On Relationships Between Substitutes Conditions

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Abstract

In the matching with contract literature, three well-known conditions (from stronger to weaker): substitutes, unilateral substitutes (US), and bilateral substitutes (BS) have proven very critical both in theory and practice. This paper aims to deepen our understanding of them by separately axiomatizing the gap between BS and the other two. We first introduce a new "doctor separability" condition (DS) and show that BS, DS, and irrelevance of rejected contracts (IRC) are equivalent to US and IRC. Due to Hatfield and Kojima (2010) and Aygün and Sönmez (2012), we know that US, "Pareto separability" (PS), and IRC are the same as substitutes and IRC. This along with our result implies that BS, DS, PS, and IRC are equivalent to substitutes and IRC. All of these results are given without IRC whenever hospitals have preferences.

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1 Introduction

Hatfield and Milgrom (2005) introduce a matching with contracts framework which admits the standard two-sided matching, package auction, and the labor market model of Kelso and Crawford (1982) as special cases.[1] They adopt the substitutes condition in the matching (without contracts) literature (see Roth and Sotomayor (1990)) to their rich setting and show the existence of a stable allocation whenever contracts are substitutes. If hospital choices are not necessarily generated by certain preferences, on the other hand, Aygün and Sönmez (2013) show that an irrelevance of rejected contracts condition (IRC) is also needed.[2]

While the substitutes and IRC conditions grant the existence of a stable allocation, Hatfield and Kojima (2008) demonstrate that the former is not necessary. Hatfield and Kojima (2010) then introduce a weaker bilateral substitutes (BS) condition guaranteeing the existence of a stable allocation. Similar to Aygün and Sönmez (2013), if hospital choices are not necessarily induced by preferences, Aygün and Sönmez (2012) reveal that IRC is needed in addition to BS. While BS and IRC together is sufficient for the existence, it is weak in that many well-known properties of stable allocations in the standard matching problem do not carry over to the matching with contracts setting under them. Among others, for instance, the doctor-optimal stable allocation[3] fails to exist. In order to restore at least some of properties, Hatfield and Kojima (2010) introduce a stronger unilateral substitutes condition (US),[4] and the existence of the doctor-optimal stable allocation is obtained under both US and IRC. Moreover, the cumulative offer process of Hatfield and Milgrom (2005) (henceforth, COP), which is a generalization of Gale and Shapley (1962)'s deferred acceptance algorithm, collapses to the doctor proposing deferred acceptance algorithm. With an additional law of aggregate demand condition of Hatfield and Milgrom (2005) (hence-

[1] Echenique (2012) shows that matching with contract problem can be embedded into Kelso and Crawford (1982)'s labor market model under a substitutes condition.

[2] In the many-to-many matching context (without contracts), Blair (1988) and Alkan (2002) use this condition. The latter refers to it as "consistency".

[3] The doctor-optimal stable allocation is the anonymously preferred stable allocation by all doctors to any other stable allocation.

[4] US is still weaker than the substitutes condition.

forth, LAD),[5] Hatfield and Kojima (2010) recover the strategy-proofness[6] (indeed group strategy-proofness)[7] of the doctor-optimal stable rule, and a version of so called "rural hospital theorem".[8] Besides, they also show that the doctor-optimal stable allocation is weakly Pareto efficient for doctors.[9] While US and IRC grant the doctor-optimal stable allocation, the set of stable outcomes still does not form a lattice. Hatfield and Milgrom (2005) obtain lattice structure under the substitutes (and IRC) condition.

Given that many well-known properties are restored by strengthening BS to US or substitutes, it is important to understand relations between them. While Hatfield and Kojima (2010) and Aygün and Sönmez (2012) clarify the difference between US and substitutes through axiomatizing the gap between them, such an analysis is yet to be done for the difference between them and BS. In this study, we pursue it and separately axiomatize the gap between BS and the other two. To this end, we introduce a *doctor separability* (henceforth, DS) condition which says that if no contract of a doctor is chosen from a set of contracts, then that doctor still should not be chosen unless a contract of a new doctor (we refer to a doctor as new doctor if he does not have any contract in the initially given set of contracts) becomes available. We then show that US and IRC are equivalent to DS, BS and IRC.[10] Hatfield and Kojima (2010) show that US and "Pareto Separability" (PS) are equivalent to substitutability. Aygün and Sönmez (2012) then extend it to a more general setting where hospital choices are primitive by additionally imposing IRC. This result along with our axiomatization gives that BS, DS, PS, and IRC are the same as substitutes and IRC.[11] As IRC is automatically satisfied whenever hospitals have preferences, all the results are given without IRC in that case.

[5] Alkan and Gale (2003) introduce a similar condition they call "size monotonicity" in a schedule matching setting.

[6] A mechanism is strategy-proof if no doctor ever has incentive to misreport his preference.

[7] A mechanism is group strategy-proof if no group of doctors ever benefit from collectively misreporting.

[8] That is, every doctor and hospital signs the same number of contracts at any stable allocation.

[9] An allocation is weakly Pareto efficient for doctors if no other allocation is strictly preferred by all doctors.

[10] Alva (2014) gives some necessary (not sufficient though) conditions for US and BS to hold.

[11] Alva (2014) provides another characterization of substitutability by using different properties, which are not directly related to the currently used ones.

As summarized above, moving from BS to US or substitutes brings important properties. Indeed, it is not only restricted to already mentioned ones above. In a recent study, Afacan (2014) shows that COP is both population and resource monotonic under US and IRC, and it respects doctors' improvements with an additional LAD. The theoretical appeal of understanding the difference between US and BS therefore is clear. In addition to that, it has a practical advantage. There is an important recent surge in the real-life market with contracts design literature including Sönmez and Switzer (2013); Sönmez (2013); Kominers and Sönmez (2013); and Aygün and Bo (2014). These papers show that US and BS conditions are also critical for the practical market design.[12] By deepening our understanding of these three substitutes conditions and providing an alternative way (possibly easier in many cases) of checking US and substitutes conditions, this note has a practical appeal as well.

2 Model and Results

There are finite sets D and H of doctors and hospitals, and a finite set of contracts X. Each contract $x \in X$ is associated with one doctor $x_D \in D$ and one hospital $x_H \in H$. Given a set of contracts $X' \subseteq X$, let $X'_D = \{d \in D : \exists\, x \in X' \text{ with } x_D = d\}$. Each hospital h has a choice function $C_h : 2^X \to 2^X$ defined as follows: for any $X' \subseteq X$:

$$C_h(X') = \{X'' \subseteq X' : \ (x \in X'' \Rightarrow x_H = h) \text{ and } (x, x' \in X'', x \neq x' \Rightarrow x_D \neq x'_D)\}.$$

Definition 1. *Contracts satisfy irrelevance of rejected contracts* (IRC) *for hospital* h *if, for any* $X' \subset X$ *and* $z \in X \setminus X'$, *if* $z \notin C_h(X' \cup \{z\})$ *then* $C_h(X') = C_h(X' \cup \{z\})$.

Definition 2. *Contracts are bilateral substitutes* (BS) *for hospital* h *if there do not exist contracts* $x, z \in X$ *and a set of contracts* $Y \subseteq X$ *such that* $x_D, z_D \notin Y_D$, $z \notin C_h(Y \cup \{z\})$, *and* $z \in C_h(Y \cup \{x, z\})$.

[12] Their respective contracts satisfy either BS or US, yet not substitutes.

Definition 3. *Contracts are unilateral substitutes* (US) *for hospital* h *if there do not exist contracts* $x, z \in X$ *and a set of contracts* $Y \subseteq X$ *such that* $z_D \notin Y_D$, $z \notin C_h(Y \cup \{z\})$, *and* $z \in C_h(Y \cup \{x, z\})$.

Below introduces our new condition.

Definition 4. *Contracts are doctor separable* (DS) *for hospital* h *if, for any* $Y \subset X$ *and* $x, z, z' \in X \backslash Y$ *with* $x_D \neq z_D = z'_D$, *if* $x_D \notin [C_h(Y \cup \{x, z\})]_D$, *then* $x_D \notin [C_h(Y \cup \{x, z, z'\})]_D$.

In words, DS says that if a doctor is not chosen from a set of contracts in the sense that no contract of him is selected, then that doctor still should not be chosen unless a contract of a new doctor (that is, doctor having no contract in the given set of contracts) becomes available. For practical purposes, we can consider DS capturing contracts where certain groups of doctors are substitutes.[13]

Theorem 1. *Contracts are* US *and* IRC *if and only if they are* BS, DS, *and* IRC.

Proof. **"If" Part.** Let contracts be BS and DS satisfying IRC. Moreover, let $Y \subset X$ and $x \in X$ such that $x_D \notin Y_D$ and $x \notin C_h(Y \cup \{x\})$. We now claim that $x \notin C_h(Y \cup \{x, z\})$ for any $z \in X$ as well. If $z_D \notin Y_D$, then by BS, the result follows. Let us now assume that $z_D \in Y_D$. Then, we can write $Y = Y' \cup \{z'\}$ for some z' where $z'_D = z_D$. This means that $x \notin C_h(Y' \cup \{x, z'\})$, and since $x_D \notin Y_D$, it in particular implies that $x_D \notin [C_h(Y' \cup \{x, z'\})]_D$. By DS then, we have $x_D \notin [C_h(Y' \cup \{x, z', z\})]_D$; in other way of writing, $x_D \notin [C_h(Y \cup \{x, z\})]_D$. Hence in particular, $x \notin C_h(Y \cup \{x, z\})$.

"Only If" Part. Let contracts be US satisfying IRC. By definition, they are BS as well. In order to show that they are also DS, let $x_D \notin [C_h(Y \cup \{x, z\})]_D$. We define $Y' = Y \setminus \{x' \in Y : \ x_D = x'_D \text{ and } x \neq x'\}$. By IRC, $C_h(Y \cup \{x, z\}) = C_h(Y' \cup \{x, z\})$. Let us now add a new contract z' where $z_D = z'_D$. By US, $x \notin C_h(Y' \cup \{x, z, z'\})$. If $x \in C_h(Y \cup \{x, z, z'\})$, then by IRC, it has to be that $C_h(Y \cup \{x, z, z'\}) = C_h(Y' \cup \{x, z, z'\})$.

[13] If $x_D \notin [C_h(Y \cup \{x, z\})]_D$, then it means that doctor x_D is not chosen. And under DS, he continues not to be chosen unless a new doctor comes. Hence, we can interpret it as the doctors in the given set of contracts are substitutes.

This, however, contradicts $x \notin C_h(Y' \cup \{x, z, z'\})$. Hence, $x \notin C_h(Y \cup \{x, z, z'\})$. For any other contract $x' \in Y$ of doctor x_D, we can define $Y' = [Y \setminus \{x'\}] \cup \{x\}$. Then, by above, $x_D \notin [C_h(Y' \cup \{x', z\})]_D$ (note that $Y' \cup \{x', z\} = Y \cup \{x, z\}$). By easily following the same steps above, we can conclude that $x' \notin C_h(Y \cup \{x, z, z'\})$ as well. Hence, $x_D \notin [C_h(Y \cup \{x, z, z'\})]_D$, showing that contracts are DS. □

Remark 1. As BS is weaker than US, Theorem 1 shows that the former does not imply DS. Moreover, DS does not imply BS either. Let $X = \{x, y, z\}$ where $x_D \neq y_D \neq z_D$ and $x_H = y_H = z_H = h$. Consider the following choices of hospital h.

$$C_h(\{x\}) = \{x\} \; ; \; C_h(\{x, y\}) = \{y\} \; ; \; C_h(\{x, y, z\}) = \{x, z\}$$

$$C_h(\{y\}) = \{y\} \; ; \; C_h(\{x, z\}) = \{x, z\} \; ; \; C_h(\{y, z\}) = \{y, z\}$$

$$C_h(\{z\}) = \{z\}.$$

We can easily verify that contracts are DS (even satisfying IRC), yet not BS as $C_h(\{x, y\}) = \{y\}$ and $C_h(\{x, y, z\}) = \{x, z\}$.

Definition 5. *Contracts are substitutes for hospital h if there do not exist contracts $x, z \in X$ and a set of contracts $Y \subseteq X$ such that $z \notin C_h(Y \cup \{z\})$ and $z \in C_h(Y \cup \{x, z\})$.*

Hatfield and Kojima (2010) introduce the following condition which has proven to be useful in understanding the difference between US and substitutes.

Definition 6. *Contracts are Pareto separable (PS) for hospital h if, for any two distinct contracts x, x' with $x_D = x'_D$ and $x_H = x'_H = h$, if $x \in C_h(Y \cup \{x, x'\})$ for some $Y \subseteq X$, then $x' \notin C_h(Y' \cup \{x, x'\})$ for any $Y' \subseteq X$.*

Hatfield and Kojima (2010) show that US and PS are equivalent to substitutes. In their setting, hospitals have preferences, inducing their choices. Aygün and Sönmez (2012) then extend their result to a more general setting where hospital choices are primitive by additionally imposing IRC. As the latter is more general and relevant to our current setting, we formally state it below.

Fact 1 (Aygün and Sönmez (2012)). *Hospital choices are US and PS satisfying IRC if and only if they are substitutes satisfying IRC.*

As a corollary of Theorem 1 and Fact 1 above, we obtain the following characterization.

Corollary 1. *Contracts are substitutes satisfying IRC if and only if they are BS, DS, PS satisfying IRC.*

Remark 2. As IRC is automatically satisfied whenever hospital choices are generated by certain preferences, all of the above results work without IRC in that case.

Remark 3. In this remark, we show that DS and PS are independent of each other. Let $X = \{x, x', y\}$ where $x_D = x'_D \neq y_D$ and $x_H = x'_H = y_H = h$. Consider the following choices of hospital h:

$$C_h(\{x\}) = \{x\} \; ; \; C_h(\{x, x'\}) = \{x'\} \; ; \; C_h(\{x, x', y\}) = \{x, y\}$$

$$C_h(\{x'\}) = \{x'\} \; ; \; C_h(\{x, y\}) = \{x, y\} \; ; \; C_h(\{x', y\}) = \{x', y\}$$

$$C_h(\{y\}) = \{y\}.$$

One can easily verify that contracts are DS (even satisfying IRC), yet not PS as $C_h(\{x, x'\}) = \{x'\}$ and $C_h(\{x, x', y\}) = \{x, y\}$. For the converse, think of the above same choices except for $C_h(\{x, y\}) = \{x\}$ and $C_h(\{x, x', y\}) = \{x', y\}$. In this case, contracts are PS (even satisfying IRC), yet not DS due to the right above choices.

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