

# FOURIER METHOD FOR ONE DIMENSIONAL SCHRÖDINGER OPERATORS WITH SINGULAR PERIODIC POTENTIALS

PLAMEN DJAKOV AND BORIS MITYAGIN

ABSTRACT. By using quasi-derivatives, we develop a Fourier method for studying the spectral properties of one dimensional Schrödinger operators with periodic singular potentials.

## 1. INTRODUCTION

Our goal in this paper is to develop a Fourier method for studying the spectral properties (in particular, spectral gap asymptotics) of the Schrödinger operator

$$(1.1) \quad L(v)y = -y'' + v(x)y, \quad x \in \mathbb{R},$$

where  $v$  is a singular potential such that

$$(1.2) \quad v(x) = v(x + \pi), \quad v \in H_{loc}^{-1}(\mathbb{R}).$$

In the case where the potential  $v$  is a real  $L^2([0, \pi])$ -function, it is well known by the Floquet–Lyapunov theory (see [5, 19, 20, 32]), that the spectrum of  $L$  is absolutely continuous and has a band–gap structure, i.e., it is a union of closed intervals separated by *spectral gaps*

$$(-\infty, \lambda_0), (\lambda_1^-, \lambda_1^+), (\lambda_2^-, \lambda_2^+), \dots, (\lambda_n^-, \lambda_n^+), \dots.$$

The points  $(\lambda_n^\pm)$  are defined by the spectra of (1.1) considered on the interval  $[0, \pi]$ , respectively, with periodic (for even  $n$ ) and anti-periodic (for odd  $n$ ) boundary conditions (*bc*):

- (a) periodic  $Per^+ : y(\pi) = y(0), y'(\pi) = y'(0);$
- (b) antiperiodic  $Per^- : y(\pi) = -y(0), y'(\pi) = -y'(0);$

So, one may consider the appropriate bases in  $L^2([0, \pi])$ , which leads to a transformation of the periodic or anti-periodic Hill–Schrödinger operator into an operator acting in an  $\ell^2$ -sequence space. This makes possible to develop a Fourier method for investigation of spectra, and especially, spectral gap asymptotics (see [14, 15], where the method has been used to estimate the gap asymptotics in terms of potential smoothness). Our papers [2, 3] (see also the survey [4]) give further development of that approach and provide a detailed analysis of (and extensive bibliography on) the intimate relationship between the smoothness of the potential  $v$  and the decay rate of the corresponding spectral gaps (and deviations of Dirichlet eigenvalues) under the assumption  $v \in L^2([0, \pi])$ .

But now singular potentials  $v \in H^{-1}$  bring a lot of new technical problems even in the framework of the same basic scheme as in [4].

First of them is to give proper understanding of the boundary conditions (a) and (b) or their broader interpretation and careful definition of the corresponding operators and their domains. This is done by using quasi-derivatives. To a great extent we follow the approach suggested (in the context of second order o.d.e.) and developed by A. Savchuk and A. Shkalikov [25, 27] (see also [26, 28, 29]) and R. Hryniv and Ya. Mykytyuk [8] (see also [9]-[13]). For specific potentials see W. N. Everitt and A. Zettl [6, 7].

E. Korotyaev [17, 18] follows a different approach but it works only in the case of a *real* potential  $v$ .

It is known (e.g., see [8], Remark 2.3, or Proposition 1 below) that every  $\pi$ -periodic potential  $v \in H_{loc}^{-1}(\mathbb{R})$  has the form

$$v = C + Q', \quad \text{where } C = \text{const}, \quad Q \text{ is } \pi\text{-periodic}, \quad Q \in L_{loc}^2(\mathbb{R}).$$

Therefore, one may introduce the “quasi-derivative“  $u = y' - Qy$  and replace the distribution equation  $-y'' + vy = 0$  by the following system of two linear equations with coefficients in  $L_{loc}^1(\mathbb{R})$

$$(1.3) \quad y' = Qy + u, \quad u' = (C - Q^2)y - Qu.$$

By the Existence–Uniqueness theorem for systems of linear o.d.e. with  $L_{loc}^1(\mathbb{R})$ -coefficients (e.g., see [1, 22]), the Cauchy initial value problem for the system (1.3) has, for each pair of numbers  $(a, b)$ , a unique solution  $(y, u)$  such that  $y(0) = a$ ,  $u(0) = b$ .

Moreover, following A. Savchuk and A. Shkalikov [25, 27], one may consider various boundary value problems on the interval  $[0, \pi]$ . In particular, let us consider the periodic or anti-periodic boundary conditions  $Per^\pm$ , where

$$\begin{aligned} (a^*) \quad Per^+ : \quad & y(\pi) = y(0), \quad (y' - Qy)(\pi) = (y' - Qy)(0). \\ (b^*) \quad Per^- : \quad & y(\pi) = -y(0), \quad (y' - Qy)(\pi) = -(y' - Qy)(0). \end{aligned}$$

R. Hryniv and Ya. Mykytyuk [8] used also the system (1.3) in order to give complete analysis of the spectra of the Schrödinger operator with real-valued periodic  $H^{-1}$ -potentials. They showed, that as in the case of periodic  $L_{loc}^2(\mathbb{R})$ -potentials, the Floquet theory for the system (1.3) could be used to explain that if  $v$  is real-valued, then  $L(v)$  is a self-adjoint operator having absolutely continuous spectrum with band-gap structure, and the spectral gaps are determined by the spectra of the corresponding Hill–Schrödinger operators  $L_{Per^\pm}$  defined in the appropriate domains of  $L^2([0, \pi])$ -functions, and considered, respectively, with the boundary conditions  $(a^*)$  and  $(b^*)$ .

In Section 2 we use the same quasi-derivative approach to define the domains of the operators  $L(v)$  for complex-valued potentials  $v$ , and to explain how their spectra are described in terms of the corresponding Lyapunov function. From a technical point of view, our approach is different from the approach of R. Hryniv and Ya. Mykytyuk [8]: they consider only the self-adjoint case and use a quadratic form to define the domain of  $L(v)$ , while we consider the non-self-adjoint case as well and use the Floquet theory and the resolvent method (see Lemma 3 and Theorem 4).

Sections 3 and 4 contains the core results of this paper. In Section 3 we define and study the operators  $L_{Per\pm}$  which arise when considering the Hill–Schrödinger operator  $L(v)$  with the adjusted boundary conditions  $(a^*)$  and  $(b^*)$ . We meticulously explain what is the Fourier representation of these operators<sup>1</sup> in Proposition 10 and Theorem 11.

In Section 4 we use the same approach as in Section 3 to define and study the Hill–Schrödinger operator  $L_{Dir}(v)$  with Dirichlet boundary conditions  $Dir : y(0) = y(\pi) = 0$ . Our main result there is Theorem 16 which gives the Fourier representation of the operator  $L_{Dir}(v)$ .

In Section 5 we use the Fourier representations of the operators  $L_{Per\pm}$  and  $L_{Dir}$  to study the localization of their spectra (see Theorem 5.1). Of course, Theorem 5.1 gives also a rough asymptotics of the eigenvalues  $\lambda_n^+, \lambda_n^-, \mu_n$  of these operators. But we are interested to find the asymptotics of spectral gaps  $\gamma_n = \lambda_n^+ - \lambda_n^-$  in the self-adjoint case, or the asymptotics of both  $\gamma_n$  and the deviations  $\mu_n - (\lambda_n^+ + \lambda_n^-)/2$  in the non-self-adjoint case, etc. Our results in that direction are presented without proofs in Section 6.

*Acknowledgment.* The authors thank Professors Rostyslav Hryniv, Andrei Shkalikov and Vadim Tkachenko for very useful discussions of many questions of spectral analysis of differential operators, both related and unrelated to the main topics of this paper.

## 2. PRELIMINARY RESULTS

1. The operator 1.1 has a second term  $vy$  with  $v \in (1.2)$ . First of all, let us specify the structure of periodic functions and distributions in  $H_{loc}^1(\mathbb{R})$  and  $H_{loc}^{-1}(\mathbb{R})$ .

The Sobolev space  $H_{loc}^1(\mathbb{R})$  is defined as the space of functions  $f(x) \in L_{loc}^2(\mathbb{R})$  which are absolutely continuous and have their derivatives  $f'(x) \in L_{loc}^2(\mathbb{R})$ . Therefore, for every  $T > 0$ ,

$$(2.1) \quad \|f\|_{1,T}^2 = \int_{-T}^T (|f(x)|^2 + |f'(x)|^2) < \infty.$$

Let  $\mathcal{D}(\mathbb{R})$  be the space of all  $C^\infty$ -functions on  $\mathbb{R}$  with compact support, and let  $\mathcal{D}([-T, T])$  be the subset of all  $\varphi \in \mathcal{D}(\mathbb{R})$  with  $supp \varphi \subset [-T, T]$ .

By definition,  $H_{loc}^{-1}(\mathbb{R})$  is the space of distributions  $v$  on  $\mathbb{R}$  such that

$$(2.2) \quad \forall T > 0 \exists C(T) : |\langle v, \varphi \rangle| \leq C(T) \|\varphi\|_{1,T} \quad \forall \varphi \in \mathcal{D}([-T, T]).$$

---

<sup>1</sup>Maybe it is worth to mention that T. Kappeler and C. Möhr [16] analyze "periodic and Dirichlet eigenvalues of Schrödinger operators with singular potential" but they never tell how these operators (or boundary conditions) are defined on the interval, i.e., in a Hilbert space  $L^2([0, \pi])$ . At some point they jump without any justification or explanation into weighted  $\ell^2$ -sequence spaces (an analog of Sobolev spaces  $H^a$ ) and consider the same sequence space operators we are used to in the regular case, i.e., if  $v \in L_{per}^2(\mathbb{R})$ . But without formulating which Sturm–Liouville problem is considered, what are the corresponding boundary conditions, what is the domain of the operator, etc., it is not possible to pass from a *non-defined* differential operator to its Fourier representation.

Of course, since

$$\int_{-T}^T |\varphi(x)|^2 dx \leq T^2 \int_{-T}^T |\varphi'(x)|^2 dx,$$

the condition (2.2) is equivalent to

$$(2.3) \quad \forall T > 0 \quad \exists C(T) : \quad |\langle v, \varphi \rangle| \leq \tilde{C}(T) \|\varphi'\|_{L^2([-T, T])} \quad \forall \varphi \in \mathcal{D}([-T, T]).$$

Set

$$(2.4) \quad \mathcal{D}_1(\mathbb{R}) = \{\varphi' : \varphi \in \mathcal{D}(\mathbb{R})\}, \quad \mathcal{D}_1([-T, T]) = \{\varphi' : \varphi \in \mathcal{D}([-T, T])\}$$

and consider the linear functional

$$(2.5) \quad q(\varphi') := -\langle v, \varphi \rangle, \quad \varphi' \in \mathcal{D}_1(\mathbb{R}).$$

In view of (2.3), for each  $T > 0$ ,  $q(\cdot)$  is a continuous linear functional defined in the space  $\mathcal{D}_1([-T, T]) \subset L^2([-T, T])$ . By Riesz Representation Theorem there exists a function  $Q_T(x) \in L^2([-T, T])$  such that

$$(2.6) \quad q(\varphi') = \int_{-T}^T Q_T(x) \varphi'(x) dx \quad \forall \varphi' \in \mathcal{D}_1([-T, T]).$$

The function  $Q_T$  is uniquely determined up to an additive constant because in  $L^2([-T, T])$  only constants are orthogonal to  $\mathcal{D}_1([-T, T])$ . Therefore, one can readily see that there is a function  $Q(x) \in L^2_{loc}(\mathbb{R})$  such that

$$q(\varphi') = \int_{-\infty}^{\infty} Q(x) \varphi'(x) dx \quad \forall \varphi' \in \mathcal{D}(\mathbb{R}),$$

where the function  $Q$  is uniquely determined up to an additive constant. Thus, we have

$$\langle v, \varphi \rangle = -q(\varphi') = -\langle Q, \varphi' \rangle = \langle Q', \varphi \rangle,$$

i.e.,

$$(2.7) \quad v = Q'.$$

A distribution  $v \in H^{-1}_{loc}(\mathbb{R})$  is called *periodic of period  $\pi$*  if

$$(2.8) \quad \langle v, \varphi(x) \rangle = \langle v, \varphi(x - \pi) \rangle \quad \forall \varphi \in \mathcal{D}(\mathbb{R}).$$

L. Schwartz [30] gave an equivalent definition of a *periodic of period  $\pi$*  distribution in the following way: Let

$$\omega : \mathbb{R} \rightarrow S^1 = \mathbb{R}/\pi\mathbb{Z}, \quad \omega(x) = x \pmod{\pi}.$$

A distribution  $F \in \mathcal{D}'(\mathbb{R})$  is *periodic* if, for some  $f \in (C^\infty(S^1))'$ , we have

$$F(x) = f(\omega(x)), \quad \text{i.e.,} \quad \langle \varphi, F \rangle = \langle \Phi, f \rangle,$$

where

$$\Phi = \sum_{k \in \mathbb{Z}} \varphi(x - k\pi).$$

Now, if  $v$  is periodic and  $Q \in L^2_{loc}(\mathbb{R})$  is chosen so that (2.7) holds, we have by (2.8)

$$\int_{-\infty}^{\infty} Q(x + \pi)\varphi'(x)dx = \int_{-\infty}^{\infty} Q(x)\varphi'(x - \pi) = \int_{-\infty}^{\infty} Q(x)\varphi'(x)dx,$$

i.e.,

$$\int_{-\infty}^{\infty} [Q(x + \pi) - Q(x)]\varphi'(x)dx = 0 \quad \forall \varphi \in \mathcal{D}(\mathbb{R}).$$

Thus, there exists a constant  $c$  such that

$$Q(x + \pi) - Q(x) = c \quad a.e.$$

Consider the function

$$\tilde{Q}(x) = Q(x) - \frac{c}{\pi}x;$$

then we have  $\tilde{Q}(x + \pi) = \tilde{Q}(x)$  a.e., so  $\tilde{Q}$  is  $\pi$ -periodic, and

$$v = \tilde{Q}' + \frac{c}{\pi}.$$

Let

$$(2.9) \quad \tilde{Q}(x) = \sum_{m \in 2\mathbb{Z}} q(m)e^{imx}$$

be the Fourier series expansion of the function  $\tilde{Q} \in L^2([0, \pi])$ . Set

$$(2.10) \quad V(0) = \frac{c}{\pi}, \quad V(m) = imq(m) \quad \text{for } m \neq 0.$$

All this leads to the following statement.

**Proposition 1.** *Every  $\pi$ -periodic distribution  $v \in H^{-1}_{loc}(\mathbb{R})$  has the form*

$$(2.11) \quad v = C + Q', \quad Q \in L^2_{loc}(\mathbb{R}), \quad Q(x + \pi) \stackrel{a.e.}{=} Q(x)$$

with

$$(2.12) \quad q(0) = \frac{1}{\pi} \int_0^\pi Q(x)dx = 0,$$

and can be written as a converging in  $H^{-1}_{loc}(\mathbb{R})$  Fourier series

$$(2.13) \quad v = \sum_{m \in 2\mathbb{Z}} V(m)e^{imx}$$

with

$$(2.14) \quad V(0) = C, \quad V(m) = imq(m) \quad \text{for } m \neq 0,$$

where  $q(m)$  are the Fourier coefficients of  $Q$ . Of course,

$$(2.15) \quad \|Q\|_{L^2([0, \pi])}^2 = \sum_{m \neq 0} \frac{|V(m)|^2}{m^2}.$$

*Remark.* R. Hryniv and Ya. Mykytyuk [8], (see Theorem 3.1 and Remark 2.3) give a more general claim about the structure of uniformly bounded  $H_{loc}^{-1}(\mathbb{R})$ -distributions.

2. In view of (2.2), each distribution  $v \in H_{loc}^{-1}(\mathbb{R})$  could be considered as a linear functional on the space  $H_{oo}^1(\mathbb{R})$  of functions in  $H_{loc}^1(\mathbb{R})$  with compact support. Therefore, if  $v \in H_{loc}^{-1}(\mathbb{R})$  and  $y \in H_{loc}^1(\mathbb{R})$ , then the differential expression  $\ell(y) = -y'' + v \cdot y$  is well-defined by

$$\langle -y'' + v \cdot y, \varphi \rangle = \langle y', \varphi' \rangle + \langle v, y \cdot \varphi \rangle$$

as a distribution in  $H_{loc}^{-1}(\mathbb{R})$ . This observation suggests to consider the Schrödinger operator  $-d^2/dx^2 + v$  in the domain

$$(2.16) \quad D(L(v)) = \{y \in H_{loc}^1(\mathbb{R}) \cap L^2(\mathbb{R}) : -y'' + v \cdot y \in L^2(\mathbb{R})\}.$$

Moreover, suppose  $v = C + Q'$ , where  $C$  is a constant and  $Q$  is a  $\pi$ -periodic function such that

$$(2.17) \quad Q \in L^2([0, \pi]), \quad q(0) = \frac{1}{\pi} \int_0^\pi Q(x) dx = 0.$$

Then the differential expression  $\ell(y) = -y'' + vy$  can be written in the form

$$(2.18) \quad \ell(y) = -(y' - Qy)' - Qy' + Cy.$$

Notice that

$$\ell(y) = -(y' - Qy)' - Qy' + Cy = f \in L^2(\mathbb{R})$$

if and only if

$$u = y' - Qy \in W_{1,loc}^1(\mathbb{R})$$

and the pair  $(y, u)$  satisfies the system of differential equations

$$(2.19) \quad \begin{cases} y' = Qy + u, \\ u' = (C - Q^2)y - Qu + f. \end{cases}$$

Consider the corresponding homogeneous system

$$(2.20) \quad \begin{cases} y' = Qy + u, \\ u' = (C - Q^2)y - Qu. \end{cases}$$

with initial data

$$(2.21) \quad y(0) = a, \quad u(0) = b.$$

Since the coefficients  $1, Q, C - Q^2$  of the system (2.20) are in  $L_{loc}^1(\mathbb{R})$ , the standard existence–uniqueness theorem for linear systems of equations with  $L_{loc}^1(\mathbb{R})$ -coefficients (e.g., see M. Naimark [22], Sect.16, or F. Atkinson [1]) guarantees that for any pair of numbers  $(a, b)$  the system (2.20) has a unique solution  $(y, u)$  with  $y, u \in W_{1,loc}^1(\mathbb{R})$  such that (2.21) holds.

On the other hand, the coefficients of the system (2.20) are  $\pi$ -periodic, so one may apply the classical Floquet theory.

Let  $(y_1, u_1)$  and  $(y_2, u_2)$  be the solutions of (2.20) which satisfy  $y_1(0) = 1, u_1(0) = 0$  and  $y_2(0) = 0, u_2(0) = 1$ . By the Caley–Hamilton theorem the Wronskian

$$\det \begin{pmatrix} y_1(x) & y_2(x) \\ u_1(x) & u_2(x) \end{pmatrix} \equiv 1$$

because the trace of the coefficient matrix of the system (2.20) is zero.

If  $(y(x), u(x))$  is a solution of (2.20) with initial data  $(a, b)$ , then  $(y(x+\pi), u(x+\pi))$  is a solution also, correspondingly with initial data

$$\begin{pmatrix} y(\pi) \\ u(\pi) \end{pmatrix} = M \begin{pmatrix} a \\ b \end{pmatrix}, \quad M = \begin{pmatrix} y_1(\pi) & y_2(\pi) \\ u_1(\pi) & u_2(\pi) \end{pmatrix}.$$

Consider the characteristic equation of the *monodromy matrix*  $M$  :

$$(2.22) \quad \rho^2 - \Delta\rho + 1 = 0, \quad \Delta = y_1(\pi) + u_2(\pi).$$

Each root  $\rho$  of the characteristic equation (2.22) gives a rise of a special solution  $(\varphi(x), \psi(x))$  of (2.20) such that

$$(2.23) \quad \varphi(x + \pi) = \rho \cdot \varphi(x), \quad \psi(x + \pi) = \rho \cdot \psi(x).$$

Since the product of the roots of (2.22) equals 1, the roots have the form

$$(2.24) \quad \rho^\pm = e^{\pm\tau\pi}, \quad \tau = \alpha + i\beta,$$

where  $\beta \in [0, 2]$  and  $\alpha = 0$  if the roots are on the unit circle or  $\alpha > 0$  otherwise.

In the case where the equation (2.22) has two distinct roots, let  $(\varphi^\pm, \psi^\pm)$  be special solutions of (2.20) that correspond to the roots (2.24), i.e.,

$$(\varphi^\pm(x + \pi), \psi^\pm(x + \pi)) = \rho^\pm \cdot (\varphi^\pm(x), \psi^\pm(x)).$$

Then one can readily see that the functions

$$\tilde{\varphi}^\pm(x) = e^{\mp\tau x} \varphi^\pm(x), \quad \tilde{\psi}^\pm(x) = e^{\mp\tau x} \psi^\pm(x)$$

are  $\pi$ -periodic, and we have

$$(2.25) \quad \varphi^\pm(x) = e^{\pm\tau x} \tilde{\varphi}^\pm(x), \quad \psi^\pm(x) = e^{\pm\tau x} \tilde{\psi}^\pm(x).$$

Consider the case where (2.22) has a double root  $\rho = \pm 1$ . If its geometric multiplicity equals 2 (i.e., the matrix  $M$  has two linearly independent eigenvectors), then the equation (2.20) has, respectively, two linearly independent solutions  $(\varphi^\pm, \psi^\pm)$  which are periodic if  $\rho = 1$  or anti-periodic if  $\rho = -1$ .

Otherwise, (if  $M$  is a Jordan matrix), there are two linearly independent vectors  $\begin{pmatrix} a^+ \\ b^+ \end{pmatrix}$  and  $\begin{pmatrix} a^- \\ b^- \end{pmatrix}$  such that

$$(2.26) \quad M \begin{pmatrix} a^+ \\ b^+ \end{pmatrix} = \rho \begin{pmatrix} a^+ \\ b^+ \end{pmatrix}, \quad M \begin{pmatrix} a^- \\ b^- \end{pmatrix} = \rho \begin{pmatrix} a^- \\ b^- \end{pmatrix} + \rho\kappa \begin{pmatrix} a^+ \\ b^+ \end{pmatrix}, \quad \rho = \pm 1, \quad \kappa \neq 0.$$

Let  $(\varphi^\pm, \psi^\pm)$  be the corresponding solutions of (2.20). Then we have

$$(2.27) \quad \begin{pmatrix} \varphi^+(x + \pi) \\ \psi^+(x + \pi) \end{pmatrix} = \rho \begin{pmatrix} \varphi^+(x) \\ \psi^+(x) \end{pmatrix}, \quad \begin{pmatrix} \varphi^-(x + \pi) \\ \psi^-(x + \pi) \end{pmatrix} = \rho \begin{pmatrix} \varphi^-(x) \\ \psi^-(x) \end{pmatrix} + \rho\kappa \begin{pmatrix} \varphi^+(x) \\ \psi^+(x) \end{pmatrix}.$$

Now, one can easily see that the functions  $\tilde{\varphi}^-$  and  $\tilde{\psi}^-$  given by

$$\begin{pmatrix} \tilde{\varphi}^-(x) \\ \tilde{\psi}^-(x) \end{pmatrix} = \begin{pmatrix} \varphi^-(x) \\ \psi^-(x) \end{pmatrix} - \frac{\kappa x}{\pi} \begin{pmatrix} \varphi^+(x) \\ \psi^+(x) \end{pmatrix}$$

are  $\pi$ -periodic (if  $\rho = 1$ ) or anti-periodic (if  $\rho = -1$ ). Therefore, the solution  $\begin{pmatrix} \varphi^-(x) \\ \psi^-(x) \end{pmatrix}$  can be written in the form

$$(2.28) \quad \begin{pmatrix} \varphi^-(x) \\ \psi^-(x) \end{pmatrix} = \begin{pmatrix} \tilde{\varphi}^-(x) \\ \tilde{\psi}^-(x) \end{pmatrix} + \frac{\kappa x}{\pi} \begin{pmatrix} \varphi^+(x) \\ \psi^+(x) \end{pmatrix},$$

i.e., it is a linear combination of periodic (if  $\rho = 1$ ), or anti-periodic (if  $\rho = -1$ ) functions with coefficients 1 and  $\kappa x/\pi$ .

The following lemma shows how the properties of the solutions of (2.19) and (2.20) depend on the roots of the characteristic equation (2.22).

**Lemma 2.** (a) *The homogeneous system (2.20) has no nonzero solution  $(y, u)$  with  $y \in L^2(\mathbb{R})$ . Moreover, if the roots of the characteristic equation (2.22) lie on the unit circle, i.e.,  $\alpha = 0$  in the representation (2.24), then (2.20) has no nonzero solution  $(y, u)$  with  $y \in L^2((-\infty, 0])$  or  $y \in L^2([0, +\infty))$ .*

(b) *If  $\alpha = 0$  in the representation (2.24), then there are functions  $f \in L^2(\mathbb{R})$  such that the corresponding non-homogeneous system (2.19) has no solution  $(y, u)$  with  $y \in L^2(\mathbb{R})$ .*

(c) *If the roots of the characteristic equation (2.22) lie outside the unit circle, i.e.,  $\alpha > 0$  in the representation (2.24), then the non-homogeneous system (2.19) has, for each  $f \in L^2(\mathbb{R})$ , a unique solution  $(y, u) = (R_1(f), R_2(f))$  such that  $R_1$  is a linear continuous operator from  $L^2(\mathbb{R})$  into  $W_2^1(\mathbb{R})$ , and  $R_2$  is a linear continuous operator in  $L^2(\mathbb{R})$  with a range in  $W_{1,loc}^1(\mathbb{R})$ .*

*Proof.* (a) In view of the above discussion (see the text from (2.22) to (2.28)), if the characteristic equation (2.22) has two distinct roots  $\rho = e^{\pm\tau\pi}$ , then each solution  $(y, u)$  of the homogeneous system (2.20) is a linear combination of the special solutions, so

$$y(x) = C^+ e^{\tau x} \tilde{\varphi}^+(x) + C^- e^{-\tau x} \tilde{\varphi}^-(x),$$

where  $\tilde{\varphi}^+$  and  $\tilde{\varphi}^-$  are  $\pi$ -periodic functions in  $H^1$ .

In the case where the real part of  $\tau$  is strictly positive, i.e.,  $\tau = \alpha + i\beta$  with  $\alpha > 0$ , one can readily see that  $e^{\tau x} \tilde{\varphi}^+(x) \notin L^2([0, \infty))$  but  $e^{\tau x} \tilde{\varphi}^+(x) \in L^2((-\infty, 0])$ , while  $e^{-\tau x} \tilde{\varphi}^-(x) \in L^2([0, \infty))$  but  $e^{-\tau x} \tilde{\varphi}^-(x) \notin L^2((-\infty, 0])$ . Therefore, if  $y \neq 0$  we have  $y \notin L^2(\mathbb{R})$ .

Next we consider the case where  $\tau = i\beta$  with  $\beta \neq 0, 1$ . The Fourier series of the functions  $\tilde{\varphi}^+(x)$  and  $\tilde{\varphi}^-(x)$

$$\tilde{\varphi}^+ \sim \sum_{k \in 2\mathbb{Z}} \tilde{\varphi}_k^+ e^{ikx}, \quad \tilde{\varphi}^- \sim \sum_{k \in 2\mathbb{Z}} \tilde{\varphi}_k^- e^{ikx}$$

converge uniformly in  $\mathbb{R}$  because  $\tilde{\varphi}^+, \tilde{\varphi}^- \in H^1$ . Therefore, we have

$$y(x) = C^+ \sum_{k \in 2\mathbb{Z}} \tilde{\varphi}_k^+ e^{i(k+\beta)x} + C^- \sum_{k \in 2\mathbb{Z}} \tilde{\varphi}_k^- e^{i(k-\beta)x},$$

where the series on the right converge uniformly on  $\mathbb{R}$ . If  $\beta$  is a rational number, then  $y$  is a periodic function, so  $y \notin L^2((-\infty, 0])$  and  $y \notin L^2([0, \infty))$ .

If  $\beta$  is an irrational number, then

$$(2.29) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T y(x) e^{-i(k \pm \beta)x} dx = C^\pm \tilde{\varphi}_k^\pm \quad \forall k \in 2\mathbb{Z}.$$

On the other hand, if  $y \in L^2([0, \infty))$ , then the Cauchy inequality implies

$$\left| \frac{1}{T} \int_0^T y(x) e^{-i(k \pm \beta)x} dx \right| \leq \frac{1}{T} \left( \int_0^T 1 \cdot dx \right)^{1/2} \left( \int_0^T |y(x)|^2 dx \right)^{1/2} \leq \frac{\|y\|_{L^2([0, \infty))}}{\sqrt{T}} \rightarrow 0.$$

But, in view of (2.29), this is impossible if  $y \neq 0$ . Thus  $y \notin L^2([0, \infty))$ . In a similar way, one can see that  $y \notin L^2((-\infty, 0])$ .

Finally, if the characteristic equation (2.22) has a double root  $\rho = \pm 1$ , then either every solution  $(y, u)$  of (2.20) is periodic or anti-periodic, and so  $y \notin L^2([0, \infty))$  and  $y \notin L^2((-\infty, 0])$ , or it is a linear combination of some special solutions (see (2.28), and the preceding discussion), so we have

$$y(x) = C^+ \varphi^+(x) + C^- \tilde{\varphi}^- + C^- \frac{\kappa x}{\pi} \varphi^+(x),$$

where the functions  $\varphi^+$  and  $\tilde{\varphi}^-$  are periodic or anti-periodic. Now one can easily see that  $y \notin L^2([0, \infty))$  and  $y \notin L^2((-\infty, 0])$ , which completes the proof of (a).

(b) Let  $(\varphi^\pm, \psi^\pm)$  be special solutions of (2.20) that correspond to the roots (2.24) as above. We may assume without loss of generalities that the Wronskian of the solutions  $(\varphi^+, \psi^+)$  and  $(\varphi^-, \psi^-)$  equals 1 because these solutions are determined up to constant multipliers.

The standard method of variation of constants leads to the following solution  $(y, u)$  of the non-homogeneous system (2.19):

$$(2.30) \quad y = v^+(x) \varphi^+(x) + v^-(x) \varphi^-(x), \quad u = v^+(x) \psi^+(x) + v^-(x) \psi^-(x),$$

where  $v^+$  and  $v^-$  satisfy

$$(2.31) \quad \frac{dv^+}{dx} \cdot \varphi^+ + \frac{dv^-}{dx} \cdot \varphi^- = 0, \quad \frac{dv^+}{dx} \cdot \psi^+ + \frac{dv^-}{dx} \cdot \psi^- = f,$$

so

$$(2.32) \quad v^+(x) = - \int_0^x \varphi^-(t) f(t) dt + C^+, \quad v^-(x) = \int_0^x \varphi^+(t) f(t) dt + C^-.$$

Assume that the characteristic equation (2.22) has roots of the form  $\rho = e^{i\beta\pi}$ ,  $\beta \in [0, 2)$ . Take any function  $f \in L^2(\mathbb{R})$  with compact support, say  $\text{supp } f \subset (0, T)$ . By (2.30) and (2.32), if  $(y, u)$  is a solution of the non-homogeneous system (2.19), then the restriction of  $(y, u)$  on the intervals  $(-\infty, 0)$  and  $[T, \infty)$  is a solution of the homogeneous system (2.20). So, by (a), if  $y \in L^2(\mathbb{R})$  then  $y \equiv 0$  on the intervals  $(-\infty, 0)$  and  $[T, \infty)$ . This may happen if only if the constants  $C^\pm$  in (2.32) are zeros, and we have

$$\int_0^T \varphi^-(t) f(t) dt = 0, \quad \int_0^T \varphi^+(t) f(t) dt = 0.$$

Hence, if  $f$  is not orthogonal to the functions  $\varphi^\pm$ , on the interval  $[0, T]$ , then the non-homogeneous system (2.19) has no solution  $(y, u)$  with  $y \in L^2(\mathbb{R})$ . This completes the proof of (b).

(c) Now we consider the case where the characteristic equation (2.22) has roots of the form (2.24) with  $\alpha > 0$ . Let  $(\varphi^\pm, \psi^\pm)$  be the corresponding special solutions. By (2.30), for each  $f \in L^2(\mathbb{R})$ , the non-homogeneous system (2.19) has a solution of the form  $(y, u) = (R_1(f), R_2(f))$ , where

$$(2.33) \quad R_1(f) = v^+(x)\varphi^+(x) + v^-(x)\varphi^-(x), \quad R_2(f) = v^+(x)\psi^+(x) + v^-(x)\psi^-(x),$$

and (2.31) holds. In order to have a solution that vanishes at  $\pm\infty$  we set (taking into account (2.25))

$$(2.34) \quad v^+(x) = \int_x^\infty e^{-\tau t} \tilde{\varphi}^-(t) f(t) dt, \quad v^-(x) = \int_{-\infty}^x e^{\tau t} \tilde{\varphi}^+(t) f(t) dt.$$

Let  $C_\pm = \max\{|\tilde{\varphi}^\pm(x)| : x \in [0, \pi]\}$ . By (2.25), we have

$$(2.35) \quad |\varphi^\pm(x)| \leq C_\pm \cdot e^{\pm\alpha x}.$$

Therefore, by the Cauchy inequality, we get

$$|v^+(x)|^2 \leq C_-^2 \left| \int_x^\infty e^{-\alpha t} |f(t)| dt \right|^2 \leq C_-^2 \left( \int_x^\infty e^{-\alpha t} dt \right) \cdot \left( \int_x^\infty e^{-\alpha t} |f(t)|^2 dt \right),$$

so

$$(2.36) \quad |v^+(x)|^2 \leq \frac{C_-^2}{\alpha} e^{-\alpha x} \int_x^\infty e^{-\alpha t} |f(t)|^2 dt.$$

Thus, by (2.35),

$$\begin{aligned} \int_{-\infty}^\infty |v^+(x)|^2 |\varphi^+(x)|^2 dx &\leq \frac{C_-^2 C_+^2}{\alpha} \int_{-\infty}^\infty e^{\alpha x} \int_x^\infty e^{-\alpha t} |f(t)|^2 dt dx \\ &\leq \frac{C_-^2 C_+^2}{\alpha} \int_{-\infty}^\infty |f(t)|^2 \left( \int_{-\infty}^t e^{\alpha(x-t)} dx \right) dt = \frac{C_-^2 C_+^2}{\alpha^2} \|f\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

In an analogous way one may prove that

$$\int_{-\infty}^\infty |v^-(x)|^2 |\varphi^-(x)|^2 dx \leq \frac{C_-^2 C_+^2}{\alpha^2} \|f\|_{L^2(\mathbb{R})}^2.$$

In view of (2.30), these estimates prove that  $R_1$  is a continuous operator in  $L^2(\mathbb{R})$ .

Next we estimate the  $L^2(\mathbb{R})$ -norm of  $y' = \frac{d}{dx} R_1(f)$ . In view of (2.31), we have

$$y'(x) = v^+(x) \cdot \frac{d\varphi^+}{dx}(x) + v^-(x) \cdot \frac{d\varphi^-}{dx}(x).$$

By (2.25),

$$v^+(x) \cdot \frac{d\varphi^+}{dx}(x) = \alpha v^+(x) \varphi^+ + v^+(x) e^{\alpha x} \frac{d\tilde{\varphi}^+}{dx}.$$

Since the  $L^2(\mathbb{R})$ -norm of  $v^+(x)\varphi^+$  has been estimated above, we need to estimate only the  $L^2(\mathbb{R})$ -norm of  $v^+(x)e^{\alpha x}d\tilde{\varphi}^+/dx$ . By (2.36), we have

$$\begin{aligned} \int_{-\infty}^{\infty} |v^+(x)e^{\alpha x}d\tilde{\varphi}^+/dx|^2 dx &\leq \frac{C_-^2}{\alpha} \int_{-\infty}^{\infty} |d\tilde{\varphi}^+/dx|^2 e^{\alpha x} \int_x^{\infty} e^{-\alpha t} |f(t)|^2 dt dx \\ &= \frac{C_-^2}{\alpha} \int_{-\infty}^{\infty} |f(t)|^2 \left( \int_{-\infty}^t |d\tilde{\varphi}^+/dx|^2 e^{\alpha(x-t)} dx \right) dt. \end{aligned}$$

Firstly, we estimate the integral in the parentheses. Notice that the function  $d\varphi^{\pm}/dx$  (and therefore,  $d\tilde{\varphi}^{\pm}/dx$ ) are in the space  $L^2([0, \pi])$  due to the first equation in (2.20). Therefore,

$$(2.37) \quad K_{\pm}^2 = \int_0^{\pi} \left| \frac{d\tilde{\varphi}^{\pm}}{dx}(x) \right|^2 dx < \infty.$$

We have

$$\begin{aligned} \int_{-\infty}^t |d\tilde{\varphi}^+/dx|^2 e^{\alpha(x-t)} dx &= \sum_{n=0}^{\infty} \int_{-(n+1)\pi}^{-n\pi} \left| \frac{d\tilde{\varphi}^+}{dx}(\xi+t) \right|^2 e^{\alpha\xi} d\xi \\ &\leq K_+^2 \cdot \sum_{n=0}^{\infty} e^{-\alpha n\pi} = \frac{K_+^2}{1 - \exp(-\alpha\pi)} < (1 + \alpha\pi) \frac{K_+^2}{\alpha\pi}. \end{aligned}$$

Thus,

$$\int_{-\infty}^{\infty} |v^+(x)e^{\alpha x}d\tilde{\varphi}^+/dx|^2 dx \leq (1 + \alpha\pi) \frac{C_-^2 K_+^2}{\alpha^2 \pi} \|f\|^2.$$

In an analogous way it follows that

$$\int_{-\infty}^{\infty} \left| v^-(x)e^{\alpha x} \frac{d\tilde{\varphi}^-}{dx} \right|^2 dx \leq (1 + \alpha\pi) \frac{C_+^2 K_-^2}{\alpha^2 \pi} \|f\|^2,$$

so the operator  $R_1$  act continuously from  $L^2(\mathbb{R})$  into the space  $W_2^1(\mathbb{R})$ .

The proof of the fact that the operator  $R_2$  is continuous in  $L^2(\mathbb{R})$  is omitted because essentially it is the same (we only replace  $\varphi^{\pm}$  with  $\psi^{\pm}$  in the proof that  $R_1$  is a continuous operator in  $L^2(\mathbb{R})$ ).  $\square$

We need also the following lemma.

**Lemma 3.** *Let  $H$  be a Hilbert space with product  $(\cdot, \cdot)$ , and let*

$$A : D(A) \rightarrow H, \quad B : D(B) \rightarrow H$$

*be (unbounded) linear operators with domains  $D(A)$  and  $D(B)$ , such that*

$$(2.38) \quad (Af, g) = (f, Bg) \quad \text{for } f \in D(A), g \in D(B).$$

*If there is a  $\lambda \in \mathbb{C}$  such that the operators  $A - \lambda$  and  $B - \bar{\lambda}$  are surjective, then*

- (i)  $D(A)$  and  $D(B)$  are dense in  $H$ ;*
- (ii)  $A^* = B$  and  $B^* = A$ , where  $A^*$  and  $B^*$  are, respectively, the adjoint operators of  $A$  and  $B$ .*

*Proof.* We need to explain only that  $D(A)$  is dense in  $H$  and  $A^* = B$  because one can replace the roles of  $A$  and  $B$ .

To prove that  $D(A)$  is dense in  $H$ , we need to show that if  $h$  is orthogonal to  $D(A)$  then  $h = 0$ . Let

$$(f, h) = 0 \quad \forall f \in D(A).$$

Since the operator  $B - \bar{\lambda}$  is surjective, there is  $g \in D(B)$  such that  $h = (B - \bar{\lambda})g$ . Therefore, by (2.38), we have

$$0 = (f, h) = (f, (B - \bar{\lambda})g) = ((A - \lambda)f, g) \quad \forall f \in D(A),$$

which yields  $g = 0$  because the range of  $A - \lambda$  is  $H$ . Thus,  $h = (B - \bar{\lambda})g = 0$ . Hence (i) holds.

Next we prove (ii). If  $g^* \in \text{Dom}(A^*)$ , then we have

$$(2.39) \quad (A - \lambda)f, g^* = (f, w) \quad \forall f \in D(A),$$

where  $w = (A^* - \bar{\lambda})g^*$ . Since the operator  $B - \bar{\lambda}$  is surjective, there is  $g \in D(B)$  such that  $w = (B - \bar{\lambda})g$ . Therefore, by (2.38) and (2.39), we have

$$((A - \lambda)f, g^*) = (f, (B - \bar{\lambda})g) = ((A - \lambda)f, g) \quad \forall f \in D(A),$$

which implies that  $g^* = g$  (because the range of  $A - \lambda$  is equal to  $H$ ) and  $(A^* - \bar{\lambda})g^* = (B - \bar{\lambda})g^*$ , i.e.,  $A^*g^* = Bg^*$ . This completes the proof of (ii).  $\square$

Consider the Schrödinger operator with a spectral parameter

$$L(v) - \lambda = -d^2/dx^2 + (v - \lambda), \quad \lambda \in \mathbb{C}.$$

In view of the formula (2.11 in Proposition 1, we may assume without loss of generality that

$$(2.40) \quad C = 0, \quad v = Q',$$

because a change of  $C$  results in a shift of the spectral parameter  $\lambda$ .

Replacing  $C$  by  $-\lambda$  in the homogeneous system (2.20), we get

$$(2.41) \quad \begin{cases} y' = Qy + u, \\ u' = (-\lambda - Q^2)y - Qu. \end{cases}$$

Let  $(y_1(x; \lambda), u_1(x; \lambda))$  and  $(y_2(x; \lambda), u_2(x; \lambda))$  be the solutions of (2.41) which satisfy the initial conditions  $y_1(0; \lambda) = 1, u_1(0; \lambda) = 0$  and  $y_2(0; \lambda) = 0, u_2(0; \lambda) = 1$ . Since these solutions depend analytically on  $\lambda \in \mathbb{C}$ , the *Lyapunov function*, or *Hill discriminant*,

$$(2.42) \quad \Delta(Q, \lambda) = y_1(\pi; \lambda) + u_2(\pi; \lambda)$$

is an entire function. Taking the conjugates of the equation in (2.41), one can easily see that

$$(2.43) \quad \Delta(\bar{Q}, \bar{\lambda}) = \overline{\Delta(Q, \lambda)}.$$

*Remark.* A. Savchuk and A. Shkalikov gave asymptotic analysis of the functions  $y_j(\pi, \lambda)$  and  $u_j(\pi, \lambda)$ ,  $j = 1, 2$ . In particular, it follows from Formula (1.5) of Lemma 1.4 in [27] that, with  $z^2 = \lambda$ ,

(2.44)

$$y_1(\pi, \lambda) = \cos(\pi z) + o(1), \quad y_2(\pi, \lambda) = \frac{1}{z}[\sin(\pi z) + o(1)], \quad u_2(\pi, \lambda) = \cos \pi z + o(1),$$

and therefore,

$$(2.45) \quad \Delta(Q, \lambda) = 2 \cos \pi z + o(1), \quad z^2 = \lambda,$$

inside any parabola

$$(2.46) \quad P_a = \{\lambda \in \mathbb{C} : |Im z| \leq a\}.$$

In the regular case  $v \in L^2([0, \pi])$  these asymptotics of the fundamental solutions and the Lyapunov function  $\Delta$  of the Hill–Schrödinger operator could be found in [21], p. 32, Formula (1.3.11), or pp. 252–253, Formulae (3.4.23'), (3.4.26).

Consider the operator  $L(v)$ , in the domain

(2.47)

$$D(L(v)) = \{y \in H^1(\mathbb{R}) : y' - Qy \in L^2(\mathbb{R}) \cap W_{1,loc}^1(\mathbb{R}), \ell_Q(y) \in L^2(\mathbb{R})\},$$

defined by

$$(2.48) \quad L(v)y = \ell_Q(y), \quad \text{with } \ell_Q(y) = -(y' - Qy)' - Qy',$$

where  $v$  and  $Q$  are as in Proposition 1.

**Theorem 4.** *Let  $v \in H_{loc}^{-1}(\mathbb{R})$  be  $\pi$ -periodic. Then*

(a) *the domain  $D(L(v))$  is dense in  $L^2(\mathbb{R})$ ;*

(b) *the operator  $L(v)$  is closed, and its conjugate operator is*

$$(2.49) \quad (L(v))^* = L(\bar{v});$$

(In particular, if  $v$  is real-valued, then the operator  $L(v)$  is self-adjoint.)

(c) *the spectrum  $Sp(L(v))$  of the operator  $L(v)$  is continuous, and moreover,*

$$(2.50) \quad Sp(L(v)) = \{\lambda \in \mathbb{C} \mid \exists \theta \in [0, 2\pi) : \Delta(\lambda) = 2 \cos \theta\}.$$

*Remark.* In the case of  $L^2$ -potential  $v$  this result is known (see Rofo–Beketov [23, 24] and V. Tkachenko [31]).

*Proof.* Firstly, we show that the operators  $L(v)$  and  $L(\bar{v})$  are formally adjoint, i.e.,

$$(2.51) \quad (L(v)y, h) = (f, L(\bar{v})h) \quad \text{if } y \in D(L(v)), \quad h \in D(L(\bar{v})).$$

Since  $y' - Qy$  and  $\bar{h}$  are continuous  $L^2(\mathbb{R})$ -functions, their product is a continuous  $L^1(\mathbb{R})$ -function, so we have

$$\liminf_{x \rightarrow \pm\infty} |(y' - Qy)\bar{h}|(x) = 0.$$

Therefore, there exist two sequences of real numbers  $c_n \rightarrow -\infty$  and  $d_n \rightarrow \infty$  such that

$$((y' - Qy)\bar{h})(c_n) \rightarrow 0, \quad ((y' - Qy)\bar{h})(d_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Now, we have

$$\begin{aligned}
(L(v)y, h) &= \int_{-\infty}^{\infty} \ell_Q(y)\bar{h}dx = \lim_{n \rightarrow \infty} \int_{c_n}^{d_n} (-(y' - Qy)' \bar{h} - Qy' \bar{h}) dx \\
&= \lim_{n \rightarrow \infty} \left( -(y' - Qy)\bar{h} \Big|_{c_n}^{d_n} + \int_{c_n}^{d_n} (y' - Qy)\bar{h}' dx - \int_{c_n}^{d_n} Qy' \bar{h} dx \right) \\
&= 0 + \int_{-\infty}^{\infty} (y' \bar{h}' - Qy \bar{h}' - Qy' \bar{h}) dx.
\end{aligned}$$

The same argument shows that

$$\int_{-\infty}^{\infty} (y' \bar{h}' - Qy \bar{h}' - Qy' \bar{h}) dx = (y, L(\bar{v})h),$$

which completes the proof of (2.51).

If the roots of the characteristic equation  $\rho^2 - \Delta(Q, \lambda)\rho + 1 = 0$  lie on the unit circle  $\{e^{i\theta}, \theta \in [0, 2\pi)\}$ , then they are of the form  $e^{\pm i\theta}$ , so we have

$$(2.52) \quad \Delta(Q, \lambda) = e^{i\theta} + e^{-i\theta} = 2 \cos \theta.$$

Therefore, if  $\Delta(Q, \lambda) \notin [-2, 2]$ , then the roots of the characteristic equation lie outside of the unit circle  $\{e^{i\theta}, \theta \in [0, 2\pi)\}$ . If so, by part (c) of Lemma 2, the operator  $L(v) - \lambda$  maps bijectively  $D(L(v))$  onto  $L^2(\mathbb{R})$ , and its inverse operator

$$(L(v) - \lambda)^{-1} : L^2(\mathbb{R}) \rightarrow D(L(v))$$

is a continuous linear operator. Thus,

$$(2.53) \quad \Delta(Q, \lambda) \notin [-2, 2] \Rightarrow (L(v) - \lambda)^{-1} : L^2(\mathbb{R}) \rightarrow D(L(v)) \text{ exists.}$$

Next we apply Lemma 3 with  $A = L(v)$  and  $B = L(\bar{v})$ . Choose  $\lambda \in \mathbb{C}$  so that  $\Delta(Q, \lambda) \notin [-2, 2]$  (in view of (2.45), see the remark before Theorem 4,  $\Delta(Q, \lambda)$  is a non-constant entire function, so such a choice is possible). Then, in view of (2.43), we have that  $\Delta(\bar{Q}, \bar{\lambda}) \notin [-2, 2]$  also. In view of the above discussion, this means that the operator  $L(v) - \lambda$  maps bijectively  $D(L(v))$  onto  $L^2(\mathbb{R})$  and  $L(\bar{v}) - \bar{\lambda}$  maps bijectively  $D(L(\bar{v}))$  onto  $L^2(\mathbb{R})$ . Thus, by Lemma 3,  $D(L(v))$  is dense in  $L^2(\mathbb{R})$  and  $L(v)^* = L(\bar{v})$ , i.e., (a) and (b) hold.

Finally, in view of (2.53), (c) follows readily from part (b) of Lemma 2.  $\square$

3. Theorem 4 shows that the spectrum of the operator  $L(v)$  is described by the equation (2.50). As we are going to explain below, this fact implies that the spectrum  $Sp(L(v))$  could be described in terms of the spectra of the operators  $L_\theta = L_\theta(v)$ ,  $\theta \in [0, \pi]$ , that arise from the same differential expression  $\ell = \ell_Q$  when it is considered on the interval  $[0, \pi]$  with the following boundary conditions:

$$(2.54) \quad y(\pi) = e^{i\theta} y(0), \quad (y' - Qy)(\pi) = e^{i\theta} (y' - Qy)(0).$$

The domains  $D(L_\theta)$  of the operators  $L_\theta$  are given by

$$(2.55) \quad D(L_\theta) = \{y \in H^1 : y' - Qy \in W_1^1([0, \pi]), \text{ (2.54) holds, } \ell(y) \in H^0\},$$

where

$$H^1 = H^1([0, \pi]), \quad H^0 = L^2([0, \pi]).$$

We set

$$(2.56) \quad L_\theta(y) = \ell(y), \quad y \in D(L_\theta).$$

Notice that if  $y \in H^1([0, \pi])$ , then  $\ell_Q(y) = f \in L^2([0, \pi])$  if and only if  $u = y' - Qy \in W_1^1([0, \pi])$  and the pair  $(y, u)$  is a solution of the non-homogeneous system (2.19).

**Lemma 5.** *Let  $\begin{pmatrix} y_1 \\ u_1 \end{pmatrix}$  and  $\begin{pmatrix} y_2 \\ u_2 \end{pmatrix}$  be the solutions of the homogeneous system (2.20) which satisfy*

$$(2.57) \quad \begin{pmatrix} y_1(0) \\ u_1(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} y_2(0) \\ u_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

If

$$(2.58) \quad \Delta = y_1(\pi) + u_2(\pi) \neq 2 \cos \theta, \quad \theta \in [0, \pi],$$

then the non-homogeneous system (2.19) has, for each  $f \in H^0$ , a unique solution  $(y, u) = (R_1(f), R_2(f))$  such that

$$(2.59) \quad \begin{pmatrix} y(\pi) \\ u(\pi) \end{pmatrix} = e^{i\theta} \begin{pmatrix} y(0) \\ u(0) \end{pmatrix}.$$

Moreover,  $R_1$  is a linear continuous operator from  $H^0$  into  $H^1$ , and  $R_2$  is a linear continuous operator in  $H^0$  with a range in  $W_1^1([0, \pi])$ .

*Proof.* By the variation of parameters method, every solution of the non-homogeneous system (2.19) has the form

$$(2.60) \quad \begin{pmatrix} y(x) \\ u(x) \end{pmatrix} = v_1(x) \begin{pmatrix} y_1(x) \\ u_1(x) \end{pmatrix} + v_2(x) \begin{pmatrix} y_2(x) \\ u_2(x) \end{pmatrix},$$

where

$$(2.61) \quad v_1(x) = - \int_0^x y_2(t) f(t) dt + C_1, \quad v_2(x) = \int_0^x y_1(t) f(t) dt + C_2.$$

We set for convenience

$$(2.62) \quad m_1(f) = - \int_0^\pi y_2(t) f(t) dt, \quad m_2(f) = \int_0^\pi y_1(t) f(t) dt.$$

By (2.60)–(2.62), the condition (2.59) is equivalent to

$$(2.63) \quad (m_1(f) + C_1) \begin{pmatrix} y_1(\pi) \\ u_1(\pi) \end{pmatrix} + (m_2(f) + C_2) \begin{pmatrix} y_2(\pi) \\ u_2(\pi) \end{pmatrix} = e^{i\theta} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}.$$

This is a system of two linear equations in two unknowns  $C_1$  and  $C_2$ . The corresponding determinant is equal to

$$\det \begin{pmatrix} y_1(\pi) - e^{i\theta} & y_2(\pi) \\ u_1(\pi) & u_2(\pi) - e^{i\theta} \end{pmatrix} = 1 + e^{2i\theta} - \Delta \cdot e^{i\theta} = e^{i\theta} (2 \cos \theta - \Delta).$$

Therefore, if (2.58) holds, then the system (2.63) has a unique solution  $\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$ , where  $C_1 = C_1(f)$  and  $C_2 = C_2(f)$  are linear combinations of  $m_1(f)$  and  $m_2(f)$ . With these values of  $C_1(f)$  and  $C_2(f)$  we set

$$R_1(f) = v_1 \cdot y_1 + v_2 \cdot y_2, \quad R_2(f) = v_1 \cdot u_1 + v_2 \cdot u_2.$$

By (2.61) and (2.62), the Cauchy inequality implies

$$|v_1(x)| \leq \int_0^x |y_2(t)f(t)|dt + |C_1(f)| \leq A \cdot \|f\|, \quad |v_2(x)| \leq B \cdot \|f\|,$$

where  $A$  and  $B$  are constants. From here it follows that  $R_1$  and  $R_2$  are continuous linear operators in  $H^0$ . Since

$$\frac{d}{dx}R_1(f) = v_1 \frac{dy_1}{dx} + v_2 \frac{dy_2}{dx}, \quad R_2(f) = v_1 \frac{du_1}{dx} + v_2 \frac{du_2}{dx} + f,$$

it follows also that  $R_1$  acts continuously from  $H^0$  into  $H^1$ , and  $R_2$  has range in  $W_1^1([0, \pi])$ , which completes the proof.  $\square$

**Theorem 6.** *Suppose  $v \in H_{loc}^{-1}(\mathbb{R})$  is  $\pi$ -periodic. Then,*

- (a) *for each  $\theta \in [0, \pi]$ , the domain  $D(L_\theta(v)) \in (2.55)$  is dense in  $H^0$ ;*
- (b) *the operator  $L_\theta(v) \in (2.56)$  is closed, and its conjugate operator is*

$$(2.64) \quad L_\theta(v)^* = L_\theta(\bar{v}).$$

*In particular, if  $v$  is real-valued, then the operator  $L_\theta(v)$  is self-adjoint.*

- (c) *the spectrum  $Sp(L_\theta(v))$  of the operator  $L_\theta(v)$  is discrete, and moreover,*

$$(2.65) \quad Sp(L_\theta(v)) = \{\lambda \in \mathbb{C} : \Delta(\lambda) = 2 \cos \theta\}.$$

*Proof.* Firstly, we show that the operators  $L_\theta(v)$  and  $L_\theta(\bar{v})$  are formally adjoint, i.e.,

$$(2.66) \quad (L_\theta(v)y, h) = (f, L_\theta(\bar{v})h) \quad \text{if } y \in D(L_\theta(v)), \quad h \in D(L_\theta(\bar{v})).$$

Indeed, in view of (2.54), we have

$$\begin{aligned} (L_\theta(v)y, h) &= \frac{1}{\pi} \int_0^\pi \ell_Q(y)\bar{h}dx = \frac{1}{\pi} \int_0^\pi (-(y' - Qy)'\bar{h} - Qy'\bar{h}) dx \\ &= -\frac{1}{\pi} (y' - Qy)\bar{h} \Big|_0^\pi + \frac{1}{\pi} \int_0^\pi (y' - Qy)\bar{h}' dx - \int_0^\pi Qy'\bar{h} dx \\ &= 0 + \frac{1}{\pi} \int_0^\pi (y'\bar{h}' - Qy\bar{h}' - Qy'\bar{h}) dx. \end{aligned}$$

The same argument shows that

$$\frac{1}{\pi} \int_0^\pi (y'\bar{h}' - Qy\bar{h}' - Qy'\bar{h}) dx = (y, L_\theta(\bar{v})h),$$

which completes the proof of (2.66).

Now we apply Lemma 3 with  $A = L_\theta(v)$  and  $B = L_\theta(\bar{v})$ . Choose  $\lambda \in \mathbb{C}$  so that  $\Delta(Q, \lambda) \neq 2 \cos \theta$  (as one can easily see from the remark before Theorem 4,  $\Delta(Q, \lambda)$  is a non-constant entire function, so such a choice is possible). Then, in

view of (2.43), we have that  $\Delta(\overline{Q}, \overline{\lambda}) \neq 2 \cos \theta$  also. By Lemma 5,  $L_\theta(v) - \lambda$  maps bijectively  $D(L_\theta(v))$  onto  $H^0$  and  $L_\theta(\overline{v}) - \overline{\lambda}$  maps bijectively  $D(L_\theta(\overline{v}))$  onto  $H^0$ . Thus, by Lemma 3,  $D(L_\theta(v))$  is dense in  $H^0$  and  $L_\theta(v)^* = L_\theta(\overline{v})$ , i.e., (a) and (b) hold.

If  $\Delta(Q, \lambda) = 2 \cos \theta$ , then  $e^{i\theta}$  is a root of the characteristic equation (2.22), so there is a special solution  $(\varphi, \psi)$  of the homogeneous system (2.20) (considered with  $C = -\lambda$ ) such that (2.23) holds with  $\rho = e^{i\theta}$ . But then  $\varphi \in D(L_\theta(v))$  and  $L_\theta(v)\varphi = \lambda\varphi$ , i.e.,  $\lambda$  is an eigenvalue of  $L_\theta(v)$ . In view of Lemma 5, this means that (2.65) holds. Since  $\Delta(Q, \lambda)$  is a non-constant entire function (as one can easily see from the remark before Theorem 11) the set on the right in (2.65) is discrete. This completes the proof of (c).  $\square$

**Corollary 7.** *In view of Theorem 4 and Theorem 6, we have*

$$(2.67) \quad Sp(L(v)) = \bigcup_{\theta \in [0, \pi]} Sp(L_\theta(v)).$$

In the self-adjoint case (i.e., when  $v$ , and therefore,  $Q$  are real-valued) the spectrum  $Sp(L(v)) \subset \mathbb{R}$  has a band-gap structure. This is a well-known result in the regular case where  $v$  is an  $L^2_{loc}(\mathbb{R})$ -function. Its generalization in the singular case was proved by R. Hryniv and Ya. Mykytiuk [8].

In order to formulate that result more precisely, let us consider the following boundary conditions (bc):

$$(a^*) \text{ periodic } Per^+ : y(\pi) = y(0), (y' - Qy)(\pi) = (y' - Qy)(0);$$

$$(b^*) \text{ antiperiodic } Per^- : y(\pi) = -y(0), (y' - Qy)(\pi) = -(y' - Qy)(0);$$

Of course, in the case where  $Q$  is a continuous function,  $Per^+$  and  $Per^-$  coincide, respectively, with the classical periodic boundary condition  $y(\pi) = y(0)$ ,  $y'(\pi) = y'(0)$  or anti-periodic boundary condition  $y(\pi) = -y(0)$ ,  $y'(\pi) = -y'(0)$  (see the related discussion in Section 6.2).

The boundary conditions  $Per^\pm$  are particular cases of (2.59), considered, respectively, for  $\theta = 0$  or  $\theta = \pi$ . Therefore, by Theorem 6, for each of these two boundary conditions, the differential expression (2.18) gives a rise of a closed (self adjoint for real  $v$ ) operator  $L_{Per^\pm}$  in  $H^0 = L^2([0, \pi])$ , respectively, with a domain

$$(2.68) \quad D(L_{Per^+}) = \{y \in H^1 : y' - Qy \in W_1^1([0, \pi]), (a^*) \text{ holds, } l(y) \in H^0\},$$

or

$$(2.69) \quad D(L_{Per^-}) = \{y \in H^1 : y' - Qy \in W_1^1([0, \pi]), (b^*) \text{ holds, } l(y) \in H^0\}.$$

The spectra of the operators  $L_{Per^\pm}$  are discrete. Let us enlist their eigenvalues in increasing order, by using even indices for the eigenvalues of  $L_{Per^+}$  and odd indices for the eigenvalues of  $L_{Per^-}$  (the convenience of such enumeration will be clear later):

$$(2.70) \quad Sp(L_{Per^+}) = \{\lambda_0, \lambda_2^-, \lambda_2^+, \lambda_4^-, \lambda_4^+, \lambda_6^-, \lambda_6^+, \dots\},$$

$$(2.71) \quad Sp(L_{Per^-}) = \{\lambda_1^-, \lambda_1^+, \lambda_3^-, \lambda_3^+, \lambda_5^-, \lambda_5^+ \dots\}.$$

**Proposition 8.** *Suppose  $v = C + Q'$ , where  $Q \in L^2_{loc}(\mathbb{R})$  is a  $\pi$ -periodic real valued function. Then, in the above notations, we have*

$$(2.72) \quad \lambda_0 < \lambda_1^- \leq \lambda_1^+ < \lambda_2^- \leq \lambda_2^+ < \lambda_3^- \leq \lambda_3^+ < \lambda_4^- \leq \lambda_4^+ < \lambda_5^- \leq \lambda_5^+ < \dots$$

Moreover, the spectrum of the operator  $L(v)$  is absolutely continuous and has a band-gap structure: it is a union of closed intervals separated by spectral gaps

$$(-\infty, \lambda_0), (\lambda_1^-, \lambda_1^+), (\lambda_2^-, \lambda_2^+), \dots, (\lambda_n^-, \lambda_n^+), \dots$$

Let us mention that A. Savchuk and A. Shkalikov [25] have studied the Sturm-Liouville operators that arise when the differential expression  $\ell_Q$ ,  $Q \in L^2([0, 1])$ , is considered with appropriate regular boundary conditions (see Theorems 1.5 and 1.6 in [27]).

### 3. FOURIER REPRESENTATION OF THE OPERATORS $L_{Per^\pm}$

Let  $L_{bc}^0$  denote the free operator  $L^0 = -d^2/dx^2$  considered with boundary conditions  $bc$  as a self-adjoint operator in  $L^2([0, \pi])$ . It is easy to describe the spectra and eigenfunctions of  $L_{bc}^0$  for  $bc = Per^\pm, Dir$ :

(a)  $Sp(L_{Per^+}^0) = \{n^2, n = 0, 2, 4, \dots\}$ ; its eigenspaces are  $E_n^0 = Span\{e^{\pm inx}\}$  for  $n > 0$  and  $E_0^0 = \{const\}$ ,  $\dim E_n^0 = 2$  for  $n > 0$ , and  $\dim E_0^0 = 1$ .

(b)  $Sp(L_{Per^-}^0) = \{n^2, n = 1, 3, 5, \dots\}$ ; its eigenspaces are  $E_n^0 = Span\{e^{\pm inx}\}$ , and  $\dim E_n^0 = 2$ .

(c)  $Sp(L_{Dir}^0) = \{n^2, n \in \mathbb{N}\}$ ; each eigenvalue  $n^2$  is simple; a corresponding normalized eigenfunction is  $\sqrt{2} \sin nx$ .

Depending on the boundary conditions, we consider as our canonical orthogonal normalized basis (o.n.b.) in  $L^2([0, \pi])$  the system  $u_k(x)$ ,  $k \in \Gamma_{bc}$ , where

$$(3.1) \quad \text{if } bc = Per^+ \quad u_k = \exp(ikx), \quad k \in \Gamma_{Per^+} = 2\mathbb{Z};$$

$$(3.2) \quad \text{if } bc = Per^- \quad u_k = \exp(ikx), \quad k \in \Gamma_{Per^-} = 1 + 2\mathbb{Z};$$

$$(3.3) \quad \text{if } bc = Dir \quad u_k = \sqrt{2} \sin kx, \quad k \in \Gamma_{Dir} = \mathbb{N}.$$

Let us notice that  $\{u_k(x), k \in \Gamma_{bc}\}$  is a complete system of unit eigenvectors of the operator  $L_{bc}^0$ .

We set

$$(3.4) \quad H_{Per^+}^1 = \{f \in H^1 : f(\pi) = f(0)\}, \quad H_{Per^-}^1 = \{f \in H^1 : f(\pi) = -f(0)\}$$

and

$$(3.5) \quad H_{Dir}^1 = \{f \in H^1 : f(\pi) = f(0) = 0\}.$$

One can easily see that  $\{e^{ikx}, k \in 2\mathbb{Z}\}$  is an orthogonal basis in  $H_{Per^+}^1$ ,  $\{e^{ikx}, k \in 1 + 2\mathbb{Z}\}$  is an orthogonal basis in  $H_{Per^-}^1$ , and  $\{\sqrt{2} \sin kx, k \in \mathbb{N}\}$  is an orthogonal basis in  $H_{Dir}^1$ .

From here it follows that

$$(3.6) \quad H_{bc}^1 = \left\{ f(x) = \sum_{k \in \Gamma_{bc}} f_k u_k(x) : \|f\|_{H^1} = \sum_{k \in \Gamma_{bc}} (1 + k^2) |f_k|^2 < \infty \right\}.$$

The following statement is well known.

**Lemma 9.** (a) If  $f, g \in L^1([0, \pi])$  and  $f \sim \sum_{k \in 2\mathbb{Z}} f_k e^{ikx}$ ,  $g \sim \sum_{k \in 2\mathbb{Z}} g_k e^{ikx}$  are their Fourier series respect to the system  $\{e^{ikx}, k \in 2\mathbb{Z}\}$ , then the following conditions are equivalent:

- (i)  $f$  is absolutely continuous,  $f(\pi) = f(0)$  and  $f'(x) = g(x)$  a.e.;
- (ii)  $g_k = ikf_k \quad \forall k \in 2\mathbb{Z}$ .

(b) If  $f, g \in L^1([0, \pi])$  and  $f \sim \sum_{k \in 1+2\mathbb{Z}} f_k e^{ikx}$ ,  $g \sim \sum_{k \in 1+2\mathbb{Z}} g_k e^{ikx}$  are their Fourier series respect to the system  $\{e^{ikx}, k \in 1+2\mathbb{Z}\}$ , then the following conditions are equivalent:

- (i\*)  $f$  is absolutely continuous,  $f(\pi) = -f(0)$  and  $f'(x) = g(x)$  a.e.;
- (ii\*)  $g_k = ikf_k \quad \forall k \in 1+2\mathbb{Z}$ .

*Proof.* An integration by parts gives the implication (i)  $\Rightarrow$  (ii) [or (i\*)  $\Rightarrow$  (ii\*)].

To prove that (ii)  $\Rightarrow$  (i) we set  $G(x) = \int_0^x g(t)dt$ . By (ii) for  $k = 0$ , we have  $G(\pi) = \int_0^\pi g(t)dt = \pi g_0 = 0$ . Therefore, integrating by parts we get

$$g_k = \frac{1}{\pi} \int_0^\pi g(x) e^{-ikx} dx = \frac{1}{\pi} \int_0^\pi e^{-ikx} dG(x) = ikG_k,$$

where  $G_k = \frac{1}{\pi} \int_0^\pi e^{-ikx} G(x) dx$  is the  $k$ -th Fourier coefficient of  $G$ . Thus, by (ii), we have  $G_k = f_k$  for  $k \neq 0$ , so by the Uniqueness Theorem for Fourier series  $f(x) = G(x) + \text{const}$ , i.e.(i) holds.

Finally, the proof of the implication (ii\*)  $\Rightarrow$  (i\*) could be reduced to part (a) by considering the functions  $\tilde{f}(x) = f(x)e^{ix} \sim \sum_{k \in 1+2\mathbb{Z}} f_{k-1} e^{ikx}$  and  $\tilde{g}(x) = g(x)e^{ix} + if(x)e^{ix}$ . We omit the details.  $\square$

The next proposition gives the Fourier representations of the operators  $L_{Per^\pm}$  and their domains.

**Proposition 10.** In the above notations, if  $y \in H_{Per^\pm}^1$ , then we have  $y = \sum_{\Gamma_{Per^\pm}} y_k e^{ikx} \in D(L_{Per^\pm})$  and  $\ell(y) = h = \sum_{\Gamma_{Per^\pm}} h_k e^{ikx} \in H^0$  if and only if

$$(3.7) \quad h_k = h_k(y) := k^2 y_k + \sum_{m \in \Gamma_{Per^\pm}} V(k-m) y_m + C y_k, \quad \sum |h_k|^2 < \infty,$$

i.e.,

$$(3.8) \quad D(L_{Per^\pm}) = \left\{ y \in H_{Per^\pm}^1 : (h_k(y))_{k \in \Gamma_{Per^\pm}} \in \ell^2(\Gamma_{Per^\pm}) \right\}$$

and

$$(3.9) \quad L_{Per^\pm}(y) = \sum_{k \in \Gamma_{Per^\pm}} h_k(y) e^{ikx}.$$

*Proof.* Since the proof is the same in the periodic and anti-periodic cases, we consider only the case of periodic boundary conditions. By (2.68), if  $y \in D(L_{Per^+})$ , then  $y \in H_{Per^+}^1$  and

$$(3.10) \quad \ell(y) = -z' - Qy' + Cy = h \in L^2([0, \pi]),$$

where

$$(3.11) \quad z := y' - Qy \in W_1^1([0, \pi]), \quad z(\pi) = z(0).$$

Let

$$y(x) = \sum_{k \in 2\mathbb{Z}} y_k e^{ikx}, \quad z(x) = \sum_{k \in 2\mathbb{Z}} z_k e^{ikx}, \quad h(x) = \sum_{k \in 2\mathbb{Z}} h_k e^{ikx}$$

be the Fourier series of  $y, z$  and  $h$ . Since  $z(\pi) = z(0)$ , Lemma 9 says that the Fourier series of  $z'$  may be obtained by differentiating term by term the Fourier series of  $z$ , and the same property is shared by  $y$  as a function in  $H_{Per^+}^1$ . Thus, (3.10) implies

$$(3.12) \quad -ikz_k - \sum_m q(k-m)imy_m + Cy_k = h_k.$$

On the other hand, by (3.11), we have  $z_k =iky_k - \sum_m q(k-m)y_m$ , so substituting that in (3.12) we get

$$(3.13) \quad -ik \left[ iky_k - \sum_m q(k-m)y_m \right] - \sum_m q(k-m)imy_m + Cy_k = h_k,$$

which leads to (3.7) because  $V(m) = imq(m)$ ,  $m \in 2\mathbb{Z}$ .

Conversely, if (3.7) holds, then we have (3.13). Therefore, (3.12) holds with  $z_k = iky_k - \sum_m q(k-m)y_m$ .

Since  $y = \sum y_k e^{ikx} \in H_{Per^+}^1$ , the Fourier coefficients of its derivative are  $iky_k$ ,  $k \in 2\mathbb{Z}$ . Thus,  $(z_k)$  is the sequence of Fourier coefficients of the function  $z = y' - Qy \in L^1([0, \pi])$ .

On the other hand, by (3.12),  $(ikz_k)$  is the sequence of Fourier coefficients of an  $L^1([0, \pi])$ -function. Therefore, by Lemma 9, the function  $z$  is absolutely continuous,  $z(\pi) = z(0)$ , and  $(ikz_k)$  is the sequence of Fourier coefficients of its derivative  $z'$ . Thus, (3.10) and (3.11) hold, i.e.,  $y \in D(L_{Per^+})$  and  $L_{Per^+}y = \ell(y) = h$ .  $\square$

Now, we are ready to explain the Fourier method for studying the spectra of the operators  $L_{Per^\pm}$ . Let

$$\mathcal{F} : H^0 \rightarrow \ell^2(\Gamma_{Per^\pm})$$

be the Fourier isomorphisms defined by corresponding to each function  $f \in H^0$  the sequence  $(f_k)$  of its Fourier coefficients  $f_k = (f, u_k)$ , where  $\{u_k, k \in \Gamma_{Per^\pm}\}$  is, respectively, the basis (3.1) or (3.2). Let  $\mathcal{F}^{-1}$  be the inverse Fourier isomorphism.

Consider the unbounded operators  $\mathcal{L}_+$  and  $\mathcal{L}_-$  acting, respectively, in  $\ell^2(\Gamma_{Per^\pm})$  as

$$(3.14) \quad \mathcal{L}_\pm(z) = (h_k(z))_{k \in \Gamma_{Per^\pm}}, \quad h_k(z) = k^2 z_k + \sum_{m \in \Gamma_{Per^\pm}} V(k-m)z_m + Cz_k,$$

respectively, in the domains

$$(3.15) \quad D(\mathcal{L}_\pm) = \{z \in \ell^2(|k|, \Gamma_{Per^\pm}) : \mathcal{L}_\pm(z) \in \ell^2(\Gamma_{Per^\pm})\},$$

where  $\ell^2(|k|, \Gamma_{Per\pm})$  is the weighted  $\ell^2$ -space

$$\ell^2(|k|, \Gamma_{Per\pm}) = \left\{ z = (z_k)_{k \in \Gamma_{Per\pm}} : \sum_k (1 + |k|^2) |z_k|^2 < \infty \right\}.$$

In view of (3.6) and Proposition 10, the following theorem holds.

**Theorem 11.** *In the above notations, we have*

$$(3.16) \quad D(L_{Per\pm}) = \mathcal{F}^{-1} (D(\mathcal{L}_{\pm}))$$

and

$$(3.17) \quad L_{Per\pm} = \mathcal{F}^{-1} \circ \mathcal{L}_{\pm} \circ \mathcal{F}.$$

If it does not lead to confusion, for convenience we will loosely use one and the same notation  $L_{Per\pm}$  for the operators  $L_{Per\pm}$  and  $\mathcal{L}_{\pm}$ .

#### 4. FOURIER REPRESENTATION FOR THE HILL-SCHRÖDINGER OPERATOR WITH DIRICHLET BOUNDARY CONDITIONS

In this section we study the Hill-Schrödinger operator  $L_{Dir}(v)$ ,  $v = C + Q'$ , generated by the differential expression  $\ell_Q(y) = -(y' - Qy)' - Qy'$  considered on the interval  $[0, \pi]$  with Dirichlet boundary conditions

$$Dir : \quad y(0) = y(\pi) = 0.$$

Its domain is

$$(4.1) \quad D(L_{Dir}(v)) = \{y \in H^1 : y' - Qy \in W_1^1([0, \pi]), y(0) = y(\pi) = 0, \ell_Q(y) \in H^0\},$$

and we set

$$(4.2) \quad L_{Dir}(v)y = \ell_Q(y).$$

**Lemma 12.** *Let  $\begin{pmatrix} y_1 \\ u_1 \end{pmatrix}$  and  $\begin{pmatrix} y_2 \\ u_2 \end{pmatrix}$  be the solutions of the homogeneous system (2.20) which satisfy*

$$(4.3) \quad \begin{pmatrix} y_1(0) \\ u_1(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} y_2(0) \\ u_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

If

$$(4.4) \quad y_2(\pi) \neq 0,$$

then the non-homogeneous system (2.19) has, for each  $f \in H^0$ , a unique solution  $(y, u) = (R_1(f), R_2(f))$  such that

$$(4.5) \quad y(0) = 0, \quad y(\pi) = 0.$$

Moreover,  $R_1$  is a linear continuous operator from  $H^0$  into  $H^1$ , and  $R_2$  is a linear continuous operator in  $H^0$  with a range in  $W_1^1([0, \pi])$ .

*Proof.* By the variation of parameters method, every solution of the non-homogeneous system (2.19) has the form

$$\begin{pmatrix} y(x) \\ u(x) \end{pmatrix} = v_1(x) \begin{pmatrix} y_1(x) \\ u_1(x) \end{pmatrix} + v_2(x) \begin{pmatrix} y_2(x) \\ u_2(x) \end{pmatrix},$$

where

$$(4.6) \quad v_1(x) = - \int_0^x y_2(x)f(t)dt + C_1, \quad v_2(x) = \int_0^x y_1(x)f(t)dt + C_2.$$

By (4.3), the condition  $y(0) = 0$  will be satisfied if and only if  $C_1 = 0$ . If so, the second condition  $y(\pi) = 0$  in (4.5) is equivalent to

$$m_1(f)y_1(\pi) + (m_2(f) + C_2)y_2(\pi) = 0,$$

where

$$m_1(f) = - \int_0^\pi y_2(x)f(t)dt, \quad m_2(f) = \int_0^\pi y_1(x)f(t)dt.$$

Thus, if  $y_2(\pi) \neq 0$ , then we have unique solution  $(y, u)$  of (2.19) that satisfies (4.5), and it is given by (4.6) with  $C_1 = 0$  and

$$(4.7) \quad C_2(f) = - \frac{y_1(\pi)}{y_2(\pi)} m_1(f) - m_2(f).$$

Thus, we have  $\begin{pmatrix} y(x) \\ u(x) \end{pmatrix} = \begin{pmatrix} R_1(f) \\ R_2(f) \end{pmatrix}$ , where

$$R_1(f) = \left( - \int_0^x y_2(x)f(t)dt \right) \cdot y_1(x) + \left( \int_0^x y_1(x)f(t)dt + C_2(f) \right) \cdot y_2(x)$$

and

$$R_2(f) = \left( - \int_0^x y_2(x)f(t)dt \right) \cdot u_1(x) + \left( \int_0^x y_1(x)f(t)dt + C_2(f) \right) \cdot u_2(x).$$

It is easy to see (compare with the proof of Lemma 5) that  $R_1$  is a linear continuous operator from  $H^0$  into  $H^1$ , and  $R_2$  is a linear continuous operator in  $H^0$  with a range in  $W_1^1([0, \pi])$ . We omit the details.  $\square$

Now, let us consider the systems (2.19) and (2.20) with a spectral parameter  $\lambda$  by setting  $C = -\lambda$  there, and let  $\begin{pmatrix} y_1(x, \lambda) \\ u_1(x, \lambda) \end{pmatrix}$  and  $\begin{pmatrix} y_2(x, \lambda) \\ u_2(x, \lambda) \end{pmatrix}$  be the solutions of the homogeneous system (2.20) that satisfy (4.3) for  $x = 0$ . Notice that

$$(4.8) \quad y_2(\bar{v}; x, \bar{\lambda}) = \overline{y_2(v; x, \lambda)}.$$

**Theorem 13.** *Suppose  $v \in H_{loc}^{-1}(\mathbb{R})$  is  $\pi$ -periodic. Then,*

- (a) *the domain  $D(L_{Dir}(v)) \in (4.1)$  is dense in  $H^0$ ;*
- (b) *the operator  $L_{Dir}(v)$  is closed, and its conjugate operator is*

$$(4.9) \quad (L_{Dir}(v))^* = L_{Dir}(\bar{v}).$$

*In particular, if  $v$  is real-valued, then the operator  $L_{Dir}(v)$  is self-adjoint.*

(c) the spectrum  $Sp(L_{Dir}(v))$  of the operator  $L_{Dir}(v)$  is discrete, and moreover,

$$(4.10) \quad Sp(L_{Dir}(v)) = \{\lambda \in \mathbb{C} : y_2(\pi, \lambda) = 0\}.$$

*Proof.* Firstly, we show that the operators  $L_{Dir}(v)$  and  $L_{Dir}(\bar{v})$  are formally adjoint, i.e.,

$$(4.11) \quad (L_{Dir}(v)y, h) = (f, L_{Dir}(\bar{v})h) \quad \text{if } y \in D(L_{Dir}(v)), \quad h \in D(L_{Dir}(\bar{v})).$$

Indeed, in view of (4.1), we have

$$\begin{aligned} (L_{Dir}(v)y, h) &= \frac{1}{\pi} \int_0^\pi \ell_Q(y)\bar{h}dx = \frac{1}{\pi} \int_0^\pi (-(y' - Qy)'\bar{h} - Qy'\bar{h}) dx \\ &= -\frac{1}{\pi}(y' - Qy)\bar{h}\Big|_0^\pi + \frac{1}{\pi} \int_0^\pi (y' - Qy)\bar{h}'dx - \int_0^\pi Qy'\bar{h}dx \\ &= 0 + \frac{1}{\pi} \int_0^\pi (y'\bar{h}' - Qy\bar{h}' - Qy'\bar{h}) dx. \end{aligned}$$

The same argument shows that

$$\frac{1}{\pi} \int_0^\pi (y'\bar{h}' - Qy\bar{h}' - Qy'\bar{h}) dx = (y, L_{Dir}(\bar{v})h),$$

which completes the proof of (4.11).

Now we apply Lemma 3 with  $A = L_{Dir}(v)$  and  $B = L_{Dir}(\bar{v})$ . Choose  $\lambda \in \mathbb{C}$  so that  $y_2(v; \pi, \lambda) \neq 0$  (in view of (2.44), see the remark before Theorem 11,  $y_2(v; \pi, \lambda)$  is a non-constant entire function, so such a choice is possible). Then, in view of (4.8), we have  $y_2(\bar{v}; \pi, \bar{\lambda}) \neq 0$  also. By Lemma 12,  $L_{Dir}(v) - \lambda$  maps bijectively  $D(L_{Dir}(v))$  onto  $H^0$  and  $L_{Dir}(\bar{v}) - \bar{\lambda}$  maps bijectively  $D(L_{Dir}(\bar{v}))$  onto  $H^0$ . Thus, by Lemma 3,  $D(L_{Dir}(v))$  is dense in  $H^0$  and  $(L_{Dir}(v))^* = L_{Dir}(\bar{v})$ , i.e., (a) and (b) hold. If  $y_2(v; \pi, \lambda) = 0$ , then  $\lambda$  is an eigenvalue of the operator  $L_{Dir}(v)$ , and  $y_2(v; x, \lambda)$  is a corresponding eigenvector. In view of Lemma 12, this means that (4.10) holds. Since  $y_2(\pi, \lambda)$  is a non-constant entire function, the set on the right in (4.10) is discrete. This completes the proof of (c).  $\square$

**Lemma 14.** (a) If  $f, g \in L^1([0, \pi])$  and  $f \sim \sum_{k=1}^\infty f_k \sqrt{2} \sin kx$ ,  $g \sim g_0 + \sum_{k=1}^\infty g_k \sqrt{2} \cos kx$  are, respectively, their sine and cosine Fourier series, then the following conditions are equivalent:

- (i)  $f$  is absolutely continuous,  $f(0) = f(\pi) = 0$  and  $g(x) = f'(x)$  a.e.;
- (ii)  $g_0 = 0$ ,  $g_k = kf_k \quad \forall k \in \mathbb{N}$ .

(b) If  $f, g \in L^1([0, \pi])$  and  $f \sim f_0 + \sum_{k=1}^\infty f_k \sqrt{2} \cos kx$  and  $g \sim \sum_{k=1}^\infty g_k \sqrt{2} \sin kx$  are, respectively, their cosine and sine Fourier series, then the following conditions are equivalent:

- (i\*)  $f$  is absolutely continuous and  $g(x) = f'(x)$  a.e.;
- (ii\*)  $g_k = -kf_k \quad k \in \mathbb{N}$ .

*Proof.* (a) We have (i)  $\Rightarrow$  (ii) because  $g_0 = \frac{1}{\pi} \int_0^\pi g(x)dx = \frac{1}{\pi}(f(\pi) - f(0)) = 0$ , and

$$g_k = \frac{1}{\pi} \int_0^\pi g(x) \sqrt{2} \cos kx dx = \frac{1}{\pi} f(x) \sqrt{2} \cos kx \Big|_0^\pi + \frac{k}{\pi} \int_0^\pi f(x) \sqrt{2} \sin kx dx = kf_k$$

for every  $k \in \mathbb{N}$ .

To prove that (ii)  $\Rightarrow$  (i), we set  $G(x) = \int_0^x g(t)dt$ ; then  $G(\pi) = G(0) = 0$  because  $g_0 = 0$ . The same computation as above shows that  $g_k = kG_k \forall k \in \mathbb{N}$ , so the sine Fourier coefficients of two  $L^1$ -functions  $G$  and  $f$  coincide. Thus,  $G(x) = f(x)$ , which completes the proof of (a).

The proof of (b) is omitted because it is similar to the proof of (a).  $\square$

Let

$$(4.12) \quad Q \sim \sum_{k=1}^{\infty} \tilde{q}(k) \sqrt{2} \sin kx$$

be the sine Fourier expansion of  $Q$ . We set also

$$(4.13) \quad \tilde{V}(0) = 0, \quad \tilde{V}(k) = k\tilde{q}(k) \quad \text{for } k \in \mathbb{N}.$$

**Proposition 15.** *In the above notations, if  $y \in H_{Dir}^1$ , then we have  $y = \sum_{k=1}^{\infty} y_k \sin kx \in D(L_{Dir})$  and  $\ell(y) = h = \sum_{k=1}^{\infty} h_k \sqrt{2} \sin kx \in H^0$  if and only if*

$$(4.14) \quad h_k = h_k(y) = k^2 y_k + \frac{1}{\sqrt{2}} \sum_{k=1}^{\infty} \left( \tilde{V}(|k-m|) - \tilde{V}(k+m) \right) y_m + C y_k, \quad \sum |h_k|^2 < \infty,$$

i. e.,

$$(4.15) \quad D(L_{Dir}) = \left\{ y \in H_{Dir}^1 : (h_k(y))_1^{\infty} \in \ell^2(\mathbb{N}) \right\}, \quad L_{Dir}(y) = \sum_{k=1}^{\infty} h_k(y) \sqrt{2} \sin kx.$$

*Proof.* By (4.1), if  $y \in D(L_{Dir})$ , then  $y \in H_{Dir}^1$  and

$$\ell(y) = -z' - Qy' + Cy = h \in L^2([0, \pi]),$$

where

$$(4.16) \quad z := y' - Qy \in W_1^1([0, \pi]).$$

Let

$$y \sim \sum_{k=1}^{\infty} y_k \sqrt{2} \sin kx, \quad z \sim \sum_{k=1}^{\infty} z_k \sqrt{2} \cos kx, \quad h \sim \sum_{k=1}^{\infty} h_k \sqrt{2} \sin kx$$

be the sine series of  $y$  and  $h$ , and the cosine series of  $z$ . Lemma 14 yields

$$z' \sim \sum_{k=1}^{\infty} (-kz_k) \sqrt{2} \sin kx, \quad y' \sim \sum_{k=1}^{\infty} ky_k \sqrt{2} \cos kx.$$

Therefore,

$$(4.17) \quad h_k = kz_k - (Qy')_k + Cy_k, \quad k \in \mathbb{N},$$

where  $(Qy')_k$  are the sine coefficients of the function  $Qy' \in L^1([0, \pi])$ .

By (4.16), we have

$$z_k = ky_k - (Qy)_k,$$

where  $(Qy)_k$  is the  $k$ -th cosine coefficient of  $Qy$ . It can be found by the formula

$$(Qy)_k = \frac{1}{\pi} \int_0^\pi Q(x)y(x)\sqrt{2} \cos kx dx = \sum_{m=1}^{\infty} a_m \cdot y_m,$$

with

$$\begin{aligned} a_m = a_m(k) &= \frac{1}{\pi} \int_0^\pi Q(x)\sqrt{2} \cos kx \sqrt{2} \sin mx dx = \frac{1}{\pi} \int_0^\pi Q(x)[\sin(m+k)x + \sin(m-k)x] dx \\ &= \frac{1}{\sqrt{2}} \begin{cases} \tilde{q}(m+k) + \tilde{q}(m-k), & m > k \\ \tilde{q}(2k), & m = k \\ \tilde{q}(m+k) - \tilde{q}(k-m) & m < k. \end{cases} \end{aligned}$$

Therefore,

(4.18)

$$(Qy)_k = \frac{1}{\sqrt{2}} \sum_{m=1}^{\infty} \tilde{q}(m+k)y_m - \frac{1}{\sqrt{2}} \sum_{m=1}^{k-1} \tilde{q}(k-m)y_m + \frac{1}{\sqrt{2}} \sum_{m=k+1}^{\infty} \tilde{q}(m-k)y_m.$$

In an analogous way we can find the sine coefficients of  $Qy'$  by the formula

$$(Qy')_k = \frac{1}{\pi} \int_0^\pi Q(x)y'(x)\sqrt{2} \sin kx dx = \sum_{m=1}^{\infty} b_m \cdot my_m,$$

where  $b_m$  are the cosine coefficients of  $Q(x)\sqrt{2} \sin kx$ , i.e.,

$$\begin{aligned} b_m = b_m(k) &= \frac{1}{\pi} \int_0^\pi Q(x)\sqrt{2} \sin kx \sqrt{2} \cos mx = \frac{1}{\pi} \int_0^\pi Q(x)[\sin(k+m)x + \sin(k-m)x] dx \\ &= \frac{1}{\sqrt{2}} \begin{cases} \tilde{q}(k+m) + \tilde{q}(k-m), & m < k, \\ \tilde{q}(2k), & m = k, \\ \tilde{q}(k+m) - \tilde{q}(m-k) & m > k. \end{cases} \end{aligned}$$

Thus we get

(4.19)

$$(Qy')_k = \frac{1}{\sqrt{2}} \sum_{m=1}^{\infty} \tilde{q}(m+k)my_m + \frac{1}{\sqrt{2}} \sum_{m=1}^{k-1} \tilde{q}(k-m)my_m - \frac{1}{\sqrt{2}} \sum_{m=k+1}^{\infty} \tilde{q}(m-k)my_m.$$

Finally, (4.18) and (4.19), imply that

$$\begin{aligned} &k^2 y_k - k(Qy)_k - (Qy')_k \\ &= k^2 y_k - \frac{1}{\sqrt{2}} \sum_{m=1}^{\infty} (m+k)\tilde{q}(m+k) + \frac{1}{\sqrt{2}} \sum_{m=k+1}^{\infty} (m-k)\tilde{q}(m-k) + \frac{1}{\sqrt{2}} \sum_{m=1}^{k-1} (k-m)\tilde{q}(k-m). \end{aligned}$$

Hence, in view of (4.13), we have

$$h_k = k^2 y_k + \frac{1}{\sqrt{2}} \sum_{m=1}^{\infty} \left( \tilde{V}(|k-m|) - \tilde{V}(k+m) \right) y_m + C y_k,$$

i.e., (4.14) holds.

Conversely, if (4.14) holds, then going back we can see, by (4.17), that  $z = y' - Qy \in L^2([0, \pi])$  has the property that  $kz_k$ ,  $k \in \mathbb{N}$ , are the sine coefficients of an  $L^1([0, \pi])$ -function. Therefore, by Lemma 14,  $z$  is absolutely continuous and those numbers are the sine coefficients of its derivative  $z'$ . Hence,  $z = y' - Qy \in W_1^1([0, \pi])$  and  $\ell(y) = h$ , i.e.,  $y \in D(L_{Dir})$  and  $L_{Dir}(y) = h$ .  $\square$

Let

$$\mathcal{F} : H^0 \rightarrow \ell^2(\mathcal{N})$$

be the Fourier isomorphisms that corresponds to each function  $f \in H^0$  the sequence  $(f_k)_{k \in \mathbb{N}}$  of its Fourier coefficients  $f_k = (f, \sqrt{2} \sin kx)$ , and let  $\mathcal{F}^{-1}$  be the inverse Fourier isomorphism.

Consider the unbounded operator  $\mathcal{L}_d$  and acting in  $\ell^2(\mathbb{N})$  as

$$(4.20) \quad \mathcal{L}_d(z) = (h_k(z))_{k \in \mathbb{N}}, \quad h_k(z) = k^2 z_k + \frac{1}{\sqrt{2}} \sum_{m \in \mathbb{N}} \left( \tilde{V}(|k-m|) - \tilde{V}(k+m) \right) z_m + C z_k$$

in the domain

$$(4.21) \quad D(\mathcal{L}_d) = \left\{ z \in \ell^2(|k|, \mathbb{N}) : \mathcal{L}_d(z) \in \ell^2(\mathbb{N}) \right\},$$

where  $\ell^2(|k|, \mathbb{N})$  is the weighted  $\ell^2$ -space

$$\ell^2(|k|, \mathbb{N}) = \left\{ z = (z_k)_{k \in \mathbb{N}} : \sum_k |k|^2 |z_k|^2 < \infty \right\}.$$

In view of (3.6) and Proposition 15, the following theorem holds.

**Theorem 16.** *In the above notations, we have*

$$(4.22) \quad D(L_{Dir}) = \mathcal{F}^{-1} (D(\mathcal{L}_d))$$

and

$$(4.23) \quad L_{Dir} = \mathcal{F}^{-1} \circ \mathcal{L}_d \circ \mathcal{F}.$$

If it does not lead to confusion, for convenience we will loosely use one and the same notation  $L_{Dir}$  for the operators  $L_{Dir}$  and  $\mathcal{L}_d$ .

## 5. LOCALIZATION OF SPECTRA

Throughout this section we need the following lemmas.

**Lemma 17.** *For each  $n \in \mathbb{N}$*

$$(5.1) \quad \sum_{k \neq \pm n} \frac{1}{|n^2 - k^2|} < \frac{2 \log 6n}{n};$$

$$(5.2) \quad \sum_{k \neq \pm n} \frac{1}{|n^2 - k^2|^2} < \frac{4}{n^2}.$$

The proof is elementary, and therefore, we omit it.

**Lemma 18.** *There exists an absolute constant  $C > 0$  such that*

(a) *if  $n \in \mathbb{N}$  and  $b \geq 2$ , then*

$$(5.3) \quad \sum_k \frac{1}{|n^2 - k^2| + b} \leq C \frac{\log b}{\sqrt{b}};$$

(b) *if  $n \geq 0$  and  $b > 0$  then*

$$(5.4) \quad \sum_{k \neq \pm n} \frac{1}{|n^2 - k^2|^2 + b^2} \leq \frac{C}{(n^2 + b^2)^{1/2}(n^4 + b^2)^{1/4}}.$$

A proof of this lemma can be found in [4], see Appendix, Lemma 79.

We study the localization of spectra of the operators  $L_{Per^\pm}$  and  $L_{Dir}$  by using their Fourier representations. By (3.14) and Theorem 11, each of the operators  $L = L_{Per^\pm}$  has the form

$$(5.5) \quad L = L^0 + V,$$

where the operators  $L^0$  and  $V$  are defined by their action on the sequence of Fourier coefficients of any  $y = \sum_{\Gamma_{Per^\pm}} y_k \exp ikx \in H^1_{Per^\pm}$ :

$$(5.6) \quad L^0 : (y_k) \rightarrow (k^2 y_k), \quad k \in \Gamma_{Per^\pm}$$

and

$$(5.7) \quad V : (y_m) \rightarrow (z_k), \quad z_k = \sum_m V(k - m) y_m, \quad k, m \in \Gamma_{Per^\pm}.$$

(We suppress in the notations of  $L^0$  and  $V$  the dependence on the boundary conditions  $Per^\pm$ .)

In the case of Dirichlet boundary condition, by (4.20) and Theorem 16, the operator  $L = L_{Dir}$  has the form (5.5), where the operators  $L^0$  and  $V$  are defined by their action on the sequence of Fourier coefficients of any  $y = \sum_{\mathbb{N}} y_k \sqrt{2} \sin kx \in H^1_{Dir}$ :

$$(5.8) \quad L^0 : (y_k) \rightarrow (k^2 y_k), \quad k \in \mathbb{N}$$

and

$$(5.9) \quad V : (y_m) \rightarrow (z_k), \quad z_k = \frac{1}{\sqrt{2}} \sum_m \left( \tilde{V}(|k - m|) - \tilde{V}(k + m) \right) y_m, \quad k, m \in \mathbb{N}.$$

(We suppress in the notations of  $L^0$  and  $V$  the dependence on the boundary conditions  $Dir$ .)

Of course, in the regular case where  $v \in L^2([0, \pi])$ , the operators  $L^0$  and  $V$  are, respectively, the Fourier representations of  $-d^2/dx^2$  and the multiplication operator  $y \rightarrow v \cdot y$ . But if  $v \in H^{-1}_{loc}(\mathbb{R})$  is a singular periodic potential, then the situation is more complicated, so we are able to write (5.5) with (5.6) and (5.7), or (5.8) and (5.9), only after having the results from Section 3 and 4 (see Theorem 11 and Theorem 16).

In view of (5.6) and (5.8) the operator  $L^0$  is diagonal, so, for  $\lambda \neq k^2$ ,  $k \in \Gamma_{bc}$ , we may consider (in the space  $\ell^2(\Gamma_{bc})$ ) its inverse operator

$$(5.10) \quad R_\lambda^0 : (z_k) \rightarrow \left( \frac{z_k}{\lambda - k^2} \right), \quad k \in \Gamma_{bc}.$$

One of the technical difficulties that arises for singular potentials is connected with the standard perturbation type formulae for the resolvent  $R_\lambda = (\lambda - L^0 - V)^{-1}$ . In the case where  $v \in L^2([0, \pi])$  one can represent the resolvent in the form (e.g., see [4], Section 1.2)

$$(5.11) \quad R_\lambda = (1 - R_\lambda^0 V)^{-1} R_\lambda^0 = \sum_{k=0}^{\infty} (R_\lambda^0 V)^k R_\lambda^0,$$

or

$$(5.12) \quad R_\lambda = R_\lambda^0 (1 - V R_\lambda^0)^{-1} = \sum_{k=0}^{\infty} R_\lambda^0 (V R_\lambda^0)^k.$$

The simplest conditions that guarantee the convergence of the series (5.11) or (5.12) in  $\ell^2$  are

$$\|R_\lambda^0 V\| < 1, \quad \text{respectively,} \quad \|V R_\lambda^0\| < 1.$$

Each of these conditions can be easily verified for large enough  $n$  if  $Re \lambda \in [n-1, n+1]$  and  $|\lambda - n^2| \geq C(\|v\|)$ , which leads to a series of results on the spectra, zones of instability and spectral decompositions.

The situation is more complicated if  $v$  is a singular potential. Then, in general, there are no good estimates for the norms of  $R_\lambda^0 V$  and  $V R_\lambda^0$ . However, one can write (5.11) or (5.12) as

$$(5.13) \quad R_\lambda = R_\lambda^0 + R_\lambda^0 V R_\lambda^0 + R_\lambda^0 V R_\lambda^0 V R_\lambda^0 + \dots = K_\lambda^2 + \sum_{m=1}^{\infty} K_\lambda (K_\lambda V K_\lambda)^m K_\lambda,$$

provided

$$(5.14) \quad (K_\lambda)^2 = R_\lambda^0.$$

We define an operator  $K = K_\lambda$  with the property (5.14) by its matrix representation

$$(5.15) \quad K_{jm} = \frac{1}{(\lambda - j^2)^{1/2}} \delta_{jm}, \quad j, m \in \Gamma_{bc},$$

where

$$z^{1/2} = \sqrt{r} e^{i\varphi/2} \quad \text{if} \quad z = r e^{i\varphi}, \quad 0 \leq \varphi < 2\pi.$$

Then  $R_\lambda$  is well-defined if

$$(5.16) \quad \|K_\lambda V K_\lambda : \ell^2(\Gamma_{bc}) \rightarrow \ell^2(\Gamma_{bc})\| < 1.$$

In view of (2.14), (5.7) and (5.15), the matrix representation of  $KVK$  for periodic or anti-periodic boundary conditions  $bc = Per^\pm$  is

$$(5.17) \quad (KVK)_{jm} = \frac{V(j-m)}{(\lambda - j^2)^{1/2} (\lambda - m^2)^{1/2}} = \frac{i(j-m)q(j-m)}{(\lambda - j^2)^{1/2} (\lambda - m^2)^{1/2}},$$

where  $j, m \in 2\mathbb{Z}$  for  $bc = Per^+$ , and  $j, m \in 1 + 2\mathbb{Z}$  for  $bc = Per^-$ . Therefore, we have for its Hilbert–Schmidt norm (which majorizes its  $\ell^2$ -norm)

$$(5.18) \quad \|KVK\|_{HS}^2 = \sum_{j, m \in \Gamma_{Per^\pm}} \frac{(j-m)^2 |q(j-m)|^2}{|\lambda - j^2| |\lambda - m^2|}.$$

By (4.13), (5.9) and (5.15), the matrix representation of  $KVK$  for Dirichlet boundary conditions  $bc = Dir$  is

$$(5.19) \quad (KVK)_{jm} = \frac{1}{\sqrt{2}} \frac{\tilde{V}(|j-m|)}{(\lambda - j^2)^{1/2} (\lambda - m^2)^{1/2}} - \frac{1}{\sqrt{2}} \frac{\tilde{V}(j+m)}{(\lambda - j^2)^{1/2} (\lambda - m^2)^{1/2}} \\ = \frac{1}{\sqrt{2}} \frac{|j-m| \tilde{q}(|j-m|)}{(\lambda - j^2)^{1/2} (\lambda - m^2)^{1/2}} - \frac{1}{\sqrt{2}} \frac{(j+m) \tilde{q}(j+m)}{(\lambda - j^2)^{1/2} (\lambda - m^2)^{1/2}},$$

where  $j, m \in \mathbb{N}$ . Therefore, we have for its Hilbert–Schmidt norm (which majorizes its  $\ell^2$ -norm)

$$(5.20) \quad \|KVK\|_{HS}^2 \leq 2 \sum_{j, m \in \mathbb{N}} \frac{(j-m)^2 |\tilde{q}(|j-m|)|^2}{|\lambda - j^2| |\lambda - m^2|} + 2 \sum_{j, m \in \mathbb{N}} \frac{(j+m)^2 |\tilde{q}(j+m)|^2}{|\lambda - j^2| |\lambda - m^2|}.$$

We set for convenience

$$(5.21) \quad \tilde{q}(0) = 0, \quad \tilde{r}(s) = \tilde{q}(|s|) \quad \text{for } s \neq 0, \quad s \in \mathbb{Z}.$$

In view of (5.20) and (5.21), we have

$$(5.22) \quad \|KVK\|_{HS}^2 \leq \sum_{j, m \in \mathbb{Z}} \frac{(j-m)^2 |\tilde{r}(j-m)|^2}{|\lambda - j^2| |\lambda - m^2|}.$$

We divide the plane  $\mathbb{C}$  into strips, correspondingly to the boundary conditions, as follows:

if  $bc = Per^+$  then  $\mathbb{C} = H_0 \cup H_2 \cup H_4 \cup \dots$ , and

if  $bc = Per^-$  then  $\mathbb{C} = H_1 \cup H_3 \cup H_5 \cup \dots$ ,

where

$$(5.23) \quad H_0 = \{\lambda \in \mathbb{C} : Re \lambda \leq 1\}, \quad H_1 = \{\lambda \in \mathbb{C} : Re \lambda \leq 4\},$$

$$(5.24) \quad H_n = \{\lambda \in \mathbb{C} : (n-1)^2 \leq Re \lambda \leq (n+1)^2\}, \quad n \geq 2;$$

- if  $bc = Dir$ , then  $\mathbb{C} = G_1 \cup G_2 \cup G_3 \cup \dots$ , where

$$(5.25) \quad G_1 = \{\lambda : Re \lambda \leq 2\}, \quad G_n = \{\lambda : (n-1)n \leq Re \lambda \leq n(n+1)\}, \quad n \geq 2.$$

Consider also the discs

$$(5.26) \quad D_n = \{\lambda \in \mathbb{C} : |\lambda - n^2| < n/4\}, \quad n \in \mathbb{N},$$

Then, for  $n \geq 3$ ,

$$(5.27) \quad \sum_{k \in n+2\mathbb{Z}} \frac{1}{|\lambda - k^2|} \leq C_1 \frac{\log n}{n}, \quad \sum_{k \in n+2\mathbb{Z}} \frac{1}{|\lambda - k^2|^2} \leq \frac{C_1}{n^2}, \quad \forall \lambda \in H_n \setminus D_n,$$

and

$$(5.28) \quad \sum_{k \in \mathbb{Z}} \frac{1}{|\lambda - k^2|} \leq C_1 \frac{\log n}{n}, \quad \sum_{k \in \mathbb{Z}} \frac{1}{|\lambda - k^2|^2} \leq \frac{C_1}{n^2}, \quad \forall \lambda \in G_n \setminus D_n,$$

where  $C_1$  is an absolute constant.

Indeed, if  $\lambda \in H_n$ , then one can easily see that

$$|\lambda - k^2| \geq |n^2 - k^2|/4 \quad \text{for } k \in n + 2\mathbb{Z}.$$

Therefore, if  $\lambda \in H_n \setminus D_n$ , then (5.1) implies that

$$\sum_{k \in n + 2\mathbb{Z}} \frac{1}{|\lambda - k^2|} \leq \frac{2}{n/4} + \sum_{k \neq \pm n} \frac{4}{|n^2 - k^2|} \leq \frac{8}{n} + \frac{8 \log 6n}{n} \leq C_1 \frac{\log n}{n},$$

which proves the first inequality in (5.27). The second inequality in (5.27) and the inequalities in (5.28) follow from Lemma 17 by the same argument.

Next we estimate the Hilbert–Schmidt norm of the operator  $K_\lambda V K_\lambda$  for  $bc = Per^\pm$  or  $Dir$ , and correspondingly,  $\lambda \in H_n \setminus D_n$  or  $\lambda \in G_n \setminus D_n$ ,  $n \in \mathbb{N}$ .

For each  $\ell^2$ -sequence  $x = (x(j))_{j \in \mathbb{Z}}$  and  $m \in \mathbb{N}$  we set

$$(5.29) \quad \mathcal{E}_m(x) = \left( \sum_{|j| \geq m} |x(j)|^2 \right)^{1/2}.$$

**Lemma 19.** *Let  $v = Q'$ , where  $Q(x) = \sum_{k \in 2\mathbb{Z}} q(k) e^{ikx} = \sum_{m=1}^{\infty} \tilde{q}(m) \sqrt{2} \sin mx$  is a  $\pi$ -periodic  $L^2([0, \pi])$  function, and let*

$$q = (q(k))_{k \in 2\mathbb{Z}}, \quad \tilde{q} = (\tilde{q}(m))_{m \in \mathbb{N}}$$

*be the sequences of its Fourier coefficients respect to the orthonormal bases  $\{e^{ikx}, k \in 2\mathbb{Z}\}$  and  $\{\sqrt{2} \sin mx, m \in \mathbb{N}\}$ . Then, for  $n \geq 3$ ,*

$$(5.30) \quad \|K_\lambda V K_\lambda\|_{HS} \leq C \left( \mathcal{E}_{\sqrt{n}}(q) + \|q\|/\sqrt{n} \right), \quad \lambda \in H_n \setminus D_n, \quad bc = Per^\pm,$$

and

$$(5.31) \quad \|K_\lambda V K_\lambda\|_{HS} \leq C \left( \mathcal{E}_{\sqrt{n}}(\tilde{q}) + \|\tilde{q}\|/\sqrt{n} \right), \quad \lambda \in G_n \setminus D_n, \quad bc = Dir,$$

where  $C$  is an absolute constant.

*Proof.* Fix  $n \in \mathbb{N}$ . We prove only (5.30) because, in view of (5.21) and (5.22), the proof of (5.31) is practically the same (the only difference is that the summation indices will run in  $\mathbb{Z}$ ).

By (5.18),

$$(5.32) \quad \|K V K\|_{HS}^2 \leq \sum_s \left( \sum_m \frac{s^2}{|\lambda - m^2| |\lambda - (m+s)^2|} \right) |q(s)|^2 = \Sigma_1 + \Sigma_2 + \Sigma_3,$$

where  $s \in 2\mathbb{Z}$ ,  $m \in n + 2\mathbb{Z}$  and

$$(5.33) \quad \Sigma_1 = \sum_{|s| \leq \sqrt{n}} \cdots, \quad \Sigma_2 = \sum_{\sqrt{n} < |s| \leq 4n} \cdots, \quad \Sigma_3 = \sum_{|s| > 4n} \cdots.$$

The Cauchy inequality implies that

$$(5.34) \quad \sum_{m \in n+2\mathbb{Z}} \frac{1}{|\lambda - m^2| |\lambda - (m+s)^2|} \leq \sum_{m \in n+2\mathbb{Z}} \frac{1}{|\lambda - m^2|^2}.$$

Thus, by (5.28) and (5.29),

$$(5.35) \quad \Sigma_1 \leq \sum_{|s| \leq \sqrt{n}} |q(s)|^2 s^2 \frac{C_1}{n^2} \leq (\sqrt{n})^2 \frac{C_1}{n^2} \|q\|^2 = \frac{C_1}{n} \|q\|^2, \quad \lambda \in H_n \setminus D_n,$$

$$(5.36) \quad \Sigma_2 \leq (4n)^2 \frac{C_1}{n^2} \sum_{|s| > \sqrt{n}} |q(s)|^2 = 16C_1 \left( \mathcal{E}_{\sqrt{n}}(q) \right)^2, \quad \lambda \in H_n \setminus D_n.$$

Next we estimate  $\Sigma_3$  for  $n \geq 3$ . First we show that if  $|s| > 4n$  then

$$(5.37) \quad \sum_m \frac{s^2}{|\lambda - m^2| |\lambda - (m+s)^2|} \leq 16 \frac{C_1 \log n}{n}, \quad \lambda \in H_n \setminus D_n.$$

Indeed, if  $|m| \geq |s|/2$ , then (since  $|s|/4 > n \geq 3$ )

$$|\lambda - m^2| \geq m^2 - |\operatorname{Re} \lambda| \geq s^2/4 - (n+1)^2 > s^2/4 - (|s|/4 + 1)^2 \geq s^2/8.$$

Thus, by (5.27),

$$\sum_{|m| \geq |s|/2} \frac{s^2}{|\lambda - m^2| |\lambda - (m+s)^2|} \leq \sum_m \frac{8}{|\lambda - (m+s)^2|} \leq 8 \frac{C_1 \log n}{n}$$

for  $\lambda \in H_n \setminus D_n$ . If  $|m| < |s|/2$ , then  $|m+s| > |s| - |s|/2 = |s|/2$ , and therefore,

$$|\lambda - (m+s)^2| \geq (m+s)^2 - |\operatorname{Re} \lambda| \geq s^2/4 - (n+1)^2 \geq s^2/8.$$

Therefore, by (5.27),

$$\sum_{|m| < |s|/2} \frac{s^2}{|\lambda - m^2| |\lambda - (m+s)^2|} \leq \sum_m \frac{8}{|\lambda - m^2|} \leq 8 \frac{C_1 \log n}{n}$$

for  $\lambda \in H_n \setminus D_n$ , which proves (5.37).

Now, by (5.37),

$$(5.38) \quad \Sigma_3 \leq 16 \frac{C_1 \log n}{n} \sum_{|s| \geq 4n} |q(s)|^2 = 16 \frac{C_1 \log n}{n} (\mathcal{E}_{4n}(q))^2.$$

Finally, (5.32), (5.35), (5.36) and (5.38) imply (5.30). □

Let  $H^N$  denote the half-plane

$$(5.39) \quad H^N = \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda < N^2 + N\}, \quad N \in \mathbb{N},$$

and let  $R_N$  be the rectangle

$$(5.40) \quad R_N = \{\lambda \in \mathbb{C} : -N < \operatorname{Re} \lambda < N^2 + N, \quad |\operatorname{Im} \lambda| < N\}.$$

**Lemma 20.** *In the above notations, for  $bc = Per^\pm$  or  $Dir$ , we have*

$$(5.41) \quad \sup \{ \|K_\lambda V K_\lambda\|_{HS}, \lambda \in H^N \setminus R_N \} \leq C \left( \frac{(\log N)^{1/2}}{N^{1/4}} \|q\| + \mathcal{E}_{4\sqrt{N}}(q) \right),$$

where  $C$  is an absolute constant, and  $q$  is replaced by  $\tilde{q}$  if  $bc = Dir$ .

*Proof.* Consider the sequence  $r = (r(s))_{s \in \mathbb{Z}}$ , defined by

$$(5.42) \quad r(s) = \begin{cases} 0 & \text{for odd } s, \\ \max(|q(s)|, |q(-s)|) & \text{for even } s. \end{cases}$$

Then, in view of (5.42), we have  $r \in \ell^2(\mathbb{Z})$  and  $\|r\| \leq 2\|q\|$ .

If  $bc = Per^\pm$ , then we have, by (5.18),

$$(5.43) \quad \|KVK\|_{HS}^2 \leq \sum_{j,m \in \mathbb{Z}} \frac{(j-m)^2 |r(j-m)|^2}{|\lambda - j^2| |\lambda - m^2|}.$$

On the other hand, if  $bc = Dir$ , then (5.22) gives the same estimate for  $\|KVK\|_{HS}^2$  but with  $r$  replaced by the sequence  $\tilde{r} \in (5.21)$ . So, to prove (5.41), it is enough to estimate the right side of (5.43) for  $\lambda \in H^N \setminus R_N$ .

If  $Re \lambda \leq -N$ , then (5.43) implies that

$$\|KVK\|_{HS}^2 \leq \sum_{j,m \in \mathbb{Z}} \frac{(j-m)^2 |r(j-m)|^2}{|N + j^2| |N + m^2|}.$$

On the other hand, for  $b \geq 1$ , the following estimate holds:

$$(5.44) \quad \sum_{j,m \in \mathbb{Z}} \frac{(j-m)^2 |r(j-m)|^2}{|b^2 + j^2| |b^2 + m^2|} \leq 4\|r\|^2 \frac{1 + \pi}{b}.$$

Indeed, the left-hand side of (5.44) does not exceed

$$\begin{aligned} & \sum_{j,m \in \mathbb{Z}} \frac{2(j^2 + m^2) |r(j-m)|^2}{|b^2 + j^2| |b^2 + m^2|} \\ & \leq 2 \sum_m \frac{1}{|b^2 + m^2|} \sum_j |r(j-m)|^2 + 2 \sum_j \frac{1}{|b^2 + j^2|} \sum_m |r(j-m)|^2 \\ & \leq 4\|r\|^2 \left( \frac{1}{b^2} + 2 \int_0^\infty \frac{1}{b^2 + x^2} dx \right) = 4\|r\|^2 \left( \frac{1}{b^2} + \frac{\pi}{b} \right) \leq 4\|r\|^2 \frac{1 + \pi}{b}. \end{aligned}$$

Now, with  $b = \sqrt{N}$ , (5.44) yields

$$(5.45) \quad \|KVK\|_{HS}^2 \leq C \frac{\|r\|^2}{\sqrt{N}} \quad \text{if } Re \lambda \leq -N,$$

where  $C$  is an absolute constant.

By (5.43) and the elementary inequality

$$|\lambda - m^2| = \sqrt{(x - m^2)^2 + y^2} \geq (|x - m^2| + |y|)/\sqrt{2}, \quad \lambda = x + iy,$$

we have

$$(5.46) \quad \|K_\lambda V K_\lambda\|_{HS}^2 \leq \sum_{s \in \mathbb{Z}} \sigma(x, y; s) |r(s)|^2,$$

where

$$(5.47) \quad \sigma(x, y; s) = \sum_{m \in \mathbb{Z}} \frac{2s^2}{(|x - m^2| + |y|)(|x - (m + s)^2| + |y|)}.$$

Now, suppose that  $\lambda = x + iy \in H^N \setminus R_N$  and  $|y| \geq N$ . By (5.25) and (5.39),

$$H^N \subset \bigcup_{1 \leq n \leq N} G_n,$$

so  $\lambda \in G_n$  for some  $n \leq N$ . Moreover,

$$(5.48) \quad \sigma(x, y; s) \leq 16\sigma(n^2, N; s) \quad \text{if } \lambda \in G_n, \quad |y| \geq N.$$

Indeed, then one can easily see that

$$|x - m^2| + |y| \geq \frac{1}{4}(|n^2 - m^2| + N), \quad m \in \mathbb{Z},$$

which implies (5.48).

By (5.47) and (5.48), if  $\lambda = x + iy \in G_n \setminus R_N$  and  $|y| \geq N$ , then

$$(5.49) \quad \|K_\lambda V K_\lambda\|_{HS}^2 \leq \sum_s \sigma(n^2, N; s) |r(s)|^2 \leq \Sigma_1 + \Sigma_2 + \Sigma_3,$$

where

$$\Sigma_1 = \sum_{|s| \leq 4\sqrt{N}} \sigma(n^2, N; s) |r(s)|^2, \quad \Sigma_2 = \sum_{4\sqrt{N} < |s| \leq 4n} \dots, \quad \Sigma_3 = \sum_{|s| > 4n} \dots.$$

If  $|s| \leq 4\sqrt{N}$ , then the Cauchy inequality and (5.4) imply that

$$\sigma(n^2, N; s) \leq 32N \cdot \sum_m \frac{1}{|n^2 - m^2|^2 + N^2} \leq 32N \frac{C}{N(n^4 + N^2)^{1/4}} \leq \frac{32C}{\sqrt{N}}.$$

Thus

$$(5.50) \quad \Sigma_1 \leq \frac{32C}{\sqrt{N}} \|r\|^2.$$

If  $4\sqrt{N} < |s| \leq 4n$  then the Cauchy inequality and (5.4) yield

$$\sigma(n^2, N; s) \leq 32n^2 \cdot \sum_m \frac{1}{|n^2 - m^2|^2 + N^2} \leq 32n^2 \frac{C}{N(n^4 + N^2)^{1/4}} \leq 32C$$

because  $n \leq N$ . Thus

$$(5.51) \quad \Sigma_2 \leq 32C \cdot \left( \mathcal{E}_{4\sqrt{N}}(r) \right)^2.$$

Let  $|s| > 4n$ . If  $|m| < |s|/2$  then  $|m + s| \geq |s|/2$ , and therefore,

$$|n^2 - (m + s)^2| \geq |m + s|^2 - n^2 \geq (|s|/2)^2 - (|s|/4)^2 \geq s^2/8.$$

Thus, by (5.3),

$$\sum_{|m| < |s|/2} \frac{s^2}{(|n^2 - m^2| + N)(|n^2 - (m + s)^2| + N)} \leq \sum_m \frac{8}{|n^2 - m^2| + N} \leq 8C \frac{\log N}{\sqrt{N}}.$$

If  $|m| \geq |s|/2$ , then we have the same estimate because  $m^2 - n^2 \geq (|s|/2)^2 - (|s|/4)^2 \geq s^2/8$ , and therefore, again by (5.3),

$$\sum_{|m| \geq |s|/2} \frac{s^2}{(|n^2 - m^2| + N)(|n^2 - (m + s)^2| + N)} \leq \sum_m \frac{8}{|n^2 - (m + s)^2| + N} \leq 8C \frac{\log N}{\sqrt{N}}.$$

Thus  $\sigma(n^2, N; s) \leq 32C(\log N)/\sqrt{N}$ , so we have

$$(5.52) \quad \Sigma_3 \leq 32C \|r\|^2 \frac{\log N}{\sqrt{N}}.$$

Now, in view of (5.42) and (5.21), the estimates (5.45) and (5.50)–(5.52) yield (5.41), which completes the proof.  $\square$

**Theorem 21.** *For each periodic potential  $v \in H_{loc}^{-1}(\mathbb{R})$ , the spectrum of the operators  $L_{bc}(v)$  with  $bc = Per^\pm$ ,  $Dir$  is discrete. Moreover, if  $bc = Per^\pm$  then, respectively, for each large enough even number  $N^+ > 0$  or odd number  $N^-$ , we have*

$$(5.53) \quad Sp(L_{Per^\pm}) \subset R_{N^\pm} \cup \bigcup_{n \in N^\pm + 2\mathbb{N}} D_n,$$

where  $R_N$  is the rectangle (5.40),  $D_n = \{\lambda : |\lambda - n^2| < n/4\}$ , and

$$(5.54) \quad \#(Sp(L_{Per^\pm}) \cap R_{N^\pm}) = \begin{cases} 2N^+ + 1 \\ 2N^- \end{cases}, \quad \#(Sp(L_{Per^\pm}) \cap D_n) = 2 \text{ for } n \in N^\pm + 2\mathbb{N},$$

where each eigenvalue is counted with its algebraic multiplicity.

If  $bc = Dir$  then, for each large enough number  $N \in \mathbb{N}$ , we have

$$(5.55) \quad Sp(L_{Dir}) \subset R_N \cup \bigcup_{n=N+1}^{\infty} D_n$$

and

$$(5.56) \quad \#(Sp(L_{Dir}) \cap R_N) = N + 1, \quad \#(Sp(L_{Dir}) \cap D_n) = 1 \text{ for } n > N.$$

*Proof.* In view of (5.13), the resolvent  $R_\lambda$  is well defined if  $\|KVK\| < 1$ . Therefore, (5.53) and (5.55) follow from Lemmas 19 and 20.

To prove (5.54) and (5.56) we use a standard method of continuous parametrization. Let us consider the one-parameter family of potentials  $v_\tau(x) = \tau v(x)$ ,  $\tau \in [0, 1]$ . Then, in the notation of Lemma 19, we have  $v_\tau = \tau \cdot Q'$ , and the assertions of Lemmas 19 and 20 hold with  $q$  and  $\tilde{q}$  replaced, respectively, by  $\tau \cdot q$  and  $\tau \cdot \tilde{q}$ . Therefore, (5.53) and (5.55) hold, with  $L_{bc} = L_{bc}(v)$  replaced by  $L_{bc}(v_\tau)$ . Moreover, the corresponding resolvents  $R_\lambda(L_{bc}(v_\tau))$  are analytic in  $\lambda$  and continuous in  $\tau$ .

Now, let us prove the first formula in (5.54) in the case  $bc = Per^+$ . Fix an even  $N^+ \in \mathbb{N}$  so that (5.53) holds, and consider the projection

$$(5.57) \quad P^N(\tau) = \frac{1}{2\pi i} \int_{\lambda \in \partial R_N} (\lambda - L_{Per^+}(v_\tau))^{-1} d\lambda.$$

The dimension  $\dim(P^N(\tau))$  gives the number of eigenvalues inside the rectangle  $R_N$ . Being an integer, it is a constant, so, by the relation (a) at the beginning of Section 3, we have

$$\dim P^N(1) = \dim P^N(0) = 2N^+ + 1.$$

In view of the relations (a)–(c) at the beginning of Section 3, the same argument shows that (5.54) and (5.56) hold in all cases.  $\square$

*Remark.* It is possible to choose the disks  $D_n = \{\lambda : |\lambda - n^2| < r_n\}$  in Lemma 19 so that  $r_n/n \rightarrow 0$ . Indeed, if we take  $r_n = n/\varphi(n)$ , where  $\varphi(n) \rightarrow \infty$  but  $\varphi(n)/\sqrt{n} \rightarrow 0$  and  $\varphi(n)\mathcal{E}_{\sqrt{n}}(W) \rightarrow 0$ , then, modifying the proof of Lemma 19, one can get that  $\|K_\lambda V K_\lambda\|_{HS} \rightarrow 0$  as  $n \rightarrow \infty$ . Therefore, Theorem 21 could be sharpened: *for large enough  $N^\pm$  and  $N$ , (5.53)–(5.56) hold with  $D_n = \{\lambda : |\lambda - n^2| < r_n\}$  for some sequence  $\{r_n\}$  such that  $r_n/n \rightarrow 0$ .*

## 6. CONCLUSION

The main goal of our paper was to bring into the framework of Fourier method the analysis of Hill–Schrödinger operators with periodic  $H_{loc}^{-1}(\mathbb{R})$  potential, considered with periodic, antiperiodic and Dirichlet boundary conditions. As soon as this is done we can apply the methodology developed in [15, 2, 3] (see a detailed exposition in [4]) to study the relationship between smoothness of a potential  $v$  and rates of decay of spectral gaps  $\gamma_n = \lambda_n^+ - \lambda_n^-$  and deviations  $\delta_n$  under a weak a priori assumption  $v \in H^{-1}$ . (In [15, 2, 3, 4] the basic assumption is  $v \in L^2([0, \pi])$ .) Still, there is a lot of technical problems; we present all the details elsewhere. But now let us give these results as stronger versions of Theorems 54 and 67 in [4].

**Theorem 22.** *Let  $L = L^0 + v(x)$  be a Hill–Schrödinger operator with a real-valued  $\pi$ -periodic potential  $v \in H_{loc}^{-1}(\mathbb{R})$ , and let  $\gamma = (\gamma_n)$  be its gap sequence. If  $\omega = (\omega(n))_{n \in \mathbb{Z}}$  is a sub-multiplicative weight such that*

$$(6.1) \quad \frac{\log \omega(n)}{n} \searrow 0 \quad \text{as } n \rightarrow \infty,$$

*then, with*

$$\Omega = (\Omega(n)), \quad \Omega(n) = \frac{\omega(n)}{n},$$

*we have*

$$(6.2) \quad \gamma \in \ell^2(\mathbb{N}, \Omega) \Rightarrow v \in H(\Omega).$$

*If  $\Omega$  is a sub-multiplicative weight of exponential type, i.e.,*

$$(6.3) \quad \lim_{n \rightarrow \infty} \frac{\log \Omega(n)}{n} > 0,$$

then there exists  $\varepsilon > 0$  such that

$$(6.4) \quad \gamma \in \ell^2(\mathbb{N}, \Omega) \Rightarrow v \in H(e^{\varepsilon|n|}).$$

The following theorem summarizes our results about the Hill–Schrödinger operator with complex-valued potentials  $v \in H^{-1}$ .

**Theorem 23.** *Let  $L = L^0 + v(x)$  be the Hill–Schrödinger operator with a  $\pi$ -periodic potential  $v \in H_{loc}^{-1}(\mathbb{R})$ .*

*Then, for large enough  $n > N(v)$  the operator  $L$  has, in a disc of center  $n^2$  and radius  $r_n = n/4$ , exactly two (counted with their algebraic multiplicity) periodic (for even  $n$ ), or antiperiodic (for odd  $n$ ) eigenvalues  $\lambda_n^+$  and  $\lambda_n^-$ , and one Dirichlet eigenvalue  $\mu_n$ .*

Let

$$(6.5) \quad \Delta_n = |\lambda_n^+ - \lambda_n^-| + |\lambda_n^+ - \mu_n|, \quad n > N(v);$$

then, for each sub-multiplicative weight  $\omega$  and

$$\Omega = (\Omega(n)), \quad \Omega(n) = \frac{\omega(n)}{n},$$

we have

$$(6.6) \quad v \in H(\Omega) \Rightarrow (\Delta_n) \in \ell^2(\Omega).$$

Conversely, in the above notations, if  $\omega = (\omega(n))_{n \in \mathbb{Z}}$  is a sub-multiplicative weight such that

$$(6.7) \quad \frac{\log \omega(n)}{n} \searrow 0 \quad \text{as } n \rightarrow \infty,$$

then

$$(6.8) \quad (\Delta_n) \in \ell^2(\Omega) \Rightarrow v \in H(\Omega).$$

If  $\omega$  is a sub-multiplicative weight of exponential type, i.e.,

$$(6.9) \quad \lim_{n \rightarrow \infty} \frac{\log \omega(n)}{n} > 0$$

then

$$(6.10) \quad (\Delta_n) \in \ell^2(\Omega) \Rightarrow \exists \varepsilon > 0 : v \in H(e^{\varepsilon|n|}).$$

2. Throughout the paper and in Theorems 22 and 23 we consider three types of boundary conditions: *Per* $^\pm$  and *Dir* in the form  $(a^*)$ ,  $(b^*)$  and  $(c^* \equiv c)$  adjusted to the differential operators (1.1) with *singular potentials*  $v \in H^{-1}$ . It is worth to observe that if  $v$  happens to be a regular potential, i.e.,  $v \in L^2([0, \pi])$  (or even  $v \in H^\alpha$ ,  $\alpha > -1/2$ ) the boundary conditions  $(a^*)$  and  $(b^*)$  *automatically* become equivalent to the boundary conditions  $(a)$  and  $(b)$  as we used to write them in the regular case. Indeed (see the paragraph after (2.67)), we have

$$(a^*) \quad \text{Per}^+ : \quad y(\pi) = y(0), \quad (y' - Qy)(\pi) = (y' - Qy)(0).$$

Therefore, with  $v \in L^2$ , both the  $L^2$ -function  $Q$  and the quasi-derivative  $u = y' - Qy$  are continuous functions, so the two terms  $y'$  and  $Qy$  can be considered separately. Then the second condition in  $(a^*)$  can be rewritten as

$$(6.11) \quad y'(\pi) - y'(0) = Q(\pi)y(\pi) - Q(0)y(0).$$

But, since  $Q$  is  $\pi$ -periodic (see Proposition 1),

$$(6.12) \quad Q(\pi) = Q(0),$$

and with the first condition in  $(a^*)$  the right side of (6.11) is  $Q(0)(y(\pi) - y(0)) = 0$ . Therefore,  $(a^*)$  comes to the form

$$(a) \quad y(\pi) = y(0), \quad y'(\pi) = y'(0).$$

Of course, in the same way the condition  $(b^*)$  *automatically* becomes equivalent to  $(b)$  if  $v \in H^\alpha$ ,  $\alpha > -1/2$ .

A.Savchuk and A. Shkalikov checked ([27]), Theorem 1.5) which boundary conditions in terms of a function  $y$  and its quasi-derivative  $u = y' - Qy$  are regular by Birkhoff–Tamarkin. Not all of them are reduced to some canonical boundary conditions in the case of  $L^2$ -potentials; the result could depend on the value of  $Q(0)$ . For example, Dirichlet–Neumann bc

$$y(0) = 0, \quad (y' - Qy)(\pi) = 0$$

would became

$$y(0) = 0, \quad y'(\pi) = Q(\pi) \cdot y(\pi).$$

Of course, one can adjust  $Q$  in advance by choosing (as it is done in [28])

$$Q(x) = - \int_x^\pi v(t)dt \quad \text{if } v \in L^2.$$

But this choice is not good if Dirichlet–Neumann bc is written with changed roles of the end points, i.e.,

$$(y' - Qy)(0) = 0, \quad y(\pi) = 0.$$

We want to restrict ourselves to such boundary conditions with  $v \in H^{-1}$  that if by chance  $v \in L^2$  then the reduced boundary conditions do not depend on  $Q(0)$ .

We consider as *good* self-adjoint bc only the following ones:

$$Dir : \quad y(0) = 0, \quad y(\pi) = 0$$

and

$$\begin{aligned} y(\pi) &= e^{i\theta}y(0) \\ (y' - Qy)(\pi) &= e^{i\theta}(y' - Qy)(0) + Be^{i\theta}y(0), \end{aligned}$$

where  $\theta \in [0, 2\pi)$  and  $B$  is real.

Observations of this subsection are quite elementary but they would be important if we would try to extend statements like Theorem 23 by finding other troikas of boundary conditions (and corresponding troikas of eigenvalues like  $\{\lambda^+, \lambda^-, \mu\}$ ) and using these spectral triangles and the decay rates of their diameters to characterize a smoothness of potentials  $v$  with a priori assumption  $v \in H^{-1}$  (or even  $v \in L^2([0, \pi])$ ).

## REFERENCES

- [1] F.V. Atkinson, *Discrete and continuous boundary problems*, Academic Press, New York, 1964.
- [2] P. Djakov and B. Mityagin, Smoothness of Schrödinger operator potential in the case of Gevrey type asymptotics of the gaps, *J. Funct. Anal.* **195** (2002), 89-128.
- [3] P. Djakov and B. Mityagin, Spectral triangles of Schrödinger operators with complex potentials. *Selecta Math. (N.S.)* **9** (2003), 495–528.
- [4] P. Djakov and B. Mityagin, Instability zones of periodic 1D Schrödinger and Dirac operators (Russian), *Uspehi Mat. Nauk* **61** (2006), no 4, 77–182 (English: *Russian Math. Surveys* **61** (2006), no 4, 663–766).
- [5] M. S. P. Eastham, *The spectral theory of periodic differential operators*, Hafner, New York 1974.
- [6] W. N. Everitt and A. Zettl, Generalized symmetric ordinary differential expressions. I. The general theory. *Nieuw Arch. Wisk. (3)* **27** (1979), 363–397.
- [7] W. N. Everitt and A. Zettl, A. Sturm–Liouville differential operators in direct sum spaces. *Rocky Mountain J. Math.* **16** (1986), 497–516.
- [8] R. O. Hryniv and Ya. V. Mykytyuk, 1-D Schrödinger operators with periodic singular potentials. *Methods Funct. Anal. Topology* **7** (2001), 31–42.
- [9] R. O. Hryniv and Ya. V. Mykytyuk, 1-D Schrödinger operators with singular Gordon potentials. *Methods Funct. Anal. Topology* **8** (2002), 36–48.
- [10] R. O. Hryniv and Ya. V. Mykytyuk, Inverse spectral problems for Sturm-Liouville operators with singular potentials. *Inverse Problems* **19** (2003), 665–684.
- [11] R. O. Hryniv and Ya. V. Mykytyuk, Transformation operators for Sturm–Liouville operators with singular potentials. *Math. Phys. Anal. Geom.* **7** (2004), 119–149.
- [12] R. O. Hryniv and Ya. V. Mykytyuk, Inverse spectral problems for Sturm-Liouville operators with singular potentials. IV. Potentials in the Sobolev space scale. *Proc. Edinb. Math. Soc.* **49** (2006), 309–329.
- [13] R. O. Hryniv and Ya. V. Mykytyuk, Eigenvalue asymptotics for Sturm-Liouville operators with singular potentials, *J. Funct. Anal.* **238** (2006), 27–57.
- [14] T. Kappeler and B. Mityagin, Gap estimates of the spectrum of Hill’s Equation and Action Variables for KdV, *Trans. AMS* **351** (1999), 619-646.
- [15] T. Kappeler and B. Mityagin, Estimates for periodic and Dirichlet eigenvalues of the Schrödinger operator, *SIAM J. Math. Anal.* **33** (2001), 113-152.
- [16] T. Kappeler and C. Möhr, Estimates for periodic and Dirichlet eigenvalues of the Schrödinger operator with singular potential, *J. Funct. Anal.* **186** (2001), 69–91.
- [17] E. Korotyaev, Characterization of the spectrum of Schroedinger operators with periodic distributions, *Int. Math. Res. Not.* **37** (2003), 2019 - 2031.
- [18] E. Korotyaev, A priori estimates for the Hill and Dirac operators, arXiv:math.SP/0701437v1 16 Jan 2007.
- [19] P. Kuchment, *Floquet theory for partial differential equations* Basel–Boston, Birkhauser Verlag, 1993.
- [20] W. Magnus and S. Winkler, "Hill’s equation", Interscience Publishers, John Wiley, 1969.
- [21] V. A. Marchenko, "Sturm-Liouville operators and applications", *Oper. Theory Adv. Appl.*, Vol. 22, Birkhäuser, 1986.
- [22] M. A. Naimark, *Linear differential operators*, Moscow, 1969.
- [23] F. S. Rofe–Beketov, On the spectrum of non-selfadjoint differential operators with periodic coefficients. (Russian) *Dokl. Akad. Nauk SSSR* **152** 1963 1312–1315; translation in *Soviet Math. Dokl.* **4** (1963), 1563–1566.
- [24] F. S. Rofe–Beketov and A. M. Kholkin, *Spectral analysis of differential operators. Interplay between spectral and oscillatory properties*. Translated from the Russian by Ognjen Milatovic and revised by the authors. With a foreword by Vladimir A. Marchenko. World

- Scientific Monograph Series in Mathematics, **7**. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2005.
- [25] A. M. Savchuk and A. A. Shkalikov, Sturm-Liouville operators with singular potentials. (Russian) *Mat. Zametki* **66** (1999), 897–912; translation in *Math. Notes* **66** (1999), 741–753 (2000).
  - [26] A. M. Savchuk and A. A. Shkalikov, The trace formula for Sturm-Liouville operators with singular potentials. (Russian) *Mat. Zametki* **69** (2001), 427–442; translation in *Math. Notes* **69** (2001), 387–400.
  - [27] A. M. Savchuk and A. A. Shkalikov, Sturm-Liouville operators with distribution potentials. (Russian) *Tr. Mosk. Mat. Obs.* **64** (2003), 159–212; translation in *Trans. Moscow Math. Soc.* 2003, 143–192.
  - [28] A. M. Savchuk and A. A. Shkalikov, Inverse problem for Sturm-Liouville operators with distribution potentials: reconstruction from two spectra. *Russ. J. Math. Phys.* **12** (2005), 507–514.
  - [29] A. M. Savchuk and A. A. Shkalikov, On the eigenvalues of the Sturm-Liouville operator with potentials in Sobolev spaces, (Russian), *Mat. Zametki* **80** (2006), 864–884.
  - [30] Laurent Schwartz, *Theorie des distributions*. (Publications de l'Institut de Mathématique de l'Université de Strasbourg, nos. 9 and 10; *Actualités Scientifiques et Industrielles*, nos. 1091 and 1122.) Vol. I, 1950, 148 pp. Vol. II, 1951, 169 pp.
  - [31] V. Tkachenko, On the spectral analysis of the one-dimensional Schrödinger operator with a periodic complex-valued potential. (Russian) *Dokl. Akad. Nauk SSSR* **155** (1964), 289–291.
  - [32] J. Weidmann, *Spectral theory of ordinary differential operators*, *Lect. Notes in Math.* **1258**, Springer, Berlin, 1987.

SABANCI UNIVERSITY, ORHANLI, 34956 TUZLA, ISTANBUL, TURKEY  
*E-mail address:* [djakov@sabanciuniv.edu](mailto:djakov@sabanciuniv.edu)

DEPARTMENT OF MATHEMATICS, THE OHIO STATE UNIVERSITY, 231 WEST 18TH AVE,  
COLUMBUS, OH 43210, USA  
*E-mail address:* [mityagin.1@osu.edu](mailto:mityagin.1@osu.edu)