Basic theory of n-local fields

by

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to my parents

okul ödevi

iki kere iki dört
iki kere dört sekiz
iki sekiz onaltı
tekrarla!
diyor öğretmen.
iki kere iki dört
iki kere dört sekiz
iki sekiz onaltı...
birden cıvıl cıvıl
lirkuşu geçiyor gökten
çocuk görüyor onu
duyuyor türküsünü kuşun
el ediyor:
kurtar beni
gel oyna benimle
minik kuş!
alçalıp iniyor kuş
ve başlıyor oynamaya çocukla.
iki kere iki dört..
tekrar et!
diyor öğretmen.
girer mi aklına çocuğun
kuş onunla oynarken...
iki kere dört
iki sekiz onaltı
onaltı onaltı daha
ne eder?
hiçbir şey etmez

hem ne diye edecekmiş ki
çekip gitmek varken serde
otuziki etmek marifet değil ki...
çocuk sıranın gözüne koyuyor kuşu
ve tüm çocuklarda
yankılanırken türküsü kuşun
alıp başını gidiyor sekizle sekiz
ardından dörtle dört ve ikiyle iki
derken birler de kırıyor kirişi
ne bir kalıyor ortada ne iki...
kuş sürdürüyor oyunu
bir türkü tutturuyor çocuk
bas bas bağırıyor öğretmen
yeter artık bu maskaralık!
umurunda değil çocukların
türküsünü dinlemek varken kuşun.
başlıyor yıkılmaya
duvarları sınıfın
camlar kum oluyor yeni baştan
mürekkepler su
sıralar ağaç
tebeşirler kaya
kalemler kuş...

–Jacques Prêvert–

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Basic theory of n-local fields

Abstract

n-local fields arise naturally in the arithmetic study of algebro-geometric objects. For example, let X be a scheme which is integral and of absolute dimension n. Let F be the field of rational functions on X. Then to any complete flag of irreducible subschemes

$$X_0 \subset X_1 \subset \cdots \subset X_{n-1} \subset X_n = X,$$

with $\dim(X_i) = i$ for $i = 0, \ldots, n$, there corresponds a completion $F(X_0, \ldots, X_n)$ of the field F introduced by Parshin, which is an example of an n-local field, in case each X_i is non-singular for $i = 0, \ldots, n$. This n-local field $F(X_0, \cdots, X_n)$ plays a central role in the class field theory of X, introduced by Parshin and Kato.

In this thesis, we develop the basic theory of n-local fields, including a complete elementary proof of Parshin's classification theorem; and for an n-local field K, introduce the sequential topology on K^+ and $K^\times$, and study the Kato-Zhukov higher ramification theory, including the Hasse-Arf theorem, for K.

Özet

Yüksek boyutlu yerel cisimler, cebirsel geometrik objelerin aritmetiğini incelerken karşımıza doğal bir biçimde çıkmaktadır. Şöyle ki, boyutu n olan integral bir cebirsel şema X içinde seçilen herhangi bir

$$X_0 \subset X_1 \subset \cdots \subset X_{n-1} \subset X_n = X,$$

indirgenemez altşemalar zinciri için Parshin, X üzerinde tanımlı olan rasyonel fonksiyonlar cismi F'nin tamlanışı $F(X_0, \cdots, X_n)$ n-yerel cismini tanımlamıştır. Elde edilen bu n-yerel cismi $F(X_0, \cdots, X_n)$, X şemasının aritmetiğini (sınıf cisim teorisini) incelerken, klasik global sınıf cisim kuramında oldğu gibi, merkezi bir rol oynamaktadır.

Bu tezde yüksek boyutlu yerel cisimlerin temel kuramı inşa edilmekte, Parshin sınıflandırma teoreminin basit bir ispatı verilmekte, K ile bir n-yerel cismini göstermek kaydıyla, K cisminin toplamsal ve çarpımsal topolojileri inşa edilmekte ve Kato-Zhukov yüksek dallanma kuramı, genelleştirilmiş Hasse-Arf teoreminin içerecek şekide incelenmektedir.

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Chapter 1

Krull valuations and valued rings

We start by reviewing the basic theory of valuations on a ring R.

1.1 Ordered groups

Definition 1.1.1. *An abelian group* $(\Gamma, +, 0)$ *is said to be totally ordered , if there exists a total ordering* $\leq$ *on* Γ *compatible with the group structure. That is, if* $x \leq y$ *then* $x + z \leq y + z$*, for all* $z \in \Gamma$*. We write* $x < y$ *if* $x \leq y$ *and* $x \neq y$*.*

Lemma 1.1.1. *An abelian group* $(\Gamma, +, 0)$ *has a total ordering* $\leq$ *compatible with the group operation* $+$ *if and only if there exists a subset* P *which is closed under* $+$*, satisfying the disjoint decomposition*

$$\Gamma = P \sqcup \{0\} \sqcup (-P),$$

where $-P = \{p \in \Gamma : -p \in P\}$*.*

Proof. Take P to be the subset of Γ consisting of positive elements with respect to $\leq$. Conversely, for $x, y \in \Gamma$, define the relation $\leq$ by

$$x \leq y \quad \text{if and only if} \quad y - x \in P \cup \{0\}.$$

$\square$

Let $\Gamma_1, \cdots, \Gamma_n$ be totally ordered abelian groups. Then $\Gamma_1 \times \cdots \Gamma_n$ is a totally ordered abelian group with respect to the lexicographic ordering. Namely

$$(a_1, \cdots, a_n) < (b_1, \cdots, b_n)$$

if and only if

$$a_1 = b_1, \cdots a_{i-1} = b_{i-1}, a_i < b_i$$

for some $1 \le i \le n$.

Let $(\Gamma, +, 0, \le)$ be a totally ordered abelian group. We add a formal element $+\infty$ to Γ and extend the order $\le$ of Γ to $\Gamma' = \Gamma \cup \{+\infty\}$ by setting $a \le +\infty$, and $+\infty \le +\infty$.

Definition 1.1.2. *Let $(\Gamma_1, \le_1)$, and $(\Gamma_2, \le_2)$ be two totally ordered abelian groups. A mapping*

$$f : \Gamma_1 \to \Gamma_2$$

is called an order homomorphism, if f is a homomorphism which respects the total orderings in the sense that

$$\alpha \le_1 \beta \Rightarrow f(\alpha) \le_2 f(\beta).$$

Given an ordered group $(\Gamma, \le)$, a subset Σ of Γ is called convex, if for every $\alpha, \beta \in \Sigma$ the set $\ell_{(\alpha,\beta)} = \{\gamma \in \Gamma : \alpha \le \gamma \le \beta\}$ is a subset of Σ.

Lemma-Definition 1.1.1. *Let $(\Gamma, \le)$ be an ordered group. Let $\mathcal{C}_\Gamma$ denote the collection of all convex subgroups of Γ. The collection $\mathcal{C}_\Gamma$ is totally ordered by inclusion, and the cardinality of the maximal chain of non-trivial proper convex subgroups of Γ is called the rank of Γ, and denoted by $rk(\Gamma)$.*

Definition 1.1.3. *An ordered group $(\Gamma, \le)$ is said to be discrete if it satisfies the following conditions:*

1. *The collection $\mathcal{C}_\Gamma$ of all convex subgroups of Γ is well ordered;*

2. *If $f : \Gamma \to \Gamma'$ is any nontrivial order homomorphism, where $(\Gamma', \le')$ is any other ordered group, then $f(\gamma)$ has an immediate successor for all $\gamma \in \Gamma$.*

Theorem 1.1.1. *Let $(\Gamma, \leq)$ be a discrete ordered group of finite rank n. Then there exists an ordered isomorphism*

$$\Gamma \xrightarrow{\sim} \mathbb{Z}^n$$

where $\mathbb{Z}^n$ is ordered lexicographically by $\leq_{lex}$.

In view of this theorem, by a rank n discrete ordered group we shall always understand $(\mathbb{Z}^n, \leq_{lex})$.

1.2 Valued rings

Definition 1.2.1. *A Γ-valued valuation v on a ring R is function*

$$v : R \to \Gamma',$$

subject to the following properties:

1. *$v(a) = \infty$ if and only if $a = 0$;*
2. *$v(ab) = v(a) + v(b)$;*
3. *$v(a + b) \geq \min(v(a), v(b))$,*

for each $a, b \in R$, where Γ is a totally ordered abelian group. We say that (R, v) is a valued ring. If R is a field, then we say that (R, v) is a valued field.

Remark 1.2.1. *Note that, if R is a ring with valuation v, then it is clear that $v(1_R) = 0_\Gamma$, since $v(1_R) = v(1_R) + v(1_R)$. Therefore $v(-1_R) = v(1_R)$. If $\alpha, \beta \in R$ with $v(\alpha) < v(\beta)$ then*

$$\begin{aligned} v(\alpha + \beta) \geq \min(v(\alpha), v(\beta)) &= v(\alpha) \\ &= v(\alpha - \beta + \beta) \\ &\geq \min(v(\alpha + \beta), v(-\beta)), \end{aligned}$$

which means $v(\alpha + \beta) = v(\alpha)$.

Lemma-Definition 1.2.1. *Let (R, v) be a valued ring. Then*

$$\mathcal{O}_v := \{\alpha \in R : v(\alpha) \geq 0\}$$

is a ring and called the maximal order of the valuation v. In case R is a field, which will be the case in our study, the ring $\mathcal{O}_v$ (which will be called the ring of integers of v) is a local ring with the maximal ideal

$$\mathcal{M}_v := \{\alpha \in R : v(\alpha) > 0\}$$

which coincides with the non-invertible elements $\mathcal{O}_v$. The multiplicative group

$$\mathcal{U}_v := \mathcal{O}_v - \mathcal{M}_v$$

of invertible elements of $\mathcal{O}_v$ is called the group of units of v.The quotient field

$$\overline{R}_v := \mathcal{O}_v / \mathcal{M}_v$$

is called the residue field of v.

Proof. Indeed, $\alpha \in \mathcal{O}_v^*$ if and only if $v(\alpha) \geq 0$ and $v(\alpha^{-1}) = -v(\alpha) \geq 0$, which means $v(\alpha) = 0$. Hence the ring of integers $\mathcal{O}_v$ is a local ring and the ideal $\mathcal{M}_v$ is maximal. $\square$

Lemma 1.2.1. *Let R be an integral domain and v_R be a valuation on R with the value group $\Gamma' = \Gamma \cup \{\infty\}$. Then the map $v : ff(R) \to \Gamma'$ given by*

$$v(\alpha/\beta) \mapsto v_R(\alpha) - v_R(\beta)$$

defines a valuation on the field of fractions $ff(R)$ of R.

Proof. We will just show that the map $v : ff(R) \to \Gamma'$ is well-defined in the sense that if $\alpha/\beta = \alpha'/\beta'$, then $v_R(\alpha) - v_R(\beta) = v_R(\alpha') - v_R(\beta')$, which is evident as $\alpha\beta' = \alpha'\beta$. $\square$

Definition 1.2.2. *Let (R, v) be a valued ring. The image $v(R^*)$ in Γ of the multiplicative group R^* of R is called the value group of v. In case $v(R^*)$ is a rank n discrete ordered group with respect to the order induced by Γ, then we say that (R, v) is a rank n discrete valued ring.*

In the next chapter we shall study rank 1 discrete valued fields, that is necessary in our investigation of n-local fields.

1.3 Examples

1. A map $\|\cdot\|$ from a ring R to $\mathbb{R}$ is called an absolute value on R if it satisfies the following conditions:

$$\|\alpha\| > 0 \; if \; \alpha \neq 0, \; \|0\| = 0,$$

$$\|\alpha\beta\| = \|\alpha\|.\|\beta\|,$$

$$\|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|.$$

An absolute value is called a non-archimedean if it satisfies the ultrametric property:

$$\|\alpha + \beta\| \leq max(\|\alpha\|, \|\beta\|).$$

One can show that for an non-archimedean absolute value $\|\|$ on $\mathbb{R}$, we have, if $\|\alpha\| \neq \|\beta\|$, then

$$\|\alpha + \beta\| = \max(\|\alpha\|, \|\beta\|).$$

2. Let R be a field with $\mathbb{Z}$-valued valuation v, and d be a real number in $(0, 1)$. For $\alpha \in R$, set $\|\alpha\|_v = d^{v(\alpha)}$. Then $\|\alpha\|_v = 0$ if and only if $\alpha = 0$ and $\|\cdot\|_v$ is positively defined. If $\alpha, \beta \in F$ then

$$\|\alpha\beta\|_v = d^{v(\alpha\beta)} = d^{v(\alpha)+v(\beta)} = d^{v(\alpha)} d^{v(\beta)} = \|\alpha\|_v \|\beta\|_v.$$

Moreover,

$$\|\alpha + \beta\|_v = d^{v(\alpha+\beta)} \leq d^{\min(v(\alpha), v(\alpha))} = \max(d^{v(\alpha)}, d^{v(\beta)}) = \max(\|\alpha\|_v \|\beta\|_v),$$

which means $\|\cdot\|_v$ is an ultrametric on F.

3. Let $F = K(X)$ and $\|\cdot\|$ be a nontrivial absolute value on F, which is trivial on the multiplicative group of the base field. If $\alpha, \beta \in F(X)$, then

$$\|\alpha + \beta\|^n \leq \|\alpha\|^n + \cdots + \|\beta\|^n \leq (n + 1) \max(\|\alpha\|^n, \|\beta\|^n).$$

By taking the n-th roots of both sides and letting n goes to infinity one gets $\|\alpha+\beta\| \leq \max(\|\alpha\|, \|\beta\|)$, which means that $\|\cdot\|$ is an ultrametric absolute value. There are two cases:

1- $\|X\| > 1$. If $f = \sum_{i=0}^{n} a_i X^i$, then $\|f\| = \|X\|^n$, thus for $\alpha = f/g \in K(X)$

$$\|\alpha\| = \|X^{-1}\|^{-(\deg(f)-\deg(g))}.$$

We define $v_\infty(\alpha)$ as $\deg(g) - \deg(f)$.

2- $\|X\| \leq 1$. It is clear that for α in $K(X)$, $\|\alpha\| \leq 1$. Let p be a monic polynomial of minimal degree satisfying the condition $\|p(X)\| < 1$. Since $\|\alpha\| < \|\beta\|$ implies $\|\alpha+\beta\| = \|\beta\|$ we have, if $p \nmid g$ then $\|g(X)\| = 1$. To see this, write $g = p.h + r$ with $0 < \deg(r) < \deg(p)$, then clearly $v_p(g) = v_p(r) = 1$. From this, one can deduce that

$$\|g(X)\| = \|p(X)\|^{v_{p(X)}(g)},$$

where $v_{p(X)}(g(X))$ is the largest integer k such that $p(X)^k$ divides $g(X)$, which defines a valuation on $K[X]$. We can extend this valuation by setting $v_{p(X)}(\alpha) = v_{p(X)}(f) - v_{p(X)}(g)$, where $\alpha = f/g$.

4. Let R be a finite ring, v be a valuation and $\|\cdot\|$ be an absolute value on R. Let $\alpha \in R$ then $0 = v(1) = v(\alpha^{|R^*|}) = |R^*|v(\alpha)$, which means $v(\alpha) = 0$, thus v is trivial. One can show in the same way that $\|\alpha\| = 1$, i.e. $\|\cdot\|$ is also trivial.

5. Let A be the subring of a field F generated by 1_F. It is clear that, if an absolute value $\|\cdot\|$ on F is an ultrametric, then $\|A\| \leq 1$. Conversely, suppose that $\|A\| \leq 1$. If $\alpha \in F$, then

$$\|(1+\alpha)^n\| = \|1+\alpha\|^n \leq \sum_{i=0}^{n} \|\binom{n}{i}\|.\|\alpha\|^i \leq (n+1)\max(\|\alpha\|^n, 1),$$

and first taking the n th roots of both sides and then letting n goes to infinity we see that $\|1+\alpha\| \leq \max(1, \|\alpha\|)$, which means that $\|\cdot\|$ is an ultrametric. In light of this fact, we can easily show that, every absolute value on a field with positive characteristic must be an ultrametric, since the subring A is a finite field in this case.

6. Let $F = \mathbb{Q}$. For a fixed prime number p, define

$$v_p(n/m) = v_p(n) - v_p(m),$$

where $v_p(n)$ is the greatest integer where $p^{v_p(n)}$ divides n. Then the ring of integers is $\{n/m : m, n \in \mathbb{Z}, p \nmid m\}$. The map

$$\phi : \mathcal{O}_{v_p} \longrightarrow \mathbb{F}_p$$

sending n/m to $\bar{n}\bar{m}^{-1}$ is a ring epimorphism with kernel $\{n/m \in \mathcal{O}_{v_p} : p|n\}$, which means that the residue field $\mathcal{O}_{v_p}$ is isomorphic to the finite field $\mathbb{F}_p$ with p elements.

7. Let F be a field, and v be a valuation on it. For $f(X) = \sum_{i=m}^{k} \alpha_i X^i$, where $\alpha_m \neq 0$, put

$$v^*(f(X)) = (m, v(\alpha_m)),$$

where $\mathbb{Z} \times v(F^*)$ is ordered lexicographically. Let $f(X) = \sum_{i=m_1}^{k_1} \alpha_i X^i$ and $g(X) = \sum_{i=m_2}^{k_2} \beta_i X^i$ be two elements in $F[X]$. Then

$$\begin{aligned} v^*(f(X)g(X)) &= (m_1 + m_2, v(\alpha_{m_1}\beta_{m_2})) \\ &= (m_1 + m_2, v(\alpha_{m_1}) + v(\beta m_2)) \\ &= (m_1, v(\alpha_{m_1})) + (m_2, v(\beta_{m_2})) \\ &= v^*(f(X)) + v^*(g(X)). \end{aligned}$$

There are two cases.

(a) $m_1 = m_2$:
In this case, either $\alpha_{m_1} + \beta_{m_2} = 0$ or $\alpha_{m_1} + \beta_{m_2} \neq 0$. If $\alpha_{m_1} + \beta_{m_2} = 0$, then clearly $v^*(f(X)) = v^*(g(X)) \geq v^*(f(X) + g(X))$. Suppose $\alpha_{m_1} + \beta_{m_2} \neq 0$. Then,

$$\begin{aligned} v^*(f(X) + g(X)) &= (m_1, v(\alpha_{m_1} + \beta_{m_2})) \\ &\geq (m_1, \min(v(\alpha_{m_1}), v(\beta_{m_1})))) \\ &= \min(v^*(f(X)), v^*(g(X))). \end{aligned}$$

(b) $m_1 > m_2$:

Then

$$v^*(f(X)+g(X)) = (m_2, v(\beta_{m_2})) = v^*(g(X)) = \min(v^*(f(X)), v^*(g(X))).$$

This means that, v^* defines a valuation on $F[X]$. We extend v^* to $F(X)$ by setting $v^*(f(X)/g(X)) = v^*(f(X)) - v^*(g(X))$. Since v^* is a group homomorphism between the multiplicative group $F[X]^*$ and $\mathbb{Z} \times v(F^\times)$, v^* is well-defined on $F(X)$. Checking that v^* is a valuation on $F(X)$ is just as same as in the case of $F[X]$.Then, the ring of integers $\mathcal{O}_{v*}$ of $v*$ is given by

$$\mathcal{O}_{v^*} = \left\{ f(X)/g(X) \in F(X) : \begin{array}{c} \deg(f(X)) > \deg(g(X)),\ \mathit{or} \\ \deg(f(X)) = \deg(g(X)) = m, v(\alpha_m) \geq v(\beta_m) \end{array} \right\},$$

and the maximal ideal $\mathcal{M}_{v*}$ of $\mathcal{O}_{v*}$ is

$$\mathcal{M}_{v^*} = \{f(X)/g(X) \in F(X) : \deg(f(X)) > \deg(g(X))\}.$$

Chapter 2

Discrete valuation fields

Throughout this chapter, by a discrete valued field (F, v), we mean a rank 1 discrete valued field.

2.1 Uniformizing elements and the ideal structure of $\mathcal{O}_v$

Definition 2.1.1. *Let F be a discrete valuation field. An element $\pi \in \mathcal{O}_v$ is called a uniformizing (or prime) element if $v(\pi)$ generates the value group $v(F^*)$. Since any nontrivial subgroup of $\mathbb{Z}$ is isomorphic to $\mathbb{Z}$ under the map*

$$\frac{1}{n} : n\mathbb{Z} \xrightarrow{\sim} \mathbb{Z},$$

we may assume that $v(F^*) = \mathbb{Z}$, that is v is normalized.

Lemma 2.1.1. *If $char(F) \neq char(\bar{F}_v)$, then $char(F) = 0$ and $char(\bar{F}_v) \neq 0$.*

Proof. Suppose $char(F) = p \neq 0$. Then $p = 0$ in F, so $\bar{p} = 0$ in $\bar{F}_v$, which means $char(\bar{F}_v) = p$. This proves the lemma. □

Lemma 2.1.2. *Let (F, v) be a valuation field and J be a non-zero ideal of the ring of integers $\mathcal{O}_v$. Let $\alpha \in J$ and $\beta \in \mathcal{O}_v$. If $v(\alpha) \leq v(\beta)$ then $\beta \in J$.*

Proof. Since $v(\beta) \geq v(\alpha)$, we have $v(\beta/\alpha) \geq 0$. Hence $\beta/\alpha \in \mathcal{O}_v$. Lemma follows from the fact that $\beta = \alpha.(\beta/\alpha)$. □

Lemma 2.1.3. *Let F be a discrete valuation field, and π be a uniformizing element. Then the ring of integers $\mathcal{O}_v$ is a principal ideal domain, and every nonzero ideal of $\mathcal{O}_v$ is generated by π^n, for some $n \in \mathbb{N}$.*

Proof. Let $\alpha \in \mathcal{O}_v$, and $n = v(\alpha)$. Then $v(\alpha\pi^{-n}) = 0$, which means $\alpha = \pi^n u$ where, u is a unit in $\mathcal{O}_v$. From this observation and by Lemma 2.1.2, one sees that, if I is an ideal of $\mathcal{O}_v$ then $I = \pi^k \mathcal{O}_v$, where $k = \min\{n \in \mathbb{N} : n = v(\alpha) \text{ for some } \alpha \in I\}$. This also shows in particular that, $\mathcal{M}_v = \pi\mathcal{O}_v$, and $\mathcal{O}_v$ has no non-trivial minimal ideal. $\square$

2.2 v-adic topology

Let F be a discrete valuation field with the valuation v. Since $\|\alpha\|_v = d^{v(\alpha)}$ is a norm on F, $d_v(\alpha, \beta) = \|\alpha - \beta\|_v = d^{v(\alpha-\beta)}$ with $d \in (0, 1)$ defines a metric on F, hence induces a Hausdorff topological space structure on F. Let $\alpha \in F$, and consider the open ball $B_n(\alpha)$ of radius d^{-n} centered at α. If $\beta \in (\alpha + \pi^{n+1}\mathcal{O}_v)$, then $d(\alpha - \beta) \leq d^{n+1}$, hence $\beta \in B_n(\alpha)$. Conversely, one can show that $B_n(\alpha)$ is contained $\alpha + \pi^n\mathcal{O}_v$. This means, the topology defined by the decreasing chain of ideals

$$(\pi) \supset (\pi^2) \supset \cdots \supset (\pi^n) \cdots, \qquad for\ n \in \mathbb{N}$$

which will be called as v-adic topology, coincides with the metric topology.

Lemma 2.2.1. *The field F with the topology defined above is a topological field. That is, the field operations $+$, $\times$ and the inversion map are continuous with respect to the above mentioned topology.*

Proof. Let $\alpha_n \to \alpha$ and $\beta_n \to \beta$. We have to show that $\alpha_n - \beta_n \to \alpha - \beta$, $\alpha_n\beta_n \to \alpha\beta$, and $\alpha_n^{-1} \to \alpha^{-1}$. Note that, $\alpha_n \to \alpha$ means $v(\alpha_n - \alpha) \to \infty$ and vice versa. But,

$$v\big((\alpha - \beta) - (\alpha_n - \beta_n)\big) \geq \min\big(v(\alpha - \alpha_n), v(\beta - \beta_n)\big) \to \infty$$

$$v(\alpha\beta - \alpha_n\beta_n) \geq \min\big(v(\alpha - \alpha_n) + v(\beta), v(\beta - \beta_n) + v(\alpha_n)\big) \to \infty$$

$$v(\alpha^{-1} - \alpha_n^{-1}) \geq v(\alpha - \alpha_n) - v(\alpha) - v(\alpha_n) \to \infty,$$

which means all the operations are continuous. ☐

Lemma 2.2.2. *Let F be a field which has a discrete valuation structure with respect to the valuations v_1 and v_2. Then the topologies induced by the valuations coincide if and only if $v_1 = v_2$. Note that $v_1 F^* = v_2 F^* = \mathbb{Z}$.*

Proof. The sufficiency is clear. So let us assume the topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ induced by the valuations v_1, v_2 respectively coincide. We know that $\alpha^n \to 0$ with respect to $\mathcal{T}_i$ if and only if $v_i(\alpha^n) = nv_i(\alpha) \to \infty$ which means $v_i(\alpha) \geq 1$. On the other hand, since the topologies coincide any sequence converging to 0 with respect to $\mathcal{T}_1$ is also converges to 0 with respect to $\mathcal{T}_2$, and vice versa. Thus, we conclude that $v_1(\alpha) > 0$ if and only if $v_2(\alpha) > 0$. Let π_1, π_2 be prime elements with respect to v_1 and v_2 respectively. Since $v_1(\pi_1) = 1$ and $v_2(\pi_2) = 1$ it follows that $v_1(\pi_2) \geq 1$ and $v_2(\pi_1) \geq 1$. If $v_2(\pi_1) > 1$ then $v_2(\pi_2^{-1}\pi_1) > 0$ hence $v_1(\pi_2^{-1}\pi_1) > 0$ which means $v_1(\pi) < 0$. This yields a contradiction, thus $v_2(\pi_1) = v_1(\pi_2) = 1$.

☐

Chapter 3

Complete discrete valuation fields: Local fields

Throughout this chapter by a discrete valued field (F, v) we mean a rank 1 discrete valued field.

Let F be a valuation field an v be the valuation on it. As we had seen that the topologies induced by the norm given by the v and the $\mathcal{O}_v$ coincide, we may say that a sequence α_n in F is a Cauchy sequence if for all $z \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that $\forall k, l > N$ $v(\alpha_k - \alpha_l) > z$. A discrete valuation field F is said to be complete, if every Cauchy sequence in F has a limit in F.

3.1 Completion

Let (α_n) be a Cauchy sequence in F. There are two cases, either $v(\alpha_n)$ is bounded or not.

1. Suppose it is unbounded, and suppose that there exists an integer z such that for infinitely many integer i, $v(\alpha_i) = z$. Let $N \in \mathbb{N}$ be such that for $r, s > N$ we have $v(\alpha_r - \alpha_s) > |z| + 1$. We know that there exists $l \in \mathbb{N}$ with $l > N$ and $v(\alpha_l) = z$. Also there exists $k > N$ such that $v(\alpha_l) > |z| + 1$. Such m exists since $v(\alpha_n)$ is unbounded. But $v(\alpha_k - \alpha_l) = z < |z| + 1$. this yields a contradiction, thus we conclude that $\lim v(\alpha_n)$ exists and equal to $+\infty$.

2. Suppose $v(\alpha_n)$ is bounded. Suppose that there exist $z_1 \neq z_2$ integers such that for infinitely many natural numbers i, j, $v(\alpha_i) = z_1$ and $v(\alpha_j) = z_2$. As above one can show that this situation yields a contradiction.

Thus if $\{\alpha_n\}$ is a Cauchy sequence then $\lim v(\alpha_n)$ exists.

Lemma 3.1.1. *The set $\mathcal{C}(F) = \mathcal{C}$ of all Cauchy sequences in F in forms a ring with respect to componentwise addition and multiplication, and the set $\mathcal{C}_0(F) = \mathcal{C}_0$ of all Cauchy sequences tending to 0 forms a maximal ideal of $\mathcal{C}$. The quotient field*

$$\hat{F} = \mathcal{C}/\mathcal{C}_0$$

is a discrete valuation field with respect to the the induced valuation

$$v : \hat{F} \to \mathbb{Z} \cup \{\infty\}$$

defined by

$$v(\alpha_n) = \lim v(\alpha_n).$$

Proof. Let $\{\alpha_n\}$,$\{\beta_n\}$be two Cauchy sequences. Let $z \in \mathbb{N}$ be given. Then there exists $N \in \mathbb{N}$ such that whenever $n_1, m_1, n_2, m_2 > N$ we have

$$v(\alpha_{n_1} - \alpha_{m_1}) > z \qquad \text{and} \qquad v(\beta_{n_2} - \beta_{m_2}) > z.$$

So we conclude that

$$v((\alpha_n - \beta_n) - (\alpha_m - \beta_m)) \geq \min(v(\alpha_n - \alpha_m), v(\beta_m - \beta_n)) > k,$$

which means sum of two Cauchy sequences is again a Cauchy sequence. On the other hand we have

$$v(\alpha_n\beta_n - \alpha_m\beta_m) \geq \min\big(v(\alpha_n - \alpha_m) + v(\beta_n), v(\beta_n - \beta_m) + v(\alpha_m))\big). \tag{3.1}$$

Since $v(\alpha_n)$ and $v(\beta_n)$ are bounded below and $v(\alpha_n - \alpha_m), v(\beta_n - \beta_m)$ tend to infinity as n, m tend to infinity, we see that the product of two Cauchy sequences is again a Cauchy sequence. Let $\{\alpha_n\}$ be a Cauchy sequence in $\mathcal{C} - \mathcal{C}_0$, which means 0 is not a limit point of $\{\alpha_n\}$, so only finitely many $\alpha_n = 0$. Consider the ideal J generated by $\mathcal{C}_0 \cup \{\alpha_n\}$. Let N be a positive integer so that for $n > N$ $\alpha_n \neq 0$. Put $\beta_n = \alpha_n^{-1}$ for $n > N$ and $\beta_n = 0$ for $n \leq N$. Then $1 - \{\alpha_n\}\{\beta_n\} \in J$, which means $J = \mathcal{C}$, hence $\mathcal{C}_0$ is maximal. $\square$

3.2 Universality

Proposition 3.2.1. *Let F be a discrete valuation field with the valuation v. Then there is a complete field $\hat{F}$ with valuation $\hat{v}$, and a continuous field embedding $i : F \hookrightarrow \hat{F}$, which is universal in the following sense. Whenever there exists a continuous field embedding $j : F \hookrightarrow K$ in to a complete field, there exists unique $\varphi : \hat{F} \to K$, such that the following diagram commutes:*

$$\begin{array}{ccc} F & \xrightarrow{i} & \hat{F} \\ & \searrow^{j} & \downarrow \varphi \\ & & K \end{array}$$

.

To prove the Proposition we have just need follow the routine procedure of completing the rational numbers $\mathbb{Q}$ to $\mathbb{R}$.

3.3 Examples

1. The completion of $\mathbb{Q}$ with respect to the valuation v_p is called the p-adic field and denoted by $\mathbb{Q}_p$.

2. The completion of $K(X)$ with respect to v_X is the Laurant series $K((X))$ with the valuation
$$v(\sum_{n \gg -\infty}^{\infty} \alpha_n X^n) = \min\{n \in \mathbb{Z} : \alpha_n \neq 0\}.$$
It is clear that $v_{\uparrow K(X)} = v_X$. We know that $K[[X]] \subset K((X))$. Moreover, if $f \in K[[X]]$ with $f(0) \neq 0$, then $1/f \in K[[X]] \subseteq K((X))$. Also $1/X \in K((X))$, thus $K(X) \subset K((X))$. An element $h \in K((X))$ can be written as $h = X^{-k} f + g$, where $f \in K[X]$, $k \in \mathbb{N}$, and $g \in K[[X]]$. We'd seen that such f is always an element of $K(X)$. Let $g = \sum_{i=0}^{\infty} \alpha_i X^i$, and put
$$g_n = \sum_{i=0}^{n} \alpha_i X^i.$$

For $n < m \in \mathbb{N}$ we have $v(g_n - g_m) = v(\sum_{i=n}^{m} \alpha_i X^i) = n$, thus $\{g_n\}$ is a Cauchy sequence converging to g. So $h_n = f + g_n \to h$, which means $K(X)$ is dense in $K((X))$. Also it it is clear that $\mathcal{O}_{v_X} = K[[X]]$ and $\mathcal{M}_{v_X} = XK[[X]]$, hence the residue field $\overline{K((X))}_v$ is K again.

3. Let F be a field with a discrete valuation v, and $\hat{F}$ be its completion. We extend the v^* on $F(X)$ to $\hat{F}((X))$ in the following way. For $f(X) = \sum_{n \geq m} \alpha_n X^n$, $\alpha_n \in \hat{F}$, $\alpha_m \neq 0$, put

$$v^*(f(X)) = (m, \hat{v}(\alpha_m)).$$

Let $f(X) \in \mathcal{O}_{v^*}$. This means either $m > 0$ or $m = 0$ and $\alpha_0 \in \mathcal{O}_{\hat{v}}$. If $m > 0$ then $f(X) \in X\hat{F}[[X]]$. If $m = 0$ and $\alpha_0 \in \mathcal{O}_{\hat{v}}$ then $f - \alpha_0 \in XK[[X]]$, thus $\mathcal{O}_{v^*} = \mathcal{O}_{\hat{v}} + X\hat{F}[[X]]$. If $f \in \mathcal{M}_{v^*}$ then either $m > 0$ or $m = 0$ and $\alpha_0 \in \mathcal{M}_{\hat{v}}$. So, $\mathcal{M}_{v^*} = \mathcal{O}_{\hat{v}} + X\hat{F}[[X]]$, thus the residue field of $\bar{\hat{F}}((X))_{v^*} = \mathcal{O}_{\hat{v}}/\mathcal{M}_{\hat{v^*}}$

Chapter 4

Structure theory of complete discrete valuation fields

Throughout this chapter by a discrete valued field (F, v) we mean a rank 1 discrete valued field.

4.1 The equal characteristic case: Teichmüller representatives

Let F be a complete discrete valuation field with ring of integers $\mathcal{O}$ and residue field $\overline{F} = k$. Let π be a prime element and T be a set of coset representatives of k in $\mathcal{O}$.

Proposition 4.1.1. *Every element $a \in \mathcal{O}$ can be written uniquely as a convergent series*

$$a = \sum_{n=0}^{\infty} \theta_n \pi^n, \quad \textit{with } \theta_n \in \mathrm{T}.$$

Similarly, every element $\alpha \in F$ can be written uniquely as

$$\alpha = \sum_{n>-\infty} \theta_n \pi^n, \quad \textit{with } \theta_n \in \mathrm{T}.$$

Proof. Since for any $\alpha \in K$, $\pi^{-v(\alpha)}\alpha \in \mathcal{O}$, the second assertion follows from the first. So, let $a \in \mathcal{O}$; by definition of T, there exists unique $\theta_0 \in S$ such that

$a - \theta_0 \equiv 0 \mod (\pi)$. Thus $a = \theta_0 + a_1\pi$ for some $a_1 \in \mathcal{O}$. Similarly $a_1 = \theta_1 + a_2\pi$ which means $a = \theta_0 + \theta_1\pi + a_2\pi^2$, and so on. Since $v(\sum_{i=n}^{\infty} a_i\pi^i) \geq n$ we have $a - \sum_{i=0}^{n} \theta_i\pi^i \to 0$, and since all series of the form $\sum \theta_n\pi^n$ is convergent, existence follows.. The uniqueness of this expression is clear. □

Observe that we can generalize the assertion of the above proposition as follows: Let F be a complete discrete valuation field with respect to the valuation v, T be a set of coset representatives of $\overline{F}_v$ and for each $i \in \mathbb{Z}$ let $\pi_i \in F$ be such that $v(\pi_i) = i$. Then every element $\alpha \in F$ can be written as a convergent series

$$\alpha = \sum_{n > -\infty} \theta_n \pi_n, \quad \text{with } \theta_n \in \mathrm{T}.$$

Lemma 4.1.1. *Let R be a local ring that is Hausdorff and complete for the topology defined by decreasing sequence $\mathfrak{a}_1 \supset \mathfrak{a}_1 \cdots$ of ideals such that $\mathfrak{a}_\mathfrak{n}.\mathfrak{a}_\mathfrak{m} \subset \mathfrak{n} + \mathfrak{m}$. Suppose that $\mathfrak{a}_1$ is the maximal ideal and let $\bar{R} = A/\mathfrak{a}_1$ is a field. Let $f(X)$ be a polynomial with coefficients in R such that the reduced polynomial $\bar{f} \in \bar{R}[X]$ has a simple root $\lambda \in \bar{R}$. Then f has unique root $x \in R$ such that $\bar{a} = \lambda$.*

Proposition 4.1.2. *Let R be a local ring that is Hausdorff and complete for the topology defined by decreasing sequence $\mathfrak{a}_1 \supset \mathfrak{a}_1 \cdots$ of ideals such that $\mathfrak{a}_\mathfrak{n}.\mathfrak{a}_\mathfrak{m} \subset \mathfrak{n} + \mathfrak{m}$. Suppose that $\overline{R} = R/\mathfrak{a}_1$ is field of characteristic zero. Then R contains a system of representatives ok $\overline{R}$ which is a field.*

Note that any discrete valuation ring R with the topology induced by the valuation on it, or equivalently the topology given by the decreasing sequence of ideals $(\pi), \cdots, (\pi^n)$, where π is a prime element satisfies the condition of the proposition.

Proof. Since characteristic of R is zero, $\phi : \mathbb{Z} \hookrightarrow R$ is injective. Since $R/\mathfrak{a}_1$ is of characteristic zero we have $\phi(\mathbb{Z}) \cap \mathfrak{a}_1 = \emptyset$, thus every element of $\phi(\mathbb{Z})$ is invertible in R, which means R contains an isomorphic copy of the field $\mathbb{Q}$. Hence by Zorn's Lemma there exists a maximal subfield T of R. Let $\overline{\mathrm{T}}$ be its image in $\overline{R}$. Since T is a subfield and $\mathfrak{a}_1 \cap \mathrm{T} = 0$, we see that the map $\varphi : \mathrm{T} \to \bar{R}$ given by $\theta \mapsto \overline{\theta}$ is injective, hence $\overline{\mathrm{T}}$ is a field. We will show that $\overline{\mathrm{T}} = \overline{R}$.

Our first claim is that $\overline{R}$ is algebraic over $\overline{\mathrm{T}}$. Suppose not! Then there exists $a \in R$ such that $\bar{a}$ is transcendental over $\overline{\mathrm{T}}$. Also $a \in R$ must be transcendental over T. Indeed if $f(X)$ is a monic polynomial in $\mathrm{T}[X]$ such that $f(a) = 0$, then $\bar{f}(\bar{a}) = 0$, which contradicts with the assumption that $\bar{a}$ is transcendental over $\overline{\mathrm{T}}$. So the bar map sends $\mathrm{T}[a]$ to $\overline{\mathrm{T}}[\bar{a}] \simeq \mathrm{T}[X]$ isomorphically. Since $\bar{a}$ is transcendental over $\overline{\mathrm{T}}$, $\mathrm{T}[a] \cap \mathfrak{a}_1 = 0$, thus a is invertible in R, which means R contains the field $\mathrm{T}(a)$. But this contradicts with the maximality of T, hence $\overline{R}$ is algebraic over $\overline{\mathrm{T}}$.

So, for any $\lambda \in \overline{R}$, there exists a unique $\bar{f}$ minimal polynomial over $\overline{\mathrm{T}}$. Since the characteristic is zero $\overline{R}$ is separable over $\overline{\mathrm{T}}$, which means λ is a simple root of $\bar{f}$. Let $f \in \mathrm{T}[X]$ be a coset representative for $\bar{f}$. By the previous lemma, there exists $a \in R$ such that $\bar{x} = \lambda$ with $f(a) = 0$. □

Proposition 4.1.3. *Let R be a ring that is Hausdorff and complete for the topology defined by decreasing sequence $\mathfrak{a}_1 \supset \mathfrak{a}_1 \cdots$ of ideals such that $\mathfrak{a}_\mathfrak{n}.\mathfrak{a}_\mathfrak{m} \subset \mathfrak{n} + \mathfrak{m}$. Suppose that the residue ring $\overline{R} = R/\mathfrak{a}_1$ is perfect of characteristic $p > 0$. Then*

1. *There exists one and only one system of representatives $f : \overline{R} \to \mathrm{T} \subset R$ which commutes with p-th powers. That is $f(\lambda^p) = f(\lambda)^p$.*

2. *An element $a \in R$ belongs to $\mathrm{T} = f(\overline{K})$ if and only if a is a p^n th power for all $n \geq 0$.*

3. *T is multiplicative, i.e. $f(\lambda\mu) = f(\lambda)f(\mu)$.*

4. *If the characteristic of R is $p > 0$, then T is additive.*

Proof. For $\lambda \in \overline{R}$ and $n \in \mathbb{N}$ put

$$\begin{aligned} L_n(\lambda) &= \{x \in R : \bar{x} = \lambda^{p^{-n}}\}, \\ U_n(\lambda) &= \{x^{p^n} \in R : x \in L_n(\lambda)\}. \end{aligned} \tag{4.1}$$

If $x \in L_n(\lambda)$ then

$$\overline{x^{p^n}} = \bar{x}^{p^n} = (\lambda^{p^{-n}})^{p^n} = \lambda,$$

thus $x^{p^n} \in L_0$, which means $U_n \subseteq L_0$. Let $a, b \in U_n(\lambda)$, then there exists $x, y \in L_n(\lambda)$ such that $a = x^{p^n}, b = y^{p^n}$. Since $\overline{x} = \overline{y}$ we have $x - y \in \mathfrak{a}_1$, by the following lemma stated below, we see that $x^{p^n} - y^{p^n} \in \mathfrak{a}_{n+1}$, whence $U_n(\lambda)$ form a Cauchy filter base. So we may set $f(\lambda) = \lim(U_n(\lambda))$. This limit exits and well-defined since U_n is Cauchy filter, R is complete and Hausdorff.

Now, we'll show that $a \in \mathrm{T} := \{\lim(U_n(\lambda) : \lambda \in \overline{K}\} = f(\overline{K})$ if and only if a is a p^n-th power for all $n \geq 0$. The necessity follows from the construction. Indeed, any element of U_n is a p^n th power and $\lim U_n = \bigcap_{n\in\mathbb{N}} U_n$. Suppose a is a p^n th power for all $n \geq 0$. Let $\lambda = \bar{a}$. By hypothesis there exists $y \in R$ such that $a = y^{p^n}$. Since $\bar{a} = \lambda$, $\overline{y^{p^n}} = \lambda$, thus $\bar{y} = \lambda^{p^{-n}}$, hence $y \in L_n(\lambda)$, which also means that $a \in U_n(\lambda)$. But $\lim U_n = \bigcap U_n$, which show the sufficiency.

If $a, b \in R$ are p^n th power then ab is also a p^n th power. So T is a multiplicatively closed set. On the other hand, if the characteristic of R is p then $a+b = x^{p^n} + y^{p^n} = (x+y)^{p^n}$. □

Lemma 4.1.2. *Under the assumptions of the above proposition $a \equiv b \pmod{\mathfrak{a}_m}$ implies $a^{p^n} \equiv b^{p^n} \pmod{\mathfrak{a}_{n+m}}$.*

Proof. We know that $(a-b)^p \in \mathfrak{a}_{pm} \subset \mathfrak{a}_{m+1}$. Since the characteristic is $p > 0$ we have

$$a^p - b^p = (a-b)^p$$

thus $a^p - b^p \in \mathfrak{a}_{m+1}$. The rest follows by induction. □

Note that in the context of valuation fields the lemma is equivalent to say that $v(\alpha - \beta) \geq n$ implies $v(\alpha^{p^m} - \beta^{p^m}) \geq n + m$.

Theorem 4.1.1. *Let F a complete discrete valuation field with respect to the valuation v. If the characteristic of the residue field $\overline{F} = 0$ or the characteristic of F is non-zero and F is perfect, then*

$$F \simeq \overline{F}(X).$$

Proof. The theorem follows from Proposition 4.1.1, Proposition 4.1.2 and the Proposition 5.2. □

Definition 4.1.1. *The set* T *is called the Teichmüller representatives of the residue field.*

4.2 Unequal characteristic case: Witt vectors

Let $A = \mathbb{Z}[X_0, X_1, \cdots, Y_0, Y_1, \cdots]$ be the ring of polynomials in variables $X_0, X_1, \cdots, Y_0, Y_1$ over the integers. We define

$$W_n(X_0, \cdots, X_n) = \sum_{i=0}^{n} p^i X_i^{p^{n-1}} \; n \geq 0,$$

in particular $W_0 = X_0$, $W_1 = X_0^p + pX_1$. Note that $W_n(X_0, \cdots, X_n) = W_{n-1}(X_0^p, \cdots, X_{n-1}^p) - p^n X_n$, and $W_n(X_0, \cdots, X_n) = X_0^{p^n} + pW_{n-1}(X_1, \cdots, X_n)$.

Proposition 4.2.1. *There exists unique polynomials*

$$w_n^*(X_0, \cdots, X_n, Y_0, \cdots, Y_n) \in A, \; n \geq 0$$

such that

$$W_n(X_0, \cdots, X_n) * W_n(Y_0, \cdots, Y_n) = W_n(w_0^*, \cdots, w_n^*),$$

where $*$ stands for $+$ or $\times$.

Proof. We observe that there exist unique polynomials w_0^* where $w_0^+ = X_0 + Y_0$ and $w_0^\times = X_0 Y_0$. For $n \geq 1$ we deduce that

$$\begin{aligned}
p^n w_n^* &= W_n(X_0, \cdots, X_n) * W_n(Y_0, \cdots, Y_n) - (p^0 w_0^{*p^n} + \cdots + p^{n-1} w_{n-1}^*) \\
&= W_{n-1}(X_0^p, \cdots, X_{n-1}^p) * W_{n-1}(Y_0^p, \cdots, Y_{n-1}^p) - W_{n-1}(w_0^{*p}, \cdots, w_{n-1}^{*p}) \\
&+ (p^n X_n * p^n Y_n).
\end{aligned}$$

The uniqueness of w_n^* is clear. Now we'll show that $p^n w_n^* \in p^n A$. Note that if $g(X_0, Y_0, \cdots) \in A$ then $g(X_0, Y_0, \cdots)^p - g(X_0^p, Y_0^p, \cdots) \in pA$. This follows from the fact that the summands of $g(X_0^p, Y_0^p, \cdots)$ are the summands of $g(X_0, Y_0, \cdots)^p$ which are not divisible by p. Moreover, if $f - g \in pA$ then $f^p - g^p \in p^2 A$ (c.f. Lemma 4.2), thus we conclude that

$$g(X_0, Y_0, \cdots)^{p^m} - g(X_0^p, Y_0^p, \cdots)^{p^{m-1}} \in p^m A.$$

So for $0 \leq i \leq n-1$ we have $w_i^*(X_0, Y_0, \cdots, X_i, Y_i)^p - w_i^*(X_0^p, Y_0^p, \cdots, X_i^p, Y_i^p) \in pA$. Thus

$$p^i (w_i^*(X_0, Y_0, \cdots, X_i, Y_i)^p)^{p^{n-1-i}} - p^i w_i^*(X_0^p, Y_0^p, \cdots, X_i^p, Y_i^p)^{p^{n-1-i}} \in p^n A,$$

which means means

$$W_{n-1}(w_0^{*p},\cdots,w_{n-1}^{*p})-W_{n-1}\big(w_0^*(X_0^p,Y_0^p),\cdots,w_0^*(X_0^p,\cdots,X_{n-1}^p,Y_0^p,\cdots,Y_{n-1}^p)\big)\in p^nA.$$

On the other hand we have $W_{n-1}(X_0^p,\cdots,X_{n-1}^p)*W_{n-1}(Y_0^p,\cdots,Y_{n-1}^p)=W_{n-1}(w_0^{*p},\cdots,w_{n-1}^{*p})$. Thus we get

$$p^nw_n^*=W_{n-1}(w_0^{*p},\cdots,w_{n-1}^{*p})-W_{n-1}(w_0^{*p},\cdots,w_{n-1}^{*p})+(p^nX_n*p^nY_n),$$

which means $p^nw_n^*\in p^nA$ hence $w_n^*\in A$. □

Corollary 4.2.1. *With the notations of the Proposition 4.2.1 we have*

$$w_n^*(X_0,\cdots,X_n,Y_0,\cdots,Y_n)^p-w_n^*(X_0^p,\cdots,X_n^p,Y_0^p,\cdots,Y_n^p)\in pA.$$

We now return to the case where the characteristics of the base field F and $\bar{F}$ are different. We know that, this means, $char(F)=0$ and $char(\bar{F})=p>0$. Let α,β be two elements in the ring of integers $\mathcal{O}$ of F, π be a prime element, and S be a set Teichmüller representatives for the residue field $\bar{F}$. We know that there exists unique $\theta_i,\gamma_i\in S$ such that

$$\alpha=\sum_{i\geq 0}\theta_i\pi^i \qquad \text{and} \qquad \beta=\sum_{i\geq 0}\gamma_i\pi^i.$$

Also there exists unique $\rho_i^+,\rho_i^\times\in S$ such that

$$\alpha+\beta=\sum_{i\geq 0}\rho_i^+\pi^i \qquad \text{and} \qquad \alpha\times\beta=\sum_{i\geq 0}\rho_i^\times\pi^i$$

We'll investigate the relation between θ_i,γ_i and ρ_i^* for $*=+$ or $*=\times$. Since an element is a Teichmüller representative if and only if it is p^nth power for all $n\in\mathbb{N}$ there exists elements $\epsilon_i,\xi_i,\lambda_i^*\in S$ such that $\epsilon_i^{p^{n-i}}=\theta_i,\xi_i^{p^{n-i}}=\gamma_i$ and $\lambda_i^{*p^{n-i}}=\rho_i^*$, where $*=+$ or $*=\times$. We observe that if $*=+$ then,

$$\sum_{i=0}^n\theta_i\pi^i+\sum_{i=0}^n\gamma_i\pi^i-\sum_{i=0}^n\rho_i^+\pi^i=\sum_{i>n}\rho_i^+\pi^i-\sum_{i>n}\theta_i\pi^i-\sum_{i>n}\gamma_i\pi^i,$$

and if $*=\times$ then,

$$\begin{aligned}\sum_{i=0}^n\theta_i\pi^i\times\sum_{i=0}^n\gamma_i\pi^i-\sum_{i=0}^n\rho_i^\times\pi^i &= \sum_{i>n}\rho_i^\times\pi^i\\ &- (\sum_{i>n}\theta_i\pi^i)\beta-\alpha(\sum_{i>n}\gamma_i\pi^i).\end{aligned}$$

But this means
$$\sum_{i=0}^{n} \theta_i \pi^i * \sum_{i=0}^{n} \gamma_i \pi^i \equiv \sum_{i=0}^{n} \rho_i^* \pi^i \mod \pi^{n+1},$$
for $* = +$ or $* = \times$. By replacing $\theta_i, \gamma_i, \rho_i^*$ by $\epsilon_i^{p^{n-i}}, \xi_i^{p^{n-i}}, \lambda_i^{*p^{n-i}}$ respectively we get
$$\sum_{i=0}^{n} \epsilon_i p^{n-i} * \sum_{i=0}^{n} \xi_i p^{n-i} \equiv \sum_{i=0}^{n} \lambda_i * p^{n-i} \mod \pi^{n+1}.$$
Note that if $\pi = p$ then the last equivalency is nothing but
$$W_n(\lambda_0^*, \cdots, \lambda_n^*) \equiv W_n(\epsilon_0 \cdots, \epsilon_n) * W_n(\xi_0, \cdots, \xi_n) \mod p^{n+1}.$$

Proposition 4.2.2. *With the above notations, we have the following identity*
$$\rho_i^* \equiv w_i^*(\theta_0^{p^{-i}}, \theta_1^{p^{-i+1}}, \cdots, \theta_i^p, \xi_0^{p^{-i}}, \xi_1^{p^{-i+1}}, \cdots, \xi_i^p,) \mod p, \ i \geq 0,$$
where w_i^* are the polynomials defined in the proof of the Proposition 4.2.1.

Proof. We'll proceed by induction. Suppose the assertion of the proposition holds for $i \leq n-1$, which means for $0 \leq i \leq n-1$ we have
$$\lambda_i^{*p^{n-i}} \equiv w_i^*(\epsilon_0^{p^{n-i}}, \epsilon_1^{p^{n-i}}, \cdots, \epsilon_i^{p^{n-i}}, \xi_0^{p^{n-i}}, \xi_1^{p^{n-i}}, \cdots, \xi_i^{p^{n-i}}).$$
In the proof of the Proposition 4.2.1 we have seen that if $g(X) \in A$ then $g(X)^p - g(X^p) \in pA$. Writing $g(X)^{p^n} - g(X^{p^n})$ as $g(X)^{p^n} - g(X^p)^{p^{n-1}} + g(X^p)^{p^{n-1}} - g(X^{p^2})^{p^{n-2}} + g(X^{p^2})^{p^{n-2}} + \cdots - g(X^{p^{n-1}})^p + g(X^{p^{n-1}})^p - g(X^{p^n})$ we see that $g(X)^{p^n} - g(X^{p^n}) \in pA$. Thus
$$w_i^*(\epsilon_0^{p^{n-i}}, \epsilon_1^{p^{n-i}}, \cdots, \epsilon_i^{p^{n-i}}, \xi_0^{p^{n-i}}, \xi_1^{p^{n-i}}, \cdots, \xi_i^{p^{n-i}}) \equiv w_i^*(\epsilon_0, \epsilon_1, \cdots, \epsilon_i, \xi_0, \xi_1, \cdots, \xi_i)^{p^{n-i}} \mod p.$$
From this we deduce that for $i \leq n-1$
$$\lambda_i^{*p^{n-i}} \equiv w_i^*(\epsilon_0, \epsilon_1, \cdots, \epsilon_i, \xi_0, \xi_1, \cdots, \xi_i)^{p^{n-i}} \mod p.$$
By the remark following the Lemma we see that for $i \leq n-1$
$$p^i \lambda_i^{*p^{n-i}} \equiv p^i w_i^*(\epsilon_0, \epsilon_1, \cdots, \epsilon_i, \xi_0, \xi_1, \cdots, \xi_i)^{p^{n-i}} \mod p^{n+1}$$

hence

$$\sum_{i=0}^{n-1} p^i \lambda_i^{*p^{n-i}} \equiv \sum_{i=0}^{n-1} p^i w_i^*(\epsilon_0, \epsilon_1, \cdots, \epsilon_i, \xi_0, \xi_1, \cdots, \xi_i)^{p^{n-i}} \mod p^{n+1}.$$

On the other hand we know that

$$\begin{aligned} W_n(\lambda_0^*, \cdots, \lambda_n^*) &\equiv W_n(\epsilon_0 \cdots, \epsilon_n) * W_n(\xi_0, \cdots, \xi_n) \\ &\equiv W_n(w_0^*(\epsilon_0, \xi_0), \cdots, w_n^*(\epsilon_0, \cdots, \epsilon_n, \xi_0, \cdots, \xi_n) \mod p^{n+1}, \end{aligned}$$

and combining these two facts we get

$$p^n \lambda_n^* \equiv p^n w_n^*(\epsilon_0, \cdots, \epsilon_n, \xi_0, \cdots, \xi_n) \mod p^{n+1},$$

which implies the assertion of the proposition. □

Corollary 4.2.2. *With the above notation we have*

$$\rho_i^* \equiv w_i^*(\theta_0, \cdots, \theta_i, \gamma_0, \cdots, \gamma_i) \mod p.$$

Proof. As we had seen in the proof of the Proposition 4.2.1 modulo p, ρ_i^* is equivalent to $w_i^*(\epsilon_0, \cdots, \epsilon_i, \xi_0, \cdots, \xi_i)^{p^{n-i}}$. □

From the proof of the proposition we deduce that

Corollary 4.2.3. *Let $(\sum \theta_i^{p^{-i}} p^i) * (\sum \gamma_i^{p^{-i}}) = \sum \rho_i^{(*)p^{-i}}$ where $\theta_i, \gamma_i, \rho^{(*)}$ are Teichmüller representatives comes from 5.2, and $* = +$ or $* = \times$. Then*

$$\rho_i \equiv \omega_i^{(*)}(\theta_0, \cdots, \theta_i, \gamma_0, \cdots, \gamma_i) \mod p.$$

Corollary 4.2.4. $(\sum \theta_i^{p^{-i}} p^i) * (\sum \gamma_i^{p^{-i}}) = \omega_i^{(*)}(\theta_0, \cdots, \theta_i, \gamma_0, \cdots, \gamma_i)$.

Definition 4.2.1. *A ring R is said to be a p-ring if it satisfies the hypothesis of the Proposition 5.2. A p-ring is said to be strict if $\mathfrak{a}_n = p^n R$ and ip p is not zero divisor in R.*

We know that, thanks to Proposition 5.2, a p-ring is always have a set of Teichmüller representatives. By the Proposition 4.1.1 for $\theta_i \in T$

$$\sum \theta_i p^i$$

converges to an element α of R. On the other hand, if R is strict then can be written in this way uniquely. The element θ_i is called the coordinates of α.

Proposition 4.2.3. *Let R and R' be two p-rings with residue fields k and k' respectively. If R is strict then for any homomorphism $\phi : k \to k'$, there exists a unique homomorphism $g : R \to R'$ such that $\phi(\overline{\alpha}) = \overline{g(\alpha)}$.*

Proof. Let T and T' be two systems of Teichmüller representatives for R and R' respectively given by the lifting maps f and f' respectively. Suppose $g : R \to R'$ satisfying the assertions of the proposition. Then for $\alpha \in R$ is equal to $\sum \theta_i p^i$ we have

$$g(\alpha) = g(\sum \theta_i p^i) = \sum g(\theta_i) p^i = \sum f'(\phi(\theta_i)) p^i.$$

Thus g is unique. By Corollary 4.2.4 it follows that the map defined in this unique way is in fact a homomorphism. □

Corollary 4.2.5. *Let R and R' are two strict p-rings. If the residue fields of are same then R and R' are canonically isomorphic.*

Lemma 4.2.1. *Let k and k' be two two perfect rings of characteristic $p > 0$. Suppose that there exists a surjective ring homomorphism $\phi : k \to k'$. If there exists a strict p-ring R with residue ring k, then there exists strict p-ring R' with residue ring k'.*

Proof. We will define an equivalence relation on R, then take R' as the quotient ring modulo this equivalence relation. For $\alpha, \beta \in R$ with coordinates θ_i and γ_i respectively, set

$$\alpha \equiv \beta \text{ if and only if } \phi(\theta_i) = \phi(\gamma_i)$$

for all i. If $\alpha_1 \equiv \alpha_2$ and $\beta_1 \equiv \beta_2$, then by the Corollary 4.2.4 the R' of R by the equivalence relation is a ring. Let $x \in R'$, and $\alpha \in R$ be a representatives for x with coordinates θ_i. Then $\xi_i = \phi(\theta_i)$ is independent of the choice of the representative α. □

Theorem 4.2.1. *(Classification theorem) For every perfect ring k of characteristic p, there exists a unique strict p-ring $W(k)$ with residue field k.*

Chapter 5

Extensions of valuation fields

Let F be a field and L be an extension of F which is discrete valuation with respect to the valuation v, with the value group Γ'. Then v induces a valuation v_0 on F in the obvious way. In this situation we say that L/F is an extension of valuation fields. It is clear that the value group $v_0(F^*)$ is a totally ordered subgroup of Γ'.

5.1 Definition of $e(L/F, v)$ and $f(L/F, v)$

Definition 5.1.1. *The number $e = \mid v(L^*)/v_0(F^*) \mid$ is called the ramification index $e(L/F, v)$ of the extension L/F.*

We know that $\alpha \in \mathcal{O}_{v_0} \subset F^*$ if and only if $v_0(\alpha) = v(\alpha) \geq 0$, which means $\alpha \in \mathcal{O}_v$, thus

$$\mathcal{O}_{v_0} = \mathcal{O}_v \cap F^*.$$

With the same way we can show that

$$\mathcal{M}_{v_0} = \mathcal{M}_v \cap F^*.$$

Now, consider the map

$$i : \overline{F}_{v_0} = \mathcal{O}_{v_0}/\mathcal{M}_{v_0} \hookrightarrow \mathcal{O}_v/\mathcal{M}_v = \overline{F_v}$$

defined by

$$\bar{\alpha} \mapsto \bar{\bar{\alpha}}.$$

If $\bar{\alpha} = \bar{\beta}$ then $\alpha - \beta \in \mathcal{M}_{v_0}$. Since $\mathcal{M}_{v_0} \subset \mathcal{M}_v$, it follows that $\bar{\bar{\alpha}} = \bar{\bar{\beta}}$ hence i is well-defined. Also we see that i is injective, thus we may view $\overline{F}_v$ as an extension of the field $\bar{F}_{v_0}$.

Definition 5.1.2. *The number $f = [\overline{F}_v : \overline{F}_{v_0}]$ is called the residue degree of the extension L/F, and denoted by $f(L/F, v)$.*

By using the very beginning results of the group theory and linear algebra, one can prove the following lemma:

Lemma 5.1.1. *Let $L \supset M \supset F$ be a chain of fields. Suppose L is a valuation field with the valuation v. Let v_M be the valuation on M induced by v. Then we have the following equalities:*

$$e(L/F, v) = e(L/M, v)e(M/F, v_M)$$

$$f(L/F, v) = f(L/M, v)f(M/F, v_M)$$

Lemma 5.1.2. *With the above notation, if L/F is finite of degree n and v_0 is discrete, then the ramification index $e(L/F, v)$ is finite, and v is discrete.*

Proof. For $e \leq e(L/F, v)$, let $\alpha_1, \cdots, \alpha_e$ be elements in $L^\times$ such that the elements

$$\overline{v(\alpha_1)}, \cdots, \overline{v(\alpha_e)}$$

are all distinct in the quotient group $v(L^*)/v(F^*)$. Since

$$e(L/F, v) = |v(L^*)/v(F^*)|,$$

such α_i's are always exist. Suppose

$$\sum_{i=1}^{e} c_i \alpha_i = 0 \text{ with } c_i \in F^*.$$

By the choice of α_i's we have $\overline{v(c_i\alpha_i)} = \overline{v(\alpha_i)}$. It follows that

$$v(c_i\alpha_i) \neq v(c_j\alpha_j),$$

whenever $i \neq j$. Thus $v(\sum_{i=1}^{e} c_i\alpha_i) = \min(v(c_i\alpha_i))$ which is on the other hand equal to infinity. Thus $c_i = 0$ for all i, i.e. α_i's are linearly independent over F. So $e \leq n$, which proves the first assertion of the lemma.

Since $v(L^*) \supset v_0(F^*) = \mathbb{Z}$ the value group of v is infinite, thus in order to prove the second assertion of the lemma, it suffices to prove that $v(L^*)$ is cyclic. Let π be a prime element of v_0. As we had shown that the ramification index is finite, we see that there is only finitely many positive elements $v(L^*)$ which are less then $v(\pi) = 1$, say $\alpha_1, \cdots, \alpha_e$, where e is the ramification index. Without loss of generality we may assume $\min(v(\alpha_i)) = v(\alpha_1)$. We claim that $v(\alpha_1)$ generates the value group $v(L^*)$. We have

$$\begin{aligned} e.v(\alpha_1) &= \underbrace{v(\alpha_1) + \cdots + v(\alpha_1)}_{e \text{ many}} = k \in \mathbb{Z}, \\ e.v(\alpha_i) &= \underbrace{v(\alpha_i) + \cdots + v(\alpha_i)}_{e \text{ many}} = l \in \mathbb{Z} \\ . \end{aligned}$$

Since $v(\alpha_1) \leq v(\alpha_i)$ it follows that $k \leq l$. So there exist positive integers s, r such that $l = sk + r$, where $0 \leq r < k$. We see that $r = v((\alpha_1^{-s}\alpha_i)^e)$. From this, we deduce that $0 \leq v(\alpha_1^{-s}\alpha_i) < v(\alpha_1)$. Thus $r = 0$, which means $l = sk$, equivalently

$$\overbrace{\underbrace{\left[v(\alpha_1) + \cdots + v(\alpha_1)\right]}_{e \text{ many}} + \cdots + \left[v(\alpha_1) + \cdots + v(\alpha_1)\right]}^{k \text{ many}} = \underbrace{v(\alpha_i) + \cdots v(\alpha_i)}_{e \text{ many}}.$$

From this, we conclude that

$$\underbrace{v(\alpha_1) + \cdots + v(\alpha_1)}_{k \text{ many}} = v(\alpha_i),$$

which proves the second assertion of the lemma. $\square$

From now on we'll deal with discrete valuations. Let $F \subset L$ be two fields with discrete valuations v and w respectively. The valuation w is said to be an extension of v, if the topology by w_0 is equivalent to the topology defined by v. In this situation, we write $w|v$ and use the notations $e(w|v)$ and $f(w|v)$. We shall assume that $w(L^* = \mathbb{Z})$ and $v(F^*) \subset \mathbb{Z}$. Let π_v and π_w be prime elements for (F, v) and (L, w). Then $w(\pi_w^e) \in v(F^*)$. Since $v(F^*)$ is cyclic it follows that $e(w|v) = w(\pi_v)$.

Lemma 5.1.3. *Let $[L : F] = n < \infty$, then $e(w|v)f(w|v) \leq n$.*

Proof. Let $e = e(w|v)$ and $f \leq f(w|v)$. Let

$$A = \{\theta_1, \cdots \theta_f\} \subset \mathcal{O}_w$$

be where A is a linearly independent set over $\mathcal{O}_v/\mathcal{M}_v$. We will show that

$$\{\theta_i \pi^j\}_{i,j},$$

where $i = 1, \cdots, f$ and $j = 0, \cdots, e$ is a linearly independent set over F. Suppose

$$\sum c_{ij}\theta_i\pi^j = 0$$

for $c_{ij} \in F$ and not all $c_{ij} = 0$. If necessary, by multiplying the expression $\sum_{i,j} c_{ij}\theta_i\pi^j$ by a suitable c_{kl}^{-1}, we may assume that some of the c_{ij}'s do not belong to $\mathcal{M}_v$. Now, by multiplying a suitable power of π, we may assume that $c_{ij} \in \mathcal{O}_v$, but not all in $\mathcal{M}_v$. We observe that if $\sum_i c_{ij}\theta_i \in \mathcal{M}_w$, then $\sum \bar{c}_{ij}\bar{\theta}_i = 0$. Since $\{\bar{\theta}_i\}_i$ is a linearly independent set, it follows that $\bar{c}_{ij} = 0$, hence $c_{ij} \in \mathcal{M}_v$, which is impossible. Thus there exists an index j such that $\sum_i c_{ij}\theta_i \notin \mathcal{M}_w$. Let j_0 be such minimal. We claim that

$$w(\sum_{j=1}^{f}(\sum_{i=1}^{e} c_{ij}\theta_i)\pi^j) = j_0.$$

Observe that the claim contradict with the fact that the above sum is equal to zero. Now, suppose $\sum c_{ij}\theta_i \notin \mathcal{M}_v$, but then

$$\sum c_{ij}\theta_i \in \mathcal{O}_v - \mathcal{M}_v,$$

thus $w(\sum c_{ij}\theta_i) = 0$, and this proves the claim. □

5.2 Extensions of complete discrete valuation fields

Let F be a discrete valuation field and $\hat{F}$ be its completion. We know that, if $\alpha \in \hat{F}$, with a representing Cauchy sequence (α_n) in F, then $\hat{v}(\alpha_n) = \lim v(\alpha_n)$ and $v(\alpha_n) \in \mathbb{Z}$ for all natural number n. Thus it follows that, $\hat{v}(\hat{F}^*) = \mathbb{Z}$. So the ramification index of the extension $\hat{F}/F$ is equal to 1. Also the residue degree of the extension is equal to 1. This means that, if F is not complete, then $[\hat{F} : F] \neq e.f$. On the contrary, we have the following proposition for the complete fields.

Proposition 5.2.1. *Let $L \supset F$ be two complete discrete valuation fields with respect to the valuations v, w respectively. Moreover suppose that $w|v$, $f = f(w|v)$, $e = e(w|v) < \infty$. If π_w is a prime element of L with respect to w and $\theta_1, \cdots, \theta_f$ are elements of $\mathcal{O}_w$ such that $\bar{\theta}_1, \cdots, \bar{\theta}_f$ form a basis for the vector space $\overline{L}_w$ over the field $\overline{F}_v$ then the set $\{\theta_i \pi_w^j\}$ is a basis for L over F, and for the $\mathcal{O}_v$-module $\mathcal{O}_w$ where $1 \leq i \leq f$ and $0 \leq j \leq e-1$. If f is finite, then $n = ef$.*

Proof. Let $S \subset \mathcal{O}_v$ be a set of coset representatives for $\overline{F}_v$ and $\bar{\alpha} \in \overline{L}_w$. Since $\{\bar{\theta}_i\}$ is basis of $\overline{F}_w$, there exists finite number of elements $\bar{s}_i \in \bar{F}_v$ with $s_i \in S$, such that $\alpha = \sum_{i=1}^{f} \bar{s}_i \bar{\theta}_i$. But this means the set

$$R' = \{\sum_{i=1}^{f} s_i \theta_i : \ s_i \in S \text{ and } s_i = 0 \text{ for all most all } i \in \mathbb{Z} \ \}$$

is a set of coset representatives for L. Let π_v be a prime element with respect to v, and for $m \in \mathbb{N}$, we set

$$\pi_m = \pi_v^k \pi_w^j,$$

where $m = ek + j, 0 \leq j < e$. Thus $w(\pi_m) = m$, so by the remark following the Proposition 4.1.1, it follows that an element $\alpha \in L$ can be expressed as a convergent series

$$\alpha = \sum_m \rho_m \pi_m \ \text{ with } \rho_m \in R'.$$

Writing ρ_m in terms of the elements of R and θ_i's

$$\rho_m = \sum_{i=1}^{f} \rho_{m,i} \theta_m \ \text{ with } \rho_{m,i} \in R,$$

we get

$$\alpha = \sum_{i,j} \Big(\sum_k \rho_{ek+j,i} \pi_v^k \Big) \theta_i \pi^j.$$

This means the set $\{\theta_i \pi_w^j\}$ is a spanning set of L over F. By the proof of the previous Lemma, we further know that $\{\theta_i \pi_w^j\}$ is a linearly independent set over F. Thus the set $\{\theta_i \pi_w^j\}$ is a basis of L over F. The assertion concerning the module part follows from this fact directly. $\square$

Theorem 5.2.1. *Let F be a complete field with respect to a discrete valuation v, and let L be an extension of F of degree n. Then there exists unique extension w of the valuation v on L, and $w = \frac{1}{f} v \circ N_{L/F}$ with $f = f(w|v)$. The field L is complete with respect to w.*

Proof. Let $w' = v \circ N_{L/F}$ and $\alpha, \beta \in L$. Since v is a valuation on F, we see that

$$w'(\alpha) = v \circ N_{L/F}(\alpha) = \infty$$

if and only if $N_{L/F}(\alpha) = 0$. But norm of an element is zero if and only if the element is zero. So w' satisfies the first property of being a valuation. Secondly, observe that

$$\begin{aligned} v \circ N_{L/F}(\alpha\beta) = v(N_{L/F}(\alpha)N_{L/F}(\beta)) &= v(N_{L/F}\alpha) + v(N_{L/F}\beta) \\ &= w'(\alpha) + w'(\beta). \end{aligned}$$

Assume that $w'(\alpha) \geq w'(\beta)$, for $\alpha, \beta \in L^*$. We shall show that $w'(\alpha + \beta) \geq w'(\beta)$. Since

$$w'(\alpha + \beta) = v(N_{L/F}(\beta)N_{L/F}(1 + \alpha/\beta)) = w'(\beta) + w'(1 + \alpha/\beta),$$

it suffices to show that $w'(1 + \eta) \geq 0$ whenever $w'(\eta) \geq 0$. Let

$$f(X) = X^m + a_{m-1}X^{m-1} + \cdots + a_0$$

be the minimal polynomial of η over F. Then $N_{F(\eta)/F}(\eta) = (-1)^m a_0$. We know that $N_{L/F}(\alpha) = \alpha^n$ if $\alpha \in F$, thus if $[L : F(\eta)] = s$, then we have

$$N_{L/F}(\eta) = N_{F(\eta)/F}\big(N_{L/F(\eta)}(\eta)\big) = ((-1)^m a_0)^s.$$

So $w'(\eta) = v(((-1)^m a_0)^s) = sv(a_0)$. From this, we deduce that $v(a) \geq 0$. Thus by the Remark 1.2.1, we get $v(a_i) \geq 0$. On the other hand we have the following equality

$$(-1)^m N_{F(\eta)/F}(1 + \eta) = f(-1) = (-1)^m + a_m(-1)^{m-1} + \cdots + a_0.$$

To see this equality, first note that norm of an element α in a field L over F is the product of elements $\sigma_i(\alpha)$, where α_i runs through the automorphisms of L which are fixing F. Thus, if $\sigma_i(\eta) = \eta_i$ then

$$N_{F(\eta)/F}(1 + \eta) = \Pi_i(1 + \eta_i).$$

On the other hand, we have the following equality

$$a_i(-1)^{m-i} = \sum_{\substack{J\subseteq\{1,\cdots,n\}\\ |J|=i}} \prod_{j\in J} \eta_j.$$

Clearly this summand occurs in the left hand side. And every summand of the left hand side can be written in this way uniquely. From this equality we deduce that

$$N_{F(\eta)/F}(1+\eta) \geq 0,$$

and

$$N_{L/F}(1+\eta) \geq 0,$$

which means $w'(1+\eta) \geq 0$, thus w' is a valuation on L.

If $\alpha \in F^*$ then

$$w'(\alpha) = v \circ N_{L/F}(\alpha) = v(\alpha^n) = nv(\alpha).$$

So the valuation $\frac{1}{n}w'$ is an extension of v. But the group $\frac{1}{n}w'(L^*)$ is not necessarily equal to $\mathbb{Z}$. Let e be the ramification index $e(\frac{1}{n}w'|v)$. By the Lemma 5.1.2, e is finite. Consider the following map on L to $\mathbb{Q}$,

$$w = \frac{e}{n}w' : L^* \longrightarrow \mathbb{Q}.$$

Let π_w be a prime element of w, note that π_w is a prime element with respect to w' also. Thus $w(\pi_w) = \frac{e}{n}w'(\pi_w)$, since e is the ramification index of $\frac{w'}{n}$. Therefore it follows that $w(\pi_w) = 1$. Hence w is a discrete valuation on L.

Now, let $\hat{L}$ be the completion of L with respect to w and $\hat{w}$ be the discrete valuation on $\hat{L}$. We know that $e(\hat{L}|L) = 1$ and $f(\hat{L}|L) = 1$. By the Lemma 5.1.1 and the remark following the Lemma 5.1.3 and bearing the Proposition 5.2.1 we see that

$$n \leq [\hat{L} : F] = e(\hat{L}|F)f(\hat{L}|F) = e(\hat{L}|L)e(L, F)f(\hat{L}|L)f(L|F) \leq n.$$

which means $[\hat{L} : F] = n$, thus $\hat{L} = L$ which means L is complete with respect to w. Also from this equality we deduce that $\frac{e}{n} = f$. $\square$

Theorem 5.2.2. *Let L_1 be a complete discrete valuation field of characteristic zero and suppose that the characteristic of the residue field $\overline{L_1}$ is $p > 0$.Let L_2 be a complete discrete valuation field of characteristic zero where p is a prime element in L_2. Moreover, suppose that the characteristic of the residue field $\overline{L_2} = p$ and there exists a field embedding $\bar{i} : \overline{L_2} \to \overline{L_1}$. Then there exists a field embedding $i : L_2 \to L_1$ such that*

$$v_{L_1} \circ i = e(L_1) v_{L_2},$$

where

$$e(L_1) = v_{L_1}(p),$$

and

$$\overline{i(\alpha)} = \bar{i}(\overline{\alpha}),$$

for every $\alpha \in \mathcal{O}_{L_2}$.

Proof. We give a proof for the theorem, where the field L_2 is perfect. For the general case (c.f. [3]). Since L_2 is perfect, by the theorem there exists a set T of Teichmüller representatives, with the corresponding function

$$f_2 : \overline{L_2} \to L_2.$$

Note that, by the lemma 4.1.1, every $\theta \in L_2$ can be written uniquely as

$$\sum f_2(\overline{\theta_s}) p^s.$$

Define

$$i : L_2 \to L_1$$

by

$$i(\sum f_2(\theta_s) p^s) = \sum \bar{i} f_1(\overline{\theta_s}) p^s,$$

where $f_1 : \overline{L_2} \to L_2$ is a lifting function in the sense of theorem 5.2. By the corollary 4.2.2 i is a field homomorphism and it satisfies the desired properties. $\square$

5.3 Elimination of wild ramification: Epp's theorem

In this section we fix a complete discrete valuation field K with residue field $\overline{K}$ of positive characteristic p.

In the mixed characteristic case, i.e. $char K = 0$, we fix the field k which consists of those elements that are algebraic over the fractional field k_0 of $W(F)$, where $F = \cap \overline{K}^{p^i}$. In the equal characteristic case, we fix a base subfield k_0 in K, which is complete with respect to the induced valuation, and has $\mathbb{F}_p$ as a residue field. It is clear that k_0 is equal to $k_0((\alpha))$, for some $\alpha \in K$, where the valuation of α is positive. In this case, k denote the algebraic closure of k_0F in K.

In the both cases, k is said to be the *constant subfield* of K.

Theorem 5.3.1. *([Epp]) Let L/K be a finite extension of complete discrete valuation fields, k the constant subfield of K. Then there exists a finite extension l/k such that $e(lL/kK) = 1$.*

Chapter 6

n-local fields

6.1 Definition of n-local fields

Definition 6.1.1. *A field K is said to be an n-local field over a finite field (resp. more generally a perfect field) K_0, if K is a complete discrete valuation field with respect to $v = v_n$ and there exists a chain of fields*

$$K = K_n, K_{n-1}, \cdots, K_1, K_0,$$

where each K_{i+1} is a complete discrete valuation field with respect to the valuation $v_{i+1}$with the residue field K_i for $1 \leq i \leq n-1$, and K_0 is finite field (resp. more generally a perfect field). K_{n-1} is also denoted by k_K or $\overline{K}_v$ and called the first residue field of K. (The finite field $\mathbb{F}_q$ with q elements is called a 0-local field.)

Examples

1. $\mathbb{F}_q((X))$ is a 1-local field.

2. For k, $n-1$ dimensional local field, we put a valuation on $k((X))$ by setting

$$v(\sum_{i \geq m} a_i X^i) = m,$$

where $a_m \neq 0$. This valuation turns $k((X))$ in to a complete discrete valuation field with residue field k.

3. For a complete discrete valuation field F, consider the following field:

$$K = F\{\{X\}\} = \{\sum_{-\infty}^{+\infty} a_i X^i : a_i \in F, \inf_{v_F}(a_i) > -\infty, \lim_{i \to -\infty}(a_i) = +\infty\}$$

Define $v_k(\sum a_i X^i) = \min v_F(a_i)$. Since $\inf_{v_F} a_i > -\infty$ such minimum always exists. Let $f = \sum f_i X^i, g = \sum g_j X^j$. We know that $v(f_i + g_i) \geq \min(v(f_i), v(g_i))$, thus

$$\min_{i \in \mathbb{N}} v_F(f_i + g_i) \geq \min_{i,j \in \mathbb{N}}(v(f_i), v(g_j)) = \min(v_F(f), v_F(g)),$$

so $v_F(f+g) \geq \min(v_F(f), v_F(g))$.

Now, suppose that $v_F(f) = v(f_n)$ and $v_F(g) = v(g_m)$ where $v_F(f_i) > v_F(f_n)$ and $v_F(g_j) > v_F(g_m)$ whenever $i < n$ and $j < m$. Let $fg = h = \sum h_s X^s$. We'll show that

$$\min v_F(h_s) = v(h_{n+m}) = v(\sum_{i+j=n+m} f_i g_j).$$

If $i < n$ then since $v(f_n)$ and $v(g_m)$ are minimal and their indexes are also minimal $v_F(f_i) > v_F(f_n)$ and $v_F(g_j) \geq v_F(g_m)$, from this we conclude the following strict inequality

$$v(f_i g_j) >_{\neq} v_F(f_n g_m) \quad \text{whenever } i \neq n.$$

So $v(h_{n+m}) = v(f_n g_m)$. It is also clear that $\min_{i,j} \geq v_F(f_n g_m)$, thus $v(h) \geq v_F(f) + v_F(g)$. Hence one gets $v(h) = v(f) + v(g)$. So v is really a valuation. Now we will show that $F\{\{X\}\}$ is complete with respect to v. Let f_n be a Cauchy sequence in $F\{\{X\}\}$. This means, as n, m tends to infinity, $v_K(f_n - f_m) = \min v_F(f_{ni} - f_{mi})$ tends to infinity. Thus for each $i \in \mathbb{Z}$, f_{ni} is a Cauchy sequence in F with respect to v_F. Completeness of F implies that f_{ni} converges to a unique point in F. Put

$$\alpha_i = \lim f_{ni} \quad \text{for } i \in \mathbb{Z}.$$

Now, one can easily show that $\sum \alpha_i X^i = \lim f_n$, which means $F\{\{X\}\}$ is complete with respect to v_K.

An element $f \in F\{\{X\}\}$ is element of ring of integers $\mathcal{O}_{v_K}$ if and only if $\min v_F(f_i) \geq 0$, and $f \in \mathcal{M}_{v_K}$ if and only if $\min v_F(f_i) > 0$. This means $\mathcal{O}_{v_K} = \mathcal{O}_{v_F}\{\{X\}\}$ and $\mathcal{M}_{v_K} = \mathcal{M}_{v_F}\{\{X\}\}$. Define

$$\varphi : \mathcal{O}_{v_K} = \mathcal{O}_{v_F}\{\{X\}\} \to \mathcal{O}_{v_F}/\mathcal{M}_{v_F}((t))$$

as $\sum a_i X^i \mapsto \sum \bar{a}_i t^i$. It is clear that φ is a ring homomorphism, which is onto and its kernel is $\mathcal{M}_v\{\{X\}\}$. Thus, the residue field of $F\{\{X\}\}$ is the Laurent series with coefficients in the residue field of F.

Let F be a complete field and consider the field $K = F\{\{X\}\}\{\{Y\}\}$. Then the residue field of K is $F'((t_1))$ where F' is the residue of $F\{\{X\}\} = \bar{F}((t_2))$. Thus the residue field of K is $\bar{F}((t_1))((t_2))$.

Lemma 6.1.1. *$K_1 = K((X))\{\{Y\}\}$ is isomorphic to $K_2 = K((Y))((X))$.*

Proof. We define $\Phi : K((X))\{\{Y\}\} \to K((Y))((X))$ as follows: For $\alpha \in K((X))\{\{Y\}\}$

$$\alpha = \sum_{-\infty}^{\infty} f_i(X)Y^i \in K((X))\{\{Y\}\},$$

where

$$f_i(X) = \sum_{j \gg -\infty}^{\infty} a_j^{(i)} X^j$$

put

$$\Phi(\alpha) = \sum_{-\infty}^{\infty} g_r(Y)X^r$$

where $g_r(Y) = \sum a_r^{(i)} Y^i$. First we will show that range of Φ is really $K((Y))((X))$. In order to do this we have to show that $g_r(Y) = 0$ for almost all negative r. Suppose for all $k \in \mathbb{Z}$ there exists $r < k$ such that $g_r(Y) = \sum a_r^{(i)} Y^i \neq 0$. This means, for some $i \in \mathbb{Z}$, the coefficient $a_r^{(i)} \neq 0$. So we conclude that $v(f_r) = r < k$, thus $\inf v(f_j) = -\infty$, which is impossible. Let

$$v_{K_1}(\sum_{-\infty}^{\infty} f_i(X)Y^i) = k,$$

which means

$$\begin{aligned}\min\{v_X(f_i(X)) : i \in \mathbb{Z}\} &= \min_{i,j\in\mathbb{Z}}\{j \in \mathbb{Z} : a_j^{(i)} \neq 0\} \\ &= v_X(f_k(X) \\ &= \min\{j \in \mathbb{Z} : a_j^{(k)} \neq 0\}.\end{aligned}$$

On the other hand

$$\begin{aligned}v_{K_2}(\Phi(\alpha)) &= v_{K_2}(\sum_{-\infty}^{\infty} g_r(Y)X^r) \\ &= \min_{r\in\mathbb{Z}}\{v_Y(g_r(Y))\} \\ &= \min_{r\in\mathbb{Z}}\{j \in \mathbb{Z} : a_r^{(j)} \neq 0\} \\ &= \min\{j \in \mathbb{Z} : a_j^{(k)} \neq 0\} \\ &= k.\end{aligned}$$

Thus we have lemma. □

Remark 6.1.1. *Let K be a field endowed with the trivial valuation v. Then clearly (K, v) is complete discrete valuation field. So we may consider the field $K\{\{X\}\}$ introduced in the previous example. Then $K\{\{X\}\} = K((X))$, as any convergent sequence in K must be constant. In particular, $\mathbb{F}_q\{\{X\}\} = \mathbb{F}_q((X))$.*

Definition 6.1.2. *For a local field k the fields*

$$k\{\{X_1\}\}\cdots\{\{X_m\}\}((X_{m+2}))\cdots((X_n)), \qquad 0 \leq m \leq n-1$$

are n-dimensional local fields and they are called the standard fields.

6.2 System of local parameters

Throughout this section K denotes an n-local field with the chain of complete discrete valuation fields

$$K = K_n, K_{n-1}, \cdots, K_1, K_0,$$

with respect to the valuations $v = v_n, v_{n-1}, \cdots, v_1$.

Definition 6.2.1. *An n-tuple* $(t_1, \cdots, t_n) \in K^n$ *is called a system of local parameters of* K *if* t_i *is a unit in* K_j *for* $j > i$ *and the residue class of* t_i *in* K_i *is a prime element for* $1 \le i \le n$.

Lemma 6.2.1. *Let* k *be a local field. For the standard field*

$$K = k\{\{X_1\}\} \cdots \{\{X_m\}\}((X_{m+2})) \cdots ((X_n)),$$

the n-tuple $(X_1, \cdots, X_m, \pi, , X_{m+2}, \cdots, X_n)$ *where* π *is a prime element of* k *forms a system of local parameters for* K.

Proof. Let i be an index where $m + 1 < n - i \le n$. Then

$$K_{n-i} = k\{\{X_1\}\} \cdots \{\{X_m\}\}((X_{m+2})) \cdots ((X_{m-i})).$$

Thus X_{m-i} is a prime element of K_{n-i}, and we also know that X_{m-i} is constant in K_{n-i+1} which means it is a unit $\mathcal{O}_j$ for $j > m - i$. □

This system of local parameters is called the canonical system of local parameters.

We will now give a non-standard definition of the lexicographic ordering on $\mathbb{Z}^n$, following Madunts and Zhukov [7],[10].

Definition 6.2.2. *The lexicographic order of* $\mathbb{Z}^n$ *is defined in the following way. For* $i = (i_1, \cdots, i_n) < j = (j_1, \cdots, j_n)$ *if and only if*

$$i_k < j_k, i_{k+1} = j_{k+1}, \cdots, i_n = j_n \quad \textit{for some } k \le n.$$

We now introduce the mapping

$$v = (v^{(1)}, \cdots, v^{(n)}) : K^* \to \mathbb{Z}^n$$

defined by

$$v^{(i)}(\alpha) = v_i(\overline{\alpha t_n^{-v^{(n)}(\alpha)} \cdots t_{i+1}^{-v^{(i+1)}(\alpha)}}), \qquad \text{for } 1 \le i \le n,$$

where the residue means the residue in the field K_i and $v^{(n)}(\alpha) = v_n(\alpha)$. Extend the mapping v to K by setting $v(0) = +\infty$

Lemma 6.2.2. *The map $v : K \to \mathbb{Z}^n \cup \{\infty\}$ defined above is a rank n discrete valuation.*

Proof. Let $\alpha, \beta \in K$. Suppose we had shown that $v^{n-j}(\alpha\beta) = v^{n-j}(\alpha) + v^{n-j}(\beta)$ for $0 = j < i \leq n-1$. Then we have

$$\begin{aligned} v^{(n-i)}(\alpha\beta) &= v_i(\overline{\alpha\beta t_n^{-v^{(n)}(\alpha\beta)} \cdots t_{i+1}^{-v^{(i+1)}(\alpha\beta)}}) \\ &= v_i(\overline{\alpha t_n^{-v^{(n)}(\alpha)} \cdots t_{i+1}^{-v^{(i+1)}(\alpha)} \beta t_n^{-v^{(n)}(\beta)} \cdots t_{i+1}^{-v^{(i+1)}(\beta)}}) \\ &= v_i(\overline{\alpha t_n^{-v^{(n)}(\alpha)} \cdots t_{i+1}^{-v^{(i+1)}(\alpha)}}) + v_i(\overline{\beta t_n^{-v^{(n)}(\beta)} \cdots t_{i+1}^{-v^{(i+1)}(\beta)}}) \\ &= v^{(i)}(\alpha) + v^{(i)}(\beta), \end{aligned} \tag{6.1}$$

which means v is a group homomorphism. Now suppose $v(\alpha) \geq v(\beta)$. We will show that $v(\alpha + \beta) \geq (\beta)$. Since v is a group homomorphism it follows that

$$v(\alpha + \beta) = v(\beta) + v(1 + \alpha/\beta).$$

Since $v(\alpha/\beta) \geq 0$, in order to prove the mentioned inequality above, it suffices to show for $x \in K$, $v(x) \geq 0$ implies $v(1 + x) \geq 0$.
Let $x \in K$ such that $v(x) \geq 0$, so $v^{(n)}(x)$ is non-negative, which means $v^{(n)}(x) \geq 0$. Since $v^{(n)} = v_n$ is a valuation $v^{(n)}(1 + x) \geq 0$. If $v^{(n)} > 0$ then clearly $v(1 + x) > 0$, if not, then

$$v^{(n-1)}(1 + x) = v_{n-1}(\overline{(1 + x)t_n^{-v^{(n)}(1+x)}}) = v_{n-1}(\overline{1 + x}).$$

Clearly, if $v_{n-1} > 0$ then $v(1+x) > 0$ and if $v_{n-1} = 0$ then $v^{(n-2)}(1+x) = v_{n-2}(1+x)$, continuing this way one concludes the desired inequality. Thus v is really a valuation on K. □

Lemma-Definition 6.2.1. *Let K be an n-local field with respect to the rank n discrete valuation v. Then the local ring $\mathcal{O}_v = \{\alpha \in K : v(\alpha) \geq 0\}$ is called the ring of integers of K with the maximal ideal $\mathcal{M}_v = \{\alpha \in K : v(\alpha) > 0\}$.*

Lemma 6.2.3. *The residue field $\mathcal{O}_K/\mathcal{M}_K$ is isomorphic to the last residue field K_0 of K.*

Proof. Consider the map

$$\begin{aligned}\phi : \mathcal{O}_K &\rightarrow K_0 \\ \alpha &\mapsto \bar{\alpha}, \quad \text{the residue of } \alpha \text{ in } K_0\ .\end{aligned}$$

It is clear that ϕ is a ring homomorphism. We will show that $\ker\phi = \mathcal{M}_v$. Let $\alpha \in \mathcal{M}_v$. So there exists an index $1 \leq i \leq n$such that

$$v^{(i)}(\alpha) > 0 \quad \text{and} \quad v^{(n)}(\alpha) = \cdots = v^{(i+1)}(\alpha) = 0,$$

thus

$$0 < v^{(i)}(\alpha) = v_i(\overline{\alpha t_n^{-v^{(n)}(\alpha)} \cdots t_{i+1}^{-v^{(i+1)}(\alpha)}}) = v_i(\bar{\alpha}),$$

which means $\bar{\alpha} \in \mathcal{M}_{v_i}$, hence its residue in K_{i-1} is zero, thus its residue in K_0 is also zero. Hence $\phi(\alpha) = 0$, and $\alpha \in \ker\phi$. Conversely, suppose that $\phi(\alpha) = 0$. We know that $v(\alpha) = (v^{(1)}(\alpha), \cdots, v^{(n)}(\alpha)) \geq 0$. Suppose that $v^{(2)}(\alpha) = \cdots = v^{(n)}(\alpha) = 0$, then

$$v^{(1)}(\alpha) = v_i(\overline{\alpha t_n^{-v^{(n)}(\alpha)} \cdots t_2^{-v^{(2)}(\alpha)}}) = v_1(\bar{\alpha}).$$

Since the residue of α in K_0 is zero, its residue in K_1 is contained in the maximal ideal $\mathcal{M}_{v_1}$, thus $v^{(1)}(\alpha) > 0$, thus $v(\alpha) > 0$. Whence $\ker\phi = \mathcal{M}_v$, which proves the Lemma. □

6.3 Ideal structure of $\mathcal{O}_K$

In what follows, we denote the the ring of integers $\mathcal{O}_v$ of a fixed n-local field K with respect to v by $\mathcal{O}_K$ and denote the the maximal ideal $\mathcal{M}_v$ of the ring of integers by $\mathcal{M}_K$.

Definition 6.3.1. *For $1 \leq l \leq n$ put $P(i_l, \cdots, i_n) = P_K(i_l, \cdots, i_n) = \{\alpha \in K : (v^{(l)}(\alpha), \cdots, v^{(n)}(\alpha)) \geq (i_l, \cdots, i_n)\}$. In particular $P(\underbrace{0, \cdots, 0}_{n \text{ many}}) = \mathcal{O}_K$ and $P(1, \underbrace{0, \cdots, 0}_{n-1 \text{ many}}) = \mathcal{M}_K$.*

Lemma 6.3.1. *For any non-zero ideal J of $\mathcal{O}_K$ there exists $(i_l, \cdots, i_n)$ such that*

$$J = P(i_l, \cdots, i_n).$$

It is clear that such $(i_l, \cdots, i_n)$ must be unique. The ring $\mathcal{O}_K$ is not Noetherian.

Proof. Let J be a non-zero ideal of $\mathcal{O}_K$ and put $i_n = \min\{v^{(n)}(\alpha) : \alpha \in J\}$. It is clear that $J \subseteq P(i_n)$. Suppose that $J \neq P(i_n)$, and for all $s \in \mathbb{Z}$ there exists $\alpha \in J$ with $v^{(n)}(\alpha) = i_n$ and $v^{(n-1)}(\alpha) < s$. Let $\beta \in P(i_n)$, then there exists $\alpha \in J$ with $(v^{(n-1)}(\alpha), v^{(n)}(\alpha)) < (v^{(n-1)}(\beta), v^{(n)}(\beta))$. Thus by the Lemma 2.1.2 it follows that $\beta \in J$, which means $J = P(i_n)$, a contradiction. This means

$$i_{n-1} = \min\{v^{(n-1)}(\alpha) : v^{(n)}(\alpha) = i_n, \alpha \in J\} > -\infty.$$

There are two cases. Either $J = P(i_{n-1}, i_n)$ or

$$i_{n-2} = \min\{v^{(n-2)}(\alpha) : v^{(n)}(\alpha) = i_n, v^{(n-1)}(\alpha) = i_{n-1}, \alpha \in J\} > -\infty.$$

We define i_{n-j} in this way. If $J \neq P(i_2, \cdots, i_n)$ then

$$i_1 = \min\{v^{(1)}(\alpha) : v^{(n)}(\alpha) = i_n, \cdots, v^{(2)}(\alpha) = i_2, \alpha \in J\} > -\infty,$$

and it is clear that $J = P(i_1, \cdots, i_n)$. If $n > 1$ then $P(i, 1) \subset P(i+1, 1)$ which means $\{P(i, 1)\}_{i \in \mathbb{N}}$ is an ascending chain of ideals which is not stationary. Hence $\mathcal{O}_K$ is not Noetherian. $\square$

6.4 The group structure of $K^\times$

Definition 6.4.1. *The multiplicative group*

$$U_K = \mathcal{O}_K^\times$$

is called the group of units with respect to v, and the multiplicative subgroup

$$V_K = 1 + \mathcal{M}_K$$

of U_K *is called the principal units with respect to* v*. For* $1 \leq l \leq n$ *the multiplicative groups*

$$U_K(i_l, \cdots, i_n) = 1 + P(i_l, \cdots, i_n),$$

for all $(i_l, \cdots, i_n) \in \mathbb{Z}^{n-l}$ *are called the higher unit groups with respect to* v*.*

Lemma 6.4.1. *Consider the Teichmüller representatives* $T = \{[\alpha] : \alpha \in K_0\}$ *of the last residue field* K_0 *in* K*. Then*

$$U_K \simeq T \oplus V_K,$$

and if $(t_n, \cdots, t_1)$ *is a local system of parameters of* K*, then*

$$K^\times \simeq \mathbb{Z}t_1 \oplus \cdots \oplus \mathbb{Z}t_n \oplus U_K.$$

Proof. This follows by Parshin's structure theorem (cf. Chapter 7), which reduces the lemma to the determination of the invertible of elements of formal power series. $\square$

6.5 Extensions of n-local fields

Throughout this section K stands for an n-local field with the corresponding chain of complete discrete valuation fields

$$K = K_n, K_{n-1}, \cdots, K_1, K_0$$

with the respecting valuations

$$v = v_n, v_{n-1}, \cdots, v_1$$

respectively.

Let L be a finite extension of K. In view of the Theorem 5.2.1, it follows that, L is a complete discrete valuation field with respect to $w = \frac{1}{f} v \circ N_{L/K}$, which extends the valuation v on K. Therefore, the residue field of $\overline{L}_w =: L_{n-1}$ is an extension field of the residue field K_{n-1} of K. In view of the Lemma 5.1.3, degree of this extension, which is equal to the residue degree $f(w|v)$ of the extension L/K, is finite. Therefore L_{n-1} is a complete discrete valuation field. Now, the following Proposition follows by induction.

Proposition 6.5.1. *If L is a finite extension of K, then there exists a canonical n-local field structure on L with the corresponding chain of complete discrete valuation fields*

$$L = L_n, L_{n-1}, \cdots, L_1, L_0$$

with the respective valuations

$$w = w_n, w_{n-1} \cdots, w_1,$$

where

$$w = w_n = \frac{1}{f(L|K)} v_n \circ N_{L_n/K_n}, w_{n-1} = \cdots, w_1 = \frac{1}{f(L_1|K_1)} v_1 \circ N_{L_1/K_1}.$$

The following proposition is easy to prove and generalizes the result of Chapter 5.

Proposition 6.5.2. *Given a finite extension L/K. Let $t_1, \cdots, t_n$ and $t'_1, \cdots, t'_n$ be local system of parameters for the fields K and L respectively. Let w and v be the valuations on L and K. Then the matrix*

$$E(L|K) := (w^{(j)}(t_i))_{1 \le i,j \le n} = \begin{pmatrix} e_1 & 0 & \cdots & 0 \\ * & e_2 & \cdots & 0 \\ * & * & \cdots & 0 \\ * & \cdots & * & e_n \end{pmatrix},$$

where $e_i = e_i(L|K) = e(L_i|K_i)$, $1 \le i \le n$, satisfies

$$[L : K] = f(L|K).det(E(L|K),$$

where $f(L|K) = [L_0 : K_0]$.

Chapter 7

Parshin's structure theorem for n-local fields

Note that for a local field K, there are three cases:

1. $char(K) = char(\overline{K}) = 0$,
2. $char(K) = char(\overline{K}) = p > 0$,
3. $char(K) = 0$, $char(\overline{K}) = p > 0$.

The following theorem, due to Parshin, classifies n- local fields in terms of standard fields.

7.1 Statement of Parshin's classification theorem

Theorem 7.1.1. *Let K be an n-local field with the corresponding chain of complete discrete valuation fields*

$$K = K_n, K_{n-1}, \cdots, K_1, K_0.$$

Then

1. *If $char(K_1) = char(\overline{K_0}) = p > 0$, then K is isomorphic to*

$$\mathbb{F}_q((X_1))\cdots((X_n));$$

2. If $char(K_1) = char(\overline{K_0}) = 0$, then K is isomorphic to

$$k((X_1))\cdots((X_{n-1}));$$

where k is a local field of characteristic 0.

3. If $char(K_{m+1}) = 0$ and $char(K_m) = p$, then K is isomorphic to a finite extension of a standard field of the form

$$k\{\{X_1\}\}\cdots\{\{X_m\}\}((X_{m+2}))\cdots((X_n)),$$

moreover, there exists a finite extension of K which is standard.

7.2 Proof: Equal characteristic case

Suppose $char(K) = p > 0$. Then K_1 is a complete discrete valuation of characteristic $p > 0$. By the structure theorem of complete discrete valuation fields of characteristic p, K_1 is isomorphic to $\overline{K_1}((X_1)) = K_0((X_1))$. But K_0 is a finite field. Thus $K_1 = \mathbb{F}_q((X_1))$, where $q = p^f$ for some positive integer f. Again using the structure theorem we deduce that $K = \mathbb{F}_q((X_1))\cdots((X_n))$. If characteristic of K_1 is zero, then by the use of the structure theorem of complete discrete valuation fields of equal characteristic, it follows that $K = K_1((X_1))\cdots((X_{n-1}))$.

7.3 Proof: unequal characteristic case

Suppose that we are in the third situation. Without loss of generality we may assume that the characteristic of K is equal to zero, and the characteristic of K_{n-1} is equal to p. Then by the previous section

$$K_{n-1} = \mathbb{F}_q((X_1))\cdots((X_{n-1})).$$

Let $k_0 = ff(W(\mathbb{F}_q))$ be the field of fractions of the Witt ring $W(\mathbb{F}_q)$ over $\mathbb{F}_q$. We know that k_0 is a complete discrete valuation field with the residue field $\mathbb{F}_q$. Now, put

$$K' = k_0\{\{t_1\}\}\cdots\{\{t_{n-1}\}\}.$$

where $t_1, \cdots, t_{n-1}, \pi$ is a system of local parameters of K. Then the residue field of K' is equal to the field

$$\overline{k}_0\{\{\overline{t_1}\}\}\cdots\{\{\overline{t_{n-1}}\}\} \simeq \mathbb{F}_q((X_1))\cdots((X_{n-1})) = K_{n-1}.$$

Thus by Theorem 5.2.2, it follows that the field K' embeds in K . Since the residue degree

$$f = f(K/K') = [\overline{K} : \overline{K'}]$$

is finite. By Proposition 5.2.1, we see that the extension K/K' is finite. Now, by Epp's theorem 5.3.1, there exists a finite extension $k = k_0(\alpha)$ of k_0, such that $e(kK/kK') = 1$. Thus, we just have to show that the field kK, which is a finite extension over K, is a standard field, Which follows from the lemma below.

Lemma 7.3.1. *Let L be a finite extension of the standard field*

$$K = k\{\{X_1\}\}\cdots\{\{X_m\}\}((X_{m+2}))\cdots((X_n)).$$

If $e(L/K) = 1$, then L is standard.

Proof. Without loss of generality, we may assume that

$$L = K(\alpha),$$

for some α in the algebraic closure of K. Observe that, the lemma follows at once, if α is algebraic over the field

$$K_{n-1} = k\{\{X_1\}\}\cdots\{\{X_m\}\}((X_{m+2}))\cdots((X_{n-1})).$$

Suppose that α is not algebraic over K_{n-1}. We know that the algebraic closure of K is contained in the field of *Puiseux* series over k^a in variables $X_1, \cdots, X_m, X_{m+2}, \cdots, X_n$ given by

$$k^a\{\{X_1^{\mathbb{Q}}\}\}\cdots\{\{X_m^{\mathbb{Q}}\}\}((X_{m+2}^{\mathbb{Q}}))\cdots((X_n)^{\mathbb{Q}}).$$

Let

$$\alpha = \sum_{q\in\mathbb{Q}} c_q X_n^q,$$

where c_q is an element of Puiseux series over K_{n-1}. Since α is not algebraic over K_{n-1}, there exists $q \in \mathbb{Q} - \mathbb{Z}$, such that $c_q \neq 0$. By multiplying a suitable power of X_n, we may assume $c_q < 0$. In this case, wee see that the valuation of α given by the norm map is a rational number, which contradicts with the fact that the ramification index $e(L/K)$ of L/K is 1.

$\square$

Chapter 8

Topologies on the additive and the multiplicative groups of an n-local field

In this chapter, we shall define the topology on K^+ and $K^\times$, where K is an n-local field (which is natural from the point of view of K-theoretic local class field theory of Kato and Parshin) inductively. These topologies are called *sequential topologies.*

8.1 Topology on K^+

In order to do define a nice topology on K^+ in the sense of K-theoretic class field theory, the strategy will be the following. Let K be an n-local field with the corresponding chain of complete discrete valuation fields

$$K = K_n, K_{n-1}, \cdots, K_1, K_0$$

with the corresponding valuations

$$v_n, v_{n-1} \cdots, v_1$$

respectively. By the definition, K_0 is a finite field, say $K_0 = \mathbb{F}_q = \mathbb{F}_{p^f}$. Therefore K_1 is either a finite extension of $\mathbb{Q}_p$ or K_1 is $\mathbb{F}_q((X))$. Thus there are two cases:

1. $K_1 = \mathbb{F}_q((X))$: In this case, in view of the Lemma 6.1.1, $K = \mathbb{F}_q((X_1))\cdots((X_n))$. The topology of such fields will be defined inductively in the first subsection of this section.

2. $K_1/\mathbb{Q}_p$ is a finite extension. In this case, we have two cases for the complete discrete valuation field K_2, which are $K_1\{\{X\}\}$ and $K_1((X))$. In the latter case the topology will be defined inductively in the first subsection, while in the former case the topology will be defined in the second subsection.

For a standard field K of mixed characteristic, that is

$$K = K_0\{\{X_1\}\}\cdots\{\{X_m\}\}((X_{m+2}))\cdots((X_n)),$$

combination of section 1 and section 2 constructs the topology on K.

Thus, we suppose in the next two subsections, that K is a field that has a topological structure on it. For some technical reasons, we will always assume that $+$ is a continuous mapping but $\times$ is sequentially continuous.

Let C be a subclass of the sequences of neighborhoods of zero in K, where a sequence of neighborhoods $(U_i)_{i\in\mathbb{Z}}$ of zero in K is contained in C if and only if $U_i = K$ for all most all positive i.

8.1.1 Topology on Laurent series $K((X))$

We construct a topology on $K((X))$ in the following way: For $(U_i)_{i\in\mathbb{Z}} \in \mathrm{C}$ put,

$$U_{\{U_i\}} := \{\sum_{i\gg-\infty}^{+\infty} a_i X^i : a_i \in U_i\}.$$

It is clear that for $(U_i)_{i\in\mathbb{Z}}, (V_i)_{i\in\mathbb{Z}} \in \mathrm{C}$ we have $(U_i \cap V_i)_{i\in\mathbb{Z}} \in \mathrm{C}$ and $U_{\{U_i\}} \cap U_{\{V_i\}} = U_{\{U_i\cap V_i\}}$, thus we may set the set

$$\mathfrak{B} := \left\{U_{(U_i)} : (U_i)_{i\in\mathbb{Z}} \in \mathrm{C}\right\}$$

as a base of open neighborhoods of 0 in $K((X))$.

Let $u^{(n)}$ be a sequence in $K((X))$ converging to zero with respect to the topology defined above. Let $k \in \mathbb{Z}$ be fixed and V be an open neighborhood of zero in K.

Put $U_i = K$ if $i \neq k$ and $U_k = V$.Since $u^{(n)} \to 0$, there exists a positive integer N such that for $m > N$ we have

$$u^{(m)} \in U_{(U_i)} = \{ \sum_{i \gg -\infty}^{\infty} a_i X^i : a_i \in U_i \},$$

which means $a_i^{(m)} \subset V$. Thus we conclude that for fixed integer k, the sequence $a_k^{(m)}$ tends to 0.

Now, we assume that the topology on K is T_0. Suppose that the set $\{i : a_i^n \neq 0\}$ is unbounded below. In this case, without loss of generality we may assume $a_{-n}^{(n)} \neq 0$. Since K is a T_0 space, for all $n \in \mathbb{N}$, there exists an open neighborhood V_{-n} of 0 such that $a_{-n}^{(n)} \notin V_{-n}$. Put,

$$U_i = \begin{cases} V_i & \text{if } i < 0, \\ K & \text{if } i \geq 0. \end{cases}$$

Then clearly, for any $n \in \mathbb{N}$, $u^{(n)} \notin U_{U_i}$ since $a_{-n}^{(n)} \notin U_{-n}$ which contradicts with the fact that $u^{(n)}$ is converging to zero. Thus $\{i : a_i^n \neq 0\}$ is bounded below.

Conversely, let

$$u^{(n)} = \sum_{i \gg -\infty}^{\infty} a_i^{(n)} X^i$$

be a sequence in $K((X))$. Suppose that for a fixed integer i, the sequence $a_i^{(n)}$ tends to 0 as n goes to infinity, and there exists $m \in \mathbb{Z}$ such that for all $n \in \mathbb{N}$, $u^{(n)} \in X^m K[[X]]$. Let $(U_i)_{i \in \mathbb{Z}} \in$ C. Then by the definition of C, there exists $M \in \mathbb{N}$ such that for $k > M$ we have $U_k = K$. Since $a_i(n)$ tends to 0 as n goes to infinity, for $m \leq i \leq M$, there exists N_i such that whenever $n > N_i$ then $a_i^{(n)} \subset U_i$. Let $N = \max\{N_i\}$, then for $n > N$ we have $a_i^{(n)} \in U_{U_i}$, which means that the sequence $a_i^{(n)}$ tends to zero in $K((X))$ with respect to the topology defined above.

Thus we have the following:

Lemma 8.1.1. *A sequence $u^{(n)} = \sum_{i \gg -\infty}^{\infty} a_i^{(n)} X^i$ in $K((X))$ converges to zero if and only if there exists an integer m such that $u^{(n)} \in X^m K[[X]]$ for all n and for each integer i, the sequence $a_i^{(n)}$ is converging to $0 \in K$ with respect to the topology on K.*

Recall the following definition from general point-set topology.

Definition 8.1.1. *Let T_1 and T_2 be two topological spaces. A function*

$$f : T_1 \to T_2$$

is said to be sequentially continuous, if for every sequence $(x_n)_{n\in\mathbb{N}}$ in T_1 converging to x the corresponding sequence $(f(x_n))_{n\in\mathbb{N}}$ in T_2 converges to $f(x)$.

Proposition 8.1.1. *Multiplication in the topology defined on $K((X))$ is a sequentially continuous map.*

Proof. Let α_n, β_n be two sequences in $K((X))((Y))$ converging to 0. We will show that the product sequence $\alpha_n\beta_n$ also converges to 0. Let

$$\begin{aligned} \alpha_n &= \sum_{i\gg-\infty}^{\infty} f_i^{(n)} X^i, \\ \beta_n &= \sum_{i\gg-\infty}^{\infty} g_i^{(n)} X^i. \end{aligned} \tag{8.1}$$

Since α_n and β_n are converging to zero, by the previous lemma, we see that for a fixed i, the sequences $f_i^{(n)}$ and $g_i^{(n)}$ converge to zero and there exists $m \in \mathbb{Z}$ such that α_n, β_n is contained in $X^m K[[X]]$, for all n. Let

$$\alpha_n\beta_n = \theta_n = \sum_{i\gg-\infty}^{\infty} \Big(\sum_{k+l=i} f_k^{(n)} g_l^{(n)} \Big) X^i.$$

It is clear $\theta_n \in X^{2m}K[[X]]$. So, in order to show that θ_n tends to zero, we just have to show that, for a fixed i, the sequence

$$\sum_{k+l=i} f_k^{(n)} g_l^{(n)}$$

converges to zero. Since K is sequentially compact we see that, for fixed l, k the sequence

$$f_k^{(n)} g_l^{(n)}$$

converges to zero. Since there is only finitely many k, l such that $k + l = i$ and $f_k^{(n)} \neq 0$ and $g_l^{(n)} \neq 0$. Thus the sum of them converges to zero. Hence we have the Proposition. $\square$

Remark 8.1.1. *Note that, the multiplication $\times$ on $K((X))((Y))$ is not a continuous binary operation.*

Lemma 8.1.2. *Let $(U_i)_{i\in\mathbb{Z}} \in \mathrm{C}$, then the topological closure $\overline{U}_{U_i}$ of the set U_{U_i} is equal to*

$$U_{\overline{U}_{i\mathbb{Z}}} := \{ \sum_{i \gg -\infty}^{\infty} a_i X^i : a_i \in \overline{U}_i \}.$$

Proof. First we will show that the set $U_{\bar{U}_i}$ is closed. In order to do this, we will show that its complement is open. Let $f = \sum_{i\gg-\infty}^{\infty} f_i X^i$ be a Laurent series in $(U_{\bar{U}_i})^c$. This means there exists an integer k such that $f_k \notin \overline{U}_k$. So there exists an open neighborhood V of zero in K such that the open set $f_k + V$ around f does not intersect with $\overline{U}_k$. So, the open set $f + U_{W_i}$ around f does not intersect with $\overline{U}_{\bar{U}_i}$, which means the complement of $U_{\bar{U}_i}$ is open, thus $U_{\overline{U}_i}$ is closed.
Let D be a closed set containing U_{U_i}. Since D is closed set containing U_{U_i} it contains all the limit points of U_{U_i}. Let $f = \sum_{i\gg-\infty}^{\infty} \bar{a}_i X^i \in U_{\bar{U}_i}$. So for each i there exists $a_i^{(n)} \in U_i$ converging to a_i. Then clearly, for a fixed i $a_i^{(n)}$ converges to zero in K and $f - f^{(n)} \in X^m K[[X]]$ for some integer m. Thus by the previous lemma $f - f^{(n)}$ converges to zero, which means $f^{(n)}$ tends to f. But this means any element in $U_{\bar{U}_i}$ is a limit point of the set U_{U_i}, thus we have the lemma. $\square$

Proposition 8.1.2. *Let K be a T_0 topological field. Then the topology on $K((X))((Y))$ defined as above is non-locally compact.*

Proof. Suppose $K((X))((Y))$ is locally compact. Then there exists an open neighborhood of 0 in $K((X))((Y))$ whose closure is compact. Let U be an open neighborhood of 0 in $K((X))((Y))$ with compact closure. Without loss of generality, we may assume that U is of the form $U_{(U_i^X)}$, where $(U_i^X) \in \mathrm{C}(K((X)))$. By the Lemma 8.1.2 we have $\overline{U} = U_{\overline{(U_i^X)}}$. Let $N \in \mathbb{N}$ be such that for all $k > N$ the set $U_k^X = K((X))$. The existence of such N follows from the definition of $\mathrm{C}(K((X)))$. Now consider the sequence

$$u^{(n)} = \sum_{i\gg-\infty}^{\infty} f_i^{(n)} Y^i$$

where $f_i^{(n)} = X^{-n}$. Since for any $n \in \mathbb{N}$ and $m \in \mathbb{Z}$ the sequence f_i^n is not contained in $X^m K[[X]]$ by the Lemma 8.1.1, f_i^n is not convergent. But this means that $u^{(n)}$ has no convergent subsequence, which contradicts with the fact that $\overline{U}$ is compact. Thus $K((X))((Y))$ is not locally compact.

$\square$

8.1.2 Topology on Laurent series $K\{\{X\}\}$

In this section, we further assume that K is a complete discrete valuation field with respect to the valuation v, with finite residue field $\overline{K}$. Moreover, we assume that the topological structure on K is define via the valuation v. Therefore, in this section K has a topological field structure.(See the introductory discussion in the beginning of this chapter.)

Note that $K\{\{X\}\}$ is 2-local field with the residue field $\overline{K\{\{X\}\}}$, which is equal to $\overline{K}((t))$. Consider the subclass $\mathrm{C}(K)$ of the sequences of neighborhoods of zero in K, where a sequence of neighborhoods $(U_i)_{i\in\mathbb{Z}}$ of zero in K is contained in C if and only if

1. The intersection of U_i contains a non-zero ideal $P_E(c)$.

2. For any ideal $P_E(l)$, there exists $s \in \mathbb{Z}$ such that $P_E(l) \subset U_i$.

For such a sequence U_i, put

$$U_{\{U_i\}} = \{\sum_{i \gg -\infty} a_i X^i : a_i \in U_i\}.$$

Then the collection of all such sets $U_{\{U_i\}}$ forms a base of neighborhoods of 0 in $K\{\{X\}\}$. The topology on $K\{\{X\}\}$ introduced in this way satisfies the properties listed in the previous section, where the proofs are almost identical.

8.1.3 Topology on a general n-local field

Let K an n-local field. We know by Parshin classification theorem that, K is a finite extension of a standard n-local field

$$K_n = k\{\{X_1\}\}\cdots\{\{X_m\}\}((X_{m+2}))\cdots((X_n)),$$

where k is a 1-local field. Introduce a topology on K to be the finite dimensional K_n-vector space topology on K. This is equivalent to say that the topology on K is homeomorphic to the product topology on $K_n^{[K:K_n]}$, induced from K_n, which is constructed inductively in the previous two subsections.

8.1.4 Properties of the sequential topology on K^+

In this subsection, we shall list the basic properties of the sequential topology introduced on the additive group K^+ of the n-local field K.

1. $(K, +, 0)$ is a complete and separated topological group.

2. If $n > 1$, then every base of neighborhoods of the identity element 0 is uncountable.

3. If $n > 1$, then the multiplication defined on K is sequentially continuous, but not continuous.

4. For each $c \in K - \{0\}$, the map

$$\begin{aligned} {}_c m : K &\rightarrow K \\ \alpha &\mapsto c\alpha, \end{aligned}$$

for every $\alpha \in K$, is a homeomorphism.

8.2 Topology on $K^\times$

As usual, we let K to be an n-local field with corresponding chain of complete discrete valuation fields

$$K = K_n, \cdots, K_1, K_0$$

with respective valuations $v = v_n, v_{n-1}, \cdots, v_1$. Furthermore, let $(t_n, \cdots, t_1)$ be a local system of parameters.

8.2.1 $char(K_{n-1}) = p$

In this case, define the topology on $K^\times$ via the isomorphism

$$K^\times \simeq \mathbb{Z}t_1 \oplus \cdots \oplus \mathbb{Z}t_n \oplus U_K$$

as follows. The topology on the piece U_K is given by the isomorphism

$$U_K \simeq T \oplus V_K,$$

where the group of principal units V_K has the induced topology from the sequential topology on K, and T has the discrete topology. Now, the topology on $K^\times$ is defined to be the product topology on

$$\mathbb{Z}t_1 \oplus \cdots \oplus \mathbb{Z}t_n \oplus U_K,$$

where the free abelian part

$$\mathbb{Z}t_1 \oplus \cdots \oplus \mathbb{Z}t_n,$$

has the discrete topology and U_K has the topology just defined above.

8.2.2 $char(K) = \cdots = char(K_{m+1}) = 0,\ char(K_m) = p$

In this case, we will introduce the topology on $K^\times$ again by the isomorphism

$$K^\times \simeq \mathbb{Z}t_1 \oplus \cdots \oplus \mathbb{Z}t_n \oplus U_K,$$

where the free abelian part

$$\mathbb{Z}t_1 \oplus \cdots \oplus \mathbb{Z}t_n,$$

has the discrete topology, and U_K has the weakest topology which makes the projection map

$$proj : U_K \to U_{K_{m+1}}$$

continuous. That is a set $U \subseteq U_K$ is open if and only if $proj(U)$ is open in $U_{K_{m+1}}$. Note that, this projection map sits in the exact sequence

$$1 \to 1 + P_K(1, \underbrace{0, \cdots, 0}_{n-m-2}) \to U_K \to U_{K_{m+1}} \to 1.$$

8.2.3 Properties of the sequential topology on $K^\times$

In this subsection, we shall list the basic properties of the sequential topology introduced on the multiplicative group $K^\times$ of the n-local field K.

1. $K^\times$ is a complete topological space.

2. Multiplication on $K^\times$ is sequentially continuous, but not continuous for $n > 2$.

3. If $n \leq 2$ then $K^\times$ is a topological group with a countable base of open subgroups.

8.3 Final remark

In this chapter, we have seen that the topologies introduced on K^+ and $K^\times$ are *not* locally compact. Thus there is no Haar measure on the n-local field K nor on $K^\times$. It is an important open problem to develop a theory of "abstract harmonic analysis" on n-local fields (that is, an alternative Tate's thesis for n-local fields).

Chapter 9

Kato-Zhukov ramification theory of n-local fields

In this chapter we shall review the higher ramification theory of an n-local field K.

In the case $n = 1$, that is K is a finite extension of the basic fields $K = \mathbb{F}_q((X))$ or $\mathbb{Q}_p$, there exists a beautiful theory of ramifications (c.f. [9]). Namely, for a finite Galois extension L/K with corresponding Galois group G, there exists a nice lower filtration $(G_i)_{i \in \mathbb{R}_{\geq -1}}$ of G, which behaves well with sub-extensions of L/K. Also, there exists an upper filtration $(G^i)_{i \in \mathbb{R}_{\geq -1}}$ which is defined by a piece-wise linear continuous function $\psi_{L/K}$ defined on $\mathbb{R}_{\geq -1}$ called the Hasse-Herbrand function of the extension L/K and the lower filtration of G, which behaves well with the sub-extensions of L/K. The most important property of lower and upper ramification filtration is that the local Artin reciprocity map which is a continuous bijection from $G = Gal(L/K)$ to $K^\times/N_{L/K}(L^\times)$ defines a bijective correspondence between upper filtration of G and the higher unit groups $U^i(K)$ of $K^\times$. (Note that both filtration $(G^i)_{i \in \mathbb{R}_{\geq -1}}$ and $(U^i(K))_{i \in \mathbb{R}_{\geq -1}}$ form a basis of neighborhoods in G and $K^\times$ respectively.)

The higher ramification theory of 2-local fields started with the investigation of V. G. Lomadze in [6] and improved by K. Kato, T. Saito in [5] and by I. Zhukov in [11]. The higher ramification theory of a general n-local field is still fragmentary. The task of this chapter is to summarize the theory for a general n-local field.

9.1 Integration on totally ordered $\mathbb{Q}$-vector spaces

Definition 9.1.1. *A totally ordered abelian group Γ with a $\mathbb{Q}$ action is called a totally ordered vector space over $\mathbb{Q}$ if the order structure of Γ is compatible with the $\mathbb{Q}$ action. That is, for all $\alpha, \beta \in \Gamma$,*

$$\alpha \leq \beta \Longleftrightarrow q\alpha \leq q\beta \quad \forall q \in \mathbb{Q}_{\geq 0}.$$

Given an ordered set Γ. For $\alpha \in \Gamma$ we denote the possibly the empty set $\{\beta \in \Gamma : \beta < \alpha\}$ by $(-\infty, \alpha)$. The set (α, ∞) is defined in the same way.
Given a step function $g : \Gamma \to \mathbb{Q}$, the support *supp*$(g)$ of g is defined to be the set

$$supp(g) = \{\alpha \in \Gamma : g(\alpha) \neq 0\}.$$

Definition 9.1.2. *Let Γ be a totally ordered $\mathbb{Q}$-vector space. A function*

$$g : \Gamma \to \mathbb{Q}$$

is called a step function, if there exists an increasing finite sequence

$$\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n$$

of elements in Γ, such that the restricted functions $g|_{I_i}$, where $I_i = (\alpha_{i-1}, \alpha_i)$ for $i = 1, \cdots, n$; and $g|_{(-\infty,\alpha_0)}$, $g|_{(\alpha_n,\infty)}$ are all constant functions.

Let $g : \Gamma \to \mathbb{Q}$ be a step function. For $\alpha, \beta \in \Gamma$, such that $\alpha \leq \beta$, we define the definite integral $\int_\alpha^\beta g(x)dx \in \Gamma$ by

$$\int_\alpha^\beta g(x)dx = \sum_{i=1}^n c_i(\alpha_i - \alpha_{i-1}),$$

where $\alpha_0 = \alpha, \cdots, \alpha_n = \beta \in \Gamma$ is any increasing finite sequence, such that the restricted function to the interval (α_{i-1}, α_i) is constant with the value $c_i \in \mathbb{Q}$ for $i = 1, \cdots, n$.

Lemma 9.1.1. *If $g : \Gamma \to \mathbb{Q}$ is any step function, and if $\alpha, \beta \in \Gamma$, such that $\alpha \leq \beta$, then there exists a finite increasing sequence $\alpha_0 =\leq \alpha_1, \cdots \leq \alpha_n = \beta$ in Γ such* *that the function $g|_{(\alpha_{i-1},\alpha_i)}$ is a constant function for $i = 1, \cdots, n$. Moreover, the definite integral $\int_\alpha^\beta g(x)dx \in \Gamma$ along α to β is independent of the choice of the finite increasing sequence $\alpha_0 = \alpha \leq \cdots \leq \alpha_n = \beta$. If $\theta \in \Gamma$ with $\beta \leq \theta$, then*

$$\int_\alpha^\theta g(x)dx = \int_\alpha^\beta g(x)dx + \int_\beta^\theta g(x)dx.$$

For $\alpha, \beta \in \Gamma$ such that $\alpha \leq \beta$, the definite integral $\int_\beta^\alpha g(x)dx \in \Gamma$ is defined by

$$\int_\beta^\alpha g(x)dx = -\int_\alpha^\beta g(x)dx.$$

Let $g : \Gamma \to \mathbb{Q}$ be a step function. Suppose that $supp(g)$ is bounded from above, that is, there exists $\beta \in \Gamma$, such that for all $\alpha \in supp(g)$ we have $\alpha \leq \beta$. In this case, for $\alpha \in \Gamma$, we can define the improper integral $\int_\alpha^\infty g(x)d(x)$ along the interval (α, ∞) as follows:

$$\int_\alpha^\infty g(x)d(x) = \int_\alpha^\beta g(x)d(x),$$

where β is an upper bound for the support of g. Similarly, for a step function $g : \Gamma \to \mathbb{Q}$ whose support is bounded below, for $\alpha \in \Gamma$, the improper integral $\int_{-\infty}^\alpha g(x)d(x)$ is defined in the obvious way.

Definition 9.1.3. *A function $h : \Gamma \to \Gamma$ is said to be a quasi-linear (piecewise linear), if there exists a finite increasing sequence $\alpha_0 \leq, \cdots \leq \alpha_n$ in Γ, such that the restricted function $h_i = h|_{(\alpha_{i-1},\alpha_i)}$ has the form*

$$h_i(\alpha) = q_i\alpha + r_i,$$

for every $\alpha \in (\alpha_{i-1}, \alpha_i)$, where $q_i, r_i \in \mathbb{Q}$.

Remark 9.1.1. *Inverse of a bijective quasi-linear map is also quasi-linear.*

Let S denote the $\mathbb{Q}$-linear space of all $\mathbb{Q}$-valued step functions on Γ. Let L denote the $\mathbb{Q}$-linear space of all quasi-linear functions on Γ. For a fixed $\alpha \in \Gamma$, consider the mapping

$$I_\alpha : S \to L$$

by

$$[I_\alpha(g)](\beta) = \int_\alpha^\beta g(x)dx$$

for every $\beta \in \Gamma$. Note that, the image, $I_\alpha(g) : \Gamma \to \Gamma$ is indeed a quasi-linear map. In fact, if g is a step function with the corresponding finite increasing chain of elements

$$\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n$$

in Γ, where g has a constant value c_i on the interval (α_{i-1}, α_i). Then

$$[I_\alpha(\theta)] = q_i\theta + r_i$$

for every $\theta \in (\alpha_{i-1}, \alpha_i)$, where $q_i, r_i \in \mathbb{Q}$ such that

$$q_i = g(\alpha_{i-1})(\alpha_{i-1}, \alpha),$$

and

$$r_i = \int_\alpha^{\alpha_{i-1}} g(x)dx,$$

which means the image $[I_\alpha]$ is really a quasi-linear map. Now, consider the composite map

$$S \xrightarrow{I_\alpha} L \xrightarrow{can.} L/L_0,$$

where L_0 is the subspace of all constant functions. We claim that the kernel S_0 of this map is equal the subspace of S consisting of all functions $\Gamma \to \mathbb{Q}$ with finite support. Let $g \in S$ be a step function, such that $g(\alpha_i) = c_i$ for $\alpha_1 \leq \cdots \leq \alpha_n$ in Γ, and $g(\alpha) = 0$ if $\alpha \neq \alpha_i$ for $i = 1, \cdots, n$. Then clearly $\int_\alpha^\beta g(x)dx = 0$, for every $\alpha, \beta \in \Gamma$. Conversely, suppose $g \in S$ be such that for all $\beta \in \Gamma$, $\int_\alpha^\beta g(x)dx = q$, for some $q \in \mathbb{Q}$, and suppose that support of g is not finite. But, then there exist $\alpha_1, \alpha_2 \in \Gamma$ with $\alpha_1 \leq \alpha_2$ such that (α_1, α_2) has infinitely many elements, and the value of g on (α_1, α_2) is constant, which is non-zero, say q. Then

$$\begin{aligned} [I_\alpha(g)](\alpha_1) - [I_\alpha(g)](\alpha_2) &= \int_\alpha^{\alpha_1} g(x)dx - \int_\alpha^{\alpha_2} g(x)dx \\ &= \int_{\alpha_1}^{\alpha_2} g(x)dx \\ &= q(\alpha_1) - \alpha_2 \\ &\neq 0, \end{aligned}$$

which means the kernel is equal to the set of those elements whose supports are finite. To sum up, we have the following lemma.

Lemma 9.1.2. *The linear map*

$$I_\alpha : S \to L$$

given by

$$[I_\alpha(g)](\beta) = \int_\alpha^\beta g(x)dx$$

induces an isomorphism of $\mathbb{Q}$-linear spaces between the spaces S/S_0 and L/L_0.

Lemma 9.1.3. *Let $g : \Gamma \to \mathbb{Q}$ be a step function, such that $g(\beta) > 0$, for all $\beta \in \Gamma$. Then the quasi-linear map $I_\alpha(g) \in L$ is a bijection.*

In the light of the lemma 9.1.2, from now on, by a $\mathbb{Q}$-valued step function on Γ, we mean an equivalence class of step functions with respect to the equivalence relation defined modulo S_0.

Remark 9.1.2. *(i) Since the induced linear map, again denoted by*

$$I_\alpha : S/S_0 \to L/L_0,$$

which is defined by

$$I_\alpha(g)(\beta) = \int_\alpha^\beta g(x)dx + L_0$$

for every $g \in S/S_0$ is bijective, it has a linear inverse D_α,

$$D_\alpha : L/L_0 \to S/S_0,$$

which we call it the derivative map.

(ii) The Lemma 9.1.3 also applies in this new formulation of the mapping I_α.

(iii) The formal properties of the usual integration and derivation theory remains valid. Namely, the chain rule, change of variables etc. formulas applies to I_α and D_α.

9.2 Upper and lower ramification groups (Abstract theory)

In order to develop higher ramification theory of n-local fields we have to generalize the Hasse-Herbrand function and introduce higher ramification groups in that general context.

In this section we fix a totally ordered $\mathbb{Q}$-vector space Γ, and a finite group G.

Definition 9.2.1. *An upper (resp. lower) filtration on G by Γ is defined to be a family of normal subgroups $(G^\alpha)_{\alpha\in\Gamma}$ (resp. $(G_\alpha)_{\alpha\in\Gamma}$) indexed by Γ subject to the following conditions: Let $\alpha, \beta \in \Gamma$,*

1. *$G^\alpha \subseteq G^\beta$ (resp. $G_\alpha \subseteq G_\beta$) whenever $\alpha \leq \beta$;*
2. *$G = G^0$ (resp. $G = G_0$);*
3. *For each $1 \neq \sigma \in G$, the set $\{\alpha \in \Gamma : \sigma \in G^\alpha\}$ (resp. $\{\alpha \in \Gamma : \sigma \in G^\alpha\}$) has a maximal element with respect the total ordering on Γ.*

Lemma 9.2.1. 1. *Let $\{G_\alpha\}_{\alpha\in\Gamma}$ be a lower filtration of G defined by Γ. Define*

$$\varphi_G = \varphi : \Gamma \to \Gamma$$

defined by

$$\varphi(\beta) = \int_0^\beta |G_\alpha| d\alpha$$

for every $\beta \in \Gamma$. Then φ is a bijective quasi-linear function.

2. *The inverse function*

$$\psi_G = \psi = \varphi^{-1} : \Gamma \to \Gamma,$$

which is a bijective quasi-linear map by Remark 9.1.1, is explicitly defined by

$$\psi(\beta) = \int_0^\beta |G^\alpha|^{-1} d\alpha,$$

for every $\beta \in \Gamma$.

3. Let $(G_\alpha)_{\alpha\in\Gamma}$ be a lower filtration on G by Γ. Define

$$G^\alpha = G_{\varphi^{-1}(\alpha)},$$

for each $\alpha \in \Gamma$. This defines an upper filtration of G by Γ.

4. Let $(G^\alpha)_{\alpha\in\Gamma}$ be an upper filtration on G by Γ. Define

$$G_\alpha = G^{\psi^{-1}(\alpha)},$$

for each $\alpha \in \Gamma$. This defines a lower filtration of G by Γ.

Proof. The first part follows by Lemma 9.1.3. For the second part observe that

$$\begin{aligned}
\int_0^{\varphi(\beta)} |G^\alpha|^{-1} d\alpha &= \int_0^{\varphi(\beta)} |G_{\varphi^{-1}(\alpha)}|^{-1} d\alpha \\
&= \int_0^{\beta} |G_a|^{-1} \varphi'(a) da \\
&= \int_0^{\beta} |G_a|^{-1} |G_a| da \\
&= \int_0^{\beta} da \\
&= \beta.
\end{aligned}$$

The first equality comes from the very definition of upper filtration. As for the second equality we use the change of variables. Namely, we substitute $\varphi^{-1}(\alpha)$ by a. The third equality follows from the Remark 9.1.2. □

Definition 9.2.2. *Let $(G_\alpha)_{\alpha\in\Gamma}$ be a lower filtration on G by Γ. The mapping*

$$\varphi : \Gamma \to \Gamma$$

defined by

$$\varphi(\beta) = \int_0^\beta |G_\alpha| d\alpha$$

for every $\beta \in \Gamma$, and the inverse map

$$\psi = \varphi^{-1} : \Gamma \to \Gamma,$$

defined by

$$\psi(\beta) = \int_0^\beta |G^\alpha|^{-1} d\alpha,$$

for every $\beta \in \Gamma$ are called the Hasse-Herbrand functions with respect to the lower filtration $(G_\alpha)_{\alpha\in\Gamma}$ of G by Γ.

Let $(G_\alpha)_{\alpha\in\Gamma}$ be a lower filtration on G by Γ, and let H be a subgroup of G. Introduce:

(i) an induced lower filtration on H by Γ as

$$H_\alpha = H \cap G_\alpha,$$

for every $\alpha \in \Gamma$;

(ii) an induced upper filtration on H by Γ as

$$(G/H)^\alpha = (G^\alpha H)/H,$$

for every $\alpha \in \Gamma$, where G^α is defined by the lower filtration and the Hasse-Herbrand function $\psi_G : \Gamma \to \Gamma$.

Proposition 9.2.1. *Let H be a normal subgroup of G. Let φ_G, ψ_G, respectively φ_H, ψ_H and $\varphi_{G/H}$, $\psi_{G/H}$ be the corresponding Hasse-Herbrand functions on Γ. Then the following transitivity laws hold:*

(i) $\varphi_G = \varphi_{G/H} \circ \varphi_H$;

(ii) $\psi_G = \psi_H \circ \varphi_{G/H}$.

We know that

$$H^{\varphi_H(\alpha)} = H_\alpha = G_\alpha \cap H,$$

which means

$$\begin{aligned} H^\alpha = H_{\psi_H(\alpha)} &= G_{\psi_H(\alpha)} \cap H \\ &= G^{\varphi_G(\psi_H(\alpha))} \cap H \\ &= G^{\varphi_{G/H} \circ \varphi_H(\psi_H(\alpha))} \cap H \\ &= G^{\varphi_{G/H}(\alpha)} \cap H. \end{aligned}$$

9.3 Kato-Swan conductor (Abstract theory)

We keep the notation and assumptions of the previous section.

Let (G_α) be a lower filtration on G by Γ, and let G^α be the associated upper filtration of G by Γ defined by the Hasse-Herbrand function $\psi_G : \Gamma \to \Gamma$. In what follows, we fix a field F, and impose the following condition on the lower filtration of G by Γ:

$$|G_\alpha| \nmid char(F)$$

for every $\alpha \in \Gamma$. (For example, if $F = \overline{\mathbb{Q}}_\ell$, $F = \overline{\mathbb{Q}}_p$, or $F = \mathbb{C}$ this condition is automatically satisfied.)

Let F be a finite dimensional F-vector space, and let

$$\rho : G \to Aut_F(V)$$

be a group homomorphism. That is, ρ is a representation of G in the vector space V over F. Then V can be viewed as a FG-module of finite type via the arrow ρ. Recall that the G_α-invariant subspace of V is defined by

$$V^{G_\alpha} = \{v \in V : \rho(v)(h) = h,\ \forall h \in G_\alpha\},$$

for every $\alpha \in \Gamma$.

Definition 9.3.1. *The Kato-Swan conductor of the representation $\rho : G \to Aut_F(V)$, with respect to the lower filtration $(G_\alpha)_{\alpha\in\Gamma}$ on G by Γ, is defined to be the value $ksw(V) \in \Gamma$ given by the integral*

$$ksw(V) = \int_0^\infty |G_\alpha| \dim_F(V/V^{G_\alpha})d\alpha.$$

Note that, for a representation $\rho : G \to Aut_F(V)$, the Kato-Swan conductor can be reformulated as

$$ksw(V) = \int_0^\infty \dim_F(V/V^{G^\alpha})d\alpha,$$

by changing the variables.

Now, in the remaining of this section, we fix a lower filtration $(G_\alpha)_{\alpha\in\Gamma}$ on G by Γ. Basic properties of the Kato-Swan conductor, with respect to this fixed filtration, of a given representation are as follows:

Lemma 9.3.1. *Consider the exact sequence of FG-modules of finite type*

$$0 \to V' \to V \to V'' \to 0.$$

Then

$$ksw(V) = ksw(V') + ksw(V'').$$

Lemma 9.3.2. *Let H be a subgroup of G. Let*

$$\xi : H \to Aut_F(W)$$

be a representation of H in a finite dimensional F-vector space W. For the induced module $V = Ind_H^G(W)$, the induced representation

$$Ind_H^G(\xi) : G \to Aut_F(V),$$

has the Kato-Swan conductor

$$ksw_G(V) = (G : H)ksw_H(W) + \dim_F(V)ksw_G(F(G/H)),$$

where $ksw_G(V)$ is defined with respect to the fixed filtration on G by Γ, and $ksw_H(W)$ is defined with respect to the induced lower filtration on H by Γ.

In the previous lemma, $F(G/H)$ denotes the F-vector space with natural basis $\{gH : g \in R(G/H)\}$, where $R(G/H)$ denotes a complete set of coset representatives for H in G. Note that G acts on the natural basis of $F(G/H)$ in the obvious way. Therefore, there exists a representation (regular representation)

$$r : G \to Aut_F(F(G/H)).$$

So the Kato-Swan conductor $ksw_G(F(G/H))$ with respect to the fixed filtration on G by Γ, is defined.

Lemma 9.3.3. *Let H be a normal subgroup of G, and let $\overline{\tau} : G/H \to Aut_F(W)$ be a representation of G/H in a finite dimensional F-vector space W. Then there exists a natural representation τ of G in W over F defined by the composition*

$$\tau : G \xrightarrow{can} G/H \xrightarrow{\overline{\tau}} Aut_F(W),$$

which has Kato-Swan conductor

$$ksw_G(W) = ksw_{G/H}(W),$$

where the Kato-Swan conductor $ksw_G(W)$ is defined with respect to the fixed filtration on G by Γ, while the Kato-Swan conductor $ksw_{G/H}(W)$ is defined with respect to the lower filtration on G/H by Γ, induced by the fixed lower filtration on G.

9.4 Case of n-local fields

In this section we fix an n-local field K with respect to the rank n discrete valuation $v = (v_n, \cdots, v_1)$, and a finite Galois extension L/K, with corresponding Galois group $Gal(L/K) = G$. Recall that, there exists a natural n-local field structure on L, given by the valuation $w = (w_n, \cdots, w_1)$, where

$$w_i = \frac{1}{f_i} v_i \circ N_{L/K},$$

for $i = 1, \cdots, n$ (cf. Proposition 6.5.1). We further assume that

(i) The integral closure W of $\mathcal{O}_K$ in L is a valuation ring;

(ii) $W = \mathcal{O}_K[a]$ for some $a \in W$.

Such extensions are called *well-ramified* extensions.

In what follws, we shall introduce a lower filtration on $G = Gal(L/K)$ by $\Gamma = \mathbb{Q}^n$, which is a totally ordered $\mathbb{Q}$-vector space with respect to the lexicographic ordering in the sense of Zhukov, as follows. For $\alpha \in \Gamma$, let G_α be the normal subgroup of G defined by

$$G_\alpha = \{\sigma \in G : w(\sigma(a)a^{-1} - 1) \geq \alpha\},$$

where $W = \mathcal{O}_K[a]$. Now, we claim that, the collection $(G_\alpha)_{\alpha \in \Gamma}$ is a lower filtration on G by Γ. Normality of G_α in G follows easily as the extension L/K is a well-ramified extension. For $\alpha, \beta \in \Gamma$, it is clear that

$$G_\alpha \subseteq G_\beta,$$

whenever $\alpha \geq \beta$. Moreover the subgroup $G_0 = G$ as $w(\sigma(a)) = w(a)$. Thus it remains to prove the following lemma:

Lemma 9.4.1. *Let $\sigma \in G - \{id_L\}$. Then the maximal element μ_σ of the set*

$$\{\alpha \in \Gamma : \sigma \in G_\alpha\}$$

is

$$\mu_\alpha = w(\sigma(a)a^{-1} - 1).$$

Proof. Clearly follows by the definition. □

Therefore, the collection $(G_\alpha)_{\alpha\in\Gamma}$ is a lower filtration on G by Γ. Now, following the lines of the abstract theory, we can define the corresponding Hasse-Herbrand functions φ_G and ψ_G on Γ, and the associated upper filtration on G by Γ, defined by the lower filtration and the Hasse-Herbrand function ψ_G.

In this setting the Kato-Swan conductor has the following form. Let

$$\rho : G \longrightarrow Aut_F(V)$$

be a Galois representation in a finite dimensional F-vector space V. Then the corresponding Kato-Swan conductor $ksw_G(V)$ with respect to the lower filtration $(G_\alpha)_{\alpha\in\mathbb{Q}^n}$ on G by $\mathbb{Q}^n$ is defined by

$$ksw_G(V) = \int_{\underline{0}}^{\infty} |G_\alpha| \dim_F(V/V^{G_\alpha})d\alpha,$$

where $\underline{0}$ is the zero vector in $\mathbb{Q}^n$.

9.5 Hasse-Arf theorem for n-local fields

In this section we shall state the generalization of the celebrated theorem of H. Hasse and C. Arf on 1-local fields to n-local fields. We shall follow the notation of the previous two sections. Furthermore, we shall assume that the extension L/K is an abelian extension. That is, the corresponding Galois group $G = Gal(L/K)$ is an abelian group.

Theorem 9.5.1 (Hasse-Arf generalized by K. Kato and T. Saito). *Let*

$$\rho : G \to Aut_F(V)$$

be any Galois representation in a finite dimensional F-vector space V. Then the Kato-Swan conductor $ksw_G(V)$ defined with respect to the lower filtration $(G_\alpha)_{\alpha \in \mathbb{Q}^n}$ introduced in the previous section satisfies

$$ksw_G(V) \in \mathbb{Z}^n.$$

Proof. For the proof in two dimensional case (cf([5])). For a sketch of proof in the general case ($cf.$[11]). $\square$

9.6 Final remark

The theory of Kato-Swan conductors is the generalization of the theory of Artin conductors to n-local fields. Artin conductors are important in the analysis of L-functions attached to Galois representations of global fields. It is conjecturally expected that, Kato-Swan conductors will play an important role in the analysis of L-functions attached to the Galois representations of higher dimensional global fields; that is, representations of the fundamental group of schemes. However, this general setting seems to be much richer than the classical theory, as the recent research predicts that, there are finer and more general ramification invariants, generalizing Kato-Zhukov theory.

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