solved for chemical potential set to the band center and for specific position-dependent potentials [18, 19]. Here, we will present a general solution.

The Bogoliubov-de Gennes (BdG) Hamiltonian of a spinless p-wave superconductor in 1d is given by:

$$H = h(p,x)\tau_z + up\,\tau_x, \tag{1}$$

where $h(p,x) = \frac{p^2}{2m} + V(x) - \mu$ is the (spinless) single particle Hamiltonian, p is the momentum operator, m the electron mass, $V(x)$ an arbitrary scalar potential, μ the chemical potential, and $u = \Delta/p_{\rm F}$ with Δ the superconducting gap and $p_{\rm F}$ the Fermi momentum. Here and below τ_i $(i = x, y, z)$ denote the Pauli matrices in the electron-hole space. In order to make use of the chiral symmetry of the Hamiltonian, we first apply a global rotation in the electron-hole space ($\tau_z \to \tau_x$, $\tau_x \to \tau_y$) and cast the Hamiltonian into off-diagonal form [20]. The main use of this form is that it is now easy to see that the zero mode solutions, i.e. Majorana fermion solutions, are either of the form $\chi_+ = \left(\begin{smallmatrix}\varphi_+\\ 0\end{smallmatrix}\right)$ or of the form $\chi_- = \left(\begin{smallmatrix}0\\ \varphi_-\end{smallmatrix}\right)$, with $(h(p,x) \pm iup)\,\varphi_\pm = 0$. The linear in momentum term can be removed by a gauge transformation with a suitably chosen imaginary parameter $\varphi_\pm = \mathrm{e}^{\pm k_u x}\psi$, where $k_u = mu/\hbar$. We then find that ψ satisfies

$$\left(-\frac{\hbar^2}{2m}\partial_x^2 + V(x) - \mu + \frac{\hbar^2 k_u^2}{2m}\right)\psi = 0, \tag{2}$$

We identify this equation as the normal state equation with an effective chemical potential $\bar{\mu} = \mu - \frac{\hbar^2 k_u^2}{2m}$, with one crucial distinction: it is $\mathrm{e}^{\pm k_u x}\psi$ that needs to be normalized, rather than ψ itself. Thus diverging solutions of Eq. (2) as $x \to \pm\infty$ lead to normalizable wavefunctions $\varphi_\pm$, provided the divergence is not faster than $\mathrm{e}^{\pm k_u x}$.

For the sake of concreteness we focus on an half infinite ($x > 0$) wire, i.e. we assume that at points $x < 0$ is the vacuum state (a normal insulator), specified by the boundary condition $\chi|_0 = 0$ (it is easy to generalize to boundary conditions of the form $a\chi(x_0) + b\frac{d\chi}{dx}|_{x_0} = 0$) and χ is normalizable, i.e. $\chi \to 0$ sufficiently fast as $x \to \infty$. From standard Sturm-Liouville theory, recall that if the solutions of the (spinless) Hamiltonian (2) are localized, then there is one exponentially decaying solution (which we choose to be f) and one exponentially increasing solution (which we choose to be g) for large x. If the spinless electron is delocalized then both f and g are oscillatory. We choose a suitable linear combination $\psi = Af + Bg$ such that $\psi(0) = 0$ and hence also χ fulfils the boundary condition. Then for large x, $\psi \sim e^{\Lambda x}$ with Λ real [22] and a function of the effective chemical potential $\Lambda = \Lambda(\bar{\mu})$. We identify three cases (i) $\Lambda < -k_u$, (ii) $|\Lambda| < k_u$, and (iii) $k_u < \Lambda$. For case (i) there are two zero modes χ_+ and $\chi_- =$. This can only happen if the decaying solution f itself accidentally fulfills the boundary condition, and the two solutions will be lifted away from zero for small perturbations, i.e. are not topologically protected. This case corresponds to an accidental crossing of energy levels at zero energy [23]. In case (ii) there is only one Majorana state, χ_- which is the topologically protected state, and in case (iii) there are no zero modes and thus no topological state. We thus obtain a formula for the topological charge:

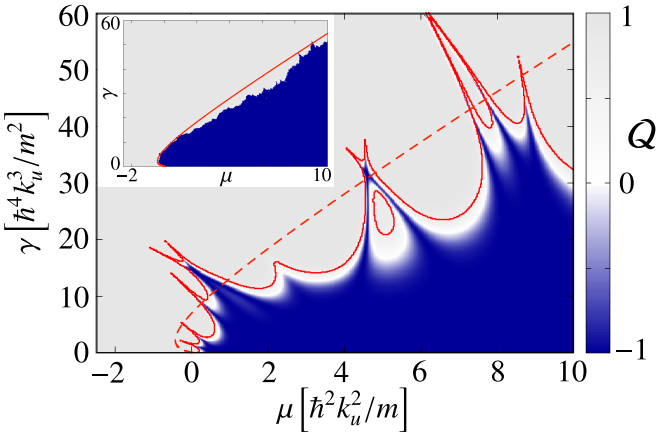

Figure 1: Topological charge $\mathcal{Q} = \det(r)$ of a disordered p-wave nanowire as a function of chemical potential μ and disorder strength γ, for a single disorder configuration in a short wire ($L = 100a$, with a the lattice constant). The inset shows a single disorder configuration in a long wire ($L = 10000a$). The red solid line in the main plot is the phase boundary computed from Eqs. (3) and the normal state conductance G, the red solid line in the inset/red dashed line in the main plot from Eqs. (3) and (4). The numerical calculation was done in a TB model with $k_u = 10a^{-1}$ and a chemcial potential in the leads $\mu_{\rm lead} = 0.5\hbar^2/2ma^2$.

$$\mathcal{Q} = \mathrm{sgn}(\hbar|\Lambda(\mu - mu^2/2)| - mu). \tag{3}$$

This the central result for the $p-$wave part of our work.

We are now at a position to demonstrate the topological robustness of the zero energy solutions. First note that it is only the asymptotic limit of the solutions ψ of the effective Schrödinger equation that matters for the existence of the solutions. Next notice that local perturbations of the potential (unless infinite) cannot change the asymptotic limit of the solutions regardless of their size and shape. Thus if there is a zero mode of the BdG hamiltonian for some potential profile (i.e.it is in the topological state) so will any other Hamiltonian that differs from the former by a local perturbation, demonstrating topological invariance.

For a disordered (normal-state) wire, Λ is usually called the Lyapunov exponent and can be estimated from the conductance as: $\Lambda = -(2/L)\log(G/G_0)$, where L is the wire length and G_0 the conductance quantum [24]. Hence, for fixed u, Eq. (3) allows one to determine the topological charge of a p-wave quantum wire from its normal state conductance alone. In short wires Λ fluctuates strongly as the chemical potential varies, and as a conse-

quence there are multiple changes of the topological properties. This is shown on the example of a single disorder realization in a short wire in Fig. 1, where we computed the topological charge within a tight-binding (TB) model numerically from $Q = \det(r)$ where r is the reflection matrix [25]. The topological phase boundary computed from Eq. (3) and the numerically computed normal state conductance agrees very well with the $\det(r)$-criterion; small deviations of the exact position of the phase boundary are due to finite size effects.

For longer wires the Lyapunov exponent is a self averaging quantity, i.e. $\Lambda(L) \to \bar{\Lambda}$, as $L \to \infty$, where $\bar{\Lambda}$ is the average Lyapunov exponent. For a wire with gaussian disorder $\langle V(x)V(y)\rangle = \gamma\delta(x-y)$ at energy ϵ, it can be obtained in closed form [26, 27]:

$$\bar{\Lambda}(\epsilon) = \frac{m^{1/2}}{\hbar\lambda}F\big(\lambda^2\epsilon\big), \quad \lambda = \left(\frac{\hbar}{\gamma m^{1/2}}\right)^{1/3}, \tag{4a}$$

$$F(x) = -\frac{1}{2}\frac{d\ln\left(\mathrm{Ai}(-2^{1/3}x)^2 + \mathrm{Bi}(-2^{1/3}x)^2\right)}{dx}. \tag{4b}$$

Then the topological transition condition Eq. (3) becomes $\hbar|\bar{\Lambda}(\mu - mu^2/2)| = mu$, valid for the entire range of μ, u, γ and shown as a red solid line in the inset of Fig. 1. The inset also shows numerics for a *single* disorder configuration for a long wire, demonstrating that due to the self-averaging long wires have a well-defined universal topological phase (similar numerics, but averaged over disorder was shown in [28]). At high energies, we have the golden rule result $\Lambda \sim 1/4\ell_{\rm tr}$, where $\ell_{\rm tr} = \hbar^2(\mu - mu^2/2)/\gamma m$ is the transport mean free path. We then obtain the condition that there is a topological transition at $k_u\ell_{\rm tr} = 1/4$, in agreement with Ref. [12, 29].

From Eq. (3) it can be also concluded that for $\bar{\mu} > 0$ any scattering is detrimental to the topological phase: Then $\Lambda = 0$ in the clean case (the normal state solutions are extended), and any scattering leads to $\Lambda \geq 0$. For $\bar{\mu} < 0$ topology can be in principle induced as seen in the inset of Fig. 1. There, a topological phase is created for $\mu < 0$ and $\gamma > 0$ due to states in the Lifshitz tail below the band bottom. This however is a relatively small effect. We shall see below this picture is drastically different for the experimentally relevant proximity nanowire systems.

We now focus on the experimentally more relevant system: a nanowire with Rashba spin-orbit coupling in proximity to an s-wave superconducter. The BdG Hamiltonian is then given as [3, 4]:

$$H = h(p,x)\tau_z + \alpha p\sigma_y\tau_z + B\sigma_x + \Delta\tau_x, \tag{5}$$

where $h(p,x) = p^2/2m + V(x) - \mu$ is the (spinless) single particle Hamiltonian, α the spin-orbit coupling strength, B the Zeeman splitting and Δ the induced s-wave order parameter. σ_i $(i = x, y, z)$ are the Pauli matrices in spin space. The topological state appears for $B^2 > \Delta^2 + \mu^2$. In this single orbital mode limit, the system is in class BDI, which is distinguished from class D by the presence of the chiral symmetry. This allows to bring the Hamiltonian into off-diagonal form [30], and the zero-energy Majorana states are of again of the form $\chi_+ = \left(\begin{smallmatrix}\varphi_+\\ 0\end{smallmatrix}\right)$ or $\chi_- = \left(\begin{smallmatrix}0\\ \varphi_-\end{smallmatrix}\right)$. After a rotation by $\pi/2$ in σ space around the x-axis and premultiplying with $\pm\sigma_x$, we find that $\varphi_\pm$ satisfies a 2×2 nonhermitian eigenvalue problem:

$$(h(p,x)\sigma_z \pm B \pm \Delta\sigma_x - i\alpha p\sigma_x)\varphi_\pm = 0 \tag{6}$$

We now construct the zero energy solution for small α. Larger values do not change the picture, but rather renormalize the topological-normal phase boundaries. We first let $\varphi \to e^{-\kappa x}\varphi$, where κ is an order α parameter that is yet to be determined. Then we have $p \to p + i\hbar\kappa$. Next we collect terms of order α and treat them as perturbations. We then have $H = H_0 + H_1$,

$$H_0 = h(p,x)\sigma_z + B + \Delta\sigma_x \tag{7a}$$

$$H_1 = -i\alpha p\sigma_x + i\frac{\hbar\kappa p}{m}\sigma_z + \hbar\kappa\alpha\sigma_x - \frac{\hbar^2\kappa^2}{2m}\sigma_z\,. \tag{7b}$$

The last two terms can be absorbed into H_0 by redefining μ and Δ. The eigenfunctions of H_0 are of the form $\xi_\pm\psi(x;\epsilon)$ where ψ is the local solution of $h\psi = \epsilon\psi$, and $\xi_\pm(\epsilon)$ is the eigenspinor of the 2×2 matrix $\epsilon\sigma_z + \Delta\sigma_x$. We now choose κ such that H_1 anticommutes with this matrix. Then it is easy to see that

$$\begin{aligned}\varphi_+ = \xi_+(\epsilon)\mathrm{e}^{-\kappa x}\big(Af(x;\epsilon) + Bg(x;\epsilon)\big)\\ +\xi_+(-\epsilon)\mathrm{e}^{\kappa x}\big(Cf(x;-\epsilon) + Dg(x;-\epsilon)\big),\end{aligned} \tag{8}$$

where $\epsilon = \sqrt{B^2 - \Delta^2}$ and $\kappa = m\alpha\Delta/\hbar\epsilon$, is a local solution of $(H_0 + H_1)\varphi = 0$ to order α^2. As in the p-wave case we have written ψ as a sum of the two linearly independent solutions f (decaying) and g (increasing). Then, φ_+ is a valid zero-energy solution (and thus a Majorana fermion) if it is normalizable and satisfies the boundary conditions. Repeating the calculation for the the other sector (φ_-), we obtain $\kappa \to -\kappa$.

We assume again without loss of generality that the system is in a normal insulator state for $x < 0$ and the boundary condition $\varphi(0) = 0$ (note that φ is a spinor). We then identify three cases: (i) If $B > \Delta$, and $|\Lambda(\mu \pm \epsilon)| < |\kappa|$ or $|\Lambda(\mu \pm \epsilon)| > |\kappa|$, there are two decaying and two diverging solutions. Then the boundary condition at $x = 0$ cannot be satisfied except in accidental cases such that $f(\pm\epsilon)$ already fulfil the boundary condition. Then there is also a second solution in the other sector, and the zero-energy states are not protected. The system is thus in the trivial state with the possibility of accidental zero modes. (ii) If $B < \Delta$, then both κ and ϵ are imaginary, hence there are always two decaying and two diverging solutions. However, there are no accidental zero modes with $f(\pm\epsilon)$ already fulfilling the boundary condition because this would mean f is an eigenfunction of (Hermitean) h with an imaginary eigenvalue. (iii) If

$B > \Delta$ and $|\Lambda(\mu \pm \epsilon)| < |\kappa| < |\Lambda(\mu \mp \epsilon)|$, there are one diverging and three decaying solutions in one sector and one decaying and three diverging solutions in the other sector. Then the boundary condition at $x = 0$ can be generally satisfied in the sector that has three decaying solutions and there is a Majorana state. As before, the solution is robust, because local perturbations do not change the asymptotic behavior of f and g. In summary we have:

$$\mathcal{Q} = \mathrm{sgn}\left(|\Lambda(\mu+\epsilon)| - \frac{m\alpha\Delta}{\hbar\epsilon}\right) \mathrm{sgn}\left(|\Lambda(\mu-\epsilon)| - \frac{m\alpha\Delta}{\hbar\epsilon}\right) \quad (9)$$

This is our central formula for the s-wave case. Note that the first term in Eq. (9) reduces to Eq. (3) in the large B-limit; in this case the second term induces new physics.

We now discuss various types of scattering using Eq. (9) and start with the example of a superlattice. In the clean case, the required odd number of channels for the topological state is only achieved if the chemical potential is within the Zeeman gap and hence close to the band bottom [3, 4]. The superlattice allows this for a larger range of chemical potentials out of the Zeeman gap. The minigaps (or equivantly, perfect backscattering) can induce topological order away from the band bottom, and the topological phase space area can be even increased as compared to the clean case, as shown in Fig. 2(a,b). Strikingly, one can even use topologically trivial pieces to induce the topological state.

In the experimentally relevant case of irregular scattering, we use the average Lyapunov exponent given by Eq. (4) to determine the phase boundaries of a long quantum wire from Eq. (9). Noting that $\bar{\Lambda}$ is a monotonous function of energy, we get:

$$\mu_\pm = F^{-1}(m^{1/2}\lambda\alpha\Delta/\sqrt{B^2-\Delta^2})/\lambda^2 \pm \sqrt{B^2-\Delta^2} \quad (10)$$

In the weak disorder limit, $\lambda \to \infty$, we recover the clean wire result:$\mu_\pm = \pm\sqrt{B^2-\Delta^2}$. In contrast to the common wisdom based on the effective p-wave model, we find that the topological region is not destroyed by disorder but merely shifted to higher chemical potentials. In fact the chemical potential (or gate) range where the wire is topological, $\mu_+ - \mu_- = 2\sqrt{B^2-\Delta^2}$, is *independent* of the disorder strength, while the total area of the topological region in the (B, μ) plane is conserved. We stress that this result is valid to all orders in disorder strength.

Fig. 2(c,d) shows our numerical results for a nanowire with disorder. In agreement with our analytical results, we observe that for a long wire the topological phase is merely shifted to larger chemical potentials. The disorder then creates a well-defined topological region for a parameter range where the clean wire is trivial. In fact, disorder is even favorable for experiments to form Majorana fermions, as the threshold magnetic field is considerably lowered for μ away from the band edge. In a short wire, the topological phase is more fragmented due to the fluctuations in the normal state conductance in agreement with Eq. (9). Nevertheless, both for a short and a long wire, a clear Majorana zero-bias peak (ZBP) appears in the tunneling conductance, as shown in Fig. 2(e,f). We note that the wire would be in the trivial phase without Majorana fermions without disorder for the whole range of parameters shown in Fig. 2(e,f).

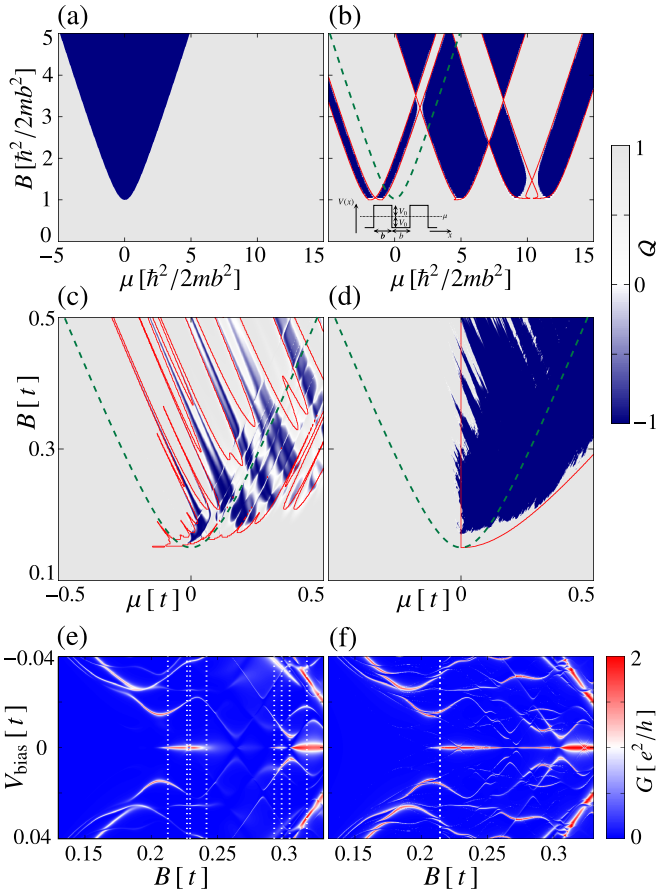

Figure 2: Topological charge $\mathcal{Q} = \det(r)$ as a function of chemical potential μ and Zeeman splitting B for a (a) clean system, (b) a superlattice, and (c, d) disorder. Red lines in (b-d) are phase boundaries calculated from Eq. (9), green dashed lines show the clean phase boundary for comparison. (b) The superlattice (see inset) parameters were $d = 3b$, $V_0 = 8\hbar^2/2mb^2$, $\Delta = \hbar^2/2mb^2$, and $k_{\mathrm{so}} = 0.05b^{-1}$ with $k_{\mathrm{so}} = m\alpha/\hbar$, and the numerical calculation was done using a transfer matrix method in Mathematica. The numerical calculations in (c-f) were done within a TB model: (c) shows the topological charge for a single disorder realisation in a short ($L = 100a$ with a the lattice constant) and (d) in a long ($L = 4000a$) wire, (e) and (f) the respective tunnel conductances for a fixed $\mu = 0.3t$, with $t = \hbar^2/2ma^2$. White dashed lines in (e, f) indicate the boundaries of the topological phase in (c, d). The remaining TB parameters were $k_{\mathrm{so}} = 0.05a^{-1}$, $\Delta = 0.15t$, $\gamma = 0.06t^2$, and the chemical potential in the leads $\mu_{\mathrm{leads}} = 0.5t$. For the tunneling conductance in (e, f) a barrier of height $1.5t$ was added on one lattice site next to one end of the wire.

Recently, it was argued that ZBPs in nanowires may appear even without Majorana fermions [31–33]. Here we caution against this interpretation. As a ZBP out of the clean topological phase boundary may well be a Majorana fermion within the dirty topological phase boundary, in particular if $B > \Delta$ and the ZBP remains for a range of magnetic field. In fact, for a dirty wire *all* accidental zero mode solutions will shift under changes in the magnetic field.

In conclusion, we studied the effects of single-particle scattering on the topological properties of a quantum wire in contact with an s-wave superconductor. We obtained analytical formulas for the phase boundaries valid for regular and irregular scattering. In the case of irregular scattering, our formulas apply to all orders in the disorder strength as well as to individual members of the ensemble. Our main result is that disorder does *not* destroy topological order, in contrast to the general expectation based on effective models, rather the topological phase is merely shifted to higher chemical potentials, while the total phase area at fixed magnetic field is conserved. Moreover one can even increase the topological phase area with periodic modulation of the gate potential.

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