On EOQ Cost Models with Arbitrary Purchase and Transportation Costs

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ABSTRACT: We analyze an economic order quantity cost model with unit out-of-pocket holding costs, unit opportunity costs of holding, fixed ordering costs, and general purchase-transportation costs. We identify the set of purchase-transportation cost functions for which this model is easy to solve and related to solving a one-dimensional convex minimization problem. For the remaining purchase-transportation cost functions, when this problem becomes a global optimization problem, we propose a Lipschitz optimization procedure. In particular, we give an easy procedure which determines an upper bound on the optimal cycle length. Then, using this bound, we apply a well-known technique from global optimization. Also for the class of transportation functions related to full truckload (FTL) and less-than-truckload (LTL) shipments and the well-known carload discount schedule, we specialize these results and give fast and easy algorithms to calculate the optimal lot size and the corresponding optimal order-up-to-level.

Keywords: Inventory; EOQ cost model; transportation cost function; purchasing cost function.

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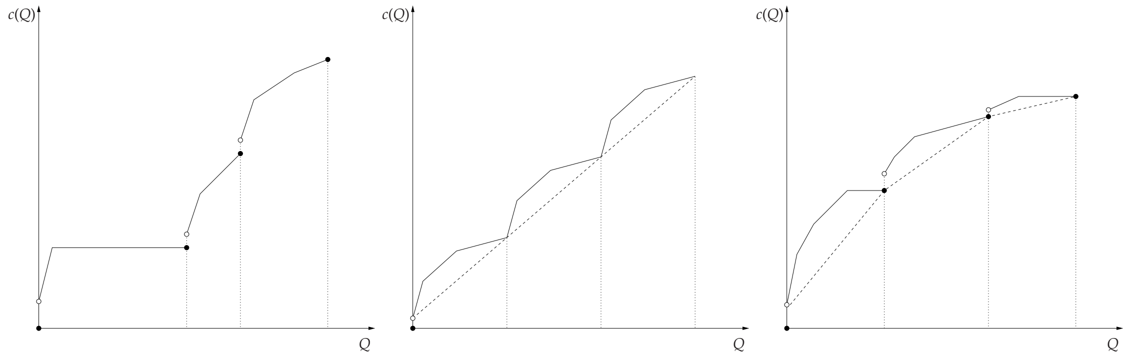
1. Introduction. In inventory control, the economic order quantity cost model (EOQ) is the most fundamental model, which dates back to the pioneering work of Harris (1913). The environment of the model is somewhat restricted. The demand for a single item occurs at a known and constant rate, shortages are not permitted, there is a fixed setup cost, and holding costs are independent of the size of the replenishment order. In this simplest form, the model describes the trade-off between the fixed setup and the holding costs. At the same time purchase and transportation costs are independent of the size of the replenishment order and due to the complete backordering assumption, these costs do not affect the optimal trade off between setup and holding costs. Though the model has several simplifying assumptions, it has been effectively used in practice. The standard EOQ cost model has also been extended to different settings, where shortages, discounts, production environments, and other extensions are considered (Hadley and Whitin, 1963; Nahmias, 1997; Silver et al., 1998; Zipkin, 2000; Muckstadt and Sapra, 2009; Drake and Marley, 2014).

In this paper, we generalize the basic assumptions of the classical EOQ cost model in the following directions. We allow, contrary to the classical model, that the holding cost per item per unit time also depends on the size of the replenishment order. In addition to the (physical) inventory holding cost per item per unit time, independent of the size of the replenishment order, we also incorporate in our model an opportunity holding cost per item per unit time dependent on the average value of an item. This average value depends on the transportation and purchase costs of a replenishment order, and thus, on the size of such an order. Also, instead of linear purchase and transportation costs, we allow arbitrary purchase and transportation costs. In the most general case we only assume that these costs are increasing in the size of the replenishment order. This means that both economies and diseconomies of scale in ordering are covered. As our literature review in Section 2 shows, a sizable list of work on EOQ cost models exist that account for the impact of the transportation costs on the lot sizing decision. This is restricted to EOQ cost models with no shortages. In particular, less-than-truckload (LTL) or full truckload (FTL) shipments have been the focal point of many studies. A special instance of the model proposed here gives an overall approach to solve all of the FTL and LTL shipment problems proposed in the literature. We start in Section 3 using only a generic purchase-transportation cost function and derive the associated EOQ cost optimization problem and study its properties. In the most general case this optimization problem is a one dimensional global optimization problem. Therefore, we first identify those purchase-transportation cost functions for which solving the original optimization problem is easy and related to solving a convex minimization problem in Section 4. It will turn out for zero opportunity costs that we need the convexity of the purchase-transportation cost function. Meanwhile, for positive opportunity costs, the class of easy instances is restricted to affine purchase-transportation cost functions. Moreover, for certain discounting schemes, like incremental discount, the associated optimization problem is also easy to solve and related to solving a finite sequence of convex programming problems. In Section 5, we consider the remaining instances of increasing purchase-transportation cost functions for which solving the optimization problem is related to solving a one-dimensional global optimization problem. The approach suggested in this section for these most general problems is the following. We first derive a so-called dominance result and use this to construct a bounded interval containing the optimal cycle length (reorder interval). If the purchase-transportation cost function is bounded from above by some affine function (quite natural for economies of scale situations) an upper bound represented by an easy analytical formula can be derived. For other purchase-transportation cost functions, it is possible to evaluate this upper bound by means of an algorithm. In the same section we will use this upper bound in combination with a general Lipschitz optimization procedure known in global optimization to solve such general EOQ cost models.

Restricting our general purchase-transportation cost functions to the so-called carload discount, FTL and LTL schedules discussed in the literature, we show that a fast algorithm exists using our dominance result. This algorithm generalizes the different algorithms shown in the literature for special subcases. To design this algorithm, we shall first show for an increasing affine purchase-transportation cost function that the resulting problem is a simple convex optimization problem that can be solved very efficiently. In particular, we shall derive analytic solutions for two special cases: (i) when there are no shortages; (ii) when there are shortages and zero opportunity costs. Having analyzed an affine purchase-transportation cost function, we shall then give a fast algorithm to solve the problem when the purchase-transportation cost function is increasing piecewise polyhedral concave as shown in Figure 1(a). This algorithm is based on solving a series of simple problems that correspond to the increasing linear pieces on the piecewise polyhedral concave function. To further improve the performance of the proposed algorithm, we shall then concentrate on two particular instances as shown in Figures 1(b) and 1(c). The former is a typical carload schedule with identical setups, and the latter represents a general carload schedule with nonincreasing truck setup costs. Both cases admit a lower bounding function, which is linear in the former case and polyhedral concave in the latter case. These lower bounding functions, shown with dashed lines in Figure 1, allow us to concentrate on solving only a few simple problems. Finally, in Section 6 we will give some numerical examples to illustrate our results.

In summary, the primary contributions of this work are (i) presenting EOQ cost models with opportunity costs and arbitrary purchase and transportation costs covering both economies and diseconomies of scale in ordering;



(a) Piecewise polyhedral concave (b) Typical carload schedule (c) General carload schedule

Figure 1: Some purchase-transportation cost functions for which fast algorithms are developed.

(ii) identifying the class of purchase and transportation cost functions for which the objective function of the EOQ cost model has a convexity-type property and so the optimization problem is easy to solve; (iii) giving efficient algorithms for the more difficult LTL and FTL shipment schemes along with decreasing truck setup costs; (iv) deriving an upper bound on the optimal replenishment cycle for *any* increasing purchase-transportation cost function and using this upper bound to give a general solution procedure for the most general case.

2. Review of Related Literature. In this section, we shall review the studies on EOQ and lot-sizing models which discuss cost functions that are more general than the linear purchase price and transportation cost functions. In most of the papers the main attention was either on the purchase price of an order or on transportation costs. Before 1970s, the transportation cost was mostly added as a fixed setup cost independent of the size of the order. The most general purchase price functions were assumed to be convex or concave. Veinott Jr (1966) gives an overview of the literature before 1966 on linear, convex and concave purchase price functions within a deterministic lot sizing environment. In this environment the purchase price of an order is replaced by production costs. For a more extensive discussion also covering the literature after 1970, the reader is referred to Porteus (1990) and Frenk et al. (2014). In these surveys, the authors also discuss the economic conditions, under which the production or purchase price functions need to be concave or convex.

At the beginning of the seventies it was realized that transportation costs cannot be modeled as a fixed setup cost. As a result, the majority of the papers appearing after 1970 and dealing with general transportation-purchase price cost functions focused on transportation costs dependent on the size of the order. In particular special freight discounts and FTL and LTL shipments used in practice were discussed. This led to special transportation cost functions with a polyhedral concavity-type structure. We refer the reader to Carter and Ferrin (1996) for an overview and an informative discussion on the role of transportation costs in inventory control, and to Das (1988) for a general discussion about various freight discounts and discounting schemes. Due to the extensive survey on purchase price functions discussed in the above survey papers our main focus in the remainder of this section is only on the implementation of special transportation costs in lot sizing and EOQ-type environments appearing after 1970.

One of the earliest works discussing the importance of transportation costs on controlling the inventory levels is given by Baumol and Vinod (1970). They try to place the freight decisions within inventory-theoretic models and point out that LTL shipments make the overall problem difficult to solve. Around the same time, Lippman (1969) considers a single-product in a multiple period setting, where charges due to multiple trucks with different sizes are taken into account. These charges create discontinuities (jumps) in the considered objective functions. Lippman obtains the optimal policies for two special cases of the objective function resulting in a monotone and concave cost model. He also analyzes the stationary, infinite horizon case and discusses the asymptotic properties of the optimal schedules. In a follow-up work, Lippman (1971) considers a similar setup for finding the economic order quantities. In this work, he assumes that the excess truck space cannot be used, and hence, the shipment cost should be incurred in the multiples of the trucks. In both of his works, no discounting scheme is present and shortages are not allowed. Iwaniec (1979) investigates the inventory model of a single product system, where the demand is stochastic and a fixed cost is charged and included in the ordering cost. The conditions, under which the full load orders minimize the total expected cost, are characterized. The multiple setup cost structure of Lippman (1971) is used also in this work. However, Iwaniec considers full backlogging, and hence, the holding and ordering costs are coupled with backlogging costs but no discounting scheme exists. Aucamp (1982) solves the continuous review case of the multiple setup problem discussed by Lippman (1971) and Iwaniec (1979). The main difference between the standard EOQ model and the Aucamp's model is the addition of vehicle costs to the setup cost. Like others above, no discounting scheme is considered. Lee (1986) discusses an EOQ model with a setup cost term that consists of fixed and freight costs. He also considers the case where the freight cost benefits from a discount scheme. The freight cost depends on the order size and added to the setup cost of placing an order. Noting that the convexity structure does not change within each interval, Lee proposes an algorithm based on finding the interval where the global minimum point resides. This algorithm is an alternate solution approach to that of Aucamp (1982), when the multiple setup cost structure of Lippman (1971) is adopted in the model.

Jucker and Rosenblatt (1985) incorporate the quantity discount schemes into the standard newsboy problem. These discounts play a role in purchasing or transporting units at the beginning of the period. Aside from the well-known all-units and incremental quantity discounts, they also discuss –what they call– carload-lot discounts. The transportation cost function is of the type shown in Figure 1(b). That is, the shipping-cost can be reduced or even exempted when the quantity of purchase is LTL. Knowles and Pantumsinchai (1988) consider an all-units discount schedule with no shortages. The products are sold in containers of various sizes. The seller offers discounts when the products are shipped in larger container sizes. They impose FTL orders by adding a restriction on the order quantity which dictates that the order quantities should be in integral multiples of the container sizes. They give a solution algorithm based on solving a series of knapsack problems. They also develop a more efficient algorithm for a restricted policy, which is based on filling the order starting from the largest container and then carrying on with smaller ones. A different perspective to transportation costs is given by Larson (1988). He introduces several models, where three stages of inventory levels are considered: at the origin, in-transit and at the destination. Then, the objective becomes minimization of total logistics costs. Hwang et al. (1990) investigate both all-units quantity and freight cost discounts within the standard EOQ context. The economies of scale realized on the freight cost is the same as that in (Lee, 1986). Recently, Toptal (2009) generalizes the work of Hwang et al. (1990) by modeling the production/inventory related net profits using a general function that features some structural properties.

A further generalization of this work appears in Konur and Toptal (2012) and combines all-units discount with both economies and diseconomies of scale into a hybrid wholesale price schedule. Tersine and Barman (1991) combine quantity and freight rate discounts from suppliers and shippers, respectively. They consider all-units and incremental quantity discount schemes both in purchasing and freight cost. However, the truck setup costs and the shortages are omitted. Arcelus and Rowcroft (1991) examine three types of freight-rate structures, where the incremental discount is applied only to purchasing. The objective function of the resulting problem is analyzed over non-overlapping intervals, and it is shown that the objective function is convex over each interval. Thus, an algorithm, which is based on identifying the local solution within each interval, is proposed to solve the overall problem.

Russell and Krajewski (1991) study the transportation cost structure for LTL shipments. They consider over-declared shipments resulting from an opportunity to reduce the total freight costs by artificially inflating the actual shipping weight to the next breakpoint. In other words, for a freight rate schedule, it may be more economical to ship LTL at a FTL rate. The decision makers then need to transform this nominal freight rate schedule into an effective one, which appropriately represents the best rate schedule for them. This effective schedule consists of intervals over which the transportation cost is determined by a polyhedral concave function consisting of a linear and a constant piece. This is again a special case of what we consider in our work as illustrated by Figure 1(a). Carter et al. (1995a) discuss in-detail the role of anomalous weight breaks in LTL shipping and examine the causes behind this anomaly with its implications in logistics management. These points occur when the discount is so large that the indifference point weight is less than even the lower rate interval. Their observation on anomalous weight breaks has led them to correct the effective freight rate schedule of Russell and Krajewski (1991) as they reported in their subsequent work (Carter et al., 1995b). Burwell et al. (1997) consider an EOQ environment under quantity and freight discounts very similar to that in Tersine and Barman (1991). Unlike Tersine and Barman, their demand is not constant but depends on the price. Therefore, the proposed algorithm to solve the model also determines the selling price besides the optimal lot size. However, they ignore the option of over-declaring the shipments and do not consider LTL or FTL freight rates. Swenseth and Godfrey (2002) carry on with a similar discussion about over-declared shipments as in (Russell and Krajewski, 1991). They do not take quantity discounts or shortages into account. Therefore, the resulting transportation cost function can be thought as a special case of the function shown in Figure 1(c). To solve the resulting problem, they propose a heuristic, which is based on evaluating two inverse functions that over- and under-shoot the optimal order quantity. Abad and Aggarwal (2005) extend the model proposed by Burwell et al. (1997) by considering both over-declaring and LTL (or FTL) shipments like Russell and Krajewski (1991) and Swenseth and Godfrey (2002). They propose a solution procedure based on solving a series of nonlinear equations to obtain the optimal order quantity as well as the selling price. In several recent works (Rieskts and Ventura, 2008; Mendoza and Ventura, 2008; Rieksts and Ventura, 2010; Toptal and Bingöl, 2011), the optimal inventory policies with both FTL and LTL transportation modes are examined. Rieskts and Ventura (2008) provide focus on both infinite and finite horizon single-stage models with no shortages. Later, Mendoza and Ventura (2008) extend the work of Rieskts and Ventura (2008) by incorporating all-units and incremental quantity discounts into their models. In the two-echelon system analyzed in Rieksts and Ventura (2010), the transportation options from a single warehouse to a single retailer include both FTL and LTL options. The authors design an optimal algorithm for this basic case and propose a heuristic algorithm for the case of multiple retailers. Toptal and Bingöl (2011) study the replenishment problem of a retailer with a particular FTL and a particular LTL carrier at its disposal. However, the setting is further complicated by explicitly modeling the truckload carrier's pricing problem. Potential savings are demonstrated, if the decisions of the retailer and the truckload carrier are coordinated.

3. Mathematical Model. We consider an EOQ-type, infinite planning horizon model with complete backordering and demand rate $\lambda > 0$. In this model the average cost criterion is used and we need to select a so-called (S, T) policy which minimizes the average cost. Before discussing the cost components determining this average cost we observe that for computing the average cost of a given (S, T) policy we may assume without loss of generality that at time 0 the net inventory level is $S \geq 0$. In any (S, T) policy every T time units (cycle) an order is issued and the size Q of the order is chosen in such a way that the net inventory level is raised to the order-up-to-level $S \geq 0$. To determine the size Q of any order, we first observe that the total demand in a cycle is λT. This implies by the complete backordering assumption and the net inventory level $S \geq 0$ at the beginning of each cycle that $Q = \lambda T$. Also, if positive, then every $t \leq T$ time units after an order the net inventory level is $S - \lambda t$; if negative, then the absolute value of $S - \lambda t$ represents the backlog at that moment. Therefore, at the end of a cycle, if the net inventory level is positive, then it is given by $S - \lambda T$; if it is negative, then its absolute value represents the maximum backlog occurring in a cycle. Only for the EOQ-type model with no shortages, there is a clear relation between the order size $Q = \lambda T$ and the order-up-to-level S. In this case, it is optimal to order at zero inventory level and so $S = Q = \lambda T$.

To select the optimal (S, T) policy, different cost components in the model need to be introduced. Each time an order is issued, we incur a fixed ordering cost $a > 0$. Also per item in stock per unit of time, the inventory holding costs consist of an out-of-pocket holding cost of $h > 0$ and an opportunity holding cost with opportunity cost rate $r \geq 0$. To penalize late deliveries, the cost of backlogging is $b \geq 0$ per backlogged item per unit of time. Clearly, for $b = 0$ we consider the extreme case of no backlogging cost, while for $b = \infty$ no backlogging is allowed and so, we do not allow shortages. Since each order also generates transportation and purchase costs, the function $p : [0, \infty) \to \mathbb{R}$ with $p(0) = 0$ represents the purchase price function. At the same time, the function $t : [0, \infty) \to \mathbb{R}$ with $t(0) = 0$, denotes the transportation cost function. Consequently, the total purchase-transportation cost of an order of size Q is given by $c(Q) := t(Q) + p(Q)$, where the function $c(.)$ denotes the purchase-transportation cost function. In most cases it is assumed that the function $c(.)$ is increasing. Since in general the more you order from a supplier the more you have to pay for transportation and ordering, this monotonicity condition $c(.)$ is quite natural. Only in special cases where the supplier uses a special discounting scheme, like all-units discount, this condition might not hold. In the remainder of this paper, we refer to the sum of the transportation and purchase costs as ordering costs and call the function $c(.)$ for simplicity the *ordering cost* function. To capture the holding-backlog costs, note that in a classical EOQ cost model the value hx represents the out of pocket holding costs per time unit when the net inventory level has value x, while $-bx$ denotes the backlog costs per time unit when the net inventory level is negative. Out of pocket holding costs represent real costs of holding inventory, such as; warehouse rental, handling, insurance and refrigeration costs. Penalty costs might occur due to fixed delivery contracts with the customers. Generalizing the standard EOQ cost model, we also introduce opportunity costs per item in stock per unit of time. This cost depends on the size $Q = \lambda T$ of the last order and applying the average cost principle we set the value of each item in stock at $Q^{-1}c(Q) = (\lambda T)^{-1}c(\lambda T)$. This means that at net inventory level $x \geq 0$, the opportunity costs are given by $r(\lambda T)^{-1}c(\lambda T)x$ per unit of time with r denoting the opportunity cost rate. Adding these costs per unit of time, we obtain for any (S, T) policy the cost rate function $f_b, 0 \leq b \leq \infty$ given by

$$f_b(T, x) = \begin{cases} \left(h + \frac{rc(\lambda T)}{\lambda T}\right)x & \text{if } x \geq 0; \\ -bx, & \text{if } x < 0. \end{cases} \tag{1}$$

Clearly this function represents the backlog-inventory holding and opportunity costs of the system per time unit at net inventory level x when using the (S, T) policy. For a detailed discussion of this cost rate function within a production environment, the reader is referred to Bayındır et al. (2006). A similar derivation for the standard EOQ model is given by, for instance, Muckstadt and Sapra (2009). To determine the range of S, it follows by the complete backordering assumption that $S \geq 0$. Also it is easy to see that for a given cycle length $T > 0$, any order-up-to-level $S > \lambda T$ is dominated in cost by $S = \lambda T$. By these observations we only derive the average cost expression for (S, T) control rules with $0 \leq S \leq \lambda T$. For such control rules, the average cost $g_b(S, T)$ has the form

$$g_b(S, T) = \frac{a + c(\lambda T) + \int_0^T f_b(T, S - \lambda t)dt}{T}. \tag{2}$$

Hence, to determine the optimal (S, T) rule, we need to solve the optimization problem

$$\min\{g_b(S, T) : T > 0, 0 \leq S \leq \lambda T\}. \tag{3}$$

We introduce for $0 \leq b \leq \infty$ the function

$$F_c(b, r, T) := \frac{a + c(\lambda T) + \varphi_b(T)}{T}, \tag{4}$$

where

$$\varphi_b(T) := \min\left\{\int_0^T f_b(T, S - \lambda t)dt : 0 \leq S \leq \lambda T\right\}. \tag{5}$$

Then, the optimization problem (3) is the same as

$$\min\{F_c(b, r, T) : T > 0\}. \tag{$P_{c,b,r}$}$$

For the inventory holding and backorder costs used in the classical EOQ model, it is easy to give an elementary expression for the value $\varphi_b(T)$. Therefore, it is possible to simplify the formula for $F_c(b, r, T)$. Since by relation (1) it follows for $0 \leq S \leq \lambda T$ that

$$\int_0^T f_b(T, S - \lambda t)dt = \frac{\left(\frac{rc(\lambda T)}{\lambda T} + h\right)S^2}{2\lambda} + \frac{b(S - \lambda T)^2}{2\lambda}, \tag{6}$$

applying standard first order conditions yields the optimal value $S(T)$ of the optimization problem in relation (5) as

$$S(T) = \begin{cases} 0, & \text{if } b = 0; \\ \frac{\lambda bT}{b + h + \frac{rc(\lambda T)}{\lambda T}}, & \text{if } 0 < b < \infty; \\ \lambda T, & \text{if } b = \infty. \end{cases}$$

Hence, we obtain by relation (6) that

$$\varphi_b(T) = \begin{cases} 0, & \text{if } b = 0; \\ \frac{\lambda b\left(\frac{rc(\lambda T)}{\lambda T}+h\right)T^2}{2\left(b+h+\frac{rc(\lambda T)}{\lambda T}\right)}, & \text{if } 0 < b < \infty; \\ \frac{\lambda\left(\frac{rc(\lambda T)}{\lambda T}+h\right)T^2}{2}, & \text{if } b = \infty. \end{cases} \tag{7}$$

This shows by relations (4) and (7) that the objective function of optimization problem ($P_{c,b,r}$) simplifies to

$$F_c(b,r,T) = \begin{cases} \frac{a+c(\lambda T)}{T}, & \text{if } b = 0; \\ \frac{a+c(\lambda T)}{T} + \frac{b\left(\frac{rc(\lambda T)}{\lambda T}+h\right)\lambda T}{2\left(b+h+\frac{rc(\lambda T)}{\lambda T}\right)}, & \text{if } 0 < b < \infty; \\ \frac{a+c(\lambda T)}{T} + \frac{\left(\frac{rc(\lambda T)}{\lambda T}+h\right)\lambda T}{2}, & \text{if } b = \infty. \end{cases} \tag{8}$$

The objective function above satisfies the following useful properties for optimization. We next introduce for any $0 \leq b \leq \infty$ the function

$$h_b(x,T) = \begin{cases} \lambda x, & \text{if } b = 0; \\ \lambda x + \frac{b(rx+h)\lambda T}{2(b+h+rx)}, & \text{if } 0 < b < \infty; \\ \lambda x + \frac{(rx+h)\lambda T}{2}, & \text{if } b = \infty. \end{cases}$$

Then, we know by relation (8) for any $0 \leq b \leq \infty$ that

$$F_c(b,r,T) = \frac{a}{T} + h_b\left(\frac{c(\lambda T)}{\lambda T}, T\right). \tag{9}$$

Clearly for every $T > 0$ and any $0 \leq b \leq \infty$, the function $x \mapsto h_b(x,T)$ is increasing. This implies for a (positive) ordering cost function $c(.)$ represented as a finite minimum of functions $c_n(.)$, $n \in J$ on some interval $\lambda I := \{\lambda T : T \in I\} \subseteq (0,\infty)$ given by $c(Q) = \min_{n\in J} c_n(Q)$, that

$$F_c(b,r,T) = \min_{n\in J} F_{c_n}(b,r,T) \tag{10}$$

for every T belonging to I. If the ordering cost function satisfies $c(Q) = \max_{n\in J} c_n(Q)$, then by a similar reasoning we obtain

$$F_c(b,r,T) = \max_{n\in J} F_{c_n}(b,r,T) \tag{11}$$

for every T belonging to I. When the set J is infinite, then the min operator in relation (10) should be replaced by the inf operator. Similarly, the max operator in relation (11) should be replaced by the sup operator.

In the formulation of the optimization problem ($P_{c,b,r}$), we assume that an optimal solution exists. To be accurate, we need to state the conditions on the function $c(.)$ so that an optimal solution to ($P_{c,b,r}$) indeed exists.

Definition 3.1 *(Aubin, 1993) A function $c : [0,\infty) \to \mathbb{R}$ is called lower semi-continuous if for every $\theta \in \mathbb{R}$ the lower level set $\{Q \geq 0 : c(Q) \leq \theta\}$ is a closed set.*

Any increasing left continuous transportation cost function $t(.)$; i.e., $\lim_{Q_n \uparrow Q} t(Q_n) = t(Q)$, is lower semi-continuous. This implies that the FTL and LTL transportation cost functions are lower semi-continuous. Also, the all units discount scheme purchase price function (Zipkin, 2000; Hadley and Whitin, 1963), being right continuous and jumping downwards at discontinuity points, is lower semi-continuous. Since the sum of lower semi-continuous functions is lower semi-continuous, the next result covers the case that both the purchase price function and the transportation cost function are lower semi-continuous. Note that we introduce for convenience the notation $T \mapsto f(T)$ denoting a function mapping T to $f(T)$.

Lemma 3.1 *For any lower semi-continuous ordering cost function $c(.)$ and every $r \geq 0, 0 < b \leq \infty$, an optimal solution $T_c(b,r)$ of optimization problem ($P_{c,b,r}$) exists.*

Proof. By relation (9) it follows for every nonnegative cost ordering function $c(.)$ and $0 < b \leq \infty$ that $F_c(b,r,T) \geq \frac{a}{T} + h_b(0,T)$. This shows using $h_b(.,.)$ positive and $\lim_{T\uparrow\infty} h_b(0,T) = \infty$ that

$$\lim_{T\downarrow 0} F_c(b,r,T) = \lim_{T\uparrow\infty} F_c(b,r,T) = \infty. \tag{12}$$

Again by relation (9), the function $T \mapsto F_c(b,r,T)$ is lower semi-continuous for every $0 < b \leq \infty$. This implies by relation (12) and the Weierstrass-Lebesque theorem (Aubin, 1993) that an optimal solution exists. □

If we do not incur any backordering costs ($b = 0$), then it follows by (8) that the existence of a finite optimal T depends on the behavior of the ratio $c(T)T^{-1}$ as T goes to infinity. Without any additional information on the growth of the function $c(.)$ nothing can be said about this existence. Fortunately, incurring no penalty cost for not delivering in time hardly occurs in the real world and therefore we only consider in the remainder of this paper the case $b > 0$.

To compute the optimal solution, we observe the following. Contrary to the classical EOQ cost models having linear ordering cost functions, the objective function as a function of the cycle length T might not be unimodal anymore for general functions $c(.)$. Hence, the objective function may contain several local minima and so, it might be difficult to find an optimal solution or guarantee that a given solution is indeed optimal. Before trying to find a way of solving these global optimization problems, we will first identify in Section 4 the classes of ordering cost functions for which solving optimization problem ($P_{c,b,r}$) reduces to solving a convex optimization problem. These easy identifiable cases will then be used in Section 5 to solve the more difficult cases. Also, despite having difficulties of computing an optimal solution for increasing $c(.)$, one can still conclude that the optimal replenishment cycle length for an EOQ cost model with positive opportunity costs is smaller than the optimal replenishment cycle length of the same model with zero opportunity costs. This result shows that an upper bound on the optimal cycle length of an EOQ cost model with positive opportunity costs is always given by the optimal cycle length of the same model with zero opportunity costs. Since it will be shown in Section 4 that EOQ cost models with zero opportunity costs are in general easier to solve, the next structural result has also practical implications. These we discuss in Section 5, where EOQ cost models with increasing ordering cost functions are explored.

Lemma 3.2 *For any increasing lower semi-continuous ordering cost function $c(.)$ and any $r > 0$ and $0 < b \leq \infty$, there exists an optimal solution $T_c(b,r)$ of the optimization problem ($P_{c,b,r}$) satisfying $T_c(b,r) \leq T_c(b,0)$.*

Proof. By Lemma 3.1 an optimal solution of optimization problem ($P_{c,b,r}$) exists for any $r \geq 0$ and $b > 0$. To show the existence of an optimal solution $T_c(b,r)$ satisfying $T_c(b,r) \leq T_c(b,0)$, it is sufficient to show for any $T \geq T_c(b,0)$ that $F_c(b,r,T) \geq F_c(b,r,T_c(b,0))$. By relation (8) we obtain after some calculations that

$$F_c(b,r,T) - F_c(b,0,T) = \begin{cases} \frac{\lambda b^2}{2(h+b)}\left(\frac{1}{T} + \frac{\lambda(h+b)}{rc(\lambda T)}\right)^{-1}, & \text{if } 0 < b < \infty; \\ \frac{r}{2}c(\lambda T), & \text{if } b = \infty. \end{cases}$$

This shows for $c(.)$ increasing that the function $T \mapsto F_c(b,r,T) - F_c(b,0,T)$ is increasing. Applying this together with the definition of $T_c(b,0)$ we obtain for every $T \geq T_c(b,0)$

$$\begin{aligned} F_c(b,r,T) &= F_c(b,0,T) + F_c(b,r,T) - F_c(b,0,T) \geq F_c(b,0,T_c(b,0)) + F_c(b,r,T_c(b,0)) - F_c(b,0,T_c(b,0)) \\ &= F_c(b,r,T_c(b,0)) \end{aligned}$$

and we have verified the result. □

4. Easily solvable instances of optimization problem ($P_{c,b,r}$) related to convex optimization problems. In this section, we will identify classes of ordering cost functions $c(.)$, for which the optimization problem ($P_{c,b,r}$) is easy to solve. In particular, we will identify for which classes of ordering cost functions solving the optimization problem ($P_{c,b,r}$) reduces to solving a convex optimization problem.

It is well-known that if $c(.)$ is convex or concave on $(0,\infty)$, then it is also continuous on $(0,\infty)$ (Bazaraa et al., 1993). Thus, we infer from Lemma 3.1 that optimization problem ($P_{c,b,r}$) for every $0 < b \leq \infty$ has an optimal solution $T_c(b,r)$. If $\lim_{t\downarrow 0} c(t) = 0$, then it is easy to see for $c(.)$ convex on $(0,\infty)$ that the function $T \mapsto \frac{c(T)}{T}$ is increasing on $(0,\infty)$. Hence, for this case, we have diseconomies of scale in ordering. For concave $c(.)$ satisfying $\lim_{t\downarrow 0} c(t) = 0$, we obtain by a similar argument that we have economies of scale in ordering.

Diseconomies of scale in ordering might happen for example when ordering items from different suppliers. Economies of scale in ordering occur when a supplier or transporter uses a discount strategy. Notice in the classical EOQ cost model one uses a linear ordering cost function and so, no diseconomies or economies of scale is considered. Since it will turn out that EOQ cost models with general convex ordering cost functions and zero opportunity costs are much easier to analyze then the same models with positive opportunity costs, we will first consider EOQ cost models with zero opportunity costs.

4.1 Easy instances with zero opportunity costs. Zero opportunity costs would be relevant when the amount of money invested into a product is of no concern. As an example we mention ice cream where the out of pocket inventory holding cost due to cooling dominates the opportunity costs. In the next lemma, we will identify all the ordering cost functions $c(.)$ for which the optimization problem ($P_{c,b,r}$) is related to a convex minimization problem.

Lemma 4.1 *For zero opportunity costs and arrival rate $\lambda > 0$, the following holds:*

(*i*) *The function $T \mapsto \frac{c(T)}{T}$ is convex on $(0,\infty)$ if and only if the function $T \mapsto F_c(b,0,T)$ is convex on $(0,\infty)$ for every $h > 0$, $a > 0$ and $0 < b \leq \infty$.*

(*ii*) *The function $T \mapsto c(T)$ is convex on $(0,\infty)$ if and only if the function $T \mapsto F_c(b,0,T^{-1})$ is convex on $(0,\infty)$ for every $h > 0$, $a > 0$ and $0 < b \leq \infty$.*

Proof. By relation (8) and the observation that the pointwise limit of convex functions is convex (take both $a \downarrow 0$ and $h \downarrow 0$), the proof of the first result is obvious. For the proof of the second result we only give the proof for $0 < b < \infty$. The proof for $b = \infty$ is similar. It follows by relation (7) that the function $\varphi_b(T) = \frac{\lambda b h T^2}{2(b+h)}$ is convex on $(0, \infty)$. This implies by the convexity of $c(.)$ that the function $T \mapsto a + c(\lambda T) + \varphi_b(T)$ is convex on $(0, \infty)$. By the perspective property of convex functions (Boyd and Vandenberghe (2004)) also the function $T \mapsto T(a + c(\lambda T^{-1}) + \varphi_b(T^{-1}))$ is convex on $(0, \infty)$ and by relation (4), the function $T \mapsto F_c(b, 0, T^{-1})$ is convex on $(0, \infty)$. To verify the reverse implication, we first observe that the pointwise limit of convex functions is again convex. Using this and taking $h \downarrow 0$ and $a \downarrow 0$ in relation (8) with T replaced by T^{-1} it follows that the function $T \mapsto Tc(\lambda T^{-1})$ is convex on $(0, \infty)$. Applying again the perspective property of convex functions yields that the function $c(.)$ is convex on $(0, \infty)$. □

To discuss the relation between parts 1 and 2 of Lemma 4.1, we observe the following. If the nonnegative increasing function $Q \mapsto c(Q)$ is convex then it does not hold that the function $Q \mapsto \frac{c(Q)}{Q}$ is convex. An example of such a function is given by $c(Q) = Q^\alpha$ with $1 < \alpha < 2$. In this case the function $c(.)$ is convex but the function $Q \mapsto \frac{c(Q)}{Q}$ is concave. Also the condition $Q \mapsto \frac{c(Q)}{Q}$ is convex does not imply that $Q \mapsto c(Q)$ is convex. If we consider the function $c(Q) = Q^\alpha$ for any $0 < \alpha < 1$, then the function $c(.)$ is concave and the function $Q \mapsto \frac{c(Q)}{Q}$ is convex. This means by Lemma 4.1 that $T \mapsto F_c(b, 0, T)$ convex is not related to $T \mapsto F_c(b, 0, T^{-1})$ being convex. However, as explained in the following for both distinct cases one can easily solve the optimization problem ($P_{c,b,r}$).

It follows for $0 < b \leq \infty$ that

$$v(P_{c,b,0}) = \min\{F_c(b, 0, T^{-1}) : T > 0\},$$

where $v(P_{c,b,r})$ denotes the optimal objective value of optimization problem ($P_{c,b,r}$). This shows for zero opportunity costs and $c(.)$ convex on $(0, \infty)$ that an optimal solution $T_c(b, 0)$ of problem ($P_{c,b,r}$) is easy to compute after a transformation of the decision variable; that is, by replacing the replenishment cycle length T by the frequency of ordering T^{-1}. By part (ii) of Lemma 4.1 the new optimization problem is a convex optimization problem and so, we can apply a standard bisection method to compute its optimal value. The optimal value $T_c(b, 0)$ is then the reciprocal of this optimal solution. Also for zero opportunity costs and $T \mapsto \frac{c(T)}{T}$ convex on $(0, \infty)$ it follows by part (i) of Lemma 4.1 that the optimization problem ($P_{c,b,r}$) is a convex programming problem. Again this is easy to solve by standard bisection. As already observed, convex ordering functions are used to model diseconomies of scale in ordering. If the ordering cost function $c(.)$ is affine given by $c(Q) = \alpha Q + \beta$, $\alpha, \beta \geq 0$ then an easy formula exists for the optimal solution $T_c(b, 0)$. By relation (8), it is easy to check that

$$T_c(b, 0) = \sqrt[2]{\zeta(b) \frac{2(a + \beta)}{\lambda h}} \tag{13}$$

with

$$\zeta(b) = \begin{cases} \frac{h+b}{b}, & \text{if } 0 < b < \infty; \\ 1, & \text{if } b = \infty \end{cases} \tag{14}$$

and optimal objective value

$$v(P_{c,b,0}) = \lambda\alpha + \sqrt[2]{\frac{2\lambda h(a + \beta)}{\zeta(b)}}. \tag{15}$$

Focusing on economies of scale in both purchase and transportation, the class of polyhedral concave functions is popular in inventory control. This class describes incremental discounting either with respect to purchase costs or transportation costs or both. Also a polyhedral concave function can be used as a lower approximation of a general concave function representing a more general discounting scheme.

Definition 4.1 (Rockafellar (1972)) *A function $c : (0,\infty) \to \mathbb{R}$ is called polyhedral concave on $(0,\infty)$, if $c(.)$ can be represented as the minimum of a finite number of affine functions on $(0,\infty)$. It is called polyhedral concave on a convex interval I, if $c(.)$ is the minimum of a finite number of affine functions on I.*

Let $c(.)$ be a *positive increasing* polyhedral concave function on $(0,\infty)$ and define $J = \{1, ..., N\}$ for some $N \in \mathbb{N}$ with N denoting the total number of affine functions. Then, we can write

$$c(Q) = \min_{n \in J} c_n(Q), \tag{16}$$

where $c_n(Q) = \alpha_n Q + \beta_n$ with $\alpha_1 > ... > \alpha_N \geq 0$, and $0 \leq \beta_1 < \beta_2 < ... < \beta_N$. If $c(.)$ is a *positive increasing* polyhedral function on some convex interval $I \subseteq (0,\infty)$, relation (16) still applies even if some β_n values are negative. Applying now relations (10) and (16) and Lemma 4.1 the next result is obtained.

Lemma 4.2 *For zero opportunity costs and arrival rate $\lambda > 0$, it follows for any positive increasing polyhedral concave function $c(.)$ given by (16) that*

$$F_c(b, 0, T) = \min_{n \in J} F_{c_n}(b, 0, T) \tag{17}$$

for every $T > 0$. Also for each $n \in J$, the function $T \mapsto F_{c_n}(b, 0, .)$ is convex on $(0,\infty)$.

By Lemma 4.2 it follows for any positive increasing polyhedral concave function $c(.)$ on $(0,\infty)$ given by relation (16) that

$$v(P_{c,b,0}) = \min_{T>0} F_c(b, 0, T) = \min_{n \in J} \min_{T>0} F_{c_n}(b, 0, T). \tag{18}$$

Also by the same lemma, the optimization problem in relation (18) can be easily solved by solving N convex optimization problems, $\min_{T>0} F_{c_n}(b, 0, T)$. Due to the affine structure of $c_n(.)$, it follows by relations (13), (15) and (18) with α replaced by α_n and β by β_n that

$$T_{c_n}(b, 0) = \sqrt[2]{\zeta(b) \frac{2(a + \beta_n)}{\lambda h}}$$

and

$$v(P_{c,b,0}) = \min_{n \in J} \left\{ \lambda \alpha_n + \sqrt[2]{\frac{2\lambda h(a + \beta_n)}{\zeta(b)}} \right\}. \tag{19}$$

Hence, the optimal $T_c(b, 0)$ is given by $T_{c_{n^*}}(b, 0)$ with n^* being the index minimizing the expression in relation (19). Hence for zero opportunity cost and a polyhedral concave ordering cost function this optimization is extremely easy and almost analytically solvable. If we replace an increasing polyhedral concave ordering cost function $c(.)$ satisfying $c(0) = 0$ by an increasing finite valued concave ordering cost function $c(.)$ satisfying $c(0) = 0$ the optimization problem ($P_{c,b,r}$) with $r = 0$ becomes, in general, a global optimization problem. To show this, we introduce for any $0 < x < \infty$ and $c(.)$ a concave increasing function on $(0,\infty)$ the right derivative and the left derivative given by

$$c'_+(x) = \lim_{y \downarrow x} \frac{c(y) - c(x)}{y - x} \text{ and } c'_-(x) = \lim_{y \uparrow x} \frac{c(y) - c(x)}{y - x},$$

respectively. For concave functions, a generalization of relation (16) is given by

$$c(Q) = \inf_{y \in J}\{yQ - c^*(y)\}$$

with $c'_+(0) := \lim_{y \downarrow 0} c'_+(y) \geq c'_-(\infty) := \lim_{y \uparrow \infty} c'_+(y) \geq 0$, $J = [c'_-(\infty), c'_+(0)]$ and

$$c^*(y) := \inf\{yx - c(x) : x \geq 0\}$$

(Rockafellar, 1972; Roberts and Varberg, 1973). Since $c(.)$ is increasing and satisfies $c(0) = 0$, the so-called conjugate function $c^*(.)$ is increasing, concave and non-positive. Applying now the observation after relation (11), we obtain the following generalization of relation (19) given by

$$v(P_{c,b,0}) = \inf_{c'_-(\infty) \leq y \leq c'_+(0)} \left\{ \lambda y + \sqrt[2]{\frac{2\lambda h(a - c^*(y))}{\zeta(b)}} \right\}. \tag{20}$$

The function $y \mapsto \frac{2\lambda h(a-c^*(y))}{\zeta(b)}$ is a decreasing convex function and the function $y \mapsto \sqrt[2]{y}$ is an increasing concave function for $y \geq 0$. Therefore the objective function $y \mapsto \left\{\lambda y + \sqrt[2]{\frac{2\lambda h(a-c^*(y))}{\zeta(b)}}\right\}$ in relation (20) has in general no nice convexity-type properties and the optimization problem ($P_{c,b,r}$) for $r = 0$ is a global optimization problem. This optimization problem can only be solved approximately unless the infimum in relation (20) is taken over a finite number as in the polyhedral concave case.

In the next subsection we will determine easy instances for EOQ cost models with positive opportunity costs.

4.2 Easy instances with positive opportunity costs. For positive opportunity costs with no shortages, one can verify under some additional monotonicity conditions, a similar result as in Lemma 4.1.

Lemma 4.3 *For positive opportunity costs with no shortages ($b = \infty$) and arrival rate $\lambda > 0$, the following holds:*

(*i*) *Let $T \mapsto \frac{c(T)}{T}$ be an increasing function, then the function $T \mapsto \frac{c(T)}{T}$ is convex on $(0, \infty)$ if and only if the function $T \mapsto F_c(\infty, r, T)$ is convex on $(0, \infty)$ for every $r > 0, h > 0$ and $a > 0$.*

(*ii*) *Let $c(.)$ be an increasing function, then the function $T \mapsto c(T)$ is convex on $(0, \infty)$ if and only if the function $T \mapsto F_c(\infty, r, T^{-1})$ is convex on $(0, \infty)$ for every $r > 0, h > 0$ and $a > 0$.*

Proof. The crucial observation in both proofs is that the product of two increasing univariate convex functions is again convex (Boyd and Vandenberghe, 2004). Applying this observation to part (*i*), we observe by the monotonicity and convexity of the function $T \mapsto \frac{c(T)}{T}$ that the function $c(.)$ is convex on $(0, \infty)$. This implies by relation (8) that the function $T \mapsto F_c(\infty, r, T)$ is convex on $(0, \infty)$ for every $r > 0, h > 0$ and $a > 0$. To prove the reverse implication in (*i*), we observe using again the pointwise limit of convex functions is convex and taking $r \downarrow 0$ that the function $T \mapsto F_c(\infty, 0, T)$ is convex on $(0, \infty)$. Applying now the first part of Lemma 4.1, we conclude that the function $T \mapsto \frac{c(T)}{T}$ is convex on $(0, \infty)$. To show part (*ii*), we observe by the monotonicity of $c(.)$ and the first observation in this proof that $T \mapsto c(T)T$ is convex on $(0, \infty)$. This shows by relation (7) that the function $\varphi_\infty(.)$ is convex on $(0, \infty)$ and so, the function $T \mapsto a + c(\lambda T) + \varphi_\infty(T))$ is convex on $(0, \infty)$. Applying now the perspective property of convex functions, we obtain that $T \mapsto T(a + c(\lambda T^{-1}) + \varphi_\infty(T^{-1}))$ is convex on $(0, \infty)$ and by (4) we have verified that $T \mapsto F_c(\infty, r, T^{-1})$ is convex on $(0, \infty)$ for every $r > 0, h > 0$ and $a > 0$. The reverse implication can be shown taking $r \downarrow 0$ thereby preserving convexity and applying part (*ii*) of Lemma 4.1. □

Again for the special case that $c(.)$ is given by $c(Q) = \alpha Q + \beta$, $\alpha, \beta \geq 0$ it follows by relation (8) that

$$T_c(\infty, r) = \sqrt[2]{\frac{2(a+\beta)}{\lambda(h+r\alpha)}} \tag{21}$$

and

$$v(P_{c,\infty,r}) = \alpha\lambda + \frac{r\beta}{2} + \sqrt{2(a+\beta)(r\alpha+h)\lambda}. \tag{22}$$

If we are dealing with economies of scale in ordering and the function $c(.)$ is a positive increasing polyhedral concave function on $(0, \infty)$ given by relation (16), we obtain again applying relation (10) that

$$F_c(b, r, T) = \min_{n \in J} F_{c_n}(b, r, T). \tag{23}$$

Since c_n is an affine function it follows by relations (10) and (22) that the optimal objective value $v(P_{c,\infty,r})$ of any EOQ cost model with positive opportunity costs and no shortages is given by

$$v(P_{c,\infty,r}) = \min_{n \in J} \left\{ \alpha_n \lambda + \frac{r\beta_n}{2} + \sqrt[2]{2(a+\beta_n)(r\alpha_n + h)\lambda} \right\}. \tag{24}$$

Also by relation (21) it follows that an optimal solution of this optimization problem is given by

$$T_{c_{n^*}}(\infty, r) = \sqrt[2]{\frac{2(a+\beta_{n^*})}{\lambda(h+r\alpha_{n^*})}}, \tag{25}$$

where n^* is the index minimizing the expression in relation (24). Unfortunately, for $c(.)$ concave, we obtain by the same technique as used for zero opportunity costs that the optimization problem is in general a global optimization problem.

For positive opportunity costs and finite backorder costs ($b < \infty$) one can only show the following convexity result for increasing affine ordering cost functions with a nonnegative constant term. Contrary to the no shortages case, it can be shown by means of a counter example that the function φ_b is not convex for increasing convex ordering cost functions, positive shortages and finite b. Therefore, we cannot apply the same proof as in Lemma 4.3.

Lemma 4.4 *For positive opportunity costs and finite backorder costs and $c(Q) = \alpha Q + \beta, \alpha \geq 0, \beta \geq 0$ for every $Q > 0$ it follows for any $\lambda > 0$ that the function $T \mapsto F_c(r, b, T^{-1})$ is convex on $(0, \infty)$.*

Proof. It follows by relation (7) for every $T > 0$ that

$$\varphi_b(T) = \frac{\lambda^2 b(h+r\alpha)T^3}{2(b+h+r\alpha)\lambda T + 2\beta} + \frac{\lambda b\beta T^2}{2(b+h+r\alpha)\lambda T + 2\beta}. \tag{26}$$

Since the ratio of a squared nonnegative convex function and a positive concave function is convex (Bector, 1968), it follows that the functions $T \mapsto T^3((b+h+r\alpha)\lambda T + \beta)^{-1}$ and $T \mapsto T^2((b+h+r\alpha)\lambda T + \beta)^{-1}$ are convex on $(0, \infty)$. Hence, using $b \geq 0$ and $\beta \geq 0$, we obtain by relation (26) that the function $\varphi_b(.)$ is convex on $(0, \infty)$. This shows that $T \mapsto a + c(\lambda T) + \varphi_b(T)$ is convex on $(0, \infty)$ and by the perspective property of convex functions the function $T \mapsto T(a + Tc(\lambda T^{-1}) + \varphi_b(T^{-1}))$ is also convex on $(0, \infty)$. By applying relation (4), the desired result follows. □

By relation (4) and (26) we obtain for positive opportunity costs, affine ordering cost and b finite that

$$F_c(b, r, T) = \alpha\lambda + \frac{\beta + a}{T} + \frac{\lambda^2 b(h+r\alpha)T^2 + \lambda b\beta T}{2(b+h+r\alpha)\lambda T + 2\beta}$$

Now it is not possible anymore as for the other cases (zero opportunity costs or b infinite) to write down an easy expression for $T_c(b,r)$ but due to Lemma 4.4 it is easy to solve the optimization problem $(P_{c,b,r})$. Again if we have economies of scale and consider a polyhedral concave function $c(.)$ given by relation (16), then it follows that

$$F_c(b,r,T) = \min_{n\in J} F_{c_n}(b,r,T).$$

Hence, by Lemma 4.4 we need to solve N convex optimization problems $\min F_{c_n}(b,r,T^{-1})$ to determine the length of an optimal replenishment cycle. As previously mentioned, due to the positive opportunity costs and b finite, there exists no easy elementary formula for the optimal solution and the optimal objective value.

In the next section we will first derive for arbitrary ordering cost functions a bounded interval containing an optimal solution. Then, we will show for several well-known examples how to use this dominance result in a solution procedure. The first example is given by the famous carload discount schedule and then we consider some generalizations. Observe for the more general ordering cost functions, the objective function $T \mapsto F_c(b,r,T)$ lacks any convexity property on $(0,\infty)$.

5. Instances of optimization problem ($P_{c,b,r}$) related to global optimization problems. In this section we first show for arbitrary ordering cost functions $c(.)$ a so-called dominance result. This dominance result implies that one can determine a bounded interval containing an optimal solution of the optimization problem $(P_{c,b,r})$. In the next lemma, it is implicitly assumed that an optimal solution for the optimization problem $(P_{c,b,r})$ exists. As shown in Lemma 3.1 this holds for $b > 0$ and $c(.)$ lower semi-continuous. Special cases are $c(.)$ increasing and left continuous on $(0,\infty)$ or $c(.)$ continuous on $(0,\infty)$.

Lemma 5.1 *The following results hold:*

(*i*) *If there exists some $d > 0$ and an ordering cost function $c_0(.)$ satisfying $c(\lambda T) \geq c_0(\lambda T)$ for every $T \geq d$ and $c(\lambda d) = c_0(\lambda d)$ and the function $T \mapsto F_{c_0}(b,r,T)$ is increasing on $[T_*,\infty)$, $T_* \leq d$, then an optimal solution of $\min_{T>0} F_c(b,r,T)$ is contained in the interval $(0,d]$.*

(*ii*) *If there exists some $d > 0$ and an ordering cost function $c_0(.)$ satisfying $c(\lambda T) \geq c_0(\lambda T)$ for every $T \leq d$ and $c(\lambda d) = c_0(\lambda d)$ and the function $T \mapsto F_{c_0}(b,r,T)$ is decreasing on $[T_{**},\infty)$, $T_{**} \geq d$, then an optimal solution of $\min_{T>0} F_c(b,r,T)$ is contained in the interval $[d,\infty)$.*

Proof. To show part (i), we observe using relation (8) and $c(\lambda T) \geq c_0(\lambda T)$ for every $T \geq d$ that $F_c(b,r,T) \geq F_{c_0}(b,r,T)$ for every $T \geq d$. Since $T \mapsto F_{c_0}(b,r,T)$ is increasing on $[T_*,\infty)$ with $T_* \leq d$ and $c(\lambda d) = c_0(\lambda d)$, it also follows for every $T \geq d$ that $F_{c_0}(b,r,T) \geq F_{c_0}(b,r,d) = F_c(b,r,d)$. Hence we have $F_c(b,r,T) \geq F_c(b,r,d)$ for every $T \geq d$ and this proves part (i). The second part can be proved similarly. □

For any ordering cost function $c_0(.)$ being a minorant of the function $c(.)$ on some interval $[d,\infty)$ or $(0,d]$ and $T \mapsto F_{c_0}(b,r,T)$ being unimodal on $(0,\infty)$, one can apply the above dominance result. It is well known that unimodality holds if the functions $T \mapsto F_{c_0}(b,r,T)$ or $T \mapsto F_{c_0}(b,r,T^{-1})$ are convex on $(0,\infty)$. Notice in Lemma 4.1, 4.3 and 4.4 examples of ordering cost functions are considered for which the objective function satisfies these convexity properties. It is then easy to solve optimization problem $(P_{c_0,b,r})$ and we obtain that the objective function is unimodal with unimodality point $T_{c_0}(b,r)$. Now we can apply Lemma 5.1 with T_* or T_{**} equal to $T_{c_0}(b,r)$. A similar approach, but with T_* or T_{**} differently chosen than the optimal solution of optimization problem $(P_{c_0,b,r})$, can also be used for the polyhedral concave ordering cost functions $c_0(.)$ considered in Lemma 4.2 and the result discussed after Lemma 4.4. In the following lemma we give an example of this approach using both Lemma 5.1 and the convexity results of Section 4.

Lemma 5.2 *If for a ordering cost function $c(.)$ there exists an affine ordering cost function $c_0(Q) = \alpha Q + \beta$, $\alpha > 0, \beta \geq 0$ satisfying $c(\lambda T) \geq c_0(\lambda T)$ for every $T > 0$ and an increasing sequence $d_n \uparrow \infty, n \in \mathbb{Z}_+, d_0 := 0$ satisfying $c(\lambda d_n) = c_0(\lambda d_n)$ for every $n \in \mathbb{N}$, then an optimal solution of problem $\min_{T>0} F_c(b, r, T)$ is contained in the interval $[d_{n_*}, d_{n_*+1}]$ with $n_* = \max\{n \in \mathbb{Z}_+ : d_n \leq T_{c_0}(b, r)\}$*

Proof. It follows for any EOQ-type model with positive or zero opportunity costs that by part (i) of Lemma 4.1, part (ii) of Lemma 4.3 and Lemma 4.4 that either the function $T \mapsto F_{c_0}(b, r, T)$ is convex or $T \mapsto F_{c_0}(b, r, T^{-1})$ is convex on $(0, \infty)$. This shows for both by a standard result for convex functions that the function $T \mapsto F_{c_0}(b, r, T)$ is decreasing on $(0, T_{c_0}(b, r)]$ and increasing on $[T_{c_0}(b, r), \infty)$. Applying now part (i) of Lemma 5.1 with $T_* = T_{c_0}(b, r)$ and $d = d_{n_*+1}$ and part (ii) of the same lemma with $T_{**} = T_{c_0}(b, r)$ and $d = d_{n_*}$, we obtain the desired result. □

By the analysis in Section 4, it is easy to calculate the optimal solution $T_{c_0}(b, r)$ for any affine function $c_0(.)$. In the next example using Lemma 5.2, we will come up with a fast algorithm for an EOQ cost model with positive or zero opportunity costs and a so-called carload discount schedule (Nahmias, 1997). This generalizes and simplifies the analysis of a less general model discussed in Section 2 of (Rieskts and Ventura, 2008). Observe in Section 2 of (Rieskts and Ventura, 2008) only an EOQ cost model with zero opportunity costs, no shortages ($b = \infty$) and a less general carload discount schedule is considered.

Example 5.1 (Carload Discount Schedule With Identical Trucks and Setup Costs) *Let $C > 0$ be the truck capacity, $g : (0, C] \to \mathbb{R}$ be an increasing polyhedral concave function satisfying $g(0) = 0$ and $s \geq 0$ be the setup cost of using one truck. Here, $g(Q)$ corresponds to the transportation cost for transporting an order of size Q with $0 < Q \leq C$. If no discount is given on the number of used (identical) trucks, then the total transportation cost function $t : [0, \infty) \to \mathbb{R}$ has the form*

$$t(Q) = \begin{cases} 0, & \text{if } Q = 0; \\ g(Q) + s, & \text{if } 0 < Q \leq C; \\ ng(C) + g(Q - nC) + (n+1)s, & \text{if } nC < Q \leq (n+1)Q, n \in \mathbb{N}. \end{cases}$$

When we use the above transportation function $t(.)$ with a linear purchase function $p(.)$, and consider $c(Q) = t(Q) + p(Q)$, then we obtain an ordering cost function $c(.)$ similar to the one shown in Figure 2. To derive a lower bounding function $c_0(.)$ for the function $c(.)$, we observe $t(Q) \geq t_0(Q)$ for every $Q \geq 0$ with

$$t_0(Q) := \frac{g(C) + s}{C} Q.$$

Also for $d_n := \lambda^{-1} nC$, $n \in \mathbb{N}$, the equality $t(\lambda d_n) = t_0(\lambda d_n)$ holds for every $n \in \mathbb{N}$. If the price of each ordered item equals $p > 0$ (no quantity discount), it follows that the ordering cost function $c(.)$ is given by $c(Q) = t(Q) + pQ$ and the lower bounding function $c_0(.)$ of $c(.)$ has the form

$$c_0(Q) = t_0(Q) + pQ = \left(\frac{g(C) + s}{C} + p \right) Q \tag{27}$$

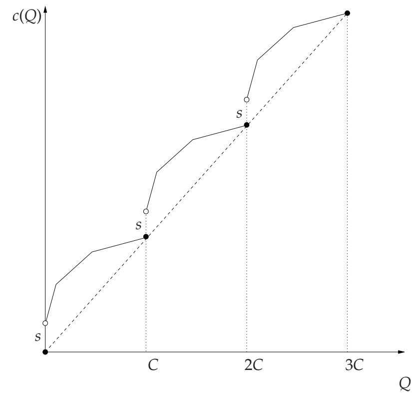


Figure 2: An ordering cost function for carload discount schedule with identical trucks.

and satisfies

$$c(\lambda d_n) = c_0(\lambda d_n) \tag{28}$$

for every $n \in \mathbb{N}$. *Applying now Lemma 5.2 (take* $\beta = 0$ *and* $\alpha = (g(C) + s)C^{-1} + p$*) an optimal solution* $T_c(b, r)$ *of the optimization problem (*$P_{c,b,r}$*) is contained within the interval* $[d_{n_*}, d_{n_*+1}]$ *with* $d_n = \lambda^{-1}nC$ *and*

$$n_* = \max\{n \in \mathbb{Z}_+ : \lambda^{-1}nC \le T_{c_0}(b, r)\} = \lfloor \lambda T_{c_0}(b, r)C^{-1} \rfloor, \tag{29}$$

where $\lfloor x \rfloor$ *denotes the largest integer, smaller than or equal to* x. *Clearly the value* $n_* + 1$ *represents the number of trucks to be used to transport the optimal order quantity. In particular, if we consider the EOQ-model with zero opportunity cost, we obtain using relation (13) with* $\beta = 0$ *and* $\alpha = C^{-1}(g(C) + s) + p$ *that*

$$T_{c_0}(b, 0) = \sqrt[2]{\zeta(b)\frac{2a}{\lambda h}}. \tag{30}$$

Also, for the no shortages case ($b = \infty$*) we obtain for* $r \ge 0$ *by relation (21) that*

$$T_{c_0}(\infty, r) = \sqrt{\frac{2a}{\lambda(h + rp + r(g(C) + s)C^{-1})}}. \tag{31}$$

Finally, for the most general EOQ-type model with shortages and positive opportunity cost rate r, *there exists a fast algorithm to compute the optimal solution* $T_{c_0}(b, r)$. *Hence, by using relation (29), it is very easy to determine the optimal number* $n_* + 1$ *of trucks to be used. If it holds additionally in relation (29) that the total order size* $\lambda T_{c_0}(b, r)$ *is a multiple of the capacity* C *and so,* $n_* = \lambda T_{c_0}(b, r)C^{-1}$, *then it follows by using* $c(\lambda d_{n_*}) = c_0(\lambda d_{n_*})$ *and* $c(.) \ge c_0(.)$ *that an optimal solution of the optimization problem (*$P_{c,b,r}$*) is given by* $T_{c_0}(b, r)$. *Otherwise, we have to solve the constrained optimization problem* $\inf_{d_{n_*} < T \le d_{n_*+1}} F_c(b, r, T)$ *with* $c(.)$ *polyhedral concave on* $I = (d_{n_*}, d_{n_*+1}]$. *Since an optimal solution is contained in the interval* $[d_{n_*}, d_{n_*+1}]$ *we need to compare the objective value of* $\inf_{d_{n_*} < T \le d_{n_*+1}} F_c(b, r, T)$ *with the single value* $F_c(b, r, d_{n_*})$ *and select that one with the lowest objective value. The associated decision variable* T *achieving this minimum value is then the optimal solution. By relation (16), it follows for every* $d_{n_*} < T \le d_{n_*+1}$ *that* $c(\lambda T) = \min_{1 \le n \le N}\{c_n(\lambda T)\}$ *with*

$$c_n(\lambda T) = \alpha_n \lambda T + \beta_n \tag{32}$$

and $\alpha_1 > ... > \alpha_N > 0$ *and* $\beta_1 < ... < \beta_N$. *Note that some* β_n *values can be negative. This implies by relation (10)*

$$\min_{d_{n_*} < T \le d_{n_*+1}} F_c(b, r, T) = \min_{1 \le n \le N} \inf_{d_{n_*} < T \le d_{n_*+1}} F_{c_n}(b, r, T).$$

Hence to determine $\inf_{d_{n_*}<T\leq d_{n_*+1}} F_c(b,r,T)$ *we need to solve N optimization problems* $\min_{d_{n_*}\leq T\leq d_{n_*+1}} F_{c_n}(b,r,T)$. *We first discuss how to solve optimization problem* $\min_{d_{n_*}\leq T\leq d_{n_*+1}} F_{c_n}(b,r,T)$ *for the zero opportunity cost case. Notice by relation* (8) *that for zero opportunity costs*

$$F_{c_n}(b,0,T) = \lambda\alpha_n + (a+\beta_n)T^{-1} + \frac{h\lambda T}{2\zeta(b)} \tag{33}$$

for every $d_{n_*} \leq T \leq d_{n_*+1}$. *Since* β_n *can be negative we distinguish two different cases. If* $a+\beta_n \leq 0$ *it is easy to see using relation* (33) *that the function* F_{c_n} *is increasing on* $(d_{n_*}, d_{n_*+1}]$ *and this shows for any* $b \leq \infty$ *that an optimal solution* T_n^* *of optimization problem* $\min_{d_{n^*}\leq T\leq d_{n^*+1}} F_{c_n}(b,0,T)$ *is given by* d_{n_*}. *Hence, we consider the case* $a+\beta_n > 0$ *and by relation* (33), *the function* $T \mapsto F_c(b,0,T)$ *is convex on* $(0,\infty)$. *Also by relation* (13), *we know that the optimal solution of optimization problem* $(P_{c_n,b,0})$ *is given by* $T_{c_n}(b,0) = \sqrt[2]{\frac{2(a+\beta_n)}{\lambda h}\zeta(b)}$. *By the convexity of the function* F_c *it follows that* F_c *is decreasing on* $(0, T_{c_n}(b,0)]$ *and increasing on* $[T_{c_n}(b,0),\infty)$. *This implies that an optimal solution* T_n^* *of optimization problem* $\min_{d_{n_*}\leq T\leq d_{n_*+1}} F_{c_n}(b,0,T)$ *is given by*

$$T_n^* = \begin{cases} d_{n_*}, & \text{if } T_{c_n}(b,0) \leq d_{n_*}; \\ T_{c_n}(b,0), & \text{if } d_{n_*} < T_{c_n}(b,0) \leq d_{n_*+1}; \\ d_{n_*+1}, & \text{if } T_{c_n}(b,0) > d_{n_*+1}. \end{cases} \tag{34}$$

If we consider an EOQ-type model with positive opportunity costs and no shortages $(b=\infty)$ *we obtain by relation* (8) *that*

$$F_{c_n}(\infty,r,T) = \lambda\alpha_n + \frac{1}{2}r\beta_n + (a+\beta_n)T^{-1} + \frac{1}{2}(r\alpha_n + h)\lambda T.$$

Hence for $a+\beta_n \leq 0$ *it follows that the optimal solution* T_n^* *of optimization problem* $\min_{d_{n_*}\leq T\leq d_{n_*+1}} F_{c_n}(\infty,0,T)$ *is given by* d_{n_*}. *Also for* $a+\beta_n > 0$ *we obtain by relation* (21) *that the optimal solution* $T_{c_n}(\infty,r)$ *of optimization problem* $(P_{c_n,\infty,r})$ *is given by*

$$T_{c_n}(\infty,r) = \sqrt[2]{\frac{2(a+\beta_n)}{\lambda(h+r\alpha_n)}}.$$

Using a similar argument as for the zero opportunity cost case this shows that an optimal solution T_n^* *of optimization problem* $\min_{d_{n_*}\leq T\leq d_{n_*+1}} F_{c_n}(b,0,T)$ *is given by*

$$T_n^* = \begin{cases} d_{n_*}, & \text{if } T_{c_n}(\infty,r) \leq d_{n_*}; \\ T_{c_n}(\infty,0), & \text{if } d_{n_*} < T_{c_n}(\infty,r) \leq d_{n_*+1}; \\ d_{n_*+1}, & \text{if } T_{c_n}(\infty,r) > d_{n_*+1}. \end{cases} \tag{35}$$

Finally, for positive opportunity costs with shortages, it follows by Lemma 4.4 *and* $c_n(.)$ *given by relation* (32) *with* $\beta_n \geq 0$ *that the function* $T \mapsto F_{c_n}(b,r,T)$ *is decreasing on* $(0, T_{c_n}(b,r)]$ *and increasing on* $[T_{c_n}(b,r),\infty)$. *Hence, as before the optimization problem* $\min_{d_{n_*}\leq T\leq d_{n_*+1}} F_{c_n}(b,r,T)$ *can be solved easily. Finally, for* β_n *negative the function* F_{c_n} *might not be unimodal on* $[d_{n_*}, d_{n_*+1}]$ *and so in this case we should apply to* $\min_{d_{n_*}\leq T\leq d_{n_*+1}} F_{c_n}(b,r,T)$ *a one dimensional Lipschitz optimization procedure common in global optimization. (Horst et al., 1995). Algorithm 1 summarizes the details of solving the carload discount schedule with identical trucks.*

For general ordering cost functions, like the well-known carload discount schedule, the objective function lacks convexity properties. However, despite the nonconvexity of the function $c(.)$, we are still able to solve this problem by solving a finite number of restricted convex optimization problems for particular cases of the carload discount

Algorithm 1: Finding $T_c(b,r)$ for carload discount schedule with identical trucks

$T^* = \arg\min_{T>0} F_{c_0}(b,r,T)$;

if *λT^* is not an integer multiple of C* **then**

  $n_* = \lfloor \lambda T^* C^{-1} \rfloor$;

  $T^* = \arg\min \left\{ \inf_{d_{n_*} < T \leq d_{n_*+1}} F_c(b,r,T), F_c(b,r,d_{n_*}) \right\}$;

$T_c(b,r) \leftarrow T^*$;

schedule. Clearly, for arbitrary increasing ordering cost functions, the problem becomes much more difficult and in general reduces to a one-dimensional global optimization problem. In the next subsection, we will propose a solution method for those problems.

5.1 Construction of upper bound on the optimal cycle length for any increasing ordering cost function. For increasing left continuous ordering cost functions $c(.)$ the problem ($P_{c,b,r}$) reduces to a univariate global optimization problem. Hence a possible strategy to solve such a problem is to determine an upper bound on the optimal order cycle length and then apply a Lipschitz optimization procedure (Horst et al., 1995) on this interval. By practical considerations it might be clear that one always will order at least once in every year and so in this case an upper bound is clear. If this holds, the optimization problem is already restricted to a bounded interval and we apply immediately the Lipschitz procedure. If this does not hold, then selecting an upper bound solely based on intuition might not guarantee that this is indeed a real upper bound. To reduce this risk, the decision maker might select a much larger upper bound than necessary and this will increase for the general case the computation time of the Lipschitz discretization procedure. Therefore, it is useful to have an easy algorithm at hand which yields an upper bound on the optimal cycle length. In the next lemma, we show that under an affine bounding condition natural for an economies of scale situation, an *elementary formula* only depending on the data of the EOQ cost can be given. Observe that this is an alternate procedure to obtain an upper bound on the optimal cycle length to that provided in Lemma 3.2. This upper bound should be applied, if the EOQ model with zero opportunity cannot be solved easily and one can easily determine the affine bounding function. Such an example is given by $c(.)$ concave. Other examples of ordering cost functions for which one can construct an affine bounding function are listed in the appendix.

Lemma 5.3 *For any EOQ type model with zero or positive opportunity costs, demand rate $\lambda > 0$, backorder cost rate $0 < b \leq \infty$ and increasing left continuous ordering cost function $c(.)$ satisfying $c(Q) \leq \alpha Q + \beta$ for some $\alpha, \beta > 0$ an elementary upper bound on the optimal replenishment cycle length $T_c(b,0)$ is given by*

$$w_{\alpha,\beta}(b) = \alpha h^{-1}\zeta(b) + \sqrt[2]{\alpha^2 h^{-2}\zeta^2(b) + 2h^{-1}\lambda^{-1}(a+\beta)\zeta(b)}, \tag{36}$$

where $\zeta(b)$ is given by (14).

Proof. By Lemma 3.2 it is sufficient to construct the upper bound for zero opportunity costs. To construct this bound, we only give the proof for finite $b > 0$. The proof for $b = \infty$ is similar. Introduce for any $d > 0$ the constant ordering cost function $c_d : [0, \infty) \mapsto \mathbb{R}$ given by $c_d(Q) = c(\lambda d)$. By relation (8) we obtain

$$F_c(b, 0, T) = T^{-1}(a + c(\lambda d)) + \frac{h\lambda T}{2\zeta(b)}$$

and so the function $T \mapsto F_{c_d}(b, 0, T)$ is convex. Also the optimal replenishment cycle length $T_{c_d}(b, 0)$ is given by

$$T_{c_d}(b, 0) = \sqrt[2]{\frac{2(a + c(\lambda d))\zeta(b)}{\lambda h}} \tag{37}$$

and this shows by the convexity that the function $T \mapsto F_{c_d}(b, 0, T)$ is increasing on $[T_{c_d}(b, 0), \infty)$. Since $c(.)$ is increasing we also obtain $c(\lambda T) \geq c_d(\lambda T)$ for every $T \geq d$ and $c(\lambda d) = c_d(\lambda d)$ and so, we may apply Lemma 5.1. Hence, for every d satisfying $T_{c_d}(b, 0) \leq d$, an optimal solution of optimization problem $(P_{c,b,0})$ is contained in the interval $[0, d]$. This shows by relation (37) that every element in set D, defined as

$$D = \left\{ d \geq 0 : \sqrt[2]{\frac{2(a + c(\lambda d))\zeta(b)}{\lambda h}} \leq d \right\} = \{d \geq 0 : c(\lambda d) \leq \frac{\lambda h d^2}{2\zeta(b)} - a\}, \tag{38}$$

is an upper bound on an optimal solution of optimization problem $(P_{c,b,0})$. Using the bounding condition $c(Q) \leq \alpha Q + \beta$ with $\alpha, \beta > 0$ we obtain by relation (38) that the closed convex set

$$D_{\alpha,\beta} = \{d \geq 0 : \alpha\lambda d + \beta \leq \frac{\lambda h d^2}{2\zeta(b)} - a\} = [w_{\alpha,\beta}(b), \infty)$$

satisfies $D_{\alpha,\beta} \subseteq D$. Hence we may conclude that also every element of $D_{\alpha,\beta}$ is an upper bound on an optimal solution of optimization problem $(P_{b,0})$. This shows the result. □

If the ordering cost function $c(.)$ does not satisfy an affine bounding condition, then it is shown in the proof of Lemma 5.3 that for D nonempty one can find a finite upper bound on $T_c(b, r)$. In particular the value

$$d_{\min} = \min\{d \geq 0 : c(\lambda d) \leq \frac{\lambda h d^2}{2\zeta(b)} - a\} \tag{39}$$

is an upper bound. Also, if the affine bounding condition on $c(.)$ holds, this upper bound d_{min} is tighter than the elementary upper bound given in Lemma 5.3. However, to compute this tighter upper bound by means of an algorithm might be time consuming unless $c(.)$ belongs to a certain class of functions. We will now give an easy algorithm for $c(.)$ given by relation (16). Since $c(.)$ is concave and increasing, and the function $d \mapsto \frac{h\lambda d^2\zeta}{2} - a$ is strictly convex and increasing on $[0, \infty)$, the region D in relation (38) is represented by the interval $[d_{min}, \infty)$. It is easy to see that algorithm 2 yields d_{min} as an output.

Algorithm 2: Finding $d_{\min}$ for polyhedral concave $c(.)$

$n_* := \max\{0 \leq n \leq N - 1 : c(k_n) > \frac{hk_n^2\zeta}{2\lambda} - a\}$;

Determine in $[k_{n_*}, k_{n_*+1}]$ or in $[k_{n_*}, \infty)$ the unique analytical solution d_* of the equation

$$\alpha_{n_*+1}\lambda d + \beta_{n_*+1} = \frac{h\lambda d^2\zeta}{2} - a$$

given by

$$d_* = \frac{\alpha_{n_*+1}\lambda + \sqrt{(\alpha_{n_*+1}\lambda)^2 + 2h\lambda\zeta(a + \beta_{n_*+1})}}{h\lambda\zeta};$$

$d_{\min} \leftarrow d_*$;

Clearly, we have

$$\min_{T>0} F_c(b,r,T) = \min_{T\in I} F_c(b,r,T) \tag{40}$$

for the constructed bounded interval I containing an optimal solution. In practice it might also happen that we only have a finite number of possibilities in the interval I and in this case we can solve our optimization problem to optimality. Since we are interested in finding an optimal solution, we could now apply a one-dimensional Lipschitz optimization algorithm (Horst et al., 1995).

In the next definition we introduce a large class of ordering cost functions for which this general Lipschitz optimization procedure proposed above for increasing ordering cost functions can be improved. As a subclass this class contains the carload discount ordering cost functions considered in Example 5.1 and the ordering cost functions discussed in Section 2. An illustration of a function belonging to this class is given in Figure 3.

Definition 5.1 *A finite valued function $c:(0,\infty)\to\mathbb{R}$ is called a piecewise polyhedral concave function, if there exists a strictly increasing sequence q_n, $n\in\mathbb{Z}_+$ with $q_0:=0$ and $q_n\uparrow\infty$ such that the function $c(\cdot)$ is polyhedral concave on $(q_n,q_{n+1}]$, $n\in\mathbb{N}$.*

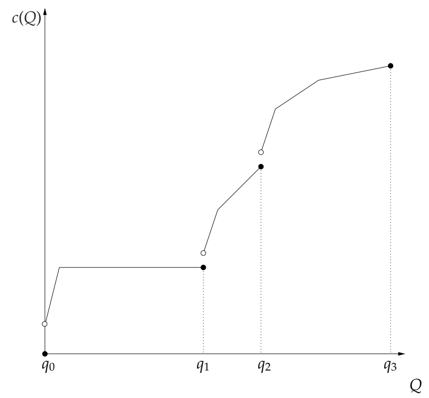


Figure 3: A piecewise polyhedral concave ordering cost function.

A piecewise concave polyhedral function might be discontinuous at the points q_n, $n\in\mathbb{Z}_+$. If $c(.)$ is a piecewise polyhedral concave function, then it follows by relation (16) that $c(Q)=\min_{1\le n\le N_k}\{\alpha_{nk}Q+\beta_{nk}\}$ for $q_{k-1}<Q\le q_k$, N_k finite and $\alpha_{1k}>\ldots>\alpha_{N_k k}$ and $\beta_{1k}<\ldots<\beta_{N_k k}$. As seen in Figure 3 it can happen that some of the constants β_{nk} are negative. By Lemma 5.3 it is now possible to determine a finite constant U being an upper bound on an optimal solution and since $q_n\uparrow\infty$ we obtain for

$$m^* := \min\{n\in\mathbb{N} : q_n > \lambda U\} < \infty \tag{41}$$

that an optimal solution is contained in the bounded interval $[0,\lambda^{-1}q_{m^*})$. For the class of piecewise polyhedral ordering cost functions satisfying some affine bounding condition, we now propose Algorithm 3. Notice in Algorithm 3, we need to solve in Step 2 many but relatively simple optimization problems. However, for m^* large this still might take some computation time.

In the next example we generalize Example 5.1 to nonidentical trucks and come up with a faster algorithm than Algorithm 3.

Algorithm 3: Finding $T_c(b,r)$ for piecewise polyhedral $c(.)$

1 Determine U and determine m^* by relation (41);

2 Solve for $k = 1, ..., m^*$ the optimization problems

$$\varphi_k := \min_{\lambda^{-1}q_{k-1} \leq T \leq \lambda^{-1}q_k} F_c(b,r,T);$$

3 $n_{opt} := \arg\min\{\varphi_k : 1 \leq k \leq m^*\}$;

4 $T_c(b,r) \leftarrow \arg\min_{\lambda^{-1}q_{n_{opt}-1} \leq T \leq \lambda^{-1}q_{n_{opt}}} F_c(b,r,T)$;

Example 5.2 (Carload Discount Schedule With nonidentical Trucks) *If we consider the carload discount schedule with nonincreasing truck setup costs as shown in Figure 4, it follows that the lower bounding function $c_0(.)$ becomes polyhedral concave. To obtain this lower bounding polyhedral concave function, we assume for $n \geq 1$ that the sequence*

$$\delta_n := \frac{c(q_n) - c(q_{n-1})}{q_n - q_{n-1}}$$

is decreasing. Then, the function $c_0 : [0, \infty) \to \mathbb{R}$ becomes

$$c_0(Q) = c(q_{n-1}) + \delta_n(Q - q_{n-1}) = \delta_n Q + \gamma_n \tag{42}$$

for $q_{n-1} \leq Q \leq q_n$, $n \geq 1$ with $\gamma_n = c(q_{n-1}) - \delta_n q_{n-1}$. As shown in Figure 4, $c(q_n) = c_0(q_n)$, $n \in \mathbb{N}$, and $c(Q) \geq c_0(Q)$ for every $Q \geq 0$.

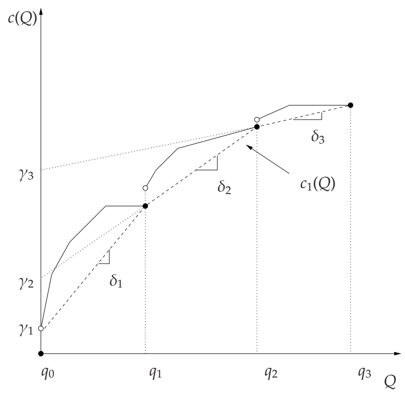


Figure 4: An ordering cost for a carload discount schedule with nonincreasing truck setup costs.

Since by construction $c(Q) \geq c_0(Q)$, it follows that $F_c(b,r,T) \geq F_{c_0}(b,r,T)$. We will now show by means of the concavity of the lower bounding function $c_0(.)$ that one can determine a better upper bound than (41). We know from relation (38) applied to $c_0(.)$ that for any d belonging to the set $D = \{d > 0 : c_0(\lambda d) \leq \frac{\lambda h d^2}{2\zeta(b)} - a\}$, we have

$$F_{c_0}(b,r,T) \geq F_{c_d}(b,r,T) \tag{43}$$

for any $T \geq d$ and $c_d(Q) = c_0(\lambda d)$. By the concavity of $c_0(.)$, this implies for

$$n_* := \max\{n \in \mathbb{N} : c_0(q_n) > \frac{hq_n^2}{2\lambda\zeta(b)} - a\}$$

that

$$F_{c_0}(b,r,T) \geq F_{c_0}(b,r,q_{n_*+1}) \tag{44}$$

for every $T \geq \lambda^{-1} q_{n_*+1}$. This implies by relation (43) and $c(q_{n_*+1}) = c_0(q_{n_*+1})$ that

$$F_c(b, r, T) \geq F_c(b, r, q_{n_*+1})$$

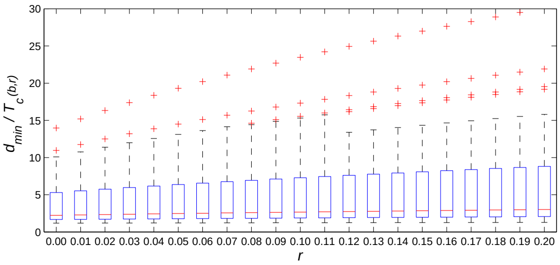
for every $T \geq \lambda^{-1} q_{n_*+1}$. Hence we have shown that any optimal solution of the original EOQ model with ordering cost function $c(\cdot)$ is contained in $[0, \lambda^{-1} q_{n_*+1}]$. By relation (41), it follows that $n_* \leq m^*$ and this shows that the newly constructed upper bound is at least as good as the constructed bound for an arbitrary piecewise polyhedral concave function. Therefore, the number of subproblems to be solved could be far less than m^*. We support this issue further in the next section.

6. Computational Study. We designed our numerical experiments with two basic goals in mind. First, we would like to demonstrate that the EOQ model is amenable to fast solution methods in the presence of a general class of ordering cost functions introduced in this paper. Second, we aim to shed some light into the dynamics of the EOQ model under the carload discount schedule, which is the most well-known transportation function in the literature. Recall that in our analysis we assumed that there exists an affine upper bound on the ordering cost function. Though straightforward, for completeness we explicitly give in Appendix A the steps to compute these affine bounds for the functions that are used in our computational experiments.

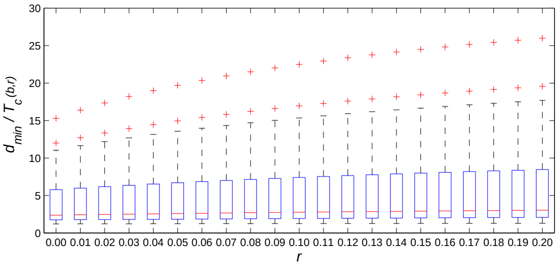
The algorithms we developed were implemented in Matlab R2008a, and the numerical experiments were performed on a Lenovo T400 portable computer with an Intel Centrino 2 T9400 processor and 4GB of memory.

6.1 Tightness of The Upper Bounds on $T_c(b, r)$ for Polyhedral Concave and Piecewise Polyhedral Concave $c(.)$. In the proof of Lemma 5.3, we already observed that the constructed upper bound $w_{\alpha,\beta}(b)$ given in relation (36) may be weak for problems with strictly positive inventory holding cost rate r because $w_{\alpha,\beta}(b)$ does not contain the value of r. Thus, in the first part of our computational study we explore the strength of the upper bounds on $T_c(b, r)$ as r changes. To this end, 100 instances are created and solved for varying values of r for both polyhedral concave and piecewise polyhedral concave ordering cost functions. For all of these instances, we set $\lambda = 1500$, $a = 200$, $h = 0.05$. Piecewise polyhedral concave functions consist of 20 intervals over which the ordering cost function $c(\cdot)$ is polyhedral concave. In this case, each polyhedral concave function is constructed by the minimum of a number of affine functions where this number is chosen randomly from the range $[2, 5]$. If $c(\cdot)$ is polyhedral concave on $[0, \infty)$, then the number of linear pieces on $c(\cdot)$ is selected randomly from the range $[2, 20]$. For both piecewise polyhedral concave and polyhedral concave $c(\cdot)$, the slope of the first affine function on each polyhedral concave function is distributed uniformly on $[0.50, 1.00]$. The following slopes are calculated by multiplying the immediately preceding slope by a random number in the range $[0.80, 1.00]$. All (truck) setup costs are identical to 50, and the distance between two breakpoints on $c(\cdot)$ is generated randomly from the range $[0.05\lambda, 0.20\lambda]$. If shortages are allowed, b takes a value of 0.25, otherwise $b = \infty$. The inventory holding cost rate r is varied in the interval $[0, 0.20]$ at increments of 0.01. The results of these experiments are summarized in figures 5 - 6 .

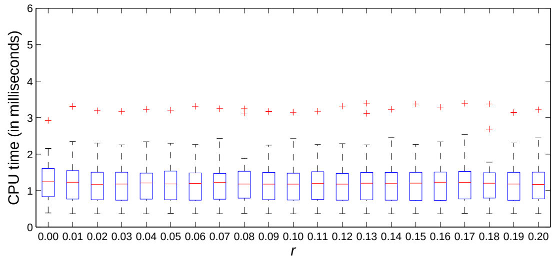
For polyhedral concave functions, the upper bound d_{min} on $T_c(b, r)$ is generally quite tight both for problems with and without shortages. See figures 5(a)-5(b). Unfortunately, for piecewise polyhedral concave functions $w_{\alpha,\beta}(b)$ relies on the existence of an affine upper bound on $c(\cdot)$ and is not particularly tight as depicted in figures 6(a)-6(b). Thus, in the future we may formulate the problem of determining the best affine upper bound as an optimization problem which would replace the approach described in Section A.2.



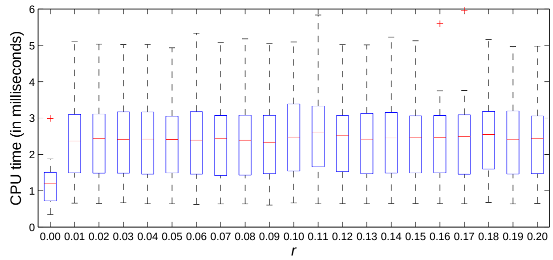
(a) No shortages are allowed.



(b) Shortages are allowed.

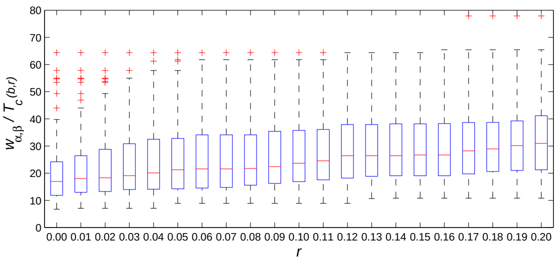


(c) No shortages are allowed.

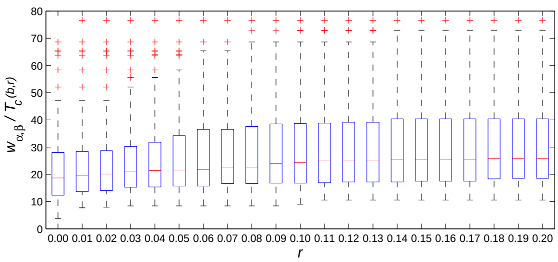


(d) Shortages are allowed.

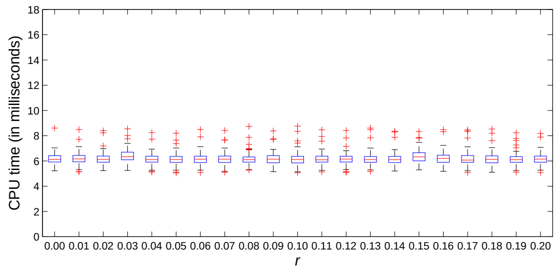
Figure 5: Quality of the upper bound on $T_c(b,r)$ for polyhedral concave functions with respect to r and associated solution times.



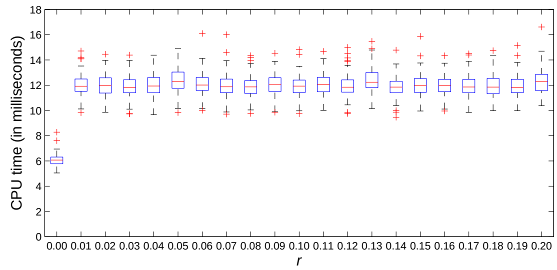
(a) No shortages are allowed.



(b) Shortages are allowed.



(c) No shortages are allowed.



(d) Shortages are allowed.

Figure 6: Quality of the upper bound on $T_c(b,r)$ for piecewise polyhedral concave functions with respect to r and associated solution times.

The values of the upper bounds d_{min} and $w_{\alpha,\beta}(b)$ on $T_c(b,r)$ are invariant to the inventory holding cost rate r; however, we observe that the ratios $d_{min}/T_c(b,r)$ and $w_{\alpha,\beta}(b)/T_c(b,r)$ are not significantly affected by increasing values of r in figures 5(a)-5(b) and 6(a)-6(b). These graphs exhibit only slightly increasing trends as r increases from zero to 0.20.

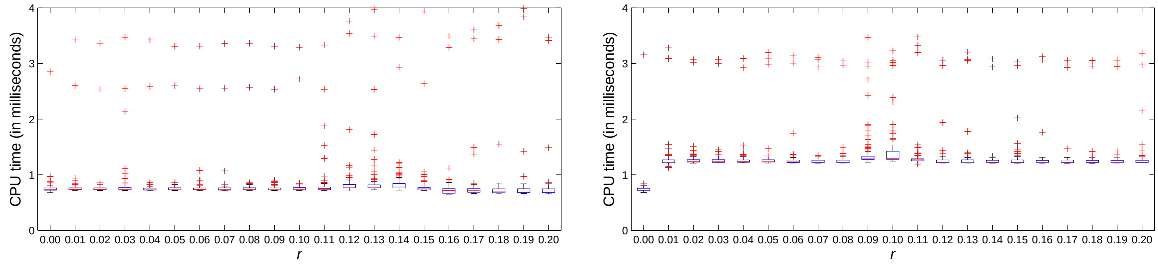
Overall, figures 5(c)-5(d) and 6(c)-6(d) demonstrate clearly that we can solve for the economic order quantity very quickly even when a general class of ordering costs as described in this paper are incorporated into the model. This is important in its own right and also suggests that decomposition approaches may be a promising direction for future research for more complex lot sizing problems with transportation costs. The algorithms proposed in this paper or their extensions may prove useful to solve the subproblems in such methods very effectively.

Two major factors determine the CPU times. First, our algorithms are built on solving many EOQ problems with linear ordering cost functions. These subproblems possess analytical solutions if no shortages are allowed or $r = 0$ when shortages are allowed. Otherwise, a line search must be employed to solve these subproblems which is computationally more costly. This fact is clearly displayed in figures 5(c)-5(d) and 6(c)-6(d). Second, the solution times depend on the number of subproblems to be solved which explains the longer solution times for piecewise polyhedral concave $c(\cdot)$ compared to those for polyhedral concave $c(\cdot)$. We will take up on this issue later again in this section.

6.2 Carload Discount Schedule. In the remainder of our computational study we focus our attention on the carload discount schedule which is widely used in the literature (Nahmias, 1997). We first start by providing a negative answer to Nahmias' claim that solving the EOQ model under the carload discount schedule with two linear pieces may be very hard, and then propose some managerial insights into the nature of the optimal order policy under this transportation cost structure. Finally, we conclude by analyzing the impact of the number of linear pieces on $c(\cdot)$ and the improved upper bound on $T_c(b, r)$ given in relation (44) on the solution times for the carload discount schedule with nonincreasing setup costs; see Example 5.2.

One hundred instances with ordering cost functions based on the carload discount schedule with two linear pieces are generated very similarly to those with piecewise polyhedral $c(\cdot)$ described previously. We only point out the differences in the data generation scheme. The ordering cost function $c(\cdot)$ is polyhedral concave over each interval $((k-1)C, kC]$, $k = 1, 2, \dots$, where $C = 250$ is the truck capacity. All truck setup costs are set to zero. The slope of the first piece of the carload discount schedule is distributed uniformly from the interval $[0.50, 1.00]$, and the cost of a truck increases linearly until the full truck load cost is incurred at a point chosen randomly in the interval $[0.25C, 0.75C]$. Any additional items do not contribute to the cost of a truck. These 100 instances are solved for varying values of r both with and without shortages. The CPU times for solving these instances are plotted in Figure 7. The median CPU time is below 1.5 milliseconds in all cases, and the maximum CPU time is about 4 milliseconds. Clearly, the economic order quantity may be identified very effectively under the classical carload discount schedule.

In the next set of experiments, our main goal is to illustrate the dynamics of the model if the transportation costs are dictated by the classical carload discount schedule. In particular, we focus on the interplay between the inventory holding costs and the structure of the classical carload discount schedule. We create ten instances for each combination of $h \in \{0.50, 1.00, 1.50, 2.00, 2.50\}$ and $b \in \{\infty, 5h\}$. For all of these instances, we set $\lambda = 1500$, $a = 100$, $r = 0$, and $C = 250$. Then, for each instance we keep the cost of a full truckload fixed at 100 but consider different slopes for the carload discount schedule as depicted in Figure 8. The main insight conveyed by the results in Figure 9 is that the optimal schedule strives to use a truck at full capacity unless holding inventory is expensive.



(a) No shortages are allowed.

(b) Shortages are allowed.

Figure 7: Solution times for the classical carload discount schedule.

For instance, in Figure 9(a) the optimal order quantity is always 3 full truckloads for $h = 0.50$ until the carload schedule turns into an (ordinary) linear transportation cost function. On the other hand, for $h = 2.50$ the optimal order quantity diverts from a full truckload if the full cost of a truck is incurred at $0.70C$ or higher.

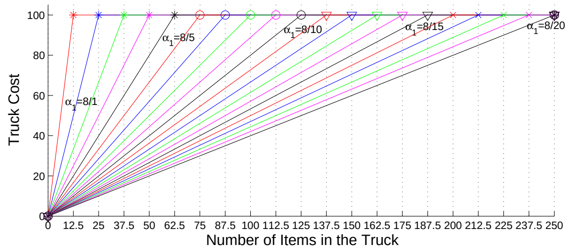
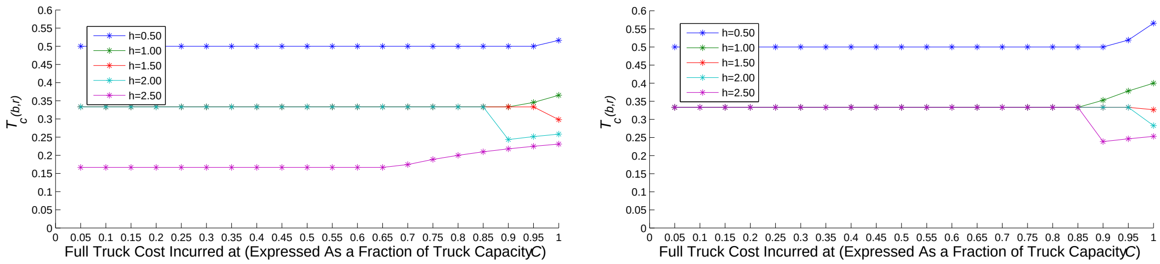
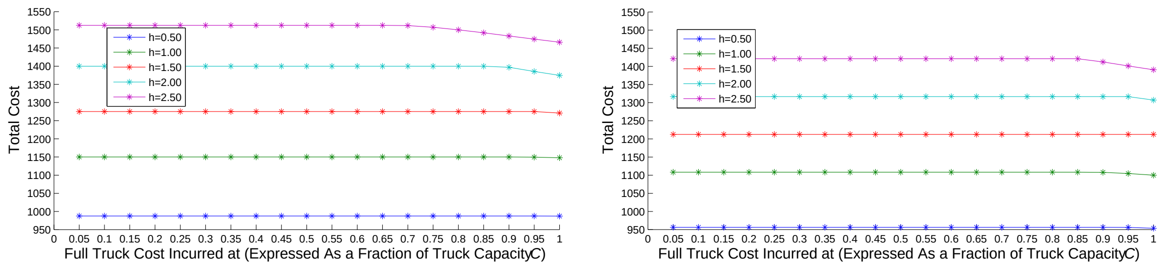


Figure 8: Alternate carload discount schedules for the same capacity and full truckload cost.



(a) No shortages are allowed.

(b) Shortages are allowed.

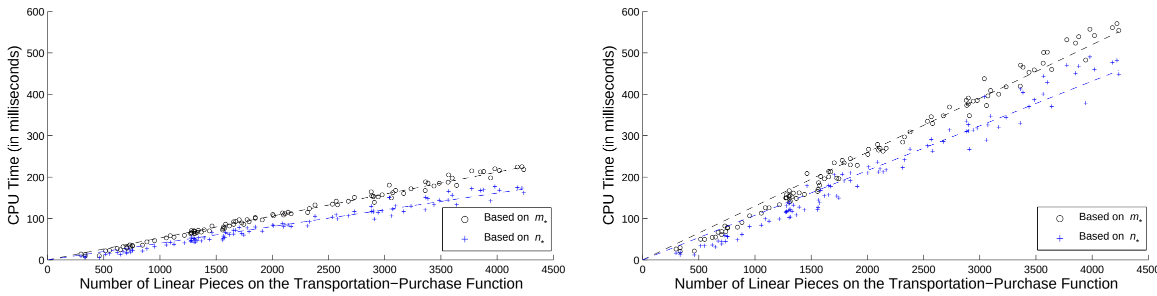


(c) No shortages are allowed.

(d) Shortages are allowed.

Figure 9: Optimal cycle length and cost for alternate carload discount schedules and different h values.

Finally, we explore how the solution times scale as a function of the number of subproblems to be solved. Recall that earlier in this section we argued that the solution times depend heavily on the number of linear pieces on the ordering cost function $c(.)$. We illustrate that this relationship is basically linear - as expected - by solving the EOQ model under a general carload discount schedule. That is, the truck setup costs are decreasing although the trucks are identical, and there may be multiple breakpoints on the ordering cost function. (See Figure 4). We generate 100 instances where we set $\lambda = 1500$, $a = 200$, $h = 0.05$, $r = 0.10$, and $b = 0.25$ if shortages are allowed, and $b = \infty$ otherwise. As before, the truck capacity is $C = 250$, and the ordering cost function $c(.)$ is polyhedral concave over each interval $((k-1)C, kC]$, $k = 1, 2, \ldots$. The setup cost of the first truck is distributed uniformly from the interval $[50, 100]$, and for each following truck the setup cost is computed by multiplying that of the previous truck with a random number in the range $[0.50, 1.00]$. For each truck, the number of breakpoints on the discount schedule is created randomly in the range $[2, 20]$, and the distance between two successive breakpoints is calculated by multiplying the remaining capacity of the truck by a random number in $[0.05, 0.20]$. The slope of the first linear piece is distributed uniformly on $[0.50, 1.00]$ and subsequent slopes are obtained by multiplying the slopes of the immediately preceding pieces by a random number in the range $[0.80, 1.00]$. The final slope is always zero. In Figure 10, we plot the solution times against the number of subproblems solved and conclude that the relationship between these two quantities is linear. The dotted lines in the figure are fitted by simple linear regression through the origin. We also observe that the relatively tighter upper bound on $T_c(b, r)$ given in relation (44) for carload discount schedules with nonincreasing setup costs provides computational savings of 22% and 28% on average for instances with and without shortages, respectively.



(a) No shortages are allowed. (b) Shortages are allowed.

Figure 10: Solution times for the carload discount schedule with nonincreasing setup costs and multiple linear pieces.

7. Conclusion and Future Research. In this work, we have analyzed the impact of general ordering cost functions in EOQ-type models. We investigated the structures of the resulting problems and derived bounds on their optimal cycle lengths. Observing that the carload discount schedule is frequently used in the real practice, we have identified a subclass of problems that also includes the well-known carload discount schedule. Due to their special structure, we have shown that the problems within this class are relatively easy to solve. Using our analysis, we have also laid down the steps of several fast algorithms. To support our analysis and results, we have set up a thorough computational study and discussed our observations from different angles. Overall, we have concluded that a large group of EOQ-type problems with general ordering cost functions can be considered as simple problems and they can be solved very efficiently in almost no time.

In the future, we intend to study the extension of the EOQ-type problems to stochastic single item inventory models with arbitrary transportation costs. There exist models in the literature, where the optimal price is determined along with the optimal order quantity. If the demand-price relationship is one-to-one (as it is the case in most of pricing studies within the realm of EOQ), then we may be able to obtain similar results at the expense of complicating the analysis. Finally, a natural follow-up work could be incorporating such general ordering costs into multi-item lot-sizing. We then need to think about consolidation of many items into a single shipment, which may yield significant savings in transportation costs without comparable increases in inventory holding costs.

Appendix A. Computing The Affine Upper Bounds. In this appendix, we demonstrate how an affine function may be computed that satisfies the affine bounding condition of Lemma 5.3 both for the carload discount schedule and the piecewise polyhedral concave ordering cost function.

A.1 The Carload Schedule. Without loss of generality, we only consider carload discount schedules with non-increasing truck setup costs which also includes trucks with identical setup costs as a special case. Similar to the construction in Example 5.1, we let $g : (0, C] \to \mathbb{R}$ be an increasing polyhedral concave function satisfying $g(0) = 0$ and s_i with $s_i \geq s_{i-1} \geq 0$, $i \geq 1$ be the setup cost of the ith truck. We then define

$$c(Q) = \begin{cases} 0, & \text{if } Q = 0; \\ g(Q) + s_1, & \text{if } 0 < Q \leq C, \end{cases}$$

where

$$g(Q) = \min_{1 \leq k \leq N} \{\alpha_k Q + \beta_k\} \tag{45}$$

with $\alpha_1 > \alpha_2 > \cdots > \alpha_N \geq 0$ and $0 = \beta_1 < \beta_2 < \cdots < \beta_N$, and

$$c(Q) = \sum_{i=1}^{n+1} s_i + ng(C) + g(Q - nC)$$

for $nC < Q \leq (n+1)C$ with integer $n \geq 1$ (see Figure 11).

Lemma A.1 *For a discount carload schedule with nonincreasing setup costs $s_i \geq 0$, $i \geq 1$ it follows that*

$$c(Q) \leq \alpha Q + \beta,$$

where $\alpha = \max(\alpha_1, c(C)C^{-1})$ *and* $\beta = s_1$.

Proof. Since $s_1 \geq 0$, we have $c(0) = 0 \leq s_1 = \beta$. For $0 < Q \leq C$, it follows by relation (45) that

$$c(Q) = \min_{1 \leq k \leq N} \{\alpha_k Q + \beta_k\} + s_1 \leq \alpha_1 Q + s_1 \leq \max(\alpha_1, c(C)C^{-1})Q + s_1 = \alpha Q + \beta.$$

For $nC < Q \leq (n+1)C$ with integer $n \geq 1$, we have

$$\begin{aligned} c(Q) = \textstyle\sum_{i=1}^{n+1} s_i + ng(C) + g(Q - nC) \quad &\leq (n+1)s_1 + ng(C) + g(Q - nC) \\ &= n(s_1 + g(C)) + g(Q - nC) + s_1 \\ &= nc(C) + g(Q - nC) + s_1 \\ &\leq \max(\alpha_1, c(C)C^{-1})nC + \min\nolimits_{1 \leq k \leq N}\{\alpha_k(Q - nC) + \beta_k\} + s_1 \\ &\leq \max(\alpha_1, c(C)C^{-1})nC + \alpha_1(Q - nC) + s_1 \\ &\leq \max(\alpha_1, c(C)C^{-1})Q + s_1 \\ &= \alpha Q + \beta. \end{aligned}$$

□

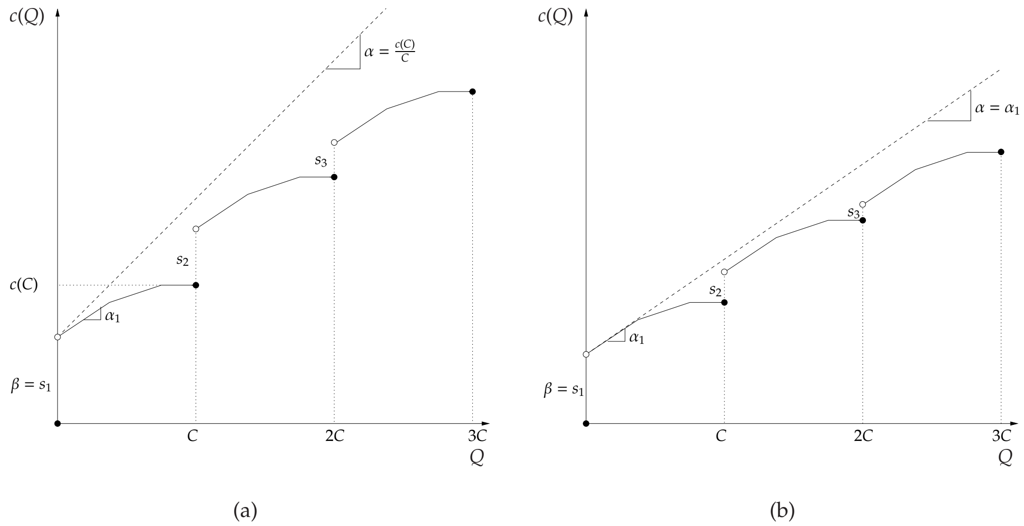


Figure 11: Construction of an upper bound for the carload discount schedule.

A.2 Piecewise Polyhedral Concave Functions. We next compute an affine bound for a piecewise polyhedral concave function over the predefined interval $[0, q_K]$, where K corresponds to the number of trucks under consideration. Let $g_k : (q_{k-1}, q_k] \to \mathbb{R}$ be an increasing polyhedral concave function satisfying $g_k(0) = 0$ and $s_i \geq 0$ be the setup cost of the ith truck. We then define

$$c(Q) = \begin{cases} 0, & \text{if } Q = 0; \\ g_1(Q) + s_1, & \text{if } 0 < Q \leq q_1; \\ \sum_{l=1}^{k-1} (g_l(q_l) + s_l) + g_k(Q - q_{k-1}) + s_k, & \text{if } q_{k-1} < Q \leq q_k, \end{cases}$$

where $2 \leq k \leq K$ and

$$g_k(Q) = \min_{1 \leq n \leq N_k} \{\alpha_{nk} Q + \beta_{nk}\}$$

with $\alpha_{1k} > \alpha_{2k} \cdots > \alpha_{Nk} \geq 0$ and $0 = \beta_{1k} < \beta_{2k} < \cdots < \beta_{Nk}$ (see Figure 12).

Lemma A.2 *Let* $u : [0, q_K] \to \mathbb{R}$ *be the piecewise linear convex function given by*

$$u(Q) = \max\left\{\alpha_{N_1 1} Q + \beta_{N_1 1} + s_1, \max_{2 \leq k \leq K}\left\{\sum_{l=1}^{k-1} (g_l(q_l) + s_l) + \alpha_{N_k k}(Q - q_{k-1}) + \beta_{N_k k} + s_k\right\}\right\}.$$

Then, it follows for $0 \leq Q \leq q_K$ *that*

$$c(Q) \leq \alpha Q + \beta,$$

where $\alpha = \frac{u(q_K) - u(0)}{q_K}$ *and* $\beta = u(0) \geq 0$.

Proof. Since $u(\cdot)$ is convex, it follows for $0 \leq Q \leq q_K$ that

$$u(Q) \leq \frac{u(q_K) - u(0)}{q_K} Q + u(0) = \alpha Q + \beta. \tag{46}$$

Clearly, $c(0) = 0 \leq u(0) = \beta$. For $0 < Q \leq q_1$, we have

$$c(Q) = \min_{1 \leq n \leq N_1} \{\alpha_{n1} Q + \beta_{n1}\} + s_1 \leq \alpha_{N_1 1} Q + \beta_{N_1 1} + s_1 \leq u(Q).$$

Similarly, for $q_{k-1} < Q \leq q_k$ with $2 \leq k \leq K$, we have

$$c(Q) = \sum_{l=1}^{k-1} (g_l(q_l) + s_l) + \min_{1 \leq n \leq N_k} \{\alpha_{nk}(Q - q_{k-1}) + \beta_{nk}\} + s_k \leq \sum_{l=1}^{k-1} (g_l(q_l) + s_l) + \alpha_{N_k k}(Q - q_{k-1}) + \beta_{N_k k} + s_k \leq u(Q).$$

The result then follows by using relation (46). □

This construction is illustrated in Figure 12 where $K = 3$.

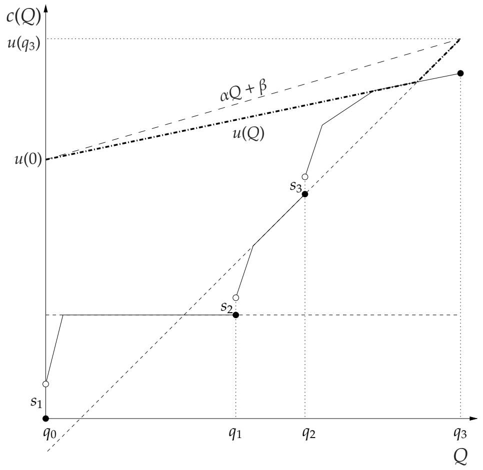


Figure 12: Construction of an upper bound for a piecewise polyhedral concave ordering cost function ($K = 3$).

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