Social Choice and Welfare

Epsilon-Ex Post Implementation

--Manuscript Draft--

Manuscript Number:	
Full Title:	Epsilon-Ex Post Implementation
Article Type:	Original Paper
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Epsilon–Ex Post Implementation*

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June 12, 2014

Abstract

We provide necessary and sufficient conditions for epsilon–ex post implementation. Our analysis extends Bergemann and Morris (2008) to allow for *epsilon–bounded rationality*. Yet our necessity condition, *epsilon–ex post monotonicity*, and Bergemann and Morris's (2008) necessity condition, *ex post monotonicity*, are *not* nested. Epsilon–ex post implementation adds another dimension of robustness to ex post implementation in terms of bounded rationality.

JEL classification: C79; D82

Keywords: Ex Post Implementation; Bounded Rationality; Epsilon Equilibrium; Interdependent values.

*We would like to thank Dirk Bergemann, Ehud Kalai, Semih Koray, Asher Wolinsky and Kemal Yıldız for their useful comments. Any remaining errors are ours.

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1 Introduction

Recent research in mechanism design with interdependent preferences has adopted the use of ex post equilibrium as a solution concept.[1] Ex post equilibrium provides not only considerable improvements in tractability but also conceptual advantages in terms of robustness with respect to informational assumptions.[2] Bergemann and Morris (2008) (BM hereafter) "analyze the problem of fully implementing a social choice set in ex post equilibrium" and they provide necessary and sufficient conditions under which ex post implementation is possible.

Building upon BM, the current paper presents a necessary condition (*epsilon–ex post monotonicity*) for the full implementation of a social choice set in epsilon–ex post equilibrium, and establishes that when coupled with *epsilon–limited veto power* and *epsilon–ex post incentive compatibility* they become sufficient, provided that there are at least three agents in the society. The set of epsilon–ex post equilibrium is a super set of ex post equilibrium when epsilon is positive.[3] However, our necessary condition, epsilon–ex post monotonicity, is not nested with the necessary condition of BM, ex post monotonicity, even though they coincide when epsilon is zero.

Our main motivation for studying the problem of epsilon–ex post implementation is *epsilon–bounded rationality*. Epsilon–ex post implementation requires that in every particular state of the world optimal outcomes (given by a social choice set) must coincide with epsilon equilibrium outcomes of the mechanism (implementing this social choice set). Therefore, the designer, possibly taking a behavioral point of view, can be thought of as considering situations in which agents do not (fully) optimize but are satisfied with a payoff that is epsilon close to their optimum. This idea was first introduced by Radner (1980). The fact that the concept of epsilon equilibrium can be considered to capture bounded rationality was also pointed out by Aumann (1997). Kalai and Lehrer (1993) and Kalai and Lehrer (1994) offer another justification for considering the epsilon equilibrium: "If players start with a vector of subjectively rational strategies, and if their individual subjective beliefs regarding opponents' strategies are compatible with the truly chosen strategies, then they must converge in finite time to play according to an epsilon Nash equilibrium of the repeated game, for

[1] See Maskin and Dasgupta (2000), Bergemann and Valimaki (2002), Reny and Perry (2002), and McLean and Postlewaite (2013), among many others.

[2] We refer the reader to the literature on robust mechanism design, e.g. Bergemann and Morris (2005) and Bergemann and Morris (2012) and the references therein, pointing out that allowing the mechanism to depend on the designer's knowledge of the type space, as Bayesian mechanisms do, seems rather unrealistic.

[3] All of our results follow when epsilon is negative. Therefore, our results can also be interpreted as providing necessary and sufficient conditions for a refinement of ex post equilibrium.

arbitrary small epsilon."[4]

Our necessary condition, epsilon–ex post monotonicity, coincides with BM's ex post monotonicity when epsilon equals zero. While the epsilon–ex post incentive compatibility condition is derived from its standard version via the use of epsilon equilibrium, we procure the epsilon–limited veto power condition from the limited veto power condition introduced by Benoit and Ok (2006) for complete information environments. Consequently, our findings provide an extension of BM's results not only by including considerations of bounded rationality but also by allowing for limited veto power.

Many papers study the problem of full implementation via different equilibrium concepts.[5] A related study is Barlo and Dalkiran (2009) which considers full implementation in epsilon Nash equilibrium in a complete information environment and provides similar conditions for epsilon Nash implementation: epsilon–monotonicity and epsilon–limited veto power. While Barlo and Dalkiran (2009) captures epsilon–bounded rationality in complete information environments, the current paper achieves a similar goal in incomplete information environments where the designer does not possess sufficient information about agents' type spaces.[6]

The next section provides the preliminaries and Section 3 presents our main results. In Section 4, we identify two examples showing that our monotonicity condition and that of BM are not nested, even though they coincide when epsilon is zero. Section 5 concludes.

[4] It is true that eventually the play of such a game will converge to a Nash equilibrium outcome, yet at any point in time – until the limit is reached – the outcome of this game is an epsilon (Nash) equilibrium outcome for some (positive) epsilon.

[5] See, for example, Maskin (1999), Jackson (1991), and Koray and Yildiz (2014), among many others.

[6] In the case of complete information the two epsilon–limited veto power conditions coincide. However, a similar observation does not hold concerning epsilon–monotonicity and epsilon–ex post monotonicity. In fact, the relation between these involve an epsilon modified version of BM's detailed discussion about how Maskin monotonicity and ex post monotonicity are related in the case of complete information: (1) ex post monotonicity does not reduce to Maskin monotonicity; and (2) in environments satisfying the single crossing property ex post monotonicity implies Maskin monotonicity under mild conditions. The first of these extends to our setting as epsilon–ex post monotonicity reduces to ex post monotonicity and epsilon–monotonicity to Maskin monotonicity in complete information environments when epsilon is set to zero. Therefore, the current paper offers a true extension of Maskin's implementation theorem: when attention is restricted to complete information environments satisfying the single crossing property and epsilon equals 0, then our epsilon ex post implementation results imply to those of Maskin under the same conditions used in BM. On the other hand, an epsilon modified version of the single crossing property is rather nonintuitive. That is why we think that providing a relation between epsilon–ex post monotonicity and epsilon–monotonicity using an epsilon modified version of the single crossing property would not present a significant contribution.

2 The Preliminaries

The environment is denoted by $G = (N, A, \Theta, \{u^\theta\}_{\{\theta\in\Theta\}})$, where $N = \{1, ..., n\}$ identifies a finite set of agents with $n > 3$; A, a compact set of alternatives; $\Theta = \Theta_1 \times \Theta_2 \times ... \times \Theta_n$, the set of all type profiles; and $u^\theta = (u_1^\theta, ..., u_n^\theta)$, the preference profile of the society when the realized state is $\theta = (\theta_1, \theta_2, ..., \theta_n)$. It is appropriate to mention that the environment is of incomplete information and payoffs are interdependent.

A *social choice function*, $f : \Theta \to A$, prescribes an alternative $f(\theta) \in A$ to each state $\theta \in \Theta$. $\mathcal{F}$ denotes the set of all social choice functions, and a subset $F \subset \mathcal{F}$ is called a *social choice set*.

A *mechanism*, alternatively a game form, describes a message/strategy space $M_i \neq \emptyset$ for each agent $i \in N$ and specifies an outcome function $g : M \to A$, where $M = \times_{i\in N} M_i$. We denote a normal–form mechanism with $\mu = (M, g)$. In state θ, the mechanism μ together with the preference profile u^θ define a game of incomplete information. In such a game, a strategy for player i is a function $s_i : \Theta_i \to M_i$. A strategy profile is denoted by $s^*(\theta) = \times_{i\in N} s_i(\theta_i)$. Below is the definition of an ε–ex post equilibrium of a mechanism μ:

Definition 1 (ε–ex post equilibrium) *Let $\varepsilon \geq 0$, and let $\mu = (M, g)$ be a normal–form mechanism. A strategy profile s^* is called an ε–*ex post equilibrium *of μ, if, for all $\theta \in \Theta$ and for all $i \in N$ and for all $m_i \in M_i$, we have*

$$u_i^\theta(g(s^*(\theta))) \geq u_i^\theta(g(m_i, s_{-i}^*(\theta_{-i}))) - \varepsilon.$$

That is, an ε–ex post equilibrium outcome of a mechanism is an ε–equilibrium outcome of this mechanism in every state of the world.

Next, we provide the definition of ε–ex post implementability:

Definition 2 (ε–ex post implementation) *Let $\varepsilon \geq 0$. F is said to be ε–*ex post implementable *(in pure strategies) if there exists a mechanism $\mu = (M, g)$ such that:*

1. *For every $f \in F$, there exists an ε–ex post equilibrium s^* of $\mu = (M, g)$ that satisfies*

$$[f = g \circ s^*], \text{ i.e. } f(\theta) = g(s^*(\theta)) \text{ for all } \theta \in \Theta;$$

2. *For every ε–ex post equilibrium s^* of $\mu = (M, g)$, there exists $f \in F$ such that:*

$$[g \circ s^* = f], \text{ i.e. } g(s^*(\theta)) = f(\theta) \text{ for all } \theta \in \Theta.$$

In words, a social choice set F is said to be ε–ex post implementable if there exists a mechanism such that ε–ex post equilibrium outcomes of this mechanism coincide exactly with the optimal alternatives given by the social choice set F.

3 ε–Ex Post Implementation

This section presents our main results, and we start with the necessity.

3.1 Necessity

We define the necessary conditions for epsilon–ex post implementation: *epsilon–ex post incentive compatibility* and *epsilon–ex post monotonicity.*

Definition 3 (ε–ex post incentive compatibility) *F is said to be ε–*ex post incentive compatible *($\varepsilon - EPIC$ henceforth) if, for every $f \in F$, it must be that*

$$u_i^\theta(f(\theta)) \geq u_i^\theta(f(\theta_i', \theta_{-i})) - \varepsilon,$$

for all $i \in N$, $\theta \in \Theta$, and $\theta_i' \in \Theta_i$.

The $\varepsilon - EPIC$ condition is weaker than the standard $EPIC$ condition when $\varepsilon > 0$. If we were interested in *partial implementation* rather than *full implementation*, $\varepsilon - EPIC$ would be a necessary and sufficient condition due to the revelation principle. Note also that when $\varepsilon > 0$ becomes large the set of social choice rules that satisfy $\varepsilon - EPIC$ enlarges. Hence, the larger the ε the less the critique of Jehiel, Moldavanu, Meyer-Ter-Vehn, and Zame (2006) stings.

Now, we wish to define the concept of a deception that will be used to define our next necessary condition,ε–ex post monotonicity: a *deception* by agent $i \in N$ is denoted by $\alpha_i : \Theta_i \to \Theta_i$. A deception α_i by agent i with a true type θ_i is interpreted as i's reported type, $\alpha_i(\theta_i)$, as a function of his true type. A *profile of deceptions* is denoted by $\alpha(\theta) = (\alpha_1(\theta_1), \alpha_2(\theta_2), ..., \alpha_n(\theta_n))$. Naturally, when agents report the deception $\alpha(\theta)$ rather than truthfully reporting θ, the resulting outcome is given by $f(\alpha(\theta))$ instead of $f(\theta)$.

The definition of ε–ex post monotonicity is as follows:

Definition 4 (ε–ex post monotonicity) *F is said to be ε–*ex post monotonic *($\varepsilon - EPM$) if, for every $f \in F$ and deception α with $f \circ \alpha \notin F$, there exists $i \in N, \theta \in \Theta, a \in A$ such that*

$$\begin{aligned} u_i^\theta(a) &> u_i^\theta(f(\alpha(\theta))) + \varepsilon, && (1) \\ a &\in A_{i,\varepsilon}^f(\alpha_{-i}(\theta_{-i})), && (2) \end{aligned}$$

where

$$A_{i,\varepsilon}^f(\alpha_{-i}(\theta_{-i})) = \left\{ a \left| u_i^{(\theta_i', \alpha_{-i}(\theta_{-i}))}(f(\theta_i', \alpha_{-i}(\theta_{-i}))) \geq u_i^{(\theta_i', \alpha_{-i}(\theta_{-i}))}(a) - \varepsilon, \forall \theta_i' \in \Theta_i \right. \right\}.$$

In words, ε–ex post monotonicity requires that when there is an attempt by the agents to deceive the designer so that a suboptimal outcome will be chosen, the following must hold: There must exist a state, a whistle blower for that state, and a reward for that whistle blower such that (1) in this state the whistle blower has an incentive strictly larger than ε to receive this reward, which alerts the designer about the deceptive behavior α; and (2) the whistle blower does not have an incentive larger than ε to falsely alert the designer if the outcome is optimal in this state or in some other state.

It must be emphasized that when $\varepsilon = 0$, ε–ex post monotonicity coincides with ex post monotonicity. But when $\varepsilon > 0$ they are independent of each other: condition (1) of the associated definition become stronger while condition (2) becomes weaker. We provide two examples proving the independence of these two conditions in the next section.

Now, we are ready to present our necessity result.

Theorem 1 (Necessity) *If F is ε–ex post implementable, then it satisfies $\varepsilon-EPIC$ and $\varepsilon-EPM$.*

Proof. Let $\mu = (M, g)$ implement F and take an $f \in F$. Then, there exists an ε–ex post equilibrium s^* of μ such that $f = g \circ s^*$. Therefore, for all $\theta \in \Theta$, $i \in N$, and for all $m_i \in M_i$ we have

$$u_i^\theta(g(s^*(\theta))) \geq u_i^\theta(g(m_i, s^*_{-i}(\theta_{-i}))) - \varepsilon\,.$$

Since $f(\theta) = g(s^*(\theta))$ for every θ, given any $\theta_i' \in \Theta_i$, letting $m_i = s_i^*(\theta_i')$, we get

$$u_i^\theta(f(\theta)) \geq u_i^\theta(g(s_i^*(\theta_i'), s^*_{-i}(\theta_{-i}))) - \varepsilon, \text{ for all } \theta_i'.$$

Because $g(s_i^*(\theta_i'), s^*_{-i}(\theta_{-i})) = g(s^*(\theta_i', \theta_{-i})) = f(\theta_i', \theta_{-i})$, we have $\varepsilon-EPIC$ satisfied as

$$u_i^\theta(f(\theta)) \geq u_i^\theta(f(\theta_i', \theta_{-i})) - \varepsilon, \text{ for all } i, \theta, \text{and } \theta_i'.$$

Next, consider a deception α with $f \circ \alpha \notin F$. Now, $s^* \circ \alpha$ cannot be an ε–ex post equilibrium of μ because otherwise $f = g \circ s^*$ implies $f \circ \alpha = g \circ s^* \circ \alpha$, which results in $f \circ \alpha \in F$, a contradiction. Therefore, for some θ, there exists i and $m_i \in M_i$ such that

$$u_i^\theta(g(m_i, s^*_{-i}(\alpha_{-i}(\theta_{-i})))) > u_i^\theta(g(s^*(\alpha(\theta)))) + \varepsilon\,.$$

Letting $a = g(m_i, s^*_{-i}(\alpha_{-i}(\theta_{-i})))$, and since $f = g \circ s$, we get condition (1) of $\varepsilon-EPM$ satisfied as $u_i^\theta(a) > u_i^\theta(f(\alpha(\theta)) + \varepsilon$. Condition (2) of $\varepsilon-EPM$ simply follows from the fact that F is implemented via μ in ε–ex post equilibrium and s^* is an ε–ex post equilibrium of μ:

$$\begin{aligned} u_i^{(\theta_i',\alpha_{-i}(\theta_{-i}))}(f(\theta_i',\alpha_{-i}(\theta_{-i}))) &= u_i^{(\theta_i',\alpha_{-i}(\theta_{-i}))}(g(s^*(\theta_i',\alpha_{-i}(\theta_{-i}))),\forall\theta_i', \\ &\geq u_i^{(\theta_i',\alpha_{-i}(\theta_{-i}))}(g(m_i,s_{-i}^*(\alpha_{-i}(\theta_{-i})))) - \varepsilon, \forall\theta_i', m_i, \\ &\geq u_i^{(\theta_i',\alpha_{-i}(\theta_{-i}))}(a) - \varepsilon, \forall\theta_i'. \end{aligned}$$

Therefore, $a \in A_{i,\varepsilon}^f(\alpha_{-i}(\theta_{-i}))$, as required by condition (2) of $\varepsilon-EPM$. ■

3.2 Sufficiency

First we introduce the epsilon–limited veto power property for environments with incomplete information and epsilon–bounded rationality. [7]

Definition 5 (ε–limited veto power) *F satisfies ε–*limited veto power at θ *if there exists a (possibly empty) set for each $i \in N$, $V_i^\theta \subseteq A$ such that*

(i) for every $f \in F$ we have $u_i^\theta(f(\theta)) \geq u_i^\theta(y) - \varepsilon$ for all $y \in V_i^\theta$, and

(ii) if $u_i^\theta(a) \geq \max_{x\in A} u_i^\theta(x) - \varepsilon$ for all $i \neq j$ for some $j \in N$ and $u_j^\theta(a) \geq u_j^\theta(y) - \varepsilon$ for all $y \in V_j^{\tilde{\theta}}$ for some $\tilde{\theta} \in \Theta$, then $a = f(\theta)$ for some $f \in F$.

*F satisfies ε–*limited veto power *($\varepsilon-LVP$) if it satisfies ε–limited veto power at every $\theta \in \Theta$.*

Our interpretation of $\varepsilon-LVP$ is as follows: agents are endowed with (possibly empty) sets of alternatives for each state that we call, with a slight abuse, *a state–dependent veto set*. An agent can veto an outcome in a state if she does not obtain a utility ε close to her best outcome in her veto set for that particular state. Furthermore, if all but one of the agents get a utility ε close to their best outcomes in some state and the odd man out gets a utility ε close to her best outcome in her veto set for some (possibly different) state, then this alternative must be optimal with respect to F.

Now we are ready to present our main sufficiency result.

Theorem 2 *If $n \geq 3$ and F satisfies $\varepsilon-EPIC$, $\varepsilon-EPM$, and $\varepsilon-LVP$, then F is ε–ex post implementable.*

[7] In this context it is useful to point out that the limited veto power property in Benoit and Ok (2006) concerns environments with complete information and full rationality and the one employed in Barlo and Dalkiran (2009) involves complete information but epsilon–bounded rationality.

Proof. Let $n \geq 3$, and F satisfy $\varepsilon - EPIC$, $\varepsilon - EPM$, and $\varepsilon - LVP$ and consider the following mechanism $\mu = (M, g)$, which is a modified version of the one proposed by BM:

The Mechanism:

Let $M_i = \Theta_i \times F \times A \times \mathbb{N}$, where $\mathbb{N}$ denotes the set of natural numbers and $m_i \in M_i$ equals $(\theta_i, f_i, a_i, n_i)$. Moreover $g : M \to A$ is defined as follows:

- **Rule 1:** If $f_i = f$ for all i, then $g(m) = f(\theta)$.
- **Rule 2:** If $f_i = f$ for all $i \neq j$ for some $j \in J$, then $g(m) = a_j$ only when $a_j \in A^f_{j,\varepsilon}(\theta_{-j})$; otherwise $g(m) = f(\theta)$.
- **Rule 3:** If neither Rule 1 nor Rule 2 applies, then $g(m) = a_j$, where $j = \text{argmax}_{i \in N} n_i$.

Below, we show that $\mu = (M, g)$ (defined above) implements F in ε–ex post equilibrium.

Claim 1 *For any $f \in F$, s^*, defined as $s^*_i(\theta) = (\theta_i, f, a, n)$ for every $\theta \in \Theta$ for all $i \in N$, is an ε–ex post equilibrium. Hence, $f = g \circ s^*$.*

Proof. Let $f \in F$ and notice that by definition we have $g(s^*(\theta)) = f(\theta)$ for every $\theta \in \Theta$.

Now, consider a deviation by agent i to $\tilde{s}_i(\theta_i) = (\tilde{\theta}_i, \tilde{f}(\theta_i), \tilde{a}(\theta_i), \tilde{n}(\theta_i))$.

If $\tilde{a}(\theta_i) \notin A^f_{i,\varepsilon}(\theta_{-i})$, then such a deviation changes the outcome to $f(\tilde{\theta}_i, \theta_{-i})$ due to Rule 1. By $\varepsilon - EPIC$ we have $u^\theta_i(f(\theta)) \geq u^\theta_i(f(\tilde{\theta}_i, \theta_{-i})) - \varepsilon$. Therefore, such a deviation cannot make agent i more than ε better at θ.

If $\tilde{a}(\theta_i) \in A^f_{i,\varepsilon}(\theta_{-i})$, then as a consequence of Rule 2 such a deviation changes the outcome at state θ to $\tilde{a}(\theta_i)$. That is, if for all $\theta'_i \in \Theta_i$

$$u_i^{(\theta'_i, \theta_{-i})}(f(\theta'_i, \theta_{-i})) \geq u_i^{(\theta'_i, \theta_{-i})}(\tilde{a}(\theta_i)) - \varepsilon,$$

we have $g(\tilde{s}_i(\theta_i), s^*_{-i}(\theta_{-i})) = \tilde{a}(\theta_i)$. Setting $\theta'_i = \theta_i$ gives

$$\begin{aligned} u^\theta_i(f(\theta)) &\geq u^\theta_i(\tilde{a}(\theta_i)) - \varepsilon \\ u^\theta_i(g(s^*(\theta))) &\geq u^\theta_i(g(\tilde{s}_i(\theta_i), s^*(\theta))) - \varepsilon \end{aligned}$$

Therefore, for all $i \in N$ and $m_i \in M_i$, $u^\theta_i(g(s^*(\theta))) \geq u^\theta_i(g(m_i, s^*_{-i}(\theta_{-i}))) - \varepsilon$ for every $\theta \in \Theta$. ■

Claim 2 *For any ε–ex post equilibrium, s^*, of $\mu = (M, g)$, there exists an $f \in F$ such that $g(s(\theta)) = f(\theta)$ for every $\theta \in \Theta$, i.e. $g \circ s^* = f$.*

Proof. The proof is presented in three cases:

Case 1: Rule 3 applies. Suppose s^* is such that for $s^*(\theta)$ there are three agents $j,k,l \in N$ with $f_j \neq f_k \neq f_l$ for some $j,k,l \in N$. Then by Rule 3 it is possible for any i to change the outcome at θ to any $a \in A$ simply by deviating to some $\tilde{s}$ with $\tilde{s}_i(\theta_i) = (\tilde{\theta}_i, \tilde{f}_i, \tilde{a}_i, \max_{i\in N} n_i(\theta_i)+1)$. Therefore, for such a s^* to be an ε–ex post equilibrium, we must have for all $i \in N$, $u_i^\theta(g(s^*(\theta))) \geq \max_{a\in A} u_i^\theta(a) - \varepsilon$. This holds for every $\theta \in \Theta$ where Rule 3 applies. Since F satisfies $\varepsilon-LVP$ at every θ by (ii) of $\varepsilon-LVP$, it must be that $g(s^*(\theta)) = f(\theta)$ for all such $\theta \in \Theta$.

Case 2: Rule 2 applies. Suppose s^* is such that for $s^*(\theta)$ we have $f_i = f$ for all $i \neq j$ for some $j \in J$, and $s_j^*(\theta_j) = (\hat{\theta}_j(\theta_j), \hat{f}(\theta_j), \hat{a}_j(\theta_j), \hat{n}_j(\theta_j))$. This means for all $i \neq j$ it is possible to change the outcome at θ to any $a \in A$ with a unilateral deviation, as in Case 1. Hence, we have for all $i \neq j$, $u_i^\theta(g(s^*(\theta))) \geq \max_{a\in A} u_i^\theta(a) - \varepsilon$.

Let s^{k*} denote the k^{th} coordinate of the vector $s^* = (s^{1*}, s^{2*}, s^{3*}, s^{4*})$. Since F satisfies $\varepsilon-LVP$ at every θ it satisfies $\varepsilon-LVP$ at $(\theta_j, s^{1*}_{-j}(\theta_{-j}))$ as well. Thus, by (i) of $\varepsilon-LVP$ we must have:

$$u_j^{(\theta_j,s^{1*}_{-j}(\theta_{-j}))}(f(\theta_j, s^{1*}_{-j}(\theta_{-j})) \geq u_i^{(\theta_j,s^{1*}_{-j}(\theta_{-j}))}(y) - \varepsilon, \text{ for all } y \in V_j^{(\theta_j,s^{1*}_{-j}(\theta_{-j}))}.$$

This implies $V_j^{(\theta_j,s^{1*}_{-j}(\theta_{-j}))} \subseteq A^f_{j,\varepsilon}(s^{1*}_{-j}(\theta_{-j}))$. Therefore, by Rule 2 agent j can change the outcome to any $y \in V_j^{(\theta_j,s^{1*}_{-j}(\theta_{-j}))}$ at θ. Since s^* is an ε–ex post equilibrium, it must be that

$$u_i^\theta(g(s^*(\theta))) \geq u_i^\theta(y) - \varepsilon, \text{ for all } y \in V_j^{(\theta_j,s^{1*}_{-j}(\theta_{-j}))}.$$

But this means, by (ii) of $\varepsilon-LVP$, that there must exist an $f \in F$ such that $g(s^*(\theta)) = f(\theta)$ for all such $\theta \in \Theta$.

Case 3: Rule 1 applies. Suppose s^* is such that for $s^*(\theta)$ we have $f_i = f$ for all $i \in N$. Therefore, by Rule 1, we have $g(s^*(\theta)) = f(s^{1*}(\theta))$ for all such θ. Hence, it is enough to show $f \circ s^{1*} \in F$. Suppose not. By $\varepsilon-EPM$ there exists $\hat{i} \in N$, $\hat{\theta} \in \Theta$, $\hat{a} \in A$ such that

$$\begin{aligned} u_{\hat{i}}^{\hat{\theta}}(\hat{a}) &> u_{\hat{i}}^{\hat{\theta}}(f(s^{1*}(\hat{\theta}))) + \varepsilon, \\ \hat{a} &\in A^f_{\hat{i},\varepsilon}(s^{1*}_{-\hat{i}}(\hat{\theta}_{-\hat{i}})). \end{aligned}$$

Subcase 3.1: Rule 3 applies at $\hat{\theta}$. $\hat{i}$ deviating to $\tilde{s}_{\hat{i}}$ with $\tilde{s}_{\hat{i}}(\hat{\theta}_i) = (\tilde{\theta}_i, \tilde{f}, \hat{a}, \max_{i\in N} n_i^*(\theta_i) + 1)$ changes the outcome at $\hat{\theta}$ to $\hat{a}$, contradicting s^* being an ε–ex post equilibrium. (Note that $n_i^*(\theta_i) := s_i^{4*}(\theta_i)$.)

Subcase 3.2: Rule 2 applies at $\hat{\theta}$. If $\hat{i}$ is not the odd man out, then by Rule 3 he can change the outcome to $\hat{a}$ at $\hat{\theta}$ by deviating to $\tilde{s}_{\hat{i}}$ of Subcase 3.1. If $\hat{i}$ is the odd-man-out, then by Rule 2 (since $\hat{a} \in A^{f}_{\hat{i},\varepsilon}(s^{1*}_{-\hat{i}}(\hat{\theta}_{-\hat{i}})))$ he can change the outcome to $\hat{a}$ at $\hat{\theta}$ by deviating to $\tilde{s}_{\hat{i}}$ with $\tilde{s}_{\hat{i}}(\hat{\theta}_i) = (\tilde{\theta}_i, \tilde{f}, \hat{a}, \tilde{n})$. Both of these contradict s^* being an ε–ex post equilibrium.

Subcase 3.3: Rule 1 applies at $\hat{\theta}$. Due to Rule 2 (since $\hat{a} \in A^{f}_{\hat{i},\varepsilon}(s^{1*}_{-\hat{i}}(\hat{\theta}_{-\hat{i}})))$, $\hat{i}$ can change the outcome to $\hat{a}$ at $\hat{\theta}$ by deviating to $\tilde{s}_{\hat{i}}$ of Subcase 3.2, delivering a contradiction. ■ ■

4 Examples

This section proves that ex post monotonicity (henceforth EPM) and $\varepsilon-EPM$ are not nested. Towards this regard, we construct two examples by modifying an example in Section 8.1 of BM.

4.1 $\varepsilon-EPM$ does not imply EPM

We show that $\varepsilon-EPM$ does not imply EPM, for any given small $\varepsilon > 0$ by employing the Pareto rule.

Suppose that $N = \{1,2,3\}$ and $\Theta_i = \{0,1\}$. Hence, a type profile is $(\theta_1,\theta_2,\theta_3) \in \Theta = \{0,1\}^3$. There are 8 possible outcomes, given by

$$A = \{000, 001, 010, 011, 100, 101, 110, 111\}.$$

The set of outcomes and the set of type profiles coincide, but the outcomes are described as strings rather than vectors. For every (true) type profile, $\theta = (\theta_1,\theta_2,\theta_3)$, of the society payoff profile corresponding to each outcome is given by the following matrix:

	$\theta_3 = 0$	$\theta_2 = 0$	$\theta_2 = 1$	$\theta_3 = 1$	$\theta_2 = 0$	$\theta_2 = 1$
$\theta = (0,0,0)$	$\theta_1 = 0$	$3,0,0$	$0,0,0$	$\theta_1 = 0$	$0,0,0$	$0,\varepsilon,0$
	$\theta_1 = 1$	$\frac{3-\varepsilon}{2},\frac{3-\varepsilon}{2},\varepsilon$	$0,0,0$	$\theta_1 = 1$	$0,0,0$	$\varepsilon,\varepsilon,\varepsilon$
	$\theta_3 = 0$	$\theta_2 = 0$	$\theta_2 = 1$	$\theta_3 = 1$	$\theta_2 = 0$	$\theta_2 = 1$
$\theta = (0,0,1)$	$\theta_1 = 0$	$0,0,0$	$0,0,0$	$\theta_1 = 0$	$0,3,0$	$\varepsilon,\frac{3-\varepsilon}{2},\frac{3-\varepsilon}{2}$
	$\theta_1 = 1$	$0,0,0$	$\varepsilon,\varepsilon,\varepsilon$	$\theta_1 = 1$	$0,0,0$	$\varepsilon,0,0$
	$\theta_3 = 0$	$\theta_2 = 0$	$\theta_2 = 1$	$\theta_3 = 1$	$\theta_2 = 0$	$\theta_2 = 1$
$\theta = (0,1,0)$	$\theta_1 = 0$	$1,1,1$	$0,3,0$	$\theta_1 = 0$	$0,0,0$	$0,0,0$
	$\theta_1 = 1$	$0,0,\varepsilon$	$0,0,\varepsilon$	$\theta_1 = 1$	$\varepsilon,\varepsilon,\varepsilon$	$0,0,0$
	$\theta_3 = 0$	$\theta_2 = 0$	$\theta_2 = 1$	$\theta_3 = 1$	$\theta_2 = 0$	$\theta_2 = 1$
$\theta = (0,1,1)$	$\theta_1 = 0$	$\varepsilon,0,0$	$1,1,1$	$\theta_1 = 0$	$0,0,0$	$0,0,3$
	$\theta_1 = 1$	$\varepsilon,\varepsilon,\varepsilon$	$\varepsilon,\varepsilon,0$	$\theta_1 = 1$	$0,0,0$	$0,\varepsilon,0$

	$\theta_3 = 0$	$\theta_2 = 0$	$\theta_2 = 1$	$\theta_3 = 1$	$\theta_2 = 0$	$\theta_2 = 1$
$\theta = (1,0,0)$	$\theta_1 = 0$	$0,0,\varepsilon$	$\varepsilon,0,\varepsilon$	$\theta_1 = 0$	$0,0,0$	$\varepsilon,\varepsilon,\varepsilon$
	$\theta_1 = 1$	$0,3,0$	$1,1,1$	$\theta_1 = 1$	$0,0,0$	$\varepsilon,0,0$
	$\theta_3 = 0$	$\theta_2 = 0$	$\theta_2 = 1$	$\theta_3 = 1$	$\theta_2 = 0$	$\theta_2 = 1$
$\theta = (1,0,1)$	$\theta_1 = 0$	$\varepsilon,0,0$	$\varepsilon,\varepsilon,\varepsilon$	$\theta_1 = 0$	$0,0,0$	$0,0,0$
	$\theta_1 = 1$	$\varepsilon,\frac{3-\varepsilon}{2},\frac{3-\varepsilon}{2}$	$0,0,0$	$\theta_1 = 1$	$0,0,3$	$0,0,0$
	$\theta_3 = 0$	$\theta_2 = 0$	$\theta_2 = 1$	$\theta_3 = 1$	$\theta_2 = 0$	$\theta_2 = 1$
$\theta = (1,1,0)$	$\theta_1 = 0$	$0,0,0$	$0,\varepsilon,0$	$\theta_1 = 0$	$\varepsilon,\varepsilon,\varepsilon$	$\varepsilon,\varepsilon,0$
	$\theta_1 = 1$	$\varepsilon,0,0$	$0,0,3$	$\theta_1 = 1$	$0,0,0$	$1,1,1$
	$\theta_3 = 0$	$\theta_2 = 0$	$\theta_2 = 1$	$\theta_3 = 1$	$\theta_2 = 0$	$\theta_2 = 1$
$\theta = (1,1,1)$	$\theta_1 = 0$	$\varepsilon,\varepsilon,\varepsilon$	$0,\varepsilon,0$	$\theta_1 = 0$	$0,0,0$	$\frac{3-\varepsilon}{2},\varepsilon,\frac{3-\varepsilon}{2}$
	$\theta_1 = 1$	$0,0,\varepsilon$	$0,0,0$	$\theta_1 = 1$	$0,0,0$	$3,0,0$

In general, the Pareto correspondence is defined by

$$PO(\theta) = \{a \in A | \forall b \in A \ \ \exists i \ \ s.t. \ \ u_i^\theta(a) \geq u_i^\theta(b)\}.$$

In the above example, there are exactly two Pareto–efficient outcomes in every state. The first Pareto outcome corresponds to the true state, $a = (\theta_1\theta_2\theta_3)$, and it favors one agent with payoff 3 and leaves the remaining two agents with payoff 0.[8] The favored agent is the one with index $\sum_i \theta_i (\text{mod} 3) + 1$. In the second Pareto outcome, either each agent receives a uniform payoff of 1, or two agents receive a payoff of $\frac{3-\varepsilon}{2}$ while the last agent receives a payoff of ε. The remaining outcomes are all Pareto inferior and the remaining payoff vectors are combinations of 0 and $\varepsilon \in (0,1)$ entries.[9]

Therefore, Pareto Rule, $PO : \Theta \to A$, for this example is given as follows:

$\theta_3 = 0$	$\theta_2 = 0$	$\theta_2 = 1$	$\theta_3 = 1$	$\theta_2 = 0$	$\theta_2 = 1$
$\theta_1 = 0$	$\{000, 100\}$	$\{010, 000\}$	$\theta_1 = 0$	$\{001, 011\}$	$\{011, 010\}$
$\theta_1 = 1$	$\{100, 110\}$	$\{110, 111\}$	$\theta_1 = 1$	$\{101, 100\}$	$\{111, 011\}$

Consider the social choice set F: $f \in F$ if and only if $f(\theta) \in PO(\theta)$ for each $\theta \in \Theta$.

Claim 3 *F does not satisfy EPM but $\varepsilon-EPM$ whenever $\varepsilon \in (0, \frac{1}{2})$.*

Proof: F is not EPM. We show that there is no triplet $\langle \theta, i, a \rangle$ satisfying (1) and (2) of EPM simultaneously. The analysis is on a case-by-case basis for each θ, but it suffices to consider only $\theta = (0,0,0)$ because a similar reasoning holds for every $\theta \in \Theta$.

[8] E.g. if the true state is (0,1,0), then $a = 010$.

[9] Note that only payoff profiles that differ from the example of BM correspond to the following 4 scenarios: 100 in (0,0,0); 100 in (1,0,1); 011 in (0,0,1); and 011 in (1,1,1).

Case 1: "$\theta = (0,0,0)$" Let $\theta = (0,0,0)$. Observe first that due to (1) of EPM the reward a must give a strictly positive payoff to the whistle blower whatever the deception α is.[10] Therefore, it is enough to consider the following four cases $a = 000$, $a = 100$, $a = 011$, $a = 111$ for any of $i = 1,2,3$ at $\theta = (0,0,0)$.

Case 1–1: "$a = 000$" $a = 000$ is the first Pareto outcome of $\theta = (0,0,0)$. It cannot work as a reward for agents $i = 2,3$ because their payoffs are 0 at $\theta = (0,0,0)$. On the other hand, $a = 000$ does not work as a reward for $i = 1$ because when $\alpha(0,0,0) = (1,1,1)$ and $f((0,\alpha_{-1}(0,0))) = 011$ we have:

$$u_1^{(0,\alpha_{-1}(0,0))}(f((0,\alpha_{-1}(0,0)))) < u_1^{(0,\alpha_{-1}(00))}(000).$$

That is, $u_1^{(0,1,1)}(011) = 0 < u_1^{(0,1,1)}(000) = \varepsilon$, which means (2) of EPM fails when $\theta_1 = 0$.

Case 1–2: "$a = 100$" $a = 100$ is the second Pareto outcome of $\theta = (0,0,0)$. Consider a deception α with $\alpha(0,0,0) = (1,1,1)$. Observe first that (1) of EPM fails for $i = 3$ when $f(\alpha(\theta)) = 111$:

$$u_3^{(0,0,0)}(100) = \varepsilon = u_3^{(0,0,0)}(111).$$

$a = 100$ satisfies the reward equality (1) of EPM only for $i = 1,2$:

$$u_i^{(0,0,0)}(100) = \frac{3-\varepsilon}{2} > \varepsilon \geq u_i^{(0,0,0)}(f(\alpha(0,0,0))),\ \ i = 1,2.$$

But, when $f((0,1,1)) = 011$, for agent 1 we have

$$u_1^{(0,\alpha_{-1}(0,0))}(f((0,\alpha_{-1}(0,0)))) = 0 < u_1^{(0,\alpha_{-1}(00))}(100) = \varepsilon,$$

and when $f((1,0,1)) = 101$, for agent 2 we have

$$u_2^{(0,\alpha_{-2}(0,0))}(f((0,\alpha_{-2}(0,0)))) = 0 < u_2^{(0,\alpha_{-2}(00))}(100) = \frac{3-\varepsilon}{2}.$$

Therefore, for the reward $a = 100$ we observe that (1) of EPM cannot be satisfied for agent $i = 3$ and (2) of EPM cannot be satisfied for $i = 1,2$.[11]

Case 1–3: "$a = 011$" $a = 011$ does not work for $i = 1,3$ because their payoffs are 0 at $\theta = (0,0,0)$. For $i = 2$, consider α with $\alpha(0,0,0) = (1,0,0)$. Now, (1) of EPM fails when $f(\alpha(\theta)) = 100$:

$$u_2^{(0,0,0)}(011) = \varepsilon < u_2^{(0,0,0)}(100) = \frac{3-\varepsilon}{2}.$$

[10] Note that there are no negative payoffs in any state.

[11] Note that we have $(0,\alpha_{-1}(0,0)) = (0,1,1)$ and $(0,\alpha_{-2}(0,0)) = (1,0,1)$.

Case 1–4: "$a = 111$" $a = 111$ gives a uniform payoff of ε to every agent at $\theta = (0,0,0)$. Consider α with $\alpha(0,0,0) = (1,0,0)$. Now, (1) of EPM fails when $f(\alpha(\theta)) = 100$ for agents $i = 1,2$:

$$u_i^{(0,0,0)}(111) = \varepsilon < u_i^{(0,0,0)}(100) = \frac{3-\varepsilon}{2},\ i = 1,2.$$

Moreover, $a = 111$ does not work for agent $i = 3$ either, because for α with $\alpha(0,0,0) = (1,1,0)$, (2) of EPM fails when $f((1,1,0)) = 110$:

$$u_3^{(0,\alpha_{-3})}(111) = 1 > u_3^{(0,\alpha_{-3})}(110) = 0.$$

That is, $u_3^{(1,1,0)}(111) = 1 > u_3^{(1,1,0)}(110) = 0$.

Hence, when $\theta = (0,0,0)$, one cannot identify a whistle-blower i and a corresponding reward a such that the triplet $\langle\theta, i, a\rangle$ satisfies (1) and (2) of EPM. By construction, the same argument goes through in every state, implying that F fails to be EPM. ■

Proof: F is $\varepsilon-EPM$ whenever $\varepsilon \in (0, \frac{1}{2})$. Consider any deception α with $f \circ \alpha \notin F$. Therefore, there exists θ such that $f(\alpha(\theta)) \notin PO(\theta)$. For each such $\theta \in \Theta$, we introduce a whistle blower $i_\theta = \sum_i \theta_i(\text{mod}3) + 1$ and a corresponding reward $a_\theta = \theta'_{i_\theta}\theta_{-i_\theta} \in F(\theta)$, where $\theta'_{i_\theta} \neq \theta_{i_\theta}$ (i.e. the second Pareto outcome in state θ). E.g. if $\theta = (1,0,0)$, then $i_\theta = 2$ and $a_\theta = 110$. We show that the $\langle\theta, i_\theta, a_\theta\rangle$ profile given below satisfies (1) and (2) of $\varepsilon-EPM$ whenever $\varepsilon \in (0, \frac{1}{2})$:

$\theta_3 = 0$	$\theta_2 = 0$	$\theta_2 = 1$	$\theta_3 = 1$	$\theta_2 = 0$	$\theta_2 = 1$
$\theta_1 = 0$	$i_\theta =$ **1;** $a_\theta =$ **100**	$i_\theta =$ **2;** $a_\theta =$ **000**	$\theta_1 = 0$	$i_\theta =$ **2;** $a_\theta =$ **011**	$i_\theta =$ **3;** $a_\theta =$ **010**
$\theta_1 = 1$	$i_\theta =$ **2;** $a_\theta =$ **110**	$i_\theta =$ **3;** $a_\theta =$ **111**	$\theta_1 = 1$	$i_\theta =$ **3;** $a_\theta =$ **100**	$i_\theta =$ **1;** $a_\theta =$ **011**

Hence, for each $\theta \in \Theta$ we either have $u^\theta_{i_\theta}(a_\theta) = 1$ or $u^\theta_{i_\theta}(a_\theta) = \frac{3-\varepsilon}{2}$. Moreover, since $f(\alpha(\theta)) \notin PO(\theta)$, we have $u^\theta_{i_\theta}(f(\alpha(\theta))) \leq \varepsilon$. Noting that $\frac{1}{2} > \varepsilon > 0$, we get $u^\theta_{i_\theta}(a_\theta) > u^\theta_{i_\theta}(f(\alpha(\theta))) + \varepsilon$. Thus, $\langle\theta, i_\theta, a_\theta\rangle$ defined above satisfies (1) of $\varepsilon-EPM$ for $\varepsilon \in (0, \frac{1}{2})$.

We are now left to show (2) of $\varepsilon-EPM$ holds with $\langle\theta, i_\theta, a_\theta\rangle$ as well: for any $(\theta, i_\theta, a_\theta)$ we need to show for all $\theta'_{i_\theta} \in \Theta_{i_\theta}$, we have

$$u_{i_\theta}^{(\theta'_{i_\theta},\alpha_{-i_\theta}(\theta_{-i_\theta}))}(f((\theta'_{i_\theta}, \alpha_{-i_\theta}(\theta_{-i_\theta})))) \geq u_{i_\theta}^{(\theta'_{i_\theta},\alpha_{-i_\theta}(\theta_{-i_\theta}))}(a_\theta) - \varepsilon\,.$$

The first Pareto outcome corresponds to one of three payoff profiles $\{(3,0,0), (0,3,0), (0,0,3)\}$, while the second corresponds either to the payoff profile $(1,1,1)$ or to one of the payoff profiles $\{(\varepsilon, \frac{3-\varepsilon}{2}, \frac{3-\varepsilon}{2}), (\frac{3-\varepsilon}{2}, \varepsilon, \frac{3-\varepsilon}{2}), (\frac{3-\varepsilon}{2}, \frac{3-\varepsilon}{2}, \varepsilon)\}$. Therefore, in a Pareto outcome there are five possible payoffs an agent can receive: $3; \frac{3-\varepsilon}{2}; 1; \varepsilon; and 0$. Because we have $f((\theta'_{i_\theta}, \alpha_{-i_\theta}(\theta_{-i_\theta}))) \in PO((\theta'_{i_\theta}, \alpha_{-i_\theta}(\theta_{-i_\theta})))$, we have the following five cases:

Case 1: $[u_{i_\theta}^{(\theta'_{i_\theta},\alpha_{-i_\theta}(\theta_{-i_\theta}))}(f(\theta'_{i_\theta},\alpha_{-i_\theta}(\theta_{-i_\theta}))) = 3]$ We are done trivially since the maximal possible payoff is 3.
Case 2: $[u_{i_\theta}^{(\theta'_{i_\theta},\alpha_{-i_\theta}(\theta_{-i_\theta}))}(f(\theta'_{i_\theta},\alpha_{-i_\theta}(\theta_{-i_\theta}))) = \frac{3-\varepsilon}{2}]$ The only payoff larger than $\frac{3-\varepsilon}{2}$ is 3. One can see from the payoff matrices that the whistle blower – reward couples $(i_\theta; a_\theta)$ given above are such that $u_{i_\theta}^{(\theta'_{i_\theta},\alpha_{-i_\theta}(\theta_{-i_\theta}))}(a_\theta) \neq 3$.
Case 3: $[u_{i_\theta}^{(\theta'_{i_\theta},\alpha_{-i_\theta}(\theta_{-i_\theta}))}(f(\theta'_{i_\theta},\alpha_{-i_\theta}(\theta_{-i_\theta}))) = 1]$ In any state, if there is an agent with a payoff of 1, then there is no agent with a payoff of $\frac{3-\varepsilon}{2}$. Furthermore, the only agent who can receive a payoff of 3 in a state is the whistle blower associated with that state. Therefore, examining the payoff matrix and whistle blower – reward couples, one can see that we must have $u_{i_\theta}^{(\theta'_{i_\theta},\alpha_{-i_\theta}(\theta_{-i_\theta}))}(a_\theta) \leq \varepsilon$.
Case 4: $[u_{i_\theta}^{(\theta'_{i_\theta},\alpha_{-i_\theta}(\theta_{-i_\theta}))}(f(\theta'_{i_\theta},\alpha_{-i_\theta}(\theta_{-i_\theta}))) = \varepsilon]$ Similarly, examining the payoff matrix and whistle blower – reward couples, one can see that we must have $u_{i_\theta}^{(\theta'_{i_\theta},\alpha_{-i_\theta}(\theta_{-i_\theta}))}(a_\theta) \leq \varepsilon$.
Case 5: $[u_{i_\theta}^{(\theta'_{i_\theta},\alpha_{-i_\theta}(\theta_{-i_\theta}))}(f(\theta'_{i_\theta},\alpha_{-i_\theta}(\theta_{-i_\theta}))) = 0]$ Notice that this can only happen when the payoff profile is one of $\{(3,0,0),(0,3,0),(0,0,3)\}$. That is, there exists some $j \neq i_\theta$ with $u_j^{(\theta'_{i_\theta},\alpha_{-i_\theta}(\theta_{-i_\theta}))}(f(\theta'_{i_\theta},\alpha_{-i_\theta}(\theta_{-i_\theta}))) = 3$. There are 8 possible outcomes where this can be the case. Examining these outcomes as the outcome of a possible deception in the above payoff matrix establishes that $u_{i_\theta}^{(\theta'_{i_\theta},\alpha_{-i_\theta}(\theta_{-i_\theta}))}(a_\theta) \leq \varepsilon$ in this case as well. ■

It is straightforward to check that F also satisfies $\varepsilon - EPIC$ and $\varepsilon - EPM$. Therefore, even though F fails to be ex post implementable, it is possible to implement F in ε–ex post equilibrium whenever $\varepsilon \in (0, \frac{1}{2})$.

4.2 EPM does not imply $\varepsilon - EPM$

This example shows that EPM does not imply $\varepsilon - EPM$ for any given small $\varepsilon > 0$. Again, suppose that $N = \{1,2,3\}$ and $\Theta_i = \{0,1\}$ and a type profile is $(\theta_1, \theta_2, \theta_3) \in \Theta = \{0,1\}^3$. As before, we have $A = \{000, 001, 010, 011, 100, 101, 110, 111\}$, where the set of outcomes and the set of type profiles coincide but the outcomes are described as strings rather than vectors. For every (true) type profile, $\theta = (\theta_1, \theta_2, \theta_3)$, of the society payoff profile corresponding to each outcome is given by the following matrix:

	$\theta_3 = 0$	$\theta_2 = 0$	$\theta_2 = 1$	$\theta_3 = 1$	$\theta_2 = 0$	$\theta_2 = 1$
$\theta = (0,0,0)$	$\theta_1 = 0$	$1-\varepsilon, 0, \varepsilon$	$0,0,0$	$\theta_1 = 0$	$0,0,0$	$0,0,0$
	$\theta_1 = 1$	$1-2\,\varepsilon, \varepsilon, \varepsilon$	$0,0,0$	$\theta_1 = 1$	$0, 2\,\varepsilon, 0$	$1-3\,\varepsilon, \varepsilon, 0$
	$\theta_3 = 0$	$\theta_2 = 0$	$\theta_2 = 1$	$\theta_3 = 1$	$\theta_2 = 0$	$\theta_2 = 1$
$\theta = (0,0,1)$	$\theta_1 = 0$	$0,0,0$	$0,0,0$	$\theta_1 = 0$	$\varepsilon, 1-\varepsilon, 0$	$\varepsilon, 1-2\,\varepsilon, \varepsilon$
	$\theta_1 = 1$	$0,0,0$	$0, 1-3\,\varepsilon, 0$	$\theta_1 = 1$	$0,0,0$	$0,0,0$
	$\theta_3 = 0$	$\theta_2 = 0$	$\theta_2 = 1$	$\theta_3 = 1$	$\theta_2 = 0$	$\theta_2 = 1$
$\theta = (0,1,0)$	$\theta_1 = 0$	$\varepsilon, 1-2\,\varepsilon, \varepsilon$	$\varepsilon, 1-\varepsilon, 0$	$\theta_1 = 0$	$0,0,0$	$0,0,0$
	$\theta_1 = 1$	$0, 0, 2\,\varepsilon$	$0,0,0$	$\theta_1 = 1$	$0, 1-3\,\varepsilon, \varepsilon$	$0,0,0$
	$\theta_3 = 0$	$\theta_2 = 0$	$\theta_2 = 1$	$\theta_3 = 1$	$\theta_2 = 0$	$\theta_2 = 1$
$\theta = (0,1,1)$	$\theta_1 = 0$	$2\,\varepsilon, 0, 0$	$\varepsilon, \varepsilon, 1-2\,\varepsilon$	$\theta_1 = 0$	$0,0,0$	$\varepsilon, 0, 1-\varepsilon$
	$\theta_1 = 1$	$\varepsilon, \varepsilon, 1-3\,\varepsilon$	$0, 2\,\varepsilon, 0$	$\theta_1 = 1$	$0,0,0$	$0,0,0$
	$\theta_3 = 0$	$\theta_2 = 0$	$\theta_2 = 1$	$\theta_3 = 1$	$\theta_2 = 0$	$\theta_2 = 1$
$\theta = (1,0,0)$	$\theta_1 = 0$	$0,0,0$	$2\,\varepsilon, 0, 0$	$\theta_1 = 0$	$0,0,0$	$\varepsilon, 1-3\,\varepsilon, 0$
	$\theta_1 = 1$	$0, 1-\varepsilon, \varepsilon$	$\varepsilon, 1-2\,\varepsilon, \varepsilon$	$\theta_1 = 1$	$0,0,0$	$2\,\varepsilon, 0, 0$
	$\theta_3 = 0$	$\theta_2 = 0$	$\theta_2 = 1$	$\theta_3 = 1$	$\theta_2 = 0$	$\theta_2 = 1$
$\theta = (1,0,1)$	$\theta_1 = 0$	$0,0,0$	$0, 0, 1-3\,\varepsilon$	$\theta_1 = 0$	$0,0,0$	$0,0,0$
	$\theta_1 = 1$	$\varepsilon, \varepsilon, 1-2\,\varepsilon$	$0,0,0$	$\theta_1 = 1$	$\varepsilon, 0, 1-\varepsilon$	$0,0,0$
	$\theta_3 = 0$	$\theta_2 = 0$	$\theta_2 = 1$	$\theta_3 = 1$	$\theta_2 = 0$	$\theta_2 = 1$
$\theta = (1,1,0)$	$\theta_1 = 0$	$0,0,0$	$0,0,0$	$\theta_1 = 0$	$0, \varepsilon, 1-3\,\varepsilon$	$0, 2\,\varepsilon, 0$
	$\theta_1 = 1$	$0,0,0$	$\varepsilon, 0, 1-\varepsilon$	$\theta_1 = 1$	$0,0,0$	$\varepsilon, \varepsilon, 1-2\,\varepsilon$
	$\theta_3 = 0$	$\theta_2 = 0$	$\theta_2 = 1$	$\theta_3 = 1$	$\theta_2 = 0$	$\theta_2 = 1$
$\theta = (1,1,1)$	$\theta_1 = 0$	$1-3\,\varepsilon, 0, \varepsilon$	$0,0,0$	$\theta_1 = 0$	$0, 0, 2\,\varepsilon$	$1-2\,\varepsilon, \varepsilon, \varepsilon$
	$\theta_1 = 1$	$0,0,0$	$0,0,0$	$\theta_1 = 1$	$0,0,0$	$1-\varepsilon, \varepsilon, 0$

As before, in every state θ there are exactly two Pareto–efficient outcomes: the first corresponds to the true state $a = \theta_1\theta_2\theta_3$, favoring the agent having an index $\sum_i \theta_i (\text{mod}3) + 1$ and a payoff of $1-\varepsilon$ (leaving the remaining two agents ε or 0); the second features the same favored agent but this time she obtains a payoff of $1-2\,\varepsilon$ while the remaining agents receive a payoff of ε each.

The payoffs are different but the Pareto outcomes coincide with the previous example. In what follows, we consider the social choice set F such that $f \in F$ if and only if $f(\theta) \in PO(\theta)$

for each $\theta \in \Theta$ where $PO : \Theta \to A$ (the Pareto rule) is given as before:

$\theta_3 = 0$	$\theta_2 = 0$	$\theta_2 = 1$	$\theta_3 = 1$	$\theta_2 = 0$	$\theta_2 = 1$
$\theta_1 = 0$	$\{000, 100\}$	$\{010, 000\}$	$\theta_1 = 0$	$\{001, 011\}$	$\{011, 010\}$
$\theta_1 = 1$	$\{100, 110\}$	$\{110, 111\}$	$\theta_1 = 1$	$\{101, 100\}$	$\{111, 011\}$

Claim 4 *F satisfies EPM but fails to satisfy $\varepsilon - EPM$ whenever $\varepsilon \in (0, \frac{1}{4})$.*

Proof: F is EPM. Consider any deception α with $f \circ \alpha \notin F$. For every θ with $f(\alpha(\theta)) \notin PO(\theta)$, let the whistle blower be given by $i_\theta = \sum_i \theta_i(\text{mod}3)+1$ and the corresponding reward $a_\theta = \theta'_{i_\theta}\theta_{-i_\theta} \in F(\theta)$, where $\theta'_{i_\theta} \neq \theta_{i_\theta}$ as in the previous example. E.g. if $\theta = (0, 0, 0)$, then $i_\theta = 1$ and $a_\theta = 100$. We show that $\langle \theta, i_\theta, a_\theta \rangle$ defined below satisfies (1) and (2) of EPM:[12]

$\theta_3 = 0$	$\theta_2 = 0$	$\theta_2 = 1$	$\theta_3 = 1$	$\theta_2 = 0$	$\theta_2 = 1$
$\theta_1 = 0$	$i_\theta =$ **1;** $a_\theta =$ **100**	$i_\theta =$ **2;** $a_\theta =$ **000**	$\theta_1 = 0$	$i_\theta =$ **2;** $a_\theta =$ **011**	$i_\theta =$ **3;** $a_\theta =$ **010**
$\theta_1 = 1$	$i_\theta =$ **2;** $a_\theta =$ **110**	$i_\theta =$ **3;** $a_\theta =$ **111**	$\theta_1 = 1$	$i_\theta =$ **3;** $a_\theta =$ **100**	$i_\theta =$ **1;** $a_\theta =$ **011**

The observation that $\langle \theta, i_\theta, a_\theta \rangle$ defined above satisfies (1) of EPM follows from analyzing the payoff matrix for each $\theta \in \Theta$, noticing that

$$u^\theta_{i_\theta}(a_\theta) = 1 - 2\,\varepsilon > 1 - 3\,\varepsilon \geq u^\theta_{i_\theta}(f(\alpha(\theta))).$$

To establish that $\langle \theta, i_\theta, a_\theta \rangle$ satisfies (2) of EPM, we have to display that for any given $(\theta, i_\theta, a_\theta)$ and for all $\theta'_{i_\theta} \in \Theta_{i_\theta}$ we have

$$u_{i_\theta}^{(\theta'_{i_\theta}, \alpha_{-i_\theta}(\theta_{-i_\theta}))}(f((\theta'_{i_\theta}, \alpha_{-i_\theta}(\theta_{-i_\theta})))) \geq u_{i_\theta}^{(\theta'_{i_\theta}, \alpha_{-i_\theta}(\theta_{-i_\theta}))}(a_\theta).$$

Checking the payoff matrix reveals that there are 4 different payoffs an agent can receive in a Pareto outcome: $1 - \varepsilon$, $1 - 2\,\varepsilon$, ε, and 0. Since $f((\theta'_{i_\theta}, \alpha_{-i_\theta}(\theta_{-i_\theta}))) \in PO((\theta'_{i_\theta}, \alpha_{-i_\theta}(\theta_{-i_\theta})))$, analyzing the following four cases suffices:

Case 1: $[u_{i_\theta}^{(\theta'_{i_\theta}, \alpha_{-i_\theta}(\theta_{-i_\theta}))}(f(\theta'_{i_\theta}, \alpha_{-i_\theta}(\theta_{-i_\theta}))) = 1 - \varepsilon]$ The result follows trivially as the maximal possible payoff is $1 - \varepsilon$.

Case 2: $[u_{i_\theta}^{(\theta'_{i_\theta}, \alpha_{-i_\theta}(\theta_{-i_\theta}))}(f(\theta'_{i_\theta}, \alpha_{-i_\theta}(\theta_{-i_\theta}))) = 1 - 2\,\varepsilon]$ The only payoff larger than $1 - 2\,\varepsilon$ is $1 - \varepsilon$. One can see from the payoff matrices that the whistle blower – reward couples $(i_\theta; a_\theta)$ given above are such that $u_{i_\theta}^{(\theta'_{i_\theta}, \alpha_{-i_\theta}(\theta_{-i_\theta}))}(a_\theta) \neq 1 - \varepsilon$.

Case 3: $[u_{i_\theta}^{(\theta'_{i_\theta}, \alpha_{-i_\theta}(\theta_{-i_\theta}))}(f(\theta'_{i_\theta}, \alpha_{-i_\theta}(\theta_{-i_\theta}))) = \varepsilon]$ In any state if there is an agent with a payoff of ε, then there is some $j \neq i_\theta$ with $u_j^{(\theta'_{i_\theta}, \alpha_{-i_\theta}(\theta_{-i_\theta}))}(f(\theta'_{i_\theta}, \alpha_{-i_\theta}(\theta_{-i_\theta}))) = 1 - \varepsilon$, or there exists

[12] Recall that (1) and (2) here correspond to (1) and (2) of $\varepsilon - EPM$ when $\varepsilon = 0$.

some $j \neq i_\theta$ with $u_j^{(\theta'_{i_\theta},\alpha_{-i_\theta}(\theta_{-i_\theta}))}(f(\theta'_{i_\theta},\alpha_{-i_\theta}(\theta_{-i_\theta}))) = 1 - 2\,\varepsilon$. Furthermore, the only agent who can obtain a payoff of $1-\varepsilon$ or $1-2\,\varepsilon$ in a state is the whistle blower associated with that state. Therefore, examining the payoff matrix and whistle blower – reward couples, one can see that in this case we always have $u_{i_\theta}^{(\theta'_{i_\theta},\alpha_{-i_\theta}(\theta_{-i_\theta}))}(a_\theta) \leq \varepsilon$.

Case 4: $[u_{i_\theta}^{(\theta'_{i_\theta},\alpha_{-i_\theta}(\theta_{-i_\theta}))}(f(\theta'_{i_\theta},\alpha_{-i_\theta}(\theta_{-i_\theta}))) = 0]$ This can only happen when the outcome is of the first type of Pareto outcome (those with a payoff profile equal to either one of the following: $(1-\varepsilon, 0, \varepsilon)$, $(\varepsilon, 1-\varepsilon, 0)$, $(\varepsilon, 0, 1-\varepsilon)$, $(0, 1-\varepsilon, \varepsilon)$, and $(1-\varepsilon, \varepsilon, 0)$). That is, there exists some $j \neq i_\theta$ with $u_j^{(\theta'_{i_\theta},\alpha_{-i_\theta}(\theta_{-i_\theta}))}(f(\theta'_{i_\theta},\alpha_{-i_\theta}(\theta_{-i_\theta}))) = 1-\varepsilon$. There are 8 possible outcomes where this can be the case. Examining all these outcomes as the outcome of a possible deception in the above payoff matrix, one can see that we must have $u_{i_\theta}^{(\theta'_{i_\theta},\alpha_{-i_\theta}(\theta_{-i_\theta}))}(a_\theta) = 0$ in this case as well. ■

Proof: F is not $\varepsilon-EPM$ when $\varepsilon \in (0, \frac{1}{4})$. We will show that there is no triplet $\langle \theta, i, a \rangle$ satisfying (1) and (2) of $\varepsilon-EPM$ simultaneously. As before, we provide a case-by-case analysis for θ. However, analyzing only the case $\theta = (0,0,0)$ suffices because similar reasoning holds for all $\theta \in \Theta$.

Case 1: "$\theta = (0,0,0)$" Let $\theta = (0,0,0)$. Observe first that due to (1) of $\varepsilon-EPM$ the reward a must give a payoff strictly greater than ε to the whistle blower, whatever the deception α.[13] Hence, it is enough to show that none of the following four cases works for any of $i = 1,2,3$ at $\theta = (0,0,0)$: $a = 000$, $a = 100$, $a = 101$, or $a = 111$

Case 1–1: "$a = 000$" $a = 000$ is the first Pareto outcome of $\theta = (0,0,0)$. It cannot work as a reward for agent $i = 2$ and $i = 3$ because their payoffs are less than (or equal to) ε at $\theta = (0,0,0)$. On the other hand, $a = 000$ does not work as a reward for $i = 1$ because when $\alpha(0,0,0) = (1,0,0)$ and $f((1,0,0)) = 100$ the strictness of the reward equality (1) of $\varepsilon-EPM$ fails:

$$u_1^{(0,0,0)}(000) = 1 - \varepsilon = u_1^{(0,0,0)}(f(\alpha(0,0,0))) + \varepsilon = 1 - 2\,\varepsilon + \varepsilon\,.$$

Case 1–2: "$a = 100$" $a = 100$ is the second Pareto outcome of $\theta = (0,0,0)$. It cannot work as a reward for agent $i = 2$ and $i = 3$ because their payoffs are less than (or equal to) ε at $\theta = (0,0,0)$. It does not work as a reward for agent $i = 1$ either, before when $\alpha(0,0,0) = (1,1,1)$ and f((1,1,1))=111 the strictness of the reward equality (1) of $\varepsilon-EPM$ fails:

$$u_1^{(0,0,0)}(100) = 1 - 2\,\varepsilon = u_1^{(0,0,0)}(f(\alpha(0,0,0))) + \varepsilon = 1 - 3\,\varepsilon + \varepsilon\,.$$

Case 1–3: "$a = 101$" $a = 101$ is not a Pareto outcome of θ. It cannot work as a reward for

[13] There are no negative payoffs in any state in this example as well.

agent $i = 1$ and $i = 3$ because their payoffs are both 0 at $\theta = (0,0,0)$. It does not work as a reward for agent $i = 2$ either, because when $\alpha(0,0,0) = (1,1,1)$ and $f(1,1,1) = 111$, the strictness of the reward equality (1) of $\varepsilon - EPM$ fails:

$$u_2^{(0,0,0)}(101) = 2\,\varepsilon = u_2^{(0,0,0)}(f(\alpha(0,0,0))) + \varepsilon = \varepsilon + \varepsilon\,.$$

Case 1–4: "$a = 111$" $a = 111$ does not work for $i = 2, 3$ because their payoffs are less than (or equal to) ε at $\theta = (0,0,0)$. For $i = 1$, consider a deception α with $\alpha(0,0,0) = (1,0,0)$. Now, (2) of $\varepsilon - EPM$ fails when $f((1,0,0)) = 100$:

$$u_1^{(1,\alpha_{-1})}(f((1,\alpha_{-1}))) = 0 < u_1^{(1,\alpha_{-1})}(111) = 2\,\varepsilon - \varepsilon = \varepsilon\,.$$

That is, $u_1^{(1,0,0)}(100) = 0 < u_1^{(1,0,0)}(111) = 2\,\varepsilon - \varepsilon = \varepsilon$.

Thus, when $\theta = (0,0,0)$, one cannot identify a whistle-blower agent i and a corresponding reward a such that $\langle \theta, i, a \rangle$ satisfies $\varepsilon - EPM$. By construction the same argument works in every state, meaning that F is not $\varepsilon - EPM$ when $\varepsilon \in (0, \frac{1}{4})$. ■

It is again straightforward to check that F also satisfies $EPIC$ and LVP.[14] Therefore, even though F fails to be ε–ex post implementable whenever $\varepsilon \in (0, \frac{1}{4})$, it is implementable in ex post equilibrium.

5 Conclusion

This paper studies the problem of full implementation via epsilon ex post equilibrium and identifies three conditions: $\varepsilon - EPIC$, $\varepsilon - EPM$, and $\varepsilon - LVP$. We prove that the first two are necessary for epsilon–ex post implementation, while all three together become sufficient whenever there are at least three agents in the society.

Our analysis extends the results of BM in two ways: i) we extend their necessary conditions to capture epsilon–bounded rationality and ii) we extend one of their sufficiency conditions to allow for limited veto power à la Benoit and Ok (2006).

Finally, our results can be interpreted as adding another dimension of robustness to ex post implementation, not in terms of informational assumptions, but in terms of bounded rationality.

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[14] What is meant by LVP here is $\varepsilon - LVP$ when $\varepsilon = 0$.

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