Proceedings of the Estonian Academy of Sciences,
2015, **64**, 3, 256–262
doi: 10.3176/proc.2015.3.08
Available online at www.eap.ee/proceedings

NONLINEAR
WAVES

Unidirectional wave motion in a nonlocally and nonlinearly elastic medium: the KdV, BBM, and CH equations

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Received 6 December 2014, accepted 20 May 2015, available online 20 August 2015

Abstract. We consider unidirectional wave propagation in a nonlocally and nonlinearly elastic medium whose constitutive equation is given by a convolution integral with a suitable kernel function. We first give a brief review of asymptotic wave models describing the unidirectional propagation of small-but-finite amplitude long waves. When the kernel function is the well-known exponential kernel, the asymptotic description is provided by the Korteweg–de Vries (KdV) equation, the Benjamin–Bona–Mahony (BBM) equation, or the Camassa–Holm (CH) equation. When the Fourier transform of the kernel function has fractional powers, it turns out that fractional forms of these equations describe unidirectional propagation of the waves. We then compare the exact solutions of the KdV equation and the BBM equation with the numerical solutions of the nonlocal model. We observe that the solution of the nonlocal model is well approximated by associated solutions of the KdV equation and the BBM equation over the time interval considered.

Key words: nonlocal elasticity, Korteweg–de Vries equation, Benjamin–Bona–Mahony equation, Camassa–Holm equation, fractional Camassa–Holm equation.

1. INTRODUCTION

In the present paper we first state a one-dimensional model of nonlocal and nonlinear elasticity and review recent progress on unidirectional asymptotic wave equations of the model. We then make numerical comparisons between the exact model and the approximate models.

Propagation of small-but-finite amplitude elastic waves has been extensively studied; however, the studies of small-but-finite amplitude waves in nonlocal elastic media are less extensive. A general theory of nonlocal elasticity and the older literature can be found in [1]. Nonlocal elastic models have been successfully used to obtain solutions for linear problems arising in various applications (see for instance [2–4] and the references cited therein). Recently, a nonlinear model of nonlocal elasticity was proposed in [5,6] in which the nonlocal stress is expressed as the weighted average of the local stress over space and consequently the nonlinear constitutive relation is based on a convolution integral over space. In subsequent papers [7,8], global existence and nonexistence of solutions of the initial-value problems posed for various generalizations of the model were investigated. In a very recent study [9], using an asymptotic expansion technique, the asymptotic equations governing unidirectional wave propagation of small-but-finite amplitude long waves in the nonlinear nonlocal elastic medium were derived. The asymptotic

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equations derived include the Korteweg–de Vries (KdV) equation [10], the Benjamin–Bona–Mahony (BBM) equation [11], and the Camassa–Holm (CH) equation [12] as well as fractional generalizations of these equations.

In this study we briefly review the nonlinear nonlocal model and its approximate models for unidirectional wave propagation and then compare numerically the solutions of the exact model with the solutions of the approximate models for an initial-value problem. In particular, we consider an initial-value problem for the nonlocal model with initial data strictly compatible with the solitary wave solution of the KdV equation or the BBM equation and then use a finite-difference scheme to solve the initial-value problem numerically. Numerical experiments demonstrate that the solutions of the nonlinear nonlocal model are well captured by the exact solutions of the approximate models over the time interval considered.

The paper is organized as follows. Section 2 presents the governing equations of one-dimensional nonlocal nonlinear elasticity theory and gives the equation of motion in dimensionless quantities for two distinct forms of the kernel function. In Section 3, unidirectional asymptotic wave equations governing small-but-finite amplitude long waves in the one-dimensional, nonlinear, nonlocal elastic models are summarized. Section 4 presents numerical comparisons of the exact model and the asymptotic models.

2. A NONLOCAL MODEL OF ONE-DIMENSIONAL ELASTIC MEDIA

In this section we briefly state the nonlinear and nonlocal model discussed in [5,6]. In the absence of body forces the equation of motion for a one-dimensional, homogeneous, elastic medium is

$$\rho_0 U_{tt} = (S(U_X))_X, \tag{1}$$

where $U(X,t)$ is the displacement of a reference point X at time t, ρ_0 is the constant density of the material, $S = S(U_X)$ is the stress and a subscript denotes partial differentiation. As in [5,6], we consider a nonlinear and nonlocal elastic medium whose constitutive equation is given by

$$S(X,t) = \int_{\mathbb{R}} \alpha(|X-Y|)\sigma(Y,t)\mathrm{d}Y, \quad \sigma(X,t) = \frac{d}{d(U_X)} W(U_X(X,t)), \tag{2}$$

where W is the strain-energy density function, Y denotes a generic point of the medium, and α is a kernel function to be specified. The main idea in writing (2) is to express the nonlocal stress as the weighted average of the local stress over space in which the kernel α acts as a weight function. Of course, if α is the Dirac delta function, (2) reduces to the classical constitutive equation of a hyperelastic material. In what follows we consider the strain-energy density function in the form

$$W(U_X) = \mu\left[\frac{1}{2}(U_X)^2 + G(U_X)\right],$$

where μ is a constant with the dimension of stress and G is a nonlinear function of U_X with $G(0) = G'(0) = 0$.

Substituting (2) into (1) and differentiating both sides of the resulting equation with respect to X, we rewrite the equation of motion for the strain U_X. Then we introduce the following non-dimensional variables

$$\bar{x} = \frac{X}{l}, \qquad \bar{t} = \frac{t}{l}\sqrt{\frac{\mu}{\rho_0}}, \qquad \bar{U} = \frac{U}{l}, \qquad \beta(x) = l\alpha(|x|),$$

where l is a characteristic length. Henceforth only the non-dimensional variables will be used and the superposed bars will be omitted. Furthermore, we use u for U_X from now on for convenience. As a result of these acts, we are able to rewrite the equation of motion for the nonlinear nonlocal model in the form

$$u_{tt} = (\beta * (u + g(u)))_{xx}, \tag{3}$$

where $g(s) = G'(s)$ with $g(0) = 0$ and the convolution operator $*$ is defined by

$$(\beta * v)(x) = \int_{\mathbb{R}} \beta(x-y)v(y)\mathrm{d}y.$$

Typically an admissible kernel function β is an even and nonnegative function of x and it monotonically decreases for $x > 0$ [1]. A few of the most commonly used ones are listed in [6]. In [9], for two particular forms of the kernel function β and the quadratic nonlinearity $g(u) = u^2$, various asymptotic equations describing unidirectional wave propagation of small-but-finite amplitude long waves have been derived. Therefore, in the remaining part of this study, we consider only the quadratic nonlinearity and focus on the two particular kernel functions: an exponential kernel and a fractional-type kernel function.

For the exponential kernel $\beta(x) = \frac{1}{2}e^{-|x|}$, it is a well-known fact that $\beta(x)$ is the Green's function of the operator $1 - D_x^2$, where D_x represents the partial derivative with respect to x [13]. Hence, for the exponential kernel, (3) becomes simply

$$u_{tt} - u_{xx} - u_{xxtt} = (u^2)_{xx}, \tag{4}$$

which is known as the improved Boussinesq (IBq) equation.

The second type of the kernel function considered in [9] is the fractional-type kernel function defined by $\widehat{\beta}(\xi) = (1 + (\xi^2)^\nu)^{-1}$, where $\widehat{\beta}(\xi)$ is the Fourier transform of $\beta(x)$ and $\nu \geq 1$ may not be an integer. As a consequence of the fact that $\beta(x)$ is the Green's function for the operator $1 + (-D_x^2)^\nu$, (3) now becomes an IBq equation of fractional type

$$u_{tt} - u_{xx} + (-D_x^2)^\nu u_{tt} = (u^2)_{xx} \,. \tag{5}$$

Here the operator $(-D_x^2)^\nu$ is defined as $(-D_x^2)^\nu u = \mathrm{F}^{-1}\left(|\xi|^{2\nu}\mathrm{F}u\right)$, where F and F^{-1} denote the Fourier transform and its inverse, respectively.

3. ASYMPTOTIC MODELS FOR UNIDIRECTIONAL WAVES IN NONLOCAL ELASTICITY

In this section we briefly review the unidirectional wave equations derived in [9] for small-but-finite amplitude long waves propagating in the one-dimensional nonlocal media governed by (4) and (5). From a wave propagation point of view, both (4) and (5) are dispersive wave equations. We underline that dispersive wave propagation is due to the internal structure of the medium but not due to the existence of the boundaries. This feature of the model differentiates the asymptotic derivation of the CH equation for elastic waves in [9] from previous work in the literature. For a derivation based on the dispersive wave propagation resulting from the existence of the boundaries, we refer the reader to [14,15] where CH-type equations are derived asymptotically for elastic waves.

In [9], a right-going solution of (4) is sought as an asymptotic power series in small parameters $\varepsilon > 0$ and $\delta > 0$ measuring nonlinear and dispersive effects, respectively. The parameter ε represents a typical (small) amplitude of waves while the parameter δ represents a typical (small) wavenumber. Following the approaches in [16–18], it was shown that, at $\mathscr{O}(\varepsilon)$ (equivalently, at $\mathscr{O}(\delta^2)$), (4) can be approximated by the KdV equation

$$w_t + w_x + ww_x + \frac{1}{2}w_{xxx} = 0 \tag{6}$$

for $w = w(x,t)$ or by the BBM equation

$$w_\tau + \kappa w_\zeta + 3ww_\zeta - w_{\zeta\zeta\tau} = 0 \tag{7}$$

for $w = w(\zeta,\tau)$, where $\kappa = \frac{6}{5}$, and

$$\zeta = \frac{2}{\sqrt{5}}\Big(x - \frac{3}{5}t\Big), \quad \tau = \frac{2}{3\sqrt{5}}t. \tag{8}$$

We note that, applying the coordinate transformation from (ζ,τ) to (x,t), (7) can be rewritten as

$$w_t + w_x + ww_x - \frac{3}{4}w_{xxx} - \frac{5}{4}w_{xxt} = 0 \tag{9}$$

for $w = w(x,t)$. We observe that even though the KdV equation and the BBM equation are valid at the same level of approximation, the group velocity of linear waves for the BBM equation is bounded for all wavenumbers while the group velocity of linear waves for the KdV equation has no lower bound. Furthermore, it was also shown that, at the next order of approximation, that is, at $\mathscr{O}(\varepsilon\delta^2)$, (4) can be approximated by the CH equation

$$w_\tau + \kappa_1 w_\zeta + 3ww_\zeta - w_{\zeta\zeta\tau} = \kappa_2(2w_\zeta w_{\zeta\zeta} + ww_{\zeta\zeta\zeta}) \tag{10}$$

for $w = w(\zeta,\tau)$, where $\kappa_1 = \frac{6}{5}$, $\kappa_2 = \frac{9}{5}$, and ζ and τ are given by (8). Again, we note that, using the coordinate transformation from (ζ,τ) to (x,t), (10) can be rewritten as

$$w_t + w_x + ww_x - \frac{3}{4}w_{xxx} - \frac{5}{4}w_{xxt} = \frac{3}{4}(2w_x w_{xx} + ww_{xxx}) \tag{11}$$

for $w = w(x,t)$. The above results are consistent with the previous literature suggesting that the KdV and BBM equations are valid at the same level of approximation while the CH equation is more accurate than the KdV and BBM equations.

In [9], the above approach is also extended to the case of the fractional-type kernel function. That is, a right-going solution of (5) is sought as an asymptotic power series in small parameters ε and δ, and fractional generalizations of the nonlinear wave equations discussed above are derived. Following similar steps as above, it was shown that, at $\mathscr{O}(\varepsilon)$ (equivalently, at $\mathscr{O}(\delta^2)$), (5) can be approximated by the fractional KdV equation [19]

$$w_t + w_x + ww_x - \frac{1}{2}(-D_x^2)^\nu w_x = 0 \tag{12}$$

for $w = w(x,t)$, or by the fractional BBM equation [20]

$$w_\tau + \kappa w_\zeta + 3ww_\zeta + (-D_\zeta^2)^\nu w_\tau = 0 \tag{13}$$

for $w = w(\zeta,\tau)$, where $\kappa = \frac{6}{5}$ and

$$\zeta = a\Big(x - \frac{3}{5}t\Big), \quad \tau = \frac{a}{3}t \tag{14}$$

with $a = (\frac{4}{5})^{1/2\nu}$. We note that, applying the coordinate transformation from (ζ,τ) to (x,t), (13) can be rewritten as

$$w_t + w_x + ww_x + \frac{3}{4}(-D_x^2)^\nu w_x + \frac{5}{4}(-D_x^2)^\nu w_t = 0 \tag{15}$$

for $w = w(x,t)$. Furthermore, it was also shown that, at $\mathscr{O}(\varepsilon\delta^2)$, (5) can be approximated by the fractional CH equation

$$w_\tau + \kappa_1 w_\zeta + 3ww_\zeta + (-D_\zeta^2)^\nu w_\tau = -\kappa_2[2(-D_\zeta^2)^\nu(ww_\zeta) + w(-D_\zeta^2)^\nu w_\zeta] \tag{16}$$

for $w = w(\zeta,\tau)$, where $\kappa_1 = 6/5$, $\kappa_2 = 3/5$, and ζ and τ are given by (14). Using the coordinate transformation from (ζ,τ) to (x,t), (16) can be rewritten as

$$w_t + w_x + ww_x + \frac{3}{4}(-D_x^2)^\nu w_x + \frac{5}{4}(-D_x^2)^\nu w_t = -\frac{1}{4}[2(-D_x^2)^\nu(ww_x) + w(-D_x^2)^\nu w_x] \tag{17}$$

for $w = w(x,t)$.

4. NUMERICAL COMPARISONS BETWEEN THE EXACT AND APPROXIMATE MODELS

The asymptotic results summarized in Section 3 can be justified numerically and/or rigorously. In this section, restricting our attention to the KdV and BBM equations, we make numerical comparisons between the exact model defined by (4) and the approximate models given by (6) and (9). It is noteworthy that the two approximate models may provide accurate predictions at least over time scales where the assumptions of the asymptotic derivation hold and that the predictions of the approximate models may not be accurate over longer time scales. We use a finite-difference method that is second-order accurate in space and time. The length of the spatial domain is chosen to be large enough so that the boundary data at the endpoints can be safely taken equal to zero on the time interval under consideration. Numerical dispersion is minimized by taking sufficiently small step sizes in time and space.

In the two numerical experiments, the exact travelling wave solutions to (6) and (9) are compared with the numerical solutions of the corresponding initial-value problems for (4). The solitary wave solutions of (6) and (9) are of the form

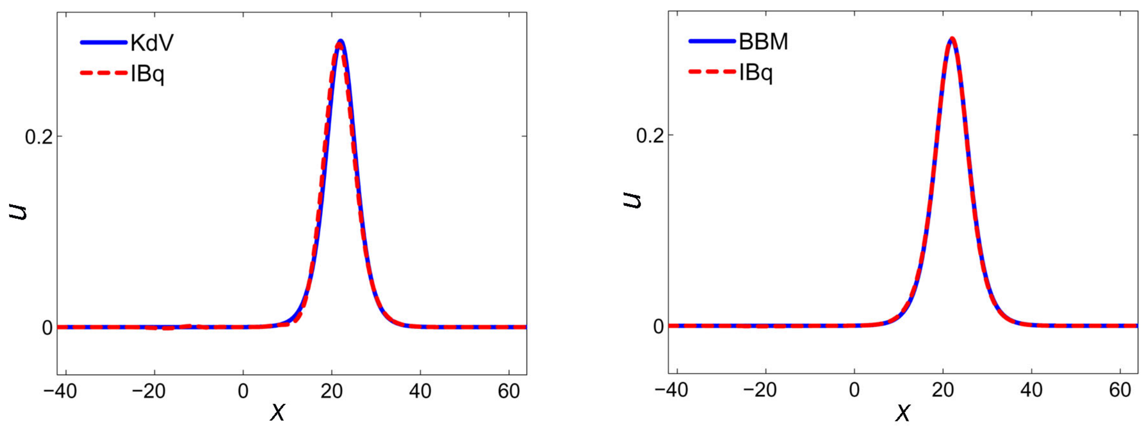
$$w(x,t) = A\ \mathrm{sech}^2(B(x-ct-x_0)), \quad A = 3(c-1), \tag{18}$$

with $B=\sqrt{\frac{c-1}{2}}$ for the KdV equation and $B=\sqrt{\frac{c-1}{5c-3}}$ for the BBM equation. The solution (18) represents a solitary wave located initially at $x=x_0$ and travelling to the right with speed $c>1$, amplitude A, width $1/B$. We solve numerically the initial-value problems for (4) with the initial data

$$u(x,0) = w(x,0), \quad u_t(x,0) = w_t(x,0), \tag{19}$$

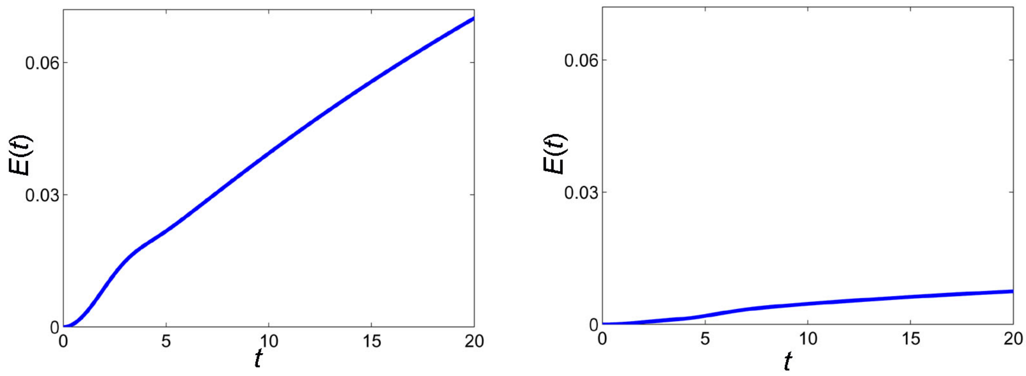
where w is given by (18).

In Fig. 1 we compare the solution profiles of (6) and (9) with those of (4) at $t=20$ for $x_0=0$ and $c=1.1$. In both cases we observe that at $t=20$ the two solution profiles are very close to each other within the resolution of the plots. We conclude that the solution of (4) is very well approximated by the associated solutions of the KdV and BBM equations over the time interval considered.



The KdV equation The BBM equation

Fig. 1. The solitary wave solution of the unidirectional wave equation (solid line) and the corresponding approximate solution of the IBq equation (dashed line) at $t=20$.



The KdV equation The BBM equation

Fig. 2. Variation of $E(t)$ with time.

In Fig. 2 we present the variation of the quantity

$$E(t) = \max_x |u(x,t) - w(x,t)| / \max_x |w(x,t)|$$

with time t for the experiments above. We note that $E(t)$ shows the relative differences between the exact solution of the unidirectional wave equation and the numerical solution of (4). Both figures indicate that $E(t)$ grows linearly with time t. We also observe that the relative difference $E(t)$ in the case of the BBM equation is significantly smaller than the one for the KdV equation. This shows that the BBM equation provides a slightly better asymptotic model for the IBq equation than the KdV equation (see [21] for a similar conclusion based on water waves).

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Ühesuunaline lainelevi mittelokaalses ja mittelineaarses elastses keskkonnas: KdV, BBM ning CH võrrandid

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On vaadeldud ühesuunalist lainelevi mittelokaalses ja mittelineaarses elastses keskkonnas, mille olekuvõrrand on esitatud läbi sobivalt valitud tuuma omava konvolutsiooniintegraali. Esmalt on esitatud ülevaade asümtootilistest mudelitest, mis kirjeldavad väikese, kuid lõpliku amplituudiga pikkade lainete ühesuunalist levimist. Kui tuumaks on hästi tuntud eksponentsiaalne tuum, siis saadakse asümptootilise mudeli rakendamisel Kortewegi-de Vriesi (KdV), Benjamini-Bona-Mahony (BBM) või Camassa-Holmi (CH) võrrand. Kui tuumafunktsiooni Fourier' teisendus omab murdarvulisi astmeid, siis selgub, et need võrrandid üldistatud murdarvulisel kujul kirjeldavad ühesuunalist lainelevi. On võrreldud KdV ja BBM võrrandite täpseid lahendeid ning mittelokaalse mudeli jaoks leitud numbrilisi lahendeid. On täheldatud, et mittelokaalse mudeli lahendit saab vaadeldud ajavahemikul aproksimeerida vastava KdV ja BBM võrrandi lahendiga.