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THE EFFICIENT PATHS TO INFINITY IN CLOSED LEONTIEF MODELS*

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For a simple Leontief technology, the results of finite-horizon turnpike theorems are extended to the case of infinite horizon. Further qualitative information is obtained about the set of efficient programs. Uniqueness is discussed, and shown by an example to be not true in general.

1. Introduction

This paper looks at efficient capital accumulation for a closed model of production in an infinite horizon. Of the vast literature on efficient behaviour, the general conclusion of the finite-horizon turnpike theorems can be stated as follows: for technologies determining a unique maximal balanced-growth path, i.e., the turnpike, efficient programs of the very long run spend most of their time near this path. As shown by McKenzie (1963) and Tsukui (1966), this result can be made more specific for Leontief models: efficient programs must remain near the turnpike and use full capacity for all but some periods in the beginning and the end. In a somewhat different setting, Furuya and Inada (1962) prove the uniqueness of efficient programs when the horizon is extended to infinity. However, their assumptions imply that only one process is profitable under the Von Neumann prices, which the Leontief model does not fulfil.

The technology assumed in this paper is of the simple Leontief type with no intermediate goods. Using the same assumptions, we extend the results of McKenzie (1963) and Tsukui (1966) to the case of infinite horizon, and as is to be expected, gain further qualitative information about the set of efficient programs. Finally we discuss uniqueness, and show by an example this is not true in general.

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2. The model and prices

We will first give definitions about efficient capital accumulation in general, state a standard proposition about supporting prices, and then introduce our model.

Notation. (z_t) stands for a sequence of vectors $(z_0, z_1, \dots)$, and $(z_t)_M^N \equiv (z_M, z_{M+1}, \dots, z_N)$. Superscripts j denote coordinates. $y \geqq z \Rightarrow y^j \geqq z^j$, $\forall j$, $y > z \Rightarrow y^j > z^j$, $\forall j$, and $y \geqslant z \Rightarrow y \geqq z$ but $y \neq z$. Also, $yz \equiv \sum y^j z^j$; and $\|z\| \equiv (zz)^{\frac{1}{2}}$.

Definitions. (1) A *technology* T is a set of pairs $(z, z') \in R_+^n x R_+^n$. T is assumed to be a closed convex cone rooted at $(0, 0)$. T has the 'free disposal' property; i.e., $(z, z') \in T$ and $z'' \leqslant z' \Rightarrow (z, z'') \in T$. Also $(0, z') \in T \Rightarrow z' = 0$; and $\exists (z, z') \in T$ with $z' > 0$.

(2) A *program* is a (z_t) such that $(z_t, z_{t+1}) \in T$, $\forall t$. A program is *efficient* if $\exists$ no other program (z_t') such that $z_0' = z_0$ and $z_t' \geqslant z_t$ for some t.

(3) The *price cone P* of T consists of all pairs (p, p') such that $pz \geqslant p'z'$ for all $(z, z') \in T$. A *price sequence* is a (p_t) such that $(p_t, p_{t+1}) \in P$, $\forall t$. A price sequence (p_t) is said to support (z_t) if $p_t z_t = p_{t+1} z_{t+1}$, $\forall t$.

(4) Let K^z be the set of all efficient programs (z_t) for which $z_0 = z$; and let Q^z be the set of all price sequences (p_t) which support some $(z_t) \in K^z$.

The following fact is standard for technologies as in Definition 1 above; see Malinvaud (1953, 1962).

The Price Proposition. (i) *Every efficient program (z_t) has a supporting price sequence (p_t) such that $p_t \geqq 0$, $\forall t$, and $p_t \geqslant 0$ for at least one t.* (ii) *If a program (z_t) has a supporting price sequence (p_t) for which $p_t > 0$, $\forall t$, then (z_t) is efficient.* ▮

Now we describe our particular technology. There are n sectors, each identified with one good. A vector $a_j \in R_+^n$ represents the stocks that must exist in each sector to add one unit of the jth good to the total capital at the end of a period. Letting A be the matrix of column vectors a_j,

$$T = \{(z, z') \mid Ax \leqslant z;\ z' = z + x;\ x, z \in R_+^n\}.$$

Equivalently,

$$T = \left\{(z, z') \middle| \begin{pmatrix} -(I+A) & A \\ I & -I \end{pmatrix} \begin{pmatrix} z \\ z' \end{pmatrix} \leqslant 0;\quad z \in R_+^n \right\},$$

the price cone of which is easily seen to be

$$P=\left\{(p,p')\middle|(p,p')\begin{pmatrix} A & I \\ -(I+A) & -I \end{pmatrix}\geqslant 0\right\}.$$

We note that the support condition in this case

$$\begin{aligned}0&=p_t z_t - p_{t+1} z_{t+1}\\ &=[p_t A - p_{t+1}(I+A)][z_{t+1}-z_t]+[p_t - p_{t+1}][(I+A)z_t - A z_{t+1}]\end{aligned}$$

yields the usual complementary slackness conditions, since each bracketed term above is $\geqslant 0$. In particular,

$$\begin{aligned} &z_{t+1}-z_t>0 \Rightarrow p_t A = p_{t+1}(I+A), \\ &p_t - p_{t+1} > 0 \Rightarrow A z_{t+1} = (I+A) z_t. \end{aligned} \tag{1}$$

We will assume throughout this paper that the initial capital $z>0$, and that A has the following properties:

(H.1) *A is indecomposable.*

By the Frobenius theory of non-negative matrices, A then has a simple positive eigenvalue λ, that has maximal absolute value among all of A's eigenvalues. To λ correspond strictly positive left and right eigenvectors, that we denote by $\bar{e}$ and e, respectively. We choose $\|\bar{e}\|=\|e\|=1$; and let $\mu=|(1+\lambda)/\lambda|$.

(H.2) *A and $(I+A)$ are non-singular.*

(H.3) *A is 'non-cyclic'*, i.e., for any other eigenvalue λ' of A, $|(1+\lambda')/\lambda'|\neq\mu$.

3. The full-capacity property

The main result of this section (Theorem 1) is an extension of the finite horizon turnpike theorems: 'All efficient programs starting from the same stocks have full-capacity production from some date on and their capital compositions uniformly converge to e.' We start the long proof with two observations. First, $(\bar{e},\mu^{-1}\bar{e})\in P$; and therefore the sequence $(\mu^{-t}\bar{e})$ is a price program. Next, define for arbitrary N,

$$F^N=\{(z^t)_0^N \mid A z_{t+1}-(I+A)z_t=0;\quad z_{t+1}-z_t\geqslant 0;\quad t=0,\dots,N-1\},$$

the convex cone in $R_+^{(N+1)n}$ consisting of all N-period full-capacity programs.

It is an easy consequence of the following identities:

$$\sum_{t=0}^{N-1} \mu^{-t}(1+\lambda)^{-1}\bar{e}(Az_{t+1}-(I+A)z_t)=\sum_{t=0}^{N-1} \mu^{-t-1}\bar{e}z_{t+1}-\mu^{-t}\bar{e}z_t$$
$$=\mu^{-N}\bar{e}z_N-\bar{e}z_0,$$

that equivalently,

$$F^N=\{(z_t)_0^N \text{ is an } N\text{-period program} \mid \mu^{-N}\bar{e}z_N-\bar{e}z_0=0\}. \qquad (2)$$

Lemma 1.[1] *For any $\varepsilon>0$ and any N, $\exists\delta>0$ such that if $(z_t)_0^N$ is a program and $d((z_t)_0^N, F^N)\geqslant\varepsilon$ then $\bar{e}z_N\leqslant-(\mu^N-\delta)\bar{e}z_0$.* ▮

Lemma 2. Let (z_t) be an efficient program. Then for any $\varepsilon>0$ and any N, $\exists M$ such that $d((z_t)_\tau^{\tau+N}, F^N)\leqslant\varepsilon$, $\forall\tau\geqslant M$. ▮

The proof of Lemma 1 follows from (2) and is similar to that of the so-called value-loss lemma in Radner (1961), while Lemma 2 is proved like Theorem 1 in McKenzie (1963). Both proofs are omitted here; for details see Alkan (1974).

Next we look at two decompositions of R^n, corresponding respectively to the right and left eigenvector structure of A, and cite some related facts: Let $E=\{\alpha e|\alpha\in R\}$, $\bar{E}=\{\alpha\bar{e}|\alpha\in R\}$, $U=\{u|\bar{e}u=0\}$, and $\bar{U}=\{u|ue=0\}$. Then $R^n=E+U=\bar{E}+\bar{U}$. Next let $C=\mu^{-1}(I+A^{-1})$ and define $V=\{v\in U|C^tv\to 0\}$. It follows from the appendix that, $\|C^tu\|\to\infty$ for all $u\in U\backslash V$. Similarly, define $\bar{V}=\{v\in\bar{U}|vC^{-t}\to 0\}$; then $\|uC^{-t}\|\to\infty$ for all $u\in\bar{U}\backslash\bar{V}$. Furthermore, $\bar{V}\perp V$; i.e., $\bar{v}v=0$, $\forall\bar{v}\in\bar{V}$ and $\forall v\in V$, since the eigenvalues corresponding to $\bar{V}$ are distinct from those to V. Finally we note that all the subspaces above are invariant under C. These facts above will be used often in the rest of this paper.

Lemma 3. Given any $\delta>0$, $\exists h$ such that $(z_t)_0^{2h}\in F^{2h}\Rightarrow d(z_h,e)\leqslant\delta$.

Proof. Consider $(z_t)_0^{2h}\in F^{2h}$. Let $y_t=\mu^{-t}z_t$; then $z_t=(I+A^{-1})^tz_0\Leftrightarrow y_t=C^ty_0$. Since $d(\alpha y_t,e)=d(z_t,e)$ for any $\alpha\neq 0$, we will study $(y_t)_0^{2h}$ such that $\|y_0\|=1$.

Suppose the lemma is not true for a $\delta>0$. Then, $\exists(y_t)_0^{2h}$ for any h such that

$$d(y_h,e)>\delta. \qquad (3)$$

Letting $y_0=ke+u$ for some $k\in R$ and some $u\in U$, we get $y_t=ke+C^tu$. Then $u\notin V$; for otherwise $C^tu\to 0$, and $d(y_t,e)\to 0$, contradicting (3). Therefore $\|C^tu\|$

[1]Define *angular distance* $d(x,y)=\|x/\|x\|-y/\|y\|\|$, $\forall x,y\neq 0$, and $d(x,S)=\inf_{y\in S}d(x,y)$.

$\to\infty$. Since $U \cap R^n_+ = \{0\}$, this implies $(C^t u)^j \to \infty$ for some component j. But $\bar{e}y = k\bar{e}e$ and hence k is bounded above. Therefore, y^j_τ is negative for some date τ. Choosing $2h \geqslant \tau$ we reach a contradiction. ▮

Let us say that a program (z_t) is 'full-capacity after N' if $z_{t+N} = (I + A^{-1})^t z_N$, $\forall t \geqslant N$, and 'full-capacity' if $N=0$.

Theorem 1. (i) *$d(z_t, e) \to 0$ uniformly for all $(z_t) \in K^z$.* (ii) *$\exists N \ni$ all $(z_t) \in K^z$ are full capacity after N.*

Proof. (i) Let $(z_t) \in K^z$. Given $\delta > 0$ pick h according to Lemma 3. By Lemma 2, $\exists M$ such that $d((z_t)^{\tau+2h}_\tau, F^{2h}) \leqslant \delta$, $\forall \tau \geqslant M$. Consider any date $\tau \geqslant M + h$. Then $\exists (\bar{z}_t)^{2h}_0 \in F^{2h}$ such that $d((z_t)^{\tau+h}_{\tau-h}, (\bar{z}_t)^{2h}_0) \leqslant \delta$ and $d(\bar{z}_h, e) \leqslant \delta$. Thus,

$$d(z_\tau, e) \leqslant d(z_\tau, \bar{z}_h) + d(\bar{z}_h, e) \leqslant d((z_t)^{\tau+h}_{\tau-h},$$

$$(\bar{z}_t)^{2h}_0) + d(\bar{z}_h, e) \leqslant 2\delta.$$

(ii) By efficiency, $\exists \rho < 1$ such that $\|z_t\| / \|z_{t+1}\| \leqslant \rho$, $\forall t$, and for all $(z_t) \in K_z$. By (i),

$$\begin{aligned} &\left\| z_{t+1}/\|z_{t+1}\| - e \right\| + \left\| z_t/\|z_{t+1}\| - (\|z_t\|/\|z_{t+1}\|)e \right\| \\ &\quad = d(z_{t+1}, e) + (\|z_t\|/\|z_{t+1}\|) d(z_t, e) \to 0. \end{aligned}$$

Let $\delta = (1-\rho) \min_j e^j$, and choose by above an M such that

$$\begin{aligned} &\max_j (e^j - z^j_{t+1}/\|z_{t+1}\|) + \max_j (z^j_t/\|z_{t+1}\| - \rho e^j) \\ &\quad \leqslant \max_j (e^j - z^j_{t+1}/\|z_{t+1}\|) \\ &\qquad + \max_j (z^j_t/\|z_{t+1}\| - \|z_t\| e^j/\|z_{t+1}\|) \leqslant \delta/2, \end{aligned}$$

$\forall t \geqslant M$, and for all $(z_t) \in K^z$. But then

$$\begin{aligned} &\min_j (z^j_{t+1}/\|z_{t+1}\| - z^j_t/\|z_{t+1}\|) \\ &\quad \geqslant \min_j (e^j - \rho e^j) - \max_j (e^j - z^j_{t+1}/\|z_{t+1}\|) \\ &\qquad - \max_j (z^j_t/\|z_{t+1}\| - \rho e^j) \geqslant \delta/2, \end{aligned}$$

implying $z_{t+1} - z_t > 0$, $\forall t \geqslant M$, and for all $(z_t) \in K^z$.

Therefore by (1), for any $(p_t)\in Q^z$, $p_tA=p_{t+1}(I+A)$, $\forall t\geqslant M$, that is $(p_t-p_{t-1})=(p_M-p_{M-1})(I+A)^{M-t}$, $\forall t\geqslant M$. But then $(p_M-p_{M+1})\in\bar{E}+\bar{V}$, for otherwise $p_\tau-p_{\tau+1}$ fails to be non-negative for some sufficiently large τ, as shown for Lemma 3. Normalizing all $(p_t)\in Q^z$ so that $\|p_M-p_{M+1}\|=1$, it is easily seen that $d((p_t-p_{t+1}),\bar{e})\to 0$ uniformly. Therefore, $\exists N$ such that $p_t-p_{t+1}>0$, $\forall t\geqslant N$, and for all $(p_t)\in Q^z$. The proof is completed by using (1) again. ∎

4. The stable cone S and other properties

Let S be the set of all stocks giving rise to full-capacity programs; i.e., the set of all z such that the sequence (z_t) where $z_t=(I+A^{-1})^t z$ satisfies $z_{t+1}-z_t=A^{-1}z_t\geqslant 0$, $\forall t$. That is

$$S=\{z\,|\,A^{-1}(I+A^{-1})^t\geqslant 0,\quad \forall t\}. \tag{4}$$

We call S the *stable cone* of A, and use it to obtain further properties of K^z in this section.

First, note that if $z\notin E+V$, then as shown for Lemma 3, $A^{-1}(I+A^{-1})^t z\notin R^n_+$ for large enough t; therefore

$$S\subset E+V.$$

Lemma 4. If (z_t) is full-capacity, then $K^z=\{(z_t)\}$.

Proof. If (z_t) is full capacity, then $z_0\in S$ by definition. It is trivial to show that $(\mu^{-t}\bar{e})$ supports (z_t), and therefore $(z_t)\in K^z$ by the price proposition.

Conversely, suppose $z_0\in S$, $(\bar{z}_t)\in K^z$, and that $(\bar{z}_t)$ is not full-capacity. Let τ be the smallest date after which $(\bar{z}_t)$ is full-capacity. By assumption $\tau\geqslant 1$, and by Theorem 1 it exists. And we have $A\bar{z}_\tau-(I+A)\bar{z}_{\tau-1}\leqslant 0$. But then $\bar{e}(A\bar{z}_\tau-(I+A)\bar{z}_{\tau-1})<0$; i.e., $\mu^{-1}\bar{e}\bar{z}_\tau<\bar{e}\bar{z}_{\tau-1}$. And since $\mu^{-(\tau-1)}\bar{e}\bar{z}_{\tau-1}\leqslant\bar{e}\bar{z}_0=\bar{e}z_0$, we have $\mu^{-\tau}\bar{e}z_N<\bar{e}z_0$. But for the full capacity program $(z_t)\in K^2$, we have $\mu^{-\tau}\bar{e}z_\tau=\bar{e}z_0$. Hence

$$\bar{e}z_\tau>\bar{e}\bar{z}_\tau. \tag{5}$$

Note that both z_τ and $\bar{z}_\tau$ belong to $S\subset E+V$. Let $z_\tau=ke+v$ and $\bar{z}_\tau=\bar{k}e+\bar{v}$ for some k, $\bar{k}\in R$ and v, $\bar{v}\in V$. Since $\bar{e}v=\bar{e}\bar{v}=0$, (5) implies $k>\bar{k}$. But for $t\geqslant\tau$,

$$z_t-\bar{z}_t=(I+A^{-1})^{t-\tau}(z_\tau-\bar{z}_\tau)=\mu^{t-\tau}(k-\bar{k})e+(I+A^{-1})^{t-\tau}(v-\bar{v});$$

i.e.,

$$\mu^{-(t-\tau)}(z_t-\bar{z}_t)=(k-\bar{k})e+C^{t-\tau}(v-\bar{v}).$$

Recalling that $\|C^{t-\tau}(v-\bar{v})\| \to 0$, we get $z_t > \bar{z}_t$ for large t, contradicting the efficiency of $(\bar{z}_t)$. ▮

Lemma 5. For any efficient (z_t), $\exists N \ni z_t \notin S$, $\forall t < N$, *and* $z_t \in S$, $\forall t \geqslant N$.

Proof. Let N be the smallest date for which $z_N \in S$. $(z_t)_N^\infty$ must belong to K^{z_N}, for otherwise (z_t) is not efficient. The proof follows from Lemma 4. ▮

Defining analogous to S a dual stable cone $\bar{S}$ for price sequences, we get the counterpart of Lemma 5 for $(p_t) \in Q^z$. It is not difficult to show that S and $\bar{S}$ are polyhedral convex cones [see Alkan (1974)]. And, provided the inequalities defining S and $\bar{S}$ are known analytically, it seems possible then to develop an algorithmic procedure based on Lemma 5 to generate $(z_t) \in K^z$. This line will not be pursued here, and we just note that Lemma 5 gives geometric information about the initial behaviour of efficient programs.

By Lemma 4, K^z consists of only one program if $z \in S$. In the next theorem we show that even when K^z contains many efficient programs, they are almost the same.

Theorem 2. (i) *Any* $(p_t) \in Q^z$ *supports any* $(z_t) \in K^z$. (ii) K^z *and* Q^z *are convex.*

Proof. (i) Let N be the date as in the proof of Theorem 1 such that $z_t = (I + A^{-1})^{t-N} z_N$ for all $(z_t) \in K^z$ and $p_t = p_N (I + A^{-1})^{N-t}$ for all $(p_t) \in Q^z$. Then $z_t \in E + V$ and $p_t \in \bar{E} + \bar{V}$, $\forall t \geqslant N$.

Now pick any (z_t), $(\bar{z}_t) \in K^z$, and let $(p_t) \in Q^z$ support (z_t). Suppose $p_\tau z_\tau > p_\tau \bar{z}_\tau$ for some $\tau \geqslant N$. Like (5), this contradicts the efficiency of $(\bar{z}_t)$. Hence $p_t z_t = p_t \bar{z}_t$. $\forall t \geqslant N$. Also, $p_0 z \geqslant p_1 \bar{z}_1 \geqslant \cdots \geqslant p_N \bar{z}_N$ by definition of a price sequence, and $p_0 z = p_1 z_1 = \cdots = p_N z_N$ since (p_t) supports (z_t). Therefore $p_t z_t = p_t \bar{z}_t$, $\forall t \leqslant N$, as well. That is, (p_t) supports (z_t).

(ii) Follows trivially from (i). ▮

Especially (i) above may lead to the conjecture that, not only for $z \in S$ as in Lemma 4 but for all z, K^z consists of a unique efficient program. It can be shown that this is true for all 2-sector technologies for example, but as the following construction shows not true in general.

An Example of Infinitely Many $(z_t) \in K^z$.

$$A = \begin{pmatrix} 1/2 & 1/2 & 13/2 \\ 1/2 & 0 & 3/2 \\ 1/4 & 3/8 & 0 \end{pmatrix}.$$

The eigenvalues of $(I + A^{-1})$ are $3/2$, $-1/3$, $-1/3$; $e = (5, 2, 1)$; $\bar{e} = (1, 1, 4)$.

Note that $C \equiv 2/3(I+A^{-1})$ has no eigenvalue of modulus greater than one. Hence $R^3 = E + V$.

Claim. $e \subset \text{Int} S$. Since $\bar{e}e > 0$, $\exists$ an ε-neighborhood G_ε of z such that $0 < k' \leqslant \bar{e}z \leqslant k''$ for some k', k'', and $\forall z \in G_\varepsilon$. Letting $z = ke + v \in E + V$, we have $\|v\| \leqslant \|z - ke\| \leqslant 1 + \varepsilon + k''$. Therefore, $C^t v \to 0$ uniformly, $\forall z \in G_\varepsilon$. Then, $\exists m$ such that $A^{-1}\mu^{-t}(1+A^{-1})^t z = A^{-1}ke + A^{-1}C^t v \geqslant 0$, $\forall t \geqslant m$; i.e., $A^{-1}(I+A^{-1})^t z \geqslant 0$, $\forall z \in G_\varepsilon$. Also since $e > 0$ and $d(A^{-1}(I+A^{-1})^t e, e) = 0$, for any m there is an ε-neighborhood G_ε of e such that $A^{-1}(I+A^{-1})^t z \geqslant 0$, $\forall t \leqslant m$ and $\forall z \in G_\varepsilon$. Choosing ε properly, by (4) $G_\varepsilon \subset S$; i.e., $e \subset \text{Int} S$.

Let
$$z_0 = \begin{pmatrix} 7 \\ 7 \\ 4 \end{pmatrix}$$

$$z_1^\varepsilon = \begin{pmatrix} 20 \\ 8 \\ 4 \end{pmatrix} + \varepsilon \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$z_t^\varepsilon = (I+A^{-1})^t z_1^\varepsilon = \left(\frac{3}{2}\right)^{t-1} \begin{pmatrix} 20 \\ 8 \\ 4 \end{pmatrix} + \varepsilon (I+A^{-1})^{t-1} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

As is easily verified, $(z_0, z_1^\varepsilon) \in T$ for ε in $[-3, 1]$. Since $\left(\begin{smallmatrix} 20 \\ 8 \\ 4 \end{smallmatrix}\right)$ belongs to the interior of S, there is a non-trivial subinterval I of $[-3, 1]$, such that

$$z_1^\varepsilon \in S \quad \text{for} \quad \varepsilon \text{ in } I.$$

By definition of S therefore, (z_t^ε) is a program for ε in I. Furthermore, the following price sequence (p_t) supports (z_t^ε) for each ε in I.

$$P_0 = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$$

$$P_t = \left(\frac{2}{3}\right)^{-t+1} \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}, \qquad t = 1, 2, \ldots$$

Hence, (z_t^ε) is efficient for each ε in I. ∎

A final comment about the assumptions: (H.1) is necessary for Lemmas 1 and 2; and beginning with Lemma 3 the rest of the results use and heavily depend on all of (H.1), (H.2), (H.3). For examples showing that (ii) of Theorem 1 fails without (H.1) and that both (i), (ii) fail without (H.3), see Alkan (1974).

Appendix

Lemma. *Let $D: E^n \to E^n$ be a linear transformation with no eigenvalues on the unit disc, where E^n is the n-complex space. Then E^n is the direct sum of two subspaces E_a and E_b; and*

$$\|D^t u\| \to 0, \quad \forall u \in E_a \quad \text{and} \quad \|D^t u\| \to \infty, \quad \forall u \in E_b, \quad u \neq 0.$$

Sketch. Consider the Jordan canonical form of D restricted to $E^n \backslash$kernel D. We let $\bar{E}_a(\bar{E}_b)$ to be the subspace corresponding to eigenvalues of modulus less than 1 (greater than 1). Then

$$E^n = E_a + E_b,$$

where

$$E_a = \text{kernel } D + \bar{E}_a \quad \text{and} \quad E_b = \bar{E}_b,$$

for which decomposition the statement above is seen to hold.

Since for any linear transformation $C: R^n \to R^n$, we can find a $D: E^n \to E^n$ such that the restriction of D to $R^n \subset E^n$ is C, the lemma holds for a $C: R^n \to R^n$ with the same properties. ∎

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