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Efficient Programmes for Polyhedral Technologies are Competitive

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1. INTRODUCTION

Efficiency of resource allocation over time was first formalized in a general setting by Malinvaud (1953). One of the major theorems in this work states that efficient programmes are competitive. This means there corresponds to every efficient sequence of decisions a non-null sequence of price vectors under which profits are maximized inter-temporally. Our purpose in this paper is to prove the existence of such prices for the class of polyhedral technologies for which this question has remained open, and also to describe and account for their non-existence in general.

The existence of prices in the finite horizon case follows from the familiar separating hyperplane theorem, and holds for all convex technologies. But there are problems when the horizon is infinite. Malinvaud's method has been to extract from the finite horizon price sequences the desired infinite one by a Cantor diagonal limit process. However, it is possible in this method that one fails in fixing a numéraire in the limit, unless prices remain bounded from one period to the next. To exclude this possibility, Malinvaud (1962) postulated technological conditions which guarantee such behaviour of prices at the start. Another approach has been to invoke the separation theorem in infinite dimensional spaces. It is interesting however that the same conditions have proved necessary for this method as well. The current status of the price theorem is in fact that it holds under the so-called " non-tightness " assumption, which need not hold for the important class of polyhedral technologies, in particular the von Neumann and Leontief models. On the other hand, the only existing counterexample to the theorem, Peleg and Yaari (1970) and McFadden (1975), is based on a special non-linear feature and gives no light on the polyhedral case.

In this paper we first give two conditions on the sequence of finite horizon prices which are necessary and sufficient for the existence of an infinite price sequence. It is precisely one of these conditions which is violated in the counterexample mentioned above, which however is always met by polyhedral technologies. We show that the second condition too, and hence the theorem, holds under polyhedrality, provided the number of goods in the economy remains bounded over time. This latter proviso is fundamental to the proof, and in fact we give a counterexample when it fails. Finally, we show by an example that an efficient programme may only have trivial competitive prices, in the sense that the corresponding value of production is zero at all times. We state a weak condition however which rules out this possibility.

2. THE GENERAL MODEL AND THE PRICE PROBLEM

For each $t = 0, 1, \ldots$ let R^t_+ be the non-negative orthant of a real vector space of dimension d_t, representing the space of goods in the economy in period t. We consider goods of different periods to be distinct. The set of all production possibilities in period t is assumed to be a closed convex set $G_t \subseteqq R^t_+ \times R^{t+1}_+$, called the *technology* of period t, where $(x, y) \in G_t$ means that the *outputs* y can be gotten from the *inputs* x. The jth good of period t is called a *producible* good if there exists a $(x, y) \in G_t$ with the jth component of y positive, and *non-producible* otherwise.

We define a *programme* to be a sequence $\{(x_t, y_{t+1})\}$ with $(x_t, y_{t+1}) \in G_t$, $\forall t$. The corresponding *consumption programme* c_t is defined by $c_0 = -x_0$ and $c_t = y_t - x_t$ for all $t \geqq 1$. A programme $\{(x_t, y_{t+1})\}$ is called *efficient* if there is no other programme $\{(x_t, y_{t+1})\}$ with $\bar{c}_t \geqq c_t$, $\forall t$ and $\bar{c}_t \neq c_t$ for some t. Finally a programme $\{(x_t, y_{t+1})\}$ is said to be *competitive* if there exists a non-null sequence $\{p_t\}$, with $p_t \in R^t_+$, such that

$$p_{t+1}y_{t+1} - p_t x_t \geqq p_{t+1}y - p_t x, \quad \forall (x, y) \in G_t. \qquad \text{...(1)}$$

A non-null sequence $\{p_t\}$ satisfying (1) is called a *price sequence*.

Now let $\{(x_t, y_{t+1})\}$ be a given efficient programme. It is well known that there exists for each T a non-zero $p^T = (p_0^T, ..., p_T^T)$ where (p_t^T, p_{t+1}^T) satisfies (1) for $t = 0, ..., T-1$. This follows from the fact that the following T-period consumption programme

$$c_0 = -x_0, \ c_t = y_t - x_t, \qquad (t = 1, ..., T-1), \ c_T = y_T$$

is an efficient point of the set of all T-period consumption programmes. This set being convex, we have the desired p^T by the separating hyperplane theorem. Let us define for each T the closed convex cone

$$\mathbb{P}^T = \{(p_0^T, ..., p_T^T) \mid (p_t^T, p_{t+1}^T) \text{ satisfies (1)}, t = 0, ..., T-1\}. \qquad \text{...(2)}$$

Now the question of whether $\{(x_t, y_{t+1})\}$ is competitive can be posed as follows: Does there exist a price sequence $\{p_t\}$ given that there is a non-zero $p^T \in \mathbb{P}^T$ for each T? The following lemma gives a first answer to this question. Let us call a sequence $\{p^T\}$, where $p^T \in \mathbb{P}^T$, $\forall T$, *normal* if there exists a τ such that $\| p^T \| = 1$, $\forall T \geqq \tau$ ($\| \ \|$ denotes the sum of coordinates). Also let us say that $\{p^T\}$ is *bounded* if for each t the sequence $\{\| p_t^T \|\}$ is bounded.

Lemma 1. *There exists a price sequence* $\{p_t\}$ *if and only if there exists a normal and bounded sequence* $\{p^T\}$, *with* $p^T \in \mathbb{P}^T$.

Proof. Let $\{p^T\}$ be a normal bounded sequence with $p^T = (p_0^T, ..., p_T^T) \in \mathbb{P}^T$. Suppose by induction $\{(p_t^T); T \in N_t\}$, where N_t is a subsequence of the integers, has limit p_t; and note that since $\{\| p_0^T \|\}$ is bounded this is true for $t = 0$. Then by boundedness, $\{(p_{t+1}^T); T \in N_t\}$ has a subsequence $\{(p_{t+1}^T); T \in N_{t+1}\}$ with a limit, say p_{t+1}. Since (p_t^T, p_{t+1}^T) satisfies (1), $\forall T \in N_{t+1}$, so does (p_t, p_{t+1}) by continuity. And since $\{p^T\}$ is normal, the sequence $\{p_t\}$ obtained in this way is non-null, hence a price sequence. The converse is trivial; given a price sequence $\{p_t\}$, simply let $p^T = \| p_\tau \|^{-1}(p_0, ..., p_T)$, where τ is some date for which $\| p_\tau \| > 0$. ||

Lemma 1 is a useful characterization of the price problem. In fact we will solve it for the polyhedral case by showing the existence of a normal bounded sequence $\{p^T\}$. To motivate the somewhat long proof of this fact, we first give counterexamples.

3. COUNTEREXAMPLES

We start with the example of McFadden (1975), including it here both for the sake of completeness and also to note the special feature it has that there exists for it no normal sequence $\{p^T\}$. The example of Peleg and Yaari (1970) is almost identical.

Example 1. Consider a one-good economy with the same technology G in each period given by

$$G = \{(x, y) \mid y \leqq 2 - 2(1-x)^2 \text{ if } 0 \leqq x \leqq 1; \ \ y = 2 \text{ if } x \geqq 1\}.$$

It can be shown that there exists an efficient programme $\{(x_t, y_{t+1})\}$ with

$$x_0 = 1, \ x_t > 0, \ \ \forall t, \ \text{ and } \ (x_t, y_{t+1}) = (1, 2)$$

for infinitely many t. Therefore by (1)

$$2p_{t+1}-p_t \geqq (2-2\varepsilon^2)p_{t+1}-(1-\varepsilon)p_t,$$

i.e.

$$2\varepsilon^2 p_{t+1} \geqq p_t \quad \text{for all } \varepsilon>0,$$

i.e. $p_t = 0$ for infinitely many t. On the other hand, since $x_t>0$, $\forall t$ and $(0, 0) \in G$, (1) implies that if $p_{t+1} = 0$ then $p_t = 0$. Therefore $p_t = 0$, $\forall t$, i.e. there exists no price sequence $\{p_t\}$, and $\{(x_t, y_{t+1})\}$ is not competitive.

In the context of Lemma 1, we note that there exists no normal sequence $\{p^T\}$ for this example, because for infinitely many T, $p^T \in \mathbb{P}^T$ implies $p_0^T = \ldots = p_{T-1}^T = 0$. As will be shown in the next section, there is no such problem for polyhedral technologies, because normal sequences $\{p^T\}$ do always exist. The question that remains then is precisely whether an efficient programme exists for which the normal sequences $\{p^T\}$ fail to be bounded. The main object of this section is the construction of such an example, which will be carried out in two steps. First, we give an efficient programme in a two-good economy, where one of the goods is forced to have a zero price.

Example 2. Consider a linear activity model with the following input and output matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

That is, $G = \{((u, v), (u+v, v)) \mid u \geqq 0, v \geqq 0\}$. Given any programme with the sequence of activity vectors $\{(u_t, v_t)\}$, the corresponding consumption programme $\{c_t\}$ has $c_t^1 = u_t - u_{t+1} + v_t$ and $c_t^2 = v_t - v_{t+1}$. Summing up over t we get

$$u_T = -\textstyle\sum_{t=0}^{T-1} c_t^1 + \sum_{t=1}^{T-1} v_t \quad \text{and } v_T = -\sum_{t=0}^{T-1} c_t^2.$$

We claim the programme given by $u_t = 2^{-t}$ and $v_t = 1$, $\forall t$ is efficient, because by the above, given any other programme $\{(\bar{u}_t, \bar{v}_t)\}$,

$$u_T - \bar{u}_T = \textstyle\sum_{t=0}^{T-1} (\bar{c}_t^1 - c_t^1) + \sum_{t=1}^{T-1} \sum_{n=0}^{t-1} (\bar{c}_n^2 - c_n^2),$$

implying that if $\{\bar{c}_t\}$ dominates $\{c_t\}$ then $u_T - \bar{u}_T \geqq \varepsilon$ for some $\varepsilon>0$ and all T greater than some t. This is a contradiction since $u_T \to 0$. Therefore $\{(u_t, v_t)\}$ is efficient.

On the other hand since u_t, v_t are positive, $\forall t$, (1) implies

$$p_{t+1}B - p_t A = 0$$

for any price sequence $\{p_t\}$. That is $p_{t+1}^1 = p_t^1 = p$ and $p_{t+1}^1 + p_{t+1}^2 = p_t^2$, $\forall t$. Equivalently $p_T^2 = p_0^2 - Tp$ for all T. Hence $p = 0$.

Next we extend this example as follows: in each period a new good enters the technology relative to which the price of the previous good is forced to zero, with the conclusion that all goods must have zero prices.

Example 3. In each period t, the economy has one non-producible good, t producible goods and t activities as follows: The first activity stores the first good, i.e. one unit of the first good in period t yields one unit of itself in period $t+1$. For each $j = 2, \ldots, t-1$, the jth activity uses one unit of the jth good to produce one unit each of the $(j-1)$st good and itself. Finally, activity t uses one unit of the non-producible good to produce one unit of the $(t-1)$st good, and a new good to be called from period t on the tth good. This technology is given formally as the following linear activity model

$$G_t = \{(A_{t-1}w, B_t w) \mid w \geqq 0\},$$

where A_{t-1} is a $t \times t$ unit matrix, B_t is a $(t+1) \times t$ matrix with $B_t^{ij} = 1$ if $i = j$ or $j-1$, and $B_t^{ij} = 0$ otherwise.

Now consider the sequence of activity vectors $\{w_t\}$ where $w_t^1 = 2^{-t}$ and $w_t^j = 1$ for all $j \geqq 1$. The corresponding consumption programme $\{c_t\}$ has $c_t^1 = 1+2^{-t}$, $c_t^j = 1$, $j = 2, \ldots, t-1$, $c_t^t = 0$ for the producible goods, and $c_t^{t+1} = -1$ for the non-producible good of period t. It can be seen by extending the argument given in Example 2 that this programme is efficient.

Therefore, since all activities operate at positive levels, any price sequence $\{p_t\}$ must satisfy $p_t B_t = p_{t-1} A_{t-1}$. That is

$$p_t^1 = p_{t-1}^1 \text{ and } p_t^j + p_t^{j-1} = p_{t-1}^j, \quad j = 2, \ldots, t,\ t \geqq j-1;$$

i.e.

$$p_t^1 = p^1 \text{ and } p_t^j = p_{j-1}^j - \textstyle\sum_{n=j-1}^{t} p_n^{j-1}, \quad j = 2, \ldots, t,\ t \geqq j-1.$$

But then $p_t^2 = p_1^2 - (t-1)p^1$, implying $p^1 = 0$ and $p_t^2 = p^2$, $\forall t$. By induction, suppose $p_t^{j-1} = p^{j-1}$, $\forall t$. Then $p_t^j = p_{j-1}^j - (t-j+1)p^{j-1}$, implying $p^{j-1} = 0$ and $p_t^j = p^j$, $\forall t$. Hence $p_t \equiv 0$, $t \geqq 0$, in other words this programme is not competitive.

Finally we give an example to show that only *trivial* price sequences $\{p_t\}$ may exist, in the sense that $p_t(y_t+x_t) = 0$, $\forall t$. In other words the goods which do have positive prices are neither inputs nor outputs of the efficient programme under consideration.

Example 4. Consider the efficient programme given by $u_t = 2^{-t}$ and $v_t = 0$ for the economy in Example 2. We have $y_t = (2^{-t}, 0)$ and $x_t = (2^{-t-1}, 0)$ $\forall t$. As in Example 2, the only price sequence $\{p_t\}$ for this programme is of the form $p_t = (0, q)$, $\forall t$. Because $p_t^1 = p$, $\forall t$ as before, and $p_{t+1}^1 + p_{t+1}^2 - p_t^1 \leqq 0$, i.e. $p_t^2 \leqq p_0^2 - Tp$, $\forall t$, implying $p = 0$. Therefore $p_t(y_t+x_t) = 0$, $\forall t$.

4. THE EXISTENCE OF PRICES FOR POLYHEDRAL TECHNOLOGIES

In this section we prove that an efficient programme $\{(x_t, y_{t+1})\}$ is competitive under the following two assumptions:

Assumption 1. The technologies G_t are polyhedral convex sets.

Assumption 2. The number of goods d_t have a maximum, say d.

As a well-known consequence of Assumption 1, the price cones $\mathbb{P}^t$ given by (2) are polyhedral and each contains a positive vector (we call a vector *positive* if all of its components are). Therefore the situation in Example 1 is no longer possible, because we can choose for each t a positive $\bar{p}^t \in \mathbb{P}^t$, define $p^t = \| \bar{p}_0^t \|^{-1} \bar{p}^t$ and thus have a normal sequence $\{p^t\}$. Now let us define for any set $S^t \subseteqq \mathbb{P}^t$ and for $t' < t$ the projection of S^t on $\mathbb{P}^{t'}$ by $S_{t'}^t$. Observe that $\mathbb{P}_{t'}^t \subseteqq \mathbb{P}^{t'}$. We state these facts in the following form:

Lemma 2. *Given an efficient programme $\{(x_t, y_{t+1})\}$, for each t*

(i) *$\mathbb{P}^t$ is a polyhedral convex cone containing a positive vector,*

(ii) $\mathbb{P}_{t'}^t \subseteqq \mathbb{P}^{t'}$ for $t' < t$.

Lemma 2 together with Assumption 2 leads to the following key result.

Lemma 3. *There exists a sequence of polyhedral convex cones $Q^t \subseteqq \mathbb{P}^t$ such that for each t*

(i) *Q_0^t contains a positive vector.*

(ii) *$Q_{t'}^t \subseteqq Q^{t'}$ for $t' < t$.*

(iii) *Q^t has at most $2d$ generators.*

Proof. Let us call a subset Q of $\mathbb{P}^t$ *continuable to* T if there exists a

$$(p_0, \ldots, p_t, \ldots, p_T) \in \mathbb{P}^T$$

with $(p_0, \ldots, p_t) \in Q$ and p_0 positive. And let us call Q *continuable* if it is continuable to all $T \geqq t$. We immediately note by Lemma 2 (ii) that if Q is continuable to infinitely many T's, then it is continuable. We will construct the Q^t's inductively with the hypothesis that each is continuable.

To start the induction, by Lemma 2(i) choose for each T a positive

$$p^T = (p_0^T, \ldots, p_T^T) \in \mathbb{P}^T.$$

By Lemma 2(ii), $(p_0^T, p_1^T) \in \mathbb{P}^1$ for all T. Since $\mathbb{P}^1$ is a polyhedral convex cone, it follows from Caratheodory's Theorem that $\mathbb{P}^1$ is the union of a finite number of convex cones each with no more than $d_0 + d_1$ generators (see Rockafellar (1970) for this theorem and other facts we use on polyhedral convexity). Therefore, one of these subcones, say Q^1, contains a subsequence of $\{(p_0^T, p_1^T)\}$. This means Q^1 is continuable to infinitely many T's, whence it is continuable. It is straightforward to check that Q^1 satisfies the lemma.

Suppose Q^{t-1} has been constructed. Define F_t to be the projection of the set $(Q^{t-1} \times R_+^t) \cap P^t$ onto $R_+^0 \times R_+^t$; i.e.

$$F_t = \{(s, r) \in R_+^0 \times R_+^t \mid \exists q \in R_+^1 \times \ldots \times R_+^{t-1} \text{ with } (s, q) \in Q^{t-1} \text{ and } (s, q, r) \in \mathbb{P}^t\}.$$

Now using the hypothesis that Q^{t-1} is continuable, choose for each $T \geqq t$ a

$$p^T = (p_0^T, \ldots, p_t^T, \ldots, p_T^T) \in \mathbb{P}^T \text{ with } (p_0^T, p_t^T) \in F_t \text{ and } p_0^T \text{ positive.}$$

Since Q^{t-1}, R_+^t, $\mathbb{P}^t$ are each polyhedral convex cones, so is $(Q^{t-1} \times R_+^t) \cap \mathbb{P}^t$. Therefore F_t is a polyhedral convex cone, and again by Caratheodory's Theorem it has a convex subcone, say C_t, with no more than $d_0 + d_t$ generators, which contains a subsequence of the vectors (p_0^T, p_t^T) chosen above. Let $(s_1, r_1), \ldots, (s_n, r_n)$ be the generators of C_t. By definition of F_t, there exists q_i with $(s_i, q_i) \in Q^{t-1}$ and $(s_i, q_i, r_i) \in \mathbb{P}^t$. Define Q^t to be the convex cone generated by (s_i, q_i, r_i), $i = 1, \ldots, n$. By construction $Q^t \subseteqq \mathbb{P}^t$, and satisfies (i). Also $Q_{t-1}^t \subseteqq Q^{t-1}$ which implies (ii) and since $n \leqq d_0 + d_t \leqq 2d$ (iii) holds. It only remains to show Q_t is continuable:

By construction there exist for each T in a subsequence N of the integers, non-negative numbers λ_i^T such that $(p_0^T, p_t^T) = \sum_{i=1}^n \lambda_i^T (s_i, r_i)$. For $T \in N$, define $\pi^T = (\bar{p}_0^T, \ldots, \bar{p}_t^T)$ by $\pi^T = \sum_{i=1}^n \lambda_i^T (s_i, q_i, r_i)$. Observe that $\pi^T \in Q^t \subseteqq \mathbb{P}^t$ and that $\bar{p}_t^T = p_t^T$. Therefore by (2), we have $(\bar{p}_0^T, \ldots, \bar{p}_t^T, p_{t+1}^T, \ldots, p_T^T) \in \mathbb{P}^T$. Finally $\bar{p}_0^T = p_0^T$, i.e. is positive. This shows Q^t is continuable to all $T \in N$, which implies it is continuable. ||

Lemma 4. *There exists a normal bounded sequence* $\{p^T\}$ with $p^T \in \mathbb{P}^T$.

Proof. Assume the contrary. Then we claim there exist a sequence $\{p^T\}$ with with $p^T = (p_0^T, \ldots, p_T^T) \in Q^T$ and integers $t(0) < t(1) < \ldots < t(n) < \ldots$ such that

$$\| p_{t(n)}^T \|^{-1} \| p_t^T \| \to 0 \text{ for each } t < t(n), \quad \forall n, \qquad \ldots(3)$$

By Lemma 3, choose for each T a $p^T \in Q^T$ with p_0^T positive. Then $\{\| p_0^T \|^{-1} p^T\}$ is normal, but by assumption not bounded. Let $t(0) = 0$, and suppose by induction on n that there exists a $t(n)$ for which (3) holds. The sequence $\{\| p_{t(n)}^T \|^{-1} p^T\}$ is normal, and by assumption not bounded. Let $t(n+1)$ be the smallest integer such that

$$\{\| p_{t(n)}^T \|^{-1} \| p_{t(n+1)}^T \|\}$$

is unbounded. By (3) $t(n+1) > t(n)$. Actually without loss of generality, we can take $\{\| p_{t(n)}^T \|^{-1} \| p_{t(n+1)}^T \|\}$ to be an increasing unbounded sequence. (For otherwise, such is the case for $T \in N$, where N is a subsequence of the integers. We can now define for each T, $\bar{p}^T = (\bar{p}_0^T, \ldots, \bar{p}_T^T)$ by $\bar{p}^T = (p_0^{T'}, \ldots, p_T^{T'})$ where T' is the smallest integer in N greater than T. By Lemma 3(ii), $\bar{p}^T \in Q^T$, $\forall T$; $\{\bar{p}^T\}$ satisfies the induction hypothesis; and now the sequence $\{\| \bar{p}^T_{t(n)} \|^{-1} \| \bar{p}^T_{t(n+1)} \|\}$ is increasing and unbounded.) Therefore

$$\| p^T_{t(n+1)} \|^{-1} \| p^T_{t(n)} \| \to 0.$$

On the other hand, by the definition of $t(n+1)$, $\{\| p^T_{t(n)} \|^{-1} \| p^T_t \|\}$ is a bounded sequence for each $t < t(n+1)$. Multiplying term by term, we get (3) for $n+1$. End of claim.

Next we show that the claim contradicts Lemma 3(iii). By Lemma 3(ii), for any N and $n < N$, $Q^T_{t(n)} \subseteqq Q^{t(N)}_{t(n)}$ for all $T \geqq t(N)$. And by (3), $\{\| p^T_{t(n)} \|^{-1} (p^T_0, \ldots, p^T_{t(n)})\}$ has a limit point, say $q_n = (0, \ldots, 0, p_{t(n)})$. Therefore q_n belongs to the closure of $Q^{t(N)}_{t(n)}$, which is $Q^{t(N)}_{t(n)}$, since it is polyhedral. Consequently, $Q^{t(N)}$ contains vectors $v_n = (0, \ldots, 0, p_{t(n)}, \ldots)$ for each $n = 1, \ldots, N$. But these v_n are linearly independent. Hence by Lemma 3(iii) N cannot exceed $2d$, implying that the claim is false. This proves the lemma. ||

Theorem. *Every efficient programme is competitive.*

Proof. By Lemmas 1 and 4. ||

We conclude with the following fact that the price sequence cannot be trivial if the programme actually uses all the non-producible goods defined in the technologies. By convention we consider the initial goods non-producible.

Corollary. *Given an efficient programme $\{(x_t, y_{t+1})\}$ such that $x^j_t > 0$ for all non-producible goods j of period t, $\forall t$, there exists a* non-trivial *price sequence $\{p_t\}$.*

Proof. Let $\{p_t\}$ be a price sequence and t the smallest integer such that $p_t \neq 0$. If $t = 0$ then $p_0 x_0 > 0$ by assumption. Otherwise by (1)

$$p_t y_t = p_t y_t - p_{t-1} x_{t-1} \geqq p_t y - p_{t-1} x = p_t y, \quad \forall (x, y) \in G_t.$$

Therefore if $p_t y_t = 0$, then $p^k_t = 0$ for all producible goods k of period t, implying $p^j_t > 0$ for some non-producible good j. Then $p_t x_t > 0$ by assumption. Therefore $p_t(y_t + x_t) > 0$. ||

5. DISCUSSION

The question of exactly what prices measure in a physical sense is a deep one, but it can be said roughly that the relative prices of two goods measure the rate of transformation between them, conditional on the consumption of all other goods staying the same. (Peleg and Yaari (1970) and Rockwell (1973) formalize this idea in different ways.) In this sense, the non-existence of prices for an efficient programme means that there exists no good (numéraire) to serve as the basis for the computation of such rates of transformation. The counterexamples in Section 3 illustrate this: corresponding to each good there is another such that the rates of transformation between them is zero (infinite). It seems intuitive that, excluding the type of capital saturation behaviour in Example 1, this phenomenon could take place only if the number of goods in the economy were to increase indefinitely. Our proof shows this to be true for polyhedral technologies.

In so far as our proof relies on Caratheodory's Theorem, it seems not possible to extend it to the case of convex but non-polyhedral price cones $\mathbb{P}^t$. This situation may arise if the technologies G_t are convex but non-polyhedral, but also in the search for the so-called " efficiency prices ", even if the technologies are polyhedral. These latter prices are competitive prices which additionally give a valuation on the capital stock in each period, such that it is not more expensive than any other capital vector which would yield as good a consumption sequence from that period on as the efficient programme itself. So the question remains open whether, under the assumptions of the existence of a normal sequence $\{p^T\}$ with $\{p^T\} \in \mathbb{P}^T$ and the boundedness of d_t, one can extract a price sequence from a sequence of convex but non-polyhedral price cones $\mathbb{P}^T$, and is interesting because it would settle the question of the existence of efficiency prices as well.

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REFERENCES

MALINVAUD, E. (1953), " Capital Accumulation and Efficient Allocation of Resources ", *Econometrica*, **21**, 233-238.

MALINVAUD, E. (1962), " A Corrigendum ", *Econometrica*, **30**, 570-573.

McFADDEN, D. (1975), " An Example of the Nonexistence of Malinvaud Prices in a Tight Economy ", *Journal of Mathematical Economics*, **2**, 17-19.

PELEG, B. and YAARI, M. (1970), " Efficiency Prices in an Infinite Dimensional Space ", *Journal of Economic Theory*, **2**, 41-85.

ROCKAFELLAR, R. T. (1970) *Convex Analysis* (Princeton University Press).

ROCKWELL, R. (1973), " The Quantitative Theory of Interest " (O.R. Center Report, U.C. Berkeley, ORC 73-9).