Bounded Operators and Isomorphisms of Cartesian Products of Fréchet Spaces

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Introduction

In [25; 26] it was discovered that there exist pairs of wide classes of Köthe spaces $(\mathcal{X}, \mathcal{Y})$ such that

$$L(X, Y) = LB(X, Y) \quad \text{if } X \in \mathcal{X},\ Y \in \mathcal{Y}, \tag{1}$$

where $LB(X, Y)$ is the subspace of all bounded operators from X to Y. If either any $X \in \mathcal{X}$ is Schwartzian or any $Y \in \mathcal{Y}$ is Montel, then this relation coincides with

$$L(X, Y) = L_c(X, Y) \quad \text{if } X \in \mathcal{X},\ Y \in \mathcal{Y}, \tag{2}$$

where $L_c(X, Y)$ denotes the subspace of all compact operators.

This phenomenon was studied later by many authors (see e.g. [1; 5; 11; 12; 13; 14; 15; 20; 21]); of prime importance are Vogt's results [24] giving a generally complete description of the relations (1) for the general case of Fréchet spaces (for further generalizations see also [3; 4]).

Originally, the main goal in [25; 26] was the isomorphism of Cartesian products (and, consequently, the quasi-equivalence property for those spaces). The papers made use of the fact that, due to Fredholm operators theory, an isomorphism of spaces $X \times Y \simeq X_1 \times Y_1$ ($X, X_1 \in \mathcal{X}$, $Y, Y_1 \in \mathcal{Y}$) that satisfies (2) also implies an isomorphism of Cartesian factors "up to some finite-dimensional subspace".

In the present paper we generalize this approach onto classes $\mathcal{X} \times \mathcal{Y}$ of products that satisfy (1) instead of (2). Although Fredholm operators theory fails, we have established that—in the case of Köthe spaces—the stability of an automorphism under a bounded perturbation still takes place, but in a weakened form: "up to some basic Banach space". In particular, we get a positive answer to Question 2 in [7]: Is it possible to modify somehow the method developed in [25; 26] in order to obtain isomorphic classification of the spaces $E_0(a) \times E_\infty(b)$ in terms of sequences a, b if $a_i \not\to \infty$ and $b_i \not\to \infty$?

Some of our results are announced without proofs in [9].

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1. Preliminaries

Let $(a_{ik})_{i,k\in\mathbb{N}}$ be a matrix of real numbers such that $0 \le a_{ik} \le a_{i,k+1}$ for all i, k. We denote by $K^p(a_{ik})$, $1 \le p < \infty$, the l^p-Köthe space defined by the matrix (a_{ik})—that is, the space of all sequences of scalars $x = (x_i)$ such that

$$|x|_k := \left(\sum_i (|x_i| a_{ik})^p\right)^{1/p} < \infty \quad \forall k \in \mathbb{N};$$

with the topology generated by the system of seminorms $\{|\cdot|_k,\ k \in \mathbb{N}\}$, it is a Fréchet space. If $a = (a_i)$ is a sequence of positive numbers, then the Köthe spaces

$$E_0^p(a) = K^p\left(\exp\left\{-\frac{1}{k}a_i\right\}\right), \qquad E_\infty^p(a) = K^p(\exp\{ka_i\})$$

are called, respectively, l^p-finite and l^p-infinite type power series spaces. These spaces are Schwartz if and only if $a_i \to \infty$. If (a_i) and (b_i) are sequences of positive numbers such that $a_i \to \infty$ and $b_i \to \infty$, then $E_0^p(a) \simeq E_0^p(b)$ (resp. $E_\infty^p(a) \simeq E_\infty^p(b)$) if and only if $a_i \asymp b_i$, that is,

$$\exists c > 0 : a_i/c \le b_i \le c \ \forall i$$

(see [18]).

Each Köthe space has a natural basis (e_j), where $e_j = (\delta_{ji})$. A subspace generated by the subsequence of the natural basis is called *basic* subspace. It is well known that $K^p(a_{ik})$ is not a Montel space if and only if there exist k_0 and a subsequence of indices (i_n) such that

$$\forall k\ \exists C_k : a_{i_n k} \le C_k a_{i_n k_0}\ \forall n.$$

Therefore we have the following proposition.

PROPOSITION 1. *An l^p-Köthe space is non-Montel if and only if it contains a basic subspace isomorphic to l^p.*

If X and Y are topological vector spaces then a linear operator $T : X \to Y$ is *bounded* (resp. *compact*) if there exists a neighborhood of zero U in X such that $T(U)$ is a bounded (precompact) set in Y. We write $(X, Y) \in \mathcal{B}$ (resp. $(X, Y) \in \mathcal{K}$) if each continuous linear operator from X into Y is bounded (resp. compact).

We say that a pair (X, Y) has the *bounded* (resp. *compact*) *factorization* property and write $(X, Y) \in \mathcal{BF}$ (resp. $(X, Y) \in \mathcal{KF}$) if each linear continuous operator $T : X \to X$ that factors through Y (i.e. $T = S_1 S_2$, where $S_2 : X \to Y$ and $S_1 : Y \to X$ are linear continuous operators) is bounded (resp. compact).

After Dragilev [10] and Bessaga [2], a Köthe matrix (a_{ik}) is said to be of type (d_1) or (d_2) if, respectively, the following condition holds:

(d_1) $\exists k_0\ \forall k\ \exists (m, C) : a_{ik}^2 \le C a_{ik_0} a_{im}$,
(d_2) $\forall k\ \exists m\ \forall l\ \exists C : a_{ik} a_{il} \le C a_{im}^2$.

Let E be a Fréchet space with basis (e_i) and fundamental system of seminorms $(|\cdot|_k)$. If for some $p \in [1, \infty)$ we have

$$x = \sum_i x_i e_i \ \Rightarrow\ \sum (|x_i||e_i|_k)^p < \infty \ \forall k,$$

then the basis (e_i) is called *l^p-absolute* and E is isomorphic to the l^p-Köthe space $K^p(|e_i|_k)$. If the corresponding Köthe matrix is of type (d_1) or (d_2) then we say that E is a (d_1)- or (d_2)-space and write $E \in (d_1)$ or $E \in (d_2)$. Recall that finite (resp. infinite) type power series spaces are (d_2) (resp. (d_1)) spaces.

Zahariuta [26] showed that $(X, Y) \in \mathcal{B}$ if X and Y are locally convex spaces, with l^1-absolute bases, satisfying respectively the conditions (d_2) and (d_1). By the results of Vogt [24] (see Satz 6.2 and Prop. 5.3) it follows that the same is true for spaces with l^p-absolute basis, so the following proposition holds.

PROPOSITION 2. *If X is (d_2)-Köthe space and Y is (d_1)-Köthe space then $(X, Y) \in \mathcal{B}$. In particular, for any $p, q \in [1, \infty)$ we have $(E_0^p(a), E_\infty^q(b)) \in \mathcal{B}$.*

2. Bounded Operators in Köthe Spaces

The following statement is crucial for our approach.

LEMMA 1. *If $X = K(a_{ik})$ is a Köthe space and $A \subset X$ is a bounded set, then for any k_0 and any $\varepsilon > 0$ there exists a Banach basic subspace B such that $A \subset B + \varepsilon U_{k_0}$, where $U_{k_0} = \{x \in X : |x|_{k_0} \le 1\}$.*

Proof. We give the proof for l^1-Köthe spaces; the case $p > 1$ can be treated similarly. Since the set A is bounded we may assume without loss of generality that

$$A = \left\{ x \in X : |x|_k = \sum_i a_{ik}|x_i| \le C_k \ \forall k \right\}.$$

Choose $C_k \nearrow \infty$ so that $a_{ik}/C_k \to 0$ for all i. Set $\gamma_i = \sum_k (a_{ik}/2^k C_k)$; then

$$\sum_i \gamma_i |x_i| = \sum_i \left(\sum_k \frac{a_{ik}}{2^k C_k} \right) |x_i| = \sum_k \frac{1}{2^k} \left(\sum_i \frac{a_{ik}}{C_k} |x_i| \right) \le 1$$

for any $x \in A$. Fix any $\varepsilon > 0$ and set

$$B = [e_i : \varepsilon\gamma_i \le a_{ik_0}], \qquad E = [e_i : \varepsilon\gamma_i > a_{ik_0}],$$

where the square brackets denote the closed linear span of the corresponding vectors. Then obviously B is a Banach space and for $x \in A \cap E$ we have

$$|x|_{k_0} = \sum_i a_{ik_0}|x_i| < \varepsilon \sum_i \gamma_i |x_i| < \varepsilon,$$

which proves the statement. □

REMARK 1. It is easy to see that, under the assumptions and notations of the lemma, if the set A is compact then for any k_0 and for any ε there exists a finite-dimensional basic subspace B such that $A \subset B + \varepsilon U_{k_0}$.

THEOREM 1. *If X is a Köthe space and $T\colon X \to X$ is a bounded* (*resp. compact*) *operator, then there exist complementary basic subspaces B and E such that*:

(i) *B is a Banach* (*resp. finite-dimensional*) *space*; *and*
(ii) *if π_E and i_E are the canonical projection onto E and embedding into X, respectively, then the operator $1_E - \pi_E T i_E$ is an automorphism on E.*

Proof. Suppose $|\cdot|_p$, $p \in \mathbb{N}$, is a fundamental system of norms in X. Since T is a bounded operator there exists a k_0 such that $T(U_{k_0})$ is a bounded set in X. Hence

$$\forall k \; \exists C_k : |Tx|_k \le C_k |x|_{k_0}.$$

By Lemma 1 (resp. Remark 1) there exists a Banach (resp. finite-dimensional) basic subspace B such that $T(U_{k_0}) \subset B + \frac{1}{2}U_{k_0}$. Let E be the basic subspace that is complementary to B. Then, setting $T_1 = \pi_E T i_E \colon E \to E$, we obtain that

$$|T_1 x|_{k_0} \le \tfrac{1}{2}|x|_{k_0} \quad \forall x \in E.$$

Now it is easy to see that the operator $1_E - T_1$ is an automorphism. Indeed, for any $x \in E$ consider the series

$$Sx = x + T_1 x + T_1^2 x + \cdots + T_1^m x + \cdots. \tag{3}$$

This series is convergent in E because, for any k, we have

$$|T_1^m x|_k \le C_k |T_1^{m-1} x|_{k_0} \le C_k \left(\tfrac{1}{2}\right)^{m-1} |x|_{k_0}, \quad m = 1, 2, \dots,$$

and so, by the Banach–Steinhaus theorem, (3) defines a linear continuous operator $S\colon E \to E$. Since $(1_E - T_1)Sx = S(1_E - T_1)x = x$, the operator S is inverse to the operator $1_E - T_1$. □

3. Isomorphisms of Cartesian Products

As usual, we identify an operator $T\colon E_1 \times E_2 \to F_1 \times F_2$ with the corresponding 2×2 matrix (T_{ij}), whose entries are operators acting between the factors of the Cartesian products.

LEMMA 2. *Let E_1, E_2, F_1, F_2 be topological vector spaces. If $T = (T_{ij})\colon E_1 \times E_2 \to F_1 \times F_2$ is an isomorphism such that $T_{11}\colon E_1 \to F_1$ is also an isomorphism, then $E_2 \simeq F_2$.*

Proof. Let $T^{-1} = (S_{ij})$. Consider the operators

$$S_{22}\colon F_2 \to E_2, \qquad H\colon E_2 \to F_2,$$

where $H = T_{22} - T_{21}T_{11}^{-1}T_{12}$. Taking into account that $T_{11}S_{12} + T_{12}S_{22} = 0$, we obtain

$$HS_{22} = T_{22}S_{22} - T_{21}T_{11}^{-1}T_{12}S_{22} = T_{22}S_{22} + T_{21}S_{12} = 1_{F_2}.$$

In an analogous way, from $S_{21}T_{11} + S_{22}T_{21} = 0$ it follows that

$$S_{22}H = S_{22}T_{22} - S_{22}T_{21}T_{11}^{-1}T_{12} = S_{22}T_{22} + S_{21}T_{12} = 1_{E_2}.$$

Hence the spaces E_2 and F_2 are isomorphic. □

The next theorem is a modification of the generalized Douady lemma in [26, Sec. 6]. In [8] an analogous modification is obtained by considering Riesz type operators instead of bounded operators.

THEOREM 2. *Suppose X_1 is a Köthe space and X_2, Y_1, Y_2 are topological vector spaces. If $X_1 \times X_2 \simeq Y_1 \times Y_2$ and $(X_1, Y_2) \in \mathcal{BF}$, then there exist complementary basic subspaces E and B in X_1 and complementary subspaces F and G in Y_1 such that B is a Banach space and*

$$F \simeq E, \qquad B \times X_2 \simeq G \times Y_2.$$

If, in addition, $(Y_1, X_2) \in \mathcal{BF}$, then G is a Banach space.

Proof. Let $T = (T_{ij})\colon X_1 \times X_2 \to Y_1 \times Y_2$ be an isomorphism, and let $T^{-1} = (S_{ij})$. Then we have $S_{11}T_{11} + S_{12}T_{21} = 1_{X_1}$. Since the operator $S_{12}T_{21}$ is bounded, by Theorem 1 there exist complementary basic subspaces E and B of X_1 such that B is a Banach space and the operator $A = \pi_E S_{11} T_{11} i_E$ is an automorphism of E. It is easy to see that the operator $P = T_{11}A^{-1}\pi_E S_{11}$ is a projection on Y_1. We set

$$F = P(Y_1), \qquad G = P^{-1}(0).$$

Obviously we have $F = T_{11}(E)$ and, moreover, the restriction of T_{11} on E is an isomorphism between E and F. From Lemma 2 it now follows that $B \times X_2 \simeq G \times Y_2$.

If, in addition, each operator acting in Y_1 that factors through X_2 is bounded, then the same is true for each operator acting in G that factors through X_2. Suppose $H\colon G \times Y_2 \to B \times X_2$ is an isomorphism and let (H_{ij}) and (R_{ij}) be operator 2×2 matrices corresponding to H and H^{-1}. Then we have $1_G = R_{11}H_{11} + R_{12}H_{21}$. Here the operator $R_{12}H_{21}$ is bounded because it factors through X_2 and the operator $R_{11}H_{11}$ is bounded because it factors through the Banach space B. Hence the operator 1_G is bounded; that is, G is a Banach space. □

REMARK 2. One can easily see by the proof and by Theorem 1 that: (a) if $(X_1, Y_2) \in \mathcal{KF}$ then the space B may be chosen to be finite-dimensional; and (b) if, in addition, $(Y_1, X_2) \in \mathcal{KF}$ then the space G also will be finite-dimensional. So, in this case we obtain a statement that is known (see [8; 26]).

4. Applications

We begin with an observation showing that an infinite-dimensional complemented Banach subspace in an l^p-Köthe space is isomorphic to l^p.

PROPOSITION 3. *Let X be an l^p-Köthe space, and let F and G be complementary subspaces in X (i.e., $X = F \oplus G$). If G is an infinite-dimensional Banach space then $G \simeq l^p$ and, moreover, F and G are isomorphic to some basic subspaces of X.*

Proof. We have $X \times \{0\} \simeq F \times G$. By Theorem 2 there exist complementary basic subspaces E and B in X and complementary subspaces F_1 and G_1 in F such that B is a Banach space and

$$F_1 \simeq E, \qquad B \simeq G_1 \times G.$$

Since every infinite-dimensional basic Banach subspace of an l^p-Köthe space is isomorphic to l^p, we obtain that $B \simeq l^p$. On the other hand, each infinite-dimensional complemented subspace of l^p, $1 \leq p < \infty$, is isomorphic to l^p (see [22] or [16]), so G is isomorphic to l^p. Finally, since $B \simeq l^p$, its complemented subspace G_1 is isomorphic to some basic subspace of B and $F \simeq E \oplus G_1$ is isomorphic to some basic subspace of X. □

This result may be considered as a partial answer to the well-known Pelczynski problem: Does a complemented subspace of a space with basis have a basis? Moreover, in this case we confirm the conjecture of Bessaga [2] that each complemented subspace of a Köthe space is isomorphic to a basic subspace.

The following theorem answers Question 2 in [7]. In fact, we consider a more general situation.

THEOREM 3. *Suppose X_1, X_2, Y_1, Y_2 are non-Montel l^p-Köthe spaces such that $X_1 \times X_2 \simeq Y_1 \times Y_2$. If $X_1, Y_1 \in (d_2)$ and $X_2, Y_2 \in (d_1)$ then $X_1 \simeq Y_1$ and $X_2 \simeq Y_2$.*

Proof. By Proposition 2, each operator acting in X_1 (resp. Y_1) that factors through Y_2 (resp. X_2) is bounded. Thus, by Theorem 2 there exist complementary basic subspaces E and B in X_1 and complementary subspaces F and G in Y_1 such that

$$F \simeq E, \qquad B \times X_2 \simeq G \times Y_2,$$

and B and G are Banach spaces. Then B (resp. G) is either a finite-dimensional space or (by Proposition 3) isomorphic to l^p.

Obviously, since $l^p \times l^p \simeq l^p$, we have $B \times l^p \simeq l^p$ and $G \times l^p \simeq l^p$. From here and Proposition 1 it follows immediately that

$$X_1 \simeq X_1 \times l^p \simeq E \times B \times l^p \simeq F \times G \times l^p \simeq Y_1 \times l^p \simeq Y_1$$

and

$$X_2 \simeq X_2 \times l^p \simeq X_2 \times B \times l^p \simeq Y_2 \times G \times l^p \simeq Y_2 \times l^p \simeq Y_2.$$

In [8], the isomorphic classification of Cartesian products $E_0^p(a) \times E_\infty^q(b)$ was studied by using strictly singular operators. Necessary and sufficient conditions were obtained for the isomorphism

$$E_0^p(a) \times E_\infty^q(b) \simeq E_0^{\tilde{p}}(\tilde{a}) \times E_\infty^{\tilde{q}}(\tilde{b})$$

in the case where $p \neq \tilde{q}$ or $q \neq \tilde{p}$. However the approach used in [8] does not work in the case where $p = \tilde{q}$ and $q = \tilde{p}$. The previous theorem covers the case $p = q = \tilde{p} = \tilde{q}$; the case where $p \neq q$, $\tilde{q} = p$, and $\tilde{p} = q$ is treated in the next theorem. We consider only the non-Montel case, since if some of the spaces are Montel then the result is known by [26].

THEOREM 4. *Suppose $p \neq q$ and the spaces $E_0^p(a)$, $E_\infty^q(b)$, $E_0^q(\tilde a)$, $E_\infty^p(\tilde b)$ are non-Montel. Then the following conditions are equivalent.*

(i) $E_0^p(a) \times E_\infty^q(b) \simeq E_0^q(\tilde a) \times E_\infty^p(\tilde b)$.

(ii) $E_0^p(a) \times E_\infty^q(b) \overset{\text{qd}}{\simeq} E_0^q(\tilde a) \times E_\infty^p(\tilde b)$ (where "qd" denotes quasi-diagonal).

(iii) *There exist complementary subsequences a', a'', b', b'', $\tilde a'$, $\tilde a''$, $\tilde b'$, $\tilde b''$ respectively of $a, b, \tilde a, \tilde b$ such that a'', b'', $\tilde a''$, $\tilde b''$ are bounded;*

$$E_0^p(a'') \simeq l^p, \quad E_0^q(\tilde a'') \simeq l^q, \quad E_\infty^q(b'') \simeq l^q, \quad E_\infty^p(\tilde b'') \simeq l^p;$$

the spaces $E_0^p(a')$, $E_\infty^q(b')$, $E_0^q(\tilde a')$, $E_\infty^p(\tilde b')$ are nuclear; and $a_i' \asymp \tilde a_i'$ and $b_i' \asymp \tilde b_i'$. That is,

$$E_0^p(a') \simeq E_0^q(\tilde a'), \qquad E_\infty^q(b') \simeq E_\infty^p(\tilde b').$$

Proof. Since (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) we prove only that (i) $\Rightarrow$ (iii). If (i) holds then, by Proposition 2 and Theorem 2, there exist complementary subsequences a' and a'' of a and complementary subspaces F_1 and G_1 of $E_0^q(\tilde a)$ such that $E_0^p(a'')$ and G_1 are Banach spaces and

$$E_0^p(a') \simeq F_1, \qquad E_0^p(a'') \times E_\infty^q(b) \simeq G_1 \times E_\infty^p(\tilde b).$$

By Proposition 3 there exist complementary subsequences $\tilde a'$ and $\tilde a''$ of $\tilde a$ such that $F_1 \simeq E_0^q(\tilde a')$, $G_1 \simeq E_0^q(\tilde a'')$, $E_0^p(a'')$ is either finite-dimensional or isomorphic to l^p, and $E_0^q(\tilde a'')$ is either finite-dimensional or isomorphic to l^q. Then $E_0^p(a') \simeq E_0^q(\tilde a')$, so by [8, Prop. 4] these spaces are nuclear. From here it follows that the spaces $E_0^p(a'')$ and $E_0^q(\tilde a'')$ are infinite-dimensional because otherwise $E_0^p(a)$ or $E_0^q(\tilde a)$ would be nuclear (hence Montel). By Mityagin's characterization of isomorphic power series spaces, we obtain that $a_i' \asymp \tilde a_i'$.

Thus we have

$$E_\infty^q(b) \times l^p \simeq E_\infty^p(\tilde b) \times l^q.$$

Now by Theorem 2 there exist complementary subsequences b' and b'' of b and complementary subspaces F_2 and G_2 in $E_\infty^p(\tilde b)$ such that $E_\infty^q(b'')$ and G_2 are Banach spaces and $E_\infty^q(b') \simeq F_2$. Using Proposition 3, we obtain that there exist complementary subsequences $\tilde b'$ and $\tilde b''$ of $\tilde b$ such that

$$E_\infty^p(\tilde b') \simeq F_2, \qquad E_\infty^p(\tilde b'') \simeq G_2.$$

Now from the same argument as before it follows that the isomorphic spaces $E_\infty^q(b')$ and $E_\infty^p(\tilde b')$ are nuclear, and Mityagin's characterization of isomorphic power series spaces shows that $b_i' \asymp \tilde b_i'$. Finally by Proposition 3 we have

$$E_\infty^q(b'') \simeq l^q \quad \text{and} \quad E_\infty^p(\tilde b'') \simeq l^p.$$

The methods presented here and in [8] may be used to study the isomorphic classification of the Cartesian products $K^p(A) \times K^q(B)$, where A is (d_2)-matrix and B is (d_1)-matrix. One can easily generalize the results of [8] in order to obtain characterizations of isomorphisms

$$K^p(A) \times K^q(B) \simeq K^{\tilde p}(\tilde A) \times K^{\tilde q}(\tilde B)$$

in the case where $p \neq \tilde{q}$ or $q \neq \tilde{p}$. In fact, our Theorem 3 treats the case $p = q = \tilde{p} = \tilde{q}$ (which is impossible to treat with the methods of [8]; see Question 2 in [7]). The next theorem is the corresponding generalization of Theorem 4.

THEOREM 5. *Let $p \neq q$. Suppose that $K^p(A)$ and $K^q(\tilde{A})$ are non-Montel (d_2)-Köthe spaces, and that $K^q(B)$ and $K^p(\tilde{B})$ are non-Montel (d_1)-Köthe spaces. Then the following statements are equivalent.*

(i) $K^p(A) \times K^q(B) \simeq K^q(\tilde{A}) \times K^p(\tilde{B})$.

(ii) *There exist complementary submatrices $A', A'', B', B'', \tilde{A}', \tilde{A}'', \tilde{B}', \tilde{B}''$ respectively of $A, B, \tilde{A}, \tilde{B}$ such that*

$$K^p(A'') \simeq l^p, \quad K^q(\tilde{A}'') \simeq l^q, \quad K^q(B'') \simeq l^q, \quad K^p(\tilde{B}'') \simeq l^p;$$

the spaces $K^p(A'), K^q(B'), K^q(\tilde{A}'), K^p(\tilde{B}')$ are nuclear; and

$$K^p(A') \simeq K^q(\tilde{A}'), \qquad K^q(B') \simeq K^p(\tilde{B}').$$

Proof. Since obviously (ii) $\Rightarrow$ (i), we need only prove that (i) $\Rightarrow$ (ii). If (i) holds then, by Proposition 2 and Theorem 2, there exist complementary submatrices A' and A'' of A and complementary subspaces F_1 and G_1 of $K^q(\tilde{A})$ such that $K^p(A'')$ and G_1 are Banach spaces and

$$K^p(A') \simeq F_1, \qquad K^p(A'') \times K^q(B) \simeq G_1 \times K^p(\tilde{B}).$$

By Proposition 3 there exist complementary submatrices $\tilde{A}'$ and $\tilde{A}''$ of $\tilde{A}$ such that $F_1 \simeq K^q(\tilde{A}')$, $G_1 \simeq K^q(\tilde{A}'')$, $K^p(A'')$ is either finite-dimensional or isomorphic to l^p, and $K^q(\tilde{A}'')$ is either finite-dimensional or isomorphic to l^q. Then $K^p(A') \simeq K^q(\tilde{A}')$, so by [8, Prop. 4] these spaces are nuclear. From here it follows that the spaces $K^p(A'')$ and $K^q(\tilde{A}'')$ are infinite-dimensional because otherwise $K^p(A)$ or $K^q(\tilde{A})$ would be nuclear (hence Montel).

Now we have

$$K^q(B) \times l^p \simeq K^p(\tilde{B}) \times l^q.$$

Repeating the same argument as before, we obtain that there exist complementary submatrices B', B'' of B and $\tilde{B}', \tilde{B}''$ of $\tilde{B}$ such that

$$K^q(B') \simeq K^p(\tilde{B}'), \quad K^q(B'') \simeq l^q, \quad K^p(\tilde{B}'') \simeq l^p,$$

and the spaces $K^q(B')$ and $K^p(\tilde{B}')$ are nuclear. $\square$

Let us note that, in [8] and in Theorem 4, a stronger result was proved: Cartesian products of power series spaces may be isomorphic if and only if they are quasi-diagonally isomorphic. The proof was based on Mityagin's results [18; 19] that two power series spaces are isomorphic if and only if they are quasi-diagonally isomorphic. In general, it is an open problem whether (d_1)- and (d_2)-Köthe spaces have this property.

5. Generalizations and Comments

In the previous section we consider applications to the isomorphic classification of Cartesian products of (d_1)- and (d_2)-spaces. Further applications may be obtained by using the results of Vogt [24] concerning the relation $(X, Y) \in \mathcal{B}$ for Fréchet spaces. Namely, Vogt proved that if X and Y are Fréchet spaces such that X has the property (LB^∞) and Y has the property (DN), then each operator from X to Y is bounded. We refer to [24] for definitions of the properties (LB^∞) and (DN) for Fréchet spaces. Here we note only that for Köthe spaces the property (DN) is equivalent to the property (d_1) and, by [24, Prop. 5.4], it is known that a Köthe space generated by a matrix (a_{ik}) has the property (LB^∞) if and only if

$$\forall \rho_k \uparrow \infty \ \forall p \ \exists q \ \forall n_0 \ \exists (N_0, C) \ \forall i \ \exists k,\ n_0 \le k \le N_0 : a_{ik} a_{ip}^{\rho_k} \le C a_{iq}^{1+\rho_k}.$$

Obviously, it is possible to generalize Theorem 3 and Theorem 5 by considering Köthe spaces with the property (LB^∞) instead of (d_2)-Köthe spaces.

There are other wide classes of Fréchet spaces for which it is possible to apply the results of Section 3. Recall that a Fréchet space X is called a *quojection* if, for any continuous seminorm $q(\cdot)$ on X the quotient space $X/\operatorname{Ker} q$ is Banach. We refer to the survey [17] for details concerning quojections.

From [3, 23] it is known that if E is a quojection then $(E, F) \in \mathcal{B}$ if and only if F has a continuous norm. Using this fact, we immediately obtain the following statement from Theorem 2.

THEOREM 6. *Suppose E_1, E_2 are quojections and F_1, F_2 are Köthe spaces admitting continuous norms. If $E_1 \times F_1 \simeq E_2 \times F_2$, then there exist complementary basic subspaces B_1, H_1 in F_1 and B_2, H_2 in F_2 such that B_1 and B_2 are Banach spaces, $H_1 \simeq H_2$, and $B_1 \times E_1 \simeq B_2 \times E_2$.*

We may generalize Theorem 6 by considering prequojections instead of quojections. Recall that a Fréchet space E is called *prequojection* if its bidual space E'' is a quojection. Each quojection is a prequojection. It is known (see [17; 23]) that $(E, F) \in \mathcal{B}$ if E is a prequojection and F is a Fréchet space with continuous norm and the bounded approximation property. Hence, in Theorem 6 we may replace the requirement "E_1, E_2 are quojections" by "E_1, E_2 are prequojections".

We suspect that the results of [3; 4] may be used to obtain further generalizations.

In all applications we consider, in fact we used the relation $\mathcal{B}$ instead of the weaker relation $\mathcal{BF}$. It is easy to give an example of a nontrivial pair (E, F) with the property $\mathcal{BF}$.

EXAMPLE. First we note that if E, F_1, F_2 are Fréchet spaces such that $(E, F_1) \in \mathcal{B}$ and $(F_2, E) \in \mathcal{B}$, then obviously we have that each operator, acting in E, that factors through $F_1 \times F_2$ is bounded. We choose E, F_1, F_2 to be appropriate Dragilev L_f-spaces. Recall that if f is a logarithmically convex function and (a_i) is a sequence of real numbers such that $a_i \uparrow \infty$, then the corresponding Dragilev space of infinite type is defined as

$$L_f((a_i), \infty) = K(\exp f(ka_i)).$$

Let f_1, f, f_2 be chosen in such a way that the functions $f_1^{-1} \circ f$ and $f^{-1} \circ f_2$ are rapidly increasing. We put

$$E = L_f((a_i), \infty), \qquad F_1 = L_{f_1}((a_i), \infty), \qquad F_2 = L_{f_2}((a_i), \infty).$$

Then it is known by [14] that $(F_1, E) \in \mathcal{K}$ and $(E, F_2) \in \mathcal{K}$ but that $(E, F_1) \notin \mathcal{K}$ and $(F_2, E) \notin \mathcal{K}$. Setting $F = F_1 \times F_2$ we obtain the desired example.

One may therefore expect further applications provided the following problem is solved.

PROBLEM 1. Characterize pairs of Fréchet spaces (E, F) with the property $\mathcal{BF}$.

Our crucial argument was the observation, stated in Lemma 1, that each bounded set in a Köthe space is "small up to a complemented Banach subspace". This argument was used to prove Theorem 1. Let us consider the following generalization of this property. We say that a Fréchet space X with a fundamental system of seminorms $(|\cdot|_k)$ has the property (SCBS) if, for any bounded set $A \subset X$ and for any k_0 and $\varepsilon > 0$, there exist complementary subspaces B and E in X such that B is a Banach space and

$$A \subset B + \varepsilon U_{k_0} \cap E. \tag{4}$$

It is easy to see that the class of Fréchet spaces with property (SCBS) is larger than the class of Köthe spaces. Recall that if (a_{ik}) is a Köthe matrix and E_i is a sequence of Banach spaces then we can consider the corresponding "Banach-valued Köthe space"

$$X = \{ x = (x_i) : x_i \in E_i,\ \|x\|_k = \textstyle\sum_i a_{ik}|x_i|_i < \infty\ \forall k \},$$

where $|\cdot|_i$ is the norm in E_i. Equipped with the system of seminorms $(\|\cdot\|_k)$, X is a Fréchet space.

PROPOSITION 4. *Each Banach-valued Köthe space has the property* (*SCBS*).

The proof is the same as for Lemma 1.

Now, repeating with slight changes the proof of Theorem 1, we obtain the following generalization.

THEOREM 7. *If X is a Fréchet space with property* (*SCBS*) *and $T: X \to X$ is a bounded operator, then there exist complementary subspaces B and E such that*:

(i) *B is a Banach space*; *and*
(ii) *if π_E and i_E are the canonical projection onto E and embedding into X, respectively, then the operator $1_E - \pi_E T i_E$ is an automorphism on E.*

It is easy to see that Theorem 2 holds if the condition "X_1 is a Köthe space" is replaced by the condition "X_1 has the property (SCBS)". Of course, this more general version of Theorem 2 may be used to obtain more general results on isomorphic classification of Cartesian products. In this context the following problem arises.

PROBLEM 2. Characterize the class of Fréchet spaces with the property (SCBS).

Finally, we may consider exact sequences instead of Cartesian products. Namely, suppose that

$$0 \longrightarrow F_1 \xrightarrow{i_1} G_1 \xrightarrow{j_1} E_1 \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow F_2 \xrightarrow{i_2} G_2 \xrightarrow{j_2} E_2 \longrightarrow 0 \quad (5)$$

are exact sequences of Fréchet spaces. As a natural generalization of the isomorphic classification problem for Cartesian products, one may consider the following question.

PROBLEM 3. Is it possible to characterize (under some conditions) the isomorphism $G_1 \simeq G_2$ in terms of the spaces F_1, F_2, E_1, E_2?

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