PRICE RULE AND VOLATILITY IN AUCTIONS WITH RESALE MARKETS

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Abstract: This paper offers a model of sealed bid auctions with resale. The policy question whether the seller would fare better under the multiprice rule (where winners pay their actual bids) or the uniprice rule (where winners pay the highest losing bid) has been on the agenda since the 60's and seen a recent revival. While theory has mostly recommended the uniprice rule, mainly on the argument that it would generate higher revenue, practice has predominantly stay with the multiprice rule. The results here recommend the multiprice rule. They say that, while expected revenue (equivalently, the price level) is an invariant throughout, the range of bids is narrower, (hence the price level less volatile) under the multiprice than under the uniprice rule, the more so the greater the resale component of participation. For auctions of treasury depth shares of public firms to be privatized, or other items, where significant proportions of the issues flow to secondary markets, these results thus support the multiprice rule on grounds of price stability next to revenue equivalence.

1 Introduction

Auctions are events whereby a seller’s item gets priced. A variety of auction formats exist. In particular, an auction may receive open bids or sealed bids and the rule that specifies what a winner pays may be various. The present paper is an exploration on whether the multiprice (each winner pays his own bid) or the uniprice (each winner pays the highest losing bid) rule is more advantageous to the seller in sealed bid auctions. Pertinent examples are auctions of treasury debt, shares of public firms to be privatized, and pollution rights.

The payment rule issue has been on the agenda ever since Milton Friedman (1960) advocated that the U.S. Treasury should switch from the multiprice to the uniprice rule. The Treasury did in fact experiment so in some auctions in the 70s but switched back not long after. Throughout the controversy that has

practise has mostly stayed with multipricing. Recently, the matter has seen a revival, and following a review of its rules (see Joint Report 1992), the U.S. Treasury has chosen to experiment once again and is currently implementing the uniprice rule in the auctions of two-year and five-year notes. This move was preceded by a strong call from Chari and Weber (1992) for a switch to the uniprice rule. More recently, in their survey of related theory and empirical work, Bikhchandani and Huang (1993) have also expressed support for the uniprice rule although in more cautious terms. The prominent contention behind all this support has been that the average payment made in auction (the auction price, or equivalently, seller’s revenue) is likely to be higher under the uniprice rule than under the multiprice rule.

A salient aspect of auctions, most prominent in the case of treasury bills, is that participants enter for profit-making in the resale markets that follow an auction. To the extent that the resale price is influenced by what bidding has occurred in an auction, participants would naturally consider and set their behavior in accordance. It is notable that few auction models have attempted any explicit treatment of this aspect. (Bikhchandani and Huang (1989), quoted below, is an exception.) In the present paper, I build and analyze a model which incorporates the resale motive and which allows scrutiny at a dimension largely ignored in previous analyses, namely the volatility of the auction price. The findings are in favor of the multiprice rule.

For background, I will briefly review the arguments put forward on why the uniprice rule would generate higher expected revenue:

(i) When buyers’ valuations of the items for sale (or estimates thereof) are positively correlated, auction theory predicts that participants shade their bids more in the multiprice auction than in the uniprice auction, to such an extent that the auction price is on average lower in the former than in the latter (Milgrom and Weber (1982)). One way to express this phenomenon is that winner’s curse is more severe in the multiprice than in the uniprice auction and (sophisticated) bidders take this into account. (Winner’s curse is owing to the fact that winners will be those bidders who have the highest valuations/estimates, who will then have to downgrade their valuations in view of the correlation, and so who would have overpaid had they bid naively not foreseeing all this.) Expected revenue is the same when and only when no correlation exists among buyers’ valuations/estimates.

(ii) Due to higher severity of the winners’ curse effect, which in general diminishes with reduced uncertainty, information gathering has a higher return in the multiprice auction. The uniprice auction necessitates and gives rise to less information acquisition by comparison. In equilibrium ex post, the cost of this incremental information is a loss of revenue to the seller (Chari and Weber (1992)).

(iii) Secondary market buyers are less informed than auction participants. They infer value by considering what bids have been made in the auction. Auction participants take this into account and signal value to future customers.
This signalling effect pulls bids higher, the more so under the uniprice rule (Bikhchandani and Huang (1989)).

(iv) Because bid preparation is simpler hence less costly in the uniprice auction, participation will be broader and so revenue will be higher. So has been put forward by Friedman (1960). Chari and Weber (1992) also subscribe to this point of view while Bikhchandani and Huang (1993) express disbelief.

The effects cited in (iii) and (iv) aside, the upshot of points (i) and (ii) above is that the resale price will exceed the average auction price by a greater margin in the multiprice auction than in the uniprice one. The differential between the two margins is to be attributed to the higher level of information acquisition activity that needs to be conducted under the multiprice rule than under the uniprice rule. Chari and Weber (1992) are of the opinion that the associated incremental cost has "dubious social value" and is "wasteful". Following change of payment rule, they forecast, the return on related investments will over time accrue to the seller.

The policy thrust of the findings in this paper is that the incremental information that needs to be acquired in the multiprice auction, at a social cost, may well have a social value, namely that of curbing the potential volatility of the auction price, which risk-averse buyers and seller would naturally be mindful about.

The model I employ is simply born by the addition of two features to the standard auction model which together intend to capture the effect of the resale motive: I hold that (i) participants resell an exogenously specified portion of their purchase in the secondary market and that (ii) resale occurs at the auction price materialized. (I defer further comments on these features to the concluding section.) I also employ the benchmark assumption that buyers' valuations (on the portion of their purchase which they keep) are uncorrelated. I then show that the ensuing games have unique equilibria at which, the higher the degree "resale orientedness", the narrower is the range of bids (shrunk ultimately to a singleton) under the multiprice rule, whereas bidding remains unaffected under the uniprice rule. Furthermore, this beneficial narrowing of uncertainty regarding the auction price comes with no loss in expected revenue: the expected auction price is invariant under the degree of resale orientedness as well as the payment rule.

I formally state the model in the next section. Section 3 contains the analysis and results. Section 4 is devoted, for illustration of the results, to two-unit auctions in which case the equilibrium (second order differential) equation turns out to be the familiar Gauss hypergeometric equation and bidding strategies are obtainable in explicit form. (Single unit auctions form the known classical case.) Section 5 contains some comments on the assumptions and features of the model.
2 The Model

There are \( m \) identical items to be auctioned and \( n + 1 \) buyer participants where \( n \geq m \). Each participant puts in a sealed bid for one item. The \( m \) highest bidders each pay according to a prespecified rule and receive an item each. (Some tie-breaking rule is applied in case of ties, not necessary to be specified here, as ties turn out to be zero probability events.) Buyers maximize expected gain, i.e., "value" less payment. The auction is called multiprice or uniprice depending on whether payment equals one's own bid or the \((m + 1)\)st highest bid, i.e., the highest losing bid.

In the so-called independent private values model, the value of an item to a buyer is a number known by the buyer not by the others, called his personal value, and drawn independently from a common distribution. The model I introduce here is essentially the same in every respect except that buyers' valuations are "semi-endogenous" in the following way: Let us call the average payment made in an auction the auction price. I stipulate that the value of an item for an auction participant \( u \) with personal value \( V(u) \) is \( \delta p + (1 - \delta)V(u) \) where \( \delta \) is some prespecified "weight" \( (\delta \in \Delta = [0, 1]) \) and \( p \) is the auction price to materialize in that auction. I suggest \( \delta \) to be interpreted as the "degree" of resale orientedness of the auction. As one parable, for example, consider that there is a post-auction market expected to clear at the auction price, and that winners sell a portion \( \delta \) of their purchase in this market while keeping the remainder for personal use.

Formally, I let \( I = [0, 1] \) be the set of all potential buyers \( u \) facing an auction, each equally likely to be one of the participants and ordered such that personal values \( V(u) \) are increasing in \( u \) with \( V(0) = 0, V(1) = 1 \). An auction game then is described by a quintuple \( \Gamma(m, n, V, \delta, R) \) where, in addition to what has already been specified, the last variable \( R \) is either \( M \) or \( U \) depending on whether the payment rule is multiprice or uniprice. A (pure) strategy of a buyer in such a game is a bid function \( b \) defined on \( I \) and employed by him in the sense that \( b(u) \) is the bid he would make if his identity is \( u \). It will be assumed that the specification of \( \Gamma(m, n, V, \delta, R) \) is common knowledge among all buyers.

One thus has, for each \( m, n \), and distribution \( V \) of personal values, a double family of auction games indexed by \( \delta \in \Delta \). The query I will pursue is on how bidding behaviour in general and the auction price in particular are affected by the size of \( \delta \), comparatively, in the multiprice and uniprice auctions. To motivate this query, let us briefly consider the two extreme degrees \( \delta = 0 \) and \( \delta = 1 \). Study of the former case, which originated more than three decades ago (Vickrey (1960)), has uncovered how a bidder \( u \) shades his value \( V(u) \) under the multiprice rule while bidding exactly \( V(u) \) under the uniprice rule and bears, in particular, the celebrated revenue equivalence theorem: the auction price is the same in expectation whether the payment rule is multiprice or uniprice. At the other extreme \( \delta = 1 \), on the other hand, indeterminacy reigns. The auction game that obtains at this limit, which our model does not allow, has
3 Analysis and results

I will throughout this section consider $m$, $n$, $V$ fixed and let $\Gamma(\delta, R) = \Gamma(m, n, V, \delta, R)$ be any auction game as described in Section 2. All results stated below are well known to hold for $m = 1$ when the model reduces to the classical case. I will further be assuming that $m \geq 2$.

3.1 Existence and uniqueness of equilibrium

3.1.1 The uniprice auction: Dominant strategy equilibrium

Proposition[1] It is a dominant strategy for any buyer $u \in I$ in a uniprice auction $\Gamma(\delta, U)$ to bid his personal value $V(u)$.

This result says that, whatever $\delta$ is and whatever the personal values of all other participants may actually be, it is impossible for any buyer to achieve a higher profit than what he would achieve by bidding his personal value. The proof is straightforward and the same as in the classical case.

3.1.2 The multiprice auction: Existence and uniqueness of symmetric equilibrium

Following standard methodology, I will restrict attention to the analysis of symmetric Nash equilibria, i.e., those equilibria where every buyer employs the same bid function. Formally, a bid function $b$ constitutes a symmetric equilibrium for $\Gamma(\delta, M)$ if every buyer $u \in I$ maximizes his expected gain by bidding $b(u)$ given that every other buyer employs $b$.

The following is the first main result of the paper.

Theorem 1 There exists for every multiprice auction $\Gamma(\delta, M)$ a unique increasing bid function which constitutes a symmetric equilibrium.

Assumption The proof I give below utilizes the assumption that $V$ is differentiable and in (Lemma 2) that $(1 - u)V$ is strictly concave, i.e., $(1 - u)V' - V$ is decreasing in $u \in I$. 
**Notation** Call $F_{mn}$ the cumulative probability distribution of the $m^{th}$ highest of $n$ independent draws from the uniform distribution on the unit interval and call $f_{mn}$ the density function of $F_{mn}$.

Towards proving Theorem 1, take any buyer $u \in I$ in the auction $\Gamma(\delta, M)$ and let $x$ be his bid. Suppose $b$ is an increasing bid function employed by all the other $n$ buyers (the *opposition*). Buyer $u$ wins an item if and only if $z \leq x$ where $z$ is the $m^{th}$ highest bid among the opposition. Thus, $u$ wins if and only if $b(y) \leq x$, i.e., $y \leq b^{-1}(x)$. Note that if $u$ wins then the $m - 1$ other winners are each uniformly distributed with density $1/(1 - y)$ on the subinterval $[y, 1]$. So the expected auction price, conditional on $u$ winning with a bid $x$ and $y$ being the $m^{th}$ largest buyer in the opposition, is given by

$$(x/m) + (m - 1)/m \int_{y}^{1} b(w)/(1 - y) \, dw.$$

The expected gain of $u$ then having made a bid $x$ is

$$G(x) = \int_{0}^{b^{-1}(x)} ((1 - \delta)V(u) + \delta(x/m + (m - 1)/m \int_{y}^{1} b(w)/(1 - y)) \, dw)$$

$$- x) f_{mn}(y) \, dy.$$

Differentiating $G(x)$ with respect to $x$ one gets,

$$G'(x) = f_{mn}(b^{-1}(x))((1 - \delta)V(u)$$

$$- (1 - \delta/m)x)/b'(b^{-1}(x)) - (1 - \delta/m)F_{mn}(b^{-1}(x)) -$$

$$\delta(m - 1)f_{mn}(b^{-1}(x))B(b^{-1}(x))/(m(1 - b^{-1}(x))b'(b^{-1}(x)))$$

where it is defined for $y \in I$ that

$$\text{(1) } B(y) = - \int_{y}^{1} b(w) \, dw.$$

Note that $b = B'$.

Suppose $b$ constitutes a symmetric equilibrium. Then $x = b(u)$. Now, setting $u = b^{-1}(x)$, defining the new parameter

$$\text{(2) } k = (\delta m - \delta)/(m - \delta),$$

and rearranging the optimality condition $G'(x) = 0$, one obtains the second order differential equation

$$\text{(3) } F_{mn}B'' + f_{mn}B' + (kf_{mn}/(1 - u))B = (1 - k)f_{mn}V$$

in $u \in I$. One has from (1) the condition

$$\text{(4) } B(1) = 0.$$
Note that the parameter \( k \) increases in \( \delta \) and takes the unit interval onto itself (for every \( m \geq 2 \)). I will refer to the differential equation (3) as \( \mathbf{B}(k) \).

To recapitulate, if \( b \) is a bid function which is increasing on \( \mathbf{I} \) and which constitutes a symmetric equilibrium for the auction \( \Gamma(\delta,M) \), then \( B \) defined by (1) is a solution of \( \mathbf{B}(k) \) that fulfills condition (4).

**Proof of Theorem 1** In view of the preceding paragraph, proof of Theorem 1 follows from Lemmas 1 and 2 stated below.

**Lemma 1** There exists a unique solution \( B \) of the differential equation \( \mathbf{B}(k) \) which is defined on \( \mathbf{I} \) and which satisfies (4).

We shall make use of expression

\[
(5) \quad f_{mn}(u) = (n!/(n-m)!(m-1)!) u^{n-m} (1-u)^{m-1}
\]

and the identity

\[
(6) \quad (n+1) F_{mn} = f_{1(n+1)} + \ldots + f_{m(n+1)}.
\]

Applying (5) and (6) to (3) and cancelling \( n!/(n-m)!(m-1)!u^{n-m} \) on both sides, one gets that \( \mathbf{B}(k) \) is equivalent to

\[
(7) \quad u\Phi_{mn}B'' + (1-u)^{m-1}B' + k(1-u)^{m-2}B = (1-k)(1-u)^{m-1}V,
\]

where

\[
\Phi_{mn} = F_{mn}/(n!/(n-m)!(m-1)!u^{n-m+1})
= (n-m)!(m-1)! \sum_{j=0}^{m-1} 1/((n-j)!j!) u^{m-1-j} (1-u)^j
\]

is positive for all \( u \in \mathbf{I} \). It is readily checked that \( \mathbf{B}(k) \) has a singularity at \( u = 0 \) and is regular elsewhere on \( \mathbf{I} \).

**Proof of Lemma 1** (Sketch; see for instance Coddington and Levinson (1995)) The general solution \( B \) of \( \mathbf{B}(k) \) is of the form

\[
B = c_1B_1 + c_2B_2 + B^*
\]

where \( B^* \) is any particular solution, \( c_1, c_2 \) are arbitrary constants, and \( B_1, B_2 \) are two independent solutions of the homogeneous equation of \( \mathbf{B}(k) \). It is straightforward to compute that 0 and \(-n-m\) are the two solutions of the indicial equation of \( \mathbf{B}(k) \). Hence \( B_1, B_2 \) have the form

\[
B_1(u) = \sum_{j=0}^{\infty} a_j u^j,
\]

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\[ B_2(u) = d \int B_1(u) \ln u + u^{-(n-m)} \left( \sum_0^\infty d_j u^j \right). \]

B defined at \( U = 0 \) implies \( c_2 = 0 \). Condition (4) determines \( c_1 \) uniquely as asserted.

**Lemma 2** If \( B \) is a solution of \( B(k) \) as in Lemma 1, then \( B' \) is increasing on \( I \).

**Proof** Let \( B \) be solution of \( B(k) \) as in Lemma 1. Then

\[
(8) \quad u\Phi_{mn}B''' + (\Phi_{mn} + u\Phi'_{mn} + (1 - u)^{m-1})B'' - \\
(m - 1 - k)(1 - u)^{m-2}B' - k(m - 2)(1 - u)^{m-3}B \\
= (1 - k)(1 - u)^{m-2}((1 - u)V' - (m - 1)V)
\]

which one obtains upon differentiating (7). The proof is in two steps.

**Step 1** \( B' \) is increasing on an interval \([0, u]\) for some \( u \in (0, 1) \).

Suppose the contrary, i.e., \( B' \) is nonincreasing on \([0, u]\) for some \( u \in (0, 1) \),
and let \( u^* \) be the maximum of all such \( u \) if a maximum exists. Then

\[
(9) \quad B''(0) \leq 0.
\]

\[
(10) \quad B''(u^*) = 0, B'''(u^*) \geq 0.
\]

It follows from (8), (9) and (10) that

\[
(11) \quad (m - 1 - k)(B'(0) - B'(u^*)) \leq (m - 2)k(B(u^*))(1 - u^*) - B(0) \\
+ (1 - k)((1 - u^*)V'(u^*) - (m - 1)V(u^*) - (V'(0) - (m - 1)V(0)).
\]

From (7), (9), and (10), on the other hand,

\[
k(B(u^*))(1 - u^*) - B(0)) \leq B'(0) - B'(u^*) + (1 - k)(V(u^*) - V(0)),
\]

using which in (11) gives

\[
(B'(0) - B'(u^*)) \leq (1 - u^*)V'(u^*) - V(u^*) - (V'(0) - V(0)).
\]

By assumption, the right hand side above is negative, which says \( B'(0) < B'(u^*) \). Upon this contradiction, therefore, it can only be that \( u^* \) does not exist, i.e., that \( B' \) in nonincreasing throughout \( I = [0, 1] \). Now evaluating (8) at \( u = 0 \),

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\[(m - 1 - k)B'(0) = (\Phi_{mn}(0) + 1)B''(0) - k(m - 2)B(0) - \]
\[(1 - k)(V'(0) - (m - 1)V(0)).\]

From (9) and the facts that \(V(0) = 0, V'(0) > 0\), then

\[B'(0) < -k(m - 2)/(m - 1 - k)B(0) \leq -B(0) = \int_0^1 B'(w) dw \leq B'(0),\]

where the last inequality follows from \(B'\) being nonincreasing on \(I\). Step 1 follows from this contradiction.

**Step 2** \(B'\) is increasing for all \(u \in I\).

Suppose the contrary. Then, in view of Step 1 and the fact that \(B''(1) = 0\) (from (7)), there exist \(u, u' \in (0, 1]\) such that \(u < u', B'(u) \geq B'(u')\), and

\[(12) \quad B''(u) = 0, \quad B'''(u) \leq 0,\]

\[(13) \quad B''(u') = 0, \quad B'''(u') \geq 0.\]

From (7), (12) and (13),

\[(14) \quad k(B(u)/(1 - u) - B(u')/(1 - u')) = B'(u') - B'(u) - (1 - k)(V(u') - V(u)).\]

Using (8), (12), (13) and (14), one gets as in Step 1 that

\[B'(u) - B'(u') \leq (1 - u)V'(u') - V(u') - ((1 - u)(V'(u) - V(u))) < 0\]

and reaches the contradiction \(B'(u') > B'(u)\).

### 3.2 Independence of expected revenue

I now turn to the second main result of the paper, stated in the form of Theorem 2 below, which says that the expected auction price is the same for all degrees \(\delta \in \Delta\) and under either the multiprice or the uniprice rule.

Let us consider first the multiprice auction, let \(B\) be the unique solution of \(B(k)\) asserted in Theorem 1, and so \(b = B'\) be the unique symmetric equilibrium bid function for \(\Gamma(\delta, M)\). When an ordered population \(y_1, \ldots, y_{n+1}\) (i.e., \(n + 1\) independent draws of buyers listed such that \(y_1 \geq \ldots \geq y_{n+1}\)) plays \(\Gamma(\delta, M)\) as predicted by the symmetric equilibrium, therefore, the auction price \(p(\delta, M) = (b(y_1) + \ldots + b(y_m))/m\). Taking expectation over all possible draws of populations, the expected auction price of \(\Gamma(\delta, M)\) is

\[(15) \quad E\rho(\delta, M) = 1/m \int_0^1 \left( f_{1(n+1)}(w) + \ldots + f_{m(n+1)}(w) \right) B'(w) dw.\]
For the uniprice auction, on the other hand, it follows from Proposition 1 that \( p(\delta, U) = y_{m+1} \), hence

\[
(16) \quad Ep(\delta, U) = \int_0^1 (f_{(m+1)(n+1)}(w)V(w) \, dw,
\]

a constant independent of \( \delta \), to which I will below refer as \( p^* \).

**Theorem 2** The expected auction price is \( p^* \) for all \( \delta \in \Delta \) under both the multiprice and the uniprice rule.

**Proof** (observed by Bernard de Meyer) Multiply \( B(k) \) on both sides by \( (n+1)(1-u)/(m(1-k)) \) to get

\[
(17) \quad \frac{n+1}{m(1-k)}(1-u)(F_{mn}B')' + \frac{(n+1)k/m(1-k)}{f_{mn}B} = \frac{(n+1)/m(1-u)}{f_{mn}V} = f_{(m+1)(n+1)}V.
\]

Upon integrating (17), the two terms on the left side by parts, and using \( F_{mn}(0) = B(1) = 0 \), one gets

\[
(n+1)/(m(1-k)) \int_0^1 F_{mn}(w)B'(w)dw - \int_0^1 f_{(m+1)(n+1)}(w)V(w) \, dw,
\]

and so

\[
(18) \quad \frac{n+1}{m} \int_0^1 F_{mn}(w)B'(w) \, dw = \int_0^1 f_{(m+1)(n+1)}(w)V(w) \, dw.
\]

The theorem now follows from (6), (15), and (16).

### 3.3 Shrinking range of bids

Having shown above that the auction price is the same in expectation for all \( \delta \in \Delta = [0,1) \) whether the auction is multiprice or uniprice, I next show in Theorem 3 below that the range of bids monotonically shrinks to 0, as the resale parameter \( \delta \) increases from 0 to 1, in the multiprice auction. Recall from Proposition 1 that bidding is unaffected by \( \delta \) in the uniprice auction.

Formally, consider the family \( \Gamma(\delta, R) \), \( \delta \in \Delta \). Let \( b_\delta \) be the unique increasing symmetric equilibrium bid function for \( \Gamma(\delta, M) \) and define \( L(\delta, M) = b_\delta(0) \), \( H(\delta, M) = b_\delta(1) \). Thus \( L(\delta, M) \), \( H(\delta, M) \) are respectively the lowest-value bid and the highest-value bid, and naturally all bids fall in the range \([L(\delta, M), H(\delta, M)]\), for \( \Gamma(\delta, M) \). Define \( L(\delta, U) \), \( H(\delta, U) \) analogously.
**Theorem 3** (i) $L(\delta, M)$ increases and $H(\delta, M)$ decreases as $\delta \in \Delta = [0, 1)$ increases.

(ii) $L(0, M) = 0$, $H(0, M) < 1$ and $\lim_{\delta \to 1} L(\delta, M) = \lim_{\delta \to 1} H(\delta, M) = p^\ast$.

(iii) $L(\delta, U) = 0$, $H(\delta, U) = 1$ for all $\delta \in \Delta$.

**Proof** As mentioned, (iii) follows from Proposition 1. Let $B_k$ be the unique solution of the differential equation $B(k)$ as asserted in Lemma 1 and recall that $b_\delta = B'_k$ where $k = (\delta m - \delta) / (m - \delta) \in \Delta$.

The assertion in (ii) for the classical multiprication $\Gamma(0, M)$ that says $L(0, M) = 0$, $H(0, M) < 1$ is well known and easily checked. For the remaining assertion in (ii), check that $B_1(u) = c + du$ is the general solution of $B(1)$ for arbitrary constants $c, d$. By continuity, $B_k(u)$ approaches $B_1^*(u) = c^* + d^* u$ for some fixed constants $c^*$, $d^*$ as $k \to 1$ and for all $u \in I$. Hence, $B_k'(u)$ approaches $B_1^{**}(u) = d^*$ as $k \to 1$ for all $u \in I$. It follows from Theorem 2 that $d^* = p^\ast$.

This concludes the proof of (ii) since $k \to 1$ as $\delta \to 1$.

To prove (i), we apply the perturbation method: Take any $k \in \Delta$ and $h = k + \varepsilon \in \Delta$ for $\varepsilon$ sufficiently small. Then $B_h(u)$ is given by the perturbed solution $B'_k(u) + \varepsilon P(u) + O(\varepsilon^2)$, where $P(u)$ is a solution on $I$ of the following differential equation (obtained by substituting for $B_h(u)$ in the perturbed equation (7) and equating the terms with coefficient $\varepsilon$),

\begin{equation}
(19) u\Phi_{mn}P'' + (1 - u)^{m-1}P' + (1 - u)^{m-2}P = -(1 - u)^{m-2}((1 - u)V(u) + B_k(u)),
\end{equation}

which additionally satisfies the condition

\begin{equation}
(20) P(1) = 0.
\end{equation}

Proof of (i), hence the theorem, now follows from application of Lemma 3 stated below.
Lemma 3

(i) There exists a unique solution \( P \) of the differential equation \( (19) \) which is defined on \( I \) and which satisfies \( (20) \). Furthermore,

(ii) \( P' \) is decreasing on \( I \).

(iii) \( P'(0) > 0, \ P'(1) < 0 \).

Proof The proof of (i) is identical to the proof of Lemma 1, and (ii) is proved in similar manner to Lemma 2. To prove (iii), simply observe that if \( P(0) \leq 0 \) then, in view of (ii), the expected auction price \( Ep(h, M) \) would be less than \( p^* \), contradicting Theorem 2. Similarly, if \( P(1) \geq 0 \) then, in view of (ii), \( Ep(h, M) \) would be greater than \( p^* \).

3.4 Illustration: Explicit solutions for two-unit auctions

This section is restricted to two-unit auctions \( \Gamma(2, n, V, \delta) \) in which case the equilibrium equation \( B(2, n, V, \delta) \) turns out to be the familiar Gauss hypergeometric equation. Taking \( V \), further, to be the identity function \( I(u) = u \), i.e., personal values to be uniformly distributed, the associated bid functions turn out to be expressible via the hypergeometric function, as I spell out below.

Let \( b_{n, \delta} \) be the symmetric equilibrium bid function for \( \Gamma(2, n, I, \delta) \). Recalling the proof of Theorem 1 and differentiating (1), \( b_{n, \delta} = B' \) where \( B \) is the unique solution of the equation \( B(2, n, I, k) \)

\[
(21) \quad (n - (n - 1)u)uB'' + n(n - 1)(1 - u)B' + n(n - 1)kB = n(n - 1)(1 - k)u(1 - u).
\]

that satisfies the associated boundary conditions.

The general solution to (21) is the sum of two solutions

\( B(u) = B^*(u) + A(u) \)

where \( B^*(u) \) is a particular solution to (21) and \( A(u) \) is the general solution to the homogeneous equation

\[
(22) \quad (n - (n - 1)u)uB'' + n(n - 1)(1 - u)B' + n(n - 1)kB = 0.
\]

It is straightforward to find

\[
B^*(u) = (1/(-n - 1)2n + 2 - nk)k(-n(n - 1)k - 2 + n(n - 1)k + 2)k(1 - k)u^2 + nk(1 - k)u.
\]

To get the general solution to (22), let \( z = ((n - 1)/n)u \) and define \( Q(z) = B(u) \). Upon this change of variable, (22) becomes

\[
(23) \quad (1 - z)zQ'' + (n - 1 - nz)Q' + nkQ = 0,
\]

which is the hypergeometric equation

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(1 - z)zQ'' + (n - 1 - (\alpha + \beta + 1)z)Q' - \alpha\beta Q = 0

where
\[
\alpha = \left[(n - 1) + ((n - 1)^2 + 4nk)^{1/2}\right]/2, \\
\beta = \left[(n - 1) - ((n - 1)^2 + 4nk)^{1/2}\right]/2. 
\]

Two linearly independent solutions of (23) are the hypergeometric series
\[ F(\alpha, \beta, n - 1, z) \text{ and } F(\alpha, \beta, n - 1, z) \ln z. \] The general solution of (22) is thus
\[
B(u) = B^*(u) + (c + d\ln((n - 1)/nu))F(\alpha, \beta, n - 1, ((n - 1)/nu)) 
\]
for arbitrary constants c, d. From the condition that our solution be defined at \( u = 0 \), it follows that \( d = 0 \), while from \( B(1) = 0 \), one has \( c = -B^*(1)/F(\alpha, \beta, n - 1, (n - 1)/n) \). Thus
\[
\begin{align*}
B^*(u) - B^*(1)F(\alpha, \beta, n - 1, ((n - 1)/nu)) \\
\quad /F(\alpha, \beta, n - 1, (n - 1)/n) 
\end{align*}
\]
is the unique solution of \( B(2, n, I, k) \) on the closed unit interval that satisfies \( B(1) = 0 \). Upon differentiating (24), one gets the bid function \( b_{n, \delta} \) in the explicit form
\[
\begin{align*}
b_{n, \delta}(u) = (1 + kn(n - 1)/2 + (1 - k)(n - 1)u + \\
(1 - k)F(1 + \alpha, 1 + \beta, n, ((n - 1)/nu)) /F(\alpha, \beta, n - 1, (n - 1)/n)) \\
/((n - 1)(n - 1 + kn/2)) 
\end{align*}
\]
where \( k = \delta/(2 - \delta). \)
The graphs below of the equilibrium bid function \( b_{n, \delta} \) for several selected values of \( n, \delta \) illustrate our results. (Legend: In Figures 1-3 \( b_{n, \delta}(0) \geq b_{n', \delta}(0) \) for \( \delta \geq \delta' \) while in Figure 4 \( b_{n, \delta}(0) \geq b_{n', \delta}(0) \) for \( n \geq n' \).

![Figure 1: Bid functions when 3 buyers bid against 2 units with \( \delta \in \{1, 5, 9\} \)](image)

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Figure 2: Bid functions when 4 buyers bid against 2 units with $\delta \in \{0.1, 0.5, 0.9\}$

Figure 3: Bid functions when 8 buyers bid against 2 units with $\delta \in \{0.1, 0.5, 0.9\}$

Figure 4: Bid functions when $n \in \{3, 4, 8\}$ buyers bid against 2 units with $\delta = 0.99$
4 Concluding Remarks

I have in this paper modelled an auction with resale as a one-period game that features an endogenous component in buyers' valuations as proxy for resale revenue and a parameter as measure of resale orientedness. I have shown that while there is a continuum of equilibria in the (purely speculative) game with full resale, a unique symmetric equilibrium exists for all other degrees of resale, under the multiprice rule. The findings then say that (i) the expected auction price is invariant for all degrees of resale under either payment rule and that (ii) the range of bids is narrower under the multiprice rule than under the uniprice rule, the more so the higher is the degree of resale. As policy implication for the seller, these results thus give support for the multiprice rule on grounds of price stability next to revenue equivalence.

To be sure, the two features in the model mentioned above are of an ad hoc nature. Regarding the first of these, let me conjecture that the model would bear nearly the same results if one were to postulate that resale occurs at the auction price plus a margin near zero. In the case of treasury auctions, Cammack (1991) has measured, for the period 1973 - 84, that the margin between the immediate post-auction market price and the auction price of the U.S. Treasury bill is 4 to 7 basis points (one basis point being one-hundredth of 1 percent.) Conjectured robustness together with empirical observation thus provides one level of the justification. One may, further, regard and tolerate the assumption that resale occurs at the auction price (plus a margin) as an equilibrium no-profit condition on participant profits.

One should also point out that expected revenue equivalence (Theorem 2) is likely to not hold if buyers' personal valuations were correlated to any degree and that expected revenue then is likely to emerge a degree higher under the uniprice rule. I conclude with a final conjecture that the shrinkage in the range of bids (Theorem 3) will then still continue to hold and offer support for the multiprice rule.

* This research was preceded and has been inspired by an empirical study of the Turkish Treasury Bill auctions (Alkan, (1989)). Previous versions of this paper have been presented in a seminar at Universite Libre de Brussels in February 1992, the Mathematical Economics Workshop at the Institute for Pure and Applied Mathematics, Rio de Janerio, in August 1993, the Bosphorous Economic Theory Workshop in September 1993, and the Economic Research Forum Financial Markets Conference, Beirut, in July 1994. I thank the participants, in particular Patrick Bolton, Hasan Ersel, Faruk Gül, and Matthew Jackson for discussions on the modelling aspect. My special thanks for his insightful efforts to Bernard de Meyer with whom I started the analysis of the model. Partial support by the Bogaziçi University Research Fund is gratefully acknowledged.
LARGE MONETARY TRADE, MARKET SPECIALIZATION AND STRATEGIC BEHAVIOUR

Meenakshi Rajeev

Abstract: This paper looks at the role of money as a medium of exchange in a competitive set-up. Together with this we have explored why, historically speaking, monetary trade and market specialization always go hand in hand. The set-up taken up for the purpose is derived from the well-known framework of Kiyotaki and Wright (1989). Our framework extends the above set-up to incorporate exchanges through trading posts for different pairs of goods. Here each agent is trying to choose his optimal strategy for trade given the best strategies of the others. The exercise reveals how a monetized trading post set-up can manifest itself through the agent's optimizing behaviour.

1. Introduction

The theory of Walrasian equilibrium yields a set of prices at which the aggregate competitive demand for each commodity equals its aggregate competitive supply. Two important issues arise in this context. The first is concerned with discovering the laws which guide the behaviour of the many economic variables, but especially prices, when the system is out of equilibrium. Walras (1890) tackled this problem by providing an algorithm for price adjustment which is well-known as the tatonnement scheme.

The other issue revolves around the function of an auctioneer as a clearing house for commodities. All agents are assumed to deposit their initial endowments with this auctioneer, who in turn reallocates them according to the pattern of excess demands. Thus, in the words of Starr (1972), "In a Walrasian pure exchange general equilibrium model, trade takes place between individual households and the market. Households do not trade directly with each other." Such an abstraction suppresses several important issues, in particular the problems of direct exchange between households due to a lack of mutual coincidence of wants even at market clearing prices. Transaction costs as well as a medium of exchange can become crucial in such cases. This paper is devoted to these issues.

When trade takes place between households in a decentralized fashion, it is likely that they would be restricted to those between pairs of agents. More importantly such pairwise meetings of a particular trader with different traders need to be separated in time. In the absence of a centralized agency, each agent going through such sequential bilateral trade will naturally insist on the value of his outgoings to be at least as large as the value of his outgoings. In other words, trades should be bilaterally balanced in value terms after each meeting, or, equivalently, maintain a quid-pro-quo condition. However, in the absence of a perfect mutual coincidence of wants between the agents, this quid-pro-quo may have to be maintained by transferring a good to the creditor for which he has no Walrasian excess demand. The need for a medium of exchange in a competitive set-up can be best appreciated

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