Cauchy problem for a higher-order Boussinesq equation

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In this study we consider global existence and well-posedness of the following Cauchy problem for a higher-order Boussinesq equation:

$$\beta u_{tt} - u_{xx} - u_{xxt} + \beta u_{xxxxxt} = (g(u))_{xx}, \quad x \in R, \quad t > 0$$

(1)

$$u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x)$$

(2)

where $\beta$ is a positive constant, $u(x, t)$ is a real-valued function of two real variables and $g(u)$ is a given function $u$. At the microscopic level the higher order Boussinesq equation was derived in [1] for the longitudinal vibrations of a dense lattice, in which a unit length of the lattice contains a large number of lattice points. Thus, in the above higher order Boussinesq equation which has been written in terms of the scaled variables, $u = u(x, t)$ is the longitudinal strain, $t$ is time and $x$ is a spatial coordinate along the length of the lattice. The same equation may also be derived at the macroscopic level using the continuum mechanics approach, in particular the nonlocal elasticity theory that includes the effect of long range interatomic forces.

The global existence of Cauchy problem for the generalized double dispersion equation was proved in [2]. It is therefore natural to ask how the higher-order dispersive term affects the global existence. In fact, the method presented in [2] for the generalized double dispersion equation was extended to the Cauchy problem (1)-(2) for the higher-order Boussinesq equation in [3]. This paper summarizes the results in [3]. Similar results also have been derived independently in [4].

In what follows $H^s = H^s(R)$ will denote the $L^2$ Sobolev space on $R$. For the $H^s$ norm we use the Fourier transform representation $\|u\|_s^2 = \int (1 + \xi^2)^s |\hat{u}(\xi)|^2 d\xi$. We use $\|u\|_{s\infty}$ to denote the $L^\infty$ norm.

1 Introduction

In this study we consider global existence and well-posedness of the following Cauchy problem for a higher-order Boussinesq equation:

$$u_{tt} - u_{xx} - u_{xxt} + \beta u_{xxxxxt} = (g(u))_{xx}, \quad x \in R, \quad t > 0$$

(1)

$$u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x)$$

(2)

In this study we establish global well-posedness of the Cauchy problem for a higher-order Boussinesq equation.

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Lemma 2.3 Suppose that \( g \in C(R), \ G(u) = \int_0^u g(p)dp, \ \varphi \in H^1, \ \psi \in H^1, \ \Lambda^{-1} \psi \in H^1 \) and \( G(\varphi) \in L^1 \). Then for the solution \( u(x,t) \) of problem (1)-(2), we have the energy identity

\[
E(t) = \|\Lambda^{-1} u_t\|^2 + \|u\|^2 + \|u_t\|^2 + \beta \|u_{xt}\|^2 + 2 \int_{-\infty}^{\infty} G(u) dx = E(0)
\]

for all \( t > 0 \) for which the solution exists.

Finally we prove the following theorem about the global existence:

**Theorem 2.4** Assume that \( s \geq 1, \ g \in C^k(R) \) with \( g(0) = 0 \) and \( k = \max \{[s-1], 1\}, \varphi \in H^s, \ \psi \in H^s, \ \Lambda^{-1} \psi \in H^s, \ G(\varphi) \in L^1, \) and \( G(u) \geq 0 \) for all \( u \in R, \) then the problem (1)-(2) has a unique global solution \( u \in C^2([0, \infty), H^s) \).

**References**