Stein manifolds M for which $O(M)$ has the property $\widetilde{\Omega}$

Aydin Aytuna

Dedicated to Mikhail Mikhaylovich Dragilev on the occation of his 90 th birthday

Abstract. In this note, we consider the linear topological invariant $\widetilde{\Omega}$ for Fréchet spaces of global analytic functions on Stein manifolds. We show that $O(M)$, for a Stein manifold M, enjoys the property $\widetilde{\Omega}$ if and only if every compact subset of M lies in a relatively compact sublevel set of a bounded plurisubharmonic function defined on M. We also look at some immediate implications of this characterization.

1. Introduction

Spaces of analytic functions, regarded as an important class of nuclear Fréchet spaces contributed amply to the development of the structure theory of Fréchet spaces. A profound example is the pioneering result of Dragilev [**6**] on the absoluteness of bases in the space of analytic functions on the unit disc with the usual topology. This paved the way to the far-reaching theorem of Dynin-Mitiagin [**7**] on the absoluteness of bases in *every* nuclear Fréchet space. Many more examples could readily be provided. Of course this influence has not been one-sided. Techniques and concepts from functional analysis were extensively used in complex analysis. Advances in the structure theory of Fréchet spaces, found some applications in the Mitiagin-Henkin [**10**] program on the linearization of basic results of the theory of analytic functions. (See, for example [**2**],[**17**],[**3**]). In order to use the results of the structure theory of Fréchet spaces effectively it is imperative to analyze the complex analytic properties shared by the complex manifolds whose analytic function spaces possess a common linear topological invariant. The present note is written from this perspective and aims to characterize Stein manifolds whose analytic function spaces possess the property $\widetilde{\Omega}$ of Vogt [**13**]. (See section 1 for the definition)

Throughout this note we will denote the space of analytic functions on a Stein manifold M with the compact-open topology by $O(M)$.

In the first section we compile some background material for the linear topological invariant $\widetilde{\Omega}$.

The second section is devoted to the proof of the characterization of Stein manifolds M for which $O(M)$ has the property $\widetilde{\Omega}$ as those manifolds with the

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property that every compact set of M lie in a precompact sublevel set of a suitably chosen bounded plurisubharmonic function (on M).

Considering the class of Stein manifolds M for which $O(M)$ has the property $\widetilde{\Omega}$, in the third section we show, as an immediate corollary of the characterization theorem, the existence of pluricomplex Green functions with certain special properties for this class. The note ends with two examples among bounded domains in $\mathbb{C}$, one in the class and one not in the class.

The manifolds considered in this note are always assumed to be connected. We will use the standard terminology and results from functional analysis and complex potential theory, as presented in [**9**] and [**8**] respectively. Throughout this note, the notation $\subset\subset$ will be used to denote relatively compact containments.

2. The linear topological invariant $\widetilde{\Omega}$

In this section we give some background material on the linear topological invariant $\widetilde{\Omega}$.

Definition 1. *(Vogt [**13**]) Let E be a Fréchet space with a generating system of seminorms $(\|.\|_k)_k$. E is said to have the property $\widetilde{\Omega}$, in case :*

$$\forall p \ \exists \ q, \ d > 0, \ \forall k \ \exists \ C > 0 \ \forall \varphi \epsilon E^* : \|\varphi\|_q^* \ \leq C \left(\|\varphi\|_p^*\right)^{\frac{d}{1+d}} \left(\|\varphi\|_k^*\right)^{\frac{1}{1+d}}$$

where $\left(\|.\|_k^*\right)_k$ are the seminorms dual to $(\|.\|_k)_k$.

Note that this property does not depend on the generating semi-norm system. If E is a nuclear Fréchet space, it turns out that the conditions below are also equivalent to the condition given in the definition of the property $\widetilde{\Omega}$:

- There exists a closed bounded absolutely convex set B in E :

  $$\forall p \ \exists \ q, \ d > 0, \ \exists \ C > 0 \ \forall \varphi \epsilon E^* : \|\varphi\|_q^* \ \leq C \left(\|\varphi\|_p^*\right)^{\frac{d}{1+d}} \left(\|\varphi\|_B^*\right)^{\frac{1}{1+d}}$$

- There exists a closed bounded absolutely convex set B in E :

  $$\begin{aligned} \forall p \ \exists \ q, \ d &> 0, \ C > 0, \text{ such that for all } r > 0 : \\ U_q &\subseteq CrB + \frac{1}{r^d} U_p \end{aligned}$$

- $\forall p \ \exists \ q$, $d > 0$, $\forall k \ \exists C > 0$, such that for all $r > 0$:

  $$U_q \subseteq CrU_k + \frac{1}{r^d} U_p$$

  where U_s denotes the unit ball of the seminorm $\|\|_s$, $s = 1, 2, ...$ (see [**12**], [**5**]).

This property is stronger than Ω, and is weaker than $\overline{\Omega}$ conditions of Vogt, and as with all Ω- type invariants, is inherited by quotients [**13**]. This invariant plays an important role in investigations of finding "non-polar" bounded sets in nuclear Fréchet spaces initiated by a question of P.Lelong. We refer reader to [**5**] for details on this matter.

Another interesting feature of nuclear Fréchet spaces with the property $\widetilde{\Omega}$ is that continuous linear operators from such a space into a nuclear weakly stable infinite type power series space are necessarily compact [**13**]. In particular nuclear weakly stable infinite type power series spaces, e.g. $O(M)$, for parabolic Stein manifolds M [**4**], cannot not have the property $\widetilde{\Omega}$. More generally we have,

PROPOSITION 1. *Let X be a nuclear Fréchet space. If the diametral dimension is equal to the diametral dimension of an nuclear weakly stable infinite type power series space the X cannot have the property $\widetilde{\Omega}$.*

PROOF. Suppose that X has the property $\widetilde{\Omega}$ and assume that the diametral dimension of X, $\Delta(X)$, satisfies $\Delta(X) = \Delta(\Lambda_\infty(\alpha_n))$ for some nuclear weakly stable exponent sequence (α_n). Choose a generating seminorm system $(\|.\|_k)_k$ so that $p+1$ is the index q assigned to p by the condition $\widetilde{\Omega}$.

Let $F \stackrel{\circ}{=} \{(x_n) : \sup |x_n| \ d_n(U_{p+1}, U_p) < \infty, \forall p\}$ with the natural Fréchet space structure.

Since F is in $\Delta(X)$, and $\Delta(\Lambda_\infty(\alpha_n)) = ((x_n) : \exists\ R \geq 1,\ \sup |x_n| R^{\alpha_n} < \infty)$; in view of Grothendieck factorization theorem there is an R_0 such that

$$\forall p\ , \quad \limsup_n \frac{-\ln d_n(U_{p+1}, U_p)}{\alpha_n} \leq \ln R_0.$$

On the other hand considering the usual topology on $\Delta(X)$ [**11**], which represents it as a projective limit of inductive limit of Banach spaces, the continuous inclusion $\Delta(X) \subseteq \Delta(\Lambda_\infty(\alpha_n))$ gives:

$$\forall R \geq 1 \text{ and } p\ \exists\ q,\ C > 0 : \sup_n R^{\alpha_n} d_n(U_q, U_p) \leq C.$$

In particular :

$$\forall R \geq 1 \text{ and } p\ \exists\ q : \ln R \leq \liminf_n \frac{-\ln d_n(U_q, U_p)}{\alpha_n}$$

We now utilize the condition $\widetilde{\Omega}$, which in our notation, reads as: There exists a closed bounded set $B \subseteq X$ such that:

$$\begin{aligned} \forall p\ \exists\ d &> 0\ ,\ C > 0, \text{ such that for all } r > 0 : \\ U_{p+1} &\subseteq CrB + \frac{1}{r^d} U_p. \end{aligned}$$

Following the argument given in [**11**], we arrive at the estimate:

$$\forall p\ \exists\ d > 0\ ,\ C > 0,\ -\ln d_n(B, U_p) \leq (1+d)(-\ln d_n(U_{p+1}, U_p)) + C, \quad n = 1, 2, ...$$

Lets fix a p and choose an $R >> (R_0)^{(1+d)}$ where d is the constant appearing in the above equation. Putting all the above implications together, we get;

$$\begin{aligned} \ln R &\leq \liminf_n \frac{-\ln d_n(U_q, U_p)}{\alpha_n} \leq \liminf_n \frac{-\ln d_n(B, U_p)}{\alpha_n} \\ &\leq \liminf_n \frac{(1+d)(-\ln d_n(U_{p+1}, U_p))}{\alpha_n} \leq (1+d) \ln R_0. \end{aligned}$$

This contradiction finishes the proof of the proposition. □

We would like to finish this section by making some immediate observations, in view of the things said above, about the class of Stein manifolds whose analytic function spaces have the property $\widetilde{\Omega}$. Smoothly bounded domains of holomorphy in $\mathbb{C}^n$, complete bounded Reinhard domains, more generally hyperconvex Stein manifolds belong to this class since their analytic function spaces possess a stronger property $\overline{\Omega}$ [**15**] [**1**]. On the other hand $\mathbb{C}^d$, $d = 1, 2, ..,$ or more generally, parabolic Stein manifolds do not belong to this class [**4**].

3. Main result

In this section we give a characterization of Stein manifolds M for which $O(M)$ has the property $\widetilde{\Omega}$.

THEOREM 1. *Let M be a Stein manifold. The Fréchet space $O(M)$ has the property $\widetilde{\Omega}$ if and only if for every compact subset K of M there exists a negative plurisubharmonic function φ on M and a $\alpha < 0$ such that*

$$K \subset (z \epsilon M : \varphi(z) < \alpha) \subset\subset M.$$

PROOF. Throughout the proof we will use the notation of Lemma 1 of [**1**]. To this end we fix a hermitian metric on M, and denote by $d\varepsilon$ the measure $cd\mu$ where μ is the measure (equivalent to the volume form) and c is the positive continuous function, respectively, of Lemma 1 [**1**]. We also choose a C^∞ strictly plurisubharmonic exhaustion function σ of M and let,

$$D_n \circeq (z \epsilon M : \sigma(z) < n), \; n = 1.2.....$$

($\Rightarrow$) It suffices to show that each $K_n \circeq \overline{D_n}$, $n = 1, 2, ...,$ is contained in a relatively compact sub-level set of a bounded plurisubharmonic function. To this end fix a K_{n_0}. Choose, as in [**16**] ([**17**]), a Hilbert space $(H_0, [.]_0)$ with continuous injections,

$$O(K_{n_0}) \hookrightarrow H_0 \hookrightarrow AC(K_{n_0})$$

where $O(K_{n_0})$ denotes the germs of analytic functions on K_{n_0} with the usual inductive limit topology and $AC(K_{n_0})$ denotes the closure, in $C(K_{n_0})$, of the restriction of $O(K_{n_0})$ to K_{n_0}. For $n > n_0$, the pair $\{K_{n_0}, D_n\}$ is a regular pair in the sense of [**16**] and hence the relative extremal function

$$\omega_n(z) \circeq \sup\{u(z) : u \epsilon PSH(D_n), \; u \leq -1 \text{ on } K_{n_0} \text{ and } u \leq 0 \text{ on } D_n\}$$

is a continuous function on D_n [**16**]. Clearly $(\omega_n)_{n > n_0}$ forms a decreasing sequence of plurisubharmonic functions. For $k = 1, 2, ...$we define a norm on $O(M)$ by:

$$[f]_k \circeq \left(\int_{D_{k+n_0}} |f|^2 \, d\varepsilon \right)^{\frac{1}{2}}, \quad f \epsilon O(M).$$

We will denote the corresponding Hilbert spaces by H_k, $k = 1, 2, ...$ The norm system $([*]_k)_{k=0}^\infty$ generates the topology of $O(M)$. Denoting the dual norms by $\left([*]_k^*\right)_{k=0}^\infty$, there exists, in view of our assumption, an index n_1 and $d > 0$ so that,

$$\forall k \; \exists \; C > 0 \;\; : \;\; [.]_{n_1}^* \leq C \left([.]_k^*\right)^{\frac{d}{1+d}} \left([.]_0^*\right)^{\frac{1}{1+d}}.$$

Fix an $m > n_1$. The inclusion $\iota_m : H_m \hookrightarrow H_0$, being a compact continuous operator, can be represented as

$$\iota_m(x) = \sum_n \lambda_n \langle x, f_n \rangle_m e_n, \quad \forall n, \;\; \lambda_n \geq 0 \;\; , \;\; \lim \lambda_n = 0,$$

for some orthonormal sequences $(f_n)_n$, $(e_n)_n$ of H_m and H_0 respectively. Let

$$d_n \circeq -\ln \lambda_n,\ n = 1, 2, ...$$

We will regard, ι_m as inclusion and identify f_n with $\lambda_n e_n, \quad n = 1, 2, ...$ It is shown in [**1**] ([**16**]) that $(e_n)_n$ forms a basis of $O(D_{m+n_0})$ and that this space can be represented as a finite center of the Hilbert scale generated by $H_m$and H_0. Moreover the coordinate functionals $(e_n^*)_n$ on $O(M)$ satisfy

$$[e_n^*]_m^* = e^{-d_n},\ n = 1, 2,\ \ .$$

In view of Proposition I.11 of [**1**] ([**16**]), the relative extremal function can be represented as:

$$1 + \omega_{n_0+m}(z) = \limsup_{\xi \to z} \limsup_n \frac{\ln |e_n(\xi)|}{d_n} \qquad \forall z \epsilon D_{n_0+m} \setminus K_{n_0}.$$

Fix an β, with $0 < \beta < \frac{d}{1+d}$. In view of Hartogs lemma (Theorem 2.6.4 [**8**]):

$$\forall \epsilon > 0 \ \ \exists\ C > 0\ :\quad |e_n|_{K_\beta} \leq C e^{\beta d_n}$$

where $|.|_{\Gamma_\beta}$ denotes the sup norm on the precompact sub-level set

$$\Gamma_\beta \circeq (z \epsilon D_{n_0+m} \quad : 1 + \omega_{n_0+m}(z) \leq \beta).$$

For a given $f \epsilon O(M)$ we estimate on Γ_β :

$$\begin{aligned} |f(z)| &\leq \sum_n |e_n^*(f)| \, |e_n(z)| \leq C \sum_n [e_n^*]_{n_1}^* [f]_{n_1} e^{\beta d_n} \\ &\leq C \sum_n \left([e_n^*]_m^*\right)^{\frac{d}{1+d}} \left([e_n^*]_0^*\right)^{\frac{1}{1+d}} [f]_{n_1} e^{\beta d_n} \\ &\leq \widetilde{C} \left(\sum_n e^{\left(\beta - \frac{d}{1+d}\right) d_n} \right) [f]_{n_1} \leq \widehat{C} [f]_{n_1} \end{aligned}$$

since $(d_n) = O\left(n^{\frac{1}{\dim M}}\right)$ ([**17**]). Moreover from the definition of $[.]_{n_1}$, there is a constant C which does not depend upon f such that

$$[f]_{n_1} \leq C |f|_{K_{n_0+n_1+1}}$$

where $|.|_{K_{n_0+n_1+1}}$ denotes the sup norm on $K_{n_0+n_1+1}$. Hence we have the estimate

$$\exists\ C_1 > 0 : |f|_{\Gamma_\beta} \leq C_1 |f|_{K_{n_0+n_1+1}}, \forall f \epsilon O(M),$$

between the sup norms. By considering powers, as usual, we can take $C_1 = 1$, and also taking into account that $K_{n_0+n_1+1} = \overline{D_{n_0+n_1+1}}$ is holomorphically convex in M, we see that

$$K_{n_0} \subseteq (z \epsilon D_{n_0+m} \quad : 1 + \omega_{n_0+m}(z) \leq \beta) \subseteq \overline{D_{n_0+n_1+1}} \subset\subset M$$

for a fixed β, with $0 < \beta < \frac{d}{1+d}$ and for every $m > n_1$.

We let

$$\omega^{n_0} \circeq \lim_m \ \omega_{n_0+m}.$$

Being the limit of a decreasing sequence of plurisubharmonic functions, ω^{n_0} is a negative plurisubharmonic function on M and is identically -1 on K_{n_0}. Moreover for any β with $0 < \beta < \frac{d}{1+d}$, we have:

$$K_{n_0} \subseteq \{z \epsilon M : \omega^{n_0} < \beta\} \subseteq \overline{D_{n_0+n_1+1}} \subset\subset M.$$

($\Leftarrow$) In this part of the proof we will follow the argument given in Th1 of [**1**] rather closely. Using the notation fixed at the beginning of the proof we fix a generating system for $O(M)$ given by the norms,

$$\|f\|_k \circeq \left(\int_{D_k} |f|^2 \, d\varepsilon \right)^{\frac{1}{2}}$$

where $D_k = (z \epsilon M : \sigma(z) \leq k)$, $k = 1, 2, ..$ .As usual we will use the notation U_k to denote the unit ball corresponding to $\|\|_k$, $k = 1, 2, ...$

Let $k_0 \epsilon \mathbb{N}$ be given. By our assumption there is a negative plurisubharmonic function ρ on M and $\alpha_1 < 0$, such that,

$$\overline{D_{k_0}} \subseteq (z \epsilon M : \rho(z) < \alpha_1) \subset\subset M.$$

Choose negative numbers $\alpha_0 < \alpha_1$, α_2 and $k_1 \epsilon \mathbb{N}$, $k_0 << k_1$ such that

$$\begin{aligned} \overline{D_{k_0}} &\subseteq (z \epsilon M : \rho(z) < \alpha_0) \subseteq (z \epsilon M : \rho(z) < \alpha_1) \\ &\subset\subset D_{k_1} \subseteq (z \epsilon M : \rho(z) < \alpha_2). \end{aligned}$$

and let

$$\Omega_- \circeq D_{k_1}, \qquad \Omega_+ \circeq \overline{(z \epsilon M : \rho(z) < \alpha_1)}^c.$$

For a fixed $t > 0$, we let,

$$\rho_t(z) \circeq -\frac{t}{\alpha_0} \rho(z) + t.$$

Clearly ρ_t is a bounded plurisubharmonic function on M.

Fix an $f \epsilon O(M)$ with $\|f\|_{k_1} \circeq \left(\int_{D_{k_1}} |f|^2 \, d\varepsilon \right)^{\frac{1}{2}} \leq 1$.

For such an f, we have the estimate,

$$\int_{\Omega_- \cap \Omega_+} |f|^2 e^{-\rho_t} d\mu \leq C \sup_{w \epsilon \Omega_- \cap \Omega_+} e^{-\rho_t(w)} \leq C e^{-\lambda t}$$

for some $C > 0$ where $\lambda \circeq 1 - \frac{\alpha_1}{\alpha_0}$.

In view of Lemma 1 of [**1**] we can decompose f on $\Omega_- \cap \Omega_+$ as $f = f_+ + f_-$ with $f_+ \epsilon O(\Omega_+)$, $f_- \epsilon O(\Omega_-)$; moreover,

$$\int_{\Omega_+} |f_+|^2 e^{-\rho_t} d\varepsilon \leq C e^{-\lambda t}, \qquad \int_{\Omega_-} |f_-|^2 e^{-\rho_t} d\varepsilon \leq C e^{-\lambda t}$$

for some constant $C > 0$ which is independent of f and t.

Hence,

$$\int_{\Omega_+} |f_+|^2 \, d\varepsilon \leq C e^{t(1-\lambda)}, \qquad \int_{\Omega_-} |f_-|^2 \, d\varepsilon \leq C e^{t(1-\lambda)}$$

Taking into account that $\rho_t \leq 0$ on D_{k_0}, we also have;

$$\int_{D_{k_0}} |f_-|^2 \, d\varepsilon \leq \int_{D_{k_0}} |f_-|^2 e^{-\rho_t} d\varepsilon \leq C e^{-\lambda t}.$$

Set

$$F = \begin{cases} f_+ & \text{on } \Omega_+ \\ f - f_- & \text{on } \Omega_- \end{cases}$$

The function F is analytic on M and from above we see that there is a $K > 0$:

$$\int |F|^2 \, d\varepsilon \leq K e^{t(1-\lambda)}.$$

Also from the above considerations we have :

$$\int_{D_{k_0}} |F-f|^2 \, d\varepsilon = \int_{D_{k_0}} |f_-|^2 \, d\varepsilon \leq Ce^{-\lambda t}$$

Now let

$$B \circeq \left(g\epsilon O(M) : \int |g|^2 \, d\varepsilon \leq 1 \right).$$

Setting $r = e^{t(1-\lambda)}$, the analysis above can be summarized as:

$$\forall \ k_0 \ \ \exists \ k_1 \ \text{ and } C > 0 : \quad U_{k_1} \subseteq \frac{1}{r^{\frac{\lambda}{1-\lambda}}} U_{k_0} + CrB \quad \forall r \geq 1.$$

Since the inclusion above is trivially true for $0 < r \leq 1$ we conclude that $O(M)$ has the property $\widetilde{\Omega}$.

This finishes the proof of the theorem. □

4. Concluding Remarks

Although the assignment $M \rightarrow O(M)$ from Stein manifolds, into Fréchet spaces is not a complete invariant, often,some complex potential theoretic properties of the given manifold M can be deduced from the knowledge of the type of the Fréchet space $O(M)$. We will look for a case in point in the context of the property $\widetilde{\Omega}$.

Let M be a Stein manifold and $z_0 \epsilon M$. Recall that the pluricomplex Green function $g_M(*, z_0)$ of M with pole at z_0 is the plurisubharmonic function on M defined as:

$$\begin{aligned} g_M(z, z_0) = \limsup_{\xi \rightarrow z} \ & \{ \sup u(\xi) : \ u \ \epsilon PSH(M) \ , u \leq 0, \text{ and} \\ & (\text{in the local coordinates}) \ \ u(w) - \log \|w - z_0\| \leq O(1) \ \text{ as } w \rightarrow z_0 \} \end{aligned}$$

(see [**8**] and the references given there). In one variable it coincides with the classical Green function and as is well known, they exists if and only if the space is not parabolic. Moreover if it exists, it is harmonic off its pole hence is a very "regular" function. The situation is rather different in higher dimensions. ([**8**], p.232). For example, denoting the unit disc by Δ, if we look at the domain $\mathbb{C} \times \Delta \subseteq \mathbb{C}^2$, we immediately see that $g_{\mathbb{C}\times\Delta}((z, w), \mathbf{0}) = \log|w|$; so the pluricomplex Green function is identically $-\infty$ on the whole complex line $\mathbb{C} \times (0)$.

Let us call a plurisubharmonic function $u : M \rightarrow [-\infty, \infty)$ *semi-proper* in case there exists a number c such that $(z \epsilon M : u(z) < c)$ is non-empty and is relatively compact in M. As a corollary of our theorem we have,

Corollary 1. *Let M be a Stein manifold and assume that $O(M)$ has the property $\widetilde{\Omega}$. Then for each $z_0 \epsilon M$, the pluricomplex Green function*

$g_M(*, z_0)$ is semi proper and satisfies $g_M(*, z_0)^{-1}(-\infty) = (z_0)$.

Proof. Fix $z_0 \epsilon M$, and choose a compact set K containing z_0 in its interior. In view of the theorem above, there exists a negative plurisubharmonic function σ on M and $c > 0$ such that

$$K \subseteq (z \epsilon M : \sigma(z) < -c) \subset\subset M.$$

Let $-c^{+} \circeq \max_{z\epsilon K}\sigma(z)$ and set $\widehat{\sigma} \circeq \sigma + c^{+}$. We choose a strictly pseudoconvex, $D \subset\subset M$ with

$$K \subseteq (z\epsilon M : \widehat{\sigma}(z) < 0) \subseteq (z\epsilon M : \widehat{\sigma}(z) < \alpha) \subset\subset D \subset\subset M$$

where $\alpha \circeq c^{+} - c$. We let $\rho \circeq g_D\ (*, z_0)$. The plurisubharmonic function ρ is a nice function, in the sense that e^{ρ} is continuous on $\overline{D}$ ([**8**, Corollary 6.2.3]). We fix $r_1 < r_2 < 0$ so that

$$(z\epsilon D : \rho < r_1) \subseteq K \subseteq (z\epsilon M : \widehat{\sigma}(z) < \alpha) \subset\subset (z\epsilon D : \rho < r_2) \subset\subset D \subset\subset M.$$

Finally set

$$\Phi \circeq \left(\frac{r_2 - r_1}{\alpha}\right)\widehat{\sigma} + r_1.$$

We will consider the open sets

$$U \circeq (z\epsilon D : \rho < r_2), \quad V \circeq \overline{(z\epsilon D : \rho < r_1)}^{c} \cap (z\epsilon D : \rho < r_2)$$

of D. For any $z\epsilon\partial V \cap U$, $\limsup_{\xi\to z}\Phi(\xi) \leq \rho(z)$, by construction. Hence in view of Corollary 2.9.15 [**8**], the function u defined by;

$$u \circeq \begin{cases} \max(\rho, \Phi) & \text{on } V \\ \rho & \text{on } U - V \end{cases}$$

is a plurisubharmonic function on $U \circeq (z\epsilon D : \rho < r_2)$.

Moreover on $\overline{(z\epsilon M : \widehat{\sigma}(z) < \alpha)}^{c} \cap (z\epsilon D : \rho < r_2)$, $\max(\rho, \Phi) = \Phi$. Hence we can extend u to a bounded plurisubharmonic function on whole of M by setting u to be equal to Φ outside $(z\epsilon D : \rho < r_2)$. Now $u - \sup_M u$, is a semi-proper negative plurisubharmonic function and since near z_0, it is equal to $g_D\ (*, z_0) - \sup_M u$,

$$g_D(*, z_0) \geq u - \sup_M u$$

on M. From this, it follows that $g_D(*, z_0)$ is a semi-proper plurisubharmonic function with $g_D(*, z_0)^{-1}(-\infty) = (z_0)$. This finishes the proof of the corollary. □

We would like to finish this note by looking at two simple, yet typical examples.

The first example we want to look at is the punctured unit disc, $\Delta - \{0\}$. Since every bounded plurisubharmonic function on it extends to a bounded plurisubharmonic function on the unit disc, it is not possible to put, say $K = \left(z\epsilon\mathbb{C} : |z| = \frac{1}{2}\right)$, into a precompact sublevel set of a bounded plurisubharmonic function on $\Delta - \{0\}$ in view of the maximum principle for plurisubharmonic functions. Actually it is not difficult to see that $O(\Delta - \{0\})$ is isomorphic to $O(\Delta) \times O(\mathbb{C})$ as Fréchet spaces. Hence $O(\Delta - \{0\})$ admits $O(\mathbb{C})$ as a quotient space and so can not have the property $\widetilde{\Omega}$.

The second example we will look at is also a subdomain of the unit disc. This time we will throw away infinite number of closed discs with radii tending to zero along with the origin from the unit disc. To this end fix an n_0 such that the closed discs;

$$K_n \circeq \left(z\epsilon\mathbb{C} : \left|z - \frac{1}{e^{n}}\right| \leq \frac{1}{e^{\frac{1}{n^3}}}\right)$$

are disjoint for $n \geq n_0$. Let

$$\Omega \doteq \Delta - \left(\bigcup_{n \geq n_0} K_n \cup \{0\} \right).$$

Fix a holomorphically convex smoothly bounded compact subset K of Ω. Choose a subdomain Θ of Ω obtained from Δ by deleting only finite number of $K_n's$ defined above such that it contains K as a holomorphically convex (in Θ), compact subset. Since Θ is hyperconvex ([**8**], p 80), the relative extremal function ω_K^Θ of K, (in Θ), is a continuous function and $\left(z\epsilon\Theta : \omega_K^\Theta(z) = -1\right) = K$ ([**16**]). For constants c near -1, the corresponding sublevel sets of ω_K^Θ restricted to Ω, are precompact in Ω and certainly they contain K. Since we can find an exhaustion of Ω by such compact sets K, the space $O(\Omega)$ has the property $\widetilde{\Omega}$, in view of the theorem above. However, $O(\Omega)$ does not have the stronger property $\overline{\Omega}$. This follows because the radii $(r_n)_n$ of the deleted discs satisfy;

$$\sum \frac{n}{\ln\left(\frac{1}{r_n}\right)} < \infty,$$

and hence, by a result of Zaharyuta [**14**], $O(\Omega) \ncong O(\Delta)$. In fact not much is known about the linear topological properties of the Fréchet space $O(\Omega)$.

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MDBF Sabanci University, Orhanli, 34956 Istanbul, Turkey
E-mail address: aytuna@sabanciuniv.edu