follows: If r is the root node of $\mathcal{T}$, we start by applying the test r_v (the test labeling r). If the outcome is 0, then the subtree rooted at the 0–successor of r is considered, otherwise (when the outcome is 1) the subtree rooted at the 1–successor of r is considered. The procedure is repeated recursively for the root of each subtree visited, until a leaf is reached. When a leaf node p is reached, the object labeling p gives the unknown object.

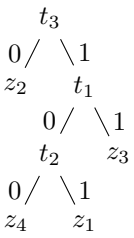


Figure 4.4.: A decision tree for the decision table of Table 4.2

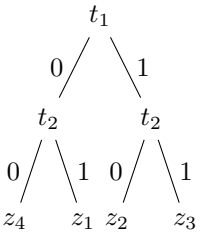


Figure 4.5.: Another decision tree for the decision table of Table 4.2

The following is immediately follows from definitions.

Lemma 17 *Let $D[T, Z]$ be a decision table, $\mathcal{T}$ be a decision tree for $D[T, Z]$, and p be a leaf node in $\mathcal{T}$. If $\bar{p}_v = t_{i_1} t_{i_2} \dots t_{i_k}$ and $\bar{p}_e = y_1 y_2 \dots y_k$, then for any $1 \leq j \leq k$, $t_{i_j}(p_v) = y_j$.*

Note that it is always possible to find such a decision tree thanks to the assumption that $D[., z]$ is unique for all $z \in Z$. There can be more than one decision tree for a

given decision table, e.g. Figure 4.4 and Figure 4.5 are two different decision trees for the decision table given in Table 4.2. Since the identification procedure for the unknown object is directly based on the decision tree used, the cost of the procedure depends on the decision tree. One may want to minimize this effort by using an appropriate decision tree. However, there can be different measures that can be used to describe the effort. Given a decision tree $\mathcal{T}$ and an object $z \in Z$, let $d_{\mathcal{T}}(z)$ be the depth of the leaf node labeled by z in $\mathcal{T}$. For the decision tree in Figure 4.4, we have $d_{\mathcal{T}}(z_1) = 3$, $d_{\mathcal{T}}(z_2) = 1$, $d_{\mathcal{T}}(z_3) = 2$, and $d_{\mathcal{T}}(z_4) = 3$. One measure can be the expected number of tests to be applied, which corresponds to minimizing the sum $\sum_{z \in Z} d_{\mathcal{T}}(z)$ assuming each object is equiprobable. Another measure can be the depth of the decision tree $\mathcal{T}$ in order to minimize the worst case behaviour of the identification procedure based on $\mathcal{T}$. The following definitions state these problems formally.

Definition 19 MinDT *problem: Given a decision table $D[T, Z]$, find a decision tree $\mathcal{T}$ such that $\sum_{z \in Z} d_{\mathcal{T}}(z)$ is minimized.*

Definition 20 MinHeightDT *problem: Given a decision table $D[T, Z]$, find a decision tree $\mathcal{T}$ such that $\max\{d_{\mathcal{T}}(z) | z \in Z\}$ is minimized.*

Another measure can be motivated as follows. Suppose that the objects are diagnoses in a medical emergency room where some binary tests are applied to reach a diagnosis. The tests all take the same amount of time, however one of the diagnosis is more important than the others, since it requires a much more urgent action to be taken. In such a case, the situation can be modeled as a binary identification problem, where one would like to find a decision tree whose root–to–leaf path corresponding to this urgent diagnosis is minimized. Definition 21 states the problem formally.

Definition 21 MinPathDT *problem: Given a decision table $D[T, Z]$ and an object $z \in Z$, find a decision tree $\mathcal{T}$ such that $d_{\mathcal{T}}(z)$ is minimized.*

Decision version of the problems MinDT, MinHeightDT, and MinPathDT are NP-complete [100, 101, 102]. Besides they are also known to be hard to approxi-

mate [103, 104, 105, 102]. In the following sections, we introduce analogous definitions as minimization metrics for ADSs.

4.3. Minimizing Adaptive Distinguishing Sequences

Given an ADS $\mathcal{A}$ and a state s, let $d_{\mathcal{A}}(s)$ be the depth of the leaf node labeled by the state s in $\mathcal{A}$. One measure that we can use to minimize ADSs can be the expected number of inputs to be applied, which corresponds to minimizing the sum $\sum_{s \in S} d_{\mathcal{A}}(s)$ assuming each state is equiprobable. Another measure can be the depth of the ADS $\mathcal{A}$ in order to minimize the worst case behaviour of the state identification based on $\mathcal{A}$. The following definitions state these problems formally.

Definition 22 MinADS *problem: Given an FSM M, find an ADS $\mathcal{A}$ for M such that $\sum_{s \in S} d_{\mathcal{A}}(s)$ is minimized.*

Definition 23 MinHeightADS *problem: Given an FSM M, find an ADS $\mathcal{A}$ for M such that $\max\{d_{\mathcal{A}}(s) | s \in S\}$ is minimized.*

For an ADS $\mathcal{A}$, let p be a leaf node labeled by a state s and let $\bar{\alpha}_s = \bar{p}_v$. The input sequence $\bar{\alpha}_s$ can be used to check whether an unknown state of M is s or not, since for any state $s' \neq s$, $\lambda(s, \bar{\alpha}_s) \neq \lambda(s', \bar{\alpha}_s)$ by Lemma 2. We call the sequence $\bar{\alpha}_s$ a *state distinguishing sequence (SDS) for s*. In order to find an ADS in which an SDS for a state s is minimized, one may want to solve the following problem.

Definition 24 MinSDS *problem: Given an FSM M and a state s of M, find an ADS $\mathcal{A}$ for M such that $d_{\mathcal{A}}(s)$ is minimized.*

It is also possible to consider a more general problem with the observation that, each SDS given by an ADS may be used different number of times. As an illustration, consider Figures 4.6, 4.7 and 4.8. There are thirteen incoming transitions of state s_1, and for each incoming transition, the SDS of s_1 will be applied in a checking sequence (see the discussion on transition verifications given in Section 4.1.1). Although the ADS

given in Figure 4.7 is better with respect to the MinADS problem, the length of the SDS for the state s_1 in this ADS is longer. Therefore, the ADS given in Figure 4.8 may be preferred. The following problem definition is motivated by such uses of an ADS.

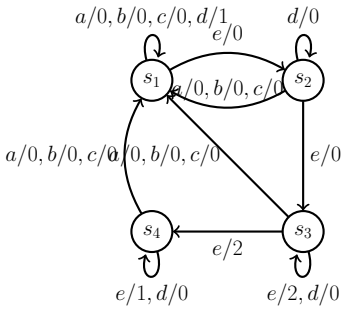


Figure 4.6.: An example FSM M_3.

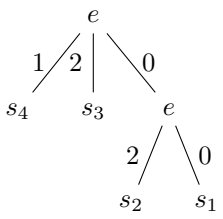


Figure 4.7.: An ADS $\mathcal{A}_1$ for M_3 of Figure 4.6.

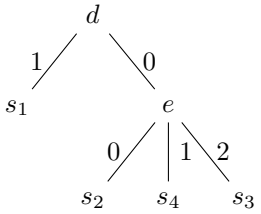


Figure 4.8.: Another ADS $\mathcal{A}_2$ for M_3 of Figure 4.6.

Definition 25 MinWeightedADS *problem: Given an FSM M, let s be a state in M and w_s be a constant representing the weight for the state s. Find an ADS $\mathcal{A}$ for M such that $\sum_{s \in S}(w_s d_{\mathcal{A}}(s))$ is minimized.*

4.4. Modeling a Decision Table as a Finite State Machine

The similarity between a decision table and an FSM is in fact clear. Objects and tests in the given decision table correspond to the states and the inputs of the FSM that will be generated. Hence applying a test will be corresponding to applying an input, and the output to be produced in this case will be the response of the object to the test. However, one major difference is the fact that objects do not change as the tests are applied whereas states can perform a transition into another state when an input is applied. In order to have the direct correspondence between the objects and the states, we will have self looping transitions with the inputs corresponding to the tests. This would yield an FSM where each state is isolated. We will add extra input symbols, one

for each state, to make the generated FSM strongly connected. The transitions with these extra inputs will be defined in such a way that they cannot play any role in forming an ADS. We now give the formal definition of the mapping of a decision table into an FSM.

4.4.1. Mapping

We call the mapping function β which is formally defined below.

Mapping β: Given a decision table $D[T, Z]$, we construct an FSM $M_D = (S, X, Y, \delta, \lambda)$ where

1. $S = \{s_z | z \in Z\}$
2. $X = X_T \cup X_Z$ where $X_T = \{x_t | t \in T\}$ and $X_Z = \{x_z | z \in Z\}$ (we assume that $T \cap Z = \emptyset$)
3. $Y = \{0, 1, 2\}$
4. For a state $s_z \in S$ and an input $x_t \in X_T$, $\delta(s_z, x_t) = s_z$ and $\lambda(s_z, x_t) = t(z)$
5. For a state $s_z \in S$ and an input $x_{z'} \in X_Z$, $\delta(s_z, x_{z'}) = s_{z'}$ and $\lambda(s_z, x_{z'}) = 2$

As an example, for the decision table given in Table 4.2, the corresponding FSM is given in Figure 4.9. The bold solid edges indicate the transitions for the inputs in X_T, and the dashed edges indicate the transitions for the inputs in X_Z.

Below, we assume that we are given a decision table $D[T, Z]$ (which we will simply refer as D) where $T = \{t_1, t_2, \ldots, t_m\}$ and $Z = \{z_1, z_2, \ldots, z_n\}$ and we assume that M_D is the FSM generated by using the mapping β given above. M_D has n states and $n(n + m)$ transitions.

4.4.2. Hardness and Inapproximability Results

An FSM M_D generated by the mapping β from a decision table D has two types of input symbols, X_T and X_Z. We now prove that no ADS of M_D would be able to use inputs in X_Z.

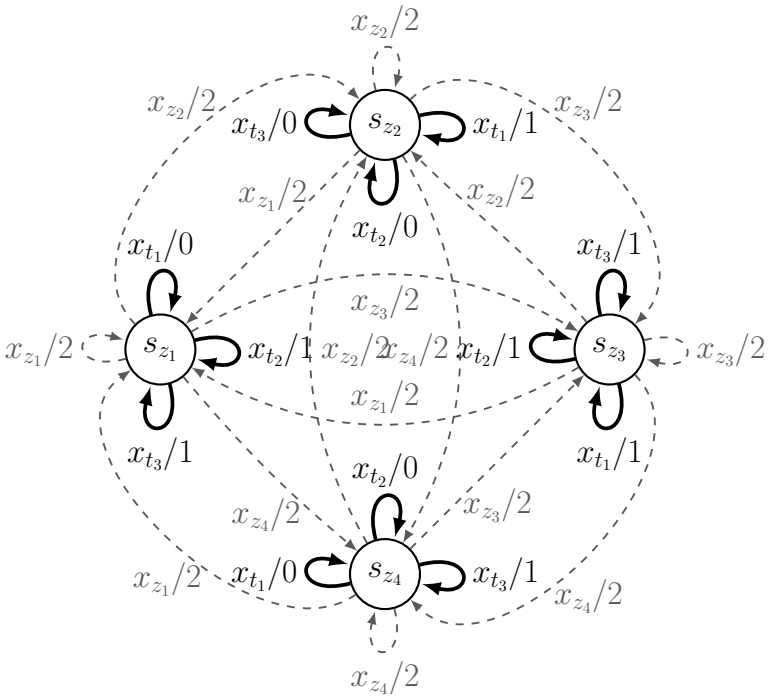


Figure 4.9.: The FSM M_D corresponding to the decision table given in Table 4.2

Lemma 18 *An input $x_z \in X_Z$ cannot appear as the label of an internal node p in an ADS $\mathcal{A}$ of M_D.*

Note that an ADS of M_D is not necessarily always branching. Figure 4.10 demonstrates such an ADS for FSM M_D of Figure 4.9. In an ADS of M_D, the internal nodes can only be labeled by the inputs in X_T and these inputs cannot change the states of M_D (Condition (4) of mapping β). Therefore, a subtree in M_D whose root node p has only one child, can safely be replaced by the subtree rooted at the only child of p. Figure 4.10 and Figure 4.11 depict an example of such a subtree replacement in an ADS which is not always branching. For this reason, without loss of generality, we assume that an ADS given for M_D is always branching.

When one considers only those ADSs of M_D that are always branching, an implication of Lemma 18 is that, there is a one–to–one correspondence between the decision trees of D and such ADSs of M_D. Since the input symbols in X_T does not change the state of M_D, the application of an input x_t in M_D to an unknown state s_z of M_D, is effectively

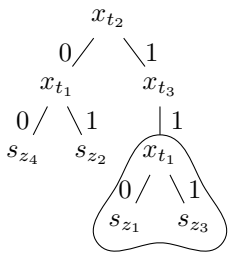


Figure 4.10.: An ADS for M_D given in Figure 4.9, which is not always branching

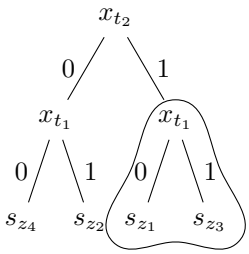


Figure 4.11.: An always branching ADS constructed from the ADS given in Figure 4.10

the same as the application of the test t to an unknown object z of D, and vice versa. Formally, we have the following results.

Lemma 19 *Given a decision tree $\mathcal{T}$ for D, there exists an isomorphic ADS $\mathcal{A}$ for M_D.*

Lemma 20 *Given an ADS $\mathcal{A}$ for M_D, there exists an isomorphic decision tree $\mathcal{T}$ for D.*

Due to this one–to–one correspondence established by Lemma 19 and Lemma 20, the problems MinADS, MinHeightADS, and MinSDS are at least as hard as the MinDT, MinHeightDT, and MinPathDT problems, respectively. Therefore, the hardness and inapproximability results existing for the problems MinDT, MinHeightDT, and MinPathDT in the literature are inherited by the problems MinADS, MinHeightADS, and MinSDS as stated by the following claims.

Theorem 11 *The decision version of the problems* MinADS, MinHeightADS, *and* MinSDS *are* NP-complete.

MinADS is a special case of MinWeightedADS (when $w_s = 1$ for all states s). Therefore the following result holds.

Theorem 12 *The decision version of the problem* MinWeightedADS *is* NP-complete.

Similarly, the inapproximability results existing in the literature for decision trees are inherited by ADS minimization.

Theorem 13 *For any constant $c > 0$, it is* NP-hard *to approximate* MinADS *and* MinWeightedADS *problems within a ratio of $(2 - c)$.*

Theorem 14 *Unless* P = NP*, there cannot be an $o(\log n)$ approximation for* MinHeightADS *and* MinSDS *problems.*

4.5. Experiment Results

As shown above, minimizing ADSs with respect to the metrics are hard. Therefore, the best we can hope to do at this point is to design heuristic algorithms.

In this section, we first briefly describe LY algorithm and explain three modified versions (GLY1, GLY2 and RLY) of LY algorithm. We compare these modifications both with respect to the size of ADSs they produce, and with respect to the length of checking sequences constructed by using these ADSs.

The experiments were carried out on a computer with an Intel Quad-Core CPU and 4GB RAM.

4.5.1. LY Algorithm

In this section, we briefly describe how LY algorithm given in [35] constructs an ADS for a given FSM $M = (S, X, Y, \delta, \lambda)$.

An input sequence $\bar{x} \in X^\star$ is said to be a *splitting sequence* for a set $S' \subseteq S$ of states, if $|\lambda(S', \bar{x})| > 1$, and for any $\bar{x}', \bar{x}'' \in X^\star$, $x \in X$ such that $\bar{x} = \bar{x}' x \bar{x}''$, x is a valid input for $\delta(S', \bar{x}')$. An input symbol $x \in X$ is called a *valid input* for a set of states $\bar{S} \subseteq S$, if the following holds: $\forall s_i, s_j \in \bar{S}, s_i \neq s_j,\ \delta(s_i, x) = \delta(s_j, x) \Rightarrow \lambda(s_i, x) \neq \lambda(s_j, x)$. Intuitively, $\bar{x}$ is an input sequence such that at least two states in S' produce different output sequences for $\bar{x}$, and no two states in S' are merged without distinguishing them.

In other words, $\bar{x}$ splits S', hence the name. We call an input symbol x a *splitting input* for S', if x is a splitting sequence of length one for S'.

LY algorithm constructs an ADS tree in two steps as explained in the following sections.

Forming a Splitting Tree

In the first step, LY algorithm constructs a tree called the *splitting tree* (ST). An ST is a rooted tree where every node is associated with a set of states called a *block* ($B \subseteq S$) and an input sequence ($\bar{x} \in X^\star$). The leaves are labeled by singleton blocks and the empty input sequence. The block of the root node is set to S. For an internal node p labeled by a block B and an input sequence $\bar{x}$, $\bar{x}$ is a splitting sequence for B. There is a child node p' of p for each $\bar{y} \in \lambda(B, \bar{x})$. The block of p' is set to be the block $B_{\bar{x}/\bar{y}}$. Therefore, the blocks of the children of an internal node p with block B is a partitioning of B.

The construction of an ST starts by creating a partial ST with only the root node with block $B = S$. This partial ST is processed iteratively, until all the leaves become nodes with singleton blocks. In each iteration, a leaf node p in the partial ST is chosen, where p has a block B such that $|B| > 1$. The algorithm finds a splitting sequence $\bar{x}$ for B. The input sequence label of p is set to be $\bar{x}$. The children of p are created as explained in the previous paragraph.

The block of a node p is set when the node p is created. However, the input sequence labeling a node is set when the algorithm processes the node p.

For our purposes, it is important to explain how a splitting sequence is found for a block B. LY algorithm, first attempts to find a splitting input by considering every input symbol $x \in X$. LY algorithm does not specify any specific order on the input symbols to be considered for this check. Therefore, a typical implementation of LY algorithm would use some fixed (possibly lexicographical) ordering of the input symbols for this check. This is exactly where our modifications on LY algorithm take place. We give the details of these suggested modifications in Section 4.5.2.

The partial ST can be expanded in this way as long as we can find leaf nodes for which there exists a splitting input. When there is no such leaf node, the algorithm finds a leaf node p in the partial ST with a block B, an input symbol x that is valid for B, an internal node p' with block B' such that $\delta(B, x) \subseteq B'$, and none of the children p'' of p' has a block B'' where $\delta(B, x) \subseteq B''$. The existence of such nodes p, p' and the valid input x are guaranteed when an ADS exists[1]. Since p' is an internal node, it has been processed by the algorithm before, and therefore there is a splitting sequence $\bar{x}$ labeling the node p'. The splitting sequence to be used for p is then obtained as $x\bar{x}$.

Forming an ADS

After the ST is constructed, LY algorithm uses the ST to construct a tree T that defines an ADS. The tree T has a similar structure and construction with an ST. The difference is that, in ST the initial states are considered, whereas in the construction of T, the final states reached are considered.

Each node p of T is associated with an input sequence $\mathcal{X}(p)$ and with a set of states $\mathcal{B}(p)$. The edges of T are labeled by output sequences. For a leaf node p of T, we have $|\mathcal{B}(p)| = 1$ and $\mathcal{X}(p) = \varepsilon$. For an internal node p, $|\mathcal{B}(p)| > 1$.

Let p be an internal node in T with $\mathcal{X}(p) = \bar{x}$ and $\mathcal{B}(p) = B$. Also let p' be a child of p, where the edge from p to p' is labeled by an output sequence $\bar{y}$. In this case, $\mathcal{B}(p') = \delta(B_{\bar{x}/\bar{y}}, \bar{x})$. Note that unlike ST, the states labeling the children of a node p do not necessarily form a partition of $\mathcal{B}(p)$.

The construction of T is performed iteratively. First a partial tree is created which only includes the root node p of T, with $\mathcal{B}(p) = S$. As long as there is a leaf node p where $\mathcal{B}(p) = B$ with $|B| > 1$ in the partial tree, p is processed in the following way. The ST is consulted to find the deepest node p' in ST such that the block B' of p' in ST includes B, i.e. $B \subseteq B'$. Let $\bar{x}$ be the splitting sequence labeling p' in ST. Then, $\mathcal{X}(p)$ is set to $\bar{x}$ and for each $\bar{y} \in \lambda(B, \bar{x})$, a child p'' of p is created in T, by setting

[1] Since the proof of correctness of LY algorithm is out of scope of this Thesis, we refer the reader to [35] to see why such nodes have to exist in the partial ST.

$\mathcal{B}(p'') = \delta(B_{\bar{x}/\bar{y}}, \bar{x})$.

Once T becomes a tree where for all the leaves p we have $|\mathcal{B}(p)| = 1$, T defines an ADS tree. We refer the reader to [35] to see why T defines an ADS.

4.5.2. Modifications on LY algorithm

We present three modified versions of LY algorithm GLY1, GLY2 and RLY. Modifications take place only in the construction of a splitting tree. As explained in Section 4.5.1, LY algorithm first tries to find a splitting input for a block B. Rather than being satisfied by the existence of such an input sequence, we identify all splitting inputs for the block B and select the one that "seems" to be the best choice.

The motivation behind the approach GLY1 comes primarily from the *balanced tree* strategy [106]. In balanced trees, it is guaranteed that the difference between the sizes of the sub-trees rooted at sibling nodes is less then some threshold to keep the height of the tree relatively small. This is in fact the same heuristic used for approximating optimal decision trees [107]. Similarly, in GLY1, for a given block B, we select a splitting input symbol that partitions B most evenly.

Let B be a block and $\mathcal{X}$ be a set of splitting sequences for B. For a splitting sequence $\bar{x}$ for B, $B_{\bar{x}}$ refers to the partitioning of B with respect to $\bar{x}$, i.e. $B_{\bar{x}} = \{B_{\bar{x}/\bar{y}} | \bar{y} \in \lambda(B, \bar{x})\}$. Among the elements of $\mathcal{X}$, we would like to pick the one that would give the most balanced partitioning of B. For this reason, we introduce the following function F that considers the differences in the cardinalities of the partitioning provided by the available splitting sequences:

$$F(B, \mathcal{X}) = \underset{\bar{x} \in \mathcal{X}}{\operatorname{argmin}} \sum_{B', B'' \in B_{\bar{x}}} \Big| |B'| - |B''| \Big| \tag{4.1}$$

In GLY1, for picking a splitting input for a block B, we consider $\mathcal{X}_B$ the set of splitting inputs for B. We define:

$$\mathbf{GLY1}(B) = rand(F(B, \mathcal{X}_B)) \tag{4.2}$$

where the function $rand(.)$ chooses an element of the set given as its input randomly[2].

In GLY2, we aim to select input symbol x in $\mathcal{X}_B$ that maximizes the size of the partitioning B_x, without concerning the sizes of the elements in B_x. If there are two or more input symbols that give the same maximum value, among these the one that gives the most even partitioning is chosen by using the function F. Formally,

$$\mathbf{GLY2}(B) = \begin{cases} x', & \text{if } \operatorname{argmax}_{x \in \mathcal{X}_B} |B_x| = \{x'\} \\ F(B, \operatorname{argmax}_{x \in \mathcal{X}_B} |B_x|), & \text{otherwise} \end{cases} \tag{4.3}$$

Besides these versions, we also consider a version of the LY algorithm in which a splitting input is selected randomly. We call this version as RLY and formally for a block B, we have

$$\mathbf{RLY}(B) = rand(\mathcal{X}_B) \tag{4.4}$$

Except these modifications, GLY1, and GLY2 are exactly the same as LY algorithm. Similarly, instead of LY, from now on we write FLY to refer to version of the LY algorithm which chooses the splitting input according to some fixed ordering (i.e. alphabetical ordering).

4.5.3. A Lookahead Based ADS Construction Algorithm

In this subsection, we explain the details of the proposed algorithm to construct a reduced ADS for an FSM $M = (S, X, Y, \delta, \lambda)$ which is minimal, deterministic, completely specified and known to have an ADS. Before presenting the details of the actual algorithm, we provide some definitions and routines that are going to be used in this section.

In [24] Hennie introduces the use of a tree, called the *successor tree*, for constructing adaptive homing/distinguishing sequences. The successor tree grows exponentially and it possesses information that can be used to find the minimum cost adaptive homing/distinguishing sequences. However since it grows exponentially (with the number of states), it becomes impractical to construct a successor tree to obtain a reduced adaptive homing/distinguishing sequences as the size of the FSM gets larger.

[2]Note that operators argmin/argmax return the set of arguments achieving the optimum value.

Our method has two phases. In the first phase, a tree called the *Enhanced Successor Tree* (EST) similar to a successor tree is generated. In the second phase, EST is used to construct an ADS.

An EST contains two types of nodes; *input nodes* $\mathcal{I}$ and *output nodes* $\mathcal{O}$. The root and the leaves of an EST are output nodes. Except the leaves, the children of an output node are input nodes, and the children of an input node are output nodes. In other words, on a path from the root to a leaf, one observes a sequence of output and input nodes alternatingly, starting and ending with an output node.

Each input node p is labeled by an input symbol $in(p)$. Similarly, each output node q is labeled by an output symbol $out(q)$, except the root node for which $out(q) = \varepsilon$. An output node q is also associated with a block $bl(q)$. For the root node q, $bl(q) = S$. An output node q is a leaf node iff $|bl(q)| = 1$. A non-leaf output node q that is associated with a block $bl(q)$, has a separate input node p as its child for each input symbol x that is valid for $bl(q)$, with $in(p) = x$. An input node p (where $x = in(p)$) with a parent output node q (where $B = bl(q)$), has a separate output node r as its child for each output symbol $y \in \lambda(B, x)$, with $out(r) = y$ and $bl(r) = \delta(B_{x/y}, x)$.

The EST of an FSM M is potentially an infinite tree. Instead of the whole tree, the algorithm constructs a limited size *partial* EST, using which an ADS can be produced. The algorithm uses heuristic approaches to explore the relevant and promising parts of the EST to find a reduced size ADS with respect to different metrics, such as the height and the external path length.

The partial EST constructed by the algorithm will be the EST where the tree is pruned at several nodes. For a leaf q in an EST we have $|bl(q)| = 1$. However, for a leaf node q in a partial EST, we have $|bl(q)| \geq 1$.

Initially, the algorithm starts with the partial EST consisting of only the root node. In each iteration, an output node q is handled and the partial EST rooted at q is expanded exhaustively upto depth k, where k is a parameter given to the algorithm. Among the children of q, an input node p that seems to be the best (according to the objective and the heuristic being used) is selected, and the search continues recursively under the

subtrees rooted at the children of p.

During the construction of the partial EST T, some nodes are marked as the nodes to be used for ADS construction later. Namely, a set of output nodes L and a set of input nodes I are marked. Each output node $q \in L$ has the property that $|bl(q)| = 1$ and it corresponds to a leaf node in the ADS that will be constructed later. Also the nodes in I will be corresponding to the non–leaf nodes of the ADS.

The algorithm constructing a partial EST is given in Algorithm 1.

Algorithm 1: Construct a partial EST for M

Input: FSM $M = (S, X, Y, \delta, \lambda)$, $k \in \mathbb{Z}_{\geq 1}$
Output: A partial EST T, a set of leaves L, and a set of input nodes I
begin

```
1   L ← ∅, I ← ∅
2   Construct an output node q0 with bl(q0) = S, out(q0) = ε
3   Initialize T to be a tree consisting of the root q0 only
4   Q ← {q0} // Q is the set of output nodes yet to be processed
5   while Q ≠ ∅ do
6       Pick an output node q ∈ Q to process
7       Q ← Q \ {q}
8       ExpandEST(q,k) // expand subtree under q exhaustively upto a certain depth
9       Choose a child node p of q // based on the objective and the heuristic used
10      I ← I ∪ {p} // The input node p will be used for the ADS
11      foreach child r of p do
            if |bl(r)| > 1 then
12              Q ← Q ∪ {r} // not a singleton yet, needs to be processed
            else
13              L ← L ∪ {r} // reached a singleton block
```

The procedure "ExpandEST(q, k)", constructs the partial EST rooted at the node q exhaustively upto the given depth k. If in this partial subtree, for every leaf node r, we have $|bl(q)| = |bl(r)|$ (which means the block $bl(q)$ could not be divided into smaller blocks by using input sequences of length upto k that are valid for $bl(q)$), the procedure increases the depth of the subtree rooted at q until it encounters a level at which there exists a leaf node r with $|bl(r)| < |bl(q)|$. Clearly, this is always possible since the FSM M has an ADS.

At line 9 of Algorithm 1, a child node p of q is chosen heuristically. This choice is based on the scores of the nodes which are calculated by processing the nodes in the subtree

rooted at q in a bottom–up manner. First the scores of the leaves in this subtree are assigned. This is followed by the score evaluation of the internal nodes in the subtree. The score of a non–leaf node depends on the scores of its children. The score of a node reflects the potential size of the ADS that will be eventually formed if that node is decided to be used in the ADS to be formed.

When the score of an output node q is computed based on the scores of its children, since we have the control over the input to be chosen, the child node of q having the minimum score is chosen. However, when the score of an input node p is computed based on the scores of its children, since we do not have the control of the output to be produced, we prepare for the worst and use the maximum score of the children of p. A similar approach is in fact also suggested by Hennie [24] (please see Chapter 3). The process of calculating the scores of the nodes depends on the heuristic used and the details are given in Section 20.

Before presenting the algorithm to construct an ADS, we will give some properties of the nodes L and I marked by Algorithm 1. For an output node q, consider the path from the root of T to q (including q). Let w and v be the concatenation of input symbols and output symbols on this path, respectively. We use below the notation $io(q)$ to refer to the input/output sequence w/v.

Proposition 3 *Let q be an output node in T, let $io(q) = w/v$. Then we have $bl(q) = \delta(S_{w/v}, w)$.*

Proposition 4 *$|L| + \sum_{q \in Q} |bl(q)| = |S|$ is an invariant of Algorithm 1 before and after every iteration the while loop.*

Proposition 4 implies the following result, since when Algorithm 1 terminates we have $Q = \emptyset$.

Corollary 1 *When Algorithm 1 terminates, $|L| = |S|$.*

Proposition 5 *Let q be an output node in T with $|bl(q)| = 1$, and let $w/v = io(q)$. There exists a unique state $s \in S$ such that $\lambda(s, w) = v$.*

Algorithm 2 describes how an ADS can be constructed based on the partial EST T, the set of marked nodes L and I in T by Algorithm 1. Note that at line 6 of Algorithm 2, $S_{w/v}$ is claimed to be a singleton, which is guaranteed by Proposition 5. In order to show that $\mathcal{A}$ which is generated by Algorithm 2 is an ADS, we also prove the following.

Proposition 6 *The leaves of $\mathcal{A}$ constructed by Algorithm 2 is labeled by distinct states.*

Theorem 15 *$\mathcal{A}$ constructed by Algorithm 2 is an ADS.*

Algorithm 2: Construct an ADS

Input: The partial EST T, the set of marked nodes L and I by Algorithm 1

Output: An ADS $\mathcal{A}$

begin

```
    // Construct and label the internal nodes of A
1   foreach node p ∈ I do
2       Construct an internal node p' in A
3       Label p' by in(p)
    // Construct and label the leaf nodes of A
4   foreach node q ∈ L do
5       Let w/v = io(q)
6       Let s be the state such that {s} = S_{w/v}
7       Construct a leaf node q' in A
8       Label q' by s
    // Construct the edges to the leaves
9   foreach leaf node q' ∈ A do
10      Let q be the corresponding node of q' in T
11      Let p be the parent of q in T
12      Let p' be the corresponding node of p in A
13      Insert an edge between p' and q' with the label out(q)
    // Construct the remaining edges
14  foreach internal node p' ∈ A do
15      Let p be the corresponding node of p' in T
16      if p has a grandparent in T then
            // except the root of A
17          Let q be the parent of p in T
18          Let r be the parent of q in T
19          Let r' be the corresponding node of p in A
20          Insert an edge between p' and r' with the label out(q)
```

Heuristics

We use different heuristic approaches to minimize the size of ADSs with respect to two different metrics, which are minimizing the height and minimizing the external path length of the ADS.

As mentioned above, the score of a node q in the partial EST constructed so far is an estimation of the size of the ADS that will be formed by using a child of q in ADS. Let $d(q)$ be the depth of q in the EST and $v(q)$ be the score of q. We also keep track of another information, $z(q)$. It is the number of singleton output nodes in the winner subtrees under q in the current partial EST, and used to break the ties as explained below.

Let us consider a leaf node q, where $|bl(q)| = 1$. This is in fact a leaf node also in the complete EST. For such leaf nodes, we set $v(q) = d(q)$ and $z(q) = 1$. However, for a leaf node q in the current partial EST with $|bl(q)| > 1$, we set $z(q) = 0$. Although q is currently a leaf node in the partial EST, if we were to expand q, there will appear a subtree under q. In order to take into account the size of the subtree rooted at q (without actually constructing this subtree), we need to estimate the size of the subtree under q. Note that $bl(q)$ is the set of states yet to be distinguished from each other.

We consider two different metrics as the size of an ADS: height or external path length. Depending on the objective, we estimate the size of the subtree that would appear under a (yet to be processed) output node q in different ways. While minimizing for height, we use two different heuristic functions $\mathcal{H}_U : \mathcal{O} \to \mathbb{R}^+$ and $\mathcal{H}_{LY} : \mathcal{O} \to \mathbb{R}^+$. Similarly, while optimizing for external path length, we use heuristic functions $\mathcal{L}_U : \mathcal{O} \to \mathbb{R}^+$ and $\mathcal{L}_{LY} : \mathcal{O} \to \mathbb{R}^+$.

For an FSM with n states the height of an ADS is bounded above by $n(n-1)/2$ [35]. We use this bound for heuristic functions $\mathcal{H}_U$ and $\mathcal{L}_U$ in the following way. The score of node q with respect to function $\mathcal{H}_U$ is given as $\mathcal{H}_U(q) = d(q) + |bl(q)|(|bl(q)| - 1)/2$, where $d(q)$ is the depth of q. On the other hand, function $\mathcal{L}_U$ multiplies the number of states with the expected height of the subtree to approximate the expected external path length. That is, $\mathcal{L}_U(q) = (d(q) + |bl(q)|(|bl(q)| - 1)/2)|bl(q)|$.

As another estimation method for the size of the subtree to appear under an output node q, one can use LY algorithm to construct an ADS for the states in $bl(q)$. The heuristic functions $\mathcal{H}_{LY}$ and $\mathcal{L}_{LY}$ use this idea. Let A' be the ADS computed by LY algorithm for the states in $bl(q)$, and let $h_{A'}$ and $l_{A'}$ be the height and the external path length of A'. Then the heuristic functions $\mathcal{H}_{LY}$ and $\mathcal{L}_{LY}$ are defined as $\mathcal{H}_{LY}(q) = d(q) + h_{A'}$ and $\mathcal{L}_{LY}(q) = d(q)|bl(q)| + l_{A'}$.

At line 9 of Algorithm 1, for the output node q being processed in that iteration, an input node p which is a child of q is chosen. Let T' refer to the subtree rooted at q in the partial EST at this point. While choosing the child input node p to be used, the scores of the nodes in T' are calculated in a bottom up manner. First, for each (current) leaf node q' (which is an output node) in T', $v(q')$ is assigned by using one of the heuristic functions ($\mathcal{H}_U(q')$ or $\mathcal{H}_{LY}(q')$ for height optimization, and $\mathcal{L}_U(q')$ or $\mathcal{L}_{LY}(q')$ for external path length optimization) and $z(q')$ is assigned. The score of the remaining nodes in T' are based on the scores of its children and are calculated as follows.

When minimizing for height, for an input node p', $v(p')$ is set to the maximum score of its children and $z(p')$ is set to the sum of singleton scores of its children. For an output node q', $v(q')$ is set to the minimum score of its children, and $z(q')$ is set to the $z(.)$ value of the winner child. When minimizing for external path length, for an input node p', $v(p')$ and $z(p')$ is set to the sum of the scores of its children. For an output node q' on the other hand, $v(q')$ is set to the minimum score of its children and $z(q')$ is set to the $z(.)$ value of the winner child. Note that, there may be ties during this process when we attempt to take minimum or maximum. Among the nodes achieving the same minimum/maximum, the tie is first tried to be broken by maximizing the number of singleton values ($z(.)$). If still there is a tie, this is broken randomly. From now on we will write LEA to refer to the Lookahead driven ADS construction algorithm on EST. We present a summary of the heuristics and algorithms used in this section in Table ??.

Table 4.3.: The list of heuristics/algorithms used to construct ADSs

Method	Abbreviation
LEA	Lookahead Algorithm that uses Heuristics $\mathcal{H}_U$,$\mathcal{H}_{LY}$,$\mathcal{L}_U$, and $\mathcal{L}_{LY}$
GLY1	Modified version of LY algorithm #1
GLY2	Modified version of LY algorithm #2
RLY	Randomized version of LY algorithm
$\mathcal{H}_U$	Heuristic for MinHeightADS that uses local information
$\mathcal{L}_U$	Heuristic for MinADS that uses local information
$\mathcal{H}_{LY}$	Heuristic for MinHeightADS that uses LY algorithm
$\mathcal{L}_{LY}$	Heuristic for MinADS that uses LY algorithm
FLY	The LY algorithm with fixed ordering of inputs
BF	The Brute–Force algorithm given in [21]

4.5.4. FSMs used in the Experiments

For our experiments, we used both randomly generated FSMs and a set of FSMs available as a benchmark as explained below.

Random FSM Generation

We randomly generate FSMs using the tool utilised in [43, 108]. An FSM M is constructed randomly as follows: First, for each input x and state s we randomly assign the values of $\delta(s, x)$ and $\lambda(s, x)$. Then we check whether M is strongly connected, minimal and has an ADS. We omit the FSMs that could not pass these tests. Consequently, all FSMs used are strongly connected, minimal, and has an ADS.

By following this procedure we generated two test suites, TS1 and TS2. In each test suite we have 6 classes of FSMs. Each class contains 100 FSMs. Thus the number of FSMs used in these experiments is $600(\text{TS1}) + 600(\text{TS2}) = 1200$. In TS1, the number of states range in $\{50, 60, \ldots, 100\}$, but the size of input and output alphabets are fixed to four. In TS2, the state and input/output alphabet cardinalities of the classes are

$(30, 4/4)$, $(30, 8/8)$, $(30, 16/16)$, $(60, 4/4)$, $(60, 8/8)$ and $(60, 16/16)$.

We use Intel Xeon E5-1650 @3.2-GHZ CPU with 16 GB RAM to carry out these tests. We implemented proposed algorithm, LY algorithm and the brute-force algorithm using C++ and compiled them using Microsoft Visual Studio .Net 2012 under 64 bit Windows 7 operating system.

Benchmark FSMs

Random FSMs allow us to perform experiments and grasp some properties of different versions of FLY algorithm, however it is possible that FSMs used in real-life situations differ from these randomly generated FSMs. Therefore we carry out some case studies on FSM specifications used in workshops between 1989-1993 [109]. The benchmark suite has 59 FSM specifications ranging from simple circuits to advanced circuits obtained from industry. We extract specifications that are minimal, deterministic, strongly connected and having an ADS. We discarded sequential circuits that have larger than 10 input bits. Note that the FSM specification of the sequential circuit having n input bits has 2^n number of inputs, that is there is an exponential growth in the number of inputs.

After post-processing, we obtain FSM specifications of circuits *DVRAM*, *Ex4*, *Log*, *Rie*, and *Shift Register*[3]. In Table 4.4 we present the size (number of states and the number of transitions) of these FSMs.

4.5.5. Results

We present the results of the experiments in three sections. We first compare the external path length, the height of the ADSs, and the running times for FLY, LEA (Lookahead Approach using heuristics $\mathcal{H}_U$, $\mathcal{H}_{LY}$, $\mathcal{L}_U$, $\mathcal{L}_{LY}$), GLY1 GLY2, and RLY. Next, we compare the length of the checking sequences constructed by using ADSs generated by these methods. Finally, we show how using minimum cost ADSs affects the length of the checking sequences.

[3]FSM specification Ex4 is partially specified. We complete the missing transitions by adding self looping transitions with a special output symbol, and do not use these inputs for ADS construction.

Table 4.4.: Size of Case Studies

Name	Number of States	Number of Transitions
Shift Register	8	16
Ex4	14	896
Log	17	8704
DVRAM	35	8960
Rie	29	14848

Comparison of External Path Lengths, Heights and Timings

In order to evaluate the relative performance of different approaches, we compute the ADSs through the $\mathcal{H}_U$, $\mathcal{H}_{LY}$, $\mathcal{L}_U$, $\mathcal{L}_{LY}$, GLY1, GLY2, FLY and brute–force algorithms separately. The brute–force algorithm (BF) is described in [21]. BF algorithm constructs EST to a depth that is sufficient to form an ADS. We also constructed ADSs with the $\mathcal{H}_U$, $\mathcal{H}_{LY}$, $\mathcal{L}_U$, $\mathcal{L}_{LY}$, GLY1 and GLY2. In the following sections we present the results of our experimental study.

In order to compare algorithms, we use two measures. The first measure ($\mathcal{M}_1$) is the mean difference between the height/external path length found by the algorithm and the optimal value found by BF. For an FSM M in a class $\mathbb{M}$, let $A(M)$ be the result returned by the algorithm A, where A is either GLY1, or GLY2, or LY, or BF, or our algorithm using the heuristics given in this work. The second measure ($\mathcal{M}_2$) is the ratio of cases in which $A(M)$ finds the optimal ADS. These measures are formally defined as follows, where the comparison operator "$\stackrel{?}{=}$" returns 1 if comparison is true, and returns 0 otherwise:

$$\mathcal{M}_1 = \frac{\sum_{\mathcal{M}\in\mathbb{M}}(size(A(M)) - size(BF(M)))}{|\mathbb{M}|}$$

$$\mathcal{M}_2 = \frac{\sum_{\mathcal{M}\in\mathbb{M}}(size(A(M)) \stackrel{?}{=} size(BF(M)))}{|\mathbb{M}|}$$

We evaluate the methods with respect to the number of states, size of input/output alphabets, and the parameter k.

The effect of number of states: To see the effect of the number of states, we performed experiments on FSMs in TS1 by using $k = 1$ for Algorithm 1. Note that, using a larger k value would intuitively increase the quality of our results at the expense of increased running time. We discuss the effect of different k values separately below. We present the results in Table 4.5.

The results are quite promising. We see that when $|M| = 50$, $\mathcal{H}_{LY}$ can find the shallowest tree in 99% of the instances. Comparing FLY and GLY2, we see that GLY2 is better than FLY. Moreover, considering the results of measure $\mathcal{M}_1$, we see that LEA constructs ADS trees closer to the optimum than the trees generated by approaches FLY, GLY1 and GLY2. We observe that this rate gradually decreases as the number of states increases. Moreover, in terms of running times, we see that $\mathcal{H}_U$ is the fastest, and $\mathcal{H}_{LY}$ is the slowest approach and $\mathcal{H}_{LY}$ is slower than FLY by nearly 3 fold.

We observe that heuristics $\mathcal{L}_U$ and $\mathcal{L}_{LY}$ cannot compute the optimum ADSs most of the times, but compared to the FLY, GLY1 and GLY2, we see that $\mathcal{L}_U$ and $\mathcal{L}_{LY}$ are able to reduce the average gap to the optimal external path length by at least 3, at most 6 fold.

The effect of the size of the input/output alphabet: We again use the measures $\mathcal{M}_1$ and $\mathcal{M}_2$ to analyze the effect of the size of the input/output alphabet on the performances. We use Algorithm 1 with $k = 1$. In this experiment we used FSMs in TS2. Table 4.6 summarizes the results.

Clearly the results indicate that the FLY algorithm computes ADSs quicker than the other methods when input size is increased. In terms of $\mathcal{M}_1$, $\mathcal{H}_{LY}$ produces the best results when the objective function is to minimize the height.

We observe that as the number of input/output symbols increases, LEA tends to produce optimum ADSs. Moreover as the number of input/output symbols increases, LEA, GLY1 and GLY2 tend to be slower. We observe that as we increment the size of input/output symbols by a factor of 4 (4 to 16) the time required to compute ADSs is

Table 4.5.: Comparison of algorithms in terms of $\mathcal{M}_1$ and $\mathcal{M}_2$ with respect to number of states.

$|M|$ is the number of states.

	$\mathcal{M}_1$						$\mathcal{M}_2$						time (nsec)				
$\|M\|$	$\mathcal{H}_u$	$\mathcal{H}_{LY}$	LY	GLY1	GLY2	RLY	$\mathcal{H}_u$	$\mathcal{H}_{LY}$	LY	GLY1	GLY2	RLY	$\mathcal{H}_u$	$\mathcal{H}_{LY}$	LY	GLY1	GLY2
50	0.05	0.01	1.02	0.98	0.41	1.34	95	99	14	16	28	7	0.82	2.61	0.95	1.07	1.45
60	0.16	0.06	1.11	1.09	0.58	1.46	84	94	6	10	26	0	0.99	3.51	1.14	1.23	1.67
70	0.18	0.07	1.38	1.12	0.69	1.56	82	93	5	16	20	0	1.17	4.35	1.39	1.78	1.98
80	0.31	0.18	1.41	1.25	0.76	1.60	69	72	2	16	19	0	1.28	5.56	1.79	2.34	2.84
90	0.40	0.34	1.47	1.38	0.98	1.73	60	66	0	9	16	0	1.13	6.58	2.19	2.85	3.04
100	0.57	0.49	1.58	1.43	0.96	1.86	53	51	3	6	14	0	1.17	8.01	2.47	2.98	3.45

$\|M\|$	$\mathcal{L}_u$	$\mathcal{L}_{LY}$	LY	GLY1	GLY2	RLY	$\mathcal{L}_u$	$\mathcal{L}_{LY}$	LY	GLY1	GLY2	RLY	$\mathcal{L}_u$	$\mathcal{L}_{LY}$	LY	GLY1	GLY2
50	6.03	3.00	18.91	12.56	10.11	20.01	2	16	0	0	4	1	0.83	2.65	0.95	1.07	1.45
60	6.89	3.49	20.99	12.78	10.98	25.36	1	9	0	0	3	0	0.92	3.39	1.14	1.23	1.67
70	7.60	3.98	25.66	14.12	11.45	29.17	1	6	0	0	1	0	1.12	4.38	1.39	1.78	1.98
80	8.50	5.3	31.04	15.47	12.60	35.34	1	0	0	0	0	0	1.31	5.42	1.79	2.34	2.84
90	9.12	5.79	33.09	16.23	12.93	42.40	0	0	0	0	0	0	1.39	6.51	2.19	2.85	3.04
100	10.05	6.72	36.49	16.89	13.37	43.67	0	0	0	0	0	0	1.58	7.91	2.57	2.98	3.45

doubled for LEA.

The effect of the value of parameter k:

Now we show the effect of the lookahead distance, i.e. the value of the parameter k used in Algorithm 1. Note that approaches GLY1, GLY2 and FLY do not use such variable. We set the number of input/output symbols to 4. We consider four classes of FSMs where the number of states in these classes are $30, 50, 70$ and 100.

Note that since the outputs are uniformly distributed during the generation of FSMs, one would expect the average depth of the ADS to be around $\lceil \log_q n \rceil$, where n and q are the number of states and the number of outputs, respectively. For our experiments with 4 outputs and the number of states ranging between 30 and 100, the height of the ADSs is expected to be 3–4 (i.e. for 4 outputs ($\lceil \log_4 30 \rceil = 3$ and $\lceil \log_4 100 \rceil = 4$)). BF algorithm reports that, out of 400 FSMs, 371 of the ADSs heights are 4 or more. Since we use $k = 2$ and $k = 3$ in these experiments, the partial ESTs formed by our algorithm are not exactly the same ESTs that would be formed by the BF algorithm.

The results are concluding: increasing the value of k, improves the quality of the results at the expense of increased running times. When the objective functions is to minimize the height (Table 4.7), we see that LEA either finds one of the optimum result or it misses the optimum result by one. When the objective function is to minimize the external path length (Table 4.8), the LEA computes an optimum ADS almost in all cases.

When we consider running times, for $k = 2$, FLY, GLY1 and GLY2 are nearly 8 times faster than the LEA that uses heuristics $\mathcal{H}_{LY}/\mathcal{L}_{LY}$, FLY, GLY1 and GLY2 are 2 times faster than when LEA uses heuristics $\mathcal{H}_U/\mathcal{L}_U$. For $k = 3$, FLY, GLY1 and GLY2 are 10 times faster than LEA when it uses $\mathcal{H}_{LY}/\mathcal{L}_{LY}$ heuristics, and finally FLY, GLY1 and GLY2 are 4 times faster than LEA when it uses $\mathcal{H}_U/\mathcal{L}_U$ heuristics.

The result of the case studies are presented in Table 4.9. Surprisingly we see that FLY and GLY1 produce same ADSs in all cases and LEA and GLY2 produce same ADSs in all cases. Besides, FLY, GLY2 and LEA produce exactly the same ADSs for FSMs Rie and Shift Register. On the other hand, we see that, in terms of height and

Table 4.6.: Comparison of algorithms in terms of $\mathcal{M}_1$ and $\mathcal{M}_2$ with respect to size of input output alphabets.
$|M|$ is the number of states.

		$\mathcal{M}_1$						$\mathcal{M}_2$						time (nsec)				
$\|M\|$	I/O	$\mathcal{H}_u$	$\mathcal{H}_{LY}$	LY	GLY1	GLY2	RLY	$\mathcal{H}_u$	$\mathcal{H}_{LY}$	LY	GLY1	GLY2	RLY	$\mathcal{H}_u$	$\mathcal{H}_{LY}$	LY	GLY1	GLY2
	4/4	0.21	0.11	0.92	0.86	0.85	1.23	79	89	24	35	37	4	0.45	1.19	0.47	0.54	0.72
30	8/8	0.30	0.20	0.64	0.69	0.68	1.12	70	80	30	39	42	0	0.55	1.14	0.40	0.69	0.99
	16/16	0	0	0.72	0.60	0.61	1.08	100	100	44	47	53	0	1.18	2.09	0.37	1.02	1.32
	4/4	0.16	0.06	1.11	1.09	0.58	1.46	84	94	6	10	26	0	0.87	1.1	1.14	1.23	1.67
60	8/8	0.03	0.02	1.16	0.99	0.54	1.41	97	98	1	13	29	0	1.31	3.17	0.80	1.96	2.23
	16/16	0	0	0.88	0.96	0.52	1.40	100	100	17	19	32	0	2.25	3.87	0.72	2.38	3.04

$\|M\|$	I/O	$\mathcal{L}_u$	$\mathcal{L}_{LY}$	LY	GLY1	GLY2	RLY	$\mathcal{L}_u$	$\mathcal{L}_{LY}$	LY	GLY1	GLY2	RLY	$\mathcal{L}_u$	$\mathcal{L}_{LY}$	LY	GLY1	GLY2
	4/4	1.95	1.46	10.2	9.03	7.27	10.8	26	34	1	2	2	0	0.46	1.16	0.47	0.51	0.63
30	8/8	0.29	0.26	9.24	8.01	6.06	10.4	86	88	1	3	4	0	0.59	1.15	0.40	0.68	0.78
	16/16	0.01	0.28	4	4.11	3.85	11.3	99	99	9	4	8	0	1.2	2.09	0.37	0.89	1.21
	4/4	6.59	3.32	22.09	13.14	11.56	26.20	1	8	1	1	2	0	0.9	1.15	1.14	1.32	1.70
60	8/8	0.71	0.17	18.20	17.23	14.21	28.38	58	92	0	0	1	0	1.31	3.16	0.80	1.44	1.98
	16/16	0.61	0.05	13.31	10.20	9.33	21.41	65	96	0	0	0	0	2.26	3.9	0.72	1.56	2.33

Table 4.7.: Comparison of algorithms in terms of $\mathcal{M}_1$ and $\mathcal{M}_2$ with respect to parameter k.
$|M|$ is the number of states.

		$\mathcal{M}_1$						$\mathcal{M}_2$						time (nsec)				
$\|M\|$	k	$\mathcal{H}_u$	$\mathcal{H}_{LY}$	LY	GLY1	GLY2	RLY	$\mathcal{H}_u$	$\mathcal{H}_{LY}$	LY	GLY1	GLY2	RLY	$\mathcal{H}_u$	$\mathcal{H}_{LY}$	LY	GLY1	GLY2
	1	0.21	0.11	0.92	0.86	0.85	1.23	79	89	24	35	37	4	0.45	1.19	0.47	0.54	0.72
30	2	0	0	0.92	0.86	0.85	1.23	100	100	24	35	37	4	0.83	2.44	0.47	0.54	0.72
	3	0	0	0.92	0.86	0.85	1.23	100	100	24	35	37	4	1.23	2.67	0.47	0.54	0.72
	1	0.05	0.01	1.02	0.98	0.41	1.34	95	99	14	16	28	7	0.82	2.61	0.95	1.07	1.98
50	2	0	0	1.02	0.98	0.41	1.34	100	100	14	16	28	7	1.56	3.95	0.95	1.07	1.98
	3	0	0	1.02	0.98	0.41	1.34	100	100	14	16	28	7	2.93	6.12	0.95	1.07	1.98
	1	0.18	0.07	1.38	1.12	0.69	1.56	82	93	5	16	20	0	1.17	4.35	1.39	1.78	1.98
70	2	0.04	0	1.38	1.12	0.69	1.56	96	100	5	16	20	0	2.28	6.62	1.39	1.78	1.98
	3	0	0	1.38	1.12	0.69	1.56	100	100	5	16	20	0	4.3	14.38	1.39	1.78	1.98
	1	0.57	0.49	1.58	1.43	0.96	1.86	53	51	3	6	14	0	1.17	8.01	2.57	2.98	3.45
100	2	0.19	0.12	1.58	1.43	0.96	1.86	81	88	3	6	14	0	3.18	21.62	2.57	2.98	3.45
	3	0	0	1.58	1.43	0.96	1.86	100	100	3	6	14	0	6.77	21.62	2.57	2.98	3.45

Table 4.8.: Comparison of algorithms in terms of $\mathcal{M}_1$ and $\mathcal{M}_2$.
$|M|$ is the number of states.

		$\mathcal{M}_1$						$\mathcal{M}_2$						time (nsec)				
$\|M\|$	k	$\mathcal{L}_u$	$\mathcal{L}_{LY}$	LY	GLY1	GLY2	RLY	$\mathcal{L}_u$	$\mathcal{L}_{LY}$	LY	GLY1	GLY2	RLY	$\mathcal{L}_u$	$\mathcal{L}_{LY}$	LY	GLY1	GLY2
	1	1.95	1.46	10.2	9.03	7.27	10.8	26	34	1	2	2	0	0.46	1.16	0.47	0.51	0.63
30	2	0.03	0.02	10.2	9.03	7.27	10.8	97	98	1	2	2	0	0.88	2.45	0.47	0.51	0.63
	3	0	0	10.2	9.03	7.27	10.8	100	100	1	2	2	0	1.25	2.58	0.47	0.51	0.63
	1	6.03	3	18.91	12.56	10.11	20.01	2	16	0	0	4	1	0.83	2.65	0.95	1.07	1.45
50	2	0.46	0.17	18.91	12.56	10.11	20.01	67	85	0	0	4	1	1.73	5.06	0.95	1.07	1.45
	3	0.01	0	18.91	12.56	10.11	20.01	99	100	0	0	4	1	3.54	8.18	0.95	1.07	1.45
	1	7.6	3.98	25.66	14.12	11.45	29.17	1	6	0	0	1	0	1.12	4.38	1.39	1.78	1.98
70	2	0.92	0.85	25.66	14.12	11.45	29.17	48	54	0	0	1	0	2.41	8.77	1.39	1.78	1.98
	3	0.06	0.06	25.66	14.12	11.45	29.17	97	94	0	0	1	0	5.15	14.53	1.39	1.78	1.98
	1	10.05	6.72	36.49	16.89	13.37	43.67	0	0	0	0	0	0	1.58	7.91	2.57	2.98	3.45
100	2	1.79	1.16	36.49	16.89	13.37	43.67	22	47	0	0	0	0	3.63	17.14	2.57	2.98	3.45
	3	0.17	0.14	36.49	16.89	13.37	43.67	85	91	0	0	0	0	8.05	29.72	2.57	2.98	3.45

Table 4.9.: Height and External Path Length Comparison for Case Studies.

Name	Method (Height-External Path Length)							Computation Time (msec)						
	FLY	GLY1	GLY2	$\mathcal{H}_U$	$\mathcal{H}_{LY}$	$\mathcal{L}_U$	$\mathcal{L}_{LY}$	FLY	GLY1	GLY2	$\mathcal{H}_U$	$\mathcal{H}_{LY}$	$\mathcal{L}_U$	$\mathcal{L}_{LY}$
Log	2-21	2-21	2-17	2-17	2-17	2-17	2-17	0.029	0.069	0.123	0.025	0.034	0.196	0.232
DVRAM	6-104	6-104	4-78	4-78	4-78	4-78	4-78	0.084	5.698	7.259	4.357	4.630	6.582	5.984
Ex4	4-29	4-29	3-22	3-22	3-22	3-22	3-22	0.001	0.0049	0.0063	0.0056	0.0058	0.023	0.018
Rie	3-46	3-46	3-46	3-46	3-46	3-46	3-46	0.257	6.256	11.590	9.454	8.348	13.450	12.674
Shift Reg.	3-24	3-24	3-24	3-24	3-24	3-24	3-24	0	0	0	0	0	0	0

external path length, GLY2 and LEA compute smaller ADSs than FLY and GLY1, which correlates with the results of experiments on the random FSMs.

Comparison of Checking Sequence Lengths

We use the checking sequence generation methods **HEN**, **UWZ**, **HIU**, **SP**, and **DY** mentioned in Section 4.1.1. We present the results as improvements in the length of the checking sequences as follows. For each FSM M, we construct a checking sequence C_1 using an ADS generated by a method (GLY1, GLY2, $\mathcal{H}_{LY}$, $\mathcal{H}_U$, $\mathcal{L}_{LY}$ and $\mathcal{L}_U$), and another checking sequence C_2 using an ADS generated by FLY. The improvement in the length of the checking sequence is $100 \times (|C_2| - |C_1|)/|C_1|$.

Results are presented in Tables 4.10, 4.11, 4.12, 4.13, 4.14,and 4.15. In comparison $\mathcal{L}_{LY}$ is better than all other approaches. In general, we note that, the improvements obtained from **SP** method are much higher than the improvements obtained from other CS generation methods. This may imply that the state identification sequences occupy larger portions of the CSs computed by the **SP** algorithm. That is the use of transfer sequences is less compared to other methods. Thus the reduction on the cost of ADSs yield dramatic reductions (upto 29.2%) on the length of the checking sequences.

There are some cases in which we obtain negative improvements, i.e. the checking sequence gets longer when a reduced cost ADS is used, please see Figures 4.6, 4.7 and 4.8. ADSs in Figure 4.7 and Figure 4.8 are generated by using GLY1 and FLY, respectively. Although the ADS of Figure 4.7 is better with respect to external path length, considering the fact that the number of incoming transitions of state s_1 is very high, ADS of Figure 4.8 having a shorter SDS for s_1 is more likely to give a shorter checking sequence.

One important and promising observation is that, as the size of FSMs gets larger (with more number of states and/or with more transitions), the improvement ratio also gets larger.

The results of the experiments on the benchmark FSMs are presented in Table 4.16. Recall that GLY1 and FLY; GLY2 and LEA compute identical ADSs (see Table 4.9),

	HEN	UWZ	HIU	SP	DY
30	3.6	3.3	3.4	3.8	3.1
50	4.8	4.6	4.6	4.1	4.4
70	5.1	5.2	5.1	7.5	4.9
100	7.6	6.4	6.3	8.0	5.7

Table 4.10.: Checking Sequence Length Comparison for FLY and GLY1

	HEN	UWZ	HIU	SP	DY
30	4.9	4.4	4.3	5.8	4.5
50	5.3	4.8	5.6	6.3	5.9
70	5.5	5.2	6.0	6.6	6.2
100	6.8	6.5	7.2	9.2	7.8

Table 4.11.: Checking Sequence Length Comparison for FLY and GLY2

	HEN	UWZ	HIU	SP	DY
30	5.3	4.1	4.8	6.0	4.5
50	6.6	5.4	6.4	7.7	6.9
70	6.9	6.8	6.7	8.5	6.9
100	7.6	6.9	7.9	9.9	7.4

Table 4.12.: Checking Sequence Length Comparison for FLY and $\mathcal{H}_U$

	HEN	UWZ	HIU	SP	DY
30	7.5	7.5	7.4	8.7	7.1
50	8.4	7.6	8.6	9.5	7.3
70	8.7	8.7	8.9	10.3	8.0
100	9.9	9.5	9.3	12.2	8.6

Table 4.13.: Checking Sequence Length Comparison for FLY and $\mathcal{H}_{LY}$

	HEN	UWZ	HIU	SP	DY
30	14.0	13.1	13.7	15.8	14.3
50	15.2	14.5	14.5	16.3	15.6
70	16.6	15.3	15.9	16.6	16.9
100	17.4	16.4	16.3	19.4	16.7

Table 4.14.: Checking Sequence Length Comparison for FLY and $\mathcal{L}_U$

	HEN	UWZ	HIU	SP	DY
30	14.8	13.8	15.2	15.2	14.5
50	17.4	16.5	16.6	19.4	15.9
70	19.5	17.4	16.8	26.7	16.7
100	22.3	18.2	17.5	29.2	17.8

Table 4.15.: Checking Sequence Length Comparison for FLY and $\mathcal{L}_{LY}$

Table 4.16.: Checking Sequence Length Comparison for Case Studies.

File	Method (Checking Sequence Length Comparison (%)))				
	HEN	**UWZ**	**HIU**	**SP**	**DY**
Log	10.21	8.21	7.68	6.45	7.38
DVRAM	21.48	12.39	9.58	11.22	10.23
Ex4	5.45	3.34	2.49	1.44	2.41

thus we compare FLY with LEA only. Moreover, for files Shift-Register and Rie, all methods compute identical ADSs, consequently we also do not analyse the results for these FSMs. The results suggest that simple modifications on the FLY algorithm can reduce the length of checking sequence up to 21.48%. We believe that this is promising and we claim that sophisticated greedy algorithms may produce better results.

Constructing Checking Sequences with minimum ADSs

In Section 4.5.5, we compare FLY, GLY1, GLY2, $\mathcal{H}_U$, $\mathcal{H}_{LY}$, $\mathcal{L}_U$ and $\mathcal{L}_{LY}$ in terms of the checking sequence lengths which are constructed by using the ADSs generated by these approaches. We see that GLY1 GLY2 do not necessarily always generate ADSs with the smallest cost (height / external path length) compared to ADSs generated by FLY. But we saw experimentally that LEA can construct almost minimum cost ADSs for sufficiently large k values. Recall that our primary motivation for this work is that using a minimum cost ADS would result in a shorter checking sequence.

In this work we suggest and propse several ADS generation approaches. Hence, for a given FSM, one could simply generate ADSs by using all of these approaches and then use the minimum cost ADS to construct a checking sequence. For this reason, in this section we show what we could gain when we use the minimum cost ADS constructed by all of the approaches while constructing the checking sequences. Moreover, we believe that comparing the length of checking sequences constructed by using ADSs generated by FLY and by the minimum cost ADS, would give a better validation of our motivation.

In order to perform this study, we use the ADSs and the checking sequences constructed in Section 4.5.5. For each FSM, and for each approach we generate an ADSs and among these ADSs, we select an ADS $\mathcal{A}_\star$ with the minimum cost. We then generate a checking sequence $C_\star$ using $\mathcal{A}_\star$ and a checking sequence C_F using the ADS constructed by FLY. We report the percentage of cases (Φ) in which $C_\star$ is shorter than C_F and also the percentage improvement (Δ) in the length of the checking sequence.

Table 4.17 and 4.18 summarise the results, where n is the number of states, p/q is the size of input and output alphabets. In Table 4.17, checking sequences are constructed by the ADSs with the minimum height and in Table 4.18, checking sequences are constructed by the ADSs with the minimum external path length.

Table 4.17.: Improvement in checking sequence length by using the shallowest ADS.

n	HEN		UWZ		HIU		SP		DY	
	Φ	Δ	Φ	Δ	Φ	Δ	Φ	Δ	Φ	Δ
30	51.8	13.7	62.4	11.5	79.9	11.5	60.5	16.1	69.1	12.3
50	65.3	14.6	73.8	13.5	82.4	13.1	76.4	15.9	70.5	13.8
70	73.2	15.4	84.5	14.1	86.8	13.5	80.3	16.4	84.3	14.2
100	83.5	16.3	88.2	15.9	87.6	14.7	83.2	17.1	85.9	15.3
Average	68.4	15.0	77.25	13.75	84.1	13.2	75.1	16.3	77.45	13.9

Comparing results in Table 4.17 and Table 4.18, we can deduce that, minimizing the external path length is a better choice for the cost of ADSs for reducing the length of CSs. We derive this conclusion by considering the average values of Φ and Δ, which are always greater.

Regardless of the checking sequence method, both the percentage of cases (Φ) and the improvement in the length of checking sequences (Δ) increase with the size of the FSM, consistently. As an extreme example, when the external path length is used as the cost of an ADS, for SP method and for FSMs with 100 states, 4 input and 4 output symbols, for 93.6% of the FSMs, using the minimum cost ADS generated shorter checking sequences,

Table 4.18.: Improvement in checking sequence length by using the ADS with minimum external path length.

n	HEN		UWZ		HIU		SP		DY	
	Φ	Δ	Φ	Δ	Φ	Δ	Φ	Δ	Φ	Δ
30	81.5	11.34	81.3	14.06	80.2	17.95	89.2	19.01	89.2	18.12
50	88.4	12.35	83.4	16.25	96.4	18.11	92.1	24.99	89.4	19.23
70	93.3	16.23	84.5	17.19	93.2	20.27	96.4	28.20	90.8	20.23
100	94.5	19.34	95.6	18.19	93.9	21.86	96.8	30.20	95.1	22.14
Average	89.4	14.8	86.2	16.4	90.9	19.5	93.6	25,6	91.1	19.9

where the improvement in the length reaching to 30.2%.

The results of experiments are manifold. (i) We observe using minimum cost ADSs reduces the length of checking sequences, in general. (ii) The reduction is higher and more probable when the cost of the ADS refers to the external path length. (iii) The reduction is higher and more probable as the size of the FSM gets larger.

Thus we can conclude that the experimental results validate our initial motivation for this work.

4.5.6. Threats to Validity

We try to identify some threats to the validity of experimental results in this section.

First, we try proposed methods for minimizing ADSs on randomly generated FSMs. It is possible that for the FSMs used in real-life situations, the performance of these methods can differ. Although using random FSMs is a general approach for the works in this field, in order to test the generalization of these methods, we also test them on some case studies obtained from benchmark FSM specifications as explained in Section 4.5.4. We see that GLY2 performs better than GLY1, and heuristic LEA that uses LY produces the best results among all approaches both in height and external

path length metrics, similar to the results we obtained from random FSMs. However, GLY1 performs similarly to FLY, which we believe is due to the fact that ADSs for these examples are quite shallow.

Another threat could be our incorrect implementation of these approaches. To eliminate this threat, we also used an existing tool that checks if a given tree is an ADS for an FSM. The ADSs generated by all approaches are all double checked with this tool to see what is produced is really an ADS.

A threat to our motivation for using minimized ADSs could be that, the actual method used for checking sequence generation may or may not support this motivation. In order to see the effect of minimizing ADSs for checking sequence construction, we considered several checking sequence construction methods. The results suggest that, in general, regardless of the checking sequence method, using minimized ADS result in shorter checking sequences. There are occasional cases where using a minimized ADS actually generates a longer checking sequence. A possible reason for such cases is explained in Section 4.5.5. As stated in that part, the number of such cases would possibly decrease if MinWeightedADS problem is considered (instead of MinADS and MinHeightADS) while minimizing ADSs.

4.6. Chapter Summary

In this work, we studied the problem of computing a minimum ADS for a given deterministic, minimal and complete FSM. We introduced several metrics with respect to which such a minimization can be defined, where each metric resulted in a definition of a separate minimization problem. For each metric defined, we showed the problem of deciding a minimum ADS with respect to that metric is NP-complete. We also presented inapproximability results for each one of these minimization problems. Since deterministic FSMs are special cases of nondeterministic FSMs, our results directly apply to nondeterministic FSMs as well.

Our initial motivation for minimizing ADSs is the use of ADSs in the context of check-

ing sequence generation. Generating a minimum ADS is important in such a context, since state recognitions and state verifications are performed by using an ADS, and a considerable part of a checking sequence consists of such state recognitions/verifications. Therefore the length of a checking sequence generated by using an ADS, correlates with the size of that ADS. Due to the hardness and inapproximability of ADS minimization, heuristic algorithms can be used to generate shorter ADSs than the ones that are generated by the only polynomial time algorithm known for ADS generation given in [35]. In order to validate our initial motivation, we considered two different modifications on LY algorithm leading to two different heuristics. The experimental results validate our initial motivation, and show that using shorter ADSs in checking sequence construction improves the length of checking sequences.

5. Using Incomplete Distinguishing Sequences when Testing from a Finite State Machine

5.1. Introduction

In this chapter, we consider constructing Checking Experiments for FSMs that do not have a PDS or an ADS. Many techniques for constructing CEs use DSs to resolve the state identification problem for two reasons: There are polynomial time algorithms that generate CEs when there is a known DS and the length of the CEs is relatively short when designed with a DS [29, 42, 43, 44, 35].

In this Chapter, we use the term *complete* PDS/ADS to denote the Definition 1 and Definition 2 respectively. Although complete DSs have a number of advantages over other approaches used to distinguish states, not all FSMs possess a complete DS, and if there is no complete DS then the state recognition task is carried out by other approaches such as UIOs or W-Sets. However, as explained in [110], using UIOs or W-Sets typically leads to significantly longer CEs. Thus the motivation for the work reported in this text comes primarily from the desire to obtain some of the benefits of complete DSs when constructing CEs for specifications that do not have complete DSs. We therefore consider the case where the FSM does not have a complete DS but instead we would like to form a collection of DSs that, between them, distinguish all of the states of FSM M.

A collection of DSs achieves this when for every pair of states s, s' with $s \neq s'$ there is some DS in the collection that distinguishes s and s'. Another way of describing this is to require a set P_S of subsets of the state set S such that: for all $s, s' \in S$ with $s \neq s'$ we have some $\bar{S} \in P_S$ such that $s, s' \in \bar{S}$; and for all $\bar{S} \in P_S$ there is a DS that distinguishes the states of $\bar{S}$. We show how a CE can be generated using a set of incomplete DSs that distinguish all of the states of the specification. We also explore problems associated with generating 'optimal' sets of incomplete DSs.

While it might seem that complete ADSs are always preferable to complete PDSs, the use of complete PDSs is beneficial in some circumstances. The key advantage of complete PDSs is that they simplify the testing process since there is no need to adapt test input to the observations made. This allows the use of a simpler/cheaper test infrastructure and is also important when the observation and processing of outputs incurs a significant cost by, for example, making testing take longer. In addition, sometimes adaptivity is not possible due to timing constraints; it might take too long for the test infrastructure to make decisions.

In Section 5.3 we consider problems associated with incomplete PDSs, motivated by the fact that sometimes we require test sequences that are not adaptive. We study the following question.

Definition 26 (MaxSubSetPDS problem) *Given FSM M with a finite set of states S and $\bar{S} \subseteq S$, find a subset $\bar{S}'$ of $\bar{S}$ that has a* PDS *such that $|\bar{S}'|$ is maximised.*

One way of expressing the problem of looking for a set of PDSs to distinguish all of the states of an FSM M with state set S is to look for a set P_S of subsets of S such that the followings hold.

- For every pair of states $s, s' \in S$ with $s \neq s'$ there is some $\bar{S} \in P_S$ such that $s, s' \in \bar{S}$; and
- for every $\bar{S} \in P_S$ there is some PDS that distinguishes all of the states of $\bar{S}$.

This leads to the following definition of the MinSetPDS problem.

Definition 27 (MinSetPDS problem) *Given FSM M and a finite set of states S, find a smallest set P_S of subsets of S such that each set in P_S has a* PDS *and for all $s, s' \in S$ with $s \neq s'$ we have that there exists $\bar{S} \in P_S$ such that $s, s' \in \bar{S}$.*

We show that the MaxSubSetPDS and MinSetPDS problems are PSPACE-complete. Moreover, we use results given in [95] to show that the MaxSubSetPDS problem is inapproximable. In Section 5.4 we adapt the problems introduced so far to incomplete ADSs.

Definition 28 (MaxSubSetADS problem) *Given FSM M with a finite set of states S and $\bar{S} \subseteq S$, find a subset $\bar{S}'$ of $\bar{S}$ that has an* ADS *such that $|\bar{S}'|$ is maximised.*

Definition 29 (MinSetADS problem) *Given FSM M and a finite set of states S, find a smallest set P_S of subsets of S such that each set in P_S has an* ADS *and for all $s, s' \in S$ with $s \neq s'$ we have that there exists $\bar{S} \in P_S$ such that $s, s' \in \bar{S}$.*

We show that the MaxSubSetADS and MinSetADS problems are PSPACE-complete. We also show that the MaxSubSetADS problem is inapproximable. This contrasts with the case where we are looking for a complete ADS, a problem that can be solved in polynomial time.

Having determined the complexity of these problems, we consider how incomplete ADSs can be used in generating checking experiments. We show how the W-method and the HSI-method can be adapted to produce checking experiments based on ADSs (and so PDSs). We also demonstrate that the optimisation problems we consider relate very naturally to test optimisation problems for these checking experiment generation methods. A combination of these results shows that the optimisation problems are also relevant to the W-method and the HSI-method. We then propose a greedy algorithm for the MinSetADS problem and report on the results of experiments that evaluated the combination of the proposed CE generation algorithm and the greedy algorithm. The experiments used a set of FSMs and compared the checking experiment size for the W-method, the HSI-method, and the HSI-method adapted to use ADSs. The experimental

subjects included randomly generated FSMs and FSMs drawn from a benchmark and suggest that the proposed method produces shorter CEs that contain fewer resets.

This Chapter is organised as follows. In the next section, we define the terminology and notation used throughout the Chapter. In Section 5.3 we present results related to PDSs and subsequently, in Section 5.4, we present results related to ADSs. In Section 5.5 we show how the W-method and HSI-method can be adapted to use ADSs and in Section 5.6 we report on the results of experiments. In Section 5.7, we summary the chapter and discuss some possible lines of future work.

5.2. Preliminaries

In this Chapter, we consider incomplete ADSs and PDSs and now we provide definitions of incomplete Preset and Adaptive Distinguishing Sequences.

Definition 30 *Given FSM* $M = (S, X, Y, \delta, \lambda)$ *and* $\bar{S} \subseteq S$*, input sequence* w *is an* incomplete Preset Distinguishing Sequence (PDS) *for* $\bar{S}$ *if for all* $s, s' \in \bar{S}$ *with* $s \neq s'$ *we have that* $\lambda(s, w) \neq \lambda(s', w)$*.*

Definition 31 *Given FSM* $M = (S, X, Y, \delta, \lambda)$ *and* $\bar{S} \subseteq S$*, an* Incomplete Adaptive Distinguishing Sequence (ADS) *for* $\bar{S}$ *is a rooted tree* $\mathcal{A}$ *such that the following hold.*

1. *Each node is labelled by a set of states and the root is labelled by* $\bar{S}$*.*
2. *Each leaf of* $\mathcal{A}$ *is labeled by a singleton set (i.e.* $\{s\}$ *for some* $s \in S$*).*
3. *Each edge is labeled by an input/output pair.*
4. *Let us suppose that node* v *has state set* $\bar{S}' \subseteq S$*. If* v *has one or more outgoing edges then these edges are labeled by the same input* x *and have the following property: if there is some* $s \in \bar{S}'$ *such that* $\lambda(s, x) = y$ *then there is a unique edge* $(v, x/y, v')$ *such that* v' *is labelled with the set* $\bar{S}'' = \{s'' \in S | \exists s' \in \bar{S}'.\lambda(s', x) = y \wedge \delta(s', x) = s''\}$ *of states that we can reach from* $\bar{S}'$ *with a transition that has label* x/y*.*

5. *If v has state set $\bar{S}' \subseteq S$ and has one or more outgoing edges then the input x on these edges satisfies the following property: for all $s, s' \in \bar{S}'$ with $s \neq s'$ we have that either $\lambda(s, x) \neq \lambda(s', x)$ or $\delta(s, x) \neq \delta(s', x)$.*

The idea is similar to the rationale given for complete ADSs: an incomplete ADS for $\bar{S}$ defines an experiment where the next input to be applied depends on the previously observed input/output sequence (and so the node reached). The last condition ensures that two states of $\bar{S}$ cannot be mapped to the same state by an incomplete ADS for $\bar{S}$ unless they have already been distinguished. If we apply $\mathcal{A}$ in a state $s \in \bar{S}$ then the resultant input/output sequence is that which labels the path of $\mathcal{A}$ from the root of $\mathcal{A}$ to a leaf and is also the label of a path of M that has starting state s. By the definition of an incomplete ADS the input/output sequences for two distinct states from $\bar{S}$ must differ and so $\mathcal{A}$ distinguishes the states from $\bar{S}$. When we set $\bar{S} = S$ these correspond to the classical notion of Preset and Adaptive Distinguishing sequences. From now on we will write PDSs/ADSs to denote incomplete PDSs/ADSs and throughout this Chapter we refer to the depth of ADS tree $\mathcal{A}$ when we write the length of $\mathcal{A}$.

We present an example FSM, which will be used throughout the Chapter in Figure 5.1. We also present manually computed incomplete ADS for states s_1, s_2 and s_4 in Figure 5.2. The input sequences retrieved from the incomplete ADS are as follows: *baab* for s_1, b for s_2 and *baab* for s_4. Note that input sequence b cannot differentiate states s_2 and s_3 but it can differentiate s_2 from s_1 and s_4, moreover input sequence *baab* can distinguish states s_1 and s_4 from all of the states of the FSM M_2.

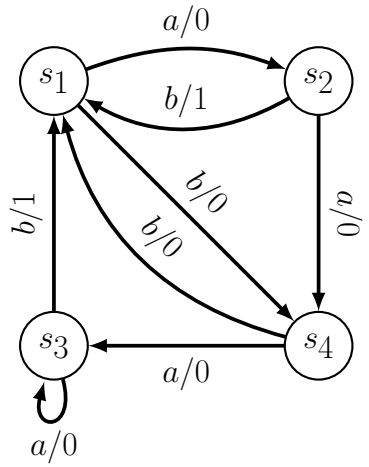


Figure 5.1.: An example FSM M_5

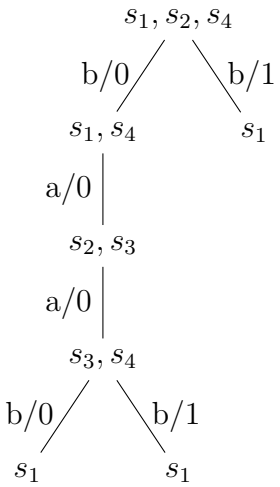


Figure 5.2.: An incomplete ADS for machine M_2 presented in Figure 5.1 where $\bar{S} = \{s_1, s_2, s_4\}$

5.3. Incomplete Preset Distinguishing Sequences

We have defined a finite automata with triple $A = \{Q, \Sigma, \delta\}$, however generally an automaton has an initial state "0" and a set "F" of accepting states and hence defined as five–tuple i.e. $A = \{Q, 0, \Sigma, \delta, F\}$. In this Chapter we refer to the automata with initial state and a set of accepting states. A word is accepted by automaton A, if it takes A from its initial states to an accepting state (a state in F). The set of all words accepted by an automaton A defines a (regular) language denoted $L(A)$.

We show that the MaxSubSetPDS problem is PSPACE-complete through relating it to the Finite Automata Intersection Problem.

It is straightforward to see that the complexity of the FA-INT problem is not altered if we restrict attention to non-empty words since we can decide whether all of the A_i accept ε in polynomial time.

Without loss of generality we assume that the automata in $\mathbb{A}$ have disjoint sets of states. Given an instance of the FA-INT problem, with a finite set $\mathbb{A} = \{A_1, A_2, \ldots, A_z\}$

of automata on a common finite alphabet Σ ($A_i = (Q_i, \Sigma, \delta_i, 0_i, F_i)$), we construct an FSM $\mathcal{M}_1(\mathbb{A}) = (S, X, Y, \delta, \lambda, s_0)$ as follows.

We introduce two new states $Sink_1$, $Sink_2$. Then we take two copies $A_i^1 = (Q_i^1, \Sigma, \delta_i^1, 0_i^1, F_i^1), A_i^2 = (Q_i^2, \Sigma, \delta_i^2, 0_i^2, F_i^2)$ of each automaton A_i and call them *pair automata*. Given $q \in Q_i$ we let q^1 and q^2 denote the corresponding states in Q_i^1 and Q_i^2 respectively. We let $\bar{S} = \{0_1^1, 0_1^2, \ldots, 0_z^1, 0_z^2\}$, which is the set of initial states of the copies of the automata. The set of states of the FSM to be constructed is given by $S = Q_1^1 \cup Q_1^2 \cup Q_2^1 \cup Q_2^2 \cup \ldots \cup Q_z^2 \cup \{Sink_1, Sink_2\}$, where the initial state is selected as 0_1^1. The input alphabet of the FSM is given by $X = \Sigma \cup \{D\}$ and the output alphabet of the FSM is given by $Y = Q_1 \cup Q_2 \cup \ldots \cup Q_z \cup \{0, 1, 2\}$.

The state transitions of the automata in $\mathbb{A}$ are inherited: if $a \in \Sigma$ and $q_i^j \in Q_i^j$ for $1 \leq i \leq z$ and $1 \leq j \leq 2$ then $\delta(q_i^j, a) = r_i^j$ for the state r_i of A_i such that $\delta_i(q_i, a) = r_i$. The state transitions with input D are as follows: $\delta(s, D) = Sink_1$ if and only if ($s = Sink_1$ or there exists i such that $s \in Q_i^1$) and $\delta(s, D) = Sink_2$ if and only if ($s = Sink_2$ or there exists i such that $s \in Q_i^2$). That is, the states of each pair automata (A_i^1 and A_i^2) end in different sink states if input D is supplied.

The output function λ of $\mathcal{M}_1(\mathbb{A})$ is defined as follows, in which $1 \leq i \leq z$.

$$\lambda(s, x) = \begin{cases} q_i, & \text{If } s = q_i^j \text{ for some } q_i^j \in Q_i^1 \cup Q_i^2 \text{ and } x \neq D, \\ q_i, & \text{If } s = q_i^j \text{ for some } q_i^j \in (Q_i^1 \cup Q_i^2) \setminus (F_i^1 \cup F_i^2) \text{ and } x = D. \\ 0, & \text{If } s = Sink_1 \text{ or } s = Sink_2, \\ 1, & \text{If } s \in F_i^1 \text{ and } x = D, \\ 2, & \text{If } s \in F_i^2 \text{ and } x = D, \end{cases}$$

We demonstrate the construction in Figure 5.3.

The basic idea is that until D is received the transitions from a state in $Q_i^1 \cup Q_i^2$ simulate the state transitions of A_i but also tell us which states of A_i are being traversed and so the value of i (the A_i have disjoint state sets). If D is received in a state q_i^j from Q_i^j then the output tells us the value of j if and only if the state q_i^j is such that q_i is an accepting state of A_i. We now explore properties of $\mathcal{M}_1(\mathbb{A})$, proving results that will be brought together in Theorem 16.

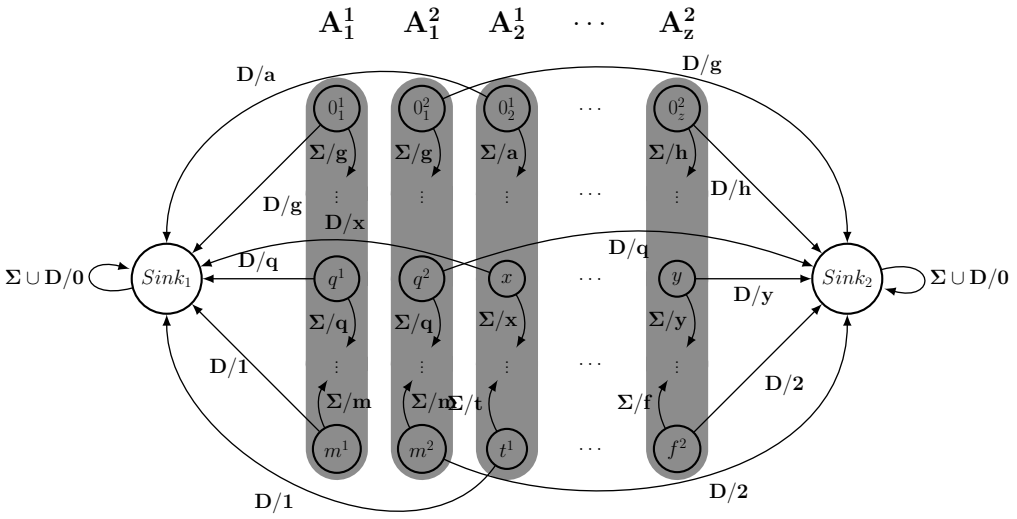


Figure 5.3.: An FSM $\mathcal{M}_1(\mathbb{A})$ constructed from an FA-INT problem instance with $\bar{S} = \{0_1^1, 0_1^2, \ldots, 0_z^1, 0_z^2\}$

Lemma 21 *Let us suppose that set $\mathbb{A} = \{A_1, A_2, \ldots, A_z\}$ of automata have a common alphabet Σ. The FSM $\mathcal{M}_1(\mathbb{A}) = (S, X, Y, \delta, \lambda, s_0)$ has a* PDS *for $\bar{S} = \{0_1^1, 0_1^2, \ldots, 0_z^1, 0_z^2\}$ if and only if there is a non-empty word $w \in \Sigma^\star$ that is accepted by all of the automata (in which case ωD is such a* PDS*).*

We now consider how a non-deterministic Turing Machine can decide whether there is a PDS for a given state set $\bar{S}$ of FSM M. In this process it guesses inputs one at a time and maintains a current set π of pairs of states such that: (s, s') is in π if and only if $s \in \bar{S}$ and the sequences of inputs received takes M from s to s'. It also maintains an equivalence relation r between states from $\bar{S}$: two states s, s'' are related under r if they have not been distinguished by the input sequence w that has been chosen: if $\lambda(s, w) = \lambda(s'', w)$. It is straightforward to see that these two pieces of information can be updated when a new input is received; we do not need to know the previous inputs received. Further, the input sequence received defines a PDS for $\bar{S}$ if and only if no two different states from $\bar{S}$ are related under r.

Lemma 22 *The problem of deciding whether a set $\bar{S}$ of states of FSM M has a* PDS *is in* PSPACE.

We claim that based on the approach used to prove Lemma 21, we can provide an inapproximability result for the MaxSubSetPDS problem but before this we explore the relationship between the optimum solutions of MaxSubSetPDS and MAX FA-INT problems.

Below, given a property P (such as distinguishing k states of an FSM) a word w is said to be a minimal word satisfying P if w satisfies P and no proper prefix of w satisfies P. The following is clear from the proof of Lemma 21.

Lemma 23 *Given set $\mathbb{A}$ of automata, let $OPT_{\mathbb{A}}$ be the set of minimal words that are accepted by the maximum number of automata from $\mathbb{A}$. Further, given $\mathcal{M}_1(\mathbb{A})$ let $OPT_{\mathcal{M}_1(\mathbb{A})}$ be the set of minimal words that maximise the size of the subset of $\bar{S}$ whose states are pairwise distinguished. Then $w \in OPT_{\mathbb{A}}$ if and only if $wD \in OPT_{\mathcal{M}_1(\mathbb{A})}$.*

We can now show that the MaxSubSetPDS problem, of finding a PDS that distinguishes the most states from some set $\bar{S}$, is PSPACE-complete and inapproximable.

Theorem 16 *The* MaxSubSetPDS *problem is* PSPACE-complete *and for any constant $\varepsilon > 0$ approximating the* MaxSubSetPDS *problem within ratio n^{ε} is* PSPACE-hard.

Finally, we consider the problem of finding a smallest set P_S of sets of states such that each set has a PDS (MinSetPDS).

Theorem 17 *The* MinSetPDS *problem is* PSPACE-complete.

5.4. Incomplete Adaptive Distinguishing Sequences

In some situations we want to use preset input sequences in testing since this requires a relatively simple test infrastructure: one that simply applies a sequence of inputs and observes the resultant outputs. However, testing can be more efficient if we use adaptive

tests, where the next input to be applied is chosen on the basis of the observations made. In addition, it is known that the problem of deciding whether an FSM has a (complete) ADS can be solved in polynomial time and there is a polynomial upper bound on the size of such an ADS [35]. These results, together with complexity results in Section 5.3, provide the motivation for considering incomplete ADSs. In this section we therefore explore incomplete ADSs and report that complexity results given for problems related to PDSs hold when we consider ADSs.

We assume that we are given a set $\mathbb{A} = \{A_1, A_2, \ldots, A_z\}$ of automata with alphabet Σ and now describe the FSM $\mathcal{M}_2(\mathbb{A})$ that we construct. We mark the initial states of the automata so that the initial state of A_i is called 0_i and will let $\bar{S} = \{0_1, 0_2 \ldots, 0_z, Sink\}$ for a state $Sink$ described below. We introduce a set $\mathcal{D} = \{d_1, d_2, \ldots, d_z\}$ of new inputs and so there exists one such input d_i for each $A_i \in \mathbb{A}$. The transitions of automata from $\mathbb{A}$ with input Σ are inherited (and given output 0) and the remaining transitions are as follows

- $\delta(Sink, x) = Sink$ for any input $x \in \Sigma \cup \mathcal{D}$.
- If $x \in \mathcal{D}$ then:
  - If $s \in F_i$ then $\delta(s, x) = s$; and
  - $\delta(s, x) = Sink$ otherwise.

The output function λ of $\mathcal{M}_2(\mathbb{A})$ is defined as follows in which $1 \leq i \leq z$.

$$\lambda(s, x) = \begin{cases} i, & \text{If } s \in F_i^1 \text{ and } x = d_i, \\ 0 & \text{For all other cases,} \end{cases}$$

Unlike the previous reduction the output function does not enable us to recognise the states of automaton A_i while we are visiting the states in $Q_i \setminus F_i$. Instead, we can only distinguish states through applying an input from $\mathcal{D}$, possibly after a sequence of previous inputs. Further, we can only distinguish a state 0_i from $Sink$ through applying an input sequence w that takes A_i to an accepting state and then apply d_i. We now

prove that we can construct an ADS for $\bar{S}$ if and only if the automata in $\mathbb{A}$ accept a common word.

In the following we represent an ADS by a set of input/output sequences: the input/output sequences produced from the different states of the FSM M being considered.

Lemma 24 *Let us suppose that set $\mathbb{A} = \{A_1, A_2, \ldots, A_z\}$ of automata have a common alphabet Σ. The FSM $\mathcal{M}_2(\mathbb{A}) = (S, X, Y, \delta, \lambda, s_0)$ has an* ADS *for $\bar{S} = \{0_1, 0_2 \ldots, 0_z, Sink\}$ if and only if there is a non-empty word $w \in \Sigma^\star$ that is accepted by all of the automata (in which case input sequences wd_1, wd_1d_2, $wd_1d_2d_3$, $\ldots$, $wd_1d_2d_3 \ldots d_z$ define an* ADS*).*

We now show that we can check in PSPACE whether a set of states has an ADS.

Lemma 25 *Given FSM M and state set $\bar{S}$, the problem of deciding whether $\bar{S}$ has an* ADS *is in* PSPACE.

The structure of $\mathcal{M}_2(\mathbb{A})$ ensures that when trying to distinguish states in $\bar{S}$ we gain nothing from adaptivity: once we have observed a non-zero output from one of the states we have distinguished this state from all other states in $\bar{S}$ (we must only observe zeros when starting in $Sink \in \bar{S}$). Thus, when exploring ADSs for $\bar{S}$ it is sufficient to consider input sequences.

We now show that the MaxSubSetADS problem, of finding an ADS that distinguishes the most states from some $\bar{S}$, is PSPACE-complete.

Theorem 18 *The* MaxSubSetADS *problem is* PSPACE-complete.

Lemma 24 implies that the optimum solution to the MAX FA-INT problem constitutes an optimum solution to the MaxSubSetADS problem and hence we can reach the following conclusion.

Lemma 26 *Given a set $\mathbb{A}$ of automata, let $OPT_{\mathbb{A}}$ be the set of minimal words accepted by the maximum number of automata from $\mathbb{A}$. Further, let $\mathcal{M}_2(\mathbb{A})$ be the FSM constructed from $\mathbb{A}$ and also let $OPT_{\mathcal{M}_2(\mathbb{A})}$ be the set of minimal* ADSs *that maximise the*

size of the subset of $\bar{S}$ whose states are pairwise distinguished by ADSs. *Then $w \in OPT_{\mathbb{A}}$ if and only if* ADS $wd_1, wd_1d_2, \ldots, wd_1 \ldots d_z$ *is in $OPT_{\mathbb{M}}$.*

Theorem 19 *For any constant $\varepsilon > 0$ approximating the* MaxSubSetADS *problem within ratio n^{ε} is* PSPACE-hard.

As with PDSs, in testing we might want a smallest set of ADSs that, between them, distinguish all states of M (MinSetADS).

Lemma 27 *The* MinSetADS *problem is in* PSPACE.

In the proof of the following, given an instance of FA-INT problem $\mathbb{A} = \{A_1, A_2, \ldots, A_k\}$, we will define an FSM $\mathcal{M}_3(\mathbb{A})$ that is the same as $\mathcal{M}_2(\mathbb{A})$ except for the following:

- For all $1 \leq i \leq z$ we add a state $0'_i$;
- We set $\bar{S} = \{0'_1, 0'_2, \ldots, 0'_z, Sink\}$;
- We introduce new input st; and
- We add the following transitions: from state $0'_i$ there is a transition to 0_i with label $st/0$ and all other inputs take $0'_i$ to $Sink$ with output 0. From all states other than the $0'_i$ the input of st leads to state $Sink$ and output 0.

The essential idea is that in order to distinguish two states from $\bar{S}$ an ADS must start with input st but this ensures that this ADS does not distinguish any two states from $S \setminus \bar{S}$ (and also does not distinguish any state in $S \setminus \bar{S}$ from $Sink$). Thus, any set of ADSs that distinguishes all of the states of $\mathcal{M}_3(\mathbb{A})$ can be partitioned into a subset that distinguishes the states of $\bar{S}$ and a subset that distinguish the states in $(S \setminus \bar{S}) \cup \{Sink\}$ and so there is an ADS for $\bar{S}$ if and only if a smallest set of ADSs for $\mathcal{M}_3(\mathbb{A})$ defines such an ADS.

Lemma 28 *The* MinSetADS *problem is* PSPACE-hard.

We therefore have the following result.

Theorem 20 *The* MinSetADS *problem is* PSPACE-complete.

5.5. Test Generation Using Incomplete DSs

The focus of test generation using complete DSs has largely been on producing checking sequences, where a checking sequence is an input sequence that distinguishes the specification from all faulty FSMs in the given fault model F. However, if we have incomplete DSs then there is a need to follow each transition by more than one DS and to ensure that this is always done from the same state of the SUT (not just the same state of the specification FSM). Algorithms that achieve this when using unique input output sequences or a characterisation set to distinguish states require the generation and use of exponentially long subsequences [82] and so are unlikely to scale well. We therefore concentrate instead on the generation of *checking experiments*.

In order to apply a checking experiment one typically requires the presence of a reliable reset r: a process or input that takes the SUT to its initial state irrespective of the state from which it was applied. Therefore a test case $\alpha \in \mathcal{T}$ will have the form $r\beta$ where r is the reset operation and $\beta \in X^*$. From now on, we will omit the reset operation from all test cases. The reliable reset is used in order to ensure that each test sequence in the checking experiment is applied from the same state of the SUT. Many systems have a reliable reset, which can often be implemented through simply turning the system off and then on again. In this section we assume that the SUT has a reliable reset but we return later to discuss this assumption further. We explore the generation of checking experiments using a set of ADSs since such methods can also be used with PDSs (a PDS defines an ADS).

We initially describe the W-method [4, 48], which is also called the Chow-Vasilevskii method. This techniques requires us to have a known upper bound m on the number of states of the SUT and returns a checking experiment. The W-method uses several components. A state cover is a set of input sequences that reach the states of M and also includes the empty sequence.

Definition 32 *A set V of input sequences is a state cover for FSM M if $\varepsilon \in V$ and for each state s_i of M there is some $v_i \in V$ such that $\delta(s_0, v_i) = s_i$.*

The W-method also uses a characterisation set W, which is a set of input sequences that between them distinguish the states of M.

Definition 33 *A set W of input sequences is a characterisation set for FSM M if for all pairs s_i, s_j of distinct states of M there is some $w \in W$ such that $\lambda(s_i, w) \neq \lambda(s_j, w)$.*

If the SUT is known to have no more states than the specification ($m = n$) then the following is the checking experiment returned.

$$VW \cup VXW$$

The more general case is given by the following

$$V(\{\varepsilon\} \cup X \cup \ldots \cup X^{m-n+1})W$$

For the FSM presented in Figure 5.1 the set VX is given as $\{a, b, aa, ab, baa, bab, ba, bb\}$ and the characterising set (according to the algorithm presented in [21]) is given as $W = \{b, ab, aab\}$. Using the algorithm from [4] the checking experiment produced is follows (after proper prefixes are removed) $\mathcal{T} = \{aab, aaab, aaaab, abb, abaab, abab, baaaab, baaab, babb, babaab, babab, baab, bbb, bbaab, bbab\}$. The length of the test suite is $|\mathcal{T}| = 66$. The number of test cases is 15 and the average test case length is 4.4.

Let us suppose that we have a set $A = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k\}$ of incomplete ADSs such that every pair of distinct states of M is distinguished by some ADS from A; such a set will be said to be *fully distinguishing*. We can find the set of input sequences that can be used by the ADSs in A when they are applied in states of M and we can use this as a characterisation set in the W-method. However, in doing so we lose the benefits of adaptivity. The Harmonized State Identifiers (HSI) method [49] gives us some clues as to how we can incorporate adaptivity. The HSI method uses separate sets of state identifiers: for a state s_i it uses a set H_i of input sequences such that if s_i, s_j are distinct states of M then there are input sequences $w_i \in H_i$ and $w_j \in H_j$ such that a common prefix of w_i and w_j distinguishes s_i and s_j. In test generation, if an input sequence $\alpha \in V(\{\varepsilon\} \cup X \cup \ldots \cup X^{m-n+1})$ reaches the state s_i of M then it is followed by all

input sequences from H_i. This leads to the following checking experiment in which $App(H)$ is a process that applies the set H_i after an input sequence α if $\delta(s_0, \alpha) = s_i$ ($H = \{H_1, \ldots, H_n\}$).

$$V(\{\varepsilon\} \cup X \cup \ldots \cup X^{m-n+1})App(H)$$

According to the algorithm given in [50] the harmonized state identifiers [1] for the FSM presented in Figure 5.1 are given as $H_1 = \{b, aab\}, H_2 = \{b, ab\}, H_3 = \{b, ab\}$ and $H_4 = \{b, aab\}$. The test suite generated with the harmonized state identifiers is as follows (after proper prefixes are removed) $\mathcal{T} = \{aab, aaaab, abb, abaab, baaab, babb, babaab, baab, bbb, bbaab\}$ the test suite length is $|\mathcal{T}| = 43$, where the number of test case is 10 and the average test case length is 4.3. In comparison with the W method, the HSI method reduces the length of the checking experiment by 34.8%.

The HSI method allows different input sequences to be applied in identifying different states. Naturally, adaptivity can also be used to achieve this and below we prove that a fully distinguishing set of ADSs defines a set of state identifiers and so we can adapt the HSI method to use ADSs. Given ADS $\mathcal{A}_j$ and state s_i, let $H(s_i, \mathcal{A}_j) \in X^*$ denote the input portion of the input/output sequence produced when $\mathcal{A}_j$ is applied in state s_i. This is also the input portion of the input/output sequence that labels both a path of $\mathcal{A}_j$ from the root to a leaf and also a path of M starting at s_i. Given state s_i, we will let the set $H_i(A)$ be $\{H(s_i, \mathcal{A}_j) | 1 \leq j \leq k\}$ with prefixes removed: this is the set of input sequences applied when using ADSs from A in state s_i. Then we obtain the following result.

Proposition 7 *Given fully distinguishing set $A = \{\mathcal{A}_1, \mathcal{A}_1, \ldots, \mathcal{A}_k\}$ for FSM M, the $H_i(A)$ are state identifiers for M.*

As a result of this we know that a fully distinguishing set of ADSs defines a set of state identifiers that can be used in the HSI method. In addition, by the definition of $H(s_i, \mathcal{A}_j)$, if we use the state identifiers then when we apply these in a state of the SUT

[1] Where the W set is as given above

that should be s_i we only apply the input sequences that we expect to require; the only condition under which we would apply different input sequences using the ADSs is if a failure is observed. Thus, whether we use the ADSs or the resultant state identifiers does not affect the ability of the test suite to find failures in an SUT that has no more than m states. As a result, given a fully distinguishing set A of ADSs we obtain a test suite in which an input sequence α followed by A involves separately applying α followed by each ADS (required) from A. This leads to the following test suite in which $App(A)$ is a process that, after input sequence α with $s_i = \delta(s_0, \alpha)$, applies the set of ADSs from A required to distinguish s_i from other states of M.

$$CE(M, A, m) = V(\{\varepsilon\} \cup X \cup \ldots \cup X^{m-n+1})App(A)$$

Theorem 21 *Given an FSM M and upper bound m on the number of states of the SUT, if A is a fully distinguishing set of* ADSs *for M then $CE(M, A, m)$ is a checking experiment for M.*

The overall size of $CE(M, A, m)$ depends both on the number of ADSs in A and the lengths of these. However, each input sequence used is followed by a reset and it has been observed that reliable resets can be hard to realise and expensive to apply since they may require a complex system to be reinitialised or may require manual involvement [111, 112]. This has motivated work that aims to minimise the number of input sequences (and so resets) used [38, 81]. In such situations we want to minimise the number of input sequences in the checking experiment and this motivates our interest in the MinSetADS problem.

The use of ADSs does have potential advantages when compared to state identifiers. First, we will typically want to avoid using a redundant set of tests to distinguish states, where redundancy corresponds to the ability to remove some tests without losing the ability to distinguish the states. For example, consider the FSM given in Figure 5.1 again. For M_1, we manually computed the fully distinguishing set $A = \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4\}$ and we present the incomplete ADSs in Figure 5.8.

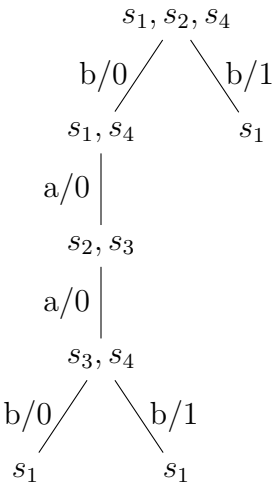


Figure 5.4.: Incomplete ADS $\mathcal{A}_1$ for $\bar{S} = \{s_1, s_2, s_4\}$

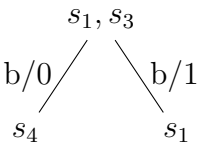


Figure 5.5.: Incomplete ADS $\mathcal{A}_2$ for $\bar{S} = \{s_1, s_3\}$

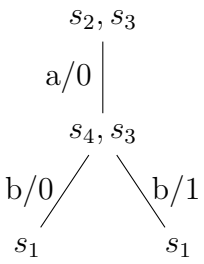


Figure 5.6.: Incomplete ADS $\mathcal{A}_3$ for $\bar{S} = \{s_2, s_3\}$

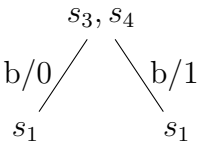


Figure 5.7.: Incomplete ADS $\mathcal{A}_4$ for $\bar{S} = \{s_3, s_4\}$

Figure 5.8.: An incomplete ADSs for machine M_1 presented in Figure 5.1

Note that the input sequence *baab* retrieved from $\mathcal{A}_1$ distinguishes states s_1 and s_4 from every other states. Similarly, the input sequence *ab* retreived from $\mathcal{A}_3$ distinguishes state s_2 from all other states consequently, the input sequence b retreived from $\mathcal{A}_1$ for state s_2 is redundant. The input sequence b retreived from ADSs $\mathcal{A}_2$ and $\mathcal{A}_4$ can distinguish state s_3 from states s_1 and s_4 and finally the input sequence *ab* can distinguish state s_3 from state s_2. Therefore, the resulting state identifiers are given as follows: $H_1(A) = \{baab\}, H_2(A) = \{ab\}, H_3(A) = \{ab, b\}$ and $H_4(A) = \{baab\}$.

Again using the algorithm given in [50] with the supplied state identifiers, the computed test suite is given as follows (after proper prefixes are removed) $\mathcal{T} = \{bbaab,$

$aabaab, abbaab, baab, babbaab, baab, bbaab\}$ $|\mathcal{T}| = 39$. There are 8 test cases and thus the average test case length is 4.8. Recall that we obtained a test suit with length 43 with HSIs. Using ADSs we able to reduce the length of the test suite by 9.3%. Notice that the average test case length is slightly elevated (from 4.3 (HSI) to 4.8 (ADS)), but the number of test cases is reduced (from 10 (HSI) to 8 (ADS)).

There are some advantages to using ADSs. If we have non-redundant ADSs then we are guaranteed to define non-redundant state identifiers; conceptually it is easier to reason about redundancy in a set of ADSs rather than a set of state identifiers that have to relate in a particular way. Second, if we apply the ADSs rather than the state identifier sets then we may obtain additional information that will be useful in debugging: if we apply the ADSs when we expect the state to be s_i and we observe the response for state s_j then we have a possible explanation for the failure (the transition took the SUT to s_j rather than s_i). The HSI method might not provide this information since it only applies the input sequences required to *check* that the state is s_i; adaptivity in the ADSs can lead to other input sequences being applied based on the response and can help identify the state reached even if it is not that expected.

We have shown that a fully distinguishing set of ADSs defines a set of identifying sets that we can use in the HSI technique. We also have the converse, that a set of identifying sets can be used to construct a fully distinguishing set of ADSs, since each sequence in an identifying set defines an ADS (in which there is no adaptivity). Thus, the complexity results in this work regarding ADSs correspond to equivalent results regarding identifying sets and so are relevant to the HSI method.

Given a set $\bar{S}$ of states and identifying sets $\{H_1, H_2, \ldots, H_n\}$, we can identify alternative subsets of the H_i that are sufficient to distinguish the states of $\bar{S}$. Let us suppose that $H'_i \subseteq H_i$ for all $s_i \in \bar{S}$. Then we will say that the H'_i form an identifying set for $\bar{S}$ if for all distinct $s_i, s_j \in \bar{S}$ we have sequences $w_i \in H'_i$ and $w_j \in H'_j$ such that a common prefix of w_i and w_j distinguishes s_i and s_j.

The following shows how the MaxSubSetADS problem relates to problems regarding identifying sets.

Proposition 8 *Let us suppose that $\bar{S}$ is a set of states of FSM M. Then the states in $\bar{S}$ can be distinguished by a single* ADS *if and only if there is an identifying set $\{H_1, H_2, \ldots, H_n\}$ for M such that we can choose subsets $H'_i \subseteq H_i$ of each identifying set, $s_i \in \bar{S}$, under which each H'_i contains only one input sequence.*

The following shows relationship between an HSI problem and MinSetADS.

Proposition 9 *If the states in S can be distinguished by k* ADS *then there is an identifying set $\{H_1, H_2, \ldots, H_n\}$ such that for all $s_i \in S$ we have that H_i has at most k input sequences.*

5.6. Practical Evaluation

In this section, we first present a greedy algorithm that aims to compute a fully distinguishing set with minimum cardinality for a given FSM M. Later we present the results of experiments using randomly generated FSMs and some benchmark FSMs. We emphasise that the main aim of the experiments was to explore the effect of using a set of ADSs instead of harmonised state identifiers. Other techniques such as H [113] and SPY [52] that use such sets of tests are likely to similarly benefit. The experiments compared the ADS method with the W and HSI methods. In order to analyse the effect of using incomplete ADSs, we study the cost of checking experiments that are constructed by these methods. As reported in [114, 53], the cost of a checking experiment is given by three properties: 1) The length of the checking experiment, 2) Average test case lengths and 3) The number of resets. In the experiments we analyse these three properties of the constructed checking experiments.

5.6.1. Greedy Algorithm

Before the actual algorithm is presented, we first have to define some important notions concerning the greedy algorithm. We present the list of symbols with their definitions in Table 5.1. In summary, the greedy algorithm receives an FSM M and integer value ℓ

Symbol	Description
$\mathbf{T}$	A set of tree structures.
T	A tree structure.
N, E	Set of nodes, set of edges.
$\mathcal{I}(n)$,$\mathcal{C}(n)$	Initial and Current sets for node n.
$x(n), y(n)$	Input sequence, output sequence for node n.
$\mathcal{M}$	A set of current sets.
$\mathcal{L}$	Set of nodes returned by the Refine procedure.
$\mathcal{V}$	Set of set of nodes used by the Refine procedure.
$\mathcal{N}$	Set of set of nodes used by the Greedy algorithm.
$\ell \in \mathbb{Z}_{\geq 1}$	Upper bound on the tree height.
$\mathcal{Q}$	A set of pairs of states.
$\phi(\mathcal{Q}, \mathcal{N}[x]) \in \mathbb{R}_{\geq 0}$	Heuristic function 1.
$\Theta_x(\mathcal{M}, \mathcal{N}[x]) \in \mathbb{Z}_{\geq 0}$	Heuristic function 2.
$\mathcal{F} : S \times S \rightarrow \in \{0, 1\}$	A function used by the Heuristic function 2.

Table 5.1.: Nomenclature for the greedy algorithm

and it returns a set of trees $\mathbf{T} = \{T_1, T_2, \dots\}$ such that all trees in this set have depth at most ℓ and set $\mathbf{T}$ defines a fully distinguishing set. The aim is to produce a set with minimum cardinality but, since a heuristic is used (a greedy algorithm), this is not guaranteed.

Basic Notation

A tree $T(E, N) \in \mathbf{T}$ consists of a set of edges (E) and nodes (N). An edge $e \in E$ has a label that is an input output pair x/y where $x \in X$ and $y \in Y$. A node $n \in N$ captures the following information: The initial set $\mathcal{I}(n)$, the current set $\mathcal{C}(n)$ and strings $x(n)$ and $y(n)$ that give the input and output sequences that label the path from the root of $T(E, N)$ to the node n. For the root node n_1 we have that $x(n_1) = y(n_1) = \varepsilon$ and $\mathcal{I}(n_1) = \mathcal{C}(n_1) = S$. Input sequence $x(n)$ is defined as $x(n) = x(n')x$ where n' is the parent node of the current node n and x is the input retrieved from the edge between n' and n. Further, $y(n) = y(n')y$ where n' is the parent node of the current node n and y is the output retrieved from the edge between n' and n. We define initial and current sets as follows: $\mathcal{I}(n) = \{s \in \mathcal{I}(n') | y(n) = \lambda(s, x(n))\}$ and $\mathcal{C}(n) = \{\delta(s, x(n)) | s \in \mathcal{I}(n)\}$. There are two types of nodes: a node is a *leaf node* if and only if it has no outgoing edges; otherwise it is an *internal node*.

The greedy algorithm repeatedly executes a routine called *refine*. The refine routine receives a node n and a single input x and produces a set of nodes $\mathcal{L}$ or returns failure. The Refine routine is summarised in Algorithm 3.

In lines 4 and 5 the refine routine forms groups of states according to the observed outputs, putting together states that lead to the same output. In lines 8-12, for each group, the refine algorithm forms a node. The key point here is that we do not ignore an input if it merges states i.e. $\delta(s, x) = \delta(s', x)$ and $\lambda(s, x) = \lambda(s', x)$. The reason for such flexibility comes primarily from the fact that instead of a single tree, the aim of the greedy algorithm is to construct a set of trees. Thus, one tree does not need to distinguish all pairs of states: a leaf node of a tree can have two or more initial states as long as other trees distinguish these.

Algorithm 3: Refine

Data: n, x

Result: $\mathcal{L}$ or failure.

begin

$\mathcal{V}[|Y|]$,$\mathcal{L} \longleftarrow \emptyset$;

if $|\mathcal{C}(n)| = 1$ **then**

  Return failure;

for $s \in \mathcal{C}(n)$ **do**

  Push $\delta(s, x)$ onto $\mathcal{V}[\lambda(s, x)]$;

$i \leftarrow 0$;

for $i < |Y|$ **do**

  if $\mathcal{V}[i] \neq \emptyset$ **then**

    Declare new node n';

    $\mathcal{C}(n') = \mathcal{V}[i]$;

    $\mathcal{I}(n') = \{s \in \mathcal{I}(n) | i = \lambda(\delta(s, x(n)), x)\}$;

    $x(n') = x(n')x$;

    $y(n') = y(n')i$;

    Push n' to $\mathcal{L}$;

Return $\mathcal{L}$;

The Algorithm

The greedy algorithm is recursive. It receives an FSM M, positive integer ℓ, a set $\mathcal{Q}$ of pairs of states not yet distinguished, and a set $\mathcal{M}$ of sets of states. The summary of the Greedy Algorithm is given in Algorithm 4. Initially $\mathcal{Q}$ contains the set of all pairs of distinct states. At each iteration, the greedy algorithm forms a tree structure T. We initiate a tree T by introducing a root node n_1 (Line 2). The root node has the following information: $\mathcal{I}(n_1) = S$, $\mathcal{C}(n_1) = S$, $x(n_1) = \varepsilon$, $y(n_1) = \varepsilon$. Then for each input $x \in X$, it executes the refine routine once and notes the obtained set of set $\mathcal{N}$ of nodes as $n_i, n_j \ldots n_k$ for some $k \geq 1$ (Lines 6 – 7).

The set $\mathcal{M}$ holds the set of current sets that belong to nodes which cannot be refined. Intuitively, if current sets of all possible sons of the current node are in set $\mathcal{M}$, there is no point for investigating this node any more, consequently, we also add the current set of such node to $\mathcal{M}$ as well (Lines 8 – 9).

Otherwise the greedy algorithm evaluates the "goodness" of inputs by calling (Line 10) a heuristic function which is defined as follows:

$$\Phi_x(\mathcal{Q}, \mathcal{N}[x]) = \sum_{n \neq n' \in \mathcal{N}[x]} |\mathcal{Q} \cap \mathcal{I}(n) \times \mathcal{I}(n')| \tag{5.1}$$

For any pair of nodes Heuristic 5.1 forms a set of pairs of states and counts the number of occurrences of pairs in set $\mathcal{Q}$. That is to say Heuristic 5.1 will return the number of pairs of states in $\mathcal{Q}$ distinguished. Intuitively a "good" input is an input that maximises this mass function: $\forall x' \in X, x \neq x'$ we have that $\Phi'_x \leq \Phi_x$.

Now consider the machine M_2 in Figure 5.9. According to Heuristic 5.1, the greedy algorithm will initially select input A to distinguish state s_3 from other states. Afterwards, the algorithm will try to distinguish states s_1, s_2 and s_4. However, according to Heuristic 5.1, there is no difference between inputs A and B and thus, the greedy algorithm can try input A repeatedly and fall into an infinite loop. To prevent, this in such cases, (i.e. if Heuristic function 5.1 cannot differentiate between inputs (Line 10)), the greedy algorithm decides the next input by the usage of the following greedy

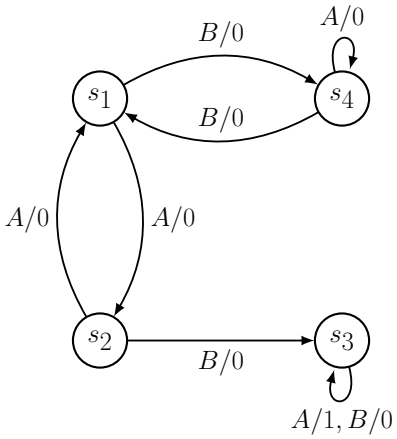


Figure 5.9.: An FSM M_6

function (Line 11): Let $\mathcal{M} = \{\mathcal{C}(1), \mathcal{C}(2), \ldots, \mathcal{C}(|\mathcal{M}|)\}$,

$$\Theta_x(\mathcal{M}, \mathcal{N}[x]) = \sum_{n \in \mathcal{N}[x], \mathcal{C}(i) \in \mathcal{M}} \mathcal{F}(\mathcal{C}(i), \mathcal{C}(n)) \tag{5.2}$$

where $\mathcal{F}$ is a binary function which returns 1 if and only if the parameters $\mathcal{C}(i)$, and $\mathcal{C}(n)$ are identical sets. Otherwise it returns 0. Intuitively function $\mathcal{F}$ is defined as follows:

$$\mathcal{F}(\mathcal{C}(a), \mathcal{C}(b)) = \begin{cases} 1 & \text{if } \mathcal{C}(a) = \mathcal{C}(b) \\ 0 & \text{Otherwise} \end{cases} \tag{5.3}$$

If the greedy choice cannot be given, the greedy algorithm declares a failure and adds the current set of this node to set $\mathcal{M}$ (Lines 14 – 15). Otherwise, if the greedy choice is made (Lines 10 – 13), the greedy algorithm adds the current nodes and edges that are obtained by the corresponding input (Lines 16 – 19) to the current tree T. While doing this the greedy algorithm checks, if the current set of the new node exist in one of its proper ancestor n' i.e. $\exists n' \in N$ such that $\mathcal{C}(n) = \mathcal{C}(n')$ and there exists a simple path from n' to n (Line 19).

Afterwards the greedy algorithm selects another unvisited node and repeats the procedure (Line 20).

The greedy algorithm repeatedly executes this scheme until no node is refineable or the depth of the tree T becomes larger than ℓ.

After the tree is constructed the greedy algorithm removes pair of states (s, s') from $\mathcal{Q}$ if s, s' are members of initial sets of different leaf nodes i.e. $s \in \mathcal{I}(n) \wedge s' \in \mathcal{I}(n')$ for $n \neq n'$ (Lines 22 – 24). Finally, if $|Q| > 0$ the algorithm calls itself (Lines 25 – 26). The Greedy algorithm stops when $|Q| = 0$ (Lines 27 – 28). Since the greedy algorithm is a heuristic the resultant tree T need not be optimal; later we report the results of experiments used to explore the effectiveness of this approach.

We now need to show that at each iteration the greedy algorithm computes an incomplete ADS. In order to achieve this we first need to emphasise some properties of tree T. First recall that the greedy algorithm selects a single set of nodes $\mathcal{N}[x]$ while constructing a tree T and since $\mathcal{N}[x]$ is constructed by a single input x, the outgoing edges are labeled by identical inputs and are labeled with different outputs. Therefore the following immediately follows from the construction of tree T.

Lemma 29 *Let n be an internal node of tree T with children $n_1, n_2, \ldots, n_p$ and let x be the input portion of the labels of the edges from node n. The following hold:*

1. $\delta(\mathcal{C}(n), x) = \cup_{i=1}^{p} \mathcal{C}(n_i)$.
2. *For all $1 \leq i \leq p$ we have that $|\lambda(\mathcal{I}(n_i), x(n_i))| = 1$.*
3. *For all $1 \leq i < j \leq p$ we have that $\lambda(\mathcal{I}(n_i), x(n)) = \lambda(\mathcal{I}(n_j), x(n))$ and $\lambda(\mathcal{I}(n_i), x(n)x) \neq \lambda(\mathcal{I}(n_j), x(n)x)$.*

Moreover, consider distinct leaf nodes (n, n') then using Corollary 29 we know that the output observed from any pair of states $s \in \mathcal{I}(n)$ and $s' \in \mathcal{I}(n')$ are different.

Lemma 30 *Let n, n' be distinct leaf nodes of tree T. If $s \in \mathcal{I}(n)$ and $s' \in \mathcal{I}(n')$ then $\lambda(s, x(n)) \neq \lambda(s', x(n))$ and $\lambda(s, x(n')) \neq \lambda(s', x(n'))$.*

Now we show that a tree T returned by the greedy algorithm defines an incomplete ADS.

Lemma 31 *Let T be a tree returned by the greedy algorithm such that $\bar{N} = \{n_1, \ldots, n_p\}$ is the set of leaf nodes of T. Let $\bar{S}$ be a set of states such that for all $1 \leq i \leq p$ we have that $|\mathcal{I}(n_i) \cap \bar{S}| \leq 1$. Then T defines an incomplete* ADS *for set $\bar{S}$.*

Algorithm 4: Greedy Algorithm

Data: FSM M, ℓ, $\mathcal{Q}$, $\mathbf{T}$, $\mathcal{M}$

Result: A set of trees $\mathbf{T}$

begin

1 $n_1 \leftarrow (S, S, \varepsilon, \varepsilon)$, $\mathcal{N} \leftarrow \emptyset$,$a := 0$, $h := 0$;

2 Add n_1 to tree $T(N, E)$, $v \leftarrow n_1$;

3 **while** $v \neq NULL$ **do**

4 $max := 0$, $index := -1$;

5 **if** $|x(v)| \leq \ell$ **then**

6 **for** $x \in X$ **do**

7 $\mathcal{N}[x]$ = *Refine(v,x)*;

8 **if** $\mathcal{N} \subseteq \mathcal{M}$ **then**

9 Add $\mathcal{C}(v)$ to $\mathcal{M}$

10 **else if** $\forall x, x'$, $\Phi(\mathcal{Q}, \mathcal{N}[x]) = \Phi(\mathcal{Q}, \mathcal{N}[x'])$ **then**

11 $index \leftarrow x$ where $\nexists x' \in X$ such that $\Theta_{x'}(\mathcal{M}, \mathcal{N}[x']) < \Theta_x(\mathcal{M}, \mathcal{N}[x])$

12 **else if** $\exists x, x'$, $\Phi(\mathcal{Q}, \mathcal{N}[x]) \neq \Phi(\mathcal{Q}, \mathcal{N}[x'])$ **then**

13 $index \leftarrow x$ where $\nexists x' \in X$ such that $\Phi_x(\mathcal{Q}, \mathcal{N}[x]) < \Phi_{x'}(\mathcal{Q}, \mathcal{N}[x'])$

14 **if** $index = -1$ **then**

15 Add $\mathcal{C}(v)$ to $\mathcal{M}$

16 **else**

17 **for** $n \in \mathcal{N}[index]$ **do**

18 **if** *No proper ancestor of node v have a current set* $\mathcal{C}(n)$ **then**

19 Add node n to N and add edge to E.

20 $v \leftarrow$ next_unvisited_node, clear B;

21 Push T onto $\mathbf{T}$;

22 **for** *All pair of leaf nodes* n, n' *where* $n \neq n'$ **do**

23 **if** $s \in \mathcal{I}(n) \wedge s' \in \mathcal{I}(n')$, **then**

24 Pop pair of states (s, s') from $\mathcal{Q}$;

25 **if** $\mathcal{M} \neq \emptyset$ **then**

26 Return Greedy(M, ℓ, $\mathcal{Q}$, $\mathbf{T}, \mathcal{M}$);

27 **if** $\mathcal{M} = \emptyset$ **then**

28 Return $\mathbf{T}$;

Although the algorithm is easy to implement, it may not compute a fully distinguishing set for a given FSM. This will happen if the upper bound on ADS length is too short. Note that for an FSM M with n states, every pair of states is distinguished by a sequence of length at most $n-1$ and it is sufficient to use at most $n-1$ such sequences in order to distinguish all of the states of M. Thus, the algorithm is guaranteed to return a fully distinguishing set if we use value of ℓ as $n-1$ or larger.

5.6.2. Experimental results

Test Generation

This section describes experiments used to explore the performance of the greedy algorithm and the effect of using fully distinguishing sets by evaluating the resultant checking experiments. We randomly generated FSMs with 4, 6, and 8 inputs and outputs using the tool utilised in [43, 108]. The FSMs were constructed as follows: First, for each input x and state s_i we randomly assigned the values of $\delta(s_i, x)$ and $\lambda(s_i, x)$. After an FSM M was generated we checked its suitability as follows. We first checked that M is synchronisable, that is whether M has a reset. In order to do this we implemented the existence check algorithm described in [34]. Then we checked whether M is strongly connected[2]. Afterwards we checked that M is minimal and then used the LY-algorithm [35] to check that M does not have a complete ADS. If the FSM failed one or more of these tests then we omitted this FSM and produced another. Consequently, all FSMs used had a reset, were strongly connected and minimal, and had no complete ADS. By following this procedure we constructed 100 FSMs with 5 states, 100 FSMs with 10 states, ..., 100 FSMs with 100 states. This was done for each size of the input and output alphabets so in total we used $6 * 10^3$ FSMs. We used an Intel $i7$ 3630 $Q3$ Ivy-Bridge CPU with 8 GB RAM to carry out these tests. We implemented W, HSI and the greedy algorithm using C++ language and compiled on Visual Studio .Net 2012.

The checking experiment generation methods considered the care where the SUT

[2] M is strongly connected if for any pair (s, s') of states of M there is some input sequence that takes M from s to s'.

has no more states than the specification FSM. The W and the HSI methods were implemented according to the descriptions presented in references [4, 50]. As explained earlier, we also implemented the HSI method adapted for a fully distinguishing set of ADSs and used the greedy algorithm to produce the ADSs. Since the outputs are uniformly distributed during the generation of FSMs, one would expect the average depth of the ADS to be around $\lceil \log_q n \rceil$, where n and q are the number of states and the number of outputs, respectively. For our experiments with 4,6 and 8 outputs and the number of states ranging between 5 and 100, the length of ℓ is expected to be 2–4 for 4 outputs ($\lceil \log_4 5 \rceil = 2$ and $\lceil \log_4 100 \rceil = 4$), 1–3 for 6 outputs ($\lceil \log_6 5 \rceil = 1$ and $\lceil \log_6 100 \rceil = 3$), and 1–3 for 8 outputs ($\lceil \log_8 5 \rceil = 1$ and $\lceil \log_8 100 \rceil = 3$). For each FSM we set the upper bound on ADS depth to be twice this value $\lceil \log_q n \rceil$ i.e. $\ell = 2 * \lceil \log_q n \rceil$. With these values, we were able to produce fully distinguishing sets.

Checking Experiment Length

We present the results using boxplot diagrams generated by ggplot2 library of the tool R [115, 116, 117]. For each box the first quartile corresponds to the lowest 25% of data, the second quartile gives the median, and the third quartile corresponds to the highest 25%. For each boxplot we added the smoothing line computed with the LOESS [118] method, and the semi-transparent ribbon surrounding the solid line is the 95% confidence interval. We also give the jitter plot[3] where each dot corresponds to the result of an execution of the corresponding method, where input is a single FSM.

Figure 5.10 presents information regarding the checking experiment length for FSMs where $p/q = 4/4$. The figure suggest that except for the FSMs with $n = 5$, the checking experiment length obtained by the modified HSI method (ADS method) is less than the HSI and W methods. Moreover, when $n \geq 40$ the third quantile of the boxplots drawn for the ADS method are lower than the first quantile of the boxplots drawn for the HSI method. These results suggest that usually the ADS method produces shorter test suites

[3]The Jitter plot adds a noise to the data to prevent occlusion in the statistical visualizations. Each data point displace on horizontal axis.

than the shortest test suite lengths produced by the HSI method. The LOESS band indicates that, for both methods, the expected mean of the checking experiment lengths are statistically different and the average mean of the test suite lengths computed by the ADS method is smaller than that of HSI method. One promising observation is that the difference between the HSI and ADS methods increases with the number of states.

We present the results of the experiments performed on FSMs with $p/q = 6/6$ and $8/8$ in Figures 5.11 and 5.12 respectively. We observe that the ADS method produces shorter checking experiments on average. Moreover we note that when $n \geq 20$ and $p/q = 6/6$ and when $n \geq 25$ and $p/q = 8/8$ the first quantiles of the boxplots drawn for the ADS method are below the third quantiles of the boxplots drawn for the HSI method. The LOESS bands indicate that as the number of inputs and outputs increases, the difference between the mean of the test suite lengths computed by the HSI and ADS methods are statistically different and increases.

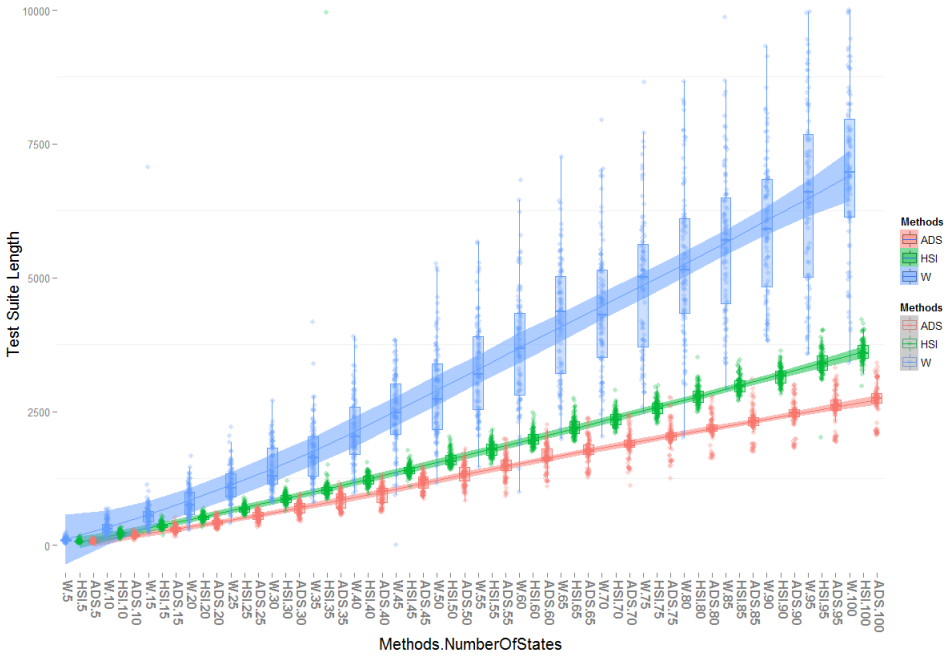


Figure 5.10.: Comparison of test suite lengths. Each boxplot summarises the distributions of 100 FSMs where $p = 4, q = 4$

To support our observations, we used R to perform a non-parametric *Kruskal-Wallis Significance* [119] test on ADS and HSI results. For each method, for each state number

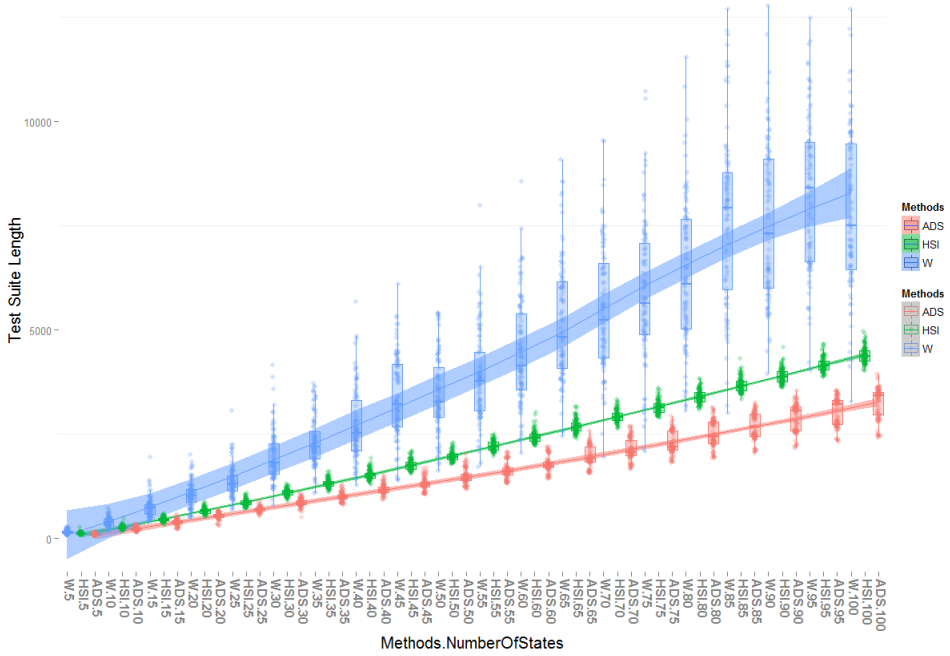


Figure 5.11.: Comparison of test suite lengths. Each boxplot summarises the distributions of 100 FSMs where $p = 6, q = 6$

(n) and for each input/output values (p/q), we constructed two sets of samples such that one set holds the results for the ADS method and the other set holds the results for the HSI method. Afterwards, we ran the Kruskal-Wallis difference test on these sets of samples. The *null hypothesis* $(H0)$ assumes that these two sets of samples have identical distributions. We selected the α value to be 0.05 and $df = 1$[4]. Therefore according to the table given for the Chi-Squared values in [120], if the null-hypothesis is correct then the Chi-Squared values $(\mathcal{X}^2)$ of these measurements should be smaller than 3.841. Otherwise, we should reject the null-hypothesis and suggest that there is a significant difference. The results in Table 5.2 suggests that except for $n = 5$, $p/q = 4$, the differences between the lengths of the checking experiments constructed by the HSI and the ADS method are statistically significant. Combining these results with the results given in Figures 5.10, 5.11 and 5.12 we can declare that, at least for the FSMs used in these experiments, replacing harmonised state identifiers with the incomplete ADSs tends to reduce checking experiment length.

[4]Here *df* stands for the *Degree of Freedom*, which is given by $k - 1$ where k is the number of samples

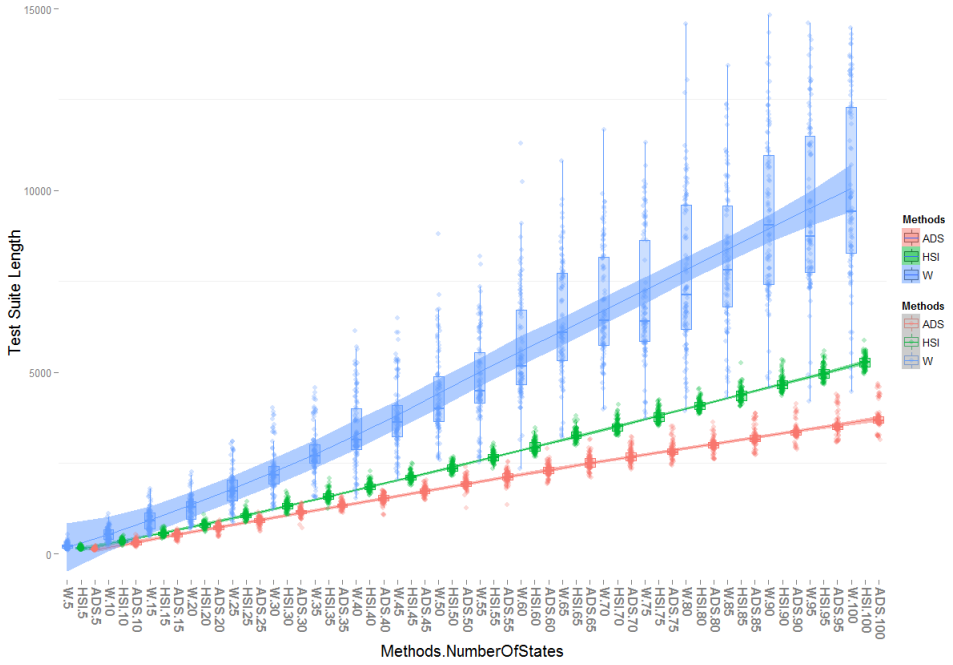


Figure 5.12.: Comparison of test suite lengths. Each boxplot summarises the distributions of 100 FSMs where $p = 8, q = 8$

Finally, for each FSM we compared the lengths of the CE produced by the ADS, W, and HSI methods. Since the averages show the ADS method producing the shortest CE and the W method the longest, for each number of states and number of inputs/outputs we counted how many results (for the 100 FSMs) did not conform to the expected pattern. Table 5.3 shows the results. Interestingly, when there are only a few states we find that many of the CEs do not follow the expected pattern. For example, for 48% of the FSMs with 5 states and 4 inputs/outputs we have that the HSI method produced a shorter CE than the ADS method, in 25% of cases the W method produced a shorter CE than the ADS method, and in 24% of cases the W method produced a shorter CE than the HSI method. However, these numbers reduce as the number of states increases and, for example, for all of the 300 FSMs with 100 states we find that the CE produced by the HSI method is no shorter than the CE produced by the ADS method. Thus, it appears that for larger FSMs the ADS method consistently produces shorter CEs than

supplied to the Kruskal-Wallis test and in our case $k = 2$.

Table 5.2.: The results of a Kruskal-Wallis Significance Tests performed on the Checking Experiments Length

Input/output (p/q) values	Corresponding $\mathcal{X}^2$–values for different number of states (n). Reject H_0 when $\mathcal{X}^2 > 3.841$																			
	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100
4/4	9e-04	16.39	52.79	63.1	73.36	88.98	73.53	105.21	100.94	91.92	97.34	97.32	111.65	123.89	127.56	140.52	140.29	145.19	141.3	147.17
6/6	47.05	21.95	58.36	100.21	128.4	137.75	142.03	139.1	144.28	140.29	148.03	149.25	147.11	148.6	148.89	149.01	149.25	149.25	149.25	149.25
8/8	55.97	14.72	10.97	26.15	87.87	81.4	122.51	140.46	145.4	142.46	147.67	146.4	147.08	147.08	149.22	149.25	149.25	149.25	149.25	149.25

the HSI and W methods.

Average Test Case Length

In the previous subsection we showed that using a fully distinguishing set can reduce the overall length of a checking experiment. In this and next subsections we investigate possible reasons for this achievement.

Figure 5.13 shows how the average test case length varied with the number of states when the number of inputs and outputs is four. Although the graphical representation suggests that the distribution and the expected mean of the average test case lengths are different, this difference is not more than 2.5 inputs on average. Moreover, as we increase the number of inputs and outputs to six and eight we observe that this difference does not change. That is both the W, HSI and the ADS methods produce comparable test cases. However, we note that as the number of inputs and outputs increases, the average lengths of test cases reduces. This is expected since having more inputs and outputs increases the number of transitions and therefore the length of a path from the initial state to another state decreases. What is more, more inputs and outputs allows the use of shorter characterising sets, harmonized state identifiers and ADSs to distinguish states.

Table 5.3.: Pairwise differences of CE Lengths. Each value corresponds to the occurrence of the comparison criteria in 100 FSMs.

Input/output (p/q) values	Comparison Criteria	Number of States.																			
		5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100
4/4	$W < HSI$	24	9	14	10	5	2	8	4	6	2	1	4	2	1	0	0	1	0	0	0
	$W < ADS$	25	6	5	7	4	0	3	0	4	1	1	2	1	0	0	2	0	0	0	0
	$HSI < ADS$	48	32	15	19	16	5	12	7	6	10	12	11	4	5	3	2	0	0	2	0
6/6	$W < HSI$	21	6	6	9	3	2	1	5	5	4	3	3	2	2	4	3	2	0	0	1
	$W < ADS$	7	6	2	7	0	1	0	0	2	1	0	1	1	1	2	0	1	0	0	1
	$HSI < ADS$	15	28	13	2	2	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0
8/8	$W < HSI$	15	10	7	7	4	6	3	3	2	0	2	1	1	0	0	1	1	0	2	3
	$W < ADS$	4	7	4	2	1	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0
	$HSI < ADS$	1	31	38	18	6	7	1	0	0	0	0	0	0	0	0	0	0	0	0	0

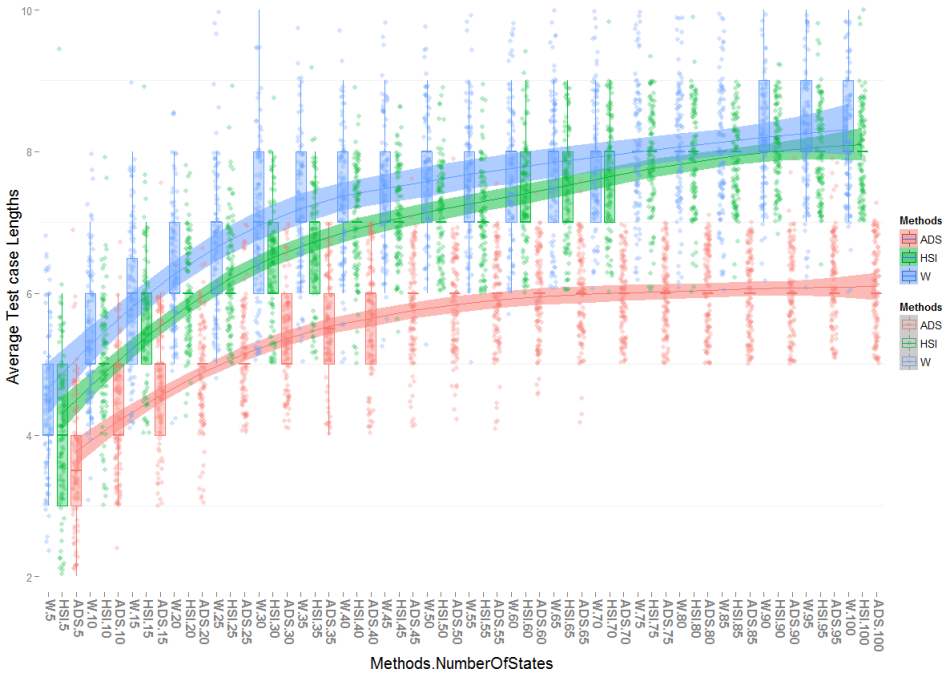


Figure 5.13.: Comparison of average test case lengths. Each boxplot summarises the distributions of 100 FSMs where $p = 4, q = 4$

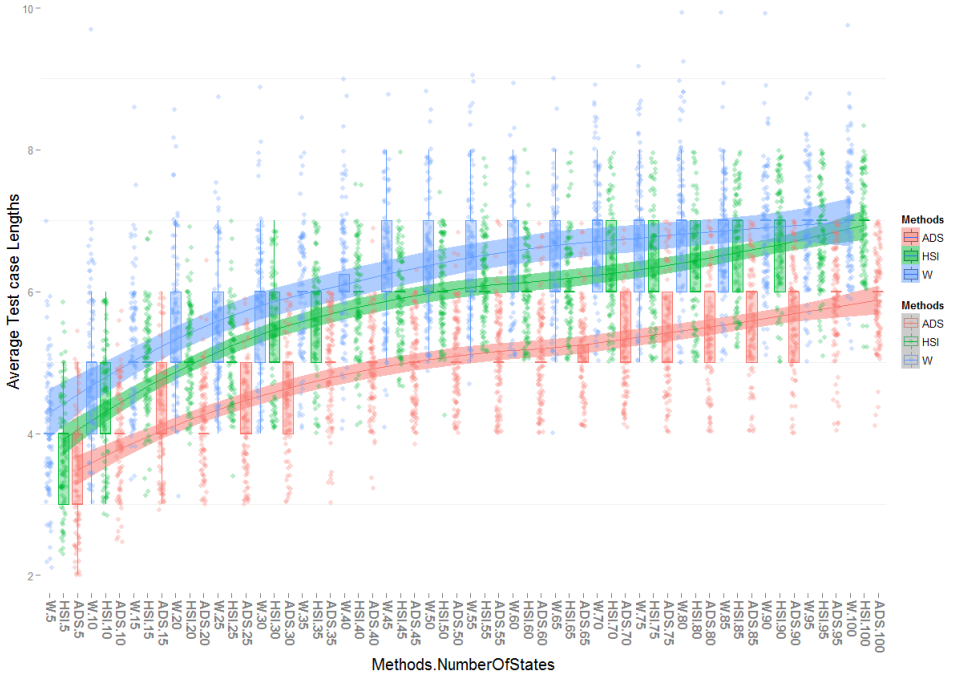


Figure 5.14.: Comparison of average test case lengths. Each boxplot summarises the distributions of 100 FSMs where $p = 6, q = 6$

Figure 5.15.: Comparison of average test case lengths. Each boxplot summarises the distributions of 100 FSMs where $p = 8, q = 8$

We performed a Kruskal-Wallis test on the test case lengths. Table 5.4 suggest that the distributions are statistically different. Therefore we can conclude that the ADS method produces shorter test cases compared to the HSI method.

Although we showed that the average test cases are shorter when ADS method is used, we claim that the small difference (2.5 inputs maximum) probably cannot explain the reduction of the checking experiment lengths. Therefore we now investigate the average number of resets of the computed checking experiments.

Number of Resets

As a reset is applied before a test case, the number of resets is the same as the number of test cases in a checking experiment. Figure 5.16 represents the results of the conducted experiments when $p/q = 4/4$.

Based on results in [53], we expect that the number of resets when using ADSs to be less than that for the W method and similar to the value for the HSI method. Although the distributions are different, the maximum difference between the means is only 3.3%. This is the case when $p/q = 4/4$. However, the difference between the HSI and ADS methods increases with the number of inputs/outputs. We observe that when $p/q = 6/6$ and $n \geq 20$ and when $p/q = 8/8$ and $n \geq 20$, the average number of resets is lower when using the ADS method. It appears that as the number of inputs/outputs increases the difference between the HSI and the ADS methods increases. These observations are similar to those for the checking experiment lengths.

Table 5.4.: Results for Non-parametric Kruskal-Wallis Significance Tests

Input/output (p/q) values	Corresponding $\mathcal{X}^2$–values for different number of states (n)																			
	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100
4/4	32.09	24.4	18.34	17.27	8.83	19.71	20.53	11.36	14.55	9.47	3.99	13.85	8.11	12.56	12.89	11.19	16.14	28.33	7.86	4.07
6/6	26.4068	13.3254	17.518	20.1425	19.73	16.33	12.14	7.18	14.8	27.76	41.72	13.94	32	19.45	12.99	14.4	17.44	15.13	12.92	12.83
8/8	46.57	37.39	17.79	10.21	7.24	20.14	13.26	12.55	23.15	10.14	12.87	11.89	16.13	15.41	13.06	12.03	5.76	8.38	24	15.66

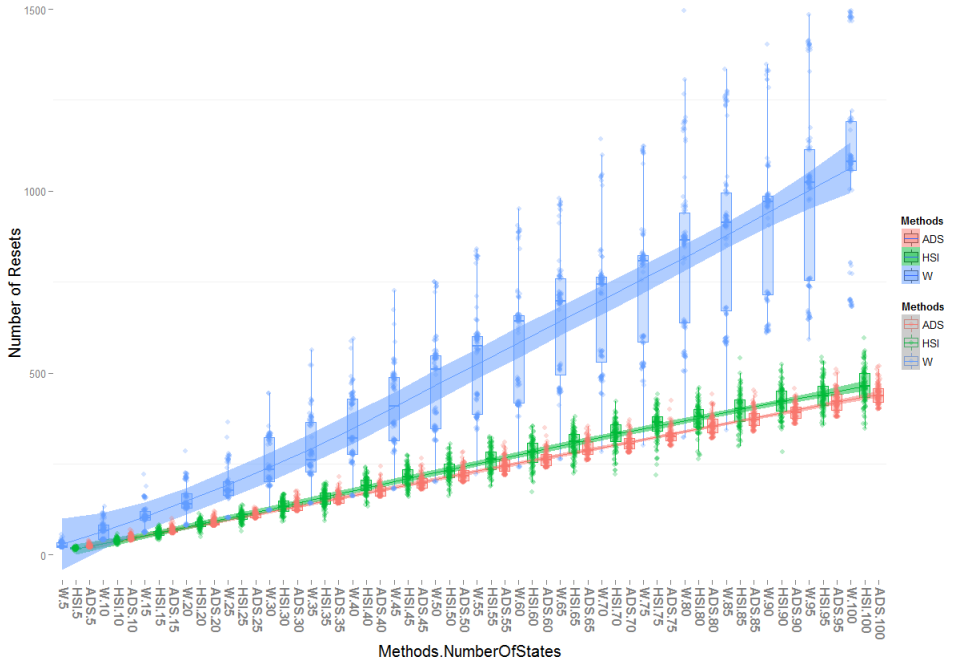


Figure 5.16.: Comparison of number of resets required for methods. Each boxplot summarises the distributions of 100 FSMs where $p = 4, q = 4$

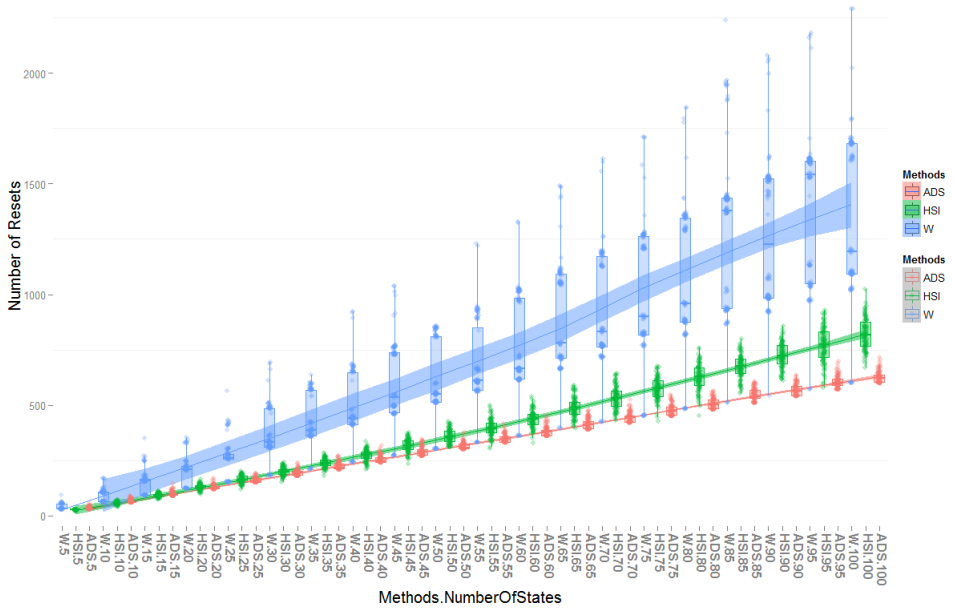


Figure 5.17.: Comparison of number of resets required for methods. Each boxplot summarises the distributions of 100 FSMs where $p = 6, q = 6$

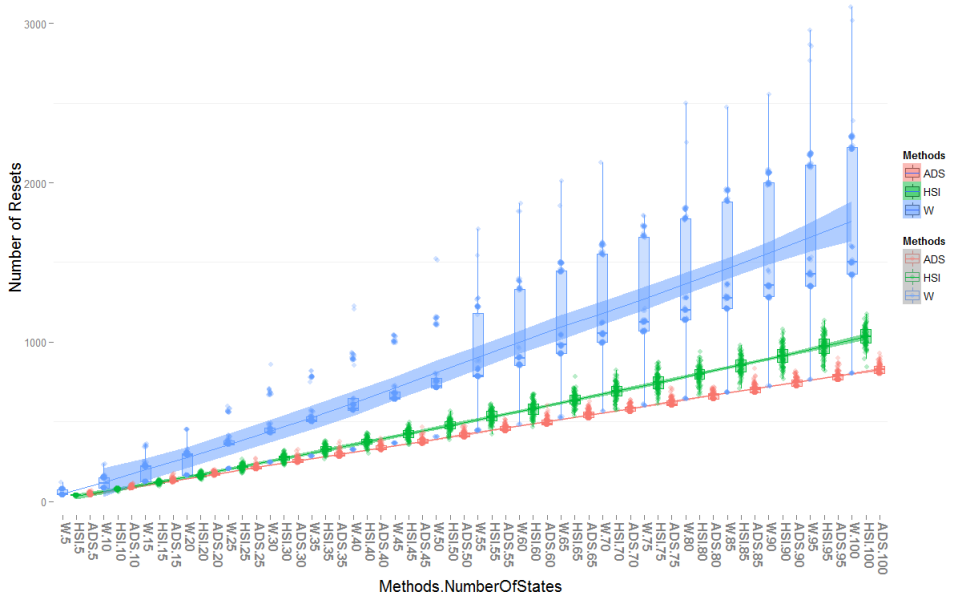


Figure 5.18.: Comparison of number of resets required for methods. Each boxplot summarises the distributions of 100 FSMs where $p = 8, q = 8$

We conducted a non-parametric Kruskal-Wallis significance test on the number of resets to support our observations. The results are given in Table 5.5 and indicate that the difference between the number of resets is statistically significant in all cases.

The HSI and the ADS methods differ only by input sequences used to distinguish states, therefore these reductions can stem from the followings: 1) For these experiments the ADS method allows us to eliminate a large number of prefixes. 2) The cardinalities (i.e. $|\mathcal{A}_i|$ for some i) of distinguishing sequences (for ADS method) and the cardinalities (i.e. $|H_i|$ for some i) of state identifiers (for HSI method) per state, are different. From now on SI depicts the average number of state identifiers and DS depicts the average number of ADSs computed for a single state.

Recall that in the transition verification phase, in order to test whether a given transition is implemented correctly or not, we first need to reach the state from which the transition originates, then execute the transition, and then check that the state reached by the transition is correct. Let us assume that the transition is initiated by an input x and we reach the transition by input sequence w that is $\delta(s_0, wx) = s_i$ then for each

Table 5.5.: The results of a Kruskal-Wallis Significance Tests performed on the Number of Resets

Input/output values (p/q)	Corresponding $\mathcal{X}^2$–values for different number of states (n). Reject H_0 when $\mathcal{X}^2 > 3.841$																			
	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100
4/4	19.83	15.65	23.54	29.91	30.64	34.12	32.42	40	33.68	42.06	49.45	51.58	48.82	43.51	46	46.3	66.34	51.2	51.03	53.24
6/6	34.47	22.95	25.5	26.08	23.7	28.51	38.83	27.83	.37.09	29.49	38.65	38.09	49.16	37.53	46.24	56.87	64.31	56.67	46.33	45.01
8/8	41.48	20.81	36.51	30.4	30.48	33.8	26.44	35.78	23.33	32	24.61	39.74	35.91	34.25	46.91	32.24	38.91	33.6	25.89	40.91

input sequences in H_i and $\mathcal{A}_i$ we need to generate test cases. For the HSI method for all $\alpha \in H_i$ we add $wx\alpha$ and for the ADS method for all $\beta \in \mathcal{A}_i$ we add $wx\beta$ to the checking experiment. Therefore the values of SI and DS can potentially affect the number of resets.

We investigated the average values of SI and DS. For each FSM M we computed the sum (over the states) of the cardinalities of sets H_i and also the sum of the cardinalities of the $\mathcal{A}_i$ and divided these sums by the number of states n. We summarise this study in Figures 5.19, 5.20 and 5.21, where $p/q = 4/4, 6/6$ and $8/8$ respectively.

In Figure 5.19, we observe that boxplot and jitterplot are similar to the results presented in Figure 5.16 and indicate that the values of DS and SI are comparable for ADS and the HSI methods. Moreover, we observe that the values of DS and SI increase with the number of states. In Figure 5.20 we see that when $n \geq 30$ the DS values is lower compared to the SI values and in Figure 5.21 we see that when $n \geq 25$ the DS values is lower compared to the SI values. Moreover we notice that as the number of inputs/outputs increases the values of SI and DS appear to decrease.

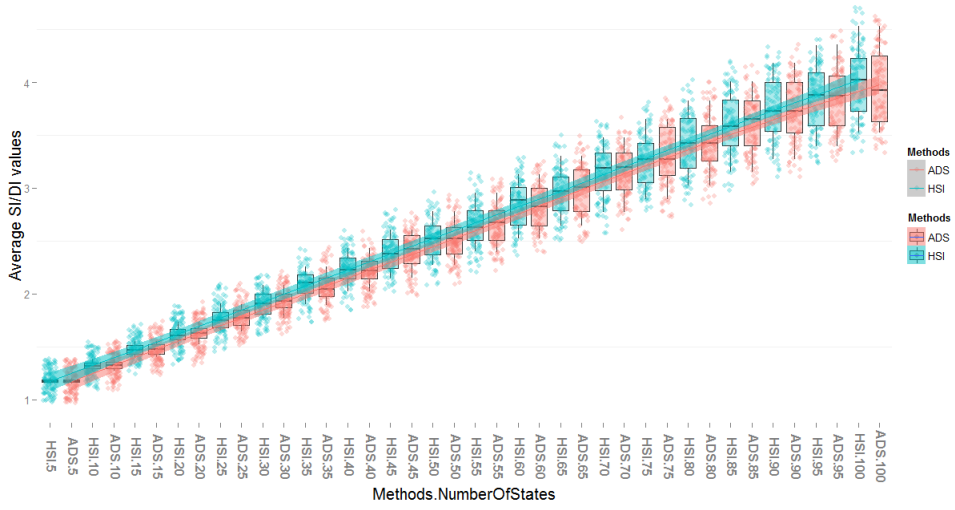


Figure 5.19.: Comparison of number of DS and SI per state. Each boxplot summarises the distributions of 100 FSMs where $p = 4, q = 4$.

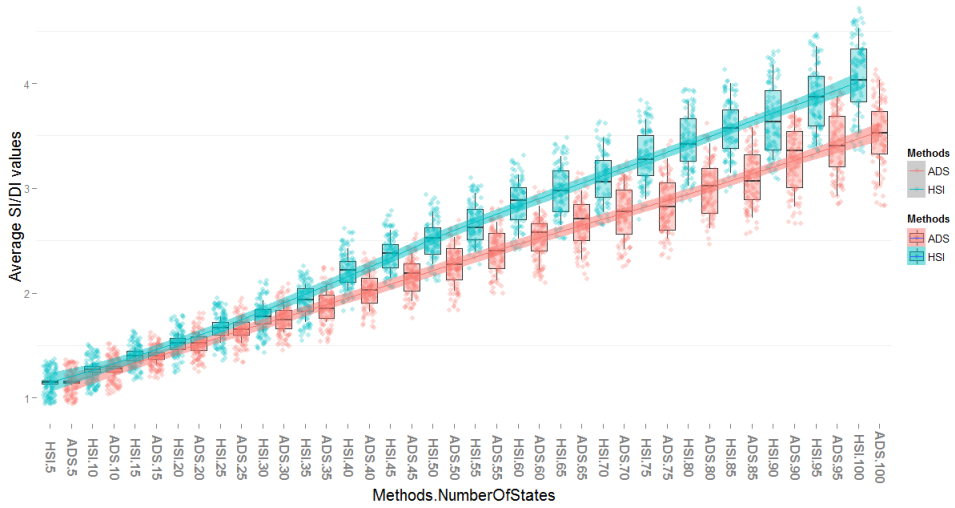


Figure 5.20.: Comparison of number of DS and SI per state. Each boxplot summarises the distributions of 100 FSMs where $p = 6, q = 6$.

We again applied the Kruskal Wallis test on the SI and DS values. The results are given in Table 5.6. We observe that in all cases we reject the null–hypothesis. These results are similar to those for the number of resets. Furthermore, it seems that the cardinalities of the inputs and outputs have an impact on the sizes of the harmonised state identifiers and the fully distinguishing sets. Therefore, according to the experimental studies, we can propose that instead of harmonised state identifiers, using a fully distinguishing set of ADSs can reduce the number of test cases used in a checking experiment.

Table 5.6.: The results of a Kruskal-Wallis Significance Tests performed on the SI and DI values

Input/output values (p/q)	Corresponding $\mathcal{X}^2$–values for different number of states (n). Reject H_0 when $\mathcal{X}^2 > 3.841$																			
	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100
4/4	46,15	65.71	76.89	61.47	63.49	65.79	77.69	69.09	73.97	65.95	75.06	53.54	65.83	67.42	64.41	74.27	53.95	69.37	80.88	60.91
6/6	46.63	58.29	69.10	69.96	52.72	63.36	68.88	81.10	68.35	51.02	59.30	65.48	63.23	60.43	68.16	70.64	68.67	60.98	71.73	61.39
8/8	34.84	70.11	68.62	75.32	57.29	59.79	64.14	64.74	64.46	69.43	69.14	63.60	66.73	59.38	76.96	64.18	69.73	74.44	62.94	79.08

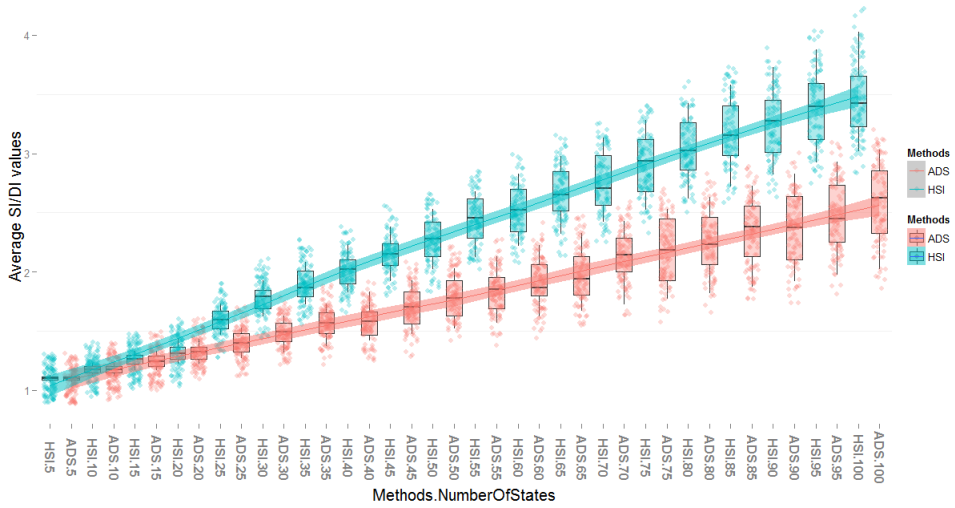


Figure 5.21.: Comparison of number of DS and SI per state. Each boxplot summarises the distributions of 100 FSMs where $p = 8, q = 8$.

As with the CE length, we compared the number of resets for the CEs for each of the FSMs. The results are summarised in Table 5.7, where we again show the numbers that do not match the expected pattern (fewest for ADS, most for W method). Interestingly, for small FSMs it appears that the HSI method normally requires fewer resets than the ADS method. However, this difference reduces as the number of states increases and appears to reduce slightly faster as the number of inputs/outputs increases. The figure seems to stabilise at around 30% for 4 inputs/outputs but for 6 and 8 inputs/outputs the figure drops to zero.

Case Studies

While using randomly generated FSMs allowed us to perform experiments with many subjects and so apply statistical tests, it is possible that FSMs used in practice differ from these randomly generated FSMs. We therefore decided to complement the experiments with some case studies. In this subsection we present the results of experiments conducted on FSM specifications retrieved from the ACM/SIGDA benchmarks, a set of test suites (FSMs) used in workshops between 1989-1993 [109]. The benchmark suite

Table 5.7.: Pairwise differences of number of resets

Input/output (p/q) values	Comparison Criteria	Number of States.																			
		5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100
4/4	$W < HSI$	1	6	6	7	6	4	9	6	4	5	1	2	3	1	1	2	1	0	0	0
	$W < ADS$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	$HSI < ADS$	91	83	72	67	57	47	49	32	35	35	32	34	41	30	23	34	35	30	30	31
6/6	$W < HSI$	0	6	9	18	13	6	7	9	7	11	8	7	3	5	10	8	3	1	3	7
	$W < ADS$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	$HSI < ADS$	98	91	64	41	39	39	28	31	22	10	13	12	5	5	2	4	2	0	3	0
8/8	$W < HSI$	0	0	5	7	13	15	11	13	12	5	13	3	6	3	7	8	9	5	4	7
	$W < ADS$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	$HSI < ADS$	93	97	86	73	23	14	13	12	11	5	5	9	4	5	1	1	0	1	1	0

has 59 FSM specifications ranging from simple circuits to advanced circuits obtained from industry. The FSM specifications are presented in the *kiss2* format. In order to precess FSMs, we converted the *kiss2* file format to our FSM specification format. We only used FSMs from the benchmark that were minimal, deterministic, had no complete ADS, were synchronisable, and had fewer than 10 input bits[5]. 19% of the FSMs had more than 10 input bits, 15% FSMs had complete ADS, 38% were not minimal and 20% had no synchronising sequence. 11% of the FSM specifications passed all of the tests. We computed checking experiments using the W, HSI and ADS methods and in Table 5.8 we present the results.

The case studies indicate that the use of ADSs led to the smallest checking experiments. We see that for FSM *bbsse* the test suite length is 64% shorter when the set of ADSs are used in the HSI method. For *planet* the reduction is 58%, for *s1* the reduction is 54%, for *dk17* the reduction is 50%, for *s386* the reduction is 48% and finally for *dk27* the reduction is 43%. The difference between the average number of resets are summarised as follows: For *bbsse* the reduction is 63%, for *s386* the reduction is 57%, for *s1* the reduction is 56%, for *planet* the reduction is 49%, for *dk17* the reduction is 46%, for *bbtas* the reduction is 38% and finally for *dk27* the reduction is 34%

Interestingly, Table 5.8 indicates that the average test case lengths for the HSI and the ADS method were similar but in one FSM ($s386$) the average test case length is shorter when the harmonized state identifiers are used (red coloured values). However, we can say that these results are similar to those obtained from randomly generated FSMs. We finally note that, as expected, it appears that as the ratio of the number of outputs to the number of inputs reduces, the number of resets and the average test case length increase.

Discussion

Based on the experimental results we can make the following main observations.

[5]Since the circuits receive inputs in bits, and since n bits correspond to 2^n inputs, we do not consider FSMs with $n \geq 10$ bits

Table 5.8.: Results of Case Studies

FSM Properties					W method			HSI method			ADS method		
Name	$\|X\|$	$\|Y\|$	$\|Q\|$	$\|Q\| * \|X\|$	Length	Number of Resets	Test Case Length	Length	Number of Resets	Test Case Length	Length	Number of Resets	Test Case Length
*dk*27	2	3	7	14	140	28	5	106	23	4	60	15	4
bbtas	4	4	6	24	964	84	11	497	42	11	228	26	8
*dk*17	4	5	8	32	302	72	4	239	62	3	118	33	3
*s*386	128	11	13	1664	886753	11700	75	239730	4006	59	123421	1687	73
bbsse	128	15	13	1664	792004	10023	79	310142	3732	83	108962	1347	80
planet	128	70	48	6144	96448	2052	47	40185	1027	39	16702	517	32
*s*1	256	20	18	5210	6497280	25600	253	2834876	11780	240	1302932	5132	240

1. **Using ADSs instead of harmonised state identifiers is advantageous:**

   We used $6 * 10^3$ randomly generated FSMs of varying number of states and inputs/outputs and found that the ADS method produced shorter checking experiments that have fewer resets. We also analysed the FSMs from the ACM/SIGDA dataset that are minimal, deterministic, have no complete ADSs, and are synchronisable. In all of these FSMs the ADS method produced shorter checking experiments with fewer resets.

2. **The ADS method computes test cases with slightly shorter lengths:**

   This results indicate that there are differences in the average test case lengths but that these do not fully explain the differences in the checking experiment lengths.

3. **The ADS method computes checking experiments with fewer resets:**

   The results suggest that ADS method produces fewer test cases. These results are similar to those for the lengths of the checking experiments except that the differences are smaller. Thus, the differences in CE length appear to come from both differences in mean test sequence length and differences in number of resets.

4. **The number of harmonised state identifiers (H_i's) computed per state is usually larger than the number of distinguishing sequences ($\mathcal{A}_i$) computed per state:**

   We see that the ADS method produces fewer distinguishing sequence per state than the harmonised state identifiers and this explains the difference in the number of resets.

5.7. Chapter Summary and Future Directions

Software testing is typically performed manually and is an expensive, error prone process. This has led to interest in automated test generation, including significant interest in model based testing (MBT). Most MBT techniques generate tests from either finite

state machines (FSMs) or labelled transition systems. Many automated FSM based test techniques use complete distinguishing sequences (DSs) to check the state of the system under test after a transition. While complete DSs have many desirable properties, an FSM M need not have a complete DS that distinguishes all of its states. However, we might still have (incomplete) DSs that distinguish some of the states of M and such DSs might be used in automated test generation.

In this work we explored the problem of constructing DSs for subsets of states of FSMs. We showed that it is PSPACE-complete to find a preset DS (PDS) that maximises the number of states distinguished and it is PSPACE-hard to approximate this problem. It is also PSPACE-complete to find a smallest set of sets of states that correspond to PDSs that distinguish all of the states of the FSM. We then explored the corresponding problems for Adaptive DSs (ADSs). It is known that we can decide in polynomial time whether an FSM has a complete ADS. However, the results for ADSs were similar to those for such PDSs: the problems considered were PSPACE-complete and it is PSPACE-hard to approximate the corresponding optimisation problem.

Having produced these results we showed that the well-known W and HSI checking experiment generation methods can be adapted to use (incomplete) ADSs and so also PDSs. In addition, we showed that the optimisation problems considered in this work are relevant to these adapted versions of the W and HSI method and also the standard HSI method. We then used experiments to explore the effect of optimisation by randomly generating FSMs and comparing the sizes of the checking experiments produced using the W-method, the HSI-method, and the HSI-method with an optimised set of ADSs. In the experiments, the proposed method, that uses ADSs, produced the smallest checking experiments and the W-method produced the largest checking experiments. In addition, the proposed method required the fewest resets. We extended these experiments to consider six FSMs from a benchmark and again found that the proposed method produced smaller test suites that required fewer resets.

There are several lines of future work. First, it would be interesting to explore realistic conditions under which the decision and optimisation problems can be solved in

polynomial time. Such conditions might lead to new notions of testability. There is also the question as to how effective is the greedy approach to generating incomplete ADSs: while the checking experiments returned were smaller than those produced using the W and HSI methods there may be approaches that return smaller sets of ADSs and smaller checking experiments. Although the results of the experiments suggest that the use of incomplete ADSs produce shorter test suites that require fewer resets, it would be interesting to extend the experiments and possibly also to consider the H and SPY algorithms. Finally, it would be interesting to extend this work to non-deterministic FSMs.

6. Distinguishing Sequences for Distributed Testing

6.1. Introduction

Early work, regarding the distributed test architecture, was motivated by protocol conformance testing [68, 62, 63]. This work identified two problems introduced by distributed testing. First, there might be a *controllability problem* in which a local tester, at a port p, cannot determine when to supply an input. Let us suppose, for example, that the tester at port 1 should start by sending input x_1, this is expected to lead to output o_1 at port 1, and the tester at port 2 should then send input x_2. The problem here is that the tester at port 2 does not observe the earlier input or output and so cannot know when to send its input. Controllability problems lead to non-determinism in testing and so there has been interest in the problem of generating test sequences that do not cause controllability problems [59, 61, 64, 69, 70, 71, 72]. *Observability problems* refer to the fact that, since we only make local observations, we may not be able to distinguish between two different behaviours (global traces). Let us suppose, for example, that the specification says that the input of x_1 at port 1 should lead to output o_1 at port 1 and that if we apply x_1 again then we should get o_1 at port 1 and o_2 at port 2. This defines the allowed global trace $x_1/\langle o_1, \varepsilon\rangle, x_1/\langle o_1, o_2\rangle$ in which ε denotes null output at a particular port. Here the tester at port 1 expects to observe $x_1o_1x_1o_1$ and the tester at port 2 expects to observe o_2. If instead the SUT produced $x_1/\langle o_1, o_2\rangle, x_1/\langle o_1, \varepsilon\rangle$ then the SUT produced a global trace not allowed by the specification but the local testers made

the expected observations: the tester at port 1 observed $x_1o_1x_1o_1$ and the tester at port 2 observed o_2. Observability problems can reduce the effectiveness of a test sequence and so there has been interest in producing test sequences that do not suffer from such observability problems [60, 63, 73, 74, 75].

This Chapter is structured as follows. In Section 6.2, we formally define the notion of a global ADS and what it means for such an ADS to be controllable and prove that a controllable ADS can be implemented using a set of distributed testers. Section 6.3 explores the complexity of problems associated with PDSs, proving that this problem is generally undecidable. Section 6.4 gives a condition under which it is decidable whether an MPFSM has a PDS while Section 6.5 then examines ADSs. Finally, Section 6.6 draws conclusions and discusses possible future work.

6.2. Test Strategies for distributed testing

Previous work has observed that when testing from an MPFSM we may require test cases that are adaptive and in this section we formalise such test cases as test strategies. We start by defining what we mean by a global test strategy, which can be seen as being a central tester that controls all of the ports, and we define what it means for such a strategy to be controllable. We also define what it means for a global strategy to be an adaptive distinguishing sequence. We then consider distributed test strategies, where we have a separate tester at each port, and show how a controllable global strategy can be mapped to such a distributed test strategy.

Before extending the notion of a strategy to distributed testing, and formally defining what we mean by an ADS in distributed testing, we briefly discuss what we mean for a test to distinguish states of an MPFSM if we have a single tester that observes the global order of events at the separate ports. Since MPFSMs only differ from the traditional notion of an FSM throughout the output being a tuple of values, the usual definitions of PDSs and ADSs apply and we will call these *traditional* PDSs and *traditional* ADSs. However, from Proposition 1 we know that, in the distributed test architecture a local

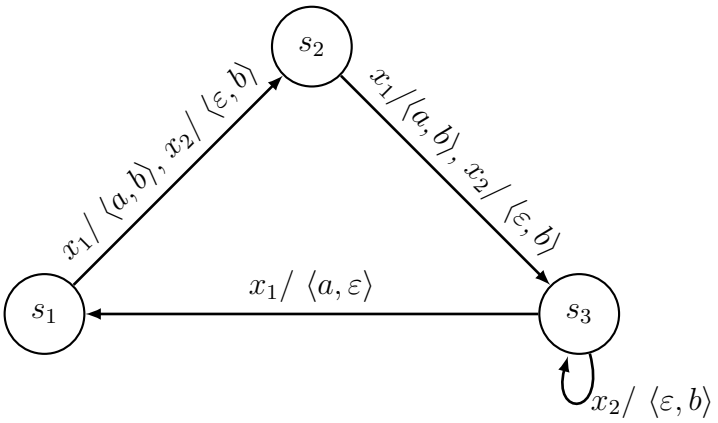


Figure 6.1.: MPFSM M_3 for Example 1

tester only observes the events at its port. This reduced observational power can lead to situations in which a traditional PDS or ADS fails to distinguish certain states.

This section builds on previous work that has discussed the notion of a test strategy for the case where the system has a single port [46] and also where we have multiple ports and a local tester has its own strategy [66]. However, as we explain below, the formalisation in this section is, by necessity, different. In addition, the notion of a strategy (for testing from an MPFSM) being controllable has not previously been discussed (the focus has been on controllable input sequences from a single state) and previous work has not considered the problem of using a strategy to distinguish more than two states in distributed testing.

6.2.1. Global Test Strategies

In this section we describe global strategies, where there is a single tester that observes all of the events and supplies all inputs. When testing from an MPFSM an observation is a trace: an input/output sequence. The tester will make a decision, regarding what to do next, on the basis of such a trace. We therefore define a global test strategy μ to be a partial function from $(X/Y)^*$ to X, where $(X/Y)^*$ denotes the set of traces (sequences of input/output pairs). That is to say, if σ is a trace produced by the SUT then $\mu(\sigma)$ determines what the tester does next: if $\mu(\sigma) = x$ ($x \in X$) then the tester applies x and

otherwise μ is not defined on σ and testing ends. We include a finiteness requirement in order to ensure that testing terminates.

Definition 34 *A* global strategy μ *is a partial function from* $(X/Y)^*$ *to* X *such that only finitely many traces from* $(X/Y)^*$ *are mapped to elements of* X*.*

When the tester applies a strategy μ one obtains what has been called an *evolution*[1] of μ [66]. We can restrict the set of evolutions if we start testing an MPFSM M when it is in state s since we must observe a trace from $L_M(s)$, and similarly we can define the set of evolutions when we know that a global strategy will be applied in a state from some set S'. The following adapts the previous notion of an evolution to testing from an MPFSM.

Definition 35 *Trace* $\sigma \in (X/Y)^*$ *is an* evolution *of global strategy* μ *if the following hold.*

1. *If* $\sigma' x/y$ *is a prefix of* σ *for* $x \in X$ *and* $y \in Y$ *then* $\mu(\sigma') = x$*.*
2. *If* σ' *is a prefix of* σ *and* $\mu(\sigma') = x$ *then there exists* $y \in Y$ *such that* $\sigma' x/y$ *is prefix of* σ*.*

Given global strategy μ*, we let* $Ev(\mu)$ *denote the set of evolutions of* μ*. Given* MPFSM M *and state* s *of* M*, we let the set of evolutions of* μ *from* s *be* $Ev(\mu, M, s) = Ev(\mu) \cap L_M(s)$*. Further, given* MPFSM M *with set of states* S *and* $S' \subseteq S$ *we let the set of evolutions of* μ *from* S' *be* $Ev(\mu, M, S') = \bigcup_{s \in S'} Ev(\mu, M, s)$*.*

This definition states that an input will only be applied after σ' if this is specified by the strategy and also that whenever the strategy can apply an input it does so. We will assume that a global strategy μ is not defined on traces that cannot occur when μ is applied and so $\sigma \notin pre(Ev(\mu))$ implies that μ is not defined on σ. Clearly, this does not reduce the effectiveness of the global strategies we consider; it simply avoids some redundancy.

[1] Previous work concerned a single (local) tester and so strategies were mappings from Σ_p^*.

While executing a global test case a controllability problem may arise. For example let $x_1, x_1' \in X_1$ be two different inputs at port 1, $x_2 \in X_2$, and let us suppose that there are traces $x_2/\langle o_1, o_2\rangle, x_1/y$ and $x_2/\langle \varepsilon, o_2\rangle, x_2/\langle o_1, \varepsilon\rangle, x_1'/y'$ in $pre(Ev(\mu))$. Since $\pi_1(x_2/\langle o_1, o_2\rangle) = \pi_1(x_2/\langle \varepsilon, o_2\rangle, x_2/\langle o_1, \varepsilon\rangle) = o_1$, tester 1 cannot differentiate between $x_2/\langle o_1, o_2\rangle$ and $x_2/\langle \varepsilon, o_2\rangle, x_2/\langle o_1, \varepsilon\rangle$, and so it cannot know which input (x_1 or x_1') to send after observing o_1. Following this observation, we define what it means for a global strategy to be controllable.

Definition 36 *Global strategy μ is* controllable *if for all $\sigma, \sigma' \in pre(Ev(\mu))$, if there exists port p such that $\pi_p(\sigma) = \pi_p(\sigma')$ and $\mu(\sigma) \in X_p$ then $\mu(\sigma') = \mu(\sigma)$.*

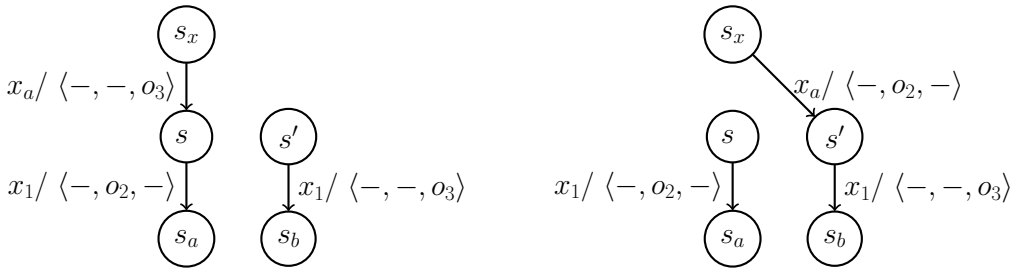
We can now adapt the notion of controllability to the case where we have a strategy and a set of states from which we might apply this.

Definition 37 *Given set S' of states of M, strategy μ is* controllable for S' *if for all $\sigma, \sigma' \in pre(Ev(\mu, M, S'))$, if there exists port p such that $\pi_p(\sigma) = \pi_p(\sigma')$ and $\mu(\sigma) \in X_p$ then $\mu(\sigma') = \mu(\sigma)$.*

The following shows that this is less restrictive than controllability: a strategy might be controllable for a given M and S' but not controllable in general.

Proposition 10 *If a strategy μ is controllable then for every* MPFSM *M and state set S' we have that μ is controllable for S'. However, it is possible that strategy μ is controllable for S' for some* MPFSM *M and state set S' and yet μ is not controllable.*

We can now define what it means for a global strategy μ to be an adaptive distinguishing sequence for an MPFSM M or for a set of states of M. Ideally, we have a single test strategy that distinguishes all of the states of M. However, even for single-port FSMs such a strategy need not exist. Where such ADSs do not exist, we might use a set of strategies that, between them, distinguish the states. In addition, in some situations we will only need to distinguish a subset S' of states since, for example, we have additional information that tells us that after a trace σ the state of the SUT must be in S'. Thus,



(a) Scenario 1 (b) Scenario 2

Figure 6.2.: Figure for Example 2

we use the following definition of what it means for a global strategy to be an ADS for an MPFSM M and for a set S' of states of M.

Definition 38 *Global strategy μ is an* adaptive distinguishing sequence (ADS) *for state set $S' \subseteq S$ of M if for all $s, s' \in S'$ with $s' \neq s$, $\sigma \in Ev(\mu, M, s)$, and $\sigma' \in Ev(\mu, M, s')$, we have that $\sigma \not\sim \sigma'$. Further, μ is an* adaptive distinguishing sequence (ADS) *for M if it is an adaptive distinguishing sequence for S.*

The main difference, when compared to ADSs for testing from a single-port MPFSM is that we compare global traces using $\sim$ rather than equality, this being an inevitable consequence of the reduced observational power of distributed testing.

In testing, an adaptive distinguishing sequence will be used to check the state of the SUT after some input sequence. For distributed testing, however, even if μ is a controllable ADS for M, we have an additional issue: there may be observability problems between the application of μ and the response of the SUT to earlier inputs. To see this, consider the following example.

Example 2 *Let us suppose that an* ADS *μ applies input x_1 at port 1, from state s the* MPFSM *M produces output o_2 at port 2 and from s' the* MPFSM *M instead produces o_3 at port 3. In each case, μ then terminates. Then clearly μ distinguishes s and s'. (Scenario 1 presented in figure 6.2a.) However, now suppose that the input before μ is applied should lead to o_3 being output and the SUT moving to state s but instead the*

SUT produces o_2 and moves to state s'. In each case, the tester at port 2 observes o_2 and the tester at port 3 observes o_3. (Scenario 2 presented in figure 6.2b.)

In Example 2, the incorrect output in the previous transition and the differences in the response to μ have masked one another but this could not have happened if there was a difference at port 1 since the tester at port 1 can determine which outputs were produced at 1 after x_1 was input. In order to avoid situations such as that given in Example 2, it is therefore sufficient to require that the ADS μ leads to differences at a particular port p and also that μ starts with an input at p: the input at p then precedes the different outputs at p (in response to μ).

Definition 39 *Given $p \in \mathcal{P}$, a global strategy μ is an* adaptive distinguishing sequence at p (a P-ADS) *for state set $S' \subseteq S$ if $\mu(\varepsilon) \in X_p$ and for all $s, s' \in S'$ with $s' \neq s$, $\sigma \in Ev(\mu, M, s)$, and $\sigma' \in Ev(\mu, M, s')$, we have that $\pi_p(\sigma) \neq \pi_p(\sigma')$. In addition, μ is an* adaptive distinguishing sequence at p (a P-ADS) *for M if μ is a* P-ADS *for S.*

We will use capital P in P-ADS when we refer to an adaptive distinguishing sequence at p for some unspecified port p; we use lowercase *p-ADS* to refer to such an ADS that explicitly starts with input at p.

When testing a single-port system a PDS is defined by a fixed input sequence: the tester simply applies this input sequence and observes the resultant output sequence. A major benefit of using such a PDS is that the test infrastructure does not have to be adaptive. To obtain such benefits in distributed testing we require that the local testers do not have to be adaptive. Consider the input sequence x_1, x_1, x_2 and the process of applying this from a state set $S' = \{s_1, s_2, s_3\}$ such that from s_1 we should obtain the trace $\sigma_1 = x_1/\langle o_1, -\rangle, x_1/\langle o_1, o_2\rangle, x_2/\langle o_1, -\rangle$, from s_2 we should obtain the trace $\sigma_2 = x_1/\langle o_1, -\rangle, x_1/\langle o_1, o_2\rangle, x_2/\langle o_1', -\rangle$ and from s_3 we should obtain the trace $\sigma_3 = x_1/\langle o_1, o_2'\rangle, x_1/\langle o_1, o_2'\rangle, x_2/\langle o_1, -\rangle$. Here we have that in σ_1 and σ_2 the tester at port 2 applies input x_2 after observing o_2 and in σ_3 the tester at port 2 applies input x_2 after observing $o_2'o_2'$. Thus, although x_1, x_1, x_2 is a fixed input sequence that causes no controllability problems for S' and distinguishes the states from S', its application

requires the tester at port 2 to be adaptive and thus it cannot be applied using local testers that are not adaptive. We therefore need to restriction the notion of a PDS to ensure that the local testers do not need to be adaptive.

Given a controllable test case μ and port p we will require that the observations made at p before an input x is supplied are identical for all states in S': the tester at p thus simply waits for this local trace to be observed before applying x.

Definition 40 *A global strategy μ that is controllable for set S' of states of* MPFSM *M is a* controllable PDS for *S' if and only if the followings hold:*

1. *Given states $s, s' \in S'$ with $s \neq s'$, if σ and σ' are the global traces that result from applying μ in states s and s' respectively then $\sigma \not\sim \sigma'$.*

2. *Given states $s, s' \in S'$ with $s \neq s'$, if σ and σ' are the global traces that result from applying μ in states s and s' respectively then for all ports p we have that the longest prefixes of $\pi_p(\sigma)$ and $\pi_p(\sigma')$ that end in an input are identical.*

When this holds each local tester follows a fixed pattern until its last input has been supplied and then it simply observes any further output. As a result, the local testers do not have to be adaptive. We now turn our attention to P-PDSs.

It is known that this problem (generating controllable PDSs) can be solved in low-order polynomial time when S' contains two states [76]. However, it will transpire that the problem becomes undecidable if we allow S' to have more states even if we restrict to there being two ports.

We can extend the definition of a PDS to P-PDSs. The only difference between the definitions of P-PDSs and PDSs is that the first input must be applied at port p and the tester at port p should observe different local traces from different states of MPFSM.

Definition 41 *A global strategy μ that is controllable for set S' of states of* MPFSM *M is a* controllable p-PDS for *S' if and only if the followings hold:*

1. $\mu(\varepsilon) \in X_p$.

2. *Given states $s, s' \in S'$ with $s \neq s'$, if σ and σ' are the global traces that result from applying μ in states s and s' respectively then $\pi_p(\sigma) \neq \pi_p(\sigma')$.*

3. *Given states $s, s' \in S'$ with $s \neq s'$, if σ and σ' are the global traces that result from applying μ in states s and s' respectively then for all ports q we have that the longest prefixes of $\pi_q(\sigma)$ and $\pi_q(\sigma')$ that end in an input are identical.*

6.2.2. Local and Distributed Test Strategies

While global strategies can be used to define what we want to happen in testing, if there are distributed testers then we need to use separate local strategies for these testers. This section defines what we mean by local and distributed strategies and proves that if a global strategy is controllable then we can implement it using a distributed strategy. This shows the value of generating controllable global strategies for use in testing, which is the problem we investigate in this work.

In the distributed test architecture, there are $|\mathcal{P}|$ physically distributed local testers that engage in executing a test strategy. A global strategy was a partial function from $(X/Y)^*$ to X and so it might appear that a local strategy for port p will be a partial function from $(X_p/Y_p)^*$ to X_p. However, the observations made by the tester at port p need not alternate between inputs and outputs. For example, there might be a trace such as $\sigma = x_1/\langle\varepsilon, o_2\rangle, x_1/\langle o_1, o_2\rangle, x_2/\langle o_1, \varepsilon\rangle$ and here $\pi_2(\sigma) = o_2 o_2 x_2$. As a result, a local strategy will be a partial function from sequences of observations at p (elements of Σ_p^*) and not from $(X_p/Y_p)^*$. We include ε in the set of values that can be returned by the local strategy, with this denoting the case where the tester waits to observe further output. In contrast to global strategies, if the local tester at port p chooses to not send an input then it might not have terminated since input can be supplied by other local testers and this can result in additional observations at p. In distributed testing the overall test system can be seen as a set of local testers and thus can be represented by a tuple of local strategies.

Definition 42 *A local strategy μ_p for port p is a function from Σ_p^* to $X_p \cup \{\varepsilon\}$ such that*

only finitely many traces from Σ_p^* *are mapped to* X_p*. Given port set* $\mathcal{P} = \{1, 2, \ldots, k\}$*, a* distributed strategy μ *is a tuple* $(\mu_1, \mu_2, \ldots, \mu_k)$ *such that for all* $p \in \mathcal{P}$ *we have that* μ_p *is a local strategy for* p*. Further,* $\sigma \in \Sigma_p^*$ *is an evolution of local strategy* μ_p *if the following hold:*

1. *If* $\sigma' x$ *is a prefix of* σ *for* $x \in X_p$ *then* $\mu_p(\sigma') = x$*.*

2. *If* σ' *is a prefix of* σ *and* $\mu_p(\sigma') = x$ *then* $\sigma' x$ *a prefix of* σ*.*

We let $Ev(\mu_p)$ *denote the set of evolutions of* μ_p*.*

Given distributed strategy $(\mu_1, \mu_2, \ldots, \mu_k)$, if σ is a global trace and $\mu_p(\pi_p(\sigma)) = x \in X_p$ then the tester at port p will apply input x whenever it observes $\pi_p(\sigma)$.

When a distributed strategy $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ is applied, each local tester makes decisions regarding when to supply input and does so on the basis of its observations. In defining the evolutions of such a strategy we need to consider a situation that could not occur with global strategies: we may get a point where more than one local strategy can supply the next input and so we have a race. This is clearly undesirable and so we define what it means for a distributed strategy to be deterministic (to not have such races) or to be deterministic from a given set of states.

Definition 43 *Given* MPFSM M *and state set* S'*, a distributed strategy* $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ *is* deterministic for S' *if there does not exist a trace* $\sigma \in L_M(S')$ *such that the following hold:*

1. *for all* $p \in \mathcal{P}$ *we have that* $\pi_p(\sigma) \in pre(Ev(\mu_p))$*; and*

2. *there exist* $p, p' \in \mathcal{P}$*,* $p \neq p'$*, such that* $\mu_p(\pi_p(\sigma)) \in X_p$ *and* $\mu_{p'}(\pi_{p'}(\sigma)) \in X_{p'}$*.*

Further, μ *is* deterministic *if there does not exist a trace* $\sigma \in (X/Y)^*$*, such that the following hold:*

1. *for all* $p \in \mathcal{P}$ *we have that* $\pi_p(\sigma) \in pre(Ev(\mu_p))$*; and*

2. *there exists* $p, p' \in \mathcal{P}$*,* $p \neq p'$*, such that* $\mu_p(\pi_p(\sigma)) \in X_p$ *and* $\mu_{p'}(\pi_{p'}(\sigma)) \in X_{p'}$*.*

This requires that we cannot have a trace σ that can occur when applying μ (all the projections of σ are prefixes of evolutions of the local strategies) after which two different testers can supply the next input. Clearly, the μ_p being functions then ensures that any next input after a global trace σ is uniquely defined.

It is straightforward to extend the notion of an evolution to deterministic distributed strategies.

Definition 44 *Trace $\sigma \in (X/Y)^*$ is an* evolution *of a (deterministic) distributed strategy $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ if the following hold:*

1. *If $\sigma' x/y$ is a prefix of σ for $x \in X_p$ then $\mu_p(\pi_p(\sigma')) = x$; and*
2. *If σ' is a prefix of σ and $\mu_p(\pi_p(\sigma')) = x$, $x \in X_p$, then there exists $y \in Y$ such that $\sigma' x/y$ is a prefix of σ.*

Given deterministic distributed strategy μ, we let $Ev(\mu)$ denote the set of evolutions of μ.

This can be extended to the case where we have a set S' of states from which a distributed strategy might be applied.

Definition 45 *Given* MPFSM *M, state set S' of M, and distributed strategy $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ that is deterministic for S', trace $\sigma \in (X/Y)^*$ is an* evolution *of μ from S' if $\sigma \in L_M(S')$ and the following hold:*

1. *If $\sigma' x/y$ is a prefix of σ for $x \in X_p$ then $\mu_p(\pi_p(\sigma')) = x$; and*
2. *If σ' is a prefix of σ and $\mu_p(\pi_p(\sigma')) = x$, $x \in X_p$, then there exists $y \in Y$ such that $\sigma' x/y$ is a prefix of σ.*

We let $Ev(\mu, M, S')$ denote the set of such evolutions of μ from S'.

Given a global strategy μ and port p we can define the projection of μ at p and, in an abuse of notation, call this $\pi_p(\mu)$. We initially define $\pi_p(\mu)$ to be a *relation* between Σ_p^* and $X_p \cup \{\varepsilon\}$: for some $\sigma_p \in \Sigma_p^*$ we may have that $\pi_p(\mu)$ maps σ_p to more than one element of $X_p \cup \{\varepsilon\}$.

Definition 46 *Given a global strategy μ and port $p \in \mathcal{P}$, $\pi_p(\mu)$ is a relation μ_p between Σ_p^* and $X_p \cup \{\varepsilon\}$ defined by: $x \in \mu_p(\sigma_p)$ for $x \in X_p \cup \{\varepsilon\}$ and $\sigma_p \in \Sigma_p^*$ if and only if there exists some $\sigma \in Ev(\mu)$ such that $x = \mu(\sigma)$ and $\pi_p(\sigma) = \sigma_p$.*

Importantly, if a global strategy is controllable then the projections form a deterministic distributed strategy.

Proposition 11 *If global strategy μ is controllable then the distributed strategy $(\pi_1(\mu), \pi_2(\mu), \ldots.\pi_k(\mu))$ is deterministic.*

This also tells us that if we have a controllable global strategy μ then the set of evolutions is defined for the distributed strategy $(\pi_1(\mu), \pi_2(\mu), \ldots, \pi_k(\mu))$ since evolutions are defined for deterministic distributed strategies.

When considering a set S' of states we restrict attention to traces in $L_M(S')$. As a result, one might think that given MPFSM M and set S' of states of M, if global strategy μ is controllable for S' then the distributed strategy $(\pi_1(\mu), \pi_2(\mu) \ldots \pi_k(\mu))$ is deterministic for S'. However, this is not necessarily the case since in forming the local strategies we consider all evolutions of a global strategy, not only those allowed from S'.

Proposition 12 *It is possible that global strategy μ is controllable for S' but the distributed strategy $(\pi_1(\mu), \pi_2(\mu), \ldots \pi_k(\mu))$ is not deterministic for S'.*

This leads us to define the projection of a global strategy in the presence of a set S' of states.

Definition 47 *Given global strategy μ, port $p \in \mathcal{P}$ and set S' of states of an* MPFSM *M, $\pi_p^{S'}(\mu)$ is a relation μ_p between Σ_p^* and $X_p \cup \{\varepsilon\}$ defined by: $x \in \mu_p(\sigma_p)$ for $x \in X_p \cup \{\varepsilon\}$ and $\sigma_p \in \Sigma_p^*$ if and only if there exists some $\sigma \in Ev(\mu, M, S')$ such that $x = \mu(\sigma)$ and $\pi_p(\sigma) = \sigma_p$.*

We can now generalise Proposition 11 for the case where we have a set S' of states. The proof of the following is equivalent to that of Proposition 11 except that we restrict attention to traces in $Ev(\mu, M, S')$.

Proposition 13 *Given state set S' of an* MPFSM *M, if global strategy μ is controllable for S' then the distributed strategy $(\pi_1^{S'}(\mu), \pi_2^{S'}(\mu), \ldots . \pi_k^{S'}(\mu))$ is deterministic for S'.*

We have shown that the projections of a controllable global strategy define a deterministic distributed strategy. We now prove that if we take the projections of a controllable global strategy then the distributed strategy we obtain has the same set of evolutions. This shows that we can safely implement a controllable global strategy using distributed testers.

Proposition 14 *Let us suppose that μ is a controllable global strategy and for all $p \in \mathcal{P}$ we have that $\mu_p = \pi_p(\mu)$. Then the distributed strategy $\mu' = (\mu_1, \mu_2, \ldots, \mu_k)$ is such that $Ev(\mu) = Ev(\mu')$.*

We now extend the above to the case where we are considering a given set of states, with the proof of the following being the same as that of Proposition 14 except that we restrict attention to traces from $Ev(\mu, M, S'))$.

Proposition 15 *Let us suppose that μ is a controllable global strategy for set S' of states of* MPFSM *M and for all $p \in \mathcal{P}$ we have that $\mu_p = \pi_p^{S'}(\mu)$. Then the distributed strategy $\mu' = (\mu_1, \mu_2, \ldots, \mu_k)$ is such that $Ev(\mu, M, S') = Ev(\mu', M, S')$.*

While we earlier represented an ADS as a single global strategy, in practice we use a distributed strategy, and so we now define what it means for a distributed strategy to be an adaptive distinguishing sequence. This is a (deterministic) distributed strategy that distinguishes the states of an MPFSM M being considered, or some specified set of states.

Definition 48 *A distributed strategy μ is an* adaptive distinguishing sequence for state set *S', $S' \subseteq S$, if μ is deterministic for S' and for all $s, s' \in S'$ with $s' \neq s$, $\sigma \in Ev(\mu, M, s)$, and $\sigma' \in Ev(\mu, M, s')$ we have that $\sigma' \not\sim \sigma$. Further, μ is an* adaptive distinguishing sequence *for M if μ is an adaptive distinguishing sequence for S.*

We can extend this to P-ADS s.

Definition 49 *Let us suppose that p is a port of M. A distributed strategy μ is an* adaptive distinguishing sequence at p (a P-ADS) for state set S', *$S' \subseteq S$, if μ is deterministic for S', $\mu(\varepsilon) \in X_p$, and for all $s, s' \in S'$ with $s' \neq s$, $\sigma \in Ev(\mu, M, s)$, and $\sigma' \in Ev(\mu, M, s')$ we have that $\pi_p(\sigma') \neq \pi_p(\sigma)$. Further, μ is an* adaptive distinguishing sequence at p (a P-ADS) *for M if μ is an adaptive distinguishing sequence at p for S.*

We can now prove that if a global strategy is controllable and is an ADS (or P-ADS) for an MPFSM M then the distributed strategy obtained by taking the projections of μ is also an ADS (or P-ADS).

Proposition 16 *Let us suppose that* MPFSM *M has port set $\mathcal{P} = \{1, 2, \ldots, k\}$. If μ is controllable and is an adaptive distinguishing sequence for M then $\mu' = (\pi_1(\mu), \pi_2(\mu), \ldots, \pi_k(\mu))$ is an adaptive distinguishing sequence for M.*

Proposition 17 *Given $S' \subseteq S$, if μ is controllable for S' and is an adaptive distinguishing sequence for S' then $\mu' = (\pi_1^{S'}(\mu), \pi_2^{S'}(\mu), \ldots, \pi_k^{S'}(\mu))$ is an adaptive distinguishing sequence for M from S'.*

The proofs of the following are almost identical to those of the two results above.

Proposition 18 *If μ is controllable and is a* P-ADS *for M then $\mu' = (\pi_1(\mu), \pi_2(\mu), \ldots, \pi_k(\mu))$ is a* P-ADS *for M.*

Proposition 19 *Given $S' \subseteq S$, if μ is controllable for S' and is a* P-ADS *for S' then $\mu' = (\pi_1^{S'}(\mu), \pi_2^{S'}(\mu), \ldots, \pi_k^{S'}(\mu))$ is a* P-ADS *for M from S'.*

These results are important since they tell us that as long as we restrict attention to controllable strategies, it is safe to generate global strategies that are ADSs/PDSs (and so also P-ADSs/P-PDSs) and then take their projections. If a global strategy μ is not controllable then there must be a port p and traces σ and σ' such that $\pi_p(\sigma) = \pi_p(\sigma')$, $\mu(\sigma) \in X_p$ and $\mu(\sigma') \neq \mu(\sigma)$. But it is straightforward to see that the set of projections of μ do not define a distributed strategy in which the local testers are deterministic: we have that $\pi_p(\mu)$ must relate $\pi_p(\sigma)$ to more than one element of $X_p \cup \{\varepsilon\}$. Thus, a local

tester cannot determine what to do next based on its observations and, in addition, μ and $(\pi_1(\mu), \pi_2(\mu) \ldots, \pi_k(\mu))$ will have different sets of evolutions. As a result, we know that it is safe to consider controllable global strategies and also that if a global strategy is not controllable then we cannot implement it using distributed testers. In the rest of the work we therefore focus on the problem of generating controllable global strategies that are ADSs/PDSs (and so also P-ADSs/P-PDSs).

6.3. Generating controllable PDSs

We have seen that a global controllable ADS can be implemented by local testers. However, a PDS is an ADS in which there is no adaptivity and so this result also holds for PDSs. Motivated by this, this section explores the problem of deciding whether an MPFSM has a controllable PDS.

We will prove that this problem is PSPACE-hard and in doing so we will use a special class of the problem of generating a PDS for a single-port MPFSMẆe then show that any algorithm that generates controllable PDSs for an MPFSM can be used to produce PDSs for this class of single-port MPFSM.

Lemma 32 *The following problem is* PSPACE-complete*: given a single-port* MPFSM *M in which no transition produces empty output, does M have a distinguishing sequence?*

We can now show that the problem of deciding whether an MPFSM has a controllable PDS that distinguishes all of its states is PSPACE-hard.

Proposition 20 *The following problem is* PSPACE-hard*: given a multi-port* MPFSM *M, is there a controllable* PDS *that distinguishes all of the states of M? In addition, this result still holds if we restrict attention to* MPFSMs *that have two ports.*

It is known that for a single port MPFSMs there are MPFSMs such that the shortest PDSs are of exponential length [35]. Relying on the reduction presented above, we deduce that this result is valid for multi–port MPFSMs.

Corollary 2 *There is a class of* MPFSMs *where the shortest* PDS *is of exponential length.*

We now explain a construction that will be used in the proof of Proposition 21. Let us assume that $M = (S, X, Y, \delta, \lambda)$ is a single-port MPFSM in which all the transitions have non-empty outputs. We construct a multi-port MPFSM $M'' = (\mathcal{P}, S', X', Y', \delta', \lambda')$ as follows. For each state $s_i \in S$, we introduce states s_i and $s_i^\star$ to give the state set $S' = S \cup \{s_i^\star | s_i \in S\}$. The MPFSM has two ports $\mathcal{P} = \{1, 2\}$. The input and output alphabets are: $X_1 = \{R\}$, $X_2 = X$, $Y_1 = Y \cup \{0, 1\}$, $Y_2 = \{\varepsilon, L\}$. For each state $s_i^\star$, we introduce the transition from $s_i^\star$ to s_i with label $R/\langle 0, L\rangle$. Moreover, for each input $x \in X$ and state $s_i^\star \in S'$, we introduce a transition from $s_i^\star$ to $\delta(s_i, x)$ with label $x/\langle\lambda(s_i, x), \varepsilon\rangle$. In addition for each input $x \in X$ and state $s_i \in S'$, we introduce a transition from s_i to $\delta(s_i, x)$ with label $x/\langle\lambda(s_i, x), \varepsilon\rangle$. Finally, for each state s_i, we introduce a self-loop transition with label $R/\langle 1, L\rangle$.

Clearly the machine M'' is completely specified. The intuition behind the reduction is as follows, in order to distinguish between states $s_i, s_i^\star$ at port 1, the first input to be applied is input R (at port 1). Otherwise, if an input $x \in X$ is applied first (at port s), states $s_i, s_i^\star$ are merged without being distinguished at port 1. Moreover the construction also guarantees that states can only be distinguished at port 1.

Proposition 21 *The problem of deciding whether a* MPFSM *has a* P-PDS *is* PSPACE-hard.

It has been shown that for a given MPFSM and a port $p \in \mathcal{P}$ it is possible to construct a separating word for two states in polynomial time and this word has length at most $k(n-2)+1$ [65]. Therefore we may consider bounded PDSs.

Definition 50 *The* Bounded PDS problem *is to decide if there is a controllable* PDS μ *for a given* MPFSM M *such that the length of the longest evolution in* $Ev(M, \mu, S)$ *is not more than* $\ell \in \mathbb{Z}_{>0}$.

We will show that the bounded PDS problem is in EXPSPACE.

Proposition 22 *The problem of deciding whether an* MPFSM M *has a* PDS *of length* ℓ *is in* EXPSPACE.

Therefore Proposition 20 and Proposition 22 together lead us to the following result.

Theorem 22 *The problem of deciding whether there is a controllable* PDS *of length* ℓ *for* MPFSM M *is in* EXPSPACE *and* PSPACE-hard. *This holds even if we restrict attention to* MPFSMs *with two ports.*

We extend this result to P-PDSs.

Theorem 23 *The problem of deciding whether there is a controllable* P-PDS *of length* ℓ *for* MPFSM M *is in* EXPSPACE *and* PSPACE-hard. *This holds even if we restrict attention to* MPFSMs *with two ports.*

We now focus on the problem of deciding whether an MPFSM M has a controllable PDS for some state set S'. We show that the problem is undecidable through reducing the undecidable Post's Correspondence Problem [121]. Post's Correspondence Problem is defined as follows:

Definition 51 *Post's Correspondence Problem (PCP) is to decide, for sequences* $\alpha_1, \alpha_2, \ldots, \alpha_b$ *and* $\beta_1, \beta_2, \ldots, \beta_b$, *whether there is a sequence* $a_1, a_2, \ldots, a_n$ *of indices in* $[1..b]$ *such that* $\alpha_{a_1}\alpha_{a_2}\ldots\alpha_{a_n} = \beta_{a_1}\beta_{a_2}\ldots\beta_{a_n}$.

Theorem 24 *The problem of deciding whether there is a controllable* PDS *for state set* S' *of* MPFSM M *is undecidable and this holds even if we restrict attention to* MPFSMs *with two ports.*

We now show that the P-PDS problem is again undecidable.

Theorem 25 *The problem of deciding whether there is a controllable* P-PDS *for state set* S' *of* MPFSM M *is undecidable and this holds even if we restrict attention to* MPFSMs *with two ports.*

Hunt et al. [122] introduced a variation of PCP that is defined as follows:

Definition 52 *The Bounded Posts Correspondence Problem (B-PCP) is to decide, for sequences* $\alpha_1, \alpha_2, \ldots, \alpha_b$ *and* $\beta_1, \beta_2, \ldots, \beta_b$*, whether there is a sequence* $a_1, a_2, \ldots, a_n$ *of indices in* $[1 \ldots b]$ *such that* $\alpha_{a_1}\alpha_{a_2}\ldots\alpha_{a_n} = \beta_{a_1}\beta_{a_2}\ldots\beta_{a_n}$ *and* $n \leq K$ *where* $K \in \mathbb{Z}_{>0}$ *is a positive integer.*

In other words in the B-PCP problem our objective is to find a solution that uses at most K sequences. In the same work the authors state that finding a solution is NP-complete.

Theorem 26 *It is* NP-complete *to decide whether an instance of the B-PCP has a solution.*

We now consider the problem of finding a bounded PDS for a subset of states i.e. we decide, given an MPFSM M, a state set S' and bound ℓ, whether there is a controllable PDS for S' that has length at most ℓ.

Theorem 27 *The Bounded* PDS *problem is in* EXPSPACE *and is* NP-hard.

Theorem 28 *The Bounded* P-PDS *problem is in* EXPSPACE *and is* NP-hard.

Finally, we consider the case where ℓ is bounded above by a polynomial function of the number of states of M.

Theorem 29 *If* ℓ *is defined by a polynomial in term of the number of states of* M *then the Bounded* PDS *problem is* NP-complete.

Theorem 30 *If* ℓ *is defined by a polynomial in term of the number of states of* M *then the Bounded* P-PDS *problem is* NP-complete.

As noted before, instead of using a PDS we could instead use a characterisation set containing controllable separating sequences and we have a polynomial upper bound on the sum of the lengths of the sequences in such a characterisation set. Thus, in practice we are likely to have a polynomial upper bound in the length of PDSs in which we are interested. This observation motivates the above result. In the following section we consider the problem of PDS generation for a class of MPFSMs we call C-MPFSMs.

6.4. PDS generation: a special case

In this section we prove that the PDS existence problem is decidable for a special class of MPFSMİn this class of MPFSM if the input of x leads to no output at port p for some state of an MPFSM M then the input of x leads to no output at p for all states of M. This can be seen as imposing a fixed pattern on the communication that occurs between the agents at the ports through interacting with M.

Definition 53 *An* MPFSM $M = (\mathcal{P}, S, s_0, X, Y, \delta, \lambda)$ *is said to be* consistent *if for all* $s, s' \in S$, $x \in X$, *and* $p \in \mathcal{P}$, $\pi_p(\lambda(s, x)) = \varepsilon$ *implies that* $\pi_p(\lambda(s', x)) = \varepsilon$.

We will call an MPFSM that is consistent a C-MPFSM. C-MPFSMs have the following important properties, the first two relating to controllability. Note that our definition of a strategy being controllable extends immediately to input sequences since an input sequence defines a strategy. Therefore, throughout this section we will use input sequence $\bar{x}$ and a strategy μ interchangeably.

Proposition 23 *Given distinct states* s *and* s' *of* C-MPFSM M, *if input sequence* $\bar{x}$ *is controllable from state* s *then* $\bar{x}$ *is controllable from* s'.

We can extend this result to a set of states.

Proposition 24 *Given* C-MPFSM M *with state set* S, *state* $s \in S$ *and set* $S' \subseteq S$ *of states of* M, *if input sequence* $\bar{x}$ *is controllable from* s *then* $\bar{x}$ *is controllable from* S'.

The following relates to observability problems.

Proposition 25 *Given* C-MPFSM M, *states* s *and* s' *of* M, *and input sequence* $\bar{x}$, *if* $\lambda(s, \bar{x}) \neq \lambda(s', \bar{x})$ *then there is some port* p *such that* $\pi_p(\lambda(s, \bar{x})) \neq \pi_p(\lambda(s', \bar{x}))$.

In other words, for a given C-MPFSM when an input sequence $\bar{x}$ 'globally distinguishes' states s, s' of M then $\bar{x}$ also 'locally distinguish' the states s, s' of M. In the following we extend this property.

Proposition 26 *Given C-*MPFSM M *with state set* S*, if* $\bar{x}$ *is a controllable input sequence and for all* $s, s' \in S'$*,* $S' \subseteq S$*, we have that* $s \neq s' \Rightarrow \lambda(s, \bar{x}) \neq \lambda(s', \bar{x})$ *then* $\bar{x}$ *is a controllable* PDS *for* S'*.*

We can therefore conclude that when testing from C-MPFSMs there are no observability problems and an input sequence is controllable for a state if and only if it is controllable for all non-empty sets of states. It will transpire that these properties simplify the PDS existence problem.

In the following we use a *canonical* MPFSM [76] $\chi^p_{min}(M)$ of a C-MPFSM M with respect to port p with the motivation that by considering $\chi^p_{min}(M)$ we avoid controllability problems (for PDSs that start with input at p). This will simplify PDS construction: by Proposition 25 we do not have to worry about observability problems and by using $\chi^p_{min}(M)$ we will also avoid controllability problems.

The canonical MPFSM $\chi^p_{min}(M)$ is constructed in two steps. First we construct a partial MPFSM $\chi_{min}(M)$, second from $\chi_{min}(M)$ we construct $\chi^p_{min}(M)$. We can now define the (incomplete) MPFSM $\chi_{min}(M) = (\mathcal{P}, S_{min}, s'_0, X, Y, \delta_{min}, \lambda_{min})$, based on [76] (which uses ideas from [37]).

For each $s_i \in S$ and $P \subseteq \mathcal{P}$ there can be vertex s_i^P representing the situation in which the state is s_i and the next input must be at a port in P (since otherwise there will be a controllability problem). In this construction we will use the following in which T is the set of transitions of M.

- $Depart^p(s_i) = \{(s_i, s_j, x/y) \in T | x \in X_p\}$ is the set of transitions from s_i whose input is at p.
- $Arrive^P(s_i) = \{(s_j, s_i, x/y) \in T | ports(x/y) = P\}$ is the set of transitions ending at s_i that involve the set P of ports.

The set S_{min} of states of $\chi_{min}(M)$ is defined by the following.

1. For all $1 \leq i \leq n$ and $P \subseteq \mathcal{P}$, $s_i^P \in S_{min}$ if $Arrive^P(s_i) \neq \emptyset$.
2. State $s_0^{\mathcal{P}}$ is in S_{min} and $s_0^{\mathcal{P}}$ is the initial state of $\chi_{min}(M)$.

The state $s_0^{\mathcal{P}}$ represents the situation before testing starts: the SUT is in the initial state and since no inputs have been applied the first input can be applied at any port without causing a controllability problem. The set T_{min} of transitions of $\chi_{min}(M)$ (and so the functions δ_{min} and λ_{min}) is defined by, for each transition $t = (s_i, s_j, x/y) \in T$ and $s_i^P \in S_{min}$ with $port(x) \in P$, including in T_{min} the transition $(s_i^P, s_j^{P_t}, x/y)$ where $P_t = ports(x/y)$.

The following results are known [76].

Proposition 27 *For each controllable path $\bar{\rho}$ in M that starts at s_0, there is a unique controllable path $\bar{\rho}'$ in $\chi_{min}(M)$ that starts at $s_0^{\mathcal{P}}$ such that $label(\bar{\rho}) = label(\bar{\rho}')$.*

Proposition 28 *For each path $\bar{\rho}'$ in $\chi_{min}(M)$ that starts at $s_0^{\mathcal{P}}$, there is a unique controllable path $\bar{\rho}$ in M that starts at s_0 such that $label(\bar{\rho}) = label(\bar{\rho}')$.*

Importantly, $\chi_{min}(M)$ captures the controllable paths of M and since an input sequence is controllable from one state if and only if it is controllable from all non-empty sets of states (Proposition 24), it captures the set of input sequences that are controllable from sets of states of M and the corresponding behaviours.

The final step is to create a completely specified MPFSM $\chi_{min}^p(M) = (\mathcal{P}, S_{min}^p, X, Y, \delta_{min}^p, \lambda_{min}^p)$ from $\chi_{min}(M)$ in which for each state s and port p there is a state $s^{\{p\}}$; this will allow us to explore controllable PDSs that start with input at p. This is achieved by applying the following.

1. For every state s of M such that $s^{\{p\}}$ is not in S_{min}.
   a) Add the state $s^{\{p\}}$.
   b) For every input x at p, if $s_i = \delta(s, x)$ and $y = \lambda(s, x)$ then add the transition $(s^{\{p\}}, s_i^P, x/y)$ such that and $P = ports(x/y)$.
2. For every input x and state $s \in S_{min}$ such that there is no transition from s with input x, add a self-loop transition from s to s with input x and output ε at all ports.

We can now show how controllable PDS construction for M relates to PDS construction for $\chi^p_{min}(M)$. Note that we require a particular type of PDS for $\chi^p_{min}(M)$: since a PDS will be applied by distributed testers we require that the trace before apply an input at p has fixed projection at p (see Definition 40). In the following, a PDS for $\chi^p_{min}(M)$ is said to be *non-redundant* if it does not lead to the execution of any of the self-loops added to make $\chi^p_{min}(M)$ completely-specified.

Proposition 29 *An input sequence $\bar{x}$ is a controllable* PDS *that starts with input at p for set $S' = \{s_1, s_2 \ldots, s_r\} \subseteq S$ of states of M if and only if $\bar{x}$ is a non-redundant* PDS *for set $S'' = \{s_1^{\{p\}}, s_2^{\{p\}}, \ldots, s_r^{\{p\}}\}$ of states of $\chi^p_{min}(M)$ such that for all $1 < i \leq |\bar{x}|$, if $x_i \in X_q$ then for all $s_i^{\{p\}}, s_j^{\{p\}} \in S''$ we have that $\pi_q(\lambda^p_{min}(s_i^{\{p\}}, \bar{x}_{i-1})) = \pi_q(\lambda^p_{min}(s_j^{\{p\}}, \bar{x}_{i-1}))$.*

The definition of $\chi^p_{min}(M)$ uses sets of ports as labels on states and this might appear to lead to a combinatorial explosion. However, the number of states of $\chi^p_{min}(M)$ is bounded above by the number of transitions of M plus one (for the initial state) plus an additional $|S||\mathcal{P}|$ states of the form $s^{\{p\}}$. Let n be the number of states, k be the number of ports and m be the number of inputs of C-MPFSM M. Also let n_{min} be the number of states of machine $\chi^p_{min}(M)$, then $n_{min} \leq nk + nm + 1$. Thus, $\chi^p_{min}(M)$ can be constructed in polynomial time.

We now present some definitions and observations related to PDSs and then we give an upper bound on the length of a minimal controllable PDSs for C-MPFSMs [2].

Given an input sequence $\bar{x}$, we will let $\mathcal{B}^{\bar{x}}$ denote the set of sets of 'current states' that can occur if we know that $\bar{x}$ has been applied from a state in S''. Thus, if one or more states in S'' leads to output $\bar{y}$ when $\bar{x}$ is applied then one of the sets in $\mathcal{B}^{\bar{x}}$ is the set of states that can be reached from states in $S'' = \{s_1^{\{p\}}, \ldots, s_n^{\{p\}}\}$ by paths with label $\bar{x}/\bar{y}$ (the set $\{\delta^p_{min}(s^{\{p\}}, \bar{x}) | \lambda^p_{min}(s^{\{p\}}, \bar{x}) = \bar{y}\}$). More formally, we have that

$$\mathcal{B}^{\bar{x}} = \{\{\delta^p_{min}(s^{\{p\}}, \bar{x}) | \lambda^p_{min}(s^{\{p\}}, \bar{x}) = \bar{y}\} | \exists s^{\{p\}} \in S''.\bar{y} = \lambda^p_{min}(s^{\{p\}}, \bar{x})\}$$

[2] This upper bound is used in the proof that the decision problem is in PSPACE; it seems likely that smaller upper bounds can be found but that is not a concern here.

While applying an input sequence $\bar{x}_i$, different states of S'' may produce different traces. This will lead to the *splitting* of the set S'' into smaller sets of states. On the other hand if a state $s\{p\}$ produces a different output trace from all other states in set $S'' \setminus \{s^{\{p\}}\}$ then $s^{\{p\}}$ is *distinguished* from states $S'' \setminus \{s^{\{p\}}\}$.

We use $\delta^p_{min}(\mathcal{B}, \bar{x}_i)$ to denote the set $\mathcal{B}^{\bar{x}_i}$. Clearly the application of a PDS $\bar{x}$ from set S'' will lead to a set $\mathcal{B}^{\bar{x}}$ of cardinality n; each set is a singleton set. We now give an upper bound on PDS length for C-MPFSMs.

Lemma 33 *Given a C-*MPFSM *M with n states, k ports, and m inputs, M has a controllable* PDS *if and only if it has one of length at most $n(n_{min})^n$ where $n_{min} = nk + nm + 1$.*

Thus we conclude that PDS existence check for C-MPFSM is decidable (it is sufficient to check all input sequences of length at most $n(n_{min})^n$).

Theorem 31 *It is decidable whether a C-*MPFSM *has a* PDS.

We will show that the problem is PSPACE-complete. First we define a nondeterministic Turing Machine $\mathcal{T}$ that can decide the existence of a PDS for a given C-MPFSM for a state set S' of C-MPFSM M. $\mathcal{T}$ will apply inputs one at a time and maintain a current set $\mathcal{C}$ of pairs of states such that (s, s') is in $\mathcal{C}$ if and only if $s \in S'$ and the sequences of inputs received takes M from s to s'. $\mathcal{T}$ also maintains an equivalence relation r defined by two states $s, s'' \in S'$ being related under r if and only if the currently guessed input sequence $\bar{x}$ does not distinguish s and s' ($\lambda(s, \bar{x}) = \lambda(s'', \bar{x})$). Finally, $\mathcal{T}$ maintains a set of ports P_c from which an input can be supplied (the ports where no differences in outputs have been observed).

Clearly, these pieces of information can be updated when a new input is received: after an input is guessed, $\mathcal{T}$ updates the current set information, the equivalence relation r and finally the set of ports (a port is removed from P_c if the latest input leads to different outputs at this port). Since the problem is to decide existence, $\mathcal{T}$ does not need to store the sequence of previous inputs received. Further, the input sequence received defines a

PDS for S' if and only if no two different states from S' are related under r. We now prove that the PDS problem is in PSPACE for C-MPFSMs.

Proposition 30 *The problem of deciding whether a C-*MPFSM *M has a* PDS *is in* PSPACE.

Thus we conclude that deciding whether a given C-MPFSM has a PDS is PSPACE-complete.

Theorem 32 *The problem of deciding whether a C-*MPFSM *has a* PDS *is* PSPACE-complete.

Likewise the MPFSM M'' constructed in Proposition 21 is a C-MPFSM and so we have the following result.

Proposition 31 *The problem of deciding whether a C-*MPFSM *has a* P-PDS *is* PSPACE-hard.

The proof of the following is identical to that of Proposition 30 except that the non-deterministic Turing Machine, in looking for a P-PDS, chooses the first input to be at p and only considers whether each pair of states produce different outputs at p.

Proposition 32 *The problem of deciding whether a C-*MPFSM *M has a p-*PDS *is in* PSPACE.

We therefore have the following result.

Theorem 33 *The problem of deciding whether a C-*MPFSM *has a* P-PDS *is* PSPACE-complete.

6.5. Generating controllable ADSs

In Section 6.2 we defined what we mean by a controllable global ADS and showed that it is sufficient to consider such ADSs. In addition, potential observability problems between an ADS and the transitions before this, which might adversely affect the effectiveness of the ADS, are avoided if we produce a P-ADS.

We now show that the problem of deciding whether an MPFSM M has an ADS that distinguishes all of its states is PSPACE-hard. We will rely on Lemma 32 that tells us that the problem of deciding whether a single-port MPFSM has a PDS is PSPACE-hard even if all transitions have non-empty output.

Theorem 34 *The following problem is* PSPACE-hard*: given a multi-port* MPFSM M*, is there a controllable ADS that distinguishes all of the states of* M*? In addition, this result still holds if we restrict attention to* MPFSMs *that have two ports.*

We have the same result if we are interested in p-ADSs.

Theorem 35 *The following problem is* PSPACE-hard*: given a multi-port* MPFSM M *and port* p *of* M*, is there a controllable* p*-ADS that distinguishes all of the states of* M*? In addition, this result still holds if we restrict attention to* MPFSMs *that only have two ports.*

We might have the situation in which there is no ADS for the MPFSM being considered but there are controllable strategies that allow us to distinguish sets of states.

Theorem 36 *The following problems are* PSPACE-hard*:*

1. *Given a multi-port* MPFSM M*, find a controllable* ADS μ *and state set* S' *where* μ *is a controllable* ADS *for* S' *and* μ *and* S' *are such that* S' *has maximal size.*

2. *Given a multi-port* MPFSM M *and port* p *of* M*, find a controllable* P-ADS μ *and state set* S' *where* μ *is a controllable* P-ADS *for* S' *and* μ *and* S' *are such that* S' *has maximal size.*

Given MPFSM M with state set S, we say that the length of the longest evolution in $Ev(\mu, M, S)$ is the *height* of μ. We now extend our observations using results given in [35].

Theorem 37 *There is a class of* MPFSMs *that contain* ADS *(or* P-ADS*) such that the shortest evolution is of exponential length.*

Finally, since existence is PSPACE-hard so are the corresponding optimisation problems.

Theorem 38 *The following problems are* PSPACE-hard*.*

1. *Given a multi-port* MPFSM M*, what is the smallest value of* ℓ *such that* M *has an* ADS *of height* ℓ*?*

2. *Given a multi-port* MPFSM M*, what is the smallest value of* ℓ *such that* M *has a* P-ADS *of height* ℓ*?*

The results in this section showed that ADS generation problems are computationally hard, in contrast to the situation with single-port FSMs, but left open the question of whether these problems are decidable. In the next section we show that we can reduce these negative results to some extent when we limit the height of the ADS (or P-ADS).

We now consider the problem of constructing a complete ADS or P-ADS μ for an MPFSM M where we bound the height of μ. We are therefore interested in the following problem.

Definition 54 *Given* MPFSM M *and natural number* ℓ*, the EXACT-HEIGHT problem for* M *and* ℓ *is to determine whether* M *has a controllable* ADS *with height* ℓ*.*

Naturally, this corresponds also to the problem of deciding whether there is a controllable ADS with height at most ℓ and it is straightforward to adapt the proofs in this section to the case where we have a bound on the height of an ADS. We therefore focus on the EXACT-HEIGHT problem. This problem is motivated by the fact that it is possible to use a set of separating sequences instead of an ADS or PDS in order to identify states. It is known that for any two states s_i and s_j of an MPFSM M with n states, k ports and m inputs, we can decide in $O(mn^2)$ time whether there is a controllable separating sequence that distinguishes s_i and s_j and if so there is such a sequence of length at most $k(n-1)$. Thus, one can construct a set of controllable separating sequences to form a characterisation set of polynomial size and this can be achieved in polynomial time.

We will show that the Directed Hamiltonian Path (DHP) problem for strongly connected directed graphs, which is NP-complete [88, 89], is polynomial time reducible to the EXACT-HEIGHT problem. An instance of a DHP problem can be defined as

follows, where a walk is said to *visit* a vertex v if v is the starting vertex or the ending vertex of at least one edge in the walk.

Definition 55 *Consider a strongly connected directed graph $G = (V, E)$ with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{e_1, e_2, \ldots, e_m\}$. We say that walk ρ of G is a Hamiltonian path if and only if the walk visits each vertex of G exactly once. The* Directed Hamiltonian Path *problem is to decide whether a strongly connected directed graph G has a Hamiltonian path.*

Given an instance G of the DHP problem, we will construct an MPFSM $M(G) = (\mathcal{P}, S, s_0, X, Y, \delta, \lambda)$. The aim will be to construct the transition structure of the MPFSM in such a way that an ADS μ simulates the rules that govern the DHP problem. We will then prove that if G has n vertices then there is an ADS μ for M whose longest evolution in $Ev(\mu, M, S)$ is of length $n - 1$ if and only if the corresponding sequence of symbols constitutes a solution to the DHP problem for G.

We now show how we construct $M(G)$. We represent vertex v_i of G by a state s_i of M and add an additional state s_e and so $S = \{s_1, \ldots, s_n\} \cup \{s_e\}$. For each edge e_i of G there will be a corresponding port i of M and unique input x_i at port i. We include an extra port 0 and so the port set of M is $\mathcal{P} = \{0\} \cup \{1, 2, \ldots, m\}$. There are no inputs at port 0. The output alphabets are: $Y_0 = \{1, 2, \ldots, n\}$ and for all $1 \leq i \leq m$, $Y_i = \{o_i\}$.

If e_i is an edge from vertex v_j to vertex v_l then the state changes associated with transitions of $M(G)$ that have input x_i are defined by the following rules:

1. We include in $M(G)$ a transition from s_j to s_l with input x_i.

2. From every state $s \in S$ with $s \neq s_j$ there is a self-loop transition from s with input x_i. These are included to make $M(G)$ completely specified and all transitions from s_e are of this form.

Thus, for each state s of $M(G)$, a walk in G has a corresponding walk in $M(G)$ that starts at s. We define the output in response to input x_i in order to ensure that in controllable testing x_i can only be followed by an input x_j if e_i can be followed by e_j

in G. As a result, controllable walks through $M(G)$ will correspond to walks of G. Let us suppose that e_i is an edge from vertex v_j to vertex v_l and in G the edges that leave v_l (and so can follow e_i) are are those in $E' \subseteq E$. Then for port p, $1 \leq p \leq m$, the transitions with input x_i produce output o_p at p if $e_p \in E'$ and otherwise they produce no output at p. As a result, input x_i can be followed by input x_j in controllable testing if and only if e_i can be followed by e_j in G. At port 0 there are two case: the input of x_i leads to output j at 0 if x_i is input when M is in state s_j (recall that e_i leaves vertex v_j) and otherwise, if x_i is input when $M(G)$ is in a state $s_{j'} \neq s_j$, it leads to no output at 0.

As a result of this construction, any input of x_i leads to the same output at all ports in $\{1, 2, \ldots, m\}$ irrespective of the state in which it is applied. Thus, only the output at port 0 can be used to distinguish states. In addition, no output can be produced at 0 when an input sequence is applied from s_e. For example consider an instance G of Directed Hamiltonian Path problem given in Figure 6.3a and corresponding MPFSM $M(G)$ given in Figure 6.3b.

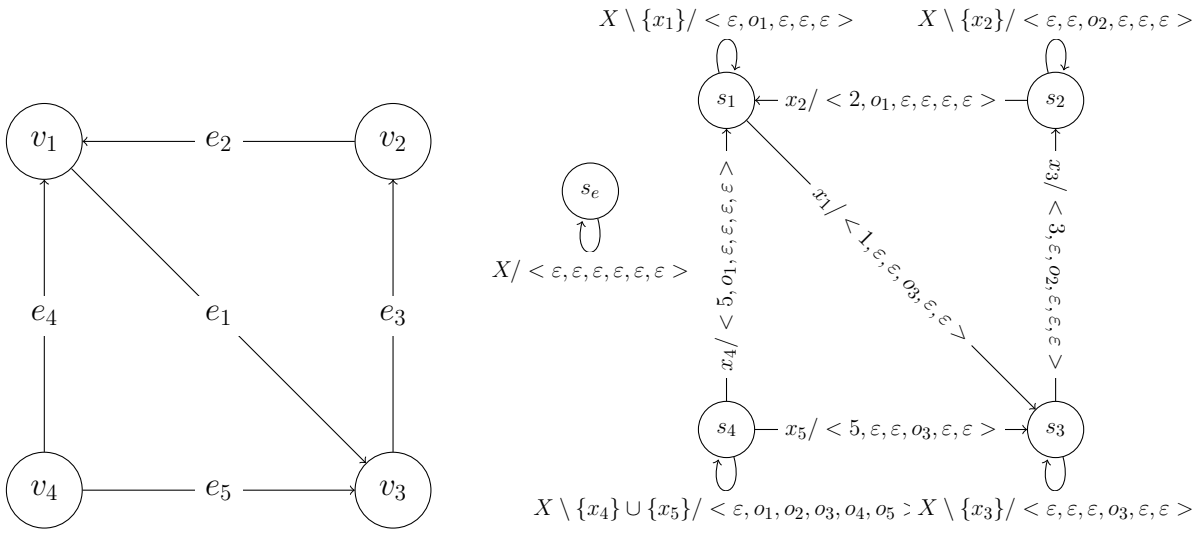
In terms of distinguishing states there is no value in following input immediately by itself (e.g. having a subsequence of the form $x_i x_i$). We will say that a strategy for $M(G)$ is *non-redundant* if it cannot lead to an input being immediately followed by itself and results will restrict attention to non-redundant strategies.

We now prove that a controllable global strategy of $M(G)$ has a particular form.

Proposition 33 *Given directed graph G and* MPFSM *$M(G)$ with state set S, if μ is a non-redundant controllable global strategy for $M(G)$ then all traces in $Ev(\mu, M(G), S \setminus \{s_e\})$ have the same input portion $x_{i_1}, \ldots, x_{i_l}$ and this has the property that $e_{i_1}, \ldots, e_{i_l}$ is a walk of G.*

We now show how the DHP problem for strongly connected G relates to the existence of an ADS μ for $M(G)$ whose longest evolution has length $\ell = n$.

Proposition 34 *Strongly connected directed graph G has a Hamiltonian path if and only if $M(G)$ has a controllable* ADS *that distinguishes all of the state of $M(G)$ and*



(a) Original Directed Hamiltonian Path problem instance G and the specified vertex is v_4.

(b) Constructed MPFSM $M(G)$ at which the specified vertices are s_4, s_e.

Figure 6.3.: An example reduction.

whose longest evolution has length $\ell = n$.

We can now prove that the problem of deciding whether an MPFSM has an ADS whose longest evolution has length ℓ is NP-hard. The following holds when we are interested in distinguishing all states in S or some subset S' of these.

Theorem 39 *The* Exact Height ADS *problem is in* EXPSPACE *and is* NP-hard.

The definition of $M(G)$ ensures that the states of $M(G)$ can only be distinguished by the output at port 0. In order to adapt the proof to the p-ADS problem we require input at 0 since a 0-ADS must start with input at 0. We can achieve this by having one input x_0 at port 0 that, irrespective of the current state, sends a constant to all ports and does not change state. It is straightforward to adapt the proof of Proposition 34 to prove that a strongly connected directed graph has a Hamiltonian path if and only if the resultant MPFSM which includes transitions with input x_0, has an ADS that starts with x_0 and whose longest evolution has length $n + 1$. Essentially, such an ADS

must start with x_0 and then apply an ADS that does not contain x_0. Since the Exact Height ADS problem is in EXPSPACE we obtain the following result[3].

Theorem 40 *The* Exact Height p-ADS *problem is in* EXPSPACE *and is* NP-hard.

Finally, if we suitably bound ℓ then the problems are NP-complete.

Theorem 41 *The* Exact Height ADS *and* Exact Height p-ADS *problems for an* MPFSM *with n states are* NP-complete *if $\ell = poly(n)$, where $poly(n)$ is a polynomial function of n.*

As noted before, in testing it is possible to use a characterisation set containing controllable separating sequences. Thus, in practice we are unlikely to be interested in ADSs that are significantly longer than the sum of the lengths of the sequences in such a characterisation set and we know that there is a polynomial upper bound on this value. This provides clear motivation for the above result: in practice we are likely to have a polynomial upper bound on the height of an ADS that we are ready to use.

6.6. Chapter Summary and Future Directions

Many automated test generation algorithms for testing from a single-port FSM M use tests that distinguish states of M and there has been particular interest in preset distinguishing sequences (PDSs) and adaptive distinguishing sequences (ADSs). There has been interest in both approaches since when applying PDSs it is possible to use a less complex test infrastructure but ADSs can be shorter, are computationally less expensive to produce, and there are FSMs that have ADSs but no PDS.

This work has shown how the concepts of PDSs and ADSs can be extended to distributed testing. We showed that if a PDS or ADS is controllable then it can be implemented by a set of distributed local testers but otherwise this is not possible. We also showed how these local testers can be devised. We then explored the problems of

[3]The problem being NP-hard follows as before since the DHP problem allows a polynomial bound.

deciding whether an MPFSM has a PDS, proving that this problem is PSPACE-hard and is generally undecidable if we consider subsets of the set of states. Having shown this we gave a condition under which the latter problem is decidable but NP-hard and a second stronger condition under which it is NP-complete.

Having explored PDSs we then considered ADSs. We proved that the problem of deciding whether an MPFSM has an ADS is PSPACE-hard but left decidability open. This situation is significantly different from the single-port MPFSM problem, which can be solved in low-order polynomial time. We also showed that the problem of deciding whether there is an ADS with height ℓ (or height at most ℓ) is NP-hard. However, we observed that in practice we can use a set of separating sequences of polynomial size and so it makes sense to assume that the upper bound ℓ is bounded above by a polynomial in the size of M. For such bounds the problem of deciding whether an MPFSM has an ADS is NP-complete.

There are several lines of future work. First, it is still open whether the ADS existence problem is decidable. There is also the problem of finding sensible conditions under which the problems studied can be solved in polynomial time. Finally, there is the problem of exploring heuristics for devising controllable PDSs and ADSs and here it is encouraging that although the PDS existence problem is PSPACE-hard for single-port FSMs it has been found that SAT solvers can be effective in solving this problem [108].

7. Conclusions

In Finite State Machine (FSM) based testing, a *Fault Detection Sequence* is applied to an implementation and the response of the implementation is analysed. If the implementation produces the expected output then the implementation is said to be a correct implementation.

We first aimed to reduce the costs of components of a CS by hoping that this will reduce the cost of the CS. The first component that we investigated is a *Reset Sequence*. The second component that we studied is *State Identification Sequence*.

In this thesis, we aim to reduce the cost of fault detection sequences. We first introduced several problems related to reset sequences: an input sequence that resets the FSM under consideration, and investigated their computational complexities. It is known that constructing one of the shortest reset sequence is an NP-complete problem [34]. However when the transition structure of a machine is monotonic (transitions of machine preserve some linear order of the states) one can construct one of the shortest reset word in polynomial time. However the complexity of constructing shortest reset sequences for monotonic partial machines was an open problem. We, therefore, investigated to reveal if this desirable complexity condition survives when the machine has partial transitions. This is interesting, because if the underlying machine is not monotonic and is partial then constructing a reset sequence is know to be PSPACE-complete. We showed that this problem is NP-hard. Currently, the upper bound is not set and we left this as an open problem. Therefore, it would be interesting to see if the problem is in NP or in PSPACE. Moreover we believe that proposing heuristics for these problems are also important.

Secondly, as state identification components form a large portion of a CS, we studied

strategies to reduce the length of state identification sequences. There are different kinds of state identification sequences: Unique Input Output (UIO) sequences, Separating Family or Distinguishing Sequences. Among others, distinguishing sequences are known to be the most efficient state identification sequences and it has been known that the use of distinguishing sequences allow to construct shorter CSs.

There are two types of distinguishing sequences, Preset Distinguishing Sequences (PDS) and Adaptive Distinguishing Sequences (ADS). For a given FSM checking the existence of a PDS is known to be PSPACE-complete and it is know that there are FSMs whose shortest PDS is given by exponential function with the number of states of the underlying FSM. On the other hand, there are polynomial time algorithms to construct ADSs and the length of the ADSs is bounded by a polynomial function of the size of the underlying FSM.

The standard ADS generation algorithm runs in polynomial time but not guaranteed to compute the minimum cost ADS. As an ADS defines a tree, the "cost" may not be immediately clear. In this thesis, we introduced several problems related to minimizing ADSs and we showed that all these problems are hard to solve and hard to approximate. We introduced several modification strategies for the exiting ADS construction algorithm and also proposed a heuristic approach for constructing reduced cost ADSs.

We further performed a set of experiments and reveal what one can gain if one use reduced cost ADSs while constructing CS. We revealed that it is possible to reduce the length of CS by 29.2% on the average.

Although distinguishing sequences are important and reduce the cost of CS, not all FSMs have a distinguishing sequence. For such cases instead of a CS a CE (set of input sequences) are applied and used to detect fault. For a given FSM with no DS, state identification problem can be solved by using either UIOs or Separating Family or enhanced version of separating family called *Harmonized State Identifiers*. In this thesis, we introduce the notion of *Incomplete Distinguishing Sequences* and show how to use these sequences while constructing CEs. The motivation of this research comes primarily from the desire to establish a tactic of utilizing desired properties of distinguishing

sequences. In order to achieve this, we first investigate the hardness of constructing such incomplete distinguishing sequences and show that construction of such sequences are PSPACE-complete. Moreover, we introduce a heuristic approach that aims to construct a set of incomplete distinguishing sequences with reduced cardinality. We experimentally showed that the use of incomplete distinguishing sequences give rise to construct shorter checking experiments.

Finally, in this thesis we studied the computational complexity of constructing distinguishing sequences for distributed testing. In distributed testing the underlying FSM has a finite set of ports (multi–port FSMs MPFSM) and each port is controlled by distinct tester. Moreover, the testers are assumed to be disconnected, that is, testers do not have a capability of using any form of communication method which means that no tester has the capability of acquiring the global trace of the underlying MPFSM. This restricted observation power of testers, necessarily change the nature of testing. Such restrictions introduces *Controllability* and *Observability* problems. Controllability problem refers to the condition in which a tester cannot decide whether it should apply an input by evaluating its local trace. Observability problem refers to the condition in which the testers cannot detect the faults of the implementation by analysing their local traces. Controllability and observability problems reduce the effectiveness of fault detection sequences designed for single–port FSMs. Similarly, these problems invalidate the state identification capabilities of distinguishing sequences. However, distinguishing sequences are still desirable for constructing fault detection experiments for MPFSMs. The formal definition of distinguishing sequences and the undecidability results for constructing distinguishing sequences for distributed testing has been set. However the hardness of computing and the definitions of controllable distinguishing sequences have not been addressed. In this thesis, we define what it means for a distinguishing sequence to be a controllable and investigate the hardness of constructing such sequences. The results are disappointing, they are all hard problems and the bounds for these sequences have not been set yet. We propose a subclass of MPFSMs for which constructing a PDS is decidable. As a future work it would be interesting to set bounds for controllable

distinguishing sequences for MPFSMs.

Appendix A

Lemma 5 *Let (U, C) be an instance of a* Set Cover *problem and $C' = \{c_1, c_2, \ldots, c_m\}$ be a cover. Then the sub-automaton $\mathbb{F}(U, C)|_{\bar{\Sigma}}$ is synchronizable, where $\bar{\Sigma} = \{x_i | c_i \in C'\}$.*

Proof 1 *Consider $\mathbb{F}(U, C)$ as defined above. If $C' = \{c_1, c_2, \ldots, c_m\}$ is a cover, we claim that for $w = x_1 x_2 \ldots x_m$, we have $\delta(Q, w) = \{S\}$, hence w is a reset word for $\mathbb{F}(U, C)$. This is evident from the fact that, for any $u \in U$, there exists a $c \in C'$ such that $u \in c$ (since C' is a cover). For a $u \in U$, let x_i be the first input symbol in w such that $u \in c_i$. In this case, $\delta(q_u, x_1 x_2 \ldots x_{i-1} x_i x_{i+1} \ldots x_m) = \delta(q_u, x_i x_{i+1} \ldots x_m) = \delta(Sink, x_{i+1} \ldots x_m) = Sink$. Now consider, $\mathbb{F}(U, C)|_{\bar{\Sigma}}$ where $\bar{\Sigma} = \{x_1, x_2, \ldots, x_m\}$. Note that $\mathbb{F}(U, C)$ is a CSA, and therefore so is $\mathbb{F}(U, C)|_{\bar{\Sigma}}$. The sequence $w = x_1 x_2 \ldots x_m$ is defined at all states of $\mathbb{F}(U, C)|_{\bar{\Sigma}}$ and therefore is a reset word for $\mathbb{F}(U, C)|_{\bar{\Sigma}}$.*

Lemma 6 *Let $\bar{\Sigma} = \{x_1, x_2, \ldots, x_m\}$ be a subset of alphabet of $\mathbb{F}(U, C)$ such that $\mathbb{F}(U, C)|_{\bar{\Sigma}}$ is synchronizable. Then $C' = \{c_1, c_2, \ldots, c_m\}$ is a cover.*

Proof 2 *Let $w = x_1 x_2 \ldots x_k$ be a reset word for the synchronizable CSA $\mathbb{F}(U, C)|_{\bar{\Sigma}}$. Since $\delta(Sink, w) = Sink$, we must have $\delta(q, w) = Sink$ for all $q_u \in Q \setminus \{Sink\}$. Due to the structure of the transition function δ, we can divide w as $w = w' x_i w''$ such that $\delta(q_u, w') = q_u$ and $\delta(q_u, x_i) = S$. This implies that $u \in c_i$.*

Theorem 1 *Given a synchronizable CSA $A = (Q, \Sigma, \delta)$ and a constant $K \in \mathbb{Z}^+$, it is* NP-complete *to decide if there exists a set $\bar{\Sigma} \subseteq \Sigma$ such that $|\bar{\Sigma}| < K$ and $A|_{\bar{\Sigma}}$ is synchronizable.*

Proof 3 *The proof is trivial using Lemma 5 and Lemma 6, and using the fact that* Set Cover *is* NP-complete.

Theorem 2 *MSS–Problem is* NP-complete.

Proof 4 *MSS is obviously in* NP. *Note that a special case of MSS is where we have the cost function W assigning costs to the input symbols is a constant function. Using Theorem 1, this special case is* NP-complete, *hence so is MSS.*

Lemma 7 *Let OPT_{sc} is the size of minimum cover for the* Set Cover *problem instance (U, C), and let $OPT_{\bar{\Sigma}}$ is the size of minimum cardinality input alphabet such that $\mathbb{F}(U, C)|_{\bar{\Sigma}}$ is synchronizable. Then $OPT_{sc} = OPT_{\bar{\Sigma}}$.*

Proof 5 *The proof is trivial using using Lemma 5 and Lemma 6.*

Theorem 3 *MSS–Problem does not admit an $o(\log n)$ approximation algorithm unless* $\mathsf{P} = \mathsf{NP}$.

Proof 6 *Consider again the special case of MSS–Problem where we have the function W as a constant function. Suppose that* $\mathsf{P} \neq \mathsf{NP}$ *and there exists a polynomial time algorithm $\mathcal{P}$ which returns an $o(\log n)$ approximation for MSS–Problem. Therefore for given* Set Cover *problem instance (U, C), one can construct automaton $\mathbb{F}(U, C)|_{\bar{\Sigma}}$, and using $\mathcal{P}$, can obtain a solution $\bar{\Sigma} \subseteq \Sigma$. In this case, Lemma 5, Lemma 6 and Lemma 7 together imply that $\bar{\Sigma}$ defines a solution C' for (U, C) which is also an $o(\log n)$ approximation for the* Set Cover *problem instance (U, C), which we know to be impossible when* $\mathsf{P} \neq \mathsf{NP}$.

Lemma 8 *MSS–Problem for PSA is in* PSPACE.

Proof 7 *For a given subset $\bar{\Sigma}$ of Σ, $A|_{\bar{\Sigma}}$ is either a CSA or a PSA. In either case, we know checking if $A|_{\bar{\Sigma}}$ is synchronizable can be performed in* PSPACE. *Using a nondeterministic Turing Machine, one can try all possible subsets $\bar{\Sigma}$, and hence the problem can be solved in* NPSPACE. *Using Savitch's Theorem [123], we conclude that the MSS–Problem for PSAs is in* PSPACE.

Lemma 9 *MSS–Problem for PSA is* PSPACE-hard.

Proof 8 *We will reduce from the problem of checking if a PSA is synchronizable or not, which is known to be* PSPACE-hard. *Let $A = (Q, \Sigma, \delta)$ be a PSA. Suppose that we construct a PSA $A' = (Q, \Sigma \cup \{a\}, \delta')$, where $\forall q \in Q, x \in \Sigma$, $\delta'(q, x) = \delta(q, x)$. However, for the input symbol a, δ' is defined in such a way that $|\delta'(Q, a)| = 1$ (that is A' is synchronizable by the sequence $w = a$). Let us define $W : \Sigma \cup \{a\} \to \mathbb{Z}^+$ as $\forall x \in \Sigma$,*

$W(x) = 0$, *and* $W(a) = \infty$. *We then consider an instance of MSS–Problem for PSA* A' *with* $K = 0$. *Note that, this instance of the MSS–Problem has a solution iff the original PSA* A *is synchronizable. Therefore, MSS–Problem for a PSA is at least as hard as deciding if a PSA is syncrhonizable.*

Theorem 4 *MSS–Problem for PSA is* PSPACE-complete.

Lemma 10 *ESW–SA problem is* PSPACE-hard.

Proof 9 *Let us consider a non–deterministic Turing Machine which starts with the state configuration* $\pi_0 = (\bar{Q}_0, \hat{Q}_0) = (\bar{Q}, \hat{Q})$. *At each step where we have the state configuration* $\pi_i = (\bar{Q}_i, \hat{Q}_i)$, *we pick an input symbol* $x \in \Sigma$ *randomly, and compute the next state configuration* π_{i+1} *as* $\pi_{i+1} = (\delta(\bar{Q}_i, x), \delta(\hat{Q}_i, x))$. *If* $\delta(\bar{Q}_i, x) \cap \delta(\hat{Q}_i, x) \neq \emptyset$ *the algorithm stops the search. Otherwise, if* $\delta(\bar{Q}_i, x) \subseteq F$, *the algorithm reports success. If this is not the case, the algorithm picks another random input symbol to proceed with the search. Since the maximum length* L *of an exclusive synchronizing sequence is less than* 2^n, *the algorithm can stop after executing at most* L *steps. The space needed to count the number of steps and handling the bookkeeping for the state configurations is polynomial in* n. *Therefore, the entire search in this way can be performed in* NPSPACE. *Based on Savitch's Theorem [123],* ESW–SA *problem is in* PSPACE *as required.*

Lemma 11 *ESW–SA problem is in* PSPACE.

Theorem 5 *ESW–SA problem is* PSPACE-complete.

Lemma 12 *Let* $\mathbb{A}'$ *be a subset of* $\mathbb{A}$ *where automata in* $\mathbb{A}'$ *has a common word and* $|\mathbb{A}'|$ *is maximized. Also let* $\bar{Q}'$ *be a subset of states* $\bar{Q}$ *of* $\mathcal{A}$, *where there exists an exclusive synchronizing word for* $\bar{Q}'$ *to* F *and* $|\bar{Q}'|$ *is maximized. Then* $|\bar{Q}'| = |\mathbb{A}'|$.

Proof 10 *Let* $w \in \Sigma^\star$ *be a common word for all the automata in* $\mathbb{A}'$. *Let* $\bar{Q}''$ *be the set of states in* $\bar{Q}$ *in* $\mathcal{A}$, *consisting of the initial states of automata in* $\mathbb{A}'$. *Using the reduction from FA–INT to ESW—SA,* $\delta(\bar{Q}'', RwS) \subseteq F$ *and* $\delta(\hat{Q}, RwS) = Sink$, *as explained in the proof of Lemma 10. Therefore, when one considers a maximum cardinality subset* $\bar{Q}'$

of $\bar{Q}$ for which there exists an exclusive synchronizing sequence, we have $|\bar{Q}'| \geq |\bar{Q}''| = |\mathbb{A}'|$.

Reversely, consider a maximum cardinality subset $\bar{Q}'$ of $\bar{Q}$ for which there exists an exclusive synchronizing sequence w. Again, as explained in the proof of Lemma 10, w is in the form $w = Rw'S$, and w' is a common word for the subset $\mathbb{A}''$ of set of automata $\mathbb{A}$, where an automaton $A_i \in \mathbb{A}''$ iff the initial state of A_i is in $\bar{Q}'$. When one considers a maximum cardinality subset $\mathbb{A}'$ of $\mathbb{A}$ for which there is a common word, we have $|\mathbb{A}'| \geq |\mathbb{A}''| = |\bar{Q}'|$.

This implies that $|\mathbb{A}'| = |\bar{Q}'|$, when maximum cardinality subsets are considered.

Theorem 6 *There exists a constant $\varepsilon > 0$ such that approximating Max ESW–SA problem within ratio n^{ε} is* PSPACE-hard.

Proof 11 *Let us assume that there exist an efficient algorithm P that approximates Max* ESW–SA *Problem within ratio n^{ε}. In this case, we can convert any given MAX FA-INT problem instance to automaton $\mathcal{A}$ and use algorithm P on $\mathcal{A}$. Relying on Lemma 12, the solution will also be an approximation for* Max FA-INT *instance for the set of automata $\mathbb{A}$, which directs us to the required contradiction.*

Lemma 13 *Let (U, C) be an arbitrary* Exact Cover *problem instance, then the states of the automaton $\mathbb{F}(U, C)$ admits a linear order for all input symbols in set Σ.*

Proof 12 *We claim that the state set Q admits the following linear order $S < q_1^0 < q_1^1 < q_2^0 < q_2^1 < \cdots < q_{|U|}^0 < q_{|U|}^1$. First consider the input symbol X, from the reduction $\mathbb{F}$, it is clear that the input X resets all states at the respective base states. Therefore for input symbol X we have:*

$$\delta(S, X) \leq \delta(q_1^0, X) \leq \delta(q_1^1, X) \leq \delta(q_2^0, X) \leq \ldots \leq \delta(q_{|U|}^0, X) \leq \delta(q_{|U|}^1, X)$$

The input symbol Y resets all the satellite states at S state and is defined only at these states. Therefore, we have:

$$- \leq - \leq \delta(q_1^1, Y) \leq - \leq \delta(q_2^1, Y) \leq \cdots \leq - \leq \delta(q_{|U|}^1, Y)$$

where $-$ denotes the cases where Y is not defined.

Finally consider the input symbols in $\Sigma \setminus \{X, Y\}$, which are only defined at some base and satellite states. Let q_i^a and q_j^b be two states such that $q_i^a < q_j^b$ and $x \in \Sigma \setminus \{X, Y\}$ be an input symbol defined at both q_i^a and q_j^b. When $i < j$, $\delta(q_i^a, x) = q_i^{a'}$ and $\delta(q_j^b, x) = q_j^{b'}$. Since $i < j$, we have $q_i^{a'} < q_j^{b'}$, or equivalently $\delta(q_i^a, x) < \delta(q_j^b, x)$. When $i = j$, then we must have $\delta(q_i^a, x) = q_i^a$ and $\delta(q_j^b, x) = q_j^b$. Again in this case, we have $q_i^a < q_j^b$, and thus $\delta(q_i^a, x) < \delta(q_j^b, x)$.

Theorem 7 Synchronizability Problem for PSMA *is* NP-hard.

Proof 13 *We first show that if there exists a solution to the* Exact Cover *problem instance (U, C), then automaton $\mathbb{F}(U, C)$ is synchronizable.*

Let (U, C) has a solution with $C' = \{c_1, c_2, \ldots, c_\ell\}$. i.e. $c_i, c_j, 1 \leq i, j, \leq \ell, i \neq j$ the followings hold: (1)$c_i \cap c_j = \emptyset$ (2) $\bigcup_{1 \leq i \leq \ell} c_i = U$.

First, we must observe that there exist single input symbol (X) that is defined at all the states. Therefore any synchronizing sequence of $\mathbb{F}(U, C)$ must begin with the input X, which leaves us with the set of all base states. Since C' is an exact cover, after input symbol X the application of the input sequence $x_1 x_2 \cdots x_\ell$ will leave us with the set of all satellite states. Therefore $X x_1 x_2 \cdots x_\ell Y$ resets $\mathbb{F}(U, C)$ at S as required.

Conversely if automaton $\mathbb{F}(U, C)$ is synchronizable then there exists a solution to the corresponding Exact Cover *problem instance.*

Let w be a rest word for machine $\mathbb{F}(U, C)$. We, without loss of generality, assume that w is minimal, i.e. no prefix and no suffix of w is a reset sequence. For such a minimal reset sequence w, it must be in the form $w = Xw'Y$, where w' has no occurrence of X and Y.

We will prove that $w' = x_1 x_2 \cdots x_\ell$ defines an exact cover $C' = \{c_1, c_2, \ldots, c_\ell\}$ for (U, C).

Since $Xw'Y$ is a reset sequence, for any $u \in U$, $\delta(q_u^0, w') = q_u^1$. This is only possible if there exists a unique input symbol x_i in w' such that $w' = w'_1 x_i w'_2$ where $\delta(q_u^0, w'_1) = q_u^0$, $\delta(q_u^0, x_i) = q_u^1$, $\delta(q_u^1, w'_2) = q_u^1$. When $\delta(q_u^0, x_i) = q_u^1$, this due to the fact that $u \in c_i$. Therefore, w' gives a unique c_i for each $u \in U$ such that $u \in c_i$. Hence if $w' = x_1 x_2 \cdots x_\ell$, then we have $C' = \{c_1, c_2, \ldots, c_\ell\}$ as an exact cover.

Theorem 8 Synchronizing Word Problem *and* Minimum Synchronizing Word Problem *for a PSMA are* NP-hard *problems.*

Lemma 14 *Let B be an arbitrary* N–Queens *puzzle instance, the states Q of the automaton $\mathbb{F}(B)$ admits a linear order.*

Proof 14 *First note that the automaton $\mathbb{F}(B)$ is not strongly connected. The automaton consist of N^2 set of states where each set has 3 states of the form $Q_{i,j} = \{q_{i,j}^0, q_{i,j}^+, q_{i,j}^-\}$. For two different set of states $Q_{i,j}$ and $Q_{i',j'}$, the set of states are disjoint and not reachable from one another. Therefore the ordering between a state $q \in Q_{i,j}$ and a state $q' \in Q_{i',j'}$ is not important, i.e. any ordering between q and q' works. On the other hand, for a given state set $Q_{i,j}$, we define the order between the states as $q_{i,j}^- < q_{i,j}^0 < q_{i,j}^+$. Due to the transition structure of $\mathbb{F}(B)$, for any input symbol $x_{k,l}$, we have the following cases:*

- *If $k = i, l = j$ then $\delta(q_{i,j}^-, x_{k,l}) = q_{i,j}^- < \delta(q_{i,j}^0, x_{k,l}) = q_{i,j}^+ = \delta(q_{i,j}^+, x_{k,l}) = q_{i,j}^+$.*
- *If (k, l) attacks (i, j) then $\delta(q_{i,j}^-, x_{k,l}) = q_{i,j}^- = \delta(q_{i,j}^0, x_{k,l}) = q_{i,j}^- < \delta(q_{i,j}^+, x_{k,l}) = q_{i,j}^+$.*
- *Otherwise, then $\delta(q_{i,j}^-, x_{k,l}) = q_{i,j}^- < \delta(q_{i,j}^0, x_{k,l}) = q_{i,j}^0 < \delta(q_{i,j}^+, x_{k,l}) = q_{i,j}^+$.*

In each case, the monotonicity condition is met by using the linear order suggested and the result follows.

Lemma 15 *Let $w \in \Sigma^\star$ be a reset word that resets $N + N^2$ states of the automaton $\mathbb{F}(B)$ at F. Then w defines a solution for the* N–Queens *puzzle instance B.*

Proof 15 *Consider that w resets $N + N^2$ states at $F = Q^+$. Note that $\delta(Q^+, w) = Q^+$ and $|Q^+| = N^2$. Since $\delta(Q^-, w) = Q^-$, the remaining N states reaching to F must be from the board states in Q^0.*

We will first show that there are exactly N effective input symbols in w. For an effective input $x_{i,j}$ in w, we have $w = w' x_{i,j} w''$ and $\delta(q^0_{i,j}, w') = q^0_{i,j}$. Based on the transition structure of $\mathbb{F}(B)$, we then have $\delta(q^0_{i,j}, w' x_{i,j}) = q^+_{i,j}$, which means one more state in F. Since, we need to accumulate N such states in F, there must be exactly N effective input symbols in w.

Let us consider two different effective input symbols $x_{i,j}$ and $x_{k,l}$ in w. Without loss of generality, assume that we have $w = w' x_{i,j} w'' x_{k,l} w'''$. Since $x_{k,l}$ is an effective input, $\delta(q^0_{k,l}, w' x_{i,j} w'') = q^0_{k,l}$. This implies that (k, l) does not attack (i, j) (or vice versa since "attacks" relation is symmetric). This means that the indices of N effective inputs in w gives N board positions for N queens that do not attack each other.

Lemma 16 *Let $\{(i_1, j_1), (i_2, j_2) \ , \ldots, (i_N, j_N)\}$ be a solution to the* N–Queens *puzzle. Then set of positions $(i_1, j_1), (i_2, j_2) \ , \ldots, (i_N, j_N)$ defines a solution for the* KFW–NSMA *Problem for automaton $\mathbb{F}(B)$.*

Proof 16 *Let us consider the input sequence $w = x_{i_1,j_1} x_{i_2,j_2} \cdots x_{i_N,j_N}$. We claim that w resets $N + N^2$ states in $F = Q^+$.*

For any prefix $w' x_{i_p,j_p}$ of w, we have $\delta(q^0_{i_p,j_p}, w') = w'$. This is due to the fact that (i_p, j_p) is not attacked by any other position. Therefore x_{i_p,j_p} is an effective input, hence the application of x_{i_p,j_p} after w' moves the state $q^0_{i_p,j_p}$ to $q^+_{i_p,j_p}$. After executing these N effective inputs, we will have N of the states in Q^0 moved into Q^+. Since also $\delta(Q^+, w) = Q^+$ and $|Q^+| = N^2$, the result thus follows.

Theorem 9 KFW–NSMA *Problem is* NP-hard.

Proof 17 *Based on the reduction given above, Lemma 15 and Lemma 16, the result is immediate.*

Theorem 10 MFW–NSMA *Problem is* NP-hard.

Appendix B

Lemma 17 *An input $x_z \in X_Z$ cannot appear as the label of an internal node p in an ADS $\mathcal{A}$ of M_D.*

Proof 18 *Using Lemma 3, $|\delta(S_{\bar{p}_v/\bar{p}_e}, \bar{p}_v)| > 1$. Lemma 4 requires that p_v be a valid input for $\delta(S_{\bar{p}_v/\bar{p}_e}, \bar{p}_v)$. However, x_z is not valid for any set of states since for any two states s and s', $\lambda(s, x_z) = \lambda(s', x_z) = 2$ and $\delta(s, x_z) = \delta(s', x_z) = s_z$.*

Lemma 18 *Given a decision tree $\mathcal{T}$ for D, there exists an isomorphic ADS $\mathcal{A}$ for M_D.*

Proof 19 *We construct an isomorphic ADS $\mathcal{A}$ by changing the label of an internal node t to x_t, and by changing the label of a leaf node from z to s_z, and by keeping the edge labels intact.*

It is easy to see that conditions (1)–(4) of Definition 2 are satisfied by $\mathcal{A}$. To show that Condition (5) of Definition 2 also holds, let p be a leaf node labeled by s_z in $\mathcal{A}$, let $\bar{p}_v = x_{t_{i_1}} x_{t_{i_2}} \dots x_{t_{i_k}}$ and $\bar{p}_e = \bar{y}$. Also consider the path from the root to the leaf labeled by z in $\mathcal{T}$. Due to the construction of $\mathcal{A}$ from $\mathcal{T}$, the sequence of node labels along this path is $t_{i_1} t_{i_2} \dots t_{i_k}$ and the sequence of edge labels is again $\bar{y}$. Using Lemma 17, we have $\bar{y} = t_{i_1}(z) t_{i_2}(z) \dots t_{i_k}(z)$. Then $\lambda(s_z, \bar{p}_v) = \lambda(s_z, x_{t_{i_1}} x_{t_{i_2}} \dots x_{t_{i_k}})$. Since s_z has self-looping transitions for input symbols in X_T, $\lambda(s_z, x_{t_{i_1}} x_{t_{i_2}} \dots x_{t_{i_k}}) = \lambda(s_z, x_{t_{i_1}}) \lambda(s_z, x_{t_{i_1}}) \dots \lambda(s_z, x_{t_{i_k}})$. Using the properties of the mapping β, this sequence is $t_{i_1}(z) t_{i_2}(z) \dots t_{i_k}(z) = \bar{y}$.

Lemma 19 *Given an ADS $\mathcal{A}$ for M_D, there exists an isomorphic decision tree $\mathcal{T}$ for D.*

Proof 20 *Lemma 18 implies that all the internal nodes have their labels from X_T. This also implies that there is no edge in $\mathcal{A}$ labeled by the output symbol 2.*

We construct an isomorphic decision tree $\mathcal{T}$ by changing the label of an internal node x_t to t, and by changing the label of a leaf node from s_z to z, and by keeping the edge labels intact. It is easy to see that Condition (1) and Condition (2) of Definition 18 are

satisfied by $\mathcal{T}$. Since $\mathcal{A}$ is always branching, and since output symbol 2 cannot appear in $\mathcal{A}$, Condition (3) of Definition 18 is also satisfied.

For Condition (4) of Definition 18, let p be a leaf node in $\mathcal{T}$ labeled by an object z, and p' be an internal node on the path from the root to p, and let t be the label of p'. Let q and q' be the corresponding nodes in $\mathcal{A}$ for p and p' respectively. Note that the label of q must be s_z and the label of q' must be x_t. If q is in the 0-successor (respectively 1–successor) of q', then by Lemma 1 we have $\lambda(\delta(s_z, \vec{q}'_v), x_t) = \lambda(s_z, x_t) = t(z)$ equal to 0 (respectively 1). Using this result and the isomorphic structure of $\mathcal{A}$ and $\mathcal{T}$, we can conclude if p is in the 0–successor of (respectively 1–successor) of q', then $t(z)$ is equal to 0 (respectively 1).

Theorem 11 *The decision version of the problems* MinADS, MinHeightADS, *and* MinSDS *are* NP-complete.

Proof 21 *It is known that if an FSM M has an ADS, there is an ADS $\mathcal{A}$ for M where the size of $\mathcal{A}$ is polynomial in the size of M [35, 99]. Therefore the cost for* MinADS, MinHeightADS, *and* MinSDS *problems can be computed in polynomial time. Hence the problems are in* NP.

Note that MinDT *[100],* MinHeightDT *[101],* MinPathDT *[102] problems are* NP-hard. *Since the problems* MinADS, MinHeightADS, MinSDS *are at least as hard, they are also* NP-hard.

MinADS is a special case of MinWeightedADS (when $w_s = 1$ for all states s). Therefore the following result holds.

Theorem 12 *The decision version of the problem* MinWeightedADS *is* NP-complete.

Theorem 13 *For any constant $c > 0$, it is* NP-hard *to approximate* MinADS *and* MinWeightedADS *problems within a ratio of $(2 - c)$.*

Proof 22 *For any constant $c > 0$, it is* NP-hard *to approximate* MinDT *problem within a ratio of $(2 - c)$ [103, 104]* [1]*, hence the same inapproximability result applies to* Mi-

[1] MinDT problem is referred to as 2-UDT problem in [103, 104].

nADS *problem. Also, since* MinADS *is a special case of* MinWeightedADS*, this inapproximability result immediately applies to* MinWeightedADS *problem as well.*

Theorem 14 *Unless* P = NP*, there cannot be an* $o(\log n)$ *approximation for* MinHeightADS *and* MinSDS *problems.*

Proof 23 *Unless* P = NP*, there cannot be an* $o(\log n)$ *approximation for* MinHeightDT *and* MinPathDT *problems [105, 102]. Hence this result applies to* MinHeightADS *and* MinSDS *problems as well.*

Proposition 3 *Let* q *be an output node in* T*, let* $io(q) = w/v$*. Then we have* $bl(q) = \delta(S_{w/v}, w)$*.*

Proof 24 *The proof is trivial by using induction on the depth of* q*.*

Proposition 4 $|L| + \sum_{q \in Q} |bl(q)| = |S|$ *is an invariant of Algorithm 1 before and after every iteration the while loop.*

Proof 25 *Before the first iteration,* $|L| = 0$ *and* Q *only has the root node* q_0 *for which we have* $bl(q_0) = S$*. In an iteration of the algorithm, an output node* q *is removed from* Q *and a child (input) node* p *of* q *is selected. It is sufficient to observe two facts. First, each state* $s \in bl(q)$ *is represented by a state* $s' \in bl(q')$ *where* q' *is a child of* p*,* $s' = \delta(s, in(p))$ *and* $out(q') = \lambda(s, in(p))$*. Second, no two states* $s_1, s_2 \in bl(q)$ *can be represented by the same state* s' *in the same child* q'*, since* $in(p)$ *is a valid input for the states in* $bl(q)$*. Therefore, when we consider all children* q' *of* p*, we have* $\sum_{q'} |bl(q')| = |bl(q)|$*. Those children* q' *of* p *with* $|bl(q')| = 1$ *are included in* L*, and those children* q' *of* p *with* $|bl(q')| > 1$ *are included in* Q*. Hence the result follows.*

Proposition 5 *Let* q *be an output node in* T *with* $|bl(q)| = 1$*, and let* $w/v = io(q)$*. There exists a unique state* $s \in S$ *such that* $\lambda(s, w) = v$*.*

Proof 26 *Suppose* s *and* s' *are two distinct states such that* $\lambda(s, w) = \lambda(s', w) = v$*. By Proposition 3, we would then have* $\delta(s, w)$ *and* $\delta(s', w)$ *in* $bl(q)$*. Since* $|bl(q)|$*, this implies* $\delta(s, w) = \delta(s', w)$*. This is not possible since at each step a valid input is applied, therefore no two states can be merged into a single state.*

Proposition 6 *The leaves of $\mathcal{A}$ constructed by Algorithm 2 is labeled by distinct states.*

Proof 27 *Assume that there are two leaf nodes q'_1 and q'_2 in $\mathcal{A}$ such that they are both labeled by the same state s. Let q_1 and q_2 be the leaf (output) nodes in T that correspond to q'_1 and q'_2. Let w_1/v_1 and w_2/v_2 be the input/output sequences $io(q_1)$ and $io(q_2)$. In this case, we would have $\lambda(s, w_1) = v_1$ and $\lambda(s, w_2) = v_2$. However, this is not possible since M is deterministic.*

Theorem 15 *$\mathcal{A}$ constructed by Algorithm 2 is an ADS.*

Proof 28 *$\mathcal{A}$ has $n = |S|$ leaves as implied by Corollary 1. We will argue that $\mathcal{A}$ satisfies the conditions of Definition 2. Condition (i) is satisfied as shown by Proposition 6. Condition (ii) and Condition (iii) are easily satisfied due to the construction in Algorithm 2. Condition (iv) is satisfied due to the fact that in T, for an input node p, each child q of p has a distinct output symbol $out(q)$. Lines 5–8 of Algorithm 2, assigns the label of leaves in such a way that Condition (v) is satisfied (see Proposition 5).*

Appendix C

Lemma 20 *Let us suppose that set* $\mathbb{A} = \{A_1, A_2, \ldots, A_z\}$ *of automata have a common alphabet* Σ. *The FSM* $\mathcal{M}_1(\mathbb{A}) = (S, X, Y, \delta, \lambda, s_0)$ *has a* PDS *for* $\bar{S} = \{0_1^1, 0_1^2, \ldots, 0_z^1, 0_z^2\}$ *if and only if there is a non-empty word* $w \in \Sigma^\star$ *that is accepted by all of the automata (in which case* ωD *is such a* PDS*).*

Proof 29 *First, let us suppose that* $\omega \neq \varepsilon$ *is in the intersections of the languages of the* A_i *and consider* $w = \omega D$. *By construction,* ω *distinguishes any two* 0_i^α *and* 0_j^β *with* $i \neq j$ *since* $\omega \in \Sigma^\star$ *is non-empty, the output in response to an element of* Σ *identifies the state of the corresponding* A_i, *and the state sets of the* A_i *are pairwise disjoint. Further, if we consider states* 0_i^1 *and* 0_i^2 *we find that* ω *takes them to accepting states from* F_i *and then* D *leads to different outputs (*1 *and* 2*). Thus, if there is some non-empty* $\omega \in \Sigma^\star$ *in the intersections of the languages of the* A_i *then* $\mathcal{M}_1(\mathbb{A})$ *has a* PDS ωD *for* $\bar{S}$.

We now prove that if $\mathcal{M}_1(\mathbb{A})$ *has a* PDS *for* $\bar{S}$ *then there is some non-empty* $\omega \in \Sigma^\star$ *in the intersections of the languages of the* A_i. *We can observe that in order to distinguish states* 0_i^1 *and* 0_i^2 *it is necessary to apply input* D *but also that after* D *has been applied the state must be in* $\{Sink_1, Sink_2\}$ *and further input cannot distinguish the states. Thus there is a* PDS *for* $\bar{S}$ *if and only if there is a* PDS *for* $\bar{S}$ *that has the form* $w = \omega D$ *where* $\omega \in \Sigma^\star$ *and we now consider such a* PDS.

Now let us suppose that $\delta(0_i^1, \omega) \notin F_i^1$ *for some* $1 \leq i \leq z$. *Then* $\delta(\delta(0_i^1, \omega), D) = Sink_1$ *and similarly* $\delta(\delta(0_i^2, \omega), D) = Sink_2$ *and it is clear that* $\lambda(0_i^1, \omega D) = \lambda(0_i^2, \omega D)$. *This contradicts* ωD *being a PDS for* $\bar{S}$. *Therefore* w *must be in the form* $w = \omega D$ *such that* ω *is non-empty and brings all the initial states to accepting states. Thus, if* $\mathcal{M}_1(\mathbb{A})$ *has a* PDS *for* $\bar{S}$ *then there is some non-empty* $\omega \in \Sigma^\star$ *in the intersections of the languages of the* A_i.

Lemma 21 *The problem of deciding whether a set* $\bar{S}$ *of states of FSM* M *has a* PDS *is in* PSPACE.

Proof 30 *We will show that a non-deterministic Turing Machine can solve this using polynomial space. Such a machine will guess inputs one at a time. It will maintain*

the set π of pairs of states and equivalence relation r as described above and this uses polynomial space. After guessing a new input x and updating π and r the machine checks whether the input sequence received defines a PDS *for $\bar{S}$: this it the case if and only if r relates no two different states of $\bar{S}$. Thus, if M has a* PDS *for $\bar{S}$ then this non-deterministic Turing Machine will find such a* PDS *using polynomial space.*

We now have to consider the case where M does not have a PDS *for $\bar{S}$: we require that the non-deterministic Turing Machine terminates. In order to ensure this we use the result that if M has n states and $\bar{S}$ has m states then M has a* PDS *for $\bar{S}$ if and only if it has such a* PDS *with length at most $B = (m-1)n^m$ [21][2]. The non-deterministic Turing Machine therefore includes a counter that counts how many inputs have been received: the machine terminates with failure if the counter exceeds the upper bound. We require additional $O(\log_2 B) = O(\log_2(m-1) + m\log_2(n)) = O(m\log_2(n))$ space for the counter and so the space required is still polynomial.*

We have defined a non-deterministic Turing Machine that requires only polynomial space in order to solve the problem and so the problem is in non-deterministic PSPACE*. We can now use Savitch's Theorem [123], which tells us that a problem is in* PSPACE *if and only if it is in non-deterministic* PSPACE*, and the result follows.*

Lemma 22 *Given set $\mathbb{A}$ of automata, let $OPT_{\mathbb{A}}$ be the set of minimal words that are accepted by the maximum number of automata from $\mathbb{A}$. Further, given $\mathcal{M}_1(\mathbb{A})$ let $OPT_{\mathcal{M}_1(\mathbb{A})}$ be the set of minimal words that maximise the size of the subset of $\bar{S}$ whose states are pairwise distinguished. Then $w \in OPT_{\mathbb{A}}$ if and only if $wD \in OPT_{\mathcal{M}_1(\mathbb{A})}$.*

Theorem 16 *The* MaxSubSetPDS *problem is* PSPACE-complete *and for any constant $\varepsilon > 0$ approximating the* MaxSubSetPDS *problem within ratio n^{ε} is* PSPACE-hard*.*

Proof 31 *The problem being* PSPACE-hard *follows from Lemma 23 and the MAX FA-INT problems being* PSPACE-hard*. To see that this problem is in* PSPACE*, first observe that it is sufficient to prove that the following problem is in* PSPACE*: for $1 \leq k \leq n$*

[2] In [21], Gill presents this result on pg: 104, Theorem 4.3. Note that Gill named the set $\bar{S}$ as the *Admissible Set* i.e. the initial states of the underlying FSM.

decide whether there is a PDS *that distinguishes k states of the FSM M. We can show that this is in* PSPACE *in a similar manner to Lemma 22, the only differences being that in a first step the non-deterministic Turing Machine guesses the set $\bar{S}'$ of k states.*

To prove that the problem of approximating the MaxSubSetPDS *is* PSPACE-hard*, let us assume that we have an algorithm $\mathcal{P}$ that belongs to a complexity class $C < PSPACE$ and returns an n^{ε} approximation for the* MaxSubSetPDS *Problem. In such a case, given an instance $\mathbb{A}$ of the MAX FA-INT problem, we can construct FSM $\mathcal{M}_1(\mathbb{A})$ and using $\mathcal{P}$ we can obtain a solution $w = \omega D$. But then Lemma 23 implies that ω defines an approximation for $\mathbb{A}$ and hence $\mathcal{P}$ defines an n^{ε} approximation for the MAX FA-INT problem. Thus the result follows.*

Theorem 17 *The* MinSetPDS *problem is* PSPACE-complete.

Proof 32 *We first prove that the problem is in* PSPACE*. Observe that in P_S we require at most one set for each pair of states of M and so if M has state set S then the set P_S of subsets has size at most $|S|(|S|-1)$.*
It is therefore sufficient to show that we can solve the problem of trying to find a set k of subsets where each subset corresponds to a PDS *($1 \leq k \leq |S|(|S|-1)$); if we can do this then a Turing Machine could start with $k = |S|(|S|-1)$ and then reduce k step by step until a set is found. Given k, a non-deterministic Turing Machine can thus initially guess such a set P_S of k subsets and for each such set $S' \in P_S$ the Turing Machine tries to build a* PDS *that distinguishes all of the states in S'. As before, for a given set S' the process terminates when the upper bound on* PDS *length is exceeded or the* PDS *being built is sufficient. Since this can be performed in polynomial space we have that the result follows from Savitch's Theorem [123].*
To see that the problem is PSPACE-hard *it is sufficient to observe that M has a complete* PDS *if and only if it has a set P_S that satisfies the conditions of the* MinSetPDS *problem and contains only one set. The result therefore follows from the complete* PDS *problem being* PSPACE-hard *[35].*

Lemma 23 *Let us suppose that set $\mathbb{A} = \{A_1, A_2, \ldots, A_z\}$ of automata have a common*

alphabet Σ. *The FSM* $\mathcal{M}_2(\mathbb{A}) = (S, X, Y, \delta, \lambda, s_0)$ *has an* ADS *for* $\bar{S} = \{0_1, \ldots, 0_z, Sink\}$ *if and only if there is a non-empty word* $w \in \Sigma^\star$ *that is accepted by all of the automata (in which case input sequences* wd_1, wd_1d_2, $wd_1d_2d_3$, $\ldots$, $wd_1d_2d_3 \ldots d_z$ *define an* ADS*).*

Proof 33 *We first show that if* w *is accepted by all the automata then input sequences* wd_1, wd_1d_2, $wd_1d_2d_3$, $\ldots$, $wd_1d_2d_3 \ldots d_z$ *define an* ADS *for* $\bar{S}$. *Since* w *is accepted by all the automata, input sequence* w *will take any initial state* 0_i *to an accepting state. We show that* $wd_1d_2 \ldots d_j$ *distinguishes states* $0_i, 0_j$ *for any* $1 \leq i \neq j \leq z$. *This follows from the fact that at state* $\delta(0_j, w)$ *the FSM will not change its state, will produce* 0 *when any input from set* $\mathcal{D} \setminus \{d_j\}$ *is applied, and will produce* j *as output if input* d_j *is applied. It is clear also that this word distinguishes all* 0_j *from Sink since it will lead to the output* j *being produced from* 0_j *but not from Sink. Therefore, if there exists a word* w *that is accepted by all the automata, then FSM* $\mathcal{M}_2(\mathbb{A})$ *has an* ADS *for set* $\bar{S}$ *in the form of* $wd_1, wd_1d_2, wd_1d_2d_3, \ldots, wd_1d_2d_3 \ldots d_z$.
Now assume that machine $\mathcal{M}_2(\mathbb{A})$ *has an* ADS *for* $\bar{S}$ *and we are required to prove that there is some* $w \in \Sigma$ *in the intersections of the languages of the automata. Let us suppose that from* $\bar{S}$ *the* ADS *applies input sequence* w *and then input* x *such that the response to* w *does not distinguish any two elements of* $\bar{S}$ *but the response to* wx *distinguishes two or more states of* $\bar{S}$. *Then the input of* x *after* w *must lead to different outputs for two or more states in* $\bar{S}$ *and so we must have that* $x \in \mathcal{D}$. *Further, the input of* w *in any state of* $\bar{S}$ *leads to a sequence of zeros since this is the response to any input sequence when in state Sink. If* w *does not take some* 0_j *to a final state then* wx *takes* 0_j *to state Sink producing only zeros as output and so the* ADS *does not distinguish* 0_j *from* $Sink \in \bar{S}$. *Thus,* w *must take each* A_i *to a final state and so by definition we have that* $w \in \Sigma^\star$ *and* w *is in the languages defined by all of the* A_i *and so the result holds.*

Lemma 24 *Given FSM* M *and state set* $\bar{S}$, *the problem of deciding whether* $\bar{S}$ *has an* ADS *is in* PSPACE.

Proof 34 *We will show that a non-deterministic Turing Machine can solve this using polynomial space. Such a machine will operate through a sequence of steps, extending*

the depth of the ADS by one in each step. It will maintain a set π of pairs of states and equivalence relation r as in the proof of Lemma 22 and again this uses polynomial space. As before, we start with $\pi = \{(s, s)|s \in \bar{S}\}$. In each step, if $(s, s') \in \pi$ then the current 'guess' takes s to s' and $(s, s') \in r$ if and only if the current 'guess' does not distinguish s and s'. A step involves the non-deterministic Turing Machine guessing a next input for each equivalence class of r and updating π and r accordingly. The machine also checks whether an ADS *has been defined for $\bar{S}$: this it the case if and only if r relates no two different states of $\bar{S}$. Thus, if M has an* ADS *for $\bar{S}$ then this non-deterministic Turing Machine will find such an* ADS *using polynomial space.*

Similar to before, we now have to consider the case where M does not have an ADS *for $\bar{S}$ and require that the non-deterministic Turing Machine terminates. This is achieved by using the result that if M has n states and $\bar{S}$ has m states then M has an* ADS *for $\bar{S}$ if and only if it has such an* ADS *with length at most $\ell \leq \Sigma_{i=2}^{i=m} C\binom{n}{i} < 2^n$ [99][3]. The non-deterministic Turing Machine thus has a counter that gives the length of the current 'guess' and terminates with failure if the counter exceeds the upper bound. We require additional $O(\log_2 \ell)$ space for the counter and so the space required is polynomial.*

The non-deterministic Turing Machine requires polynomial space in order to solve the problem and so the problem is in non-deterministic PSPACE*; the result again follows from Savitch's Theorem [123].*

Theorem 18 *The* MaxSubSetADS *problem is* PSPACE-complete.

Proof 35 *The problem being* PSPACE-hard *follows from Lemma 24 and the MAX FA-INT problem being* PSPACE-hard. *In order to see that the problem is in* PSPACE*, we prove that the following problem is in* PSPACE*: "for $1 \leq k \leq n$, decide whether there is an* ADS *that distinguishes k states". We can deduce that this problem is in* PSPACE*, by considering the algorithm presented in Lemma 25. This time as a preprocessing step the Turing Machine will guess a set of states $\bar{S}$ with cardinality k then the Turing Machine continues to implement the procedure that we describe in the proof of Lemma 25. Therefore the* MaxSubSetADS *problem is in* PSPACE.

[3] Theorem 1, bound (2).

Lemma 25 *Given a set $\mathbb{A}$ of automata, let $OPT_{\mathbb{A}}$ be the set of minimal words accepted by the maximum number of automata from $\mathbb{A}$. Further, let $\mathcal{M}_2(\mathbb{A})$ be the FSM constructed from $\mathbb{A}$ and also let $OPT_{\mathcal{M}_2(\mathbb{A})}$ be the set of minimal* ADSs *that maximise the size of the subset of $\bar{S}$ whose states are pairwise distinguished by* ADSs*. Then $w \in OPT_{\mathbb{A}}$ if and only if* ADS *$wd_1, wd_1d_2, \ldots, wd_1 \ldots d_z$ is in $OPT_{\mathbb{M}}$.*

Theorem 19 *For any constant $\varepsilon > 0$ approximating the* MaxSubSetADS *problem within ratio n^{ε} is* PSPACE-hard.

Proof 36 *To prove that the problem of approximating* MaxSubSetADS *is* PSPACE-hard, *we consider an algorithm $\mathcal{P}$ that belongs to a complexity class $C < PSPACE$ and returns an n^{ε} approximation for the* MaxSubSetADS *Problem. In such a case, given a MAX FA-INT problem instance $\mathbb{A}$, we can construct FSM $\mathcal{M}_2(\mathbb{A})$ and using $\mathcal{P}$ we can obtain a solution $wd_1, wd_1d_2, \ldots, wd_1d_2 \ldots d_z$. But then Lemma 26 implies that w defines an approximation for $\mathbb{A}$ and hence $\mathcal{P}$ is also an approximation for the MAX FA-INT problem. The result thus follows.*

Lemma 26 *The* MinSetADS *problem is in* PSPACE.

Proof 37 *We can show that this problem is in* PSPACE *by following a procedure that is similar to the one we present in the proof of Theorem 18. As before, if the FSM M has state set S then P_S requires at most $|S|(|S|-1)$ sets. As a result, it is sufficient to prove that given k the following problem can be solved in* PSPACE*: is there a collection $P_S = \{\bar{S}_1, \bar{S}_2 \ldots \bar{S}_k\}$ of subsets of S such that for every pair s, s' of states there is some $\bar{S}_i$ such that $s, s' \in \bar{S}_i$ and for each $\bar{S}_i$ $(1 \leq i \leq k)$ there is an* ADS *that distinguishes the states of $\bar{S}_i$. The Turing Machine guesses such a P_S and then performs the remaining steps for each of the $\bar{S}_i$ separately. Clearly, this procedure takes polynomial space. The Turing Machine will return failure when it exceeds the bound given for the maximum depth, while if suitable* ADSs *are found then the Turing Machine returns success.*

Lemma 27 *The* MinSetADS *problem is* PSPACE-hard.

Proof 38 *We again consider an instance* $\mathbb{A} = \{A_1, \ldots, A_k\}$ *of the FA-INT problem with a common alphabet* Σ *and we construct* $\mathcal{M}_3(\mathbb{A})$*. From Lemma 24 and the FA-INT problem being PSPACE-hard we know that the problem of deciding whether there is an* ADS *for* $\{0_1, \ldots, 0_z, Sink\}$ *is* PSPACE-hard*. We will prove that any solution to the* MinSetADS *problem for* $\mathcal{M}_3(\mathbb{A})$ *also determines whether there is an* ADS *for* $\{0_1, \ldots, 0_z, Sink\}$*.*

Let us suppose that P_S *is a smallest set of subsets of* S *such that for every pair of states* s, s' *with* $s \neq s'$ *there is some* $\bar{S}' \in P_S$ *that contains both* s *and* s' *and for every set* $\bar{S}' \in P_S$ *there is an* ADS *that distinguishes the states in* $\bar{S}'$*. By construction, any set* $\bar{S}' \in P_S$ *that contains* $Sink$ *and a state* $s \in \bar{S} \setminus \{Sink\}$ *must correspond to* ADSs *that start with* st*. Similarly, if* $\bar{S}' \in P_S$ *contains* $Sink$ *and some* $s \in S \setminus \bar{S}$ *then it must correspond to* ADSs *that do not start with* st*.*

We will let P'_S *denote the set of subsets of* P_S *that contain* $Sink$ *and at least one state* $s \in \bar{S} \setminus \{Sink\}$*. We will prove that there is an* ADS *that distinguishes all of the states of* $\bar{S}$ *if and only if* P'_S *contains only one set.*

First assume that P'_S *contains only one set. Thus, the one set in* P'_S *contains all states from* $\bar{S}$ *and this implies that there is an* ADS *for* $\bar{S}$ *as required.*

Now assume that $\bar{S}$ *has an* ADS*. By definition, no set in* P'_S *contains a state* $s \notin \bar{S}$ *and for all* $s \notin \bar{S}$ *we have that* $P_S \setminus P'_S$ *contain a set that has both* s *and* $Sink$*. Thus, for each* $s, s' \in (S \setminus \bar{S}) \cup \{Sink\}$ *with* $s \neq s'$ *we have that* $P_S \setminus P'_S$ *has a set that contains both* s *and* s'*. As a result, it is sufficient for the sets in* P'_S *to contain all pairs* s, s' *from* $\bar{S}$ *with* $s \neq s'$*. Since there is an* ADS *that achieves this, by the minimality of* P_S *we must have that* P'_S *contains only one set.*

We now know that $\bar{S}$ *has an* ADS *if and only if* P'_S *contains only one set and so if we can solve the* MinSetADS *problem for* $\mathcal{M}_3(\mathbb{A})$ *then we can decide whether* $\bar{S}$ *has an* ADS*. We can now note that* $\bar{S}$ *has an* ADS *if and only if the state set* $\{0_1, \ldots, 0_z, Sink\}$ *of* $\mathcal{M}_3(\mathbb{A})$ *has an* ADS*: the* ADS *for* $\bar{S}$ *in* $\mathcal{M}_3(\mathbb{A})$ *starts with* st *and then applies an* ADS *for state set* $\{0_1, \ldots, 0_z, Sink\}$ *of* $\mathcal{M}_3(\mathbb{A})$*. The result thus follows from Lemma 24 and the FA-INT problem being PSPACE-hard.*

Theorem 20 *The* MinSetADS *problem is* PSPACE-complete.

Proposition 7 *Given fully distinguishing set $A = \{\mathcal{A}_1, \ldots, \mathcal{A}_k\}$ for FSM M, the $H_i(A)$ are state identifiers for M.*

Proof 39 *It is sufficient to prove that if s_i, s_j are distinct states of M then there are input sequences $w_i \in H_i(A)$ and $w_j \in H_j(A)$ such that there is a common prefix w of w_i and w_j that distinguishes s_i and s_j. First observe that since A is fully distinguishing there is some $\mathcal{A}_l \in A$ that distinguishes s_i and s_j. But by definition this means that the application of $\mathcal{A}_l$ from s_i and s_j leads to different input/output sequences. However, the input sequence can only differ once a different output has been observed. Thus, $\mathcal{A}_l$ has a node v such that the following hold:*

1. *The path from the root of $\mathcal{A}_l$ to v has a label α/β that labels paths from both s_i and s_j; and*
2. *There are edges with labels x/y_i and x/y_j from v with $y_i \neq y_j$ such that $\alpha x/\beta y_i$ labels a path from s_i and $\alpha x/\beta y_j$ labels a path from s_j.*

However, this means that $w = \alpha x$ is a prefix of input sequences in H_i and H_j and also that w distinguishes s_i and s_j. The result therefore follows.

Theorem 21 *Given an FSM M and upper bound m on the number of states of the SUT, if A is a fully distinguishing set of* ADSs *for M then $CE(M, A, m)$ is a checking experiment for M.*

Proof 40 *This follows from Proposition 7 and the HSI method returning a checking experiment.*

Proposition 8 *Let us suppose that $\bar{S}$ is a set of states of FSM M. Then the states in $\bar{S}$ can be distinguished by a single* ADS *if and only if there is an identifying set $\{H_1, \ldots, H_n\}$ for M such that we can choose subsets $H'_i \subseteq H_i$ of each identifying set, $s_i \in \bar{S}$, under which each H'_i contains only one input sequence.*

Proof 41 *First assume that the states in $\bar{S}$ can be distinguished by a single ADS $\mathcal{A}$. Given state $s_i \in \bar{S}$ let H'_i denote the set containing one input sequence: the input portion of the input/output sequence produced when $\mathcal{A}$ is applied in state s_i. It is straightforward to check that the argument used in the proof of Proposition 7 applies and so the H'_i are state identifiers for $\bar{S}$ as required.*

Now let us suppose that we have state identifiers $\{H'_i | s_i \in \bar{S}\}$ for $\bar{S}$ such that each H'_i contains only one input sequence, which we call w_i. Form a deterministic automaton $\mathcal{A}$ that is a tree with $|\bar{S}|$ leaves such that for each state $s_i \in \bar{S}$ the tree $\mathcal{A}$ has a path from the root to a leaf such that this path has label $w_i/\lambda(s_i, w_i)$. Since the H'_i define identifying sets for $\bar{S}$, for distinct states $s_i, s_j \in \bar{S}$ we have that the input in w_i, w_j can only differ after a different output in response to a common prefix of w_i, w_j. Thus, $\mathcal{A}$ is an ADS *for $\bar{S}$ as required.*

Proposition 9 *If the states in S can be distinguished by k* ADS *then there is an identifying set $\{H_1, \ldots, H_n\}$ such that for all $s_i \in S$ we have that H_i has at most k input sequences.*

Proof 42 *We will assume that the states in S can be distinguished by a set A of k* ADSs*. Given state $s_i \in \bar{S}$ let H_i denote the set containing the input portion of the input/output sequence produced when $\mathcal{A} \in A$ is applied in state s_i. By Proposition 7 the H_i are state identifiers for S as required.*

Lemma 28 *Let n be an internal node of tree T with children $n_1, n_2, \ldots, n_p$ and let x be the input portion of the labels of the edges from node n. The following hold:*

1. *$\delta(\mathcal{C}(n), x) = \cup_{i=1}^{p} \mathcal{C}(n_i)$.*
2. *For all $1 \leq i \leq p$ we have that $|\lambda(\mathcal{I}(n_i), x(n_i))| = 1$.*
3. *For all $1 \leq i < j \leq p$ we have that $\lambda(\mathcal{I}(n_i), x(n)) = \lambda(\mathcal{I}(n_j), x(n))$ and $\lambda(\mathcal{I}(n_i), x(n)x) \neq \lambda(\mathcal{I}(n_j), x(n)x)$.*

Lemma 29 *Let n, n' be distinct leaf nodes of tree T. If $s \in \mathcal{I}(n)$ and $s' \in \mathcal{I}(n')$ then $\lambda(s, x(n)) \neq \lambda(s', x(n))$ and $\lambda(s, x(n')) \neq \lambda(s', x(n'))$.*

Lemma 30 *Let T be a tree returned by the greedy algorithm such that $\bar{N} = \{n_1, \ldots, n_p\}$ is the set of leaf nodes of T. Let $\bar{S}$ be a set of states such that for all $1 \leq i \leq p$ we have that $|\mathcal{I}(n_i) \cap \bar{S}| \leq 1$. Then T defines an incomplete* ADS *for set $\bar{S}$.*

Proof 43 *We will show that T can be used to construct an incomplete* ADS *$\mathcal{A}$ for $\bar{S}$. Take a copy of tree T and for every node n remove from $\mathcal{I}(n)$ all states not in $\bar{S}$. Now remove all nodes with empty initial sets to form $\mathcal{A}$.*

Now we need to show that $\mathcal{A}$ is an incomplete ADS *for $\bar{S}$. By the construction of $\mathcal{A}$, each leaf node must be labeled by a singleton set. To see this, assume that an initial set of a leaf node contains two or more states. Since we drop states that are not in $\bar{S}$ this implies that there exist distinct $s, s' \in \bar{S}$ such that $s, s' \in \mathcal{I}(n_a)$ for some a, providing a contradiction.*

Moreover, it is easy to see that each internal node is labeled by a set of states, and each edge is labeled by an input output pair. Therefore conditions (1)-(3) of incomplete ADS *given in Definition 2 are satisfied. For an internal node in $\mathcal{A}$ there are at most $|Y|$ outgoing edges and due to the refine algorithm edges from a common node have identical input labels and different output labels. Thus, using Corollary 29, Condition (4) of Definition 2 is satisfied. Moreover due to the Corollary 30 we can deduce that Condition (5) of Definition 2 also holds. Therefore T defines an incomplete* ADS *for $\bar{S}$.*

Appendix D

Proposition 10 *If a strategy μ is controllable then for every* MPFSM *M and state set S' we have that μ is controllable for S'. However, it is possible that strategy μ is controllable for S' for some* MPFSM *M and state set S' and yet μ is not controllable.*

Proof 44 *The first part is immediate from the definitions. For the second part consider a global strategy μ that:*

- *starts by supplying input x_1 at port 1;*
- *if $\langle o_1, \varepsilon \rangle$ is output then the strategy supplies input x_2 at port 2 and terminates;*
- *for every other output the strategy terminates.*

This strategy is not controllable since it should send input x_2 to port 2 after $x_1/\langle o_1, \varepsilon \rangle$ but not after the empty sequence ε, even though $x_1/\langle o_1, \varepsilon \rangle$ and ε have the same projections at port 2. However, μ is controllable for any S' from which the input of x_1 cannot lead to output $\langle o_1, \varepsilon \rangle$. The result thus follows.

Proposition 11 *If global strategy μ is controllable then the distributed strategy $(\pi_1(\mu), \pi_2(\mu), \ldots . \pi_k(\mu))$ is deterministic.*

Proof 45 *We are required to prove that $\mu' = (\pi_1(\mu), \pi_2(\mu) \ldots . \pi_k(\mu))$ is deterministic. We will use proof by contradiction and assume that μ' is non-deterministic. By Definition 43 there therefore exists global trace $\sigma_1 \in (X/Y)^*$ such that for all $p \in \mathcal{P}$ we have that $\pi_p(\sigma_1) \in Ev(\mu_p)$ and ports p, p' with $p \neq p'$ such that $\mu_p(\pi_p(\sigma_1)) \in X_p$ and $\mu_{p'}(\pi_{p'}(\sigma_1)) \in X_{p'}$. By Definition 46, there exist global traces σ and σ' in $Ev(\mu)$ such that:*

- *$\pi_p(\sigma) = \pi_p(\sigma_1)$ and $\mu(\sigma) = x$, $x \in X_p$; and*
- *$\pi_{p'}(\sigma') = \pi_{p'}(\sigma_1)$ and $\mu(\sigma') = x'$, $x' \in X_{p'}$.*

By Definition 36, since μ is controllable and $\pi_p(\sigma) = \pi_p(\sigma_1)$ we must have that μ supplies input x after σ_1. Similarly, by Definition 36, since $\pi_{p'}(\sigma') = \pi_{p'}(\sigma_1)$ we must have that μ supplies input x' after σ_1. This contradicts the definition of a global strategy, since μ applies two different inputs after σ_1, and so the result follows.

Proposition 12 *It is possible that global strategy μ is controllable for S' but the distributed strategy $(\pi_1(\mu), \pi_2(\mu), \ldots \pi_k(\mu))$ is not deterministic for S'.*

Proof 46 *Consider a global strategy μ that initially supplies input x_1 at port 1 and only supplies another input if the output is $\langle 1, \varepsilon \rangle$, in which case the input is x_2 at port 2. Further let S' be some set of states from which the input of x_1 cannot lead to $\langle 1, \varepsilon \rangle$. Then clearly μ is controllable for S' since from S' it simply supplies x_1, observes an output, and then terminates. However, if we take the projections we find that $\pi_1(\mu)$ starts by supplying input x_1 and $\pi_2(\mu)$ can initially supply input x_2 (since μ can supply x_2 after $x_1/\langle 1, \varepsilon \rangle$ and $\pi_2(x_1/\langle 1, \varepsilon \rangle = \varepsilon)$. Thus, the distributed strategy $(\pi_1(\mu), \pi_2(\mu))$ is not deterministic for S' as required.*

Proposition 13 *Given state set S' of an* MPFSM *M, if global strategy μ is controllable for S' then the distributed strategy $(\pi_1^{S'}(\mu), \pi_2^{S'}(\mu), \ldots . \pi_k^{S'}(\mu))$ is deterministic for S'.*

Proposition 14 *Let us suppose that μ is a controllable global strategy and for all $p \in \mathcal{P}$ we have that $\mu_p = \pi_p(\mu)$. Then the distributed strategy $\mu' = (\mu_1, \mu_2, \ldots, \mu_k)$ is such that $Ev(\mu) = Ev(\mu')$.*

Proof 47 *First we prove that $Ev(\mu) \subseteq Ev(\mu')$. Proof by contradiction: assume that there is some $\sigma \in Ev(\mu) \setminus Ev(\mu')$. Let σ' denote the longest prefix of σ that is in $pre(Ev(\mu'))$ and so there exists an input/output pair x/y such that $\sigma' x/y$ is a prefix of σ and $\sigma' x/y \notin pre(Ev(\mu'))$. Let p be such that $x \in X_p$. Since $\sigma' x/y \in Ev(\mu)$ we have that $\mu(\sigma') = x$ and so $\mu_p(\pi_p(\sigma')) = x$. Thus, μ' can supply input x after σ' and so $\sigma' x/y \in pre(Ev(\mu'))$, providing a contradiction as required.*
Now we prove that $Ev(\mu') \subseteq Ev(\mu)$. Proof by contradiction: assume that there is some $\sigma \in Ev(\mu') \setminus Ev(\mu)$. Let σ' denote the longest prefix of σ that is in $pre(Ev(\mu))$ and

so there exists an input/output pair x/y such that $\sigma' x/y$ is a prefix of σ and $\sigma' x/y \notin pre(Ev(\mu))$. Let p be such that $x \in X_p$. Since $\sigma' x/y \in Ev(\mu')$ we have that $\mu_p(\pi_p(\sigma')) = x$. Thus, by the definition of $\pi_p(\mu)$, there exists some $\sigma'' \in Ev(\mu, M, S'))$ such that $\pi_p(\sigma'') = \pi_p(\sigma')$ and $\mu(\sigma'') = x$. However, since $\pi_p(\sigma'') = \pi_p(\sigma')$ and μ is controllable we must have that $\mu(\sigma') = \mu(\sigma'')$. Thus, $\mu(\sigma') = x$ and so $\sigma' x/y \in pre(Ev(\mu))$, providing a contradiction as required.

Proposition 15 *Let us suppose that μ is a controllable global strategy for set S' of states of* MPFSM *M and for all $p \in \mathcal{P}$ we have that $\mu_p = \pi_p^{S'}(\mu)$. Then the distributed strategy $\mu' = (\mu_1, \mu_2, \ldots, \mu_k)$ is such that $Ev(\mu, M, S') = Ev(\mu', M, S')$.*

Proposition 16 *Let us suppose that* MPFSM *M has port set $\mathcal{P} = \{1, 2, \ldots, k\}$. If μ is controllable and is an adaptive distinguishing sequence for M then $\mu' = (\pi_1(\mu), \pi_2(\mu), \ldots, \pi_k(\mu))$ is an adaptive distinguishing sequence for M.*

Proof 48 *By Proposition 11, since μ is controllable we know that μ' is deterministic. In addition, by Proposition 14 we know that $Ev(\mu) = Ev(\mu')$ and so for every state s of M we have that $Ev(\mu, M, s) = Ev(\mu', M, s)$. The result now follows from μ being an adaptive distinguishing sequence for M.*

Proposition 17 *Given $S' \subseteq S$, if μ is controllable for S' and is an adaptive distinguishing sequence for S' then $\mu' = (\pi_1^{S'}(\mu), \pi_2^{S'}(\mu), \ldots, \pi_k^{S'}(\mu))$ is an adaptive distinguishing sequence for M from S'.*

Proof 49 *By Proposition 13, since μ is controllable for S' we know that μ' is deterministic for S'. In addition, by Proposition 15, for every state s of M we have that $Ev(\mu, M, s) = Ev(\mu', M, s)$. The result now follows from μ being an adaptive distinguishing sequence for S'.*

Proposition 18 *If μ is controllable and is a* P-ADS *for M then $\mu' = (\pi_1(\mu), \pi_2(\mu), \ldots, \pi_k(\mu))$ is a* P-ADS *for M.*

Proposition 19 *Given $S' \subseteq S$, if μ is controllable for S' and is a* P-ADS *for S' then $\mu' = (\pi_1^{S'}(\mu), \pi_2^{S'}(\mu), \ldots, \pi_k^{S'}(\mu))$ is a* P-ADS *for M from S'.*

Lemma 31 *The following problem is* PSPACE-complete*: given a single-port* MPFSM *M in which no transition produces empty output, does M have a distinguishing sequence?*

Proof 50 *The problem being in* PSPACE *is a consequence of the general* PDS *existence problem for single-port FSMs being in* PSPACE*and so we focus on proving that the problem is* PSPACE-hard*. We will show that any algorithm that can solve this problem can also solve the general problem of deciding whether a single-port* MPFSM *has a distinguishing sequence. Let $M = (S, s_0, X, Y, \delta, \lambda)$ be a single-port* MPFSM *in which some transitions may have output ε. We construct an* MPFSM *$M' = (S, s_0, X, Y \cup \{y\}, \delta, \lambda')$ where $y \notin Y$ is a new output and the function λ' is defined by: given $s \in S$ and $x \in X$, if $\lambda(s, x) \neq \varepsilon$ then $\lambda'(s, x) = \lambda(s, x)$ and otherwise $\lambda'(s, x) = y$. It is now sufficient to observe that an input sequence is a distinguishing sequence for M if and only if it is a distinguishing sequence for M'. The result now follows from the problem of deciding whether a single-port* MPFSM *has a distinguishing sequence being PSPACE-complete [35].*

Proposition 20 *The following problem is* PSPACE-hard*: given a multi-port* MPFSM *M, is there a controllable* PDS *that distinguishes all of the states of M? In addition, this result still holds if we restrict attention to* MPFSMs *that have two ports.*

Proof 51 *Assume that we have been given a single-port* MPFSM *$M_1 = (S, s_0, X, Y, \delta, \lambda)$ such that all of the transitions of M_1 have non-empty output. We will construct a multi-port* MPFSM *M that has two ports 1 and 2. The state set of M will be S and the initial state will be s_0. Port 1 will have input alphabet $X_1 = X$ and output alphabet $Y_1 = \emptyset$. Port 2 will have input alphabet $X_2 = \emptyset$ and output alphabet $Y_2 = Y$. Given state s and input x such that $\delta(s, x) = s'$ and $\lambda(s, x) = y$, we will include in M the transition from s to s' that has input $x \in X_1$ and produces output $\langle \varepsilon, y \rangle$.*

Now consider controllable PDSs *for M. First observe that all input sequences are controllable. In addition, since no output is produced at port 1, the restriction that no input*

can follow a difference in output is always satisfied. Thus, we can consider all input sequences. Finally, an input sequence distinguishes two states of M if and only if it distinguishes two states of M_1. Thus, an input sequence is a controllable PDS *for M if and only if it is a* PDS *for M_1. The result now follows from Lemma 32.*

Corollary 3 *There is a class of* MPFSMs *where the shortest* PDS *is of exponential length.*

Proof 52 *Consider a single-port* MPFSM *that has a* PDS *with exponential length [35], now reapply the reduction given in Proposition 20.*

Proposition 21 *The problem of deciding whether a* MPFSM *has a* P-PDS *is* PSPACE-hard.

Proof 53 *Given a single-port* MPFSM M *that has no transitions with output ε, we construct the multi-port* MPFSM M'' *as described above. We show that the single-port* MPFSM M *has a* PDS *if and only if M'' has a* P-PDS *where $p = 1$.*
First let us suppose that the single-port MPFSM M *has a* PDS $\bar{x}$*. We show that $R\bar{x}$ is a* P-PDS *for M''. First note that after input R the tester at port 2 sees no differences in outputs (it observes L from all the states) and the tester at port 1 sees 0 from states of the form $s_i^\star$ and 1 from states of the form s_i and so can differentiate state s_i from $s_i^\star$. In addition, when tester 2 observes L it starts applying $\bar{x}$ to the* MPFSM *and since $\bar{x}$ produces different outputs from different states of* MPFSM M *the tester at port 1 can distinguishes each pair of states of the* MPFSM M''.
Now let us suppose that M'' has a P-PDS *and we need to show that a minimal* P-PDS *must be in the form of $R\bar{x}$ where $\bar{x}$ is a* PDS *for single-port* MPFSM M*. First assume that the* PDS *starts with input x rather than R. Then such an attempt causes each pair $s_i, s_i^\star$ of states to merge at state $\delta(s_i, x)$ without being distinguished. Therefore the first input must be $R \in X_1$.*
After input R, the tester at port 1 observes either 0 (at state $s_i^\star$) or 1 (at state s_i). Note that after R the MPFSM *must be at state s_i for some i (rather than $s_i^\star$). Then if $R\bar{x}$ is a* P-PDS *for M'', then $\bar{x}$ must distinguish any pair of states $s_i, s_j \in S$.*

We know that single-port MPFSM M *has a* PDS *if and only if* M'' *has a* P-PDS *where* $p = 1$. *Thus the result follows from Lemma 32.*

Proposition 22 *The problem of deciding whether an* MPFSM M *has a* PDS *of length* ℓ *is in* EXPSPACE.

Proof 54 *We now show that the problem is in* EXPSPACE. *A non-deterministic Turing machine can guess an input sequence* $\bar{x}$ *of length* ℓ *and it can compute and store each* $\lambda(s, \bar{x})$ *in space that is polynomial in* ℓ. *In order to check that* $\bar{x}$ *is controllable the Turing machine can simply compare prefixes of the sequences of the form* $\lambda(s, \bar{x})$*: there are controllability problems if there are prefixed* σ_1 *and* σ_1' *of such traces that have the same projection at a port* p *such that after* σ *and* σ' *the behaviour at* p *differs. This can be checked in time that is polynomial in terms of* ℓ. *Finally, the Turing machine can check in time that is polynomial in terms of* ℓ *whether* $\bar{x}$ *is a* PDS. *Thus, a non-deterministic Turing machine can check whether a guess* $\bar{x}$ *is a controllable* PDS *in space that is polynomial in terms of* ℓ *and so exponential in terms of the representation of* ℓ *(that takes* $O(\log_2 \ell)$ *space). Finally, using Savitch's Theorem [123] we know that a deterministic Turing machine can also solve the problem in exponential space. We therefore have that the problem is in* EXPSPACE.

Theorem 22 *The problem of deciding whether there is a controllable* PDS *of length* ℓ *for* MPFSM M *is in* EXPSPACE *and* PSPACE-hard. *This holds even if we restrict attention to* MPFSMs *with two ports.*

Theorem 23 *The problem of deciding whether there is a controllable* P-PDS *of length* ℓ *for* MPFSM M *is in* EXPSPACE *and* PSPACE-hard. *This holds even if we restrict attention to* MPFSMs *with two ports.*

Proof 55 *The hardness result follows from Proposition 21. The proof of being in* EXPSPACE *is identical to the proof of Proposition 22 except that the non-deterministic Turing machine guesses an input sequence* $\bar{x}$ *such that the first input of* $\bar{x}$ *is an element of the input alphabet of port* p.

Theorem 24 *The problem of deciding whether there is a controllable* PDS *for state set S' of* MPFSM *M is undecidable and this holds even if we restrict attention to* MPFSMs *with two ports.*

Proof 56 *We will prove this by showing that any algorithm that solves this problem can be used to solve Post's Correspondence Problem. Let us suppose that we have an instance of PCP defined by sequences $\alpha_1, \alpha_2, \ldots, \alpha_b$ and $\beta_1, \beta_2, \ldots, \beta_b$. We will now define an* MPFSM *M such that there is a controllable* PDS *for $S' = \{s_1, s_2, s_3, s_4\}$ if and only if there is a solution to this instance of PCP.*

Let m denote the length of the longest sequence in $\alpha_1, \alpha_2, \ldots, \alpha_b, \beta_1, \beta_2, \ldots, \beta_b$ (ie. $m = max\{|\alpha_1|, |\alpha_2|, \ldots, |\alpha_b|, |\beta_1|, |\beta_2|, \ldots, |\beta_b|\}$). For all $1 \leq j \leq b$ there is an input x_j at port 1. We will structure M such that a sequence of m consecutive inputs of x_j leads to output sequence α_j at port 2 from states s_1 and s_2 and output sequence β_j at port 2 from states s_3 and s_4. The first such sequence takes states s_1, s_2, s_3, s_4 to s'_1, s'_2, s'_3, s'_4 respectively and after that these input sequences lead to cycles. From state s'_j the input of x' at port 2 leads to output j at port 2 and M moves to state s_e from which there are only self-loop transitions labelled with inputs and no output. If an input sequence does not follow this pattern (a sequence of m instances of some x_{a_1} followed by m instances of some x_{a_2} etc.) then the states reached from the s_i are all mapped to state s_e and we cannot then distinguish s_1 from s_2 or s_3 from s_4.

Now consider the conditions under which a controllable PDS *can distinguish the states in S'. This can only be achieved if x' is applied when the states reached from s_1, s_2, s_3, s_4 are s'_1, s'_2, s'_3, s'_4 respectively since this is the only way of distinguishing s_1 and s_2 (and also the only way of distinguishing s_3 and s_4). By definition, the input sequence must be in the form $x_{a_1}^m x_{a_2}^m \ldots x_{a_n}^m$, where x^m denotes x repeated m times, and the output sequence at port 2 must be the following:*

- *From states s_1 and s_2 the sequence $\alpha_{a_1}\alpha_{a_2}\ldots\alpha_{a_n}$*
- *From states s_3 and s_4 the sequence $\beta_{a_1}\beta_{a_2}\ldots\beta_{a_n}$.*

By the definition of a controllable PDS*, we require there to be a common sequence of*

outputs at port 2 before x' is applied and so we require that $\alpha_{a_1}\alpha_{a_2}\ldots\alpha_{a_n} = \beta_{a_1}\beta_{a_2}\ldots\beta_{a_n}$. There is thus an input sequence that has the required properties, and so defines a controllable PDS*, if and only if there is a solution to the given instance of PCP. The result therefore follows from the PCP being undecidable.*

Theorem 25 *The problem of deciding whether there is a controllable* P-PDS *for state set S' of* MPFSM *M is undecidable and this holds even if we restrict attention to* MPFSMs *with two ports.*

Proof 57 *Let us assume that we have an instance of PCP defined by sequences $\alpha_1, \alpha_2, \ldots, \alpha_b$ and $\beta_1, \beta_2, \ldots, \beta_b$. Consider the* MPFSM *$M = (\mathcal{P}.S, X, Y, \delta, \lambda)$ constructed from the PCP instance as presented in the proof of Theorem 24. From M we construct* MPFSM *M' as follows: we introduce four new states $\{s_1^0, s_2^0, s_3^0, s_4^0\}$ to the state set of M. We introduce a new input symbol R to the input alphabet of port 2 and we introduce a new output symbol o_1 to the output alphabet of port 1. For $1 \leq i \leq 4$, we introduce a transition labeled by $R/(o_1, \varepsilon)$ from state s_i^0 to s_i. For all other input, the* MPFSM *produces ε at each port when the* MPFSM *is in state s_i^0. Now we set $S' = \{s_1^0, s_2^0, s_3^0, s_4^0\}$.*
Clearly, in order to distinguish states S' at port 2, the input $R \in X_2$ should be applied. Moreover, the states S' can only be distinguished at port 2. The rest of the proof is identical to the proof of Theorem 24, and thus the result follows.

Theorem 26 *It is* NP-complete *to decide whether an instance of the B-PCP has a solution.*

Theorem 27 *The Bounded* PDS *problem is in* EXPSPACE *and is* NP-hard.

Proof 58 *The problem being* NP-hard *follows from the construction presented in the proof of Theorem 24 and the bounded PCP being* NP-hard *(for bound ℓ of the bounded PCP we use bound $m\ell$ for the bounded* PDS *problem).*
The problem being in EXPSPACE *follows in the same way as the proof of Proposition 22 except that the non-deterministic Turing machine considers a subset S' of the state set of the* MPFSM *M.*

Theorem 28 *The Bounded* P-PDS *problem is in* EXPSPACE *and is* NP-hard.

Proof 59 *The problem being* NP-hard *follows from the construction presented in the proof of Theorem 24 and the bounded PCP being* NP-hard. *The proof of the problem being in* EXPSPACE *is similar to the proof of Theorem 23.*

Theorem 29 *If ℓ is defined by a polynomial in term of the number of states of M then the Bounded* PDS *problem is* NP-complete.

Proof 60 *Now let us assume that there exist a polynomial time solvable algorithm A which solves the bounded* PDS *problem for FSM with n number of states where ℓ is bounded by some polynomial. Let us assume that we are given a bounded PCP problem instance $\alpha_1, \alpha_2, \ldots, \alpha_b$ and $\beta_1, \beta_2, \ldots, \beta_b$ where K is bounded by polynomial function of $m\ell$.*

Then we can use the reduction given in Theorem 24 and use algorithm A on the PCP instance where $\ell = Km$ and obtain the PDS. *But this implies that we solve the PCP problem for K which is not possible since bounded PCP is* NP-complete[4].

We now show that the problem is in NP. *We have seen in the proof of Theorem 27 that the bounded* PDS *problem can be solved by a non-deterministic Turing machine in space and time that is polynomial in terms of the size of M and ℓ. However, since ℓ is a polynomial in terms of the number of states of M, we have that the problem can be solved by a non-deterministic Turing machine in time and space that polynomial in terms of the size of M. Thus, the problem is in* NP *as required.*

Theorem 30 *If ℓ is defined by a polynomial in term of the number of states of M then the Bounded* P-PDS *problem is* NP-complete.

Proof 61 *Now let us assume that there exist a polynomial time solvable algorithm A which solves the bounded* P-PDS *problem for FSM with n number of states where ℓ is bounded by some polynomial. Let us assume that we are given a bounded PCP problem*

[4]Note that For Bounded PCP the exact algorithm requires time $O(2^K)$ and thus there is no limitations on K

instance $\alpha_1, \alpha_2, \ldots, \alpha_b$ *and* $\beta_1, \beta_2, \ldots, \beta_b$ *where* K *is bounded by polynomial function of* $m\ell$.

Then we can use the reduction given in Theorem 25 and use algorithm A *on the PCP instance where* $\ell = Km$ *and obtain the* P-PDS. *But this implies that we solve the PCP problem for* K *which is not possible since bounded PCP is* NP-complete.

We now show that the problem is in NP. *Recall that in Theorem 23 we show that a non-deterministic Turing machine guess an input sequence of length* ℓ. *Since the length of the* P-PDS *is polynomial, a non-deterministic Turing machine can non-determinisitically guess an input sequence* $\bar{x}$ *and check if* $\bar{x}$ *defines a* P-PDS *or not. Thus, the problem is in* NP *as required.*

Proposition 23 *Given distinct states* s *and* s' *of* C-MPFSM M, *if input sequence* $\bar{x}$ *is controllable from state* s *then* $\bar{x}$ *is controllable from* s'.

Proof 62 *Let us suppose that* $\bar{x} = x_1, x_2, \ldots, x_r$, $x_1/y_1, x_2/y_2, \ldots, x_r/y_r$ *labels a path from state* s *and* $x_1/y'_1, x_2/y'_2, \ldots, x_r/y'_r$ *labels a path from state* s'. *By definition,* $\bar{x}$ *is controllable from* s *if and only if for all* $1 < i \leq r$ *we have that if* $port(x_i) = p$ *then* $\pi_p(x_{i-1}/y_{i-1}) \neq \varepsilon$. *But since* M *is a* C-MPFSM *this is the case if and only if for all* $1 < i \leq r$ *we have that if* $port(x_i) = p$ *then* $\pi_p(x_{i-1}/y'_{i-1}) \neq \varepsilon$. *By definition, this is the case if and only if* $\bar{x}$ *is controllable from* s' *and so the result follows.*

Proposition 24 *Given* C-MPFSM M *with state set* S, *state* $s \in S$ *and set* $S' \subseteq S$ *of states of* M, *if input sequence* $\bar{x}$ *is controllable from* s *then* $\bar{x}$ *is controllable from* S'.

Proof 63 *Let us suppose that* $\bar{x} = x_1, x_2, \ldots, x_r$ *and* $x_1/y_1, x_2/y_2, \ldots, x_r/y_r$ *labels a path from state* s. *By Definition 37, we require to prove that for all* $s_i, s_j \in S'$ *and proper prefixes* $\bar{x}_i$ *and* $\bar{x}_j$ *of* $\bar{x}$ *with* $|\bar{x}_i| \leq |\bar{x}_j|$, *if* $port(x_{j+1}) = p$ *then* $(\pi_p(\lambda(s_i, \bar{x}_i)) = \pi_p(\lambda(s_j, \bar{x}_j))) \implies (\bar{x}_i = \bar{x}_j)$. *Since* M *is a* C-MPFSM $\pi_p(\lambda(s_i, \bar{x}_i)) = \pi_p(\lambda(s_j, \bar{x}_j))$ *implies that there are no inputs or outputs at* p *in the subsequence* $x_{i+1}/y_{i+1}, x_{i+2}/y_{i+2}, \ldots, x_j/y_j$ *of* $x_1/y_1, x_2/y_2, \ldots, x_r/y_r$. *Since* $\bar{x}$ *is controllable from* s *and the input* x_{j+1} *is at port* p, *we have that* $x_{i+1}/y_{i+1}, x_{i+2}/y_{i+2}, \ldots, x_j/y_j = \varepsilon$ *and so* $\bar{x}_i = \bar{x}_j$ *as required.*

Proposition 25 *Given C-*MPFSM *M, states s and s' of M, and input sequence $\bar{x}$, if $\lambda(s,\bar{x}) \neq \lambda(s',\bar{x})$ then there is some port p such that $\pi_p(\lambda(s,\bar{x})) \neq \pi_p(\lambda(s',\bar{x}))$.*

Proof 64 *Let $\bar{x}'$ be the shortest prefix of $\bar{x}$ such that $\lambda(s,\bar{x}') \neq \lambda(s',\bar{x}')$. Then $\bar{x}' = \bar{x}''x$ for some $x \in X$ and by the minimality of $\bar{x}'$ we have that $\lambda(s,\bar{x}'') = \lambda(s',\bar{x}'')$. Since $\lambda(s,\bar{x}') \neq \lambda(s',\bar{x}')$, $\lambda(\delta(s,\bar{x}''),x) \neq \lambda(\delta(s',\bar{x}''),x)$. Thus, there must be some port p such that $\pi_p(\lambda(s,\bar{x}')) \neq \pi_p(\lambda(s',\bar{x}'))$. But if $\bar{x} = \bar{x}'\bar{x}'''$ then since M is a C-*MPFSM *we know that $\pi_p(\lambda_p(\delta(s,\bar{x}'),\bar{x}'''))$ and $\pi_p(\lambda_p(\delta(s,\bar{x}'),\bar{x}'''))$ contain the same number of outputs and so $\pi_p(\lambda(s,\bar{x})) \neq \pi_p(\lambda(s',\bar{x}))$ as required.*

Proposition 26 *Given C-*MPFSM *M with state set S, if $\bar{x}$ is a controllable input sequence and for all $s,s' \in S'$, $S' \subseteq S$, we have that $s \neq s' \Rightarrow \lambda(s,\bar{x}) \neq \lambda(s',\bar{x})$ then $\bar{x}$ is a controllable* PDS *for S'.*

Proof 65 *If S' contains at most one state then the result holds immediately. We therefore assume that S' contains two or more states and it is sufficient to consider two distinct states s,s' from S'. However, we know that $\lambda(s,\bar{x}) \neq \lambda(s',\bar{x})$ and so the result follows from Proposition 25.*

Proposition 27 *For each controllable path $\bar{\rho}$ in M that starts at s_0, there is a unique controllable path $\bar{\rho}'$ in $\chi_{min}(M)$ that starts at $s_0^{\mathcal{P}}$ such that $label(\bar{\rho}) = label(\bar{\rho}')$.*

Proposition 28 *For each path $\bar{\rho}'$ in $\chi_{min}(M)$ that starts at $s_0^{\mathcal{P}}$, there is a unique controllable path $\bar{\rho}$ in M that starts at s_0 such that $label(\bar{\rho}) = label(\bar{\rho}')$.*

Proposition 29 *An input sequence $\bar{x}$ is a controllable* PDS *that starts with input at p for set $S' = \{s_1, s_2 \ldots, s_r\} \subseteq S$ of states of M if and only if $\bar{x}$ is a non-redundant* PDS *for set $S'' = \{s_1^{\{p\}}, s_2^{\{p\}}, \ldots, s_r^{\{p\}}\}$ of states of $\chi_{min}^p(M)$ such that for all $1 < i \leq |\bar{x}|$, if $x_i \in X_q$ then for all $s_i^{\{p\}}, s_j^{\{p\}} \in S''$ we have that $\pi_q(\lambda_{min}^p(s_i^{\{p\}}, \bar{x}_{i-1})) = \pi_q(\lambda_{min}^p(s_j^{\{p\}}, \bar{x}_{i-1}))$.*

Proof 66 *First let us suppose that $\bar{x}$ is a controllable* PDS *for $S' = \{s_1, s_2, \ldots, s_r\}$ that starts with input at p. Since $\bar{x}$ is controllable, the label of the path in M from s_i that has input portion $\bar{x}$ is the same as the label of the path in $\chi_{min}^p(M)$ from $s_i^{\{p\}}$ that has input*

portion $\bar{x}$. Thus, since $\bar{x}$ distinguishes the states in S' it also distinguishes the states in S''. Further, since $\bar{x}$ is a controllable PDS *for S', by definition for all $1 < i \leq |\bar{x}|$, if $x_i \in X_q$ then for all $s_i^{\{p\}}, s_j^{\{p\}} \in S''$ we have that $\pi_q(\lambda^p_{min}(s_i^{\{p\}}, \bar{x}_{i-1})) = \pi_q(\lambda^p_{min}(s_j^{\{p\}}, \bar{x}_{i-1}))$. Finally, since $\bar{x}$ is controllable in M, it is non-redundant in $\chi^p_{min}(M)$ and so the result holds.*

Now let us suppose that $\bar{x}$ is a non-redundant PDS *for set $S'' = \{s_1^{\{p\}}, s_2^{\{p\}}, \ldots, s_r^{\{p\}}\}$ of states of $\chi^p_{min}(M)$ such that for all $1 < i \leq |\bar{x}|$, if $x_i \in X_q$ then for all $s_i^{\{p\}}, s_j^{\{p\}} \in S''$ we have that $\pi_q(\lambda^p_{min}(s_i^{\{p\}}, \bar{x}_{i-1})) = \pi_q(\lambda^p_{min}(s_j^{\{p\}}, \bar{x}_{i-1}))$. Similar to before, the label of the path from $s_i^{\{p\}}$ in $\chi^p_{min}(M)$ that has input portion $\bar{x}$ is the same as the label of the path from s_i in M that has input portion $\bar{x}$. Thus, $\bar{x}$ distinguishes the states in S' and so, by Proposition 25, $\bar{x}$ distinguishes the states from S' in distributed testing.*

Lemma 32 *Given a C-*MPFSM *M with n states, k ports, and m inputs, M has a controllable* PDS *if and only if it has one of length at most $n(n_{min})^n$ where $n_{min} = nk + nm + 1$.*

Proof 67 *Consider the* MPFSM *$\chi^p_{min}(M)$ of M with set S_{min} of states. Let $S'' = \{s_1^{\{p\}}, s_2^{\{p\}}, \ldots, s_n^{\{p\}}\}$ and let $\bar{x}$ be a shortest* PDS *for S'' of $\chi^p_{min}(M)$.*

Now let us assume that $\mathcal{B} = S''$ and $\bar{x} = \bar{x}_a\bar{x}'x_b\bar{x}_b$ where $\bar{x}' = x_0, x_1, \ldots x_{|\bar{x}'|}$ is a fragment of PDS *$\bar{x}$ such that $|\delta^p_{min}(\mathcal{B}, \bar{x}_a)| < |\delta^p_{min}(\mathcal{B}, \bar{x}_a x_0)| = |\delta^p_{min}(\mathcal{B}, \bar{x}_a\bar{x}')| < |\delta^p_{min}(\mathcal{B}, \bar{x}_a\bar{x}'x_b)|$. We will prove that for any distinct proper prefixes $\bar{x}''$, $\bar{x}'''$ of $\bar{x}'$ we have that $\delta^p_{min}(\mathcal{B}, \bar{x}_a\bar{x}'') \neq \delta^p_{min}(\mathcal{B}, \bar{x}_a\bar{x}''')$. We will use proof by contradiction and assume that there exist proper prefixes $\bar{x}''$, $\bar{x}'''$ of $\bar{x}'$ such that $\delta^p_{min}(\mathcal{B}, \bar{x}_a\bar{x}'') = \delta^p_{min}(\mathcal{B}, \bar{x}_a\bar{x}''')$ and $|\bar{x}''| < |\bar{x}'''|$.*

Now let us suppose that $\bar{x}''' = \bar{x}''\bar{x}_0$ and $\bar{x}' = \bar{x}'''\bar{x}_1 = \bar{x}''\bar{x}_0\bar{x}_1$. Since $\bar{x}$ is a PDS *for S'' we must have that $\bar{x}_1 x_b \bar{x}_b$ distinguishes any two state that are in the same set in $\delta^p_{min}(\mathcal{B}, \bar{x}_a\bar{x}''\bar{x}_0)$. From Propositions 24 and 25 we know that no path of $\chi^p_{min}(M)$ causes controllability and observability problems in M. Hence, since $\delta^p_{min}(\mathcal{B}, \bar{x}_a\bar{x}'') = \delta^p_{min}(\mathcal{B}, \bar{x}_a\bar{x}''\bar{x}_0)$, we can replace $\bar{x}'$ (which equals $\bar{x}''\bar{x}_0\bar{x}_1$) by $\bar{x}''\bar{x}_1$ in $\bar{x}$. But this leads to a shorter controllable* PDS*, which contradicts the minimality of $\bar{x}$.*

We can now note that the number of possible values for a set $\mathcal{B}^{\bar{x}_a}$ is bounded above by the number of possible mappings from the states S'' to the states reached from S'' by $\bar{x}_a$

and so by $(n_{min})^n$. *Further,* $\mathcal{B}^{\bar{x}}$ *initially contains only one set and finally contains* n *sets and so there are at most* $n-1$ *points at which the size of* $\mathcal{B}^{\bar{x}}$ *increases. We can therefore conclude that a minimal* PDS *can be seen as being a sequence of at most* $n-1$ *subsequences each of length at most* $(n_{min})^n$. *Moreover, Propositions 29 implies that* $\bar{x}$ *is a* PDS *for state set* S'' *of* $\chi^p_{min}(M)$ *if and only if* $\bar{x}$ *is a* PDS *for* MPFSM M *and so the result follows.*

Theorem 31 *It is decidable whether a* C-MPFSM *has a* PDS.

Proposition 30 *The problem of deciding whether a* C-MPFSM M *has a* PDS *is in* PSPACE.

Proof 68 *We will show that a non-deterministic Turing Machine* $\mathcal{T}$ *can decide whether there is a* PDS *using polynomial space.* $\mathcal{T}$ *will guess inputs one at a time. It will maintain the set* $\mathcal{C}$ *of pairs of states, equivalence relation* r*, the set of allowable ports* P_c *as described above and this uses polynomial space. The machine first inspects set* P_c *and guesses an input symbol* x *from a port in* P_c.

After guessing a new input x *and updating* $\mathcal{C}$*,* r*, and* P_c *the machine checks whether the input sequence received defines a* PDS *for* S*: this is the case if and only if* r *relates no two different states of* S. *Thus, if* M *has a* PDS *for* S *then* $\mathcal{T}$ *will find such a* PDS *using polynomial space.*

Consider the case where M *does not have a* PDS *for* S*: we require that the non-deterministic Turing Machine terminates. In order to ensure this we use the result that if* C-MPFSM M *has* n *states then* M *has a* PDS *for* S *if and only if it has such a* PDS *with length at most* $n(n_{min})^n$. $\mathcal{T}$ *therefore includes a counter that counts how many inputs have been received: the machine terminates with failure if the counter exceeds the upper bound. Therefore we need additional* $O(\log_2(n(n_{min})^n)) = O(n\log_2(n_{min}))$ *space for the counter and so the space required is still polynomial.*

We have defined a non-deterministic Turing Machine that requires only polynomial space in order to solve the problem and so the problem is in non-deterministic PSPACE. *We*

can now use Savitch's Theorem [123], which tells us that a problem is in PSPACE *if and only if it is in non-deterministic* PSPACE*, and the result follows.*

Theorem 32 *The problem of deciding whether a* C-MPFSM *has a* PDS *is* PSPACE-complete.

Proof 69 *First observe that the reduction presented in Proposition 20 generates a* C-MPFSM*. Thus we know that deciding whether an* MPFSM *has a controllable* PDS *is* PSPACE-hard*. Consequently Propositions 20 and 30 together imply that the problem is* PSPACE-complete.

Proposition 31 *The problem of deciding whether a* C-MPFSM *has a* P-PDS *is* PSPACE-hard.

Proposition 32 *The problem of deciding whether a* C-MPFSM M *has a* p-PDS *is in* PSPACE.

Theorem 33 *The problem of deciding whether a* C-MPFSM *has a* P-PDS *is* PSPACE-complete.

Theorem 34 *The following problem is* PSPACE-hard*: given a multi-port* MPFSM M*, is there a controllable ADS that distinguishes all of the states of* M*? In addition, this result still holds if we restrict attention to* MPFSMs *that have two ports.*

Proof 70 *Assume that we have been given a single-port* MPFSM $M_1 = (S, s_0, X, Y, \delta, \lambda)$ *such that all of the transitions of* M_1 *have non-empty output. We will construct a multi-port* MPFSM M *that has two ports* 1 *and* 2*. The state set of* M *will be* S *and the initial state will be* s_0*. Port* 1 *will have input alphabet* $X_1 = X$ *and output alphabet* $Y_1 = \emptyset$*. Port* 2 *will have input alphabet* $X_2 = \emptyset$ *and output alphabet* $Y_2 = Y$*. Given state* s *and input* x *such that* $\delta(s, x) = s'$ *and* $\lambda(s, x) = y$*, we will include in* M *the transition from* s *to* s' *that has input* $x \in X_1$ *and produces output* $\langle \varepsilon, y \rangle$*.*
Now consider controllable adaptive test cases for M*. Since all inputs are at port* 1 *and no outputs are produced at port* 1*, there is no opportunity for a controllable adaptive test case to lead to different input sequences from different states: the tester choosing the next input will have observed no output irrespective of the state that the adaptive test case was applied in. Thus, all controllable adaptive test cases for* M *correspond to fixed*

input sequences. In addition, since every transition produces non-empty output at port 2 *and no output at port* 1*, if the tester applies an input sequence* $x_1, x_2, \ldots, x_m$ *at port* 1 *and the tester at port* 2 *observes output sequence* $y_1, y_2, \ldots, y_m$ *then we know that for all* $1 \leq i \leq m$*,* y_i *was produced in response to input* x_i*. Thus, there are no observability problems.*

To summarise, a controllable adaptive test case for M *corresponds to a fixed input sequence* $\bar{x}$ *and the output sequence observed at port* 2 *when* $\bar{x}$ *is applied from state* $s \in S$ *is exactly* $\lambda(s, \bar{x})$*. Thus, an adaptive test case for* M *is a controllable ADS for* M *if and only if it corresponds to an input sequence* $\bar{x}$ *such that for all* $s, s' \in S$ *with* $s \neq s'$ *we have that* $\lambda(s, \bar{x}) \neq \lambda(s', \bar{x})$*. This is the case if and only if* $\bar{x}$ *is a distinguishing sequence for* M_1*. The result now follows from Lemma 32.*

Theorem 35 *The following problem is* PSPACE-hard*: given a multi-port* MPFSM M *and port* p *of* M*, is there a controllable* p*-ADS that distinguishes all of the states of* M*? In addition, this result still holds if we restrict attention to* MPFSMs *that only have two ports.*

Proof 71 *Given a single-port* MPFSM M_1 *that has no transitions with output* ε*, we construct a multi-port* MPFSM *in the same way as in the proof of Theorem 34 except that we set* $X_2 = \{x_2\}$ *and* $Y_1 = \{o_1\}$*, we add a new output* o_2 *to* Y_2*, and for every state* s *we have that* $\delta(s, x_2) = s$ *and* $\lambda(s, x_2) = \langle o_1, o_2 \rangle$*. Thus, the input of* x_2 *does not change state and does not help distinguish states.*

Let us suppose that we have the situation in which there is a trace σ *and two different sequences* $\sigma_1, \sigma_2 \in \{\langle o_1, o_2 \rangle\}^*$ *such that the tester at port* 1 *sends different inputs after* $\sigma\sigma_1$ *and* $\sigma\sigma_2$ *and* $|\sigma_1| < |\sigma_2|$*. After observing* $\pi_1(\sigma)$ *followed by* $|\sigma_1|$ *occurrences of* o_1 *the tester at port* 1 *does not know whether to apply the input that should follow* $\sigma\sigma_1$ *or wait for further instances of* o_1*. Thus, any such situation causes a controllability problem and so, since we are considering controllable* P-ADS*s, such situations cannot occur.*

We therefore know that the tester at port 2 *cannot provide the tester at port* 1 *with additional information, through applying* x_2*, that allows the* ADS *to adapt the input*

supplied at port 1 *based to output produced at port* 2. *Thus, an input sequence* $\bar{x}$ *is a* 2-*ADS for* M *if and only if* $\bar{x}$ *with all instances of* x_2 *removed is a distinguishing sequence for* M_1. *The result therefore follows from Lemma 32.*

Theorem 36 *The following problems are* PSPACE-hard*:*

1. *Given a* MPFSM M*, find a controllable* ADS μ *and state set* S' *where* μ *is a controllable* ADS *for* S' *and* μ *and* S' *are such that* S' *has maximal size.*

2. *Given a* MPFSM M *and port* p *of* M*, find a controllable* P-ADS μ *and state set* S' *where* μ *is a controllable* P-ADS *for* S' *and* μ *and* S' *are such that* S' *has maximal size.*

Proof 72 *If we have an algorithm that solves the first part and are given* MPFSM M*, then* M *has a controllable* ADS *if and only if the algorithm returns such an* ADS. *The first part thus follows from Theorem 34. Similarly, the second part follows from Theorem 35.*

Theorem 37 *There is a class of* MPFSMs *that contain* ADS *(or* P-ADS*) such that the shortest evolution is of exponential length.*

Proof 73 *Consider a single-port* MPFSM *that has a* PDS *with exponential length [35], now reapply the reduction given in Theorem 34.*

Theorem 38 *The following problems are* PSPACE-hard.

1. *Given a* MPFSM M*, what is the smallest value of* ℓ *such that* M *has an* ADS *of height* ℓ*?*

2. *Given a* MPFSM M*, what is the smallest value of* ℓ *such that* M *has a* P-ADS *of height* ℓ*?*

Proof 74 *An* MPFSM *has an* ADS/P-ADS *if and only if it has a minimum height* ADS/P-ADS. *Thus, any algorithm that returns the smallest* ℓ *such that* M *has an* ADS/P-ADS *of height* ℓ *also decides whether* M *has an* ADS/P-ADS. *The result thus follows from the existence problems being* PSPACE-hard.

Proposition 33 *Given directed graph G and* MPFSM *$M(G)$ with state set S, if μ is a non-redundant controllable global strategy for $M(G)$ then all traces in $Ev(\mu, M(G), S \setminus \{s_e\})$ have the same input portion $x_{i_1}, \ldots, x_{i_l}$ and this has the property that $e_{i_1}, \ldots, e_{i_l}$ is a walk of G.*

Proof 75 *First observe that all transitions of $M(G)$ with input x_i, $1 \leq i \leq m$, produce the same output at all ports of $M(G)$ except 0. In addition, $M(G)$ has no inputs at port 0. We will prove that the input portions are the same for all traces in $Ev(\mu, M(G), S \setminus \{s_e\})$ and will use proof by contradiction: assume that the input portions of $Ev(\mu, M(G), s)$ and $Ev(\mu, M(G), s')$ are different for some states $s, s' \in S \setminus \{s_e\}$. Let $\bar{x}$ denote the longest common prefix of the input portions of $Ev(\mu, M(G), s)$ and $Ev(\mu, M(G), s')$. Without loss of generality, assume that $Ev(\mu, M(G), s)$ has an input portion that follows $\bar{x}$ with input x_p at port p. However, since $M(G)$ has no input at port 0 we have that $p \neq 0$ and so the responses to $\bar{x}$ in states s and s' have the same outputs at p. Thus, since μ is controllable, $Ev(\mu, M, s')$ must have an input portion that follows $\bar{x}$ with input x_p. However, this contradicts the definition of $\bar{x}$ as required. Thus, all traces in $Ev(\mu, M(G), S \setminus \{s_e\})$ have the same input portion $x_{i_1}, \ldots, x_{i_l}$. Further, by the definition of $M(G)$, in a non-redundant controllable global strategy an input x_i can only be followed by input x_j if in G we have that e_i can be followed by e_j. The result therefore follows.*

Proposition 34 *Strongly connected directed graph G has a Hamiltonian path if and only if $M(G)$ has a controllable* ADS *that distinguishes all of the state of $M(G)$ and whose longest evolution has length $\ell = n$.*

Proof 76 *First we prove that if G has a Hamiltonian path $\rho = e_1, \ldots, e_{n-1}$ then $M(G)$ has an* ADS *whose longest evolution has length n. Choose an edge e_n of G that can follow e_{n-1} in G: since G is strongly connected there must be some such edge. By the definition of $M(G)$, the input sequence $x_1, \ldots, x_n$ defines a controllable global strategy for $M(G)$. In addition, since ρ is a Hamiltonian path, for every state s_i of $M(G)$, $s_i \neq s_e$, the application of input sequence $x_1, \ldots, x_n$ from s_i includes an input that corresponds to an edge with starting vertex v_i and so leads to an output sequence at port 0 that starts*

with i. Finally, the application of $x_1, \ldots, x_n$ in state s_e leads to no output being produced at 0. Thus, $x_1, \ldots, x_n$ defines an ADS *and its longest evolution has length n as required. Now we assume that $M(G)$ has a controllable* ADS *whose longest evolution has length $\ell = n$ and we are required to prove that G has a Hamiltonian path. By Proposition 33 we know that there is some input sequence $x_1, \ldots, x_n$ such that all traces of $Ev(\mu, M(G), S \setminus \{s_e\})$ have input portion $x_1, \ldots, x_n$. Further, since μ is an* ADS *for $M(G)$ we must have that for every state $s_i \neq s_e$, $x_1, \ldots, x_n$ contains an input x_j such that v_i is the starting vertex of e_j. In addition, since μ is controllable we must have that $e_1, \ldots, e_n$ is a walk of G. To conclude, all vertices of G start edges in walk $e_1, \ldots, e_n$ of G and so $e_1, \ldots, e_{n-1}$ is a Hamiltonian path of G.*

Theorem 39 *The* Exact Height ADS *problem is in* EXPSPACE *and is* NP-hard.

Proof 77 *We will first show that a non-deterministic Turning machine T can decide the* Exact Height ADS *problem using exponential space. We can allow T to initially guess an* ADS *μ with height at most ℓ. Since this defines a finite tree with at most n leaves there is an upper bound on the size of the tree that is polynomial in terms of n and ℓ and so this take space that is polynomial in ℓ and n.*

In order to check whether μ is controllable it is sufficient to compute the traces that can be produced by applying μ from states of M and for any two traces σ and σ' check whether there are corresponding controllability problems. There are corresponding controllability problems if there are prefixed σ_1 and σ'_1 of σ and σ' respectively that have the same projection at a port p such that after σ and σ' the behaviour of the tester at p differs. Thus, the Turing machine can check this in polynomial time. Finally, the Turing machine can check in polynomial time whether μ distinguishes the states of M. The Turing machine takes space that is polynomial in n and ℓ and so exponential in the description of the problem (since ℓ can be described in $O(\log_2 \ell)$ space). Thus, we have that a non-deterministic Turing machine can solve the problem in exponential space. Finally, using the Savitch's Theorem [123] we know that a deterministic Turing machine can also solve the problem in exponential space. We therefore have that the problem is in EXPSPACE.

The problem being NP-hard *follows from Proposition 34 and the fact that the* Directed Hamiltonian Path *problem with strongly connected directed graphs is* NP-hard.

Theorem 40 *The* Exact Height p-ADS *problem is in* EXPSPACE *and is* NP-hard.

Theorem 41 *The* Exact Height ADS *and* Exact Height p-ADS *problems for an* MPFSM *with n states are* NP-complete *if $\ell = poly(n)$, where $poly(n)$ is a polynomial function of n.*

Bibliography

[1] G.H. Mealy. A method for synthesizing sequential circuits. *Bell System Technical Journal*, 34(5):1045–1079, 1955.

[2] E. P. Moore. Gedanken-experiments. In C. Shannon and J. McCarthy, editors, *Automata Studies*. Princeton University Press, 1956.

[3] A.D. Friedman and P.R. Menon. *Fault detection in digital circuits*. Computer Applications in Electrical Engineering Series. Prentice-Hall, 1971.

[4] T. S. Chow. Testing software design modelled by finite state machines. *IEEE Transactions on Software Engineering*, 4:178–187, 1978.

[5] Gerard J. Holzmann. *Design and validation of computer protocols*. Prentice-Hall software series. Prentice Hall, 1991.

[6] E. Brinksma. A theory for the derivation of tests. In *Proceedings of Protocol Specification, Testing, and Verification VIII*, pages 63–74, Atlantic City, 1988. North-Holland.

[7] W. Y. L. Chan, C. T. Vuong, and M. R. Otp. An improved protocol test generation procedure based on uios. *SIGCOMM Comput. Commun. Rev.*, 19(4):283–294, August 1989.

[8] A.T. Dahbura, K.K. Sabnani, and M.U. Uyar. Formal methods for generating protocol conformance test sequences. *Proceedings of the IEEE*, 78(8):1317–1326, Aug.

[9] D. Lee, K.K. Sabnani, D.M. Kristol, and S. Paul. Conformance testing of protocols specified as communicating finite state machines-a guided random walk based approach. *Communications, IEEE Transactions on*, 44(5):631–640, May.

[10] D. Lee and M. Yannakakis. Principles and methods of testing finite-state machines - a survey. *Proceedings of the IEEE*, 84(8):1089–1123, 1996.

[11] S.H. Low. Probabilistic conformance testing of protocols with unobservable transitions. In *Network Protocols, 1993. Proceedings., 1993 International Conference on*, pages 368–375, Oct.

[12] Milena Mihail and Christos H. Papadimitriou. On the random walk method for protocol testing. In *In Proc. Computer-Aided Verification (CAV 1994), volume 818 of LNCS*, pages 132–141. Springer, LNCS, 1994.

[13] K. Sabnani and A. Dahbura. A protocol test generation procedure. *Computer Networks*, 15(4):285–297, 1988.

[14] D. P. Sidhu and T.-K. Leung. Formal methods for protocol testing: A detailed study. *IEEE Transactions on Software Engineering*, 15(4):413–426, 1989.

[15] R. V. Binder. *Testing Object-Oriented Systems: Models, Patterns, and Tools*. Addison-Wesley, 1999.

[16] M. Haydar, A. Petrenko, and H. Sahraoui. Formal verification of web applications modeled by communicating automata. In *Formal Techniques for Networked and Distributed Systems (FORTE 2004)*, volume 3235 of *Springer Lecture Notes in Computer Science*, pages 115–132, Madrid, September 2004. Springer-Verlag.

[17] A. V. Aho, A. T. Dahbura, D. Lee, and M. U. Uyar. An optimization technique for protocol conformance test generation based on UIO sequences and Rural Chinese Postman Tours. In *Protocol Specification, Testing, and Verification VIII*, pages 75–86, Atlantic City, 1988. Elsevier (North-Holland).

[18] Aysu Betin-Can and Tevfik Bultan. Verifiable concurrent programming using concurrency controllers. In *Proceedings of the 19th IEEE international conference on Automated software engineering*, pages 248–257. IEEE Computer Society, 2004.

[19] I. Pomeranz and S. M. Reddy. Test generation for multiple state-table faults in finite-state machines. *IEEE Transactions on Computers*, 46(7):783–794, 1997.

[20] Mark Utting, Alexander Pretschner, and Bruno Legeard. A taxonomy of model-based testing approaches. *Software Testing, Verification and Reliability*, 22(5):297–312, 2012.

[21] A. Gill. *Introduction to The Theory of Finite State Machines*. McGraw-Hill, New York, 1962.

[22] Z. Kohavi. *Switching and Finite State Automata Theory*. McGraw-Hill, New York, 1978.

[23] F. C. Hennie. Fault-detecting experiments for sequential circuits. In *Proceedings of Fifth Annual Symposium on Switching Circuit Theory and Logical Design*, pages 95–110, Princeton, New Jersey, November 1964.

[24] F.C. Hennie. *Finite-state models for logical machines*. Wiley, 1968.

[25] C.R. Kime. An organization for checking experiments on sequential circuits. *IEEE Trans. Electron. Comput. (Short Notes)*, EC-15:p.113, 1966.

[26] G. Gonenc. A method for the design of fault detection experiments. *IEEE Transactions on Computers*, 19:551–558, 1970.

[27] K. Inan and H. Ural. Efficient checking sequences for testing finite state machines. *Information and Software Technology*, 41(11–12):799–812, 1999.

[28] H. Ural and K. Zhu. Optimal length test sequence generation using distinguishing sequences. *IEEE/ACM Transactions on Networking*, 1(3):358–371, 1993.

[29] Rob M. Hierons and Hasan Ural. Optimizing the length of checking sequences. *IEEE Trans. Comput.*, 55:618–629, May 2006.

[30] H. Ural, X. Wu, and F. Zhang. On minimizing the lengths of checking sequences. *IEEE Transactions on Computers*, 46(1):93–99, 1997.

[31] R. M. Hierons and H. Ural. Reduced length checking sequences. *IEEE Transactions on Computers*, 51(9):1111–1117, 2002.

[32] Jessica Chen, Robert Hierons, Hasan Ural, and Husnu Yenigun. Eliminating redundant tests in a checking sequence. In Ferhat Khendek and Rachida Dssouli, editors, *Testing of Communicating Systems*, volume 3502 of *Lecture Notes in Computer Science*, pages 371–371. Springer Berlin / Heidelberg, 2005. 10.1007/11430230_11.

[33] Adenilso da Silva Simão and Alexandre Petrenko. Generating checking sequences for partial reduced finite state machines. In *TestCom/FATES*, pages 153–168, 2008.

[34] David Eppstein. Reset sequences for monotonic automata. 1990.

[35] D. Lee and M. Yannakakis. Testing finite-state machines: State identification and verification. *IEEE Transactions on Computers*, 43(3):306–320, 1994.

[36] T. Y. Chen and P. L. Poon. On the effectiveness of classification trees for test case construction. *Information and Software Technology*, 40(13):765–775, 1998.

[37] R. M. Hierons and H. Ural. Synchronized checking sequences based on UIO sequences. *Information and Software Technology*, 45(12):793–803, 2003.

[38] R. M. Hierons. Minimizing the number of resets when testing from a finite state machine. *Information Processing Letters*, 90(6):287–292, 2004.

[39] B. Yang and H. Ural. Protocol conformance test generation using multiple UIO sequences with overlapping. In *ACM SIGCOMM 90: Communications, Architec-*

tures, and Protocols, pages 118–125, Twente, The Netherlands, September 24-27 1990.

[40] Y. N. Shen, F. Lombardi, and A. T. Dahbura. Protocol conformance testing using multiple UIO sequences. In *Proceedings of Protocol Specification, Testing, and Verification IX*, pages 131–143, Twente, Netherlands, 1990. North-Holland.

[41] A.V. Aho, A.T. Dahbura, D. Lee, and M.U. Uyar. An optimization technique for protocol conformance test generation based on uio sequences and rural chinese postman tours. *Communications, IEEE Transactions on*, 39(11):1604 –1615, nov 1991.

[42] R. T. Boute. Distinguishing sets for optimal state identification in checking experiments. *IEEE Trans. Comput.*, 23:874–877, 1974.

[43] Robert M. Hierons, Guy-Vincent Jourdan, Hasan Ural, and Husnu Yenigun. Checking sequence construction using adaptive and preset distinguishing sequences. In *Proceedings of the 2009 Seventh IEEE International Conference on Software Engineering and Formal Methods*, SEFM '09, pages 157–166, Washington, DC, USA, 2009. IEEE Computer Society.

[44] Guy-Vincent Jourdan, Hasan Ural, Husnu Yenigun, and Ji Zhang. Lower bounds on lengths of checking sequences. *Formal Aspects of Computing*, 22(6):667–679, 2010.

[45] Manfred Broy, Bengt Jonsson, and Joost-Pieter Katoen. *Model-Based Testing of Reactive Systems: Advanced Lectures (Lecture Notes in Computer Science)*. Springer, 2005.

[46] Rajeev Alur, Costas Courcoubetis, and Mihalis Yannakakis. Distinguishing tests for nondeterministic and probabilistic machines. In *Proceedings of the Twenty-seventh Annual ACM Symposium on Theory of Computing*, STOC '95, pages 363–372, New York, NY, USA, 1995. ACM.

[47] Natalia Kushik, Khaled El-Fakih, and Nina Yevtushenko. Adaptive homing and distinguishing experiments for nondeterministic finite state machines. 8254:33–48, 2013.

[48] M. P. Vasilevskii. Failure diagnosis of automata. *Cybernetics*, 4:653–665, 1973.

[49] G. Luo, A. Petrenko, and G. v. Bochmann. Selecting test sequences for partially-specified nondeterministic finite state machines. In *The 7th IFIP Workshop on Protocol Test Systems*, pages 95–110, Tokyo, Japan, November 8–10 1994. Chapman and Hall.

[50] Gang Luo, Alexandre Petrenko, and Gregor Von Bochmann. Selecting test sequences for partially-specified nondeterministic finite state machines. In Tadanori Mizuno, Teruo Higashino, and Norio Shiratori, editors, *Protocol Test Systems*, IFIP The International Federation for Information Processing, pages 95–110. Springer US, 1995.

[51] Rita Dorofeeva, Khaled El-Fakih, and Nina Yevtushenko. An improved conformance testing method. In *Proceedings of the 25th IFIP WG 6.1 International Conference on Formal Techniques for Networked and Distributed Systems*, FORTE'05, pages 204–218, Berlin, Heidelberg, 2005. Springer-Verlag.

[52] Adenilso da Silva Simão, Alexandre Petrenko, and Nina Yevtushenko. On reducing test length for fsms with extra states. *Software Testing, Verification and Reliability*, 22(6):435–454, 2012.

[53] Andre Takeshi Endo and Adenilso Simao. Evaluating test suite characteristics, cost, and effectiveness of FSM-based testing methods. *Information and Software Technology*, 55(6):1045 – 1062, 2013.

[54] Mustafa Emre Dinçtürk. A two phase approach for checking sequence generation. Master's thesis, Sabanci University, 2009.

[55] ISO/IEC. *Information technology - Opens Systems Interconnection, 9646 Parts 1-7*. 1995.

[56] Eduardo Cunha de Almeida, Joo Eugenio Marynowski, Gerson Suny, Yves Le Traon, and Patrick Valduriez. Efficient distributed test architectures for large-scale systems. In Alexandre Petrenko, Adenilso da Silva Simo, and Jos Carlos Maldonado, editors, *Testing Software and Systems - 22nd IFIP WG 6.1 International Conference, ICTSS 2010, Natal, Brazil, November 8-10, 2010. Proceedings*, volume 6435 of *Lecture Notes in Computer Science*, pages 174–187. Springer, 2010.

[57] L. Cacciari and O. Rafiq. Controllability and observability in distributed testing. *Information and Software Technology*, 41(11–12):767–780, 1999.

[58] O. Rafiq and L. Cacciari. Coordination algorithm for distributed testing. *The Journal of Supercomputing*, 24(2):203–211, 2003.

[59] S. Boyd and H. Ural. The synchronization problem in protocol testing and its complexity. *Information Processing Letters*, 40(3):131–136, 1991.

[60] J. Chen, R. M. Hierons, and H. Ural. Conditions for resolving observability problems in distributed testing. In *24rd IFIP International Conference on Formal Techniques for Networked and Distributed Systems (FORTE 2004)*, volume 3235 of *Lecture Notes in Computer Science*, pages 229–242. Springer-Verlag, 2004.

[61] W.-H. Chen and H. Ural. Synchronizable test sequences based on multiple UIO sequence. *IEEE/ACM Transactions on Networking*, 3(2):152–157, 1995.

[62] R. Dssouli and G. von Bochmann. Error detection with multiple observers. In *Protocol Specification, Testing and Verification V*, pages 483–494. Elsevier Science (North Holland), 1985.

[63] R. Dssouli and G. von Bochmann. Conformance testing with multiple observers. In *Protocol Specification, Testing and Verification VI*, pages 217–229. Elsevier Science (North Holland), 1986.

[64] S. Guyot and H. Ural. Synchronizable checking sequences based on UIO sequences. In *Protocol Test Systems, VIII*, pages 385–397, Evry, France, September 1995. Chapman and Hall.

[65] R. M. Hierons and H. Ural. The effect of the distributed test architecture on the power of testing. *The Computer Journal*, 52(4):497–510, 2007.

[66] R. M. Hierons. Reaching and distinguishing states of distributed systems. *SIAM Journal on Computing*.

[67] G. V. Jourdan, H. Ural, and H. Yenigun. Minimizing coordination channels in distributed testing. In *26th IFIP International Conference on Formal Techniques for Networked and Distributed Systems (FORTE 2006)*, volume 4229 of *Lecture Notes in Computer Science*, pages 451–466. Springer-Verlag, 2006.

[68] B. Sarikaya and G. v. Bochmann. Synchronization and specification issues in protocol testing. *IEEE Transactions on Communications*, 32:389–395, April 1984.

[69] Robert Hierons, Mercedes Merayo, and Manuel Nunez. Controllable test cases for the distributed test architecture. In *Automated Technology for Verification and Analysis*, volume 5311 of *Lecture Notes in Computer Science*, pages 201–215. Springer Berlin / Heidelberg, 2008.

[70] R.M. Hierons. Controllable testing from nondeterministic finite state machines with multiple ports. *Computers, IEEE Transactions on*, 60(12):1818 –1822, dec. 2011.

[71] G. Luo, R. Dssouli, and G. v. Bochmann. Generating synchronizable test sequences based on finite state machine with distributed ports. In *The 6th IFIP Workshop on Protocol Test Systems*, pages 139–153. Elsevier (North-Holland), 1993.

[72] K.-C. Tai and Y.-C. Young. Synchronizable test sequences of finite state machines. *Computer Networks and ISDN Systems*, 30(12):1111–1134, 1998.

[73] A. Khoumsi. A temporal approach for testing distributed systems. *Software Engineering, IEEE Transactions on*, 28(11):1085 – 1103, nov 2002.

[74] C. Wang and M. Schwartz. Fault detection with multiple observers. *IEEE/ACM Transactions on Networking*, 1:48–55, 1993.

[75] Y. C. Young and K. C. Tai. Observational inaccuracy in conformance testing with multiple testers. In *IEEE 1st workshop on application-specific software engineering and technology*, pages 80–85, 1998.

[76] Robert M. Hierons. Canonical finite state machines for distributed systems. *Theoretical Computer Science*, 411(2):566–580, 2010.

[77] D.S. Ananichev. The annulation threshold for partially monotonic automata. *Russian Mathematics*, 54(1):1–9, 2010.

[78] Dimitry S. Ananichev and Mikhail V. Volkov. Synchronizing monotonic automata. In *Proceedings of the 7th international conference on Developments in language theory*, DLT'03, pages 111–121, Berlin, Heidelberg, 2003. Springer-Verlag.

[79] B. K. Natarajan. An algorithmic approach to the automated design of parts orienters. In *Foundations of Computer Science, 1986., 27th Annual Symposium on*, pages 132 –142, oct. 1986.

[80] Y. Benenson, T. Paz-Elizur, R. Adar, E. Keinan, Z. Livneh, and E. Shapiro. Programmable and autonomous computing machine made of biomolecules. *Nature*, 414(6862):430–434, November 2001.

[81] Robert M. Hierons and Hasan Ural. Generating a checking sequence with a minimum number of reset transitions. *Automated Software Engineering*, 17(3):217–250, 2010.

[82] A. Rezaki and H. Ural. Construction of checking sequences based on characterization sets. *Computer Communications*, 18(12):911–920, 1995.

[83] Yaakov Benenson, Rivka Adar, Tamar Paz-Elizur, Zvi Livneh, and Ehud Shapiro. Dna molecule provides a computing machine with both data and fuel. *Proceedings of the National Academy of Sciences*, 100(5):2191–2196, 2003.

[84] Milan N. Stojanovic and Darko Stefanovic. A deoxyribozyme-based molecular automaton. *Nature Biotechnology*, 21(9):1069–1074, August 2003.

[85] Yaakov Benenson and Ehud Shapiro. Molecular computing machines, 2004.

[86] I. K. Rystsov. Polynomial complete problems in automata theory. *Inf. Process. Lett.*, 16(3):147–151, 1983.

[87] Pavel V. Martyugin. Complexity of problems concerning carefully synchronizing words for pfa and directing words for nfa. pages 288–302, 2010.

[88] R. M. Karp. Reducibility among combinatorial problems. In R. E. Miller and J. W. Thatcher, editors, *Complexity of Computer Computations*. Plenum Press, New York-London, 1972. 85–103.

[89] M. R. Garey and D. S. Johnson. *Computers and Intractability*. W. H. Freeman and Company, New York, 1979.

[90] Carsten Lund and Mihalis Yannakakis. On the hardness of approximating minimization problems. *J. ACM*, 41(5):960–981, September 1994.

[91] Uriel Feige. A threshold of ln n for approximating set cover. *J. ACM*, 45(4):634–652, July 1998.

[92] Ran Raz and Shmuel Safra. A sub-constant error-probability low-degree test, and a sub-constant error-probability pcp characterization of np. In *Proceedings of the twenty-ninth annual ACM symposium on Theory of computing*, STOC '97, pages 475–484, New York, NY, USA, 1997. ACM.

[93] Noga Alon, Dana Moshkovitz, and Shmuel Safra. Algorithmic construction of sets for k-restrictions. *ACM Trans. Algorithms*, 2(2):153–177, April 2006.

[94] Dexter Kozen. Lower bounds for natural proof systems. In *Proceedings of the 18th Annual Symposium on Foundations of Computer Science*, SFCS '77, pages 254–266, Washington, DC, USA, 1977. IEEE Computer Society.

[95] A. Condon, J. Feigenbaum, C. Lund, and P. Shor. Probabilistically checkable debate systems and nonapproximability of PSPACE-hard functions. *Chicago Journal of Theoretical Computer Science*, 19, 1995.

[96] A.M. Turky and A. Ahmad. Using genetic algorithm for solving n-queens problem. In *Information Technology (ITSim), 2010 International Symposium in*, volume 2, pages 745–747, 2010.

[97] D.E. Knuth. Dancing links. In *Millennial Perspectives in Computer Science*, pages 187–214. Palgrave, 2000.

[98] Jordan Bell and Brett Stevens. A survey of known results and research areas for n-queens. *Discrete Mathematics*, 309(1):1 – 31, 2009.

[99] M. N. Sokolovskii. Diagnostic experiments with automata. *Cybernetics and Systems Analysis*, 7:988–994, 1971.

[100] L. Hyafil and R. L. Rivest. Constructing Optimal Binary Decision Trees is NP-complete. *Information Processing Letters*, 5:15–17, 1976.

[101] M. R. Garey and D. S. Johnson. *Computers and Intractability.* W. H. Freeman and Company, New York, 1979.

[102] Uraz Cengiz Türker and Hüsnü Yenigün. Hardness and inapproximability results for minimum verification set and minimum path decision tree problems. Technical Report 19826, Sabancı University, Istanbul, Turkey, 2012.

[103] Venkatesan T. Chakaravarthy, Vinayaka Pandit, Sambuddha Roy, Pranjal Awasthi, and Mukesh Mohania. Decision trees for entity identification: approximation algorithms and hardness results. In *Proceedings of the twenty-sixth ACM*

SIGMOD-SIGACT-SIGART symposium on Principles of database systems, PODS '07, pages 53–62. ACM, 2007.

[104] Venkatesan T. Chakaravarthy, Vinayaka Pandit, Sambuddha Roy, Pranjal Awasthi, and Mukesh K. Mohania. Decision trees for entity identification: Approximation algorithms and hardness results. *ACM Trans. Algorithms*, 7(2):15:1–15:22, March 2011.

[105] Eduardo S. Laber and Loana Tito Nogueira. On the hardness of the minimum height decision tree problem. *Discrete Applied Mathematics*, 144(1-2):209–212, November 2004.

[106] Evengii Landis Georgy, Adelson-Velskii. An Algorithm for the Organization of Information. *Doklady Akademii Nauk USSR*, 146(2):263–266, 1962.

[107] Micah Adler and Brent Heeringa. Approximating optimal binary decision trees. In *Proceedings of the 11th international workshop, APPROX 2008, and 12th international workshop, RANDOM 2008 on Approximation, Randomization and Combinatorial Optimization: Algorithms and Techniques*, APPROX '08 / RANDOM '08, pages 1–9, Berlin, Heidelberg, 2008. Springer-Verlag.

[108] C. Gunicen, U. C. Türker, H. Ural, and H. Yenigün. Generating preset distinguishing sequences using sat. In Erol Gelenbe, Ricardo Lent, and Georgia Sakellari, editors, *Computer and Information Sciences II*, pages 487–493. Springer London, 2012. 10.1007/978-1-4471-2155-8_62.

[109] Franc Brglez. ACM/SIGMOD benchmark dataset, available online at http://cbl.ncsu.edu:16080/benchmarks/Benchmarks-upto-1996.html.

[110] Hasan Ural. Formal methods for test sequence generation. *Computer Communications*, 15(5):311 – 325, 1992.

[111] M. Yao, A. Petrenko, and G. v. Bochmann. Conformance testing of protocol

machines without reset. In *Protocol Specification, Testing and Verification, XIII (C-16)*, pages 241–256. Elsevier (North-Holland), 1993.

[112] S. Fujiwara, G. v. Bochmann, F. Khendek, M. Amalou, and A. Ghedamsi. Test selection based on finite state models. *IEEE Transactions on Software Engineering*, 17(6):591–603, 1991.

[113] Rita Dorofeeva, Khaled El-Fakih, and Nina Yevtushenko. An improved conformance testing method. In *FORTE*, pages 204–218, 2005.

[114] Rita Dorofeeva, Khaled El-Fakih, Stephane Maag, Ana R. Cavalli, and Nina Yevtushenko. Fsm-based conformance testing methods: A survey annotated with experimental evaluation. *Information and Software Technology*, 52(12):1286 – 1297, 2010.

[115] Hadley Wickham. *ggplot2: Elegant Graphics for Data Analysis (Use R!)*. Springer, 1st ed. 2009. corr. 3rd printing 2010 edition, August 2009.

[116] Sarah Stowell. *Instant R: An Introduction to R for Statistical Analysis*. Jotunheim Publishing, 2012.

[117] Paul Teetor. *R Cookbook*. O'Reilly, first edition, 2011.

[118] William S. Cleveland. Robust Locally Weighted Regression and Smoothing Scatterplots. *Journal of the American Statistical Association*, 74(368):829–836, 1979.

[119] William H. Kruskal and W. Allen Wallis. Use of ranks in one-criterion variance analysis. *Journal of the American Statistical Association*, 47(260):pp. 583–621, 1952.

[120] M. G. Bulmer. *Principles of Statistics*. Dover Publications, March 1979.

[121] E. L. Post. A variant of a recursively unsolvable problem. *Bulletin of the American Mathematical Society*, 52:264–268, 1946.

[122] H. Hunt III, R. Constable, and S. Sahni. On the computational complexity of program scheme equivalence. *SIAM Journal on Computing*, 9(2):396–416, 1980.

[123] Walter J. Savitch. Relationships between nondeterministic and deterministic tape complexities. *Journal of Computer and System Sciences*, 4(2):177 – 192, 1970.