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The Core of the Matching Game

AHMET ALKAN*

Department of Management, Bogazici University, 80815 Istanbul, Turkey

AND

DAVID GALE†

Department of Mathematics, University of California, Berkeley, California 94720

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In the matching game there are two sets of players P and Q and for each pair (p, q) there is a set of payoffs (u_p, v_q) which can be obtained if the players p and q decide to collaborate. We give a constructive proof that the core of this game is nonempty and show that it has a strong connectedness property which may be thought of as a nonlinear generalization of convexity. © 1990 Academic Press, Inc.

1. INTRODUCTION

In the Matching Game there are two sets of players or agents, for example, firms and workers, or men and women, who may form a partnership and then choose some joint activity which will yield utility to each of the partners. The problem is to find a set of partnerships and a choice of activities for each pair such that no two players who are not paired could form a partnership and choose an activity which would make both of them better off. The set of such arrangements constitutes the core of the game (a rigorous definition is given in the next section). The properties of the core were studied extensively by Demange and Gale (1985) where it was shown to have various structural properties which are not present in the cores of more general market type games. However, no proof of nonemptiness was given there.

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Proofs of existence have been given by Quinzii (1984) and Kaneko (1982) making use of the balancedness result of Scarf (1967). Crawford and Knoer (1981) describe a proof based on a limiting argument from the discrete case, the so-called stable marriage problem, but leave the details to the reader.

Our purpose here is to provide a complete proof which is elementary, meaning that it does not use fixed point theorems or their equivalents and which is constructive in that it can be used as the basis for an algorithm which will find a point in the core in a finite number of steps if utility functions are piecewise linear and will approximate such points to any desired degree of accuracy for general continuous utilities (Alkan, 1983). The technique of proof is quite different from any others we have seen and depends only on linear programming duality for the special case of the well-known optimal assignment problem.

The main virtue of our approach, however, is that it enables us to analyze the geometric structure of the core which turns out to have a strong connectedness property. Namely, we show that for any two points in the core there is a *monotone path* in the core connecting them, where monotone means that every coordinate along the path either increases, decreases, or remains constant. We call such a set an *M-set.* Convex sets correspond to the special case in which the monotone paths are segments, and *M*-sets have some properties analogous to those of convex sets, as we will show.

2. The Model

There are two sets of players, P and Q. Given any pair (p, q) in $P \times Q$ it is assumed that p and q can engage in a continuum of activities, each of which yields a *payoff* (say in utility) of u units to p and v units to q. The set of such payoffs is called the *payoff set* of the pair (p, q). As an example, p might be a worker, q a firm such that p working for q could generate income and the payoff set would correspond to all possible ways of dividing this income, or p and q might be a couple who gets satisfaction from living together which varies for each of the members depending on how they share the housekeeping chores, etc.

We make two assumptions about the payoff set, individual rationality and Pareto optimality. The first implies that all payoff sets lie in the nonnegative quadrant. The idea is that a person who does not find a partner gets payoff zero. Therefore a pair (p, q) for which there is no point yielding a positive payoff for both will behave as though the members were not partners and receive payoff (0, 0).

The second condition is that no pair will choose to engage in activities which are Pareto dominated. Thus, we will assume that, if (u, v) and (u', v') are points of a payoff set, then $u' > u$, if and only if $v > v'$.

The above conditions lead to the following simple analytical formulation of the model.

> The payoff set for the pair (p, q) is the graph of a continuous, nonnegative decreasing function f_{pq} defined on some interval $0 \le u \le c_{pq}$, where $f(c_{pq}) = 0$. (1)

Condition (1) simply states that, if p receives payoff u, then q receives payoff $f_{pq}(u)$. We do not exclude the case $c_{pq} = 0$, which means the payoff set is simply the origin (0, 0).

For mathematical purposes it is convenient to extend the functions f_{pq} to negative values of u by the rule

$$f_{pq}(u) = f_{pq}(0) - u \qquad \text{for } u \le 0. \tag{2}$$

A typical payoff set is shown in Fig. 1.

We also assume P and Q have the same number of members since, if, say, $|P| > |Q|$, one may add $|P| - |Q|$ fictitious q's such that $f_{pq}(u) = -u$ for all p and fictitious q. Thus, being paired with a fictitious q is equivalent to being unpaired.

DEFINITIONS. A *matching* μ is a partition of $P \cup Q$ into pairs $\{p, q\}$.

A *feasible payoff* is a P-vector $\mathbf{u}$ and a Q-vector $\mathbf{v}$ such that there is a matching μ such that

$$f_{pq}(u_p) = v_q \qquad \text{for all } \{p, q\} \text{ in } \mu. \tag{3}$$

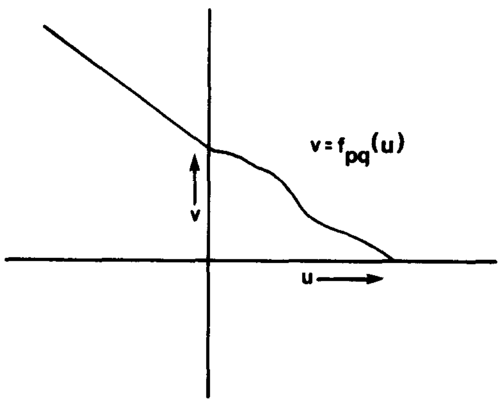

FIG. 1. A typical payoff set.

Thus, a feasible payoff is the result of a matching of the players together with a point in the payoff set of each matched pair.

The fundamental notion of the model is the following: a feasible payoff is called *stable* if

$$f_{pq}(u_p) \leq v_q \qquad \text{for all } (p, q). \tag{4}$$

The interpretation is familiar. If (4) is violated, we would have a pair (p, q) such that $f_{pq}(u_p) > v_q$. In this case the pair (p, q) is said to *block* the payoff $(\mathbf{u}, \mathbf{v})$ because by becoming partners p and q could achieve payoffs $(u'_p, v'_p) \gg (u_p, v_q)$ and hence both would be better off. A stable payoff is one which cannot be blocked.

It will be convenient to combine (3) and (4) as

$$f_{pq}(u_p) \leq f_{\bar{p}q}(u_{\bar{p}}) \qquad \text{for all } (\bar{p}, q) \text{ in } \mu \text{ and all } p. \tag{5}$$

Finally a *core payoff* is a stable payoff $(\mathbf{u}, \mathbf{v})$ with $\mathbf{u}, \mathbf{v} \geq 0$.

Our purpose is to prove the existence and give properties of the set of core payoffs.

3. Background

As mentioned, the proofs to be presented here depend only on the duality theorem for the Optimal Assignment Problem which we now recall.

If $A = (a_{pq})$ is a $P \times Q$-matrix, a matching μ is called *maximal* (*minimal*) if

$$\sum_{(p,q)\in\mu} a_{pq} \geq (\leq) \sum_{(p,q)\in\mu'} a_{pq},$$

where μ' is any other matching.

Duality Theorem. *The matching μ is maximal (minimal) if and only if there is a P-vector $\mathbf{u}$ and a Q-vector $\mathbf{v}$ such that*

$$\begin{aligned} a_{pq} &\leq (\geq)\, u_p + v_q && \text{for all } (p, q) \text{ and} \\ a_{pq} &= u_p + v_q && \text{for } (p, q) \in \mu. \end{aligned} \tag{6}$$

This can be written in the more convenient form

$$a_{pq} - u_p \leq (\geq)\, a_{\bar{p}q} - u_{\bar{p}} \qquad \text{for } (\bar{p}, q) \text{ in } \mu \text{ and all } p. \tag{7}$$

Note that (7) for the maximal case is precisely the statement (5) that $(\mathbf{u}, \mathbf{v})$ is a stable payoff for the special case where the functions f_{pq} are given by

$$f_{pq}(u_p) = a_{pq} - u_p. \tag{8}$$

We now have

LEMMA 1. *For the functions f_{pq} satisfying* (2) *there exists a stable payoff.*

Let $a_{pq} = f_{pq}(0)$. Then for $u \leq 0$, $f_{pq}(u) = a_{pq} - u$. Let $(\mathbf{u}, \mathbf{v})$ be dual variables for the optimal assignment problem with $A = (a_{pq})$. We may suppose $\mathbf{u} \leq 0$ since (7) remains valid if one substracts a positive constant k from all u_p. But then (7) is the same as (5) so $(\mathbf{u}, \mathbf{v})$ is stable. ∎

We remark that $(\mathbf{u}, \mathbf{v})$ above is clearly not a core payoff since $\mathbf{u}$ is negative. The idea of our proof is to "perturb" the above stable solution until $\mathbf{u}$ becomes nonnegative.

An immediate consequence of (7) is the following multiplicative version.

LEMMA 2. *If $B = (b_{pq})$ is a positive $P \times Q$-matrix, then there is a matching μ and a positive P-vector $\mathbf{d}$ such that*

$$d_p b_{pq} \geq d_{\bar{p}} b_{\bar{p}q} \qquad \textit{for } (\bar{p}, q) \textit{ in } \mu \textit{ and all } p. \tag{9}$$

Let μ be a minimal matching for the matrix $A = \log B = \log(b_{pq})$. From (6) there exists $\mathbf{u}$ such that

$$\log b_{pq} - u_p \geq \log b_{\bar{p}q} - u_{\bar{p}} \qquad \text{for } (\bar{p}, q) \text{ in } \mu, \text{ all } p.$$

Taking exponentials and defining $d_p = e^{-u_p}$ gives (9). ∎

4. EXISTENCE OF CORE PAYOFFS

We will prove the existence of core payoffs when the functions f_{pq} are piecewise linear. Since any continuous decreasing function is a uniform limit of such functions, the existence will then follow for the general case.

For f_{pq} piecewise linear, we denote by $f_{pq}^+(u)$ the derivative from the right of f_{pq} at u.

Note that $f_{pq}^+(u)$ is negative since f_{pq} is decreasing. Also we have

$$f_{pq}(u + d) = f_{pq}(u) + d f_{pq}^+(u) \qquad \text{for } d \text{ positive} \tag{10}$$

but sufficiently small by piecewise linearity.

The main tool for all that follows is the

PERTURBATION LEMMA. *Let* $(\mathbf{u}, \mathbf{v})$ *be a stable payoff with* $\mathbf{v} \gg 0$. *Then for any* $\varepsilon > 0$ *there is a stable payoff* $(\mathbf{u}', \mathbf{v}')$ *such that*

$$0 < u_p' - u_p < \varepsilon,\ 0 < v_q - v_q' < \varepsilon \qquad \text{for all } (p, q).$$

Let μ be a matching giving the payoff $(\mathbf{u}, \mathbf{v})$ and let D be all pairs (p, q) such that $f_{pq}(u_p) = v_q$. Note that $\mu \subset D$. Now define $B = (b_{pq})$ by

$$\begin{aligned} b_{pq} &= -f_{pq}^{+}(u_p) && \text{for } (p, q) \text{ in } D \\ &= M, \text{ very large,} && \text{otherwise.} \end{aligned}$$

Since $\mu \subset D$ and M is large, it follows that a minimal matching ν for B is contained in D. Since B is positive (because $f_{pq}^{+}(u_p)$ is negative), we have a positive vector $\bar{\mathbf{d}}$ satisfying (9), so

$$d_p\, f_{pq}^{+}(u_p) \le d_{\bar{p}}\, f_{\bar{p}q}^{+}(u_{\bar{p}}) \qquad \text{for } (\bar{p}, q) \text{ in } \nu. \tag{11}$$

Let $\mathbf{u}' = \mathbf{u} + \mathbf{d}$ and define $\mathbf{v}'$ by

$$v_q' = f_{\bar{p}q}(u_p') \qquad \text{for } (\bar{p}, q) \text{ in } \nu.$$

Note that $f_{\bar{p}q}(u_p') < f_{\bar{p}q}(u_p) \le v_q$ by stability of (u, v), so $v_q > v_q'$.

Let $|d| = \max_{p \in P} d_p$. We claim that for $|d|$ sufficiently small $(\mathbf{u}', \mathbf{v}')$ is stable. Thus, from (10) and (5), we must show

$$\begin{gathered} f_{pq}(u_p) + d_p f_{pq}^{+}(u_p) \le v_q' = f_{\bar{p}q}(u_{\bar{p}}) + d_{\bar{p}} f_{\bar{p}q}^{+}(u_{\bar{p}}) \\ \text{for } (p, q) \text{ in } \nu \text{ and all } p. \end{gathered} \tag{12}$$

There are two cases.

Case I. $(p, q) \in D$. Then $f_{pq}(u_p) = f_{\bar{p}q}(u_{\bar{p}}) = v_q$ for (p, q) in ν since ν is contained in D. So the first terms on both sides of (12) are equal and therefore from (11), (12) is satisfied.

Case II. $(p, q) \notin D$. Then by definition of D and the stability of $(\mathbf{u}, \mathbf{v})$ it follows that $f_{pq}(u_p) < v_q = f_{\bar{p}q}(u_{\bar{p}})$ for $(\bar{p}, q)$ in ν. Because of this strict inequality one can choose $|d|$ sufficiently small so that again inequality (12) holds. We can further decrease $|d|$ if necessary to make $u_p' - u_p < \varepsilon$ and $v_q - v_q' < \varepsilon$. ■

THEOREM 1. *There exists a core payoff.*

From Lemma 1 there exists a stable payoff $(\mathbf{u}, \mathbf{v})$ for $\mathbf{u} \le 0$. Let W be the set of all stable $(\mathbf{u}, \mathbf{v})$ with $\mathbf{v} \ge 0$. This set is bounded below for $\mathbf{v}$ and above for $\mathbf{u}$ from condition (1). Therefore there is a solution in W with $\mathbf{v}$ minimal (in the standard partial order of Q-space). We claim the corresponding stable payoff $(\mathbf{u}, \mathbf{v})$ is a core payoff; i.e., $\mathbf{u}$ is nonnegative, for if, say $u_{\bar{p}} < 0$ for some $\bar{p}$ in P, then $f_{\bar{p}q}(u_{\bar{p}}) > 0$ for all q in Q, but by stability $f_{\bar{p}q}(u_{\bar{p}}) \leq v_q$ so we would have $v \gg 0$. But then from the Perturbation Lemma we could find a stable $(\mathbf{u}, \mathbf{v}')$ with $0 \leq \mathbf{v}' \ll \mathbf{v}$ contradicting minimality of $\mathbf{v}$. ■

5. The Core Is An M-Set

Definitions. If $\mathbf{a}, \mathbf{b}, \mathbf{x}$ are distinct points in R^n, we say $\mathbf{x}$ is *between* $\mathbf{a}$ and $\mathbf{b}$ if x_i is between a_i and b_i for all i (i.e., either $a_i \leq x_i \leq b_i$ or $b_i \leq x_i \leq a_i$).

A set X in R^n is a *B-set* if for any $\mathbf{a} \neq \mathbf{b}$ in X there is an $\mathbf{x}$ in X, where $\mathbf{x}$ is between $\mathbf{a}$ and $\mathbf{b}$.

A *monotone path* in R^n is a function φ: $[0, 1] \to R^n$, such that for $0 < t_1 < t < t_2 < 1$, $\varphi(t)$ is between $\varphi(t_1)$ and $\varphi(t_2)$.

A set X in R^n is an *M-set* if for every $\mathbf{a} \neq \mathbf{b}$ in X there is a monotone path from $\mathbf{a}$ to $\mathbf{b}$.

Theorem 2. *A closed B-set is an M-set.*

A proof of this purely point-set topological fact is given in the appendix.

Let U be the set of all $\mathbf{u}$, such that there exists $\mathbf{v}$ such that $(\mathbf{u}, \mathbf{v})$ is a core payoff. We will prove that U is a B-set (hence by Theorem 2 an M-set). By symmetry the result also follows for V, the set of all core payoffs $\mathbf{v}$ to players Q. We first consider a special case.

We denote by $\mathbf{e}$ the point of R^n all of whose coordinates are 1.

Lemma 3. *If* $\mathbf{u}^1, \mathbf{u}^2$ *are in* U *and* $\mathbf{u}^2 \geq \mathbf{u}^1 + \varepsilon\mathbf{e}$ *for* $\varepsilon > 0$, *then there is* $\mathbf{u}'$ *in* U *between* $\mathbf{u}^1$ *and* $\mathbf{u}^2$ *with* $\mathbf{u}' \leq \mathbf{u}^1 + \varepsilon\mathbf{e}$.

We first consider the piecewise linear case. Let U' be all $\mathbf{u}$ in U, such that $\mathbf{u}^{-1} \leq \bar{\mathbf{u}} \leq \bar{\mathbf{u}}^1 + \varepsilon\bar{\mathbf{e}}$. Then U' is nonempty (it contains $\mathbf{u}^1$) and closed, so it has a maximal element $\mathbf{u}'$. We claim $u'_p = u^1_p + \varepsilon$ for some p in P for if not $\mathbf{u}' \ll \mathbf{u}^1 + \varepsilon\mathbf{e} \leq \mathbf{u}^2$, so $\mathbf{v}' \gg \mathbf{v}^2 \geq 0$. Hence by the Perturbation Lemma, one can find $\mathbf{u}''$ such that $\mathbf{u}' \ll \mathbf{u}'' \leq \mathbf{u}^1 + \varepsilon\mathbf{e}$, contradicting maximality of $\mathbf{u}'$. Thus $\mathbf{u}'$ is between $\mathbf{u}^1$ and $\mathbf{u}^2$. For the case of general function f_{pq} and given $\varepsilon > 0$, one approximates by piecewise linear functions. Since all the approximating $\mathbf{u}'$ satisfy $u'_p = u^1_p + \varepsilon$ for some p, it follows that this must hold also in the limit. ■

The following lemma, also proved in Demange and Gale (1985), is basic for the general theory of the matching game.

Let $(\bar{\mathbf{u}}^1, \mathbf{v}^1)$ and $(\mathbf{u}^2, \mathbf{v}^2)$ be core payoffs with corresponding matchings μ_1 and μ_2. We here write $\mu_i(p) = q$ to mean $(p, q) \in \mu_i$, $i = 1, 2$.

We define

$$P_1 = \{p | u_p^1 > u_p^2\}, \qquad P_2 = \{p | u_p^2 > u_p^1\}, \qquad P_0 = \{p | u_p^1 = u_p^2\}.$$

Q_1, Q_2, and Q_0 are defined similarly.

LEMMA 4. *The mappings μ_1 and μ_2 are bijections between P_1 and Q_2, P_2 and Q_1, P_0 and Q_0.*

First $\mu_1(P_1) \subset Q_2$ for if $p \in P_1$ and $\mu_1(p) = q$, then $v_q^1 = f_{pq}(u_p^1) < f_{pq}(u_p^2) \leq v_q^2$ by stability of $(\mathbf{u}^2, \mathbf{v}^2)$. Therefore $|P_1| \leq |Q_2|$. Similarly $\mu_2(P_2 \cup P_0) \subset Q_1 \cup Q_0$, for if $q = \mu_2(p)$, $v_q^2 = f_{pq}(u_p^2) \leq f_{pq}(u_p^1) \leq v_q^1$ by stability of $(\mathbf{u}^1, \mathbf{v}^1)$. Therefore $|P_2 \cup P_0| \leq |Q_1 \cup Q_0|$, but $|P| = |Q|$, hence $|P_1| = |Q_2|$ and $|P_2 \cup P_0| = |Q_1 \cup Q_0|$. Symmetrically $|P_2| = |Q_1|$ so $|P_0| = |Q_0|$ and the mappings are bijections. ■

LEMMA 5. *If $p \in P_1 \cup P_0$ and $q \in Q_1$, then $f_{pq}(u_p^1) < v_p^1$.*

From the lemma $q \in Q_1$, so

$$\begin{aligned} v_q^1 > v_q^2 &\geq f_{pq}(u_p^2) && \text{(by stability)} \\ &\geq f_{pq}(u_p^1) && \text{(since } u_p^1 \geq u_p^2\text{).} \quad ■ \end{aligned}$$

THEOREM 3. *U is a B-set.*

For $\mathbf{u}^1$, $\mathbf{u}^2$ in U let $\hat{\mathbf{u}}_1$, $\hat{\mathbf{u}}_2$ be $\mathbf{u}^1$ and $\mathbf{u}^2$ restricted to the set P_2. Then from Lemma 4 the corresponding $\mathbf{v}^1$ and $\mathbf{v}^2$ are restricted to Q_1. It follows from Lemma 3 that there is $\hat{\mathbf{u}}$ between $\hat{\mathbf{u}}^1$ and $\hat{\mathbf{u}}^2$ with $\hat{u}' \leq \hat{u}' + \varepsilon e$ (e restricted to P_2).

Define $\mathbf{u}'$ by

$$\begin{aligned} u_p' &= \hat{u}_p' && \text{for } p \in P_2 \\ &= u_p^1 && \text{otherwise.} \end{aligned}$$

We claim for ε sufficiently small $\mathbf{u}'$ is in U, for suppose (p, q) blocks $\mathbf{u}'$ so that $f_{pq}(u_p') > v_q'$. This is only possible for $p \in P_1 \cup P_0$ and $q \in Q_1$, but from Lemma 5 above $f_{pq}(u_p^1) < v_q^1$. So by choosing ε sufficiently small, we can assure that this inequality still holds with v_q^1 replaced by v_q' so $(\mathbf{u}', \mathbf{v}')$ is a core payoff and $\mathbf{u}'$ is between $\mathbf{u}^1$ and $\mathbf{u}^2$. ■

APPENDIX

THEOREM 2. *A closed B-set is an M-set.*

Given $\mathbf{a} \neq \mathbf{b}$ in X we must find a monotone path from $\mathbf{a}$ to $\mathbf{b}$. First, it suffices to consider the special case $\mathbf{a} < \mathbf{b}$ for if $a_i > b_i$ for some i we make the transformation of R^n replacing x_i by $-x_i$. We may clearly also assume $\mathbf{a} = 0$, $\mathbf{b} = \mathbf{e}$.

Next, for any $\mathbf{x}$ in R^n let $\bar{x} = (1/n) \sum_{i=1}^n x_i$.

LEMMA 5. *For any $t \in [0, 1]$ there is $\mathbf{x}_t$ in X such that $\bar{x}_t = t$.*

Suppose there is no such x. Let $X' = \{x \in X \cap [0, \mathbf{e}] \mid \bar{x} \leq t\}$ and let $\mathbf{x}'$ maximize $\bar{x}$ in X'. Now define $X'' = \{\mathbf{x} \in X \mid \mathbf{x}' < \mathbf{x} \leq n\}$ and let $\mathbf{x}''$ minimize $\bar{x}$ in this set. Then we have $\mathbf{x}' < \mathbf{x}''$. But since X is a B-set there must be $\mathbf{x}'''$ between $\mathbf{x}'$ and $\mathbf{x}''$, but then $\mathbf{x}' < \mathbf{x}''' < \mathbf{x}''$ which contradicts either the maximality of $\mathbf{x}'$ in X' or the minimality of x'' in X''. ■

The construction of the monotone path is now a standard exercise. Choose a point $x_{1/2}$ in $X \cap [0, \mathbf{e}]$. Then choose points $0 \leq \mathbf{x}_{1/4} \leq \mathbf{x}_{1/2} \leq \mathbf{x}_{3/4} \leq e$. In this way we get points for all diadic rationals and the correspondences is Lipschitzian with Lipschitz constant n for if $\mathbf{x}' \geq \mathbf{x}$ then $|\mathbf{x}' - \mathbf{x}| \leq n(\bar{x}' - \bar{x})$. Then by uniform continuity the function extends to the whole interval [0, 1] giving the desired path. ■

We conjecture that M-sets, like convex sets, are contractable, but we have only succeeded in proving this in two dimensions. However, there is an analogue for M-sets of the "separating hyperplane property" of convex sets. Recall that if X is a closed convex set then for any point y not in X there is a hyperplane in the complement of X which contains y.

DEFINITION. For every subset S of $\{1, \ldots, n\}$ the *S-orthant*, O_S consists of all $\mathbf{x}$ in R^n such that $x_i \geq 0$ for i in S and $x_i \leq 0$ for i in $N - S$.

SEPARATING ORTHANT THEOREM. *If X is an M-set and $\mathbf{y}$ is not in X then there is an orthant O_S such that $\mathbf{y} + O_S$ is in the complement of X.*

Induction on n. For $n = 1$, X is an interval and the proof is immediate. Assume that this is true for n. We suppose y is the origin of R^{n+1} and argue by contradiction. Suppose each of the 2^{n+1} orthants of R^{n+1} contains a point of X. For each $S \subset N$ let O_S^+ (O_S^-) to be all points $(\mathbf{x}, x_{n+1})$ in X such that $\mathbf{x}$ is in O_S and $x_{n+1} \geq 0$ (≤ 0). By assumption there is a point $\mathbf{x}^+$ in O_S^+ ($\mathbf{x}^-$ in O_S^-). Now since $\mathbf{x}^+$ and $\mathbf{x}^-$ are connected by a monotone path this path must cross the hyperplane $R^n = \{\mathbf{x} \in R^{n+1} \mid x_{n+1} = 0\}$ at some point $(\mathbf{x}', 0)$ and $\mathbf{x}'$ must be in O_S because the path is monotone. Thus we have shown that there is a point of X in every orthant O_S of R^n. But the intersection of X with R^n is an M-set, so by induction hypothesis y (the origin) is in $X \cap R^n \subset X$. ■

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