Bari–Markus property for Riesz projections of 1D periodic Dirac operators

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Dedicated to the memory of Erhard Schmidt

The Dirac operators

$$Ly = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{dy}{dx} + v(x)y, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad x \in [0, \pi],$$

with L^2-potentials

$$v(x) = \begin{pmatrix} 0 & P(x) \\ Q(x) & 0 \end{pmatrix}, \quad P, Q \in L^2([0, \pi]),$$

considered on $[0, \pi]$ with periodic, antiperiodic or Dirichlet boundary conditions (bc), have discrete spectra, and the Riesz projections

$$S_N = \frac{1}{2\pi i} \int_{|z|=N-\frac{1}{2}} (z - L_{bc})^{-1} \, dz, \quad P_n = \frac{1}{2\pi i} \int_{|z-n|=\frac{1}{2}} (z - L_{bc})^{-1} \, dz$$

are well-defined for $|n| \geq N$ if N is sufficiently large. It is proved that

$$\sum_{|n|>N} \left\| P_n - P_n^0 \right\|^2 < \infty,$$

where P_n^0, $n \in \mathbb{Z}$, are the Riesz projections of the free operator.

Then, by the Bari–Markus criterion, the spectral Riesz decompositions

$$f = S_N f + \sum_{|n|>N} P_n f, \quad \forall f \in L^2;$$

converge unconditionally in L^2.

1 Introduction

The question for unconditional convergence of the spectral decompositions is one of the central problems in Spectral Theory of Differential Operators [2, 3, 20, 23, 26, 27].

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In the case of ordinary differential operators on a finite interval, say $I=[0,\pi]$,

$$\ell(y)=\frac{d^m y}{dx^m}+\sum_{k=0}^{m-2} q_k(x)\frac{d^k y}{dx^k}, \quad q_k\in W_k^2(I), \tag{1.1}$$

with strongly regular boundary conditions (bc) the eigenfunction decompositions

$$f(x)=\sum_k c_k(f)u_k(x), \quad \ell(u_k)=\lambda_k u_k,\ u_k\in(bc), \tag{1.2}$$

converge unconditionally for every $f\in L^2(I)$ (see [3,17,22]).

If (bc) are *regular but not strictly regular* the system of root functions (eigenfunctions and associated functions) in general is not a basis in L^2. But if the root functions are combined properly in disjoint groups $B_n,\ \bigcup B_n=\mathbb{N}$, then the series

$$f(x)=\sum_n P_n f, \quad \text{where} \quad P_n f=\sum_{k\in B_n} c_k(f)u_k(x), \tag{1.3}$$

converges unconditionally in L^2 (see [29,30]).

Let us be more specific in the case of operators of second order

$$\ell(y)=y''+q(x)y, \quad 0\le x\le\pi. \tag{1.4}$$

Then, Dirichlet $bc=Dir:\ \ y(0)=y(\pi)=0$ is *strictly regular*; however, Periodic $bc=Per^+:\ \ y(0)=y(\pi),\ y'(0)=y'(\pi)$ and Antiperiodic $bc=Per^-:\ y(0)=-y(\pi),\ y'(0)=-y'(\pi)$ are *regular, but not strictly regular.*

Analysis—even if it becomes more difficult and technical—could be extended to singular potentials $q\in H^{-1}$. A. Savchuk and A. Shkalikov showed ([28], Theorems 2.7 and 2.8) that for both Dirichlet bc or (properly understood) Periodic or Antiperiodic bc, the spectral decomposition (1.3) converges unconditionally. An alternative proof of this result is given in [10].

For Dirac operators (2.1) the results on unconditional convergence are sparse and not complete so far [13,14, 18,19,30–32].

The case of separate boundary conditions, at least for smooth potential v, has been studied in detail in [13,14, 18,19]. For periodic (or antiperiodic) bc B. Mityagin [24,25] proved unconditional convergence of the series (1.3) with $\dim P_n=2,\ |n|\ge N(v)$, for potentials $v\in H^b,\ b>1/2$—see Theorem 8.8 [25] for a precise statement.

Our techniques from [10] to analyze the resolvents $(\lambda-L_{bc})^{-1}$ of Hill operators with the weakest (in Sobolev scale) assumption $v\in H^{-1}$ on "smoothness" of the potential are adjusted and extended in the present paper to Dirac operators with potentials in L^2. We prove (see Theorem 3.1 for a precise statement) that if $v\in L^2$ and $bc=Per^{\pm}, Dir$ the sequence of deviations $\|P_n-P_n^0\|$ is in ℓ^2. Then, the Bari–Markus criterion (see [1,21] or [12], Ch.6, Sect.5.3, Theorem 5.2)) shows that the spectral decomposition

$$f=S_N f+\sum_{|n|>N} P_n f, \quad \forall f\in L^2, \tag{1.5}$$

where, for $|n|\ge N(v)$,

$$\dim P_n=\begin{cases}2, & bc=Per^{\pm},\\ 1, & bc=Dir,\end{cases} \tag{1.6}$$

converge unconditionally. This is Theorem 5.1, the main result of the present paper.

Further analysis requires thorough discussion of the algebraic structure of *regular* and *strictly regular* bc for Dirac operators. Then we can claim a general statement which is an analogue of (1.5)–(1.6), or Theorem 5.1, with $bc=Dir$ in case of strictly regular boundary conditions, and $bc=Per^{\pm}$ in case of regular but not strictly regular boundary conditions. We will give all the details in another paper.

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2 Preliminary results

Consider the Dirac operator on $I = [0, \pi]$

$$Ly = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{dy}{dx} + v(x)y, \tag{2.1}$$

where

$$v(x) = \begin{pmatrix} 0 & P(x) \\ Q(x) & 0 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \tag{2.2}$$

and v is an L^2-potential, i.e., $P, Q \in L^2(I)$.

We equip the space H^0 of $L^2(I)$-vector functions $F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ with the scalar product

$$\langle F, G \rangle = \frac{1}{\pi} \int_0^\pi \left(f_1(x)\overline{g_1(x)} + f_2(x)\overline{g_2(x)} \right) dx.$$

Consider the following boundary conditions (bc) :
(a) *periodic* Per^+ : $y(0) = y(\pi)$, i.e., $y_1(0) = y_1(\pi)$ and $y_2(0) = y_2(\pi)$;
(b) *anti-periodic* Per^- : $y(0) = -y(\pi)$, i.e., $y_1(0) = -y_1(\pi)$ and $y_2(0) = -y_2(\pi)$;
(c) *Dirichlet* Dir : $y_1(0) = y_2(0), \; y_1(\pi) = y_2(\pi)$.
The corresponding closed operator with a domain

$$\Delta_{bc} = \left\{ f \in (W_1^2(I))^2 \; : \; f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in (bc) \right\} \tag{2.3}$$

will be denoted by L_{bc}, or respectively, by $L_{Per^\pm}$ and L_{Dir}. If $v = 0$, i.e., $P \equiv 0, Q \equiv 0$, we write L^0_{bc} (or simply L^0), or $L^0_{Per^\pm}, L^0_{Dir}$ respectively. Of course, it is easy to describe the spectra and eigenfunctions for L^0_{bc}.

(a) $Sp\big(L^0_{Per^+}\big) = \{n \text{ even}\} = 2\mathbb{Z}$; each number $n \in 2\mathbb{Z}$ is a double eigenvalue, and the corresponding eigenspace is

$$E^0_n = Span\{e^1_n, e^2_n\}, \quad n \in 2\mathbb{Z}, \tag{2.4}$$

where

$$e^1_n(x) = \begin{pmatrix} e^{-inx} \\ 0 \end{pmatrix}, \quad e^2_n(x) = \begin{pmatrix} 0 \\ e^{inx} \end{pmatrix}; \tag{2.5}$$

(b) $Sp\big(L^0_{Per^-}\big) = \{n \text{ odd}\} = 2\mathbb{Z}+1$; the corresponding eigenspaces E^0_n are given by (2.4) and (2.5) but with $n \in 2\mathbb{Z}+1$;

(c) $Sp\big(L^0_{Dir}\big) = \{n \in \mathbb{Z}\}$; each eigenvalue n is simple. The corresponding normalized eigenfunction is

$$g_n(x) = \frac{1}{\sqrt{2}} \left(e^1_n + e^2_n \right), \quad n \in \mathbb{Z}, \tag{2.6}$$

so the corresponding (one-dimensional) eigenspace is

$$G^0_n = Span\{g_n\}. \tag{2.7}$$

We study the spectral properties of the operators $L_{Per^\pm}$ and L_{Dir} by using their Fourier representations with respect to the eigenvectors of the corresponding free operators given above in (2.4)–(2.7).

Let

$$P(x) = \sum_{m \in 2\mathbb{Z}} p(m) e^{imx}, \qquad Q(x) = \sum_{m \in 2\mathbb{Z}} q(m) e^{imx}, \tag{2.8}$$

and

$$P(x) = \sum_{m \in 1+2\mathbb{Z}} p_1(m) e^{imx}, \qquad Q(x) = \sum_{m \in 1+2\mathbb{Z}} q_1(m) e^{imx}, \tag{2.9}$$

be, respectively, the Fourier expansions of the functions P and Q about the systems $\{e^{imx},\, m \in 2\mathbb{Z}\}$ and $\{e^{imx},\, m \in 1+2\mathbb{Z}\}$.

Then

$$\|v\|^2 = \sum_{m \in 2\mathbb{Z}} \left(|p(m)|^2 + |q(m)|^2\right) = \sum_{m \in 1+2\mathbb{Z}} \left(|p_1(m)|^2 + |q_1(m)|^2\right). \tag{2.10}$$

Let V be the operator of multiplication by the matrix potential $v(x)$. The Fourier representation of V is defined by its action on vectors e_n^1 and e_n^2, with $n \in 2\mathbb{Z}$ for $bc = Per^+$ and $n \in 1+2\mathbb{Z}$ for $bc = Per^-$. In view of (2.2) and (2.8), we have

$$Ve_n^1 = \sum_{k \in n+2\mathbb{Z}} q(k+n) e_k^2, \quad Ve_n^2 = \sum_{k \in n+2\mathbb{Z}} p(-k-n) e_k^1, \tag{2.11}$$

so, the matrix representation of V is

$$V \sim \begin{pmatrix} 0 & V^{12} \\ V^{21} & 0 \end{pmatrix}, \quad (V^{12})_{kn} = p(-k-n), \quad (V^{21})_{kn} = q(k+n). \tag{2.12}$$

In the case of Dirichlet boundary conditions the operator L^0 is diagonal as well. The matrix representation of V given by the following lemma.

Lemma 2.1 *Let $(g_n)_{n\in\mathbb{Z}}$ be the orthogonal normalized basis (2.6) of eigenfunctions of L^0 in the case of Dirichlet boundary conditions. Then*

$$V_{kn} := \langle V g_n, g_k \rangle = W(k+n), \quad k, n \in \mathbb{Z}, \tag{2.13}$$

with

$$W(m) = \begin{cases} (p(-m) + q(m))/2, & m \text{ even}, \\ (p_1(-m) + q_1(m))/2, & m \text{ odd}. \end{cases} \tag{2.14}$$

The proof follows from a direct computation of $\langle V g_n, g_k \rangle$. Let us mention, that the sequences $p_1(m)$ and $q_1(m)$ in (2.14) are Hilbert transforms of $p(n)$ and $q(n)$ (see [6], Lemma 2 in Section 1.3) but we do not need this fact. In the following only the relation (2.10) is essential.

In view of (2.4)–(2.7) the operator $R_\lambda^0 = (\lambda - L^0)^{-1}$ is well defined, respectively, for $\lambda \notin 2\mathbb{Z}$ if $bc = Per^+$, $\lambda \notin 1+2\mathbb{Z}$ if $bc = Per^-$, and $\lambda \notin \mathbb{Z}$ if $bc = Dir$. The operator R_λ^0 is diagonal, and we have

$$R_\lambda^0 e_n^1 = \frac{1}{\lambda - n} e_n^1, \quad R_\lambda^0 e_n^2 = \frac{1}{\lambda - n} e_n^2 \quad \text{for} \quad bc = Per^\pm, \tag{2.15}$$

and

$$R_\lambda^0 g_n = \frac{1}{\lambda - n} g_n \quad \text{for} \quad bc = Dir. \tag{2.16}$$

The standard perturbation type formulae for the resolvent $R_\lambda = (\lambda - L^0 - V)^{-1}$ are

$$R_\lambda = (1 - R_\lambda^0 V)^{-1} R_\lambda^0 = \sum_{k=0}^{\infty} (R_\lambda^0 V)^k R_\lambda^0, \tag{2.17}$$

and

$$R_\lambda = R_\lambda^0 (1 - V R_\lambda^0)^{-1} = \sum_{k=0}^{\infty} R_\lambda^0 (V R_\lambda^0)^k. \tag{2.18}$$

The simplest conditions that guarantee convergence of the series (2.17) or (2.18) in ℓ^2 are

$$\|R^0_\lambda V\| < 1, \quad \text{respectively,} \quad \|V R^0_\lambda\| < 1.$$

In the case of Dirac operators there are no such good estimates but there are good estimates for the norms of $(R^0_\lambda V)^2$ and $(V R^0_\lambda)^2$ (see [4, 5] and [6], Section 1.2, for more comments).

But now we are going to suggest another approach that is borrowed from the study of Hill operators with periodic singular potentials (see [8–10]). Notice, that one can write (2.17) or (2.18) as

$$R_\lambda = R^0_\lambda + R^0_\lambda V R^0_\lambda + \cdots = K^2_\lambda + \sum_{m=1}^{\infty} K_\lambda (K_\lambda V K_\lambda)^m K_\lambda, \tag{2.19}$$

provided

$$(K_\lambda)^2 = R^0_\lambda. \tag{2.20}$$

In view of (2.15) and (2.16), we define an operator $K = K_\lambda$ with the property (2.20) by

$$K_\lambda e^1_n = \frac{1}{\sqrt{\lambda - n}} e^1_n, \quad K_\lambda e^2_n = \frac{1}{\sqrt{\lambda - n}} e^2_n \quad \text{for} \quad bc = Per^\pm, \tag{2.21}$$

and

$$K_\lambda g_n = \frac{1}{\sqrt{\lambda - n}} g_n \quad \text{for} \quad bc = Dir, \tag{2.22}$$

where

$$\sqrt{z} = \sqrt{r} e^{i\varphi/2} \quad \text{if} \quad z = re^{i\varphi}, \ -\pi \le \varphi < \pi.$$

Then R_λ is well-defined if

$$\|K_\lambda V K_\lambda\|_{\ell^2 \to \ell^2} < 1. \tag{2.23}$$

In view of (2.11) and (2.21), for periodic or anti–periodic boundary conditions $bc = Per^\pm$, we have

$$\begin{aligned} (K_\lambda V K_\lambda) e^1_n &= \sum_k \frac{q(k+n)}{(\lambda - k)^{1/2} (\lambda - n)^{1/2}} e^2_k, \\ (K_\lambda V K_\lambda) e^2_n &= \sum_k \frac{p(-k-n)}{(\lambda - k)^{1/2} (\lambda - n)^{1/2}} e^1_k, \end{aligned} \tag{2.24}$$

so, the Hilbert–Schmidt norm of the operator $K_\lambda V K_\lambda$ is given by

$$\|K_\lambda V K_\lambda\|^2_{HS} = \sum_{k,m} \frac{|q(k+m)|^2}{|\lambda - k||\lambda - m|} + \sum_{k,m} \frac{|p(-k-m)|^2}{|\lambda - k||\lambda - m|}, \tag{2.25}$$

where $k, m \in 2\mathbb{Z}$ for $bc = Per^+$ and $k, m \in 1 + 2\mathbb{Z}$ for $bc = Per^-$.

In an analogous way (2.13), (2.14) and (2.22) imply, for Dirichlet boundary conditions $bc = Dir$,

$$(K_\lambda V K_\lambda) g_n = \sum_k \frac{W(k+n)}{(\lambda - k)^{1/2} (\lambda - n)^{1/2}} g_k, \quad k, n \in \mathbb{Z}, \tag{2.26}$$

and therefore, we have

$$\|K_\lambda V K_\lambda\|^2_{HS} = \sum_{k,m} \frac{|W(k+m)|^2}{|\lambda - k||\lambda - m|}, \quad k, m \in \mathbb{Z}. \tag{2.27}$$

For convenience, we set

$$r(m) = \max(|p(m)|, |p(-m)|) + \max(|q(m)|, |q(-m)|), \quad m \in 2\mathbb{Z}, \tag{2.28}$$

if $bc = Per^{\pm}$, and

$$r(m) = |W(m)|, \quad m \in \mathbb{Z}, \tag{2.29}$$

if $bc = Dir$. Now we define operators $\bar{V}$ and $\bar{K}_\lambda$ which dominate, respectively, V and K_λ, as follows:

$$\bar{V} e_n^1 = \sum_{k \in n+2\mathbb{Z}} r(k+n) e_k^2, \quad \bar{V} e_n^2 = \sum_{k \in n+2\mathbb{Z}} r(k+n) e_k^1 \quad \text{for} \quad bc = Per^{\pm}, \tag{2.30}$$

$$\bar{V} g_n = \sum_{k \in \mathbb{Z}} r(k+n) g_k \quad \text{for} \quad bc = Dir, \tag{2.31}$$

and

$$\bar{K}_\lambda e_n^1 = \frac{1}{\sqrt{|\lambda - n|}} e_n^1, \quad \bar{K}_\lambda e_n^2 = \frac{1}{\sqrt{|\lambda - n|}} e_n^2 \quad \text{for} \quad bc = Per^{\pm}, \tag{2.32}$$

$$\bar{K}_\lambda g_n = \frac{1}{\sqrt{|\lambda - n|}} g_n \quad \text{for} \quad bc = Dir. \tag{2.33}$$

Since the matrix elements of the operator $K_\lambda V K_\lambda$ do not exceed, by absolute value, the matrix elements of $\bar{K}_\lambda \bar{V} \bar{K}_\lambda$, we estimate from above the Hilbert–Schmidt norm of the operator $K_\lambda V K_\lambda$ by one and the same formula:

$$\|K_\lambda V K_\lambda\|_{HS}^2 \leq \|\bar{K}_\lambda \bar{V} \bar{K}_\lambda\|_{HS}^2 = \sum_{i,k} \frac{|r(i+k)|^2}{|\lambda - i||\lambda - k|}, \tag{2.34}$$

where $i, k \in 2\mathbb{Z}$ if $bc = Per^+$ and $i, k \in 1 + 2\mathbb{Z}$ if $bc = Per^-$, or $i, k \in \mathbb{Z}$ if $bc = Dir$. Next we estimate the Hilbert–Schmidt norm of the operator $\bar{K}_\lambda \bar{V} \bar{K}_\lambda$ for $\lambda \in C_n = \{\lambda : \ |\lambda - n| = 1/2\}$.

For each ℓ^2-sequence $x = (x(j))_{j \in \mathbb{Z}}$ and $m \in \mathbb{N}$ we set

$$\mathcal{E}_m(x) = \left(\sum_{|j| \geq m} |x(j)|^2 \right)^{1/2}. \tag{2.35}$$

Lemma 2.2 *In the above notations, if $n \neq 0$, then*

$$\|\bar{K}_\lambda \bar{V} \bar{K}_\lambda\|_{HS}^2 = \sum_{i,k} \frac{|r(i+k)|^2}{|\lambda - i||\lambda - k|} \leq 60 \left(\frac{\|r\|^2}{\sqrt{|n|}} + (\mathcal{E}_{|n|}(r))^2 \right), \quad \lambda \in C_n. \tag{2.36}$$

Proof. Since

$$2|\lambda - i| \geq |n - i| \quad \text{if} \quad i \neq n, \qquad \lambda \in C_n = \{\lambda : \ |\lambda - n| = 1/2\}, \tag{2.37}$$

the sum in (2.36) does not exceed

$$4|r(2n)|^2 + 4 \sum_{k \neq n} \frac{|r(n+k)|^2}{|n-k|} + 4 \sum_{i \neq n} \frac{|r(n+i)|^2}{|n-i|} + 4 \sum_{i,k \neq n} \frac{|r(i+k)|^2}{|n-i||n-k|}.$$

In view of (4.2) and (4.3) in Lemma 4.1, each of the above sums does not exceed the right-hand side of (2.36), which completes the proof. □

Corollary 2.3 *There is $N \in \mathbb{N}$ such that*

$$\|K_\lambda V K_\lambda\| \leq 1/2 \quad \text{for} \quad \lambda \in C_n, \quad |n| > N. \tag{2.38}$$

3 Core results

By our Theorem 18 in [6] (about spectra localization), for sufficiently large $|n|$, say $|n| > N$, the operator $L_{Per^\pm}$ has exactly two (counted with their algebraic multiplicity) periodic (for even n) or antiperiodic (for odd n) eigenvalues inside the disc with a center n of radius $1/2$. The operator L_{Dir} has, for all sufficiently large $|n|$, one eigenvalue in every such disc.

Let P_n and P_n^0 be the Riesz projections corresponding to L and L^0, i.e.,

$$P_n = \frac{1}{2\pi i}\int_{C_n} (\lambda - L)^{-1}\, d\lambda, \quad P_n^0 = \frac{1}{2\pi i}\int_{C_n} \left(\lambda - L^0\right)^{-1} d\lambda,$$

where $C_n = \{\lambda : \ |\lambda - n| = 1/2\}$.

Theorem 3.1 *Suppose L and L^0 are, respectively, the Dirac operator* (2.1) *with an L^2 potential v and the free Dirac operator, subject to periodic, antiperiodic or Dirichlet boundary conditions $bc = Per^\pm$ or Dir. Then, there is $N \in \mathbb{N}$ such that for $|n| > N$ the Riesz projections P_n and P_n^0 corresponding to L and L^0 are well defined and we have*

$$\sum_{|n|>N} \left\|P_n - P_n^0\right\|^2 < \infty. \tag{3.1}$$

P r o o f. Now we present the proof of the theorem up to a few technical inequalities. They will be proved later in Section 4, Lemmas 4.1 and 4.2.

1. Let us notice that the operator-valued function K_λ is analytic in $\mathbb{C} \setminus \mathbb{R}$. But (2.19), (3.2) below and all formulas of this section, which are essentially variations of (2.19), always have even powers of K_λ, and $K_\lambda^2 = R_\lambda^0$ is analytic on $\mathbb{C} \setminus Sp(L^0)$. Certainly, this justifies the use of Cauchy formula or Cauchy theorem when warranted.

In view of (2.38), the corollary after the proof of Lemma 2, if $|n|$ is sufficiently large then the series in (2.19) converges. Therefore,

$$P_n - P_n^0 = \frac{1}{2\pi i}\int_{C_n} \sum_{s=0}^{\infty} K_\lambda (K_\lambda V K_\lambda)^{s+1} K_\lambda \, d\lambda. \tag{3.2}$$

Remark 3.2 We are going to prove (3.1) by estimating the Hilbert–Schmidt norms $\left\|P_n - P_n^0\right\|_{HS}$ which dominate $\left\|P_n - P_n^0\right\|$. Of course, these norms are equivalent as long as the dimensions $dim\left(P_n - P_n^0\right)$ are uniformly bounded because for any finite dimensional operator T we have

$$\|T\| \le \|T\|_{HS} \le (dim\, T)^{1/2}\|T\|$$

but in the context of this paper for all projections $dim\, P_n,\ dim\, P_n^0 \le 2$.

2. If $bc = Dir$, then, by (2.6),

$$\left\|P_n - P_n^0\right\|_{HS}^2 = \sum_{m,k\in\mathbb{Z}} \left|\left\langle \left(P_n - P_n^0\right) g_m, g_k\right\rangle\right|^2.$$

By (3.2), we get

$$\left\langle \left(P_n - P_n^0\right) g_m, g_k\right\rangle = \sum_{s=0}^{\infty} I_n(s,k,m),$$

where

$$I_n(s,k,m) = \frac{1}{2\pi i}\int_{C_n} \langle K_\lambda (K_\lambda V K_\lambda)^{s+1} K_\lambda g_m, g_k\rangle \, d\lambda.$$

Therefore,

$$\sum_{|n|>N} \left\|P_n - P_n^0\right\|_{HS}^2 \le \sum_{s,t=0}^{\infty} \sum_{|n|>N} \sum_{m,k\in\mathbb{Z}} |I_n(s,k,m)| \cdot |I_n(t,k,m)|.$$

Now, the Cauchy inequality implies

$$\sum_{|n|>N} \left\|P_n - P_n^0\right\|_{HS}^2 \le \sum_{s,t=0}^{\infty} (A(s))^{1/2}(A(t))^{1/2}, \tag{3.3}$$

where

$$A(s) = \sum_{|n|>N} \sum_{m,k\in\mathbb{Z}} |I_n(s,k,m)|^2. \tag{3.4}$$

Notice that $A(s)$ depends on N but this dependence is suppressed in the notation.
From the matrix representation of the operators K_λ and V we get

$$\langle K_\lambda (K_\lambda V K_\lambda)^{s+1} K_\lambda g_m, g_k\rangle = \sum_{j_1,\dots,j_s} \frac{W(k+j_1)W(j_1+j_2)\cdots W(j_s+m)}{(\lambda-k)(\lambda-j_1)\cdots(\lambda-j_s)(\lambda-m)}, \tag{3.5}$$

and therefore,

$$I_n(s,k,m) = \frac{1}{2\pi i}\int_{C_n} \sum_{j_1,\dots j_s} \frac{W(k+j_1)W(j_1+j_2)\cdots W(j_s+m)}{(\lambda-k)(\lambda-j_1)\cdots(\lambda-j_s)(\lambda-m)}\, d\lambda. \tag{3.6}$$

In view of (2.29), we have

$$\left|\frac{W(k+j_1)W(j_1+j_2)\cdots W(j_s+m)}{(\lambda-k)(\lambda-j_1)\cdots(\lambda-j_s)(\lambda-m)}\right| \le B(\lambda,k,j_1,\dots,j_s,m), \tag{3.7}$$

where

$$B(\lambda,k,j_1,\dots,j_s,m) = \frac{r(k+j_1)r(j_1+j_2)\cdots r(j_{s-1}+j_s)r(j_s+m)}{|\lambda-k||\lambda-j_1|\cdots|\lambda-j_s||\lambda-m|}, \quad s>0, \tag{3.8}$$

and

$$B(\lambda,k,m) = \frac{r(m+k)}{|\lambda-k||\lambda-m|} \tag{3.9}$$

in the case when $s=0$ and there are no j-indices. Moreover, by (2.29), (2.31) and (2.33), we have

$$\sum_{j_1,\dots,j_s} B(\lambda,k,j_1,\dots,j_s,m) = \langle \bar K_\lambda (\bar K_\lambda \bar V \bar K_\lambda)^{s+1} \bar K_z g_m, g_k\rangle. \tag{3.10}$$

Lemma 3.3 *In the above notations, we have*

$$A(s) \le B_1(s) + B_2(s) + B_3(s) + B_4(s), \tag{3.11}$$

where

$$B_1(s) = \sum_{|n|>N} \sup_{\lambda\in C_n} \left(\sum_{j_1,\dots,j_s} B(\lambda,n,j_1,\dots,j_s,n)\right)^2; \tag{3.12}$$

$$B_2(s) = \sum_{|n|>N} \sum_{k\neq n} \sup_{\lambda\in C_n} \left(\sum_{j_1,\dots,j_s} B(\lambda, k, j_1, \dots, j_s, n) \right)^2; \tag{3.13}$$

$$B_3(s) = \sum_{|n|>N} \sum_{m\neq n} \sup_{\lambda\in C_n} \left(\sum_{j_1,\dots,j_s} B(\lambda, n, j_1, \dots, j_s, m) \right)^2; \tag{3.14}$$

$$B_4(s) = \sum_{|n|>N} \sum_{m,k\neq n} \sup_{\lambda\in C_n} \left(\sum_{j_1,\dots,j_s}^{*} B(\lambda, k, j_1, \dots, j_s, m) \right)^2, \quad s \geq 1, \tag{3.15}$$

where the symbol $*$ over the sum in the parentheses means that at least one of the indices $j_1, \dots, j_s$ is equal to n.

Proof. Indeed, in view of (3.4), we have

$$A(s) \leq A_1(s) + A_2(s) + A_3(s) + A_4(s),$$

where

$$A_1(s) = \sum_{|n|>N} |I_n(s,n,n)|^2, \quad A_2(s) = \sum_{|n|>N} \sum_{k\neq n} |I_n(s,k,n)|^2,$$

$$A_3(s) = \sum_{|n|>N} \sum_{m\neq n} |I_n(s,n,m)|^2, \quad A_4(s) = \sum_{|n|>N} \sum_{m,k\neq n} |I_n(s,k,m)|^2.$$

By (3.6)–(3.9) we get immediately that

$$A_\nu(s) \leq B_\nu(s), \quad \nu = 1, 2, 3.$$

On the other hand, by the Cauchy formula,

$$\int_{C_n} \frac{W(k+j_1)W(j_1+j_2)\cdots W(j_s+m)}{(\lambda-k)(\lambda-j_1)\cdots(\lambda-j_s)(\lambda-m)} \, d\lambda = 0 \quad \text{if} \quad k, j_1, \dots, j_s, m \neq n.$$

Therefore, removing from the sum in (3.6) the terms with zero integrals, and estimating from above the remaining sum, we get

$$|I_n(s,k,m)| \leq \sup_{\lambda\in C_n} \left(\sum_{j_1,\dots,j_s}^{*} B(\lambda, k, j_1, \dots, j_s, m) \right), \qquad m, k \neq n.$$

From here it follows that $A_4(s) \leq B_4(s)$, which completes the proof. □

3. If $bc = Per^{\pm}$, then using the orthonormal system of eigenvectors of the free operator L^0 given by (2.5), we get

$$\|P_n - P_n^0\|_{HS}^2 = \sum_{\alpha,\beta=1}^{2} \sum_{m,k} \left|\left\langle (P_n - P_n^0) e_m^\alpha, e_k^\beta \right\rangle\right|^2, \tag{3.16}$$

where $m, k \in 2\mathbb{Z}$ if n is even or $m, k \in 1 + 2\mathbb{Z}$ if n is odd. By (3.2), we have

$$\left\langle (P_n - P_n^0) e_m^\alpha, e_k^\beta \right\rangle = \sum_{s=0}^{\infty} I^{\alpha\beta}(n,s,k,m), \tag{3.17}$$

where

$$I^{\alpha\beta}(n,s,k,m) = \frac{1}{2\pi i} \int_{C_n} \left\langle K_\lambda (K_\lambda V K_\lambda)^{s+1} K_\lambda e_m^\alpha, e_k^\beta \right\rangle d\lambda. \tag{3.18}$$

Therefore,

$$\sum_{|n|>N} \|P_n - P_n^0\|_{HS}^2 \leq \sum_{\alpha,\beta=1}^{2} \sum_{t,s=0}^{\infty} \sum_{|n|>N} \sum_{m,k} |I^{\alpha\beta}(n,s,k,m)| \cdot |I^{\alpha\beta}(n,t,k,m)|.$$

Now, the Cauchy inequality implies

$$\sum_{|n|>N} \|P_n - P_n^0\|_{HS}^2 \leq \sum_{\alpha,\beta=1}^{2} \sum_{t,s=0}^{\infty} \big(A^{\alpha\beta}(s)\big)^{1/2} \big(A^{\alpha\beta}(t)\big)^{1/2}, \tag{3.19}$$

where

$$A^{\alpha\beta}(s) = \sum_{|n|>N} \sum_{m,k} |I^{\alpha\beta}(n,s,k,m)|^2. \tag{3.20}$$

Lemma 3.4 *In the above notations, with r given by* (2.28)*, $B(\lambda,k,j_1,\dots,j_s,m)$ defined in* (3.8)*,* (3.9)*, and $B_j(s),\ j=1,\dots,4$, defined by* (3.12)–(3.15)*, we have*

$$A^{\alpha\beta}(s) \leq B_1(s) + B_2(s) + B_3(s) + B_4(s), \quad \alpha,\beta = 1,2. \tag{3.21}$$

P r o o f. The matrix representations of the operators V and K_λ given in (2.12) and (2.21) imply that if s is even, then $\langle K_\lambda(K_\lambda V K_\lambda)^{s+1} K_\lambda e_m^\alpha, e_k^\alpha\rangle = 0$ for $\alpha = 1,2$, and if s is odd then

$$\langle K_\lambda(K_\lambda V K_\lambda)^{s+1} K_\lambda e_m^1, e_k^1\rangle = \sum_{j_1,\dots,j_s} \frac{p(-k-j_1)q(j_1+j_2)\cdots p(-j_{s-1}-j_s)q(j_s+m)}{(\lambda-k)(\lambda-j_1)\cdots(\lambda-j_s)(\lambda-m)}, \tag{3.22}$$

$$\langle K_\lambda(K_\lambda V K_\lambda)^{s+1} K_\lambda e_m^2, e_k^2\rangle = \sum_{j_1,\dots,j_s} \frac{q(k+j_1)p(-j_1-j_2)\cdots q(j_{s-1}+j_s)p(-j_s-m)}{(\lambda-k)(\lambda-j_1)\cdots(\lambda-j_s)(\lambda-m)}. \tag{3.23}$$

In analogous way it follows that if s is odd then

$$\langle K_\lambda(K_\lambda V K_\lambda)^{s+1} K_\lambda e_m^1, e_k^2\rangle = 0 \quad \text{and} \quad \langle K_\lambda(K_\lambda V K_\lambda)^{s+1} K_\lambda e_m^2, e_k^1\rangle = 0,$$

and if s is even then

$$\langle K_\lambda(K_\lambda V K_\lambda)^{s+1} K_\lambda e_m^1, e_k^2\rangle = \sum_{j_1,\dots,j_s} \frac{q(k+j_1)p(-j_1-j_2)\cdots p(-j_{s-1}-j_s)q(j_s+m)}{(\lambda-k)(\lambda-j_1)\cdots(\lambda-j_s)(\lambda-m)}, \tag{3.24}$$

$$\langle K_\lambda(K_\lambda V K_\lambda)^{s+1} K_\lambda e_m^2, e_k^1\rangle = \sum_{j_1,\dots,j_s} \frac{p(-k-j_1)q(j_1+j_2)\cdots q(j_{s-1}+j_s)p(-j_s-m)}{(\lambda-k)(\lambda-j_1)\cdots(\lambda-j_s)(\lambda-m)}. \tag{3.25}$$

From (2.28), (3.12)–(3.15) and the above formulas it follows that

$$\big|\langle K_\lambda(K_\lambda V K_\lambda)^{s+1} K_\lambda e_m^\alpha, e_k^\beta\rangle\big| \leq \sum_{j_1,\dots,j_s} B(\lambda,k,j_1,\dots,j_s,m),$$

which implies immediately

$$|I_n^{\alpha\beta}(s,k,m)| \leq \sup_{\lambda\in C_n} \left(\sum_{j_1,\dots,j_s} B(\lambda,k,j_1,\dots,j_s,m) \right). \tag{3.26}$$

By (3.20),

$$A^{\alpha\beta}(s) \leq A_1^{\alpha\beta}(s) + A_2^{\alpha\beta}(s) + A_3^{\alpha\beta}(s) + A_4^{\alpha\beta}(s),$$

where

$$A_1^{\alpha\beta}(s) = \sum_{|n|>N} |I_n^{\alpha\beta}(s,n,n)|^2, \quad A_2^{\alpha\beta}(s) = \sum_{|n|>N} \sum_{k\neq n} |I_n^{\alpha\beta}(s,k,n)|^2,$$

$$A_3^{\alpha\beta}(s) = \sum_{|n|>N} \sum_{m\neq n} |I_n^{\alpha\beta}(s,n,m)|^2, \quad A_4^{\alpha\beta}(s) = \sum_{|n|>N} \sum_{m,k\neq n} |I_n^{\alpha\beta}(s,k,m)|^2.$$

Therefore, in view of (3.26) and (3.12)–(3.14), we get

$$A_\nu^{\alpha\beta}(s) \leq B_\nu(s), \quad \nu = 1,2,3.$$

Finally, as in the proof of Lemma 3.3, we take into account that in the sums (3.22)–(3.25) the terms with indices $j_1,\ldots,j_s,m,k \neq n$ have zero integrals over the contour C_n. Therefore,

$$|I_n^{\alpha\beta}(s,k,m)| \leq \sup_{\lambda\in C_n} \left(\sum_{j_1,\ldots,j_s}^{*} B(\lambda,k,j_1,\ldots,j_s,m) \right), \quad m,k\neq n.$$

In view of (3.15), this yields $A_4^{\alpha\beta}(s) \leq B_4(s)$, which completes the proof. □

4. In view of (3.3) and (3.11), Theorem 3.1 will be proved if we get "good estimates" of the sums $B_\nu(s),\ \nu = 1,\ldots,4$, that are defined by (3.12)–(3.15). Such estimates are given in the next proposition. For convenience, we set for any ℓ^2-sequence $r = (r(j))$

$$\rho_N = 8\left(\frac{\|r\|^2}{\sqrt{N}} + (\mathcal{E}_N(r))^2 \right)^{1/2}. \tag{3.27}$$

Proposition 3.5 *In the above notations,*

$$B_\nu(s) \leq C\|r\|^2\rho_N^{2s}, \quad \nu=1,2,3, \qquad B_4(s) \leq Cs\|r\|^4\rho_N^{2(s-1)}, \quad s\geq 1, \tag{3.28}$$

where C is an absolute constant.

Remark: For convenience, here and thereafter we denote by C any absolute constant.

Proof. *Estimates for $B_1(s)$.* By (3.9) and (3.12), we have

$$B_1(0) = \sum_{|n|>N} \sup_{\lambda\in C_n} \frac{|r(2n)|^2}{|\lambda-n|^2} = 4(\mathcal{E}_N(r))^2 \leq 4\|r\|^2,$$

so (3.28) holds for $B_1(s)$ if $s=0$.

If $s=1$, then by (3.8), the sum $B_1(1)$ from (3.12) has the form

$$B_1(1) = \sum_{|n|>N} \sup_{\lambda\in C_n} \left| \sum_j \frac{r(n+j)r(j+n)}{|\lambda-n||\lambda-j||\lambda-n|} \right|^2.$$

By (2.37), and since $|\lambda-n| = 1/2$ for $\lambda\in C_n$, we get

$$B_1(1) \leq \sum_{|n|>N} \left(8\sum_{j\neq n} \frac{|r(j+n)|^2}{|j-n|} + 8|r(2n)|^2 \right)^2$$

$$\leq 128 \sum_{|n|>N} \left(\sum_{j\neq n} \frac{|r(j+n)|^2}{|j-n|} \right)^2 + 128 \sum_{|n|>N} |r(2n)|^4.$$

By the Cauchy inequality and (4.5) in Lemma 4.2, we have

$$\sum_{|n|>N}\left(\sum_{j\neq n}\frac{|r(j+n)|^2}{|j-n|}\right)^2 \leq \sum_{|n|>N}\sum_{j\neq n}\frac{|r(j+n)|^2}{|j-n|^2}\|r\|^2 \leq C\|r\|^2\rho_N^2.$$

On the other hand, $\sum_{|n|>N}|r(2n)|^4 \leq \|r\|^2(\mathcal{E}_N(r))^2 \leq \|r\|^2\rho_N^2$, so (3.28) holds for $B_1(s)$ if $s=1$.

Next, we consider the case $s>1$. In view of (3.8), since $|\lambda-n|=1/2$ for $\lambda\in C_n$, the sum $B_1(s)$ from (3.12) can be written as

$$B_1(s)=\sum_{|n|>N}4\sup_{\lambda\in C_n}\left|\sum_{j_1,\dots,j_s}\frac{r(n+j_1)r(j_1+j_2)\cdots r(j_s+n)}{|\lambda-j_1||\lambda-j_2|\cdots|\lambda-j_s|}\right|^2.$$

Therefore, we have (with $j=j_1, k=j_s$)

$$B_1(s)=4\sum_{|n|>N}\sup_{\lambda\in C_n}\left|\sum_{j,k}\frac{r(n+j)}{|\lambda-j|^{1/2}}\cdot H_{jk}(\lambda)\cdot\frac{r(k+n)}{|\lambda-k|^{1/2}}\right|^2,$$

where $(H_{jk}(\lambda))$ is the matrix representation of the operator $H(\lambda)=(\bar{K}_\lambda\bar{V}\bar{K}_\lambda)^{s-1}$. By (2.36) in Lemma 2.2,

$$\|H(\lambda)\|_{HS}=\left(\sum_{j,k}|H_{jk}(\lambda)|^2\right)^{1/2} \leq \|\bar{K}_\lambda\bar{V}\bar{K}_\lambda\|_{HS}^{s-1}\leq\rho_N^{s-1} \quad \text{for} \quad \lambda\in C_n, \quad |n|>N.$$

Therefore, the Cauchy inequality implies

$$B_1(s)\leq 4\sup_{\lambda\in C_n}\|H(\lambda)\|_{HS}^2\cdot\sigma\leq 4\rho_N^{2(s-1)}\cdot\sigma,$$

where

$$\sigma=\sum_{|n|>N}\sup_{\lambda\in C_n}\sum_{j,k}\frac{|r(n+j)|^2}{|\lambda-j|}\cdot\frac{|r(k+n)|^2}{|\lambda-k|}.$$

By (2.37) and since $|\lambda-n|=1/2$ for $\lambda\in C_n$, we have

$$\sigma\leq 4\sum_{|n|>N}\sum_{j,k\neq n}\frac{|r(n+j)|^2|r(n+k)|^2}{|n-j||n-k|}+4\sum_{|n|>N}|r(2n)|^2\sum_{k\neq n}\frac{|r(n+k)|^2}{|n-k|}$$
$$+4\sum_{|n|>N}|r(2n)|^2\sum_{j\neq n}\frac{|r(n+j)|^2}{|n-j|}+4\sum_{|n|>N}|r(2n)|^4.$$

In view of (4.6) in Lemma 4.2, the triple sum does not exceed $C\|r\|^2\rho_N^2$. By (4.2) in Lemma 4.1, each of the double sums can be estimated from above by

$$C\sum_{|n|>N}|r(2n)|^2\rho_N^2\leq C\|r\|^2\rho_N^2,$$

and the same estimate holds for the single sum. Therefore,

$$B_1(s)\leq C\rho_N^{2(s-1)}\cdot\|r\|^2\rho_N^2,$$

which completes the proof of (3.28) for $B_1(s)$.

Estimates for $B_2(s)$. By (3.9) and (3.12), we have

$$B_2(0) = \sum_{|n|>N} \sum_{k\neq n} \sup_{\lambda\in C_n} \frac{|r(k+n)|^2}{|\lambda-k|^2|\lambda-n|^2}.$$

Taking into account that $|\lambda - n| = 1/2$ for $\lambda \in C_n$, we get, in view of (2.37) and (4.5) in Lemma 4.2,

$$B_2(0) \leq 16 \sum_{|n|>N} \sum_{k\neq n} \frac{|r(k+n)|^2}{|n-k|^2} \leq C\|r\|^2.$$

So, (3.28) holds for $B_2(s)$ if $s = 0$.

If $s = 1$, then, by (3.8), the sum $B_2(s)$ in (3.28) has the form

$$B_2(1) = \sum_{|n|>N} \sum_{k\neq n} \sup_{\lambda\in C_n} \left| \sum_j \frac{r(k+j)r(j+n)}{|\lambda-k||\lambda-j||\lambda-n|} \right|^2.$$

Since $|\lambda - n| = 1/2$ for $\lambda \in C_n$, we get, in view of (2.37),

$$B_2(1) \leq \sum_{|n|>N} \sum_{k\neq n} \left| \sum_{j\neq n} 8\frac{r(k+j)r(j+n)}{|n-k||n-j|} + 8r(2n)\frac{r(k+n)}{|n-k|} \right|^2.$$

Therefore,

$$B_2(1) \leq 128\sigma_1 + 128\sigma_2,$$

where (by the Cauchy inequality and (4.5) in Lemma 4.2)

$$\begin{aligned}
\sigma_1 &= \sum_{|n|>N,k\neq n} \left(\sum_{j\neq n} \frac{r(k+j)r(j+n)}{|n-k||n-j|} \right)^2 \\
&\leq \sum_{|n|>N,k\neq n} \frac{1}{|n-k|^2} \left(\sum_{j\neq n} \frac{|r(n+j)|^2}{|n-j|^2} \right) \cdot \|r\|^2 \\
&= \sum_{|n|>N,j\neq n} \frac{|r(n+j)|^2}{|n-j|^2} \sum_{k\neq n} \frac{\|r\|^2}{|n-k|^2} \\
&\leq C\rho_N^2 \|r\|^2,
\end{aligned}$$

and

$$\sigma_2 = \sum_{|n|>N,k\neq n} |r(2n)|^2 \frac{|r(n+k)|^2}{|n-k|^2} \leq C\rho_N^2\|r\|^2.$$

Thus, (3.28) holds for $B_2(s)$ if $s = 1$.

If $s > 1$, then by (3.8) and $|\lambda - n| = 1/2$ for $\lambda \in C_n$, we have

$$B_2(s) = \sum_{|n|>N,k\neq n} 2 \sup_{\lambda\in C_n} \left| \sum_{j_1,\dots,j_s} \frac{r(k+j_1)r(j_1+j_2)\cdots r(j_s+n)}{|\lambda-k||\lambda-j_1||\lambda-j_2|\cdots|\lambda-j_s|} \right|^2.$$

In view of (2.31) and (2.32), we get (with $j = j_1, i = j_s$)

$$B_2(s) = 2 \sum_{|n|>N,k\neq n} \sup_{\lambda\in C_n} \left| \sum_{j,i} \frac{r(k+j)}{|\lambda-k||\lambda-j|^{1/2}} \cdot H_{ji}(\lambda) \cdot \frac{r(i+n)}{|\lambda-i|^{1/2}} \right|^2,$$

where $H_{ji}(\lambda)$ is the matrix representation of the operator $H(\lambda) = (\bar{K}_\lambda \bar{V} \bar{K}_\lambda)^{s-1}$. Therefore, by the Cauchy inequality and (2.36) in Lemma 2.2,

$$B_2(s) \leq 2 \sup_{\lambda \in C_n} \|H(\lambda)\|_{HS}^2 \cdot \tilde{\sigma} \leq 2\rho_N^{2(s-1)} \cdot \tilde{\sigma}, \tag{3.29}$$

where

$$\tilde{\sigma} = \sum_{|n|>N, k \neq n} \sup_{\lambda \in C_n} \sum_{i,j} \frac{|r(k+j)|^2 |r(i+n)|^2}{|\lambda - k|^2 |\lambda - j||\lambda - i|}.$$

From $|\lambda - n| = 1/2$ for $\lambda \in C_n$ and (2.37) it follows that

$$\tilde{\sigma} \leq 8(\tilde{\sigma}_1 + \tilde{\sigma}_2 + \tilde{\sigma}_3 + \tilde{\sigma}_4),$$

with

$$\tilde{\sigma}_1 = \sum_{|n|>N} \sum_{k \neq n} \sum_{j,i \neq n} \frac{|r(k+j)|^2 |r(i+n)|^2}{|n-k|^2 |n-j||n-i|} \leq C\|r\|^2 (\mathcal{E}_{2N}(r))^2 \leq C\|r\|^2 \rho_N^2$$

(by (4.8) in Lemma 4.2);

$$\begin{aligned}
\tilde{\sigma}_2 &= \sum_{|n|>N} \sum_{k \neq n} \sum_{j \neq n} \frac{|r(k+j)|^2 |r(2n)|^2}{|n-k|^2 |n-j|} \\
&\leq \sum_{|n|>N} |r(2n)|^2 \sum_{k \neq n} \frac{1}{|n-k|^2} \sum_j |r(k+j)|^2 \\
&\leq C\|r\|^2 (\mathcal{E}_{2N}(r))^2 \\
&\leq C\|r\|^2 \rho_N^2;
\end{aligned}$$

$$\begin{aligned}
\tilde{\sigma}_3 &= \sum_{|n|>N} \sum_{k \neq n} \sum_{i \neq n} \frac{|r(k+n)|^2 |r(n+i)|^2}{|n-k|^2 |n-i|} \\
&\leq \sum_{|n|>N} \sum_{k \neq n} \frac{|r(k+n)|^2}{|n-k|^2} \cdot \sum_i |r(n+i)|^2 \\
&\leq C\|r\|^2 \rho_N^2
\end{aligned}$$

(by (4.5) in Lemma 4.2);

$$\tilde{\sigma}_4 = \sum_{|n|>N, k \neq n} \frac{|r(k+n)|^2 |r(2n)|^2}{|n-k|^2} \leq \|r\|^2 \sum_{|n|>N, k \neq n} \frac{|r(k+n)|^2}{|n-k|^2} \leq C\|r\|^2 \rho_N^2$$

(by (4.5) in Lemma 4.2). These estimates imply the inequality $\tilde{\sigma} \leq C\|r\|^2 \rho_N^2$, which completes the proof of (3.28) for $\nu = 2, s > 1$.

Estimates for $B_3(s)$. The sums $B_3(s)$ can be estimated in a similar way because the indices k and m play symmetric roles. More precisely, since

$$B(\lambda, k, i_1, \ldots, i_s, n) = B(\lambda, n, j_1, \ldots, j_{\tau-1}, k)$$

if $j_1 = i_{s-1}, \ldots, j_{s-1} = i_1$, we have $B_3(s) = B_2(s)$. Thus, (3.28) holds for $\nu = 3$.

Estimates for $B_4(s)$. Here $s \geq 1$ by the definition of $B_4(s)$.

Fix $s \geq 1$ and consider the sum in (3.15) that defines $B_4(s)$; then at least one of the indices $j_1, \ldots, j_s$ is equal to n. Let $\tau \leq t$ be the least integer such that $j_\tau = n$. Then, by (3.8) or (3.9), and since $|\lambda - n| = 1/2$ for $\lambda \in C_n$, we have

$$B(\lambda, k, j_1, \ldots, j_{\tau-1}, n, j_{\tau+1}, \ldots, j_s, m)$$
$$= \frac{1}{2} B(\lambda, k, j_1, \ldots, j_{\tau-1}, n) \cdot B(\lambda, n, j_{\tau+1}, \ldots, j_s, m).$$

Therefore,

$$B_4(s) \leq \sum_{\tau=1}^{s} \sum_{|n|>N} \sum_{k \neq n} \sup_{\lambda \in C_n} \left| \sum_{j_1,\ldots,j_{\tau-1}} B(\lambda, k, j_1, \ldots, j_{\tau-1}, n) \right|^2$$
$$\times \sum_{m \neq n} \sup_{\lambda \in C_n} \left| \sum_{j_{\tau+1},\ldots,j_s} B(\lambda, n, j_{\tau+1}, \ldots, j_s, m) \right|^2.$$

On the other hand, by the estimate of $B_3(s)$ given by (3.28),

$$\sum_{m \neq n} \sup_{\lambda \in C_n} \left| \sum_{j_{\tau+1},\ldots,j_s} B(\lambda, n, j_{\tau+1}, \ldots, j_s, m) \right|^2 \leq C\|r\|^2 \rho_N^{2(s-\tau)}, \quad |n| > N.$$

Thus, we have

$$B_4(s) \leq C\|r\|^2 \sum_{\tau=1}^{s} \rho_N^{2(s-\tau)} \sum_{|n|>N} \sum_{k \neq n} \sup_{\lambda \in C_n} \left| \sum_{j_1,\ldots,j_{\tau-1}} B(\lambda, k, j_1, \ldots, j_{\tau-1}, n) \right|^2.$$

Now, by (3.28) for $\nu = 2$,

$$\sum_{|n|>N} \sum_{k \neq n} \sup_{\lambda \in C_n} \left| \sum_{j_1,\ldots,j_{\tau-1}} B(\lambda, k, j_1, \ldots, j_{\tau-1}, n) \right|^2 \leq C\|r\|^2 \rho_N^{2(\tau-1)}.$$

Hence,

$$B_4(s) \leq C\|r\|^4 \sum_{\tau=1}^{s} \rho_N^{2(s-1)} = Cs\|r\|^4 \rho_N^{2(s-1)},$$

which completes the proof of (3.28). □

5. Now, we can complete the proof of Theorem 3.1. Lemma 3.4, (3.21) together with the inequalities (3.28) and (3.27) in Proposition 3.5 imply that

$$A^{\alpha\beta}(s) \leq 4C\|r\|^2 \big(1 + \|r\|^2/\rho_N^2\big)(1+s)\rho_N^{2s}, \tag{3.30}$$

$$\big(A^{\alpha\beta}(s) A^{\alpha\beta}(t)\big)^{1/2} \leq 4C\|r\|^2 \big(1 + \|r\|^2/\rho_N^2\big)(1+s)(1+t)\rho_N^{s+t}. \tag{3.31}$$

With $\rho_N \leq 1/2$ by (3.27) the inequality (3.31) guarantees that the series on the right side of (3.19) converges and

$$\sum_{n>N} \big\|P_n - P_n^0\big\|^2 \leq \sum_{n>N} \big\|P_n - P_n^0\big\|_{HS}^2 \leq C_1 \|r\|^2 \big(1 + \|r\|^2/\rho_N^2\big) < \infty.$$

So, Theorem 3.1 is proven subject to Lemmas 4.1 and 4.2 in the next section. □

4 Technical lemmas

In this section we use that

$$\sum_{n>N} \frac{1}{n^2} < \sum_{n>N} \left(\frac{1}{n-1} - \frac{1}{n} \right) = \frac{1}{N}, \quad N \geq 1. \tag{4.1}$$

Lemma 4.1 *If $r = (r(k)) \in \ell^2(2\mathbb{Z})$ (or $r = (r(k)) \in \ell^2(\mathbb{Z})$), then*

$$\sum_{k\neq n} \frac{|r(n+k)|^2}{|n-k|} \leq \frac{\|r\|^2}{|n|} + (\mathcal{E}_{|n|}(r))^2, \quad |n| \geq 1; \tag{4.2}$$

$$\sum_{i,k\neq n} \frac{|r(i+k)|^2}{|n-i||n-k|} \leq 12 \left(\frac{\|r\|^2}{\sqrt{|n|}} + (\mathcal{E}_{|n|}(r))^2 \right), \quad |n| \geq 1, \tag{4.3}$$

where $n \in \mathbb{Z},\ i,k \in n + 2\mathbb{Z}$ (or, respectively, $i, k \in \mathbb{Z}$).

P r o o f. If $|n-k| \leq |n|$, then we have $|n+k| \geq 2|n| - |n-k| \geq |n|$. Therefore,

$$\sum_{k\neq n} \frac{|r(n+k)|^2}{|n-k|} \leq \sum_{0<|n-k|\leq |n|} |r(n+k)|^2 + \sum_{|n-k|>|n|} \frac{|r(n+k)|^2}{|n|} \leq (\mathcal{E}_{|n|}(r))^2 + \frac{\|r\|^2}{|n|},$$

which proves (4.2).

Next we prove (4.3). We have

$$\sum_{i,k\neq n} \frac{|r(i+k)|^2}{|n-i||n-k|} \leq \sum_{(i,k)\in J_1} + \sum_{(i,k)\in J_2} + \sum_{(i,k)\in J_3}, \tag{4.4}$$

where $J_1 = \{(i,k) :\ 0 < |n-i| \leq |n|/2,\ |n-k| \leq |n|/2\}$,

$$J_2 = \left\{ (i,k) :\ i \neq n,\ |n-k| > \frac{|n|}{2} \right\}, \quad J_3 = \left\{ (i,k) :\ |n-i| > \frac{|n|}{2},\ k \neq n \right\}.$$

For $(i,k) \in J_1$ we have $|i+k| = |2n - (n-i) - (n-k)| \geq 2|n| - |n-i| - |n-k| \geq |n|$. Therefore, by the Cauchy inequality,

$$\sum_{(i,k)\in J_1} \leq \left(\sum_{(i,k)\in J_1} \frac{|r(i+k)|^2}{|n-i|^2} \right)^{1/2} \left(\sum_{(i,k)\in J_1} \frac{|r(i+k)|^2}{|n-k|^2} \right)^{1/2} \leq 4(\mathcal{E}_{|n|}(r))^2.$$

On the other hand, again by the Cauchy inequality,

$$\begin{aligned} \sum_{(i,k)\in J_2} = \sum_{(i,k)\in J_3} &\leq \left(\sum_{(i,k)\in J_3} \frac{|r(i+k)|^2}{|n-i|^2} \right)^{1/2} \left(\sum_{(i,k)\in J_3} \frac{|r(i+k)|^2}{|n-k|^2} \right)^{1/2} \\ &\leq \left(\sum_{|n-i|>\frac{|n|}{2}} \frac{1}{|n-i|^2} \sum_k |r(i+k)|^2 \right)^{1/2} \left(\sum_{k\neq n} \frac{1}{|n-k|^2} \sum_i |r(i+k)|^2 \right)^{1/2} \\ &\leq 4 \frac{\|r\|^2}{\sqrt{|n|}}, \end{aligned}$$

which completes the proof. □

Lemma 4.2 *If $r = (r(k)) \in \ell^2(2\mathbb{Z})$ (or $r = (r(k)) \in \ell^2(\mathbb{Z})$), then*

$$\sum_{|n|>N, k\neq n} \frac{|r(n+k)|^2}{|n-k|^2} \leq C\left(\frac{\|r\|^2}{N} + (\mathcal{E}_N(r))^2\right); \tag{4.5}$$

$$\sum_{|n|>N}\sum_{i,p\neq n} \frac{|r(n+i)|^2|r(n+p)|^2}{|n-i||n-p|} \leq C\left(\frac{\|r\|^2}{N} + (\mathcal{E}_N(r))^2\right)\|r\|^2; \tag{4.6}$$

$$\sum_{|n|>N, j,p\neq n} \frac{|r(j+p)|^2}{|n-j|^2|n-p|^2} \leq C\left(\frac{\|r\|^2}{N} + (\mathcal{E}_N(r))^2\right); \tag{4.7}$$

$$\sum_{|n|>N}\sum_{i,j,p\neq n} \frac{|r(n+i)|^2|r(j+p)|^2}{|n-i||n-j||n-p|^2} \leq C\left(\frac{\|r\|^2}{N} + (\mathcal{E}_N(r))^2\right)\|r\|^2, \tag{4.8}$$

where C is an absolute constant.

Proof. With $\tilde{k} = n-k$ and $\tilde{n} = n+k$ it follows that whenever $|\tilde{k}| \leq |n|$ we have $|\tilde{n}| = |2n-\tilde{k}| \geq 2|n| - |\tilde{k}| \geq |n|$. Therefore,

$$\begin{aligned}
\sum_{|n|>N}\sum_{k\neq n} \frac{|r(n+k)|^2}{|n-k|^2} &= \sum_{|n|>N}\sum_{0<|n-k|\leq|n|} \frac{|r(n+k)|^2}{|n-k|^2} + \sum_{|n|>N}\sum_{|n-k|>|n|} \frac{|r(n+k)|^2}{|n-k|^2} \\
&\leq \sum_{|\tilde{k}|>0} \frac{1}{|\tilde{k}|^2} \sum_{|\tilde{n}|>N} |r(\tilde{n})|^2 + \sum_{|n|>N} \frac{1}{n^2} \sum_k |r(n+k)|^2 \\
&\leq C\left((\mathcal{E}_N(r))^2 + \frac{\|r\|^2}{N}\right),
\end{aligned}$$

which proves (4.5).

Since $\frac{1}{|n-i||n-p|} \leq \frac{1}{2}\left(\frac{1}{|n-i|^2} + \frac{1}{|n-p|^2}\right)$, the sum in (4.6) does not exceed

$$\frac{1}{2}\sum_{|n|>N, i\neq n} \frac{|r(n+i)|^2}{|n-i|^2} \sum_p |r(n+p)|^2 + \frac{1}{2}\sum_{|n|>N, p\neq n} \frac{|r(n+p)|^2}{|n-p|^2} \sum_i |r(n+i)|^2.$$

In view of (4.5), the latter is less than $C\left(\frac{\|r\|^2}{N} + (\mathcal{E}_N(r))^2\right)\|r\|^2$, which proves (4.6).

In order to prove (4.7), we set $\tilde{j} = n-j$ and $\tilde{p} = n-p$. Then

$$\begin{aligned}
\sum_{|n|>N; j,p\neq n} \frac{|r(j+p)|^2}{|n-j|^2|n-p|^2} &= \sum_{\tilde{j},\tilde{p}\neq 0} \frac{1}{\tilde{j}^2}\frac{1}{\tilde{p}^2} \sum_{|n|>N} |r(2n-\tilde{j}-\tilde{p})|^2 \\
&\leq \sum_{0<|\tilde{j}|,|\tilde{p}|\leq N/2} \frac{1}{\tilde{j}^2}\frac{1}{\tilde{p}^2} \sum_{n>N} |r(2n-\tilde{j}-\tilde{p})|^2 + \sum_{|\tilde{j}|>N/2}\sum_{|\tilde{p}|\neq 0} \cdots + \sum_{|\tilde{j}|\neq 0}\sum_{|\tilde{p}|>N/2} \cdots \\
&\leq C(\mathcal{E}_N(r))^2 + \frac{C}{N}\|r\|^2 + \frac{C}{N}\|r\|^2,
\end{aligned}$$

which completes the proof of (4.7).

Let σ denote the sum in (4.8). The inequality $ab \leq (a^2+b^2)/2$, considered with $a = 1/|n-i|$ and $b = 1/|n-j|$, implies that $\sigma \leq (\sigma_1 + \sigma_2)/2$, where

$$\sigma_1 = \sum_{|n|>N, i\neq n} \frac{|r(n+i)|^2}{|n-i|^2} \sum_{p\neq n} \frac{1}{|n-p|^2} \sum_j |r(j+p)|^2 \leq C\left((\mathcal{E}_N(r))^2 + \frac{\|r\|^2}{N}\right)\|r\|^2$$

(by (4.5)), and

$$\sigma_2 = \sum_{|n|>N} \sum_{j,p\neq n} \frac{|r(j+p)|^2}{|n-j|^2|n-p|^2} \sum_i |r(n+i)|^2 \leq C \left((\mathcal{E}_N(r))^2 + \frac{\|r\|^2}{N} \right) \|r\|^2$$

(by (4.7)). Thus (4.8) holds. □

5 Conclusions

1. The convergence of the series (3.1) is the analytic core of Bari–Markus Theorem (see [12], Ch. 6, Sect. 5.3, Theorem 5.2) which guarantees that the series $\sum_{|n|>N} P_n f$ converges unconditionally in L^2 for every $f \in L^2$. But in order to have the identity

$$f = S_N f + \sum_{|n|>N} P_n f,$$

we need to check the "algebraic" hypotheses in Bari–Markus Theorem:

(a) The system of projections

$$\{S_N;\;\; P_n,\;\; |n| > N\} \tag{5.1}$$

is *complete*, i.e., the linear span of the system of subspaces

$$\{E^*;\;\; E_n,\;\; |n| > N\}, \quad E^* = Ran\, S_N, \quad E_n = Ran\, P_n, \tag{5.2}$$

is dense in $L^2(I)$.

(b) The system of subspaces (5.2) is *minimal,* i.e., there is no vector in one of these subspaces that belongs to the closed linear span of all other subspaces. Condition (b) holds because the projections in (5.1) are continuous, commute and

$$P_n S_N = 0, \quad P_n P_m = 0 \quad \text{for} \quad m \neq n, \quad |m|, |n| > N.$$

The system (5.1) is complete; this fact is well-known since the early 1950's (see details in [12, 15, 16]). More general statements are proven in [19] and [25], Theorems 6.1 and 6.4 or Proposition 7.1.

Therefore, all hypotheses of Bari–Markus Theorem hold, and we have the following theorem.

Theorem 5.1 *Let L be the Dirac operator* (2.1) *with an L^2-potential v, subject to the boundary conditions $bc = Per^{\pm}$ or Dir. Then there is $N \in \mathbb{N}$ such that the Riesz projections*

$$S_N = \frac{1}{2\pi i} \int_{|z|=N-\frac{1}{2}} (z - L_{bc})^{-1}\, dz, \quad P_n = \frac{1}{2\pi i} \int_{|z-n|=\frac{1}{2}} (z - L_{bc})^{-1}\, dz$$

are well-defined, and

$$f = S_N f + \sum_{|n|>N} P_n f, \quad \forall f \in L^2;$$

moreover, this series converges unconditionally in L^2.

2. General *regular* boundary conditions for the operator L^0 (or L) (2.1)–(2.2) are given by a system of two linear equations

$$\begin{aligned} &y_1(0) + b y_1(\pi) + a y_2(0) = 0, \\ &d y_1(\pi) + c y_2(0) + y_2(\pi) = 0, \end{aligned} \tag{5.3}$$

with the restriction

$$bc - ad \neq 0. \tag{5.4}$$

A regular boundary condition is *strictly regular,* if additionally

$$(b-c)^2 + 4ad \neq 0, \tag{5.5}$$

i.e., the characteristic equation

$$z^2 + (b+c)z + (bc - ad) = 0 \tag{5.6}$$

has two *distinct* roots.

As we noticed in Introduction our main results (Theorem 5.1) can be extended to the cases of both strictly regular (SR) and regular but not strictly regular $(R \setminus SR)$ bc. More precisely, the following statements hold.

(SR) case. Let L_{bc} be an operator (2.1)–(2.2) with $(bc) \in$ (5.3)–(5.4). Then its spectrum $SP\,(L_{bc}) = \{\lambda_k,\ k \in \mathbb{Z}\}$ is discrete, $\sup |Im\, \lambda_k| < \infty,\ |\lambda_k| \to \infty$ as $k \to \pm\infty$, and all but finitely many eigenvalues λ_k are simple, $L_{bc} u_k = \lambda_k u_k,\ |k| > N = N(v)$. Put

$$S_N = \frac{1}{2\pi i} \int_C (z - L_{bc})^{-1}\, dz,$$

where the contour C is chosen so that all $\lambda_k,\ |k| \leq N$, lie inside of C, and $\lambda_k,\ |k| > N$, lie outside of C. Then the spectral decompositions

$$f = S_N f + \sum_{|k|>N} c_k(f) u_k, \quad \forall f \in L^2$$

are well–defined and *converge unconditionally* in L^2.

$(R \setminus SR)$ case. Let bc be regular, i.e., (5.3)–(5.4) hold, but not strictly regular, i.e.,

$$(b-c)^2 + 4ad = 0, \tag{5.7}$$

and $z_* = \exp(i\pi\tau)$ be a double root of (5.6).

Then its spectrum $SP\,(L_{bc}) = \{\lambda_k,\ k \in \mathbb{Z}\}$ is discrete; it lies in $\Pi_N \cup \bigcup_{m>N} D_m,\ N = N(v)$, where

$$\Pi_N = \{z \in \mathbb{C} :\ |Im\,(z-\tau)|, |Re\,(z-\tau)| < N - 1/2\}$$

and $D_m = \{z \in \mathbb{C} :\ |(z - m - \tau)| < 1/2\}$. The spectral decompositions

$$f = S_N f + \sum_{|m|>N} P_m f, \quad \forall f \in L^2$$

are well–defined if we set

$$S_N = \frac{1}{2\pi i} \int_{\partial \Pi_N} (z - L_{bc})^{-1}\, dz, \quad P_m = \frac{1}{2\pi i} \int_{\partial D_m} (z - L_{bc})^{-1}\, dz, \quad |m| > N,$$

and they *converge unconditionally* in L^2.

Complete presentation and proofs of these general results will be given elsewhere.

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