CAUCHY PROBLEMS FOR A CLASS OF NONLOCAL NONLINEAR BI-DIRECTIONAL WAVE EQUATIONS

by

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Submitted to the Graduate School of Engineering and Natural Sciences
in partial fulfillment of
the requirements for the degree of
Doctor of Philosophy

Sabancı University
Spring 2011

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Mathematics, Doctorate Thesis, 2011

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Keywords: Global well-posedness, finite time blow-up, nonlocal elasticity, Boussinesq type wave equations.

Abstract

In this thesis study, two nonlocal models governing nonlinear wave motions in a continuous medium are proposed and the Cauchy problems corresponding to these nonlocal nonlinear wave equations are considered. Both of the models involve convolution integral operators with general kernel functions whose Fourier transforms are nonnegative. One of the models is based on a single equation governing the longitudinal wave propagation, whereas the other model is based on two coupled equations governing the propagation of transverse waves. Some well-known examples of nonlinear wave equations, such as Boussinesq-type equations, follow from the proposed models for suitable choices of the kernel functions. The main aim of this thesis is to discuss well-posedness of the Cauchy problems. For this purpose, global existence of solutions of the models assuming enough smoothness on the initial data together with some positivity conditions on the nonlinear term are established. Furthermore, sufficient conditions for finite time blow-up are provided.

İki Yönlü Dalga Denklemlerinin Yerel ve Doğrusal Olmayan Bir Sınıfı İçin Cauchy Problemleri

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Matematik, Doktora Tezi, 2011

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Anahtar Kelimeler: Global varlık, sonlu zamanda patlama, yerel olmayan elastisite, Boussinesq tipi dalga denklemleri.

Özet

Bu tez çalışmasında elastik bir sürekli ortamdaki doğrusal olmayan dalga hareketini yöneten iki yerel olmayan model önerilmiş olup, bu yerel ve doğrusal olmayan dalga denklemlerine karşı gelen Cauchy problemleri ele alınmıştır. Her iki model de Fourier dönüşümleri negatif olmayan genel çekirdek fonksiyonları ile tanımlı konvolüsyon integral operatörleri içermektedir. Modellerden bir tanesi boyuna dalga yayılımını yöneten tek denklem üzerine inşa edilirken, diğer model enine dalgaların yayılımını yöneten iki kuple denklem üzerine inşa edilir. Boussinesq tipi denklemler gibi doğrusal olmayan dalga yayılımının iyi bilinen denklem örnekleri, çekirdek fonksiyonlarının uygun seçimleri için önerilen modellerden elde edilebilirler. Bu tezin temel amacı Cauchy problemlerinin iyi tanımlılığını tartışmaktır. Bu amaçla, başlangıç koşullarının yeterince düzgün olduğu ve doğrusal olmayan terimin bazı pozitiflik özelliklerine sahip olduğu varsayımları altında, modellerin çözümlerinin global varlıkları ispatlanmıştır. Buna ek olarak, çözümlerin sonlu zamanda patlaması icin yeter koşullar elde edilmiştir.

Acknowledgements

I wish to express my gratitude to my thesis advisors Albert Erkip and Hüsnü Ata Erbay for their care, love and patience. I would never succeed without them. I also want to thank my study group members Saadet Erbay, Ceni Babaoğlu, Handan Borluk and Gülçin Muslu one by one for their kindness and support. I feel so honored to have the chance working together. Besides, being a member of Sabancı University was also a major chance of mine. The assistance of the academicians during my doctorate study period cannot be ignored.

I am proud of being a member of both Duruk family and Mutlubaş family since they did their best to contribute my success. I specially want to thank my dear husband Behçet Mutlubaş for calming me down tirelessly and making every moment we share significant.

I was supported by the Scientific and Technological Research Council of Turkey (TUBITAK) under the project TBAG-110R002 during the last term.

Contents

Chapter 1

Introduction

Initial value problems for wave equations characterizing the wave propagation in continuum mechanics have drawn attention of the mathematicians all the time. Structural or geometrical properties of the continuous medium may cause the waves to disperse during the propagation. In order to explain the concept of dispersion, consider the one dimensional single linear partial differential equation with constant coefficients: $P(\frac{\partial}{\partial x}, \frac{\partial}{\partial t})u = 0$, where P is a polynomial. Substituting elementary plane wave solutions of the form $u(x,t) = Ae^{i(kx-\omega t)}$, where k is the wave number, ω is the frequency and A is the amplitude, into the partial differential equation, we get an algebraic equation of the form $P(ik, -i\omega) = 0$. Assume that this equation has solutions as $\omega = W(k)$. Afterwards, phase velocity for a wave motion is defined as $c = \omega/k$. Here, if c is dependent on k, then the wave is called dispersive. For example, the linear wave equation

$$u_{tt} - u_{xx} = 0 \qquad (c^2 = 1) \tag{1.1}$$

is nondispersive. On the other hand, the linear Boussinesq equations

$$u_{tt} - u_{xx} \pm u_{xxxx} = 0 \qquad (c^2 = 1 \pm k^2) \tag{1.2}$$

are dispersive wave equations. That is, the wave propagates without changing its shape for (1.1) contrary to (1.2). Another example for dispersive wave equations is the linear improved Boussinesq equation:

$$u_{tt} - u_{xx} - u_{xxtt} = 0 \qquad (c^2 = \frac{1}{1+k^2}).$$

It is a well-known fact that nondispersive nonlinear wave equations lead to singularities in finite time. In Chapter 2, this result and the related concepts will be discussed in detail. However, it has been observed that dispersive nonlinear wave propagation either delays or totally prevents this kind of singularities. A typical example is the Korteweg-de-Vries (KdV) equation which models shallow water wave propagation:

$$u_t + uu_x + u_{xxx} = 0.$$

Physically, the second term represents the nonlinear effects and the last term expresses the dispersive effects. Steeping effect of the nonlinearity and smoothing effect of dispersion reveals a balance with each other. There occurs a kind of waves called soliton which propagates without changing its shape even after collisions and there are numerous studies about this kind of solutions. Global well-posedness of the solutions for initial value problems of the KdV equation has been extensively studied since it describes physical events arising in different contexts. Kenig, Ponce and Vega [24] derived a fundamental result on local well-posedness in the Sobolev space H^s for $s > 3/4$. Later, Kappeler and Topalov [23] also showed that the KdV equation is globally well-posed for $u_0 \in H^{-1}$. Another nonlinear dispersive wave equation also characterizing shallow water wave propagation is the Boussinesq equation. On the contrary to the KdV equation describing the unidirectional wave propagation, it describes bi-directional wave propagation. Two basic forms of the Boussinesq equation are

$$u_{tt} - u_{xx} - u_{xxxx} + (u^2)_{xx} = 0 \tag{1.3}$$

and

$$u_{tt} - u_{xx} + u_{xxxx} + (u^2)_{xx} = 0 \tag{1.4}$$

which are called "bad" and "good" Boussinesq equation, respectively. For the "good" Boussinesq equation (1.4) and its various generalized forms, there have been a lot of research results from the local and global well-posedness to the blow-up of solutions of their initial value problems [18, 20, 22, 31, 38]. However, for the "bad" Boussinesq equation (1.3), solutions for the initial value problem blow up [22].

In 1967, G. Whitham studied the mathematical aspects of nonlinear nonlocal equations in order to model water waves and hence proposed the following very general nonlinear nonlocal equation later named as Whitham's equation [32]:

$$u_t + uu_x + \int_{-\infty}^{\infty} \beta(x-y)u_x(y,t)dy = 0. \tag{1.5}$$

Here $\beta(x)$ is a kernel function and the integral term is in fact the convolution $\beta * u_x$. Dispersive effect arises from the convolution term, unlike the KdV equation where this effect was due to the term u_{xxx}. This fact can be observed from the linear dispersion relation $c = \widehat{\beta}(k)$ where $\widehat{\beta}$ is the Fourier transform of β. Whitham's motivation for introducing (1.5) was to have a simple equation suitable for describing such typical water wave phenomena as sharp crests and breaking of waves, which the KdV model fails to do. The point is that the convolution smooths out the possible singularities which can occur for u_x. In 1972, Benjamin, Bona and Mahony [1] introduced a model equation for water waves

$$u_t + uu_x - u_{xxt} = 0. \tag{1.6}$$

Since the nonlocal character of (1.6) is not sufficiently clear, we now rewrite it in order to make its nonlocality more apparent. Arranging (1.6) as
$(1 - \partial_x^2)u_t + \partial_x(\frac{1}{2}u^2) = 0$, we get

$$u_t + \int_{-\infty}^{\infty} \beta(x-y)(\frac{1}{2}u^2)_x(y,t)dy = 0$$

where $\beta(x) = \frac{1}{2}e^{-|x|}$ is the Green function of $(1 - \partial_x^2)$. Comparing with Whitham's equation, nonlocality in (1.6) appears on the nonlinear term. Camassa and Holm [2, 3] derived

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx} \tag{1.7}$$

which can also be rewritten as a nonlocal wave equation

$$u_t + uu_x + \int_{-\infty}^{\infty} \beta(x-y)(u^2 + \frac{1}{2}u_x^2)_x(y,t)dy = 0$$

with $\beta(x) = \frac{1}{2}e^{-|x|}$. Another model for shallow water wave dynamics was given by Degasperis and Procesi [7]:

$$u_t - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx}. \tag{1.8}$$

As we did for the Camassa-Holm equation, (1.8) can be written in the following form:

$$u_t + uu_x + \int_{-\infty}^{\infty} \beta(x-y)(\frac{3}{2}u^2)_x(y,t)dy = 0.$$

The Cauchy problems of (1.7) and (1.8) have been studied extensively (see, for instance, [17] for (1.7) and [5, 41] for (1.8) and references cited therein).

The nonlocal equations given above characterize the unidirectional wave propagation. Similarly, certain Boussinesq-type equations characterizing bi-directional wave propagation have a nonlocal nature. Consider the so-called improved Boussinesq (IMBq) equation

$$u_{tt} - u_{xx} - u_{xxtt} = (g(u))_{xx}. \tag{1.9}$$

It can be converted into

$$u_{tt} = \int_{-\infty}^{\infty} \beta(x-y)(u+g(u))_{xx}(y,t)dy \tag{1.10}$$

where $\beta(x) = \frac{1}{2}e^{-|x|}$ as before. Nonlocality again becomes clearer with (1.10). Existence and uniqueness of solutions, locally and globally in time, and non-existence of global solutions to the initial-boundary-value problem for the IMBq equation (1.9) were discussed in [4, 19, 42]. Rosenau [36] derived the higher-order Boussinesq equation

$$u_{tt} - u_{xx} - u_{xxtt} + \gamma u_{xxxxtt} = g(u)_{xx} \tag{1.11}$$

using the quasi-continuum approximation for longitudinal vibrations of a dense lattice. If the kernel function in (1.10) is taken as

$$\beta(x) = \frac{1}{2(c_1^2 - c_2^2)}(c_1 e^{-|x|/c_1} - c_2 e^{-|x|/c_2}),$$

with certain positive constants c_1 and c_2, then it can be observed that (1.11) is equivalent to (1.10). Global well-posedness of the Cauchy problem for (1.11) was studied in [8].

Consider a convolution type nonlinear wave equation

$$u_{tt} = (\beta * (u + g(u))_{xx} \tag{1.12}$$

with a general kernel function $\beta(x)$ and a general nonlinear function $g(u)$. Note that both (1.10) and (1.11) are special cases of (1.12). This equation describes a general class of nonlocal nonlinear wave equations characterizing bi-directional wave propagation in a continuous medium. A generalization of (1.12) to a coupled system of two nonlocal nonlinear wave equations is given by

$$u_{tt} = (\beta_1 * (u + g_1(u,v)))_{xx}, \tag{1.13}$$

$$v_{tt} = (\beta_2 * (u + g_2(u,v)))_{xx} \tag{1.14}$$

which characterize nonlinear and nonlocal interaction of two coupled waves propagating in a continuous medium.

The unusual interaction between the nonlinear nature and the nonlocal nature of both (1.12) and (1.13)-(1.14) pose some interesting open problems regarding qualitative features of solutions of these equations. The main purpose of this thesis study is to discuss well-posedness of the Cauchy problems defined for (1.12) and (1.13)-(1.14). In particular, this thesis is concerned with global existence and blow-up of solutions to the Cauchy problems associated with (1.12) and (1.13)-(1.14).

The rest of this thesis study is composed of the chapters given as follows. In Chapter 2, it is shown that the propagation of longitudinal (or two transverse) strain waves through a nonlocal nonlinear elastic medium is governed by (1.12) (or (1.13)-(1.14)). Before giving the derivation of (1.12) and (1.13)-(1.14) from the equations of motion of the nonlocal nonlinear elastic medium, the propagation of longitudinal waves in the classical (local) theory of nonlinear elasticity is briefly discussed. The chapter ends with an introduction of a general class of kernel functions, which covers the most common used kernels in the literature.

In Chapter 3, local existence results for the Cauchy problems associated with (1.12) and (1.13)-(1.14) are given. Depending on the values of the parameter characterizing the smoothness of the kernel function, two different forms of the local existence theorem are presented.

Chapter 4 is devoted to global well-posedness results. Conservation of energy is supplied for both nonlocal nonlinear single model and coupled model.

In Chapter 5 conditions for finite time blow-up of solutions to the two Cauchy problems are provided.

Finally, conclusions and some related open problems that are planned to investigate afterwards are presented.

Basic concepts, inequalities and theorems needed during this thesis study are given in the appendix part.

In what follows $H^s = H^s(\mathbb{R})$ will denote the Sobolev space on $\mathbb{R}$. For the H^s norm we use the Fourier transform representation $\|u\|_s^2 = \int_{\mathbb{R}}(1+\xi^2)^s|\widehat{u}(\xi)|^2 d\xi$. We use $\|u\|_\infty$, $\|u\|$ and $\langle u, v\rangle$ to denote the L^∞ and L^2 norms and the inner product in L^2, respectively. The symbol ∂_x stands for the classical partial derivative with respect to x.

Chapter 2

Two Models of Wave Propagation in Nonlocal Elasticity

In this chapter, the nonlocal nonlinear single wave equation (1.12) and the coupled system (1.13)-(1.14) are derived from the equations of motion of one-dimensional nonlocal elasticity. In that respect, we first look at the classical (local) model of one-dimensional elasticity and the related nonlinear wave equation. We refer to the literature to discuss how this case leads to singularities in finite time. Secondly, we show that (1.12) and (1.13)-(1.14) model propagation of longitudinal and transverse waves, respectively, propagating in a nonlocal elastic medium. Finally, introducing a smoothness condition, we describe the general class of kernel functions that will be used throughout this thesis and show that the most commonly used kernel functions belong to this class.

2.1 Nonlinear Wave Propagation in Classical Elasticity

Consider a one-dimensional, homogenous, nonlinearly elastic infinite medium. Let a scalar-valued function $U(X,t)$ be the displacement of a reference point X at time t. In the absence of body forces the equation of motion for the displacement is

$$\rho_0 U_{tt} = (\sigma(U_X))_X, \tag{2.1}$$

where ρ_0 is the mass density of the medium, $\sigma = \sigma(U_X)$ is the local stress and subscripts denote partial derivatives. In classical theory of elasticity, stress at a spatial point depends on the "strain" U_X at the same point. The relation between stress and strain is given by the following equation:

$$\sigma(U_X) = \frac{\partial F(U_X)}{\partial U_X}. \tag{2.2}$$

Here, F is a scalar valued function, called strain energy density function, which relates the stress and the strain at a given point by requiring that the stress can be obtained by taking the derivative of F with respect to strain. Hence by using (2.2), (2.1) becomes

$$\rho_0 U_{tt} = (F'(U_X))_X. \tag{2.3}$$

For convenience, it is assumed that there is neither initial energy, $F(0) = 0$, nor initial stress, $F'(0) = 0$. Now, if both sides of (2.3) is differentiated with respect to X, then it becomes

$$\rho_0 U_{Xtt} = (F'(U_X))_{XX}.$$

After some non-dimensionalization process equivalent to taking $\rho_0 = 1$ and for simplicity, replacing X with x and U_X with u, the equation of motion for the strain can be obtained as follows:

$$u_{tt} = (F'(u))_{xx} \tag{2.4}$$

which is a second-order nonlinear partial differential equation. Thus, from partial differential equation viewpoint, some questions arise such as the type of the equation, local existence of a solution for a given initial data, possibility to extend the solution to all times and dependence of the solution on the initial data.

A first order system of quasilinear partial differential equations in two independent variables is of the form

$$W_t + A(W)W_x = 0, \tag{2.5}$$

where W is a vector function of x and t, A is a matrix function of W, hence of x and t. The system (2.5) is called *strictly hyperbolic* if $A(W)$ has real and distinct eigenvalues $\lambda_k(W)$. Let r_k be the right eigenvector corresponding to an eigenvalue λ_k. The characteristic k-th field of the system is called *genuinely nonlinear* if $r_k \cdot grad(\lambda_k) \neq 0$ whereas it is called *linearly degenerate* if $r_k \cdot grad(\lambda_k) = 0$. In other words, if the directional derivative of λ_k never vanishes, then the characteristic field is genuinely nonlinear. When the kth field is genuinely nonlinear, different waves propagate with different speeds. Waves of linearly degenerate fields behave almost linearly and converge to traveling waves as time goes to infinity. In this case, smooth initial data never brings out discontinuous solutions or vice versa.

Now, we define the velocity $v(x,t) = \int^x u_t(y,t)dy$ and convert (2.4) into a first-order system:

$$\begin{aligned} u_t &= v_x \\ v_t &= (F'(u))_x = F''(u)u_x. \end{aligned}$$

This system known as p-system [28] is equivalent to the following vector-valued equation:

$$W_t - \begin{pmatrix} 0 & 1 \\ F''(u) & 0 \end{pmatrix} W_x = 0, \tag{2.6}$$

where $W(x,t) = (u(x,t), v(x,t))^T$. Since (2.6) is of the form (2.5), we make some observations depending on the definitions given above. Eigenvalues of the 2×2 matrix are $\lambda_{1,2}(u) = \pm\sqrt{F''(u)}$. In order to have real and distinct eigenvalues $F''(u)$ must be strictly positive. Hence, we say that the p-system is strictly hyperbolic if $F''(u) > 0$. Moreover, each characteristic field is genuinely nonlinear if $F'''(u) \neq 0$. In this system, we have wavelike solutions propagating forward and backward with wavespeeds $\lambda_1(u)$ and $\lambda_2(u)$, respectively.

The Cauchy problem for hyperbolic wave equation (2.4) has some particular features that makes it difficult to work. The most important property which was proved by P. D. Lax in 1964 is the fact that the solution to the Cauchy problem will blow up in finite time even for small initial data. There also occurs discontinuous solutions no matter how smooth the initial data are. Unless some effects such as dissipation, dispersion or nonlocal interactions, or extra dimension arise, there is no possibility to get continuous solution for this Cauchy problem.

Without loss of generality, we decompose the strain-energy density function into harmonic and anharmonic parts such as:

$$F(u) = \frac{1}{2}u^2 + G(u) \tag{2.7}$$

with

$$G(u) = \int_0^u g(s)\mathrm{d}s\ . \tag{2.8}$$

Observe that $G'(u) = g(u)$. Thus, the local stress becomes $\sigma(u) = F'(u) = u + g(u)$. If we insert $F'(u)$ into (2.4), the nonlinear wave equation

$$u_{tt} = (u + g(u))_{xx} \tag{2.9}$$

is obtained. We assumed at the beginning that there is no initial energy, $F(0) = 0$, and no initial stress, $F'(0) = 0$. So, $F'(0) = g(0) = 0$. This equation will be strictly hyperbolic if $F''(u) = 1 + g'(u) > 0$ which gives $g'(u) > -1$. Also if $F'''(u) = g''(u) \neq 0$, then it will be genuinely nonlinear. Therefore, singularities will always occur.

We note that in the absence of nonlinear terms, the wave equation $u_{tt} - u_{xx} = 0$ is nondispersive. One of the main points of this thesis study will be to show how the dispersive effect resulting from nonlocality regularizes (2.9).

2.2 Nonlinear Wave Propagation in Nonlocal Elasticity

There are some disadvantages of classical theory of nonlinear elasticity mentioned in the previous section. One of the major drawbacks is that it does not include any intrinsic length scale and consequently does not take into account the long range forces that become increasingly important at small scales. As a result, the local theory of elasticity is incapable of predicting, for instance, (i) the dispersive nature of harmonic waves in crystal lattices and (ii) the boundedness of the stress field near the tip of a crack. In order to overcome such deficiencies various generalizations of the local theory of elasticity have been proposed. One such generalization is the theory of nonlocal elasticity which has been developed by Kröner [25], Eringen and Edelen [13], Kunin [26], Rogula [35], Eringen [15, 16] over the last several decades. The theory of nonlocal elasticity differs from the local theory of elasticity as the stress at a point depends on the strain field at every point in the body. Although there has been a considerable amount of research done on small scale effects within the context of the theory of nonlocal elasticity, they are mostly restricted to linear models. In this thesis study, we will study two Cauchy problems based on a one-dimensional nonlinear model of nonlocal elasticity.

2.2.1 A Nonlocal Model for Longitudinal Waves: Single Equation

Since in the nonlocal theory of elasticity the stress at a point depends on the strain field at every point in the body, it is written as a functional of the strain field (see [16] and the references cited therein). We now derive the dimensionless form of the equation governing the resulting dynamics in one space dimension.

Consider a one-dimensional, homogeneous, non-linearly and *non-locally* elastic infinite medium. This time, the local stress $\sigma(u) = F'(u)$ is replaced with the nonlocal stress $S(u)$ in the equation of motion (2.4):

$$u_{tt} = (S(u))_{xx}.$$

In contrast with classical elasticity, we employ a nonlocal model of constitutive equation, which gives the stress S as a general nonlinear nonlocal function of the strain u. The constitutive equation for nonlinear nonlocal elastic response considered here has the following form

$$S = S(x,t) = \int_{\mathbb{R}} \beta(x-y)\sigma(u(y,t))\mathrm{d}y, \qquad \sigma(x,t) = F'(u(x,t)) \tag{2.10}$$

where σ is the classical (local) stress, F is the local strain-energy density function, β is the kernel function [8]. The kernel β serves as a weight on the relative contribution of the local stress $\sigma(y,t)$ at a point y in a neighborhood of x to the nonlocal stress $S(x,t)$. So, when the kernel becomes the Dirac delta measure, the classical constitutive relation of elasticity is recovered and hence (2.4) are recovered.

If a stress-free undistorted state is considered as the reference configuration, the strain energy function must satisfy $F(0) = F'(0) = 0$. Using (2.7) and (2.8), the nonlocal stress becomes

$$S(u) = \int_{\mathbb{R}} \beta(x-y)(u(y,t) + g(u(y,t)))\mathrm{d}y \tag{2.11}$$

which is in fact the convolution $\beta * (u + g(u))$. Thus, the equation of motion for the strain in nonlocal elasticity is

$$u_{tt} = (\beta * (u + g(u)))_{xx}. \tag{2.12}$$

It can be observed from (2.11) that our proposed constitutive relation differ from the standard constitutive relations by its property that nonlocality both affects linear

and nonlinear parts of the model. We observe that the corresponding linear form of (2.12) is a dispersive equation with $c^2 = \widehat{\beta}(k)$.
We can also show that (2.12) models propagation of longitudinal waves. Let (X, Y, Z) be the position of a reference point in three-dimensional space and (x, y, z) be its position at time t in the body. Consider a one-dimensional wave motion such as

$$x = X + U(X,t), \quad y = Y, \quad z = Z$$

which reveals that the displacement field is $(U(X,t), 0, 0)$ and the wave propagates in the x direction. Since the displacement of particles is *parallel* to the direction of wave propagation, this wave is called *longitudinal wave* and obeys (2.12).

2.2.2 A Nonlocal Model for Transverse Waves: Coupled System

The motion corresponding to *transverse* waves is described by

$$x = X, \quad y = Y + U(X,t), \quad z = Z + V(X,t).$$

Hence, the displacement field is $(0, U(X,t), V(X,t))$. In other words, the displacement of particles is *perpendicular* to the direction of propagation.
The stress components P and Q can be expressed in terms of the strains $u = U_x$ and $v = V_X$. The equations of motion in this case are:

$$u_{tt} = (P(u,v))_{xx},$$
$$v_{tt} = (Q(u,v))_{xx}.$$

The constitutive equations for the above transverse motion become:

$$P(u,v) = \beta_1 * \frac{\partial F}{\partial u},$$
$$Q(u,v) = \beta_2 * \frac{\partial F}{\partial v}.$$

Here, $F(u,v)$ is the strain energy density function with the properties $F(0,0) = 0$ and $\nabla F(0,0) = 0$. Thus, nonlocal nonlinear two coupled partial differential equations governing the propagation of transverse waves are

$$u_{tt} = (\beta_1 * \frac{\partial F}{\partial u})_{xx},$$
$$v_{tt} = (\beta_2 * \frac{\partial F}{\partial v})_{xx}.$$

This system may be viewed as a natural generalization of the single equation (2.12) to a coupled system of two nonlocal nonlinear equations.
As a special case, assume that $F(u,v) = \frac{1}{2}(u^2+v^2) + G(u,v)$ with

$$g_1 = \frac{\partial G}{\partial u}, \qquad g_2 = \frac{\partial G}{\partial v}. \tag{2.13}$$

Then our coupled equations become,

$$u_{tt} = (\beta_1 * (u + g_1(u,v)))_{xx}, \tag{2.14}$$

$$v_{tt} = (\beta_2 * (v + g_2(u,v)))_{xx}. \tag{2.15}$$

We note that the nonlinear functions g_i (i=1,2) satisfy the exactness condition

$$\frac{\partial g_1}{\partial u} = \frac{\partial g_2}{\partial v}. \tag{2.16}$$

Remark 2.2.1 As we mentioned above, (2.14)-(2.15) may be regarded as the system governing the one-dimensional propagation of two "pure" transverse nonlinear waves in a nonlocal elastic isotropic homogeneous medium [8]. From the modelling point of view we want to remark that, in general, the system will also contain a third equation characterizing the propagation of a longitudinal wave. Nevertheless, with some further restrictions imposed on the form of F, one may get transverse waves without a coupled longitudinal wave [21]. We also want to note that, in the general case, the exactness condition (2.16) is necessary in order to obtain the conservation law of energy.

2.3 Kernel Functions

An important open question in the nonlocal theory of elasticity is how to choose the kernel functions appearing in (2.12) and (2.14)-(2.15) which represent the details of the atomic scale effects. The triangular kernel, the exponential kernel, and the Gaussian kernel (see, for instance, equations (3.3), (3.4) and (3.5) of [14], respectively) are examples of only the most commonly used kernel functions. In general it is assumed that the kernel function β is a nonnegative even function monotonically decreasing for $x > 0$. We refer to [33] for an example of a non-monotone, sign changing kernel function. In this study, we attempt to cover both types of the kernels used in the literature by a general class of kernel functions.

The kernels used in the literature all satisfy the positivity condition $\widehat{\beta}(\xi) \geq 0$. This is a natural consequence of the wave character of the equation of motion, which means that the phase velocity c is real. On the other hand, in the literature the following two conditions are imposed on the kernel: $\beta(0) \geq \beta(x)$ and $\beta(-x) = \beta(x)$. In fact, these conditions are implied by the positivity of $\widehat{\beta}(\xi)$ through Bochner's theorem [34]. In addition, we assume that the kernel β is an integrable function whose Fourier transform satisfies

$$0 \leq \widehat{\beta}(\xi) \leq C(1+\xi^2)^{-r/2} \quad \text{for all } \xi \tag{2.17}$$

for a suitable constant $C > 0$. This inequality corresponds to the decay rate of the Fourier transform of the kernel function, which in turn is related to the smoothness of β. Here the exponent r can be any real number (not necessarily an integer) and it determines the regularizing effect of the convolution in the model. In this study we only consider kernels with $r \geq 2$. When $r < 2$, the model is linearly unstable with unbounded growth rate at short wavelengths and thus this case seems to be of a different nature.

In the next subsection we present several examples of kernel functions, showing how the general class of kernels defined by (2.17) covers the most common used kernels in the literature.

2.3.1 Examples for the Kernel

The following list of kernels contains the most commonly used kernels.

1. *The Dirac measure*: $\beta = \delta$. In this case $r = 0$, and we recover the wave equation (2.9) of one-dimensional elasticity.

2. *The triangular kernel* [14]:

$$\beta(x) = \begin{cases} 1-|x|, & |x| \leq 1 \\ 0, & |x| \geq 1 \ . \end{cases}$$

   Since

$$\widehat{\beta}(\xi) = \frac{4}{\xi^2}\sin^2(\frac{\xi}{2}),$$

   we have $r = 2$. Note that

$$(\beta * v)_{xx} = v(x-1) - 2v(x) + v(x+1)$$

and (2.12) becomes a differential-difference equation.

3. *The exponential kernel* [14]: $\beta(x) = \frac{1}{2}e^{-|x|}$. Since $\widehat{\beta}(\xi) = (1+\xi^2)^{-1}$, we have $r = 2$. Note that β is the Green's function for the operator $1 - \partial_x^2$ so that

$$(\beta * v)_{xx} = (1-\partial_x^2)^{-1} v_{xx} = \beta * v - v \ .$$

(2.12) becomes the IMBq equation

$$u_{tt} - u_{xx} - u_{xxtt} = (g(u))_{xx} \ .$$

4. *The double-exponential kernel* [29]:

$$\beta(x) = \frac{1}{2(c_1^2 - c_2^2)}(c_1 e^{-|x|/c_1} - c_2 e^{-|x|/c_2})$$

where c_1 and c_2 are real and positive constants. Since $\widehat{\beta}(\xi) = (1+\gamma_1\xi^2+\gamma_2\xi^4)^{-1}$ where $\gamma_1 = c_1^2 + c_2^2$ and $\gamma_2 = c_1^2 c_2^2$, we have $r = 4$. As above, β is the Green's function for the operator $1 - \gamma_1\partial_x^2 + \gamma_2 D_x^4$ and

$$(\ \beta * v)_{xx} = (1 - \gamma_1\partial_x^2 + \gamma_2\partial_x^4)^{-1} v_{xx} \ .$$

(2.12) becomes the higher-order Boussinesq equation

$$u_{tt} - u_{xx} - \gamma_1 u_{xxtt} + \gamma_2 u_{xxxxtt} = (g(u))_{xx} \ .$$

5. *The Gaussian kernel* [14]:

$$\beta(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

Note that $\widehat{\beta}(\xi) = e^{-\xi^2/2}$.

6. *A sign-changing kernel* [33]:

$$\beta(x) = \frac{1}{\sqrt{2\pi}}(1-x^2)e^{-x^2/2}.$$

Note that $\widehat{\beta}(\xi) = \xi^2 e^{-\xi^2/2}$.

In these last two examples the kernel function β, hence its Fourier transform $\widehat{\beta}$ is rapidly decreasing, and we can take any r in (2.17). The equation of motion, (2.12), is an integro-differential equation.

For suitable choices of the kernel functions, the system (2.14)-(2.15) also reduces to some well-known coupled systems of nonlinear wave equations. To illustrate this we consider the exponential kernel $\beta_1(x) = \beta_2(x) = \frac{1}{2}e^{-|x|}$. (2.14)-(2.15) yields the coupled improved Boussinesq equations

$$u_{tt} - u_{xx} - u_{xxtt} = (g_1(u, v))_{xx}, \tag{2.18}$$

$$v_{tt} - v_{xx} - v_{xxtt} = (g_2(u, v))_{xx}. \tag{2.19}$$

Similarly, if the kernels $\beta_1(x)$ and $\beta_2(x)$ are chosen as the double-exponential kernel, then (2.14)-(2.15) reduces to the coupled higher-order Boussinesq system

$$u_{tt} - u_{xx} - au_{xxtt} + bu_{xxxxtt} = (g_1(u, v))_{xx}, \tag{2.20}$$

$$v_{tt} - v_{xx} - av_{xxtt} + bv_{xxxxtt} = (g_2(u, v))_{xx}. \tag{2.21}$$

These examples make it obvious that choosing the kernels $\beta_i(x)$ in (2.14)-(2.15) as the Green's functions of constant coefficient linear differential operators in x will yield similar coupled systems describing the bi-directional propagation of nonlinear waves in dispersive medium.

Chapter 3

Local Existence

In the previous chapter, two nonlocal nonlinear models governing longitudinal and transverse wave motions in a nonlocal nonlinear continuous medium have been derived, respectively. In this chapter, the Cauchy problems corresponding to these nonlocal nonlinear wave equations are considered and the existence of solutions locally in time to the Cauchy problems are proved under some suitable assumptions on the initial data, the nonlinear function g and the regularity of the kernel functions. Before that, as a preliminary for the rest of this chapter, we state a basic theorem which is about the local existence and uniqueness of the solution of the initial-value problem for an ordinary differential equation and a basic result on convolutions, known as Young's inequality.

Theorem 3.0.1 [27] Consider the initial-value problem

$$u_t = F(u),$$
$$u(0) = u_0.$$

Let $F : B \to B$ be a locally Lipschitz continuous function in $u(t)$ from a Banach space B into itself. Then for given initial data $u_0 \in B$, there is some $T > 0$ such that the initial-value problem above is well-posed with solution $u \in C^1([0, T), B)$.

Lemma 3.0.2 Let $1 \leq p \leq \infty$ and $f \in L^1(\mathbb{R}), g \in L^p(\mathbb{R})$. The convolution $(f * g)(x) = \int_{\mathbb{R}} f(y - x)g(y)dy$ is well-defined and $f * g \in L^p(\mathbb{R})$ with

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p.$$

3.1 Local Existence for the Single Equation

Consider the Cauchy problem of the single nonlocal equation describing longitudinal wave propagation in a nonlocal elastic medium

$$u_{tt} = (\beta * (u + g(u)))_{xx}, \quad x \in \mathbb{R}, \quad t > 0, \tag{3.1}$$

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x). \tag{3.2}$$

Here, β is an integrable function whose Fourier transform satisfies (2.17). The function space we choose the initial data from certainly affects the solution space. To illustrate the approach, we start with an existence theorem where the initial data are in $L^p \cap L^\infty$. Then, we study the problem in Sobolev space H^s and obtain the results about the local well-posedness depending on the smoothness properties of the kernel function β.

3.1.1 Local Existence in $L^p \cap L^\infty$

First, we will obtain some estimates on the nonlinear term:

Lemma 3.1.1 Let $g \in C^1(\mathbb{R})$ with $g(0) = 0$. Then for any $u \in L^p \cap L^\infty$, we have $g(u) \in L^p \cap L^\infty$. Moreover there is some constant $a(M)$ depending on M such that for all $u \in L^p \cap L^\infty$ with $\|u\|_\infty \leq M$

$$\|g(u)\|_\infty \leq a(M)\|u\|_\infty \tag{3.3}$$

$$\|g(u)\|_p \leq a(M)\|u\|_p \tag{3.4}$$

Proof: Using the mean value theorem,

$$|g(u) - g(0)| \leq a(M)|u| \tag{3.5}$$

where $a(M) = \sup_{|\theta| \leq M} |g'(\theta)|$. Since $g(0) = 0$, (3.3) and (3.4) holds. □

Lemma 3.1.2 Let $g \in C^1(\mathbb{R})$. Then for any $M > 0$ there is some constant $b(M)$ such that for all $u, v \in L^p \cap L^\infty$ with $\|u\|_\infty \leq M$, $\|v\|_\infty \leq M$ and $\|u\|_p \leq M$, $\|v\|_p \leq M$, we have

$$\|g(u) - g(v)\|_\infty \leq b(M)\|u - v\|_\infty, \quad \text{and} \quad \|g(u) - g(v)\|_p \leq b(M)\|u - v\|_p.$$

Proof: The proof follows from the mean value theorem estimate:

$$|g(u) - g(v)| \leq \sup_{|\theta| \leq 2M} |g'(\theta)||u - v|.$$

□

Theorem 3.1.3 Let $1 \leq p \leq \infty$. Let $\beta_{xx} \in L^1$ and $\varphi, \psi \in L^p \cap L^\infty$. Then there is some $T > 0$ such that the Cauchy problem (3.1)-(3.2) is well-posed with solution $u \in C^2([0,T), L^p \cap L^\infty)$.

Proof: To use Theorem 3.0.1, we convert (3.1) into an $L^p \cap L^\infty$ valued system of ordinary differential equations given below:

$$\begin{aligned} u_t &= v & u(0) &= \varphi \\ v_t &= (\beta * f(u))_{xx} & v(0) &= \psi, \end{aligned}$$

where $f(u) = u + g(u)$ for simplicity. We must show that the right-hand side of the system is Lipschitz on $L^p \cap L^\infty$. Since the first component is linear, it is enough to prove the Lipschitz condition only for the map $K(u) = (\beta * (f(u)))_{xx} = \beta_{xx} * f(u)$. To this end, we estimate:

$$\|\beta_{xx} * f(u)\|_p \leq \|\beta_{xx}\|_1 \|f(u)\|_p \leq a(M)\|\beta_{xx}\|_1 \|u\|_p$$

and

$$\|\beta_{xx} * f(u)\|_\infty \leq \|\beta_{xx}\|_1 \|f(u)\|_\infty \leq a(M)\|\beta_{xx}\|_1 \|u\|_\infty.$$

Thus, we observe that $\beta_{xx} * f(u)$ maps $L^p \cap L^\infty$ into itself. A similar estimate using Lemma 3.1.2 implies that $\beta_{xx} * f(u)$ is locally Lipschitz on $L^p \cap L^\infty$. □

Remark 3.1.4 The triangular kernel and the exponential kernel given in Section 2.3.1 do not satisfy the condition $\beta_{xx} \in L^1$. However, since for each of them β_{xx} is the sum of an L^1 function and some δ functions, we can also obtain local existence in $L^p \cap L^\infty$ the same way. The key step is:

$$\|(\beta * f(u))_{xx}\|_p = \|h * f(u) + \textit{some shifts of } f(u)\|_p \leq \|h * f(u)\|_p + C(\|f(u)\|_p)$$

where $h \in L^1(\mathbb{R})$. From the proof of Theorem 3.1.3, solution exists locally in $L^p \cap L^\infty$.

3.1.2 Local Existence in H^s

Now, we take the initial data from the Sobolev space H^s. In order to prove the related theorems, we first need two lemmas referring to boundedness and Lipschitz property of the nonlinear term [6, 39].

Lemma 3.1.5 Let $s \geq 0$, $g \in C^{[s]+1}(\mathbb{R})$ with $g(0) = 0$. Then for any $u \in H^s \cap L^\infty$, we have $g(u) \in H^s \cap L^\infty$. Moreover, there is some constant $A(M)$ depending on M and s such that for all $u \in H^s \cap L^\infty$ with $\|u\|_\infty \leq M$

$$\|g(u)\|_s \leq A(M)\|u\|_s .$$

Lemma 3.1.6 Let $s \geq 0$, $g \in C^{[s]+1}(\mathbb{R})$. Then for any $M > 0$ there is some constant $B(M)$ such that for all $u, v \in H^s \cap L^\infty$ with $\|u\|_\infty \leq M$, $\|v\|_\infty \leq M$ and $\|u\|_s \leq M$, $\|v\|_s \leq M$ we have

$$\|g(u) - g(v)\|_s \leq B(M)\|u - v\|_s \quad \text{and} \quad \|g(u) - g(v)\|_\infty \leq B(M)\|u - v\|_\infty .$$

For $s > \frac{1}{2}$, by the Sobolev embedding theorem, $H^s \subset L^\infty$. Then the bounds on L^∞ norms in Lemma 3.1.6 become redundant and we get:

Corollary 3.1.7 Let $s > \frac{1}{2}$, $g \in C^{[s]+1}(\mathbb{R})$. Then for any $M > 0$ there is some constant $B(M)$ such that for all $u, v \in H^s$ with $\|u\|_s \leq M$, $\|v\|_s \leq M$ we have

$$\|g(u) - g(v)\|_s \leq B(M)\|u - v\|_s .$$

From now on, we always assume that $g \in C^N(\mathbb{R})$ with $g(0) = 0$, where $N \geq \max(1, s)$ is an integer. Theorems 3.1.8 and 3.1.9 given below show the local well-posedness of the Cauchy problem (3.1)-(3.2) in H^s:

Theorem 3.1.8 Let $s > 1/2$ and $r \geq 2$. Then there is some $T > 0$ such that the Cauchy problem (3.1)-(3.2) is well posed with solution in $C^2([0, T], H^s)$ for initial data $\varphi, \psi \in H^s$.

Proof: We again use Theorem 3.0.1 and convert (3.1) into an H^s valued system of ordinary differential equations given below:

$$\begin{aligned} u_t &= v & u(0) &= \varphi \\ v_t &= (\beta * f(u))_{xx} & v(0) &= \psi, \end{aligned}$$

where $f(u) = u + g(u)$ for simplicity. Fourier transform representation for H^s norm gives

$$\|(\beta * w)_{xx}\|_s = \left\|(1+\xi^2)^{s/2}(-\xi^2)\widehat{\beta}(\xi)\widehat{w}(\xi)\right\|. \tag{3.6}$$

Using decay condition on β, we get

$$|-\xi^2\widehat{\beta}(\xi)| \leq C\xi^2(1+\xi^2)^{-r/2}.$$

Here $\xi^2(1+\xi^2)^{-r/2} \leq 1$ since $r \geq 2$. Inserting these into (3.6) gives

$$\begin{aligned}\|(\beta * w)_{xx}\|_s &\leq C\left\|(1+\xi^2)^{s/2}\widehat{w}(\xi)\right\| \\ &= C\|w\|_s. \end{aligned} \tag{3.7}$$

Recalling that $(\beta * w)_{xx} = \beta * w_{xx}$ in the distribution sense, we observe that $\beta * (\ \ )_{xx}$ is a bounded linear map on H^s. Then since Corollary 3.1.7 applies, $\beta * (f(u))_{xx}$ is locally Lipschitz on H^s. □

Removing the restriction $s > 1/2$, we observe that the L^∞ estimate will be needed to control the nonlinear term. By the way, even if we start with the L^∞ data, the term $\beta * (\ \ )_{xx}$ may not stay in L^∞. The following theorem gives the necessary assumptions dealing with such case:

Theorem 3.1.9 Let $s \geq 0$ and $r > 5/2$. Then there is some $T > 0$ such that the Cauchy problem (3.1)-(3.2) is well posed with solution in $C^2([0,T], H^s \cap L^\infty)$ for initial data $\varphi, \psi \in H^s \cap L^\infty$.

Proof: As in the proof of Theorem 3.1.8 we convert the problem into an ODE system on $H^s \cap L^\infty$ where the space is endowed with the norm $\|w\|_{s,\infty} = \|w\|_s + \|w\|_\infty$. Then all we need is to show that $\beta * (f(u))_{xx}$ is Lipschitz on $H^s \cap L^\infty$. Since

$$\left|-\xi^2\widehat{\beta}(\xi)\right| \leq C\xi^2(1+\xi^2)^{-r/2} \leq C(1+\xi^2)(1+\xi^2)^{-r/2} = C(1+\xi^2)^{-(r-2)/2},$$

we have

$$\begin{aligned}\|\beta * w_{xx}\|_{s+r-2} &= \left\|(1+\xi^2)^{(s+r-2)/2}(-\xi^2)\widehat{\beta}(\xi)\widehat{w}(\xi)\right\| \\ &\leq C\left\|(1+\xi^2)^{((s+r-2)-(r-2))/2}\widehat{w}(\xi)\right\| \\ &= C\left\|(1+\xi^2)^{s/2}\widehat{w}(\xi)\right\| = C\|w\|_s.\end{aligned}$$

Then $\beta * (\ \ )_{xx}$ is a bounded linear map from H^s into H^{s+r-2}. Since $s \geq 0$ and $r > \frac{5}{2}$ we have $s + r - 2 > \frac{5}{2} - 2 = \frac{1}{2}$. Again the Sobolev embedding theorem implies that $\beta * (\ \ )_{xx}$ is a bounded linear map from $H^s \cap L^\infty$ into $H^s \cap L^\infty$. Lemma 3.1.6 implies the Lipschitz condition on $H^s \cap L^\infty$.

□

Remark 3.1.10 Increasing the decay rate r results in the more regularization of the Cauchy problem. Theorem 3.1.9 shows that the more regularization of the Cauchy problem (3.1)-(3.2) allows less smooth initial data.

Remark 3.1.11 In Theorems 3.1.8 and 3.1.9 we have not used the assumption $\widehat{\beta}(\xi) \geq 0$ given in (2.17); so in fact these results hold for more general kernel functions.

Remark 3.1.12 Going back to the kernels given in Subsection 2.3.1, we see that for the double-exponential kernel, the Gaussian kernel and the sign-changing kernel, $r > \frac{5}{2}$ so Theorem 3.1.9 applies. Although Theorem 3.1.9 does not apply for the triangular kernel and the exponential kernel where $r = 2$, in Subsection 4.1.3 we show that it is still possible to allow for less smooth data for such kernels.

3.2 Local Existence for the Coupled System

This time consider the Cauchy problem of the coupled nonlocal system governing propagation of two transverse waves in a nonlocal elastic medium

$$u_{tt} = (\beta_1 * (u + g_1(u,v)))_{xx}, \qquad x \in \mathbb{R}, \quad t > 0 \tag{3.8}$$

$$v_{tt} = (\beta_2 * (v + g_2(u,v)))_{xx}, \tag{3.9}$$

$$u(x,0) = \varphi_1(x), \quad u_t(x,0) = \psi_1(x), \tag{3.10}$$

$$v(x,0) = \varphi_2(x), \quad v_t(x,0) = \psi_2(x). \tag{3.11}$$

We assume that nonlinear functions $g_i(u,v)$ satisfy the exactness condition (2.16) or equivalently there exists a function $G(u,v)$ satisfying (2.13).

Let $U = (u,v)$ be a vector function. Vector-valued versions of Lemmas 3.1.5 and 3.1.6 (see also [6, 37, 40]) are required to evaluate the nonlinear terms in the coupled system. We define the norms $\|U\|_s = \|u\|_s + \|v\|_s$ and $\|U\|_\infty = \|u\|_\infty + \|v\|_\infty$.

Lemma 3.2.1 Let $s \geq 0$, $h \in C^{[s]+1}(\mathbb{R}^2)$ with $h(0) = 0$. Then for any $U = (u, v) \in (H^s \cap L^\infty)^2$, we have $h(U) \in H^s \cap L^\infty$. Moreover there is some constant $A(M)$ depending on M such that for all $U \in (H^s \cap L^\infty)^2$ with $\|U\|_\infty \leq M$

$$\|h(U)\|_s \leq A(M)\|U\|_s .$$

Lemma 3.2.2 Let $s \geq 0$, $h \in C^{[s]+1}(\mathbb{R}^2)$. Then for any $M > 0$ there is some constant $B(M)$ such that for all $U, V \in (H^s \cap L^\infty)^2$ with $\|U\|_\infty \leq M$, $\|V\|_\infty \leq M$ and $\|U\|_s \leq M$, $\|V\|_s \leq M$ we have

$$\|h(U) - h(V)\|_s \leq B(M)\|U - V\|_s \quad \text{and} \quad \|h(U) - h(V)\|_\infty \leq B(M)\|U - V\|_\infty .$$

For $s > \frac{1}{2}$, $H^s \subset L^\infty$ by the Sobolev embedding theorem. Then the bounds on L^∞ norms in Lemma 3.2.2 become unnecessary and we get:

Corollary 3.2.3 Let $s > \frac{1}{2}$, $h \in C^{[s]+1}(\mathbb{R}^2)$. Then for any $M > 0$ there is some constant $B(M)$ such that for all $U, V \in (H^s)^2$ with $\|U\|_s \leq M$, $\|V\|_s \leq M$ we have

$$\|h(U) - h(V)\|_s \leq B(M)\|U - V\|_s .$$

Throughout this study we assume that $g_1, g_2 \in C^N(\mathbb{R}^2)$ with $g_1(0) = g_2(0) = 0$, where $N \geq \max(1, s)$ is an integer. Local well-posedness of the Cauchy problem (3.8)-(3.11), similar with the ones in Section 3.1, are proved below:

Theorem 3.2.4 Let $s > 1/2$ and $r_1, r_2 \geq 2$. Then there is some $T > 0$ such that the Cauchy problem (3.8)-(3.11) is well posed with solution u, v in $C^2([0, T], H^s)$ for initial data $\varphi_i, \psi_i \in H^s$ $(i = 1, 2)$.

Proof: The coupled system can also be converted into an H^s valued system of ordinary differential equations

$$\begin{aligned} u_t &= w_1 & u(0) &= \varphi_1, \\ v_t &= w_2 & v(0) &= \varphi_2, \\ w_{1t} &= (\beta_1 * f_1(u, v))_{xx} & w_1(0) &= \psi_1, \\ w_{2t} &= (\beta_2 * f_2(u, v))_{xx} & w_2(0) &= \psi_2, \end{aligned}$$

where $f_1(u, v) = u + g_1(u, v)$ and $f_2(u, v) = v + g_2(u, v)$ to shorten the notation. As we mentioned before, we must check whether the right-hand side of the system is

Lipschitz on H^s. We have

$$\begin{aligned}\|(\beta_i * w)_{xx}\|_s &= \left\|(1+\xi^2)^{s/2}(-\xi^2)\widehat{\beta}_i(\xi)\widehat{w}(\xi)\right\| \\ &\leq \left\|(1+\xi^2)^{s/2}C_i\xi^2(1+\xi^2)^{-r_i/2}\widehat{w}(\xi)\right\| \\ &\leq C_i\left\|(1+\xi^2)^{s/2}\widehat{w}(\xi)\right\| = C_i\|w\|_s\end{aligned}$$

since $r_i \geq 2$ for $i = 1, 2$. So, $\beta_i * (\ \ )_{xx}$ is also a bounded linear map on H^s. By Corollary 3.2.3, $\beta_i * (f_i(u,v))_{xx}$ is locally Lipschitz on H^s. □

As in Theorem 3.1.9, Theorem 3.2.4 can be extended to the case of $H^s \cap L^\infty$ for $0 \leq s \leq 1/2$.

Theorem 3.2.5 Let $s \geq 0$ and $r_i > \frac{5}{2}$. Then there is some $T > 0$ such that the Cauchy problem (3.8)-(3.11) is well posed with solution u, v in $C^2([0,T], H^s \cap L^\infty)$ for initial data $\varphi_i, \psi_i \in H^s \cap L^\infty$ $(i = 1, 2)$.

Proof: Following the process in the proof of Theorem 3.1.9, it can be observed that

$$\|\beta_i * w_{xx}\|_{s+r_i-2} \leq C_i\|w\|_s$$

for $i = 1, 2$. This implies that $\beta_i * (\ \ )_{xx}$ is a bounded linear map from H^s into H^{s+r_i-2}. Since $s \geq 0$ and $r_i > \frac{5}{2}$ we have $s + r_i - 2 > \frac{5}{2} - 2 = \frac{1}{2}$. Hence, $\beta_i * (\ \ )_{xx}$ is a bounded linear map from $H^s \cap L^\infty$ into $H^s \cap L^\infty$. Lemma 3.2.2 implies the Lipschitz condition on $H^s \cap L^\infty$.

□

Remark 3.2.6 Local existence results do not depend on the assumption $\widehat{\beta}_i(\xi) \geq 0$ and the exactness condition (2.16). Thus, they hold for more general forms of both the kernel functions and the nonlinear terms.

Chapter 4

Global Existence

A typical local-in-time existence theorem asserts that either a solution exists for all time or else there is a time $T^* < \infty$ such that some norm of the solution u becomes unbounded as $t \nearrow T^*$. Here, we give a basic condition closely related to this fact. The solution in Theorems 3.1.8 and 3.1.9 can be extended to a maximal interval $[0, T_{\max})$ where finite $T_{\max}$ is characterized by the blow up condition

$$\limsup_{t \to T_{\max}^-} \left(\|u(t)\|_{s,\infty} + \|u_t(t)\|_{s,\infty} \right) = \infty .$$

Clearly $T_{\max} = \infty$, i.e. there is a global solution iff

$$\text{for any } T < \infty, \text{ we have } \limsup_{t \to T^-} \left(\|u(t)\|_{s,\infty} + \|u_t(t)\|_{s,\infty} \right) < \infty .$$

If we replace u in this condition by a vector-valued function U, the same condition holds for the Cauchy problem (3.8)-(3.11) and the solution in Theorems 3.2.4 and 3.2.5 can also be extended to a maximal time interval of existence $[0, T_{\max})$. The lemma given below follows from this result:

Lemma 4.0.7 Suppose the conditions of Theorem 3.1.8 or 3.1.9 hold and u is the solution of the Cauchy problem (3.1)-(3.2). Then there is a global solution if and only if

$$\text{for any } T < \infty, \text{ we have } \limsup_{t \to T^-} \|u(t)\|_{\infty} < \infty .$$

Proof: Clearly if $\limsup_{t \to T^-} \left(\|u(t)\|_{s,\infty} + \|u_t(t)\|_{s,\infty} \right) < \infty$ then $\limsup_{t \to T^-} \|u(t)\|_{\infty} < \infty$. Conversely, assume the solution exists for $t \in [0, T)$ and $\|u(t)\|_{\infty} \leq M$ for all $0 \leq t \leq T$. Integrating the equation twice and calculating the resulting double integral as an iterated integral, we have

$$u(x,t) = \varphi(x) + t\psi(x) + \int_0^t (t-\tau)(\beta * f(u))_{xx}(x,\tau)\,d\tau, \tag{4.1}$$

$$u_t(x,t) = \psi(x) + \int_0^t (\beta * f(u))_{xx}(x,\tau)\,d\tau . \tag{4.2}$$

Hence for all $t \in [0,T)$

$$\|u(t)\|_s \leq \|\varphi\|_s + T\|\psi\|_s + T\int_0^t \|(\beta * f(u))_{xx}(\tau)\|_s\,d\tau,$$

$$\|u_t(t)\|_s \leq \|\psi\|_s + \int_0^t \|(\beta * f(u))_{xx}(\tau)\|_s\,d\tau .$$

But $\|(\beta * f(u))_{xx}(\tau)\|_s \leq C\|(f(u))(\tau)\|_s \leq CA(M)\|u(\tau)\|_s$ where the first inequality follows from (3.7) and the second from Lemma 3.1.5. Then

$$\|u(t)\|_s + \|u_t(t)\|_s \leq \|\varphi\|_s + (T+1)\|\psi\|_s + (T+1)CA(M)\int_0^t \|u(\tau)\|_s\,d\tau,$$

and Gronwall's Lemma implies

$$\|u(t)\|_s + \|u_t(t)\|_s \leq (\|\varphi\|_s + (T+1)\|\psi\|_s)\,e^{(T+1)CA(M)T} < \infty$$

for all $t \in [0,T)$ and consequently $\limsup_{t\to T^-}\left(\|u(t)\|_{s,\infty} + \|u_t(t)\|_{s,\infty}\right) < \infty$. □

As in Lemma 4.0.7, the following lemma says that it is enough to control $\|U(t)\|_\infty$ to prove the global existence of a solution of the Cauchy problem (3.8)-(3.11). The proof which is almost the same as that of the previous lemma is omitted.

Lemma 4.0.8 Suppose the conditions of Theorem 3.2.4 or 3.2.5 hold and U is the solution of the Cauchy problem (3.8)-(3.11). Then there is a global solution if and only if

$$\text{for any } T < \infty, \text{ we have } \limsup_{t\to T^-}\|U(t)\|_\infty < \infty .$$

The rest of this chapter consists of the results obtained under some suitable assumptions so that the solutions of both Cauchy problems (3.1)-(3.2) and (3.8)-(3.11) exist globally in time. The proofs refer to the conservation of energy which will be given for both problems.

4.1 Global Existence for the Single Equation

We will assume that $\widehat{\beta}(\xi)$ has only isolated zeros which gives

$$0 < \widehat{\beta}(\xi) \leq C(1+\xi^2)^{-r/2} .$$

We will use an unbounded operator P as $Pv = \mathcal{F}^{-1}\left(|\xi|^{-1}(\widehat{\beta}(\xi))^{-1/2}\widehat{v}(\xi)\right)$ with the inverse Fourier transform $\mathcal{F}^{-1}$. Although P may fail to be a bounded operator, for $s \geq 0$ its inverse $P^{-1} : H^{s+1-\frac{r}{2}} \to H^s$ is bounded:

$$\begin{aligned}\left\|P^{-1}v\right\|_s &= \left\|(1+\xi^2)^{s/2}|\xi|\left(\widehat{\beta}(\xi)\right)^{1/2}\widehat{v}(\xi)\right\| \\ &\leq \left\|(1+\xi^2)^{s/2}|\xi|(1+\xi^2)^{-r/4}\widehat{v}(\xi)\right\| \\ &\leq \left\|(1+\xi^2)^{s/2}(1+\xi^2)^{1/2}(1+\xi^2)^{-r/4}\widehat{v}(\xi)\right\| \\ &= \left\|(1+\xi^2)^{\frac{1}{2}(s+1-\frac{r}{2})}\widehat{v}(\xi)\right\| = \|v\|_{s+1-\frac{r}{2}}\,.\end{aligned}$$

Moreover it is one-to-one:

$$\ker(P^{-1}) = \{v \in H^{s+1-\frac{r}{2}} : P^{-1}v = 0\}$$

and, since $\widehat{\beta}(\xi)$ has isolated zeros and $|\xi| \neq 0$,

$$P^{-1}v = \mathcal{F}^{-1}\left(|\xi|\left(\widehat{\beta}(\xi)\right)^{1/2}\widehat{v}(\xi)\right) = 0 \quad \Leftrightarrow \quad v = 0.$$

Then P is well defined with $domain(P) = range(P^{-1})$. Observe that

$$P^2(\beta * v)_{xx} = \mathcal{F}^{-1}\left(|\xi|^{-2}\left(\widehat{\beta}(\xi)\right)^{-1}\left(-|\xi|^2\right)\left(\widehat{\beta}(\xi)\right)\widehat{v}(\xi)\right) = -v. \qquad (4.3)$$

4.1.1 Conservation of Energy

Lemma 4.1.1 Suppose the conditions of Theorem 3.1.8 or 3.1.9 hold and the solution of the Cauchy problem (3.1)-(3.2) exists in $C^2([0,T), H^s \cap L^\infty)$ for some $s \geq 0$. If $P\psi \in L^2$, then $Pu_t \in C^1([0,T), L^2)$. If moreover $P\varphi \in L^2$, then $Pu \in C^2([0,T), L^2)$.

Proof: From (4.2) we get

$$Pu_t(x,t) = P\psi(x) - \int_0^t \left(P^{-1}f(u)\right)(x,\tau)\,d\tau\ .$$

But for fixed t, we have $f(u) \in H^s$. Also $P^{-1}v = \mathcal{F}^{-1}\left(|\xi|\left(\widehat{\beta}(\xi)\right)^{1/2}\widehat{v}(\xi)\right)$ thus $P^{-1}(f(u)) \in H^{s+\frac{r}{2}-1} \subset L^2 = H^0$ since $s + \frac{r}{2} - 1 \geq 0$. The second statement follows similarly from (4.1). □

Lemma 4.1.2 Suppose the conditions of Theorem 3.1.8 or 3.1.9 hold and u satisfies (3.1)-(3.2) on some interval $[0,T)$. If $P\psi \in L^2$ and the function $G(\varphi)$ defined by

(2.8) belongs to L^1, then for any $t \in [0, T)$ the energy

$$E(t) = \|Pu_t(t)\|^2 + \|u(t)\|^2 + 2\int_{\mathbb{R}} G(u)\,dx$$

is constant in $[0, T)$.

Proof: By Lemma 4.1.1 $Pu_t(t) \in L^2$. The equation of motion, (3.1), can be rewritten as $P^2 u_{tt} + u + g(u) = 0$ using (4.3). Multiplying by $2u_t$ and integrating over $\mathbb{R}$ with respect to x, we get

$$2\int_{\mathbb{R}} ((P^2 u_{tt})u_t + uu_t + g(u)u_t)dx = 0,$$

or

$$2\int_{\mathbb{R}} (Pu_{tt}Pu_t + uu_t + g(u)u_t)dx = 0,$$

or, using Parseval's identity,

$$\frac{d}{dt}(\|Pu_t(t)\|^2 + \|u(t)\|^2 + 2\int_{\mathbb{R}} G(u)\,dx) = 0.$$

Hence, we get that $\frac{dE}{dt} = 0$. □

The following two subsections are devoted to global existence results for two different classes of kernel functions.

4.1.2 Sufficiently Smooth Kernels: $r > 3$

Theorem 4.1.3 Let $s \geq 0$ and $r > 3$. Let $\varphi, \psi \in H^s \cap L^\infty$, $P\psi \in L^2$ and $G(\varphi) \in L^1$. If there is some $k > 0$ so that $G(u) \geq -ku^2$ for all $u \in \mathbb{R}$, then the Cauchy problem (3.1)-(3.2) has a global solution in $C^2([0,\infty), H^s \cap L^\infty)$.

Proof: Since $r > 3$, by Theorem 3.1.9 we have local existence (in $C^2([0,T), L^2 \cap L^\infty)$ for some $T > 0$). The hypothesis implies that $E(0) < \infty$. Assume that u exists on $[0, T)$. Since $G(u) \geq -ku^2$, we get for all $t \in [0, T)$

$$\|Pu_t(t)\|^2 + \|u(t)\|^2 \leq E(0) + 2k\|u(t)\|^2 \ . \tag{4.4}$$

Since $\widehat{\beta}(\xi) \leq C(1+\xi^2)^{-r/2}$; we have

$$\begin{aligned} \|Pu_t(t)\|^2 = \left\|\widehat{Pu_t}(t)\right\|^2 &= \int_{\mathbb{R}} \xi^{-2}\left(\widehat{\beta}(\xi)\right)^{-1}(\widehat{u}_t(\xi,t))^2\,d\xi \\ &\geq C^{-1}\int_{\mathbb{R}} \xi^{-2}(1+\xi^2)^{r/2}(\widehat{u}_t(\xi,t))^2\,d\xi \\ &\geq C^{-1}\int_{\mathbb{R}} (1+\xi^2)^{(r-2)/2}(\widehat{u}_t(\xi,t))^2\,d\xi \\ &\geq C^{-1}\|u_t(t)\|^2_{\frac{r}{2}-1} \ . \end{aligned} \tag{4.5}$$

By the triangle inequality, for any Banach space valued differentiable function v we have

$$\frac{d}{dt}\left\|v(t)\right\| \leq \left\|\frac{d}{dt}v(t)\right\| .$$

Then putting together (4.8)-(4.5)

$$\begin{aligned}
\frac{d}{dt}\left\|u(t)\right\|_{\frac{r}{2}-1}^{2} &= 2\left\|u(t)\right\|_{\frac{r}{2}-1}\frac{d}{dt}\left\|u(t)\right\|_{\frac{r}{2}-1} \\
&\leq 2\left\|u_t(t)\right\|_{\frac{r}{2}-1}\left\|u(t)\right\|_{\frac{r}{2}-1} \\
&\leq \left\|u_t(t)\right\|_{\frac{r}{2}-1}^{2} + \left\|u(t)\right\|_{\frac{r}{2}-1}^{2} \\
&\leq C\left\|Pu_t(t)\right\|^{2} + \left\|u(t)\right\|_{\frac{r}{2}-1}^{2} \\
&\leq C\left(E(0) + 2k\left\|u(t)\right\|^{2}\right) + \left\|u(t)\right\|_{\frac{r}{2}-1}^{2} \\
&\leq CE(0) + (2Ck+1)\left\|u(t)\right\|_{\frac{r}{2}-1}^{2} .
\end{aligned}$$

Gronwall's lemma implies that $\left\|u(t)\right\|_{\frac{r}{2}-1}$ stays bounded in $[0,T)$.
But since $\frac{r}{2}-1>\frac{1}{2}$, we conclude that $\left\|u(t)\right\|_{\infty}$ also stays bounded in $[0,T)$. By Lemma 4.0.7 this implies a global solution. □

4.1.3 Kernels with Singularity

In the next theorem we will consider kernels of the form $\beta(x)=\mu(|x|)$ where μ is a sufficiently smooth and rapidly decreasing function with $\mu\in C^2([0,\infty))$, $\mu(0)>0$, $\mu'(0)<0$ and $\mu''\in L^1\cap L^\infty$. Then the β will have a jump in the first derivative. The typical example we have in mind is the Green's function $\frac{1}{2}e^{-|x|}$. For such a kernel we have

$$\widehat{\beta}(\xi) \leq C\left(1+\xi^2\right)^{-1}$$

so $r=2$. Due to the jump in β' at $x=0$, the distributional derivative will satisfy

$$\beta'' = \mu'' + 2\mu'(0)\delta,$$

where δ is the Dirac measure and we use the notation $\mu''(x)=\mu''(|x|)$. Then we have

$$(\beta * w)_{xx} = \mu'' * w - \lambda w,$$

where $\lambda=-2\mu'(0)>0$. We will call this type of kernels *mildly singular*. To be more precise, we say that β is mildly singular if $\beta_{xx}*w=h*w-\lambda w$ for some $\lambda>0$ and for some $h\in L^1\cap L^\infty$.

Theorem 4.1.4 Let $s \geq 0$ and let the kernel β be mildly singular. Let $\varphi, \psi \in H^s \cap L^\infty$, $P\psi \in L^2$ and $G(\varphi) \in L^1$. If there is some $C > 0$ and $q > 1$ so that $|g(u)|^q \leq CG(u)$ for all $u \in \mathbb{R}$; then the Cauchy problem (3.1)-(3.2) has a global solution in $C^2([0,\infty), H^s \cap L^\infty)$.

Proof: By Theorem 3.1.8, we already have local existence for $s > 1/2$. Nevertheless, similar to Remark 3.1.4, we can improve the local existence result for $s \geq 0$ as follows: Since $\beta_{xx} * w = h * w - \lambda w$ and $h \in L^1$, it follows from Lemma 3.0.2, for $w \in L^\infty$, $\beta_{xx} * w \in L^\infty$. Moreover, $\|h * w\|_s \leq \|h\|_1 \|w\|_s$. As a conclusion, $\beta_{xx} * (\ )$ is a bounded linear map from $H^s \cap L^\infty$ into $H^s \cap L^\infty$. Thus, by Theorem 3.1.8 we have the local existence of the solution. Now, we follow the idea in [6]. Suppose the solution u exists for $t \in [0, T)$. For fixed $x \in \mathbb{R}$ let

$$e(t) = \frac{1}{2}(u_t(x,t))^2 + \lambda\left(\frac{1}{2}(u(x,t))^2 + G(u(x,t))\right) .$$

Then

$$\begin{aligned} e'(t) &= (u_{tt} + \lambda(u + g(u)))\, u_t \\ &= ((\beta * (u + g(u)))_{xx} + \lambda(u + g(u)))\, u_t \\ &= (h * u)\, u_t + (h * g(u))\, u_t \\ &\leq u_t^2 + \frac{1}{2}\left(\|h * u\|_\infty^2 + \|h * g(u)\|_\infty^2\right) . \end{aligned}$$

Since $h \in L^1 \cap L^\infty$, we have $h \in L^P$ for all $p \geq 1$. By Young's inequality

$$e'(t) \leq u_t^2 + \frac{1}{2}\left(\|h\|^2 \|u\|^2 + \|h\|_{L^p}^2 \|g(u)\|_{L^q}^2\right)$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Now the last two terms may be estimated as $\|u\|^2 \leq E(0)$ and

$$\|g(u)\|_{L^q}^2 = \left(\int_{\mathbb{R}} |g(u)|^q\, dx\right)^{2/q} \leq \left(C \int_{\mathbb{R}} G(u)\, dx\right)^{2/q} \leq (CE(0))^{2/q}$$

so that

$$e'(t) \leq D + 2e(t)$$

for some constant D depending on $\|h\|_{L^p}$, $\|h\|$ and $E(0)$. This inequality holds for all $x \in \mathbb{R}$, $t \in [0, T)$. Gronwall's lemma then implies that $e(t)$ and thus $u(x,t)$ stays bounded. □

Remark 4.1.5 When $g(u) = cu^{2n+1}$ with $c > 0$ and positive integer n, both the condition $G(u) \geq -ku^2$ in Theorem 4.1.3 and the condition $|g(u)|^q \leq CG(u)$ in Theorem 4.1.4 are satisfied.

Remark 4.1.6 The estimate (4.5) shows that when $P\psi \in L^2$, we will have $\psi \in H^{\frac{r}{2}-1}$. The converse is not necessarily true since, without any further assumptions on the kernel, the factor $\left(\widehat{\beta}(\xi)\right)^{-1/2}$ may be quite large. For the Gaussian kernel and the sign-changing kernel given in Subsection 2.3.1, $P\psi \in L^2$ implies a very strong smoothness condition on ψ.

4.2 Global Existence for the Coupled System

As in the previous section, we assume that

$$0 < \widehat{\beta}_i(\xi) \leq C_i(1+\xi^2)^{-r_i/2} \qquad i = 1, 2.$$

Also, we define the operators P_i as $P_i w = \mathcal{F}^{-1}(|\xi|^{-1}(\widehat{\beta}_i(\xi))^{-1/2}\widehat{w}(\xi))$ with the inverse Fourier transform $\mathcal{F}^{-1}$. Besides, $P_i^2(\beta_i * w)_{xx} = -w$.

4.2.1 Conservation of Energy

Lemma 4.2.1 Suppose the conditions of Theorem 3.2.4 or 3.2.5 hold and the solution of the Cauchy problem (3.8)-(3.11) exists with u and v in $C^2([0,T), H^s \cap L^\infty)$ for some $s \geq 0$. If $P_1\psi_1, P_2\psi_2 \in L^2$, then $P_1u_t, P_2v_t \in C^1([0,T), L^2)$. If moreover $P_1\varphi_1, P_2\varphi_2 \in L^2$, then $P_1u, P_2v \in C^2([0,T), L^2)$.

Proof: Since

$$\begin{aligned} u_t(x,t) &= \psi_1(x) + \int_0^t (\beta_1 * f_1(u,v))_{xx}(x,\tau)d\tau, \\ v_t(x,t) &= \psi_2(x) + \int_0^t (\beta_2 * f_2(u,v))_{xx}(x,\tau)d\tau, \end{aligned}$$

we get

$$\begin{aligned} P_1u_t(x,t) &= P_1\psi_1(x) - \int_0^t \left(P_1^{-1}f_1(u,v)\right)(x,\tau)\,d\tau, \\ P_2v_t(x,t) &= P_2\psi_2(x) - \int_0^t \left(P_2^{-1}f_2(u,v)\right)(x,\tau)\,d\tau\ . \end{aligned}$$

It is clear from Lemma 3.2.1 that $f_i(u,v) \in H^s$. Also $P_i^{-1}w = \mathcal{F}^{-1}(|\xi|\,(\widehat{\beta}_i(\xi))^{1/2}\widehat{w}(\xi))$ thus $P_i^{-1}(f_i(u,v)) \in H^{s+\frac{r_i}{2}-1} \subset L^2$ and hence $P_1u_t, P_2v_t \in L^2$. The continuity and differentiability of P_1u, P_2v in t follows from the integral representation above. The second statement can be proved with a similar approach using the following equations:

$$\begin{aligned} u(x,t) &= \varphi_1(x) + t\psi_1(x) + \int_0^t (t-\tau)(\beta_1 * f_1(u,v))_{xx}(x,\tau)d\tau, \\ v(x,t) &= \varphi_2(x) + t\psi_2(x) + \int_0^t (t-\tau)(\beta_2 * f_2(u,v))_{xx}(x,\tau)d\tau. \end{aligned}$$

□

Lemma 4.2.2 Suppose the conditions of Theorem 3.2.4 or 3.2.5 hold and u, v satisfies (3.8)-(3.11) on some interval $[0,T)$. If $P_1\psi_1, P_2\psi_2 \in L^2$ and the function $G(\varphi_1, \varphi_2)$ defined by (2.13) belongs to L^1, then for any $t \in [0,T)$ the energy

$$\begin{aligned} E(t) &= \|P_1u_t(t)\|^2 + \|P_2v_t(t)\|^2 + \|u(t)\|^2 + \|v(t)\|^2 + 2\int_{\mathbb{R}} G(u,v)dx \\ &= \|P_1u_t(t)\|^2 + \|P_2v_t(t)\|^2 + 2\int_{\mathbb{R}} F(u,v)dx \end{aligned}$$

is constant in $[0,T)$.

Proof: By Lemma 4.2.1, $P_1u_t(t), P_2v_t(t) \in L^2$. Equations (3.8)-(3.9) can be rewritten as

$$P_1^2u_{tt} + u + g_1(u,v) = 0, \tag{4.6}$$

$$P_2^2v_{tt} + v + g_2(u,v) = 0. \tag{4.7}$$

Multiplying (4.6) by $2u_t$ and (4.7) by $2v_t$, integrating both equations over $\mathbb{R}$ with respect to x, adding the two equalities and using Parseval's identity we obtain $\frac{dE}{dt} = 0$. □

We analyzed two different classes of kernel functions for the global existence of solutions of (3.1)-(3.2) in the previous section. Now, we again categorize our kernel functions as sufficiently smooth kernels and kernels with singularity and prove global well-posedness of the Cauchy problem (3.8)-(3.11) for each case.

4.2.2 Sufficiently Smooth Kernels: $r_1, r_2 > 3$

Theorem 4.2.3 Let $s \geq 0$, $r_1, r_2 > 3$. Let $\varphi_i, \psi_i \in H^s$, $P_i\psi_i \in L^2$ $(i = 1,2)$ and $G(\varphi_1, \varphi_2) \in L^1$. If there is some $k > 0$ so that $G(a,b) \geq -k(a^2+b^2)$ for all

$a, b \in \mathbb{R}$, then the Cauchy problem (3.8)-(3.11) has a global solution with u and v in $C^2([0,\infty), H^s)$.

Proof: Since $r_1, r_2 > 3$, by Theorem 3.2.5 we have local existence. The hypothesis implies that $E(0) < \infty$. Assume that u, v exist on $[0,T)$ for some $T > 0$. Since $G(u,v) \geq -k(u^2+v^2)$, we get for all $t \in [0,T)$

$$\|P_1 u_t(t)\|^2 + \|P_2 v_t(t)\|^2 \leq E(0) + (2k-1)(\|u(t)\|^2 + \|v(t)\|^2). \tag{4.8}$$

Noting that $\widehat{\beta}_i(\xi) \leq C_i(1+\xi^2)^{-r_i/2}$ for $i = 1, 2$; we have

$$\begin{aligned} \|P_1 u_t(t)\|^2 = \left\|\widehat{P_1 u_t}(t)\right\|^2 &= \int \xi^{-2}(\widehat{\beta}_1(\xi))^{-1}(\widehat{u}_t(\xi,t))^2 d\xi \\ &\geq C_1^{-1} \int \xi^{-2}(1+\xi^2)^{r_1/2}(\widehat{u}_t(\xi,t))^2 d\xi \\ &\geq C_1^{-1} \int (1+\xi^2)^{(r_1-2)/2}(\widehat{u}_t(\xi,t))^2 d\xi \\ &= C_1^{-1} \|u_t(t)\|_{\rho_1}^2 , \end{aligned} \tag{4.9}$$

and similarly,

$$\|P_2 v_t(t)\|^2 \geq C_2^{-1} \|v_t(t)\|_{\rho_2}^2 \tag{4.10}$$

where $\rho_i = \frac{r_i}{2} - 1, \ i = 1, 2$. By the triangle inequality, for any Banach space valued differentiable function w we have

$$\frac{d}{dt}\|w(t)\| \leq \left\|\frac{d}{dt}w(t)\right\| .$$

Combining (4.8), (4.9) and (4.10),

$$\begin{aligned} \frac{d}{dt}(\|u(t)\|_{\rho_1}^2 &+ \|v(t)\|_{\rho_2}^2) \\ &= 2(\|u(t)\|_{\rho_1}\frac{d}{dt}\|u(t)\|_{\rho_1} + \|v(t)\|_{\rho_2}\frac{d}{dt}\|v(t)\|_{\rho_2}) \\ &\leq 2(\|u_t(t)\|_{\rho_1}\|u(t)\|_{\rho_1} + \|v_t(t)\|_{\rho_2}\|v(t)\|_{\rho_2}) \\ &\leq \|u_t(t)\|_{\rho_1}^2 + \|u(t)\|_{\rho_1}^2 + \|v_t(t)\|_{\rho_2}^2 + \|v(t)\|_{\rho_2}^2 \\ &\leq C(\|P_1 u_t(t)\|^2 + \|P_2 v_t(t)\|^2) + \|u(t)\|_{\rho_1}^2 + \|v(t)\|_{\rho_2}^2 \\ &\leq C[E(0) + (2k-1)(\|u(t)\|^2 + \|v(t)\|^2)] + \|u(t)\|_{\rho_1}^2 + \|v(t)\|_{\rho_2}^2 \\ &\leq CE(0) + (C(2k-1)+1)(\|u(t)\|_{\rho_1}^2 + \|v(t)\|_{\rho_2}^2) \end{aligned}$$

where $C = \max(C_1, C_2)$. Gronwall's lemma implies that $\|u(t)\|_{\rho_1} + \|v(t)\|_{\rho_2}$ stays bounded in $[0,T)$. Since $\rho_i = \frac{r_i}{2} - 1 > \frac{1}{2}$, $\|u(t)\|_\infty + \|v(t)\|_\infty$ also stays bounded in $[0,T)$. By Lemma 4.0.8, a global solution exists. □

4.2.3 Kernels with Singularity

We extend the global existence result for mildly singular kernels in [6] to the coupled system.

Theorem 4.2.4 Let $s \geq 0$ and let the kernels $\beta_1 = \beta_2$ be mildly singular. Suppose that $\varphi_1, \varphi_2, \psi_1, \psi_2 \in H^s$, $P_1\psi_1, P_2\psi_2 \in L^2$ and $G(\varphi_1, \varphi_2) \in L^1$. If there are some $C > 0$, $k \geq 0$ and $q_i > 1$ so that

$$|g_i(a,b)|^{q_i} \leq C[G(a,b) + k(a^2 + b^2)]$$

for all $a, b \in \mathbb{R}$ and $i = 1, 2$; then the Cauchy problem (3.8)-(3.11) has a global solution with u and v in $C^2([0,\infty), H^s)$.

Proof: By Theorem 3.2.4 we have a local solution for $s > 1/2$. Nevertheless, as was done in the proof of Theorem 4.1.4 local existence can be shown for $s \geq 0$. Suppose the solution (u, v) exists for $t \in [0, T)$. For fixed $x \in \mathbb{R}$ we define

$$e(t) = \frac{1}{2}[(u_t(x,t))^2 + (v_t(x,t))^2] + \frac{\lambda}{2}[(u(x,t))^2 + (v(x,t))^2 + 2G(u(x,t), v(x,t))].$$

Then

$$\begin{aligned} e'(t) &= [u_{tt} + \lambda(u + g_1(u,v))]u_t + [v_{tt} + \lambda(v + g_2(u,v))]v_t \\ &= [(\beta_1 * (u + g_1(u,v)))_{xx} + \lambda(u + g_1(u,v))]u_t \\ &\quad + [(\beta_2 * (v + g_2(u,v)))_{xx} + \lambda(v + g_2(u,v))]v_t \\ &= (h * u)u_t + (h * g_1(u,v))u_t + (h * v)v_t + (h * g_2(u,v))v_t \\ &\leq (u_t)^2 + (v_t)^2 + \frac{1}{2}(\|h * u\|_\infty^2 + \|h * v\|_\infty^2) \\ &\quad + \frac{1}{2}(\|h * g_1(u,v)\|_\infty^2 + \|h * g_2(u,v)\|_\infty^2). \end{aligned}$$

Since $h \in L^1 \cap L^\infty$ we have $h \in L^p$ for all $p \geq 1$. By Young's inequality

$$\begin{aligned} e'(t) &\leq (u_t)^2 + (v_t)^2 + \frac{1}{2}\|h\|^2(\|u\|^2 + \|v\|^2) \\ &\quad + \frac{1}{2}\|h\|_{L^{p_1}}^2 \|g_1(u,v)\|_{L^{q_1}}^2 + \frac{1}{2}\|h\|_{L^{p_2}}^2 \|g_2(u,v)\|_{L^{q_2}}^2, \end{aligned}$$

where $1/p_i + 1/q_i = 1$ $(i = 1, 2)$. Now the terms may be estimated as

$$\|u\|^2 + \|v\|^2 \leq E(0)$$

and for $i = 1, 2$

$$\begin{aligned} \|g_i(u,v)\|_{L^{q_i}}^2 &= \left(\int |g_i(u,v)|^{q_i} \, dx \right)^{2/q_i} \\ &\leq \left(C \int [G(u,v) + k(u^2+v^2)] dx \right)^{2/q_i} \leq [C(1+k)E(0)]^{2/q_i} \end{aligned}$$

so that

$$e'(t) \leq D + 2e(t)$$

for some constant D depending on $\|h\|_{L^{p_i}}$, $\|h\|$ and $E(0)$ $(i = 1, 2)$. This inequality holds for all $x \in \mathbb{R}$, $t \in [0, T)$. Gronwall's lemma then implies that $e(t)$ and thus $u(x,t)$ and $v(x,t)$ stay bounded. Thus by Lemma 4.0.8 we have global solution. □

Chapter 5

Finite Time Blow-up

In this chapter, we will prove blow-up of solutions in finite time for the Cauchy problems (3.1)-(3.2) and (3.8)-(3.11). For this purpose, we will use the following well-known lemma.

Lemma 5.0.5 [22, 30] Suppose $\Phi(t)$, $t \geq 0$, is a positive, twice differentiable function satisfying $\Phi''\Phi - (1+\nu)(\Phi')^2 \geq 0$ where $\nu > 0$. If $\Phi(0) > 0$ and $\Phi'(0) > 0$, then $\Phi(t) \to \infty$ as $t \to t_1$ for some $t_1 \leq \Phi(0)/\nu\Phi'(0)$.

5.1 Blow-up for the Single Equation

We first rewrite the energy identity as

$$E(t) = \|Pu_t(t)\|^2 + 2\int_{\mathbb{R}} F(u)\,dx = E(0)$$

where $F(u) = \int_0^u f(\rho)\,d\rho$ with $f(u) = u + g(u)$ as before.

Theorem 5.1.1 Suppose that the conditions of Theorem 3.1.8 or 3.1.9 hold, $P\varphi, P\psi \in L^2$ and $G(\varphi) \in L^1$. If there is some $\nu > 0$ such that

$$uf(u) \leq 2(1+2\nu)F(u), \tag{5.1}$$

and

$$E(0) = \|P\psi\|^2 + 2\int_{\mathbb{R}} F(\varphi)\,dx < 0\ ,$$

then the solution u of the Cauchy problem (3.1)-(3.2) blows up in finite time.

Proof: Assume that there is a global solution. Then $Pu(t), Pu_t(t) \in L^2$ for all $t > 0$. Let $\Phi(t) = \|Pu(t)\|^2 + b(t+t_0)^2$ for some positive b and t_0 that will be determined later. We have

$$\begin{aligned}\Phi'(t) &= 2\langle Pu, Pu_t\rangle + 2b(t+t_0),\\ \Phi''(t) &= 2\|Pu_t\|^2 + 2\langle Pu, Pu_{tt}\rangle + 2b.\end{aligned}$$

Note that

$$\begin{aligned}2\langle Pu, Pu_{tt}\rangle &= 2\langle u, P^2u_{tt}\rangle = -2\langle u, f(u)\rangle = -2\int_{\mathbb{R}} uf(u)\,dx\\ &\geq -4(1+2\nu)\int_{\mathbb{R}} F(u)\,dx\\ &= 2(1+2\nu)\left(\|Pu_t\|^2 - E(0)\right),\end{aligned}$$

so that

$$\Phi''(t) \geq 4(1+\nu)\|Pu_t\|^2 - 2(1+2\nu)E(0) + 2b.$$

On the other hand, we have

$$\begin{aligned}(\Phi'(t))^2 &= 4\left[\langle Pu, Pu_t\rangle + b(t+t_0)\right]^2\\ &\leq 4\left[\|Pu\|\,\|Pu_t\| + b(t+t_0)\right]^2\\ &= 4\left[\|Pu\|^2\|Pu_t\|^2 + 2\|Pu\|\,\|Pu_t\|\,b(t+t_0) + b^2(t+t_0)^2\right]\\ &\leq 4\left[\|Pu\|^2\|Pu_t\|^2 + b\|Pu\|^2 + b\|Pu_t\|^2(t+t_0)^2 + b^2(t+t_0)^2\right].\end{aligned}$$

Thus

$$\begin{aligned}&\Phi''(t)\Phi(t) - (1+\nu)(\Phi'(t))^2\\ &\quad\geq \left[4(1+\nu)\|Pu_t\|^2 - 2(1+2\nu)E(0) + 2b\right]\left[\|Pu\|^2 + b(t+t_0)^2\right]\\ &\qquad -4(1+\nu)\left[\|Pu\|^2\|Pu_t\|^2 + b\|Pu\|^2 + b\|Pu_t\|^2(t+t_0)^2 + b^2(t+t_0)^2\right]\\ &\quad= \left[-2(1+2\nu)E(0) + 2b - 4b(1+\nu)\right]\left[\|Pu\|^2 + b(t+t_0)^2\right]\\ &\quad= -2(1+2\nu)(b+E(0))\Phi(t).\end{aligned}$$

Now if we choose $b \leq -E(0)$, this gives

$$\Phi''(t)\Phi(t) - (1+\nu)(\Phi'(t))^2 \geq 0.$$

Moreover

$$\Phi'(0) = 2\langle P\varphi, P\psi\rangle + 2bt_0 \geq 0$$

for sufficiently large t_0. According to Lemma 5.0.5, this implies that $\Phi(t)$, and thus $\|Pu(t)\|^2$ blows up in finite time contradicting the assumption that the global solution exists. □

5.2 Blow-up for the Coupled System

Theorem 5.2.1 Suppose that the conditions of Theorem 3.2.4 or 3.2.5 hold, $P_1\varphi_1,\ P_2\varphi_2,\ P_1\psi_1,\ P_2\psi_2 \in L^2$ and $G(\varphi_1, \varphi_2) \in L^1$. If there is some $\nu > 0$ such that

$$uf_1(u,v) + vf_2(u,v) \leq 2(1+2\nu)F(u,v),$$

and

$$E(0) = \|P_1\psi_1\|^2 + \|P_2\psi_2\|^2 + 2\int_{\mathbb{R}} F(\varphi_1, \varphi_2)dx < 0,$$

then the solution (u, v) of the Cauchy problem (3.8)-(3.11) blows up in finite time.

Proof: This time take

$$\Phi(t) = \|P_1u(t)\|^2 + \|P_2v(t)\|^2 + b(t+t_0)^2$$

for some positive b and t_0 that will be specified later and again assume that the maximal time of existence of the solution of the Cauchy problem (3.8)-(3.11) is infinite. Then $P_1u(t), P_1u_t(t), P_2v(t), P_2v_t(t) \in L^2$ for all $t > 0$; thus $\Phi(t)$ must be finite for all t. However, we will show below that $\Phi(t)$ blows up in finite time.
We have

$$\Phi'(t) = 2\langle P_1u, P_1u_t\rangle + 2\langle P_2v. P_2v_t\rangle + 2b(t+t_0),$$
$$\Phi''(t) = 2\|P_1u_t\|^2 + 2\|P_2v_t\|^2 + 2\langle P_1u, P_1u_{tt}\rangle + 2\langle P_2v, P_2v_{tt}\rangle + 2b\ .$$

Since

$$\begin{aligned}\langle P_1u, P_1u_{tt}\rangle &= \langle u, P_1^2u_{tt}\rangle = -\langle u, f_1(u,v)\rangle,\\ \langle P_2v, P_2v_{tt}\rangle &= \langle v, P_2^2v_{tt}\rangle = -\langle v, f_2(u,v)\rangle\end{aligned}$$

and

$$\begin{aligned}-\int[uf_1(u,v) + vf_2(u,v)]dx &\geq -2(1+2\nu)\int F(u,v)dx\\ &= (1+2\nu)(\|P_1u_t(t)\|^2 + \|P_2v_t(t)\|^2 - E(0)),\end{aligned}$$

we get

$$\begin{aligned}\Phi''(t) &\geq 2\|P_1u_t\|^2 + 2\|P_2v_t\|^2 + 2b - 2(1+2\nu)(E(0) - \|P_1u_t\|^2 - \|P_2v_t\|^2)\\ &= -2(1+2\nu)E(0) + 2b + 4(1+\nu)(\|P_1u_t\|^2 + \|P_2v_t\|^2).\end{aligned}$$

By the Cauchy-Schwarz inequality we have

$$\begin{aligned}(\Phi'(t))^2 &= 4[\langle P_1u, P_1u_t\rangle + \langle P_2v, P_2v_t\rangle + b(t+t_0)]^2 \\ &\leq 4[\|P_1u\|\,\|P_1u_t\| + \|P_2v\|\,\|P_2v_t\| + b(t+t_0)]^2\ .\end{aligned}$$

For the mixed terms we use the inequalities

$$2\,\|P_1u\|\,\|P_1u_t\|\,\|P_2v\|\,\|P_2v_t\| \leq \|P_1u\|^2\,\|P_2v_t\|^2 + \|P_2v\|^2\,\|P_1u_t\|^2$$

and

$$\begin{aligned}2\,\|P_1u\|\,\|P_1u_t\|\,(t+t_0) &\leq \|P_1u\|^2 + \|P_1u_t\|^2\,(t+t_0)^2, \\ 2\,\|P_2v\|\,\|P_2v_t\|\,(t+t_0) &\leq \|P_2v\|^2 + \|P_2v_t\|^2\,(t+t_0)^2\end{aligned}$$

to obtain

$$(\Phi'(t))^2 \leq 4\Phi(t)(\|P_1u_t\|^2 + \|P_2v_t\|^2 + b)\ .$$

Therefore,

$$\begin{aligned}&\Phi(t)\Phi''(t) - (1+\nu)(\Phi'(t))^2 \\ &\quad\geq \Phi(t)[-2(1+2\nu)E(0) + 2b + 4(1+\nu)(\|P_1u_t\|^2 + \|P_2v_t\|^2)] \\ &\qquad - 4(1+\nu)\Phi(t)(\|P_1u_t\|^2 + \|P_2v_t\|^2 + b) \\ &\quad= -2(1+2\nu)(E(0)+b)\Phi(t)\ .\end{aligned}$$

If we choose $b \leq -E(0)$, then $\Phi(t)\Phi''(t) - (1+\nu)(\Phi'(t))^2 \geq 0$. Moreover

$$\Phi'(0) = 2\,\langle P_1\varphi_1, P_1\psi_1\rangle + 2\,\langle P_2\varphi_2. P_2\psi_2\rangle + 2bt_0 \geq 0$$

for sufficiently large t_0. According to Lemma 5.0.5, we observe that $\Phi(t)$ blows up in finite time. This contradicts with the assumption of the existence of a global solution. □

5.3 Remarks

From the above proofs of Theorems 5.1.1 and 5.2.1, we observe that we may prove blow-up even if $E(0) > 0$. In this case, all we need is to choose b and t_0 so that $\Phi(0) > 0$ and $\Phi'(0) > 0$. Note that such b and t_0 may fail to be positive. The first two remarks are about such extensions of Theorems 5.1.1 and 5.2.1.

Remark 5.3.1 In Theorem 5.1.1, let $E(0) > 0$. By choosing $b = -E(0)$ we still get blow up if there is some t_0 so that the initial data satisfies

$$\Phi(0) = \|P\varphi\|^2 - E(0)t_0^2 > 0, \qquad \Phi'(0) = 2\langle P\varphi, P\psi\rangle - 2E(0)t_0 > 0.$$

When $\langle P\varphi, P\psi\rangle > 0$, taking $t_0 = 0$ works. When $\langle P\varphi, P\psi\rangle \leq 0$, then t_0 must be chosen negative. The two inequalities can be rewritten as

$$E(0)^{-2}(\langle P\varphi, P\psi\rangle)^2 < t_0^2, \qquad t_0^2 < E(0)^{-1}\|P\varphi\|^2.$$

Putting these two inequalities together, we say that such a t_0 exists and hence there is a blow-up if the initial data satisfies

$$(\langle P\varphi, P\psi\rangle)^2 < E(0)\|P\varphi\|^2.$$

Remark 5.3.2 In Theorem 5.2.1, let $E(0) > 0$. To shorten the notation let

$$A = \langle P_1\varphi_1, P_1\psi_1\rangle + \langle P_2\varphi_2, P_2\psi_2\rangle, \qquad B = \|P_1\varphi_1\|^2 + \|P_2\varphi_2\|^2.$$

Again letting $b = -E(0)$, we will get blow up if there is some t_0 so that the initial data satisfies

$$A - E(0)t_0 > 0, \qquad B - E(0)t_0^2 > 0.$$

When $A \leq 0$, then t_0 must be chosen negative. As a conclusion, such a t_0 exists if and only if $A^2 < E(0)B$. Hence there is blow-up if the initial data satisfies

$$(\langle P_1\varphi_1, P_1\psi_1\rangle + \langle P_2\varphi_2, P_2\psi_2\rangle)^2 < E(0)\left(\|P_1\varphi_1\|^2 + \|P_2\varphi_2\|^2\right).$$

The final remark is about the relation between the type of the nonlinearity and the blow-up condition.

Remark 5.3.3 For the nonlocal single equation, consider a typical nonlinearity of the form $G(u) = c|u|^q$ for some $q > 2$. From (2.8), $g(u) = cq|u|^{q-2}u$. We proved global existence of the solution of the Cauchy problem via Section 4.1 when $c > 0$. On the other hand, when $c < 0$, the blow-up condition (5.1) holds. Moreover, since $E(0) = \|P\psi\|^2 + \|\varphi\|^2 + 2c\int_{\mathbb{R}}|\varphi|^q dx$ holds, $E(0)$ can be made negative by choosing φ sufficiently large and ψ sufficiently small. The above results about global existence and blow-up are sharp in this sense.

Chapter 6

Conclusions

In this thesis study, we have proposed two general classes of nonlocal nonlinear wave equations arising in longitudinal and transverse wave propagation in a nonlocal elastic medium. Pointing out the singularities arising in the classical theory of nonlinear elasticity regarding nondispersive wave propagation, we have discussed the regularization property of dispersion brought with nonlocality. We have showed how the dispersive effects provide global well-posedness in the nonlocal theory of nonlinear elasticity. A general form of the kernel functions appear in (2.12) and (2.14)-(2.15) governing longitudinal and transverse wave motion, respectively. We point out that this form of the kernel functions covers the most commonly used kernels appearing in the literature. We have discussed the relation between the regularizing effect of the kernels and smoothness of the initial data. We have naturally analyzed the global well-posedness and finite time blow-up of the solutions of the two Cauchy problems. The nonlocal nonlinear wave equations we have proposed and the relevant contents appear in the recent papers [9, 10] as well.

Although there has been a considerable amount of research done on bi-directional wave propagation within the context of the theory of nonlinear partial differential equations, they are mostly restricted to local models. There is growing interest in using nonlocal models of various physical phenomena arising in different areas. This thesis takes a step in this direction by studying qualitative properties of two nonlocal models proposed for bi-directional nonlinear wave propagation in a continuous medium. A further step in this direction has been given in a recent study [11] where the analysis given here for the nonlocal single equation is extended to the two-dimensional case. We also refer to [12] for a similar analysis of a nonlocal single equation derived in the context of the peridynamic formulation of elasticity.

Nonlocality poses new interesting research problems to study in the future. We can shortly mention some of these problems as follows. An obvious question is what happens when the decay rate r of the Fourier transform of β is less than 2. Global existence of small amplitude solutions and derivation of non-linear scattering results for the Cauchy problems (3.1)-(3.2) and (3.8)-(3.11) with small initial data are also open questions. Recalling that nonlocality affects both linear and nonlinear parts of the two models considered in this thesis, it is also interesting to investigate the case where nonlocality affects linear part only. Moreover, initial-boundary value problems corresponding to (3.1) and (3.8)-(3.9) can be considered. This requires a correct interpretation of boundary conditions. Such an investigation will lead to a wide area of research.

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