A Note on "A LP-based Heuristic for a Time-Constrained Routing Problem"

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Abstract

In their paper, Avella et al. (2006) investigate a time-constrained routing problem. The core of the proposed solution approach is a large-scale linear program that grows both row- and column-wise when new variables are introduced. Thus, a column-and-row generation algorithm is proposed to solve this linear program optimally, and an optimality condition is presented to terminate the column-and-row generation algorithm. We demonstrate by using Lagrangian duality that this optimality condition is incorrect and may lead to a suboptimal solution at termination.

Keywords: large-scale optimization, column generation, column-and-row generation, time-constrained routing

1. Introduction

Avella et al. (2006) study a time-constrained routing problem motivated by an application that schedules the visit of a tourist to a given geographical area. The problem is to send the tourist to one tour (a feasible sequence of sites) on each day of the vacation period by maximizing her total satisfaction while ensuring that each attraction is visited no more than once. The authors refer to this problem as the Intelligent Tourist Problem (ITP) and formulate it as a set packing problem with side constraints:

$$
\begin{aligned}
\text{maximize} \quad & \sum_{j \in J} r_j y_j, && && (1)\\
\text{subject to} \quad & \sum_{j \in D(i)} y_j \leq 1, && i \in V, && (2)\\
& y_j - \sum_{t \in S(j)} x_{jt} = 0, && j \in J, && (3)\\
& \sum_{j \in F(t)} x_{jt} = 1, && t \in T, && (4)\\
& y_j, x_{jt} \in \{0, 1\}, && j \in J, t \in T, && (5)
\end{aligned}
$$

where the set of sites that may be visited by a tourist in a vacation period T is denoted by V, and J represents the set of daily tours that originate from and terminate at the same location. The total satisfaction of the tourist from participating in tour j is given by r_j, and the binary variable y_j is set to one, if tour j is incorporated into the itinerary of the tourist. If tour j is performed on day t, the binary variable x_{jt} takes the value one. Here, $D(i) \subset J$ denotes the subset of the tours containing site i, $S(j) \subset T$ represents the set of days on which tour j can be performed, and $F(t) \subset J$ denotes the subset of the tours allowed on day t. By constraints (2), at most one tour in the selected itinerary includes site i. Constraints (3) impose that a tour to be included in the itinerary is assigned to one of the days in T on which the tour is allowed. It is also required that exactly one tour is selected on each day of the vacation period as prescribed by constraints (4). Finally, the objective function (1) maximizes the total satisfaction of the tourist over the vacation period T.

Avella et al. (2006) solve the linear programming (LP) relaxation of (1)-(5) by a column-and-row generation approach due to a potentially huge number of tours. A subset $\bar{J} \subset J$ of these tours is selected to form the restricted master problem (RMP) for the column-and-row generation procedure. At each iteration, a set of new tours $j \in L \subseteq (J \setminus \bar{J})$ is introduced to the RMP. For each $j \in L$, this implies adding $x_{jt}, t \in S(j)$, and the associated linking constraint $y_j - \sum_{t \in S(j)} x_{jt} = 0$ to the RMP. Now, let $\pi_i, i \in V$, $\gamma_j, j \in J$, and $\lambda_t, t \in T$, denote the dual variables associated with the constraints (2)-(4) in the LP relaxation of (1)-(5), respectively. The following theorem, given in Avella et al. (2006) without a proof, defines the stopping condition for the column-and-row generation algorithm applied to ITP by the authors.

Theorem 1 *The solution of the current RMP is optimal for the LP relaxation of* (1)-(5) *if* $\bar{r}_j = r_j - \sum_{i: j \in D(i)} \pi_i - \sum_{t: j \in F(t)} \lambda_t \leq 0$, *for each* $j \in J$.

The statement of the theorem above corrects two typos in the original Theorem 3.1 given by Avella et al. (2006), where the termination condition appears as $\bar{r}_j = r_j - \sum_{i: j \in D(i)} \pi_i - \sum_{t: j \in F(t)} \lambda_t \geq 0$, for each $t \in T$ and $j \in D(t)$. We demonstrate that the stopping condition in Theorem 1 is incorrect and may lead to a suboptimal solution when used to terminate the column-and-row generation algorithm as proposed by Avella et al. (2006).

Given the optimal dual variables associated with the current RMP, the resulting pricing subproblem to be solved becomes $\max_{j \in J \setminus \bar{J}} \bar{c}_j$, where $\bar{c}_j = r_j - \sum_{i: j \in D(i)} \pi_i - \gamma_j$ denotes the reduced cost of tour j. If the optimal objective function value of this subproblem is positive with $\bar{c}_j > 0$, the variables y_j, $x_{jt}, t \in S(j)$, and the primal constraint $y_j - \sum_{t \in S(j)} x_{jt} = 0$ should be added to the RMP. Otherwise, the optimal solution of the current RMP is declared as optimal for the LP relaxation of (1)-(5), and the algorithm terminates. The challenge here is that the value of the dual variable $\gamma_j, j \in J \setminus \bar{J}$ is unknown because the corresponding constraint is currently absent from the RMP. Hence, in order to design an optimal column-and-row generation algorithm for ITP, we must devise a method that anticipates the correct value of $\gamma_j, j \in J \setminus \bar{J}$ to be incorporated into the pricing subproblem. According to Theorem 1, $\sum_{t: j \in F(t)} \lambda_t = \sum_{t \in S(j)} \lambda_t$ is considered as an appropriate estimate for $\gamma_j, j \in J \setminus \bar{J}$. However, observe that in the dual of (1)-(5) we would like to set γ_j as large as possible and the dual constraints $\gamma_j \leq \lambda_t$ associated with $x_{jt}, t \in S(j)$, collectively imply that $\gamma_j \leq \min_{t\in S(j)} \lambda_t$. In the sequel, we show that $\min_{t\in S(j)} \lambda_t$ is indeed the correct value of γ_j for $j \in J \setminus \bar{J}$.

Consider an iteration of the column-and-row generation algorithm, where the optimal dual solution associated with the current RMP is denoted by $\pi_i, i \in V$, $\gamma_j, j \in \bar{J}$, and $\lambda_t, t \in T$. Suppose that $y_{j'}, j' \in J\setminus\bar{J}$ is to be priced out, where $\bar{r}_{j'} = r_{j'} - \sum_{i:j'\in D(i)} \pi_i - \sum_{t\in S(j')} \lambda_t \leq 0$. If $\mid S(j') \mid > 1$ and $\max_{t\in S(j')} \lambda_t > 0$, then we may have $\gamma_{j'} \leq \min_{t\in S(j')} \lambda_t < \sum_{t\in S(j')} \lambda_t$. Clearly, this may lead to $\bar{r}_{j'} = r_{j'} - \sum_{i:j'\in D(i)} \pi_i - \sum_{t\in S(j')} \lambda_t \leq 0 < r_{j'} - \sum_{i:j'\in D(i)} \pi_i - \gamma_{j'} = \bar{c}_{j'}$. Thus, we conclude that while $\bar{r}_j \leq 0$ for all $j \in J$ as required by Theorem 1 due to Avella et al. (2006), there may still exist a tour j' with $\bar{c}_{j'} > 0$. To determine the value of $\gamma_j, j \in J \setminus \bar{J}$ to be employed in the reduced cost calculations in a column-and-row generation scheme, we construct a Lagrangian relaxation of (1)-(5) by dualizing the complicating constraints (2) and (4) by multipliers $\pi_i \geq 0, i \in V$ and $\lambda_t, t \in T$, respectively. The resulting Lagrangian subproblem stated below allows us to set the values of the unknown dual variables $\gamma_j, j \in J \setminus \bar{J}$, correctly:

$$
\begin{aligned}
\mathcal{L}(\pi,\lambda) = \text{maximize} \quad & \sum_{j\in J}(r_j - \sum_{i:j\in D(i)} \pi_i)y_j - \\
& \sum_{j\in J}\sum_{t\in S(j)} \lambda_t x_{jt} + \sum_{i\in V}\pi_i + \sum_{t\in T}\lambda_t, \\
\text{subject to} \quad & (3), (5).
\end{aligned}
$$

For given values of $\pi_i, i \in V$, and $\lambda_t, t \in T$, this subproblem decomposes into $\mid J \mid$ subproblems, one for each tour $j \in J$. The optimal solution of the subproblem for $j \in J$ is identified by considering the cases $y_j = 0$ and $y_j = 1$ separately. We observe that $x_{jt} = 0, t \in S(j)$ when $y_j = 0$, and $x_{jt'} = 1, t' = \text{argmin}_{t\in S(j)} \lambda_t$, and $x_{jt} = 0, t \in S(j) \setminus \{t'\}$ when $y_j = 1$. Therefore, we trivially determine the optimal solution for the subproblem for $j \in J$ as $y_j = 1$, if $(r_j - \sum_{i:j\in D(i)} \pi_i - \min_{t\in S(j)} \lambda_t) > 0$; $y_j = 0$, otherwise. Note that we obtain the same optimal solution even if the integrality constraints on the $y-$ and $x-$variables are relaxed. Thus, the Lagrangian relaxation gives the same result as the LP relaxation of (1)-(5). We conclude that $r_j - \sum_{i:j\in D(i)} \pi_i - \min_{t\in S(j)} \lambda_t$ is indeed the reduced cost $\bar{c}_j = r_j - \sum_{i:j\in D(i)} \pi_i - \gamma_j$ for $j \in J$. We next formulate the correct termination criterion for a column-and-row generation algorithm for ITP in Theorem 2, and conclude with a counterexample demonstrating the error in Theorem 1.

Theorem 2 *The solution of the current RMP is optimal for the LP relaxation of (1)-(5) if* $\bar{c}_j = r_j - \sum_{i:j\in D(i)} \pi_i - \min_{t\in S(j)} \lambda_t \leq 0$, *for each* $j \in J$.

Example 1 *Consider an instance of ITP with 3 sites, 4 tours, 2 time periods. The RMP is initialized with the first three tours. All other data are specified in Table 1, where the optimal dual solution of the initial RMP is provided in columns "π_i^0," "γ_j^0," and "λ_t^0." The corresponding optimal primal solution is* $y_1^0 = 1, x_{11}^0 = 1, y_2^0 = 0, y_3^0 = 1, x_{32}^0 = 1$ *with an objective function value of 7.*

Table 1: The data for the counterexample.

i	$D(i)$	π_i^0	π_i^1	j	r_j	$S(j)$	γ_j^0	γ_j^1	t	λ_t^0	λ_t^1
1	{1, 4}	0	1	**1**	5	{1}	3	3	**1**	3	3
2	{1, 2}	2	1	**2**	4	{1, 2}	2	3	**2**	2	3
3	{3, 4}	0	0	**3**	2	{2}	2	2			
				4	4	{1, 2}		3			

Using Theorem 1, $\bar{r}_4 = r_4 - \sum_{i:4\in D(i)} \pi_i^0 - \sum_{t:4\in F(t)} \lambda_t^0 = r_4 - (\pi_1^0 + \pi_3^0) - (\lambda_1^0 + \lambda_2^0) = -1$, and the column-and-row generation terminates. However, the correct reduced cost of y_4 is given by $\bar{c}_4 = r_4 - \sum_{i:4\in D(i)} \pi_i^0 - \min_{t\in S(4)} \lambda_t^0 = r_4 - (\pi_1^0 + \pi_3^0) - \min(\lambda_1^0, \lambda_2^0) = 2$ according to Theorem 2. Then, augmenting the RMP with y_4 (and the associated x-variables and a single linking constraint) and re-solving it provides us with an optimal solution $y_1^1 = 0, y_2^1 = 1, x_{21}^1 = 1, y_3^1 = 0, y_4^1 = 1, x_{42}^1 = 1$, and an associated objective value of 8. The corresponding optimal dual solution is displayed in columns "π_i^1," "γ_j^1," and "λ_t^1" in Table 1.

Decomposition methods, such as Lagrangian relaxation, column generation, Dantzig-Wolfe reformulation, are among the most successful and classical topics in mathematical programming with several recent results focusing on generalizations and extensions, e.g., Vanderbeck and Savelsbergh (2006); Liang and Wilhelm (2010). Furthermore, column-and-row generation, which was first employed by Zak (2002) to solve a multi-stage cutting stock problem, emerges as a new area of research in different contexts. Recently, two generic column-and-row generation schemes based on Lagrangian relaxation have been developed by Sadykov and Vanderbeck (2011) and Frangioni and Gendron (2010), where the latter approach was previously applied successfully to a multi-commodity capacitated network design problem in Frangioni and Gendron (2009). In fact, applying the frameworks of both Frangioni and Gendron (2010) and Sadykov and Vanderbeck (2011) to ITP would give rise to the same Lagrangian subproblem $\mathcal{L}(\pi, \lambda)$ developed in this paper. On the other hand, ITP is also subsumed by the generic column-and-row generation scheme of Muter et al. (2011) which solves the LP relaxations of problems with column-dependent-rows based on concepts from LP duality.

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