ON CONTINUITY OF MASKIN'S IMPLEMENTATION RESULT

by

NUH AYGÜN DALKIRAN

Submitted to the Social Sciences Institute
in partial fulfillment of the requirements for the degree of
Master of Art

Sabancı University

July 2006

ON CONTINUITY OF MASKIN'S IMPLEMENTATION RESULT

APPROVED BY

Assist. Prof. Dr. Mehmet BARLO
(Thesis Supervisor)

Prof. Dr. Mehmet BAÇ

Assoc. Prof. Dr. Ali ÇARKOĞLU

DATE OF APPROVAL: 10/07/2006

to my family

&

to bitanem

Acknowledgements

First of all, I would like to thank to my thesis supervisor, Mehmet Barlo, for his invaluable guidance throughout this thesis. My work would not have been possible without his motivation and brilliant ideas. I would like to also express my gratitude to him for our enjoyable off-class conversations. I am also appreciative to my thesis jury members, Mehmet Baç and Ali Çarkoglu for their helpful comments about my thesis.

Special thanks also go to my professor Özgür Kıbrıs for his motivation and encouragement during my graduate study at Sabancı University. All of my other Sabancı University Professors are also deserving of my gratitude for everything they have ever taught me.

Alexandru Balas, my roommate, additionally deserves a strong thank-you for both our academic and non-academic conversations of the last two years. All colleagues and friends I made at Sabancı University are equally entitled to my appreciation for their invaluable contribution to the two years of my life at Sabancı University.

I am also thankful to TÜBİTAK "The Scientific & Technological Research Council of Turkey" for their financial support as a scholarship.

Finally, my family deserves infinite thanks for their encouragement and endless support throughout my education. Last but not least, final thanks goes to Fatma Nur Çaplı for her understanding and patience throughout the preparation of this thesis.

ON CONTINUITY OF MASKIN'S IMPLEMENTATION RESULT

Nuh Aygün DALKIRAN
Economics, MA Thesis, 2006
Supervisor: Mehmet BARLO

Abstract

This thesis displays that the seminal results of Maskin (1999) on Nash implementation are continuous with respect to a specific measure when attention is restricted to the domain of preferences representable by cardinal utility functions. Our continuity measure is associated with three modified components of Maskin's results: epsilon-implementability, epsilon-monotonicity and epsilon-no veto power. Employing cardinal utility functions, we define epsilon-neighborhoods around Maskin's standard components and show that his results continue to hold with this epsilon-approximation.

Keywords: Nash implementation, epsilon-equilibrium, Maskin monotonicity.

MASKİN'İN UYGULAMA SONUCUNUN SÜREKLİLİĞİ ÜZERİNE

Nuh Aygün DALKIRAN
Ekonomi, Yüksek Lisans Tezi, 2006
Tez Danşmanı: Mehmet BARLO

Özet

Bu tezde, Maskin'in Nash uygulaması için bulduğu temel sonucların sürekliliği incelenmektedir. Bunun için kullandığımız süreklilik ölçüsü Maskin (1999)'in sonuçlarının kaynağı olan üç temel unsurun uygun şekilde değiştirilmesi ile elde edilmiş üç yeni nosyona dayanmaktadır. Bu üç yeni unsur sırasıyla: epsilon-dengesi, epsilon-monotonisite ve epsilon-veto hakkı olmamasıdır. Bu unsurlar standard unsurların epsilon komşuluklarıdır. Bulduğumuz sonuç göstermektedir ki kardinal fayda fonksiyonları kullanıldığında Maskin'in sonucu bu epsilon yaklaşım için de geçerli olmaktadır.

Anahtar Sözcükler: Nash uygulaması, epsilon-dengesi, Maskin monotonisitesi.

Contents

Chapter 1

Introduction

A society is a group of individuals distinguishable from other groups by mutual interests, characteristic relationships, shared institutions, and a common culture. Here what is meant by a society is merely a collection of individuals with certain characteristics. We often refer to these individuals as agents of the society. A society faces many economic, social, and political situations where individuals must interact to make decisions that may affect them collectively. Voting to elect representatives, choosing a public policy, as well as production and allocation of private and/or public goods are some common examples. These kind of situations in which a society has to decide among the available alternatives are known as *social decision problems.*

We will assume that the objectives of a society are represented by a social choice rule depending on the social decision problem and the characteristics of the society. A *social choice rule* can be regarded as a rule agreed by the members of the society or designed by a social planner according to some normative characteristics; and it selects a feasible set of alternatives among all alternatives available to the society depending on the characteristics of the society. One can regard the social choice rule as the set of socially optimal alternatives. Examples range from the Pareto rule which selects Pareto optimal alternatives, to the Walrasian rule which is a social choice rule selecting the competitive equilibrium allocations.

If the relevant characteristics of the society are publicly known, then social choice rule outcomes can be obtained easily. The problem of implementation arises because the true preference profile of the society is generally not common knowledge. Hence, a social planner may have to elicit preferences of individuals in the society. However, there is then the problem of misrepresentation of preferences. Depending on the preference profile of the society and the social choice rule, individuals may act strategically to influence the outcome of a social decision problem to their advantage. Hence, the design of the institution through which individuals of a society interact has a profound effect on the strategic behavior of the individuals of that society.

A social choice rule is said to be implementable if a mechanism exists so that the equilibrium of this mechanism and the socially optimal alternatives indicated by the social choice rule coincide. To be more precise, given a normative goal characterized by a social choice rule, implementation theory deals with the characterization of mechanisms that will create outcomes consistent with the given normative goal hence with the social choice rule.[1]

Obviously, game theory plays a central role in implementation theory,

[1] More information on implementation theory can be found in the following surveys; Allen (1997), Corchón (1996), Groves and Ledyard (1987), Jackson (2001), Maskin (1985), Maskin and Sjöström (2002), Moore (1992), Moulin (1982), Palfrey (1992), Palfrey (2001), Palfrey and Srivastava (1993), and Postlewaite (1985)

since an institution is modeled as a mechanism which is nothing but a non-cooperative game form. In implementation theory, unlike the many applications in game theory, a game is not given but is to be identified, that is, rather than fixing a game and looking for the set of outcomes given by some solution concept, we fix a set of outcomes and look for a game that yields that set of outcomes as equilibria.

Another important point we should mention about implementation is the revelation principle which demonstrates that if standard concepts of equilibrium are used (Nash equilibrium, weak dominance or Bayesian Nash equilibrium) it is always possible to define a mechanism for an implementable social choice rule such that truthful revelation of preferences is an equilibrium of this mechanism. Such mechanisms are called *direct revelation mechanisms.*

An example at this point would clarify what implementation theory deals with. Consider a society that has to select a project among a set of projects. Each member of the society has a preference ranking over the set of projects. The society may have formed a certain normative goal which forms a social choice rule defining the project to be selected as a function of the preference profile of the society. In particular, the society may be unwilling to select a project ranked lower than another project by all members of the society. (i.e. a project which is Pareto dominated). The society may also wish to select a Condorcet winner[2] if it exists. Then the implementation problem would be: "Does there exist a procedure where for every possible preference profile of the society, the equilibrium outcome of the procedure would be Pareto

[2] An alternative is a Condorcet winner if it defeats any other alternative in a (pairwise) majority voting election.

efficient and Condorcet consistent?"

In order to render a positive answer to this question, one has to come up with a mechanism whose equilibria coincide with outcomes identified by the social choice rule for all possible preference profiles the agents may have. It should be noted that by a mechanism we mean a game form which specifies a set of possible actions to the members of the society and specifies the outcome as a function of these actions.

One of the most important problems considered by implementation theory is the full characterization of implementable social choice rules. That is to say: "Can we identify properties that precisely identify the social choice rules which are implementable and which are not implementable?"

As we have mentioned previously, the implementability of a social choice rule depends on the game theoretic equilibrium concept employed. Implementation theory has considered several equilibrium concepts so far. If we assume that the individuals behave in a non-cooperative manner, the equilibrium concept must be chosen among non-cooperative equilibrium concepts[3]. An important point in modeling the non-cooperative mode of behavior of the society is its information structure. If the information is incomplete, it is natural to restrict attention to weak dominance or Bayesian Nash equilibrium. However, if we assume complete information, one of the most prominent equilibrium concepts is Nash Equilibrium.

One of the main results regarding implementation in Nash equilibrium

---

[3] A cooperative equilibrium concept can also be used; see for example Dutta and Sen (1991a) for strong Nash equilibrium, and Bernheim and Whinston (1987) for coalition proof equilibrium

is due to Maskin (1999).[4] He has found out that a condition called "monotonicity" is necessary for a social choice rule to be Nash-implementable and, with at least three agents, monotonicity coupled with a condition called "no veto power" is sufficient for Nash implementability[5].

The *monotonicity* condition says that in case of a change in the preference profile of the society, if in all agents' preference orderings, a socially optimal alternative does not fall below relative to any other alternative that it was not below before, then it remains socially optimal. The *no veto power* property, on the other hand, is a condition of near unanimity which says that if all but one agent has the same alternative top ranked, then that alternative must be socially optimal.

In this thesis, we prove the continuity of Maskin's main results when preferences of the society can all be represented by cardinal utility functions. The restrictions we put on the domain of preferences are due to the essence of our continuity measure; that is, the domain we consider is almost the most general domain of preferences where this continuity measure for implementation can be defined.

The continuity measure we define for Nash-implementability is due to two new conditions we define, namely, epsilon-monotonicity and epsilon-no veto power. These two conditions generalize the standard monotonicity and no veto power conditions. We define an equilibrium concept which we call epsilon-equilibrium and we prove that these two new conditions mimic the properties of monotonicity and no veto power in Nash implementation

[4]Maskin's article was circulated as a working paper in 1977.

[5]We will give a brief survey on Nash Implementation later in Chapter 3.

for epsilon-equilibrium. That is, epsilon-monotonicity turns out to be a necessity condition for epsilon-implementation (implementation in epsilon-equilibrium), and epsilon-no veto power coupled with monotonicity is sufficient for a social choice rule to be epsilon-implemented when there are at least three agents in the society.

The results we obtained are important because they may be used to define a distance notion for social choice rules in terms of Nash implementability. This may lead us to a notion which, given the mechanism, measures the sacrifice of a society when a non-Nash implementable social choice rule is to be implemented. Finally, we must confess that the construction of an appropriate example to present our results remains to be done.

The paper is organized as follows : Chapter 2 gives the definitions and the notation used throughout this thesis. Chapter 3 provides a short survey on Nash implementation. Chapter 4, introduces the preference domain we deal with, and defines our continuity measure for implementation on this domain. Finally, chapter 5 concludes the paper.

Chapter 2

Preliminaries

This chapter offers the basic definitions and the notations to be used later in this thesis.

Let $N = \{1, 2, ..., n\}$ denote a society with n agents where $i \in N$ denotes i^{th} agent in the society and A denote the non-empty set of alternatives (or outcomes) available to the society. (Note that A may be finite, denumerable or uncountable.)

The set of all complete preorders on A is denoted by $\mathcal{R}_{\mathcal{A}}$ (It is sometimes called the **unrestricted domain of preferences**.) where an element $R_i \in \mathcal{R}_{\mathcal{A}}$ is called the **preference ordering** of agent i on A.[1] (The set of all strict preference orderings on A is denoted by $\mathcal{P}_A$ [2]). A preference profile of the society is denoted by $R = \{R_1, R_2, ..., R_n\}$ where $R_i \in \mathcal{R}_A$ for all $i \in N$. The set of **all possible preference profiles** of the society is denoted

[1] aR_ib means agent i weakly prefers a to b. (i.e a is at least as high as b in the ordering R_i.

[2] A strict preference ordering is a negatively transitive and asymmetric binary relation. i.e. It ranks no two alternatives as indifferent.

by $\mathcal{R} = \prod_{i \in N} \mathcal{R}_i$ where $\mathcal{R}_i \subseteq \mathcal{R}_A$ is the set of **all possible preference orderings** of agent i on A.

We start with the definition of a social choice rule.

Definition 1 *A **social choice rule** (an **SCR**) $F : \mathcal{R} \twoheadrightarrow A$ is a correspondence from $\mathcal{R}$ into A, that is it selects a subset of A for each possible preference profile of the society: $F(R) \subseteq A$ for all $R \in \mathcal{R}$. If an alternative $a \in A$ is chosen by the social choice rule F under a preference profile $R \in \mathcal{R}$ i.e. $a \in F(R)$, we say that a is F**-optimal** with respect to R.*

As mentioned previously, a social choice rule is interpreted as selecting the "welfare optimal" alternatives $F(R)$ for each possible preference profile $R \in \mathcal{R}$ of the society.[3] Prominent examples of social choice rules include the Pareto Rule, $F^{PO} = \{a \in A|$ for all $b \in A$ there exists i such that $aR_ib\}$ which selects all Pareto Optimal alternatives given the preference profile R, the Condorcet Rule, $F^{CON} = \{a \in A|$ for all $b \in A$ $\#\{aR_ib\} \geqq \#\{aR_ib\}\}$, where $\#\{aR_ib\}$ denotes the number of individuals who prefer a to b, and it selects a (pairwise) majority voting winner for each profile R of **strict** preferences, and in a pure exchange economy of l goods, where an alternative means an allocation of goods across individuals (i.e. $a = (a_1, ..., a_n)$, where $a_i \in \mathbb{R}^l_+$), the Walrasian Rule F^W which, given individuals' endowments $(e_1, ..., e_n)$, chooses the set of competitive equilibrium allocations.

[3] A social choice rule differs from a **social welfare function** of Arrow (1951) in that it does not rank non-optimal alternatives. However, a social choice welfare function f induces a natural social choice rule, that is the correspondence which selects the alternatives top-ranked by f for each profile.

We continue with the definition of a mechanism and the definition of implementability of a social choice rule by a mechanism via an equilibrium concept.

Definition 2 *A (normal form)* ***mechanism*** μ *(or game form) is a pair* $\mu = (S = \times_{i\in N} S_i, g)$ *where* S_i *is the non-empty set denoting strategy space for each agent* $i \in N$ *and* $g : S \to A$ *is the outcome function. Note also that* (N, μ, R) *defines a normal form game to be played by the society if the preference profile of the society is* $R \in \mathcal{R}$

Definition 3 *A social choice rule* $F : \mathcal{R} \twoheadrightarrow A$ *is* ***implementable*** *by a* ***mechanism*** μ *via the* ***equilibrium concept*** Σ *if*

$$\Sigma(N, \mu, R) = F(R) \text{ for all } R \in \mathcal{R}$$

The definitions below give some of the technical terms which are mostly game theoretic and employed later in this thesis.

Definition 4 *An n-person* ***normal form game*** *is* $\Gamma = \langle N, (S_i)_{i\in N}, (u_i)_{i\in N}\rangle$ *where* S_i *is a non-empty set of strategies of player* $i \in N$*, and* $u_i : \times_{i\in N} S_i \to \mathbb{R}$ *is agent* i*'s utility or payoff function.*

Definition 5 *Let* $\mu = (S, g)$ *be a normal form mechanism. A strategy profile* $s^* \in \times_{i\in N} S_i$ *is called a* ***Nash Equilibrium of*** μ ***at*** R*, if for all* $i \in N$*, and for all* $s_i \in S_i$

$$g(s^*) R_i g(s_i, s^*_{-i}).$$

Definition 6 *Let* $\mu = (S, g)$ *be a normal form mechanism. A strategy* $s_i \in S_i$ *is a* ***dominant strategy*** *for* i *in the game* (N, μ, R) *if*

$$g(s_i, s_{-i})R_i g(s'_i, s_{-i}) \textit{ for all } s'_i \in S_i \textit{ and for all } s_-i \in \times_{j \neq i} S_j$$

*A strategy profile $s \in S$ is a **dominant strategy equilibrium** of the game (N, μ, R) if s_i is a dominant strategy for each i in the game (N, μ, R).*

Definition 7 *The **lower contour set** $L_i(a, R)$ of i at $a \in A$ under R is defined by*

$$L_i(a, R) = \{b \in A | aR_i b\}.$$

*The **strict lower contour set** $SL_i(a, R)$ of i at $a \in A$ under R is defined by*

$$SL_i(a, R) = \{b \in A | aP_i b\}.$$

where P_i is the strict preference ordering induced by R_i.

Definition 8 *A social choice rule $F : \mathcal{R} \twoheadrightarrow A$ is **dictatorial** if there exists $i \in N$ such that for all $R \in \mathbb{R}$ and $a \in A$, one has $aR_i b$ for all $b \in A$ implies $a \in F(R)$.*

Definition 9 *An outcome $a \in A$ is **Pareto optimal** with respect to the preference profile $R \in \mathcal{R}$ if there exists an outcome b such that $bP_i a$ for some $i \in N$ then there exists $i \neq j \in N$ such that $aP_j b$. A social choice rule $F : \mathcal{R} \twoheadrightarrow A$ is said to be **Pareto optimal** if, for all $R \in \mathcal{R}$ and $a \in F(R)$, the outcome a is Pareto optimal with respect to R.*

Definition 10 *A function $u_i : A \to \mathbb{R}$ from the alternative set into real numbers, **represents the preference ordering** R_i on A if; $aR_i b$ if and only if $u_i(a) \geq u_i(b)$ holds, for all $a, b \in A$. If there exists a function $u_i : A \to \mathbb{R}$ that represents a preference relation R_i on A then we say that R_i is **representable by the utility function** u.*

Chapter 3

Implementation in Nash Equilibrium

3.1 Introduction

This chapter provides a brief survey on Nash Implementation.

Nash implementation was first studied by Groves and Ledyard (1977), Hurwicz and Schmeidler (1978), and Maskin (1999)[1]. However, the most general results were obtained by Maskin. The following quotation from Jackson (2001) explains the importance of his works:

> The seminal work on Nash implementation, not only provides us an understanding of what is Nash implementable, but it also provides a blueprint for the techniques and approach that underlie many of the general characterization results in the literature.

[1]Recall that; this article was circulated as a working paper as Maskin (1977) and reprinted in 1999.

We mentioned before that Maskin identified two conditions, namely monotonicity and no veto power, where monotonicity turns out to be a necessary condition for a social choice rule to be Nash implementable and, when there are at least three agents in the society, monotonicity together with no veto power suffice for a social choice rule to be Nash implementable.

Two other important works on Nash implementation are Moore and Repullo (1990) and Danilov (1992). Moore and Repullo (1990) define a condition, which is called condition μ, and which turns out to be a necessary and sufficient condition for Nash implementability in case of three or more agents. Although their condition closes the gap between Maskin's necessity and sufficiency conditions, to determine whether or not a social choice rule satisfies this condition is difficult. Danilov (1992) gives an explicit formula for the system of sets which condition μ of Moore and Repullo is based on and he introduces the condition called essential monotonicity (a.k.a Danilov monotonicity) which is also a necessary and sufficiency condition for Nash implementability when there are at least three agents in the society. However, it should be noted that Moore and Repullo's approach works in a more general setting[2].

Our work in this thesis is based on the seminal work of Maskin (1999). Hence, in the next section, we will explore the conditions identified by Maskin thoroughly and we will present the proofs of the necessity and sufficiency theorems of Maskin (1999).

[2]For further information see Moore and Repullo (1990) and Danilov (1992)

3.2 Necessity and Sufficiency

We start with two equivalent definitions of "monotonicity". The first one is the original definition in Maskin (1999). We include the second one since it is more commonly used in the literature.

Definition 11 [3] *A social choice rule $F : \mathcal{R} \twoheadrightarrow A$ is **monotonic** if for all $a \in A$,*

$$\forall R, R' \in \mathcal{R} \text{ if } a \in F(R) \text{ and } \forall i \in N, \forall b \in A, aR_i b \Rightarrow aR'_i b, \text{ then } a \in F(R').$$

Definition 12 *A social choice rule $F : \mathcal{R} \twoheadrightarrow A$ is **monotonic** if and only if for all $a \in A$, for all $R, R' \in \mathcal{R}$ the following is true:*

$$\text{if } a \in F(R) \text{ and } L_i(a, R) \subseteq L_i(a, R') \ \forall i \in N, \text{ then } a \in F(R').$$

In other words, monotonicity calls for the social choice rule to satisfy the following property: If an alternative is chosen under a given preference profile by the social choice rule, then it must also be chosen when the preference profile is altered so that none of the alternatives beaten by the original one gets to be ranked higher than the original one in any of the agents' preference orderings; i.e. if the lower contour set of a socially optimal alternative does not shrink for any agent, then this alternative must remain being socially optimal. This seems to be an intuitive condition. It is also reasonable in the sense that, it is satisfied by the prominent social choice rules mentioned above, which are the Pareto Rule, F^{PO}; the Condorcet Rule, F^{CON}; and

[3] Monotonicity condition was called as "strong positive association" by Muller and Satterthwaite (1977).

the Walrasian Rule, F^W.[4] To be more precise, let us give a small argument which explains why the Pareto Rule satisfies monotonicity. Let $a \in A$ be a Pareto optimal alternative with respect to preference profile R, hence chosen by the Pareto Rule under R. This means for any other alternative $b \in N$, there exists an agent $i^* \in N$ such that,$aR_{i^*}b$. If we replace the preference profile R with R' such that for all $i \in N$, aR_ib implies aR'_ib, then $aR'_{i^*}b$ holds, therefore a is Pareto optimal with respect to R' as well, and hence it is chosen under R' by the Pareto Rule, F^{PO}.

On the other hand, some well-known social choice rules do not satisfy monotonicity. For example, the Borda Count Rule, F^{BC}, (i.e rank-order voting) fails to satisfy monotonicity. The Borda Count Rule works as follows: each individual assigns points to every alternative in the alternative set A so that the best alternative of each player get $\#A$ points, the second best of each individual gets $\#A - 1$ points, and so on. The alternatives who get the highest points in total are chosen by the Borda Count Rule. Now, to see why the Borda Count Rule fails to satisfy monotonicity, consider the following example:

Example 1 *Let* N=$\{1,2\}$ *and* A=$\{a,b,c,d\}$ *and let the preference profile* R *be as follows;*

- $aR_1bR_1cR_1d$*;*
- $dR_2cR_2aR_2b$*;*

[4]The Walrasian Rule is not monotonic in general but it is monotonic on a domain of preferences such that all competitive equilibria occur in the interior of the feasible set, see Hurwicz and E. Maskin (1995) for more detail.

Here, a gets 6 points and it is the only alternative chosen by F^{BC}. Now, consider the following preference profile R' defined as follows;

- $aR_1cR_1bR_1d$;
- $cR_2dR_2aR_2b$;

Note that, aR_ib implies aR'_ib but now c is the only alternative chosen by F^{BC}. This is a violation to monotonicity.

One may have doubts for the monotonicity condition, but as the theorem we will present after defining Nash implementability suggests, for a social choice rule to be Nash Implementable monotonicity is inescapable.

Definition 13 *A social choice rule $F : \mathcal{R} \twoheadrightarrow A$ is **implementable in Nash Equilibrium** if there exists a mechanism $\mu = (S, g)$ such that:*

1. *For every $R \in \mathcal{R}$ and for every $a \in F(R)$ there exists $s^* \in S$ such that s^* is a Nash equilibrium of μ at R and $g(s^*) = a$.*
2. *For every $R \in \mathcal{R}$ and for every $b \notin F(R)$, there does not exist $s^* \in S$ such that s^* is a Nash equilibrium of μ at R and $g(s^*) = b$.*

Requirement (1) in the definition of Nash implementability of a social choice rule F means that, there is a Nash equilibrium of μ corresponding to each F-optimal alternative. On the other hand requirement (2) means, every Nash equilibrium of μ is F-optimal.[5]. Together they imply that if F is Nash implementable by a mechanism μ then Nash equilibria of μ and F-optimal alternatives coincide.

[5]This is the contrapositive of what is stated in requirement (2) in the definition of Nash implementability.

Theorem 1 *If a social choice rule $F : R \twoheadrightarrow A$ is implementable in Nash Equilibrium then it is monotonic.*

Proof. The proof here is a modified version of the proof in Maskin (1999) Let $F : \mathcal{R} \twoheadrightarrow A$ be a Nash implementable social choice rule. Take $R, R' \in \mathcal{R}$ such that $a \in F(R)$ and assume aR_ib implies aR'_ib for all $i\ inN$ and for all $b \in A$

Since F is implementable via Nash Equilibrium, there exists a mechanism $\mu = (S, g)$ such that $g : S \rightarrow A$ where there exists $s \in S$ with $g(s) = a$ and $g(s)R_ig(s_i, s_{-i})$ for all $s'_i \in S_i$ for all $i \in N$.

By assumption, this implies $g(s)R'_ig(s_i, s_{-i})$ for all $s'_i \in S_i$ for all $i \in N$ which means s is a Nash Equilibrium with respect to R' and hence $g(s) \in F(R')$ i.e $a \in F(R')$. Therefore F is monotonic. ■

Monotonicity in terms of Nash implementability can be interpreted in two ways. The first is: if an alternative is to be implemented at one profile but not another, then it must have fallen in someone's rankings in order to break the Nash equilibrium via some deviation. Whereas the second interpretation is: if an alternative is implemented at one profile and rises in each individual's rankings at another preference profile, then the strategy profile leading to the alternative which forms a Nash equilibrium at the first profile must still be a Nash equilibrium profile at the second profile. These conditions are equivalent since they are the contra-positive of each other. Both of these interpretations are important. The first implies that there must exist some preference reversal if an equilibrium at one profile is broken at another. The second emphasizes that if the ranking of an equilibrium alternative improves for each agent then it must remain an equilibrium outcome, which is often used for checking if monotonicity is satisfied.

We continue with the sufficiency conditions for Nash implementation; we restrict ourselves to the case, where there are at least three agents[6].

The example below which is due to Maskin (1985) indicates that monotonicity itself is not sufficient for a social choice rule to be Nash implementable.

Example 2 ***A monotonic social choice rule which is not Nash implementable:***
Let $n = 3$, $A = \{a, b, c\}$, $\mathcal{R} = \mathcal{P}_{\mathcal{A}} \times \mathcal{P}_{\mathcal{A}} \times \mathcal{P}_{\mathcal{A}}$. *Consider the social choice rule* $F : \mathcal{R} \twoheadrightarrow A$ *such that, for any* $R \in \mathcal{R}$ *and* $x, y \in A$ *the following holds:* $x \in F(R)$ *if and only if*

$$x \text{ is Pareto optimal,}$$
$$\text{if } x \in a, b \text{ and } xP_1y \text{ for all } y \neq x;$$
$$\text{if } x = c \text{ then there exists } y \in A \text{ such that } xP_1y.$$

It is clear that F *is monotonic. Assume,* F *is Nash implementable by a mechanism* $\mu = (S, g)$. *Now consider the following profiles* $P, P', P'' \in \mathcal{R}$ *such that;*

- bP_1cP_1a; cP_2aP_2b; cP_3aP_3b
  $F(P) = \{b, c\}$
- $aP_1'bP_1'c$; $cP_2'bP_2'a$; $cP_3'aP_3'b$
  $F(P') = \{a\}$
- $bP_1''aP_1''c$; $aP_2''bP_2''c$; $aP_3''bP_3''c$
  $F(P'') = \{b\}$

[6] We will give a short discussion for the case of two agents later.

Since μ Nash implements F, there exists $s \in S$ such that s is a Nash equilibrium under P and $g(s) = c$. Because bP_1c, there does not exist $s'_1 \in S_1$ such that $g(s'_1, s_{-1}) = b$. Moreover, there does not exist $s'_1 \in S$ such that $g(s'_1, s_{-1}) = a$ because if exists, then (s'_1, s_{-1}) would be a Nash equilibrium under P'' with $g(s'_1, s_{-1}) = a$ contradicting $F(P'') = b$, but this implies s is a Nash equilibrium under P' with $g(s) = c$ contradicting $F(P') = a$. Therefore, F is not Nash implementable.

As the above example shows, monotonicity is not sufficient for a social choice rule to be Nash implementable. Thus we need additional conditions. Below we will define a new condition called "no veto power", and then, we will prove that monotonicity together with no veto power is sufficient for a social choice rule to be Nash implemented when there are at least three agents in the society.

Definition 14 *A social choice rule is said to satisfy the* ***no veto power*** *(NVP) property if there exists a player $j \in N$ such that for all the other players $i \neq j$, and for all $b \in A$, aR_ib implies $a \in F(R)$.*

No veto power is a near unanimity condition as mentioned before; it basically says that if all but one agent rank an alternative as first (i.e. as one of their best alternatives), then that alternative must be optimal for the society; hence it must be chosen by the social choice rule.

Now, we present the sufficiency theorem for Nash implementability when there are at least three agents in the society.

Theorem 2 *Let $n \geq 3$, if a social choice rule $F : \mathcal{R} \twoheadrightarrow A$ is monotonic and satisfies no veto power then F is Nash implementable.*

Proof. [7] The proof here is a modified version of Repullo (1987) and will be used as a blue print for the sufficiency proof in Chapter 4. We will construct a game form which implements F in Nash Equilibrium[8]. For each player $i \in N$ define the strategy space

$$S_i = \mathcal{R} \times A \times \mathbb{N}$$

That is, each agent $i \in N$ announces a triple consisting of a preference profile $R^i \in \mathcal{R}$ for the society (not necessarily the true one), an alternative $a^i \in A$ and a natural number $m \in \mathbb{N}$ (the numbers are for breaking the ties). Define the outcome function $g : S \to A$ as follows:

(i) If $s_i = (R, a, m)$ for all $i \in N$ and $a \in F(R)$, then $g(s) = a$.

That is if players are unanimous in their strategy, and their proposed alternative is F-optimal with respect to the proposed preference profile R, the outcome is a.

(ii) If there exists a player $j \in N$ such that for all the other players $i \neq j$ $s_i = (R, a, m)$ and $s_j = (R^j, a^j, m^j)$ and $a \in F(R)$ then $g(s) = a^j$ if $a^j \in L(a, R_i)$, and $g(s) = a$ otherwise.

In other words, if all the players but one play the same strategy, and their proposed alternative a is F-optimal with respect to their proposed

[7] This theorem has been proved by Williams (1986) with stronger assumptions than in Maskin (1999), also Repullo (1987), Saijo (1988) and McKelvey (1989) have proved this sufficiency theorem for Nash implementability.

[8] Here, we construct a mechanism whose all pure strategy Nash equilibria satisfy (1) and (2) in the definition of Nash implementation, but the construction can be extended to handle mixed strategies. See Maskin (1999) for details.

profile R, the odd-man-out gets his proposed alternative only if it is in the lower contour set of a under the preference ordering that the other players propose for him; otherwise outcome is a.

(iii) If neither (i) nor (ii) applies, then $g(s) = a^k$ where $k = max\{j|j \in \operatorname{argmax}_{i \in N} m^i\}$.

That is, when neither (i) nor (ii) applies, the outcome is the alternative proposed by the player with the highest index among those whose proposed number is maximal.

It remains to show that the mechanism defined above implements any F, which satisfies monotonicity and no veto power, in Nash equilibrium. To make the proof more understandable, we divide it into claims.

Claim 1 *For all $R \in \mathcal{R}$ and all $a \in A$, if $a \in F(R)$, for any $m \in \mathbb{N}$ $s = (s_1, ..., s_n)$ such that $s_i = (R, a, m)$ for all $i \in N$, constitutes a Nash equilibrium with respect to R. i.e for all $i \in N$, $(g(s))R_i(g(s_i', s_{-i}))$, for all $s_i' \in S_i$.*

Proof. To understand why, consider a unilateral deviation of agent j. Then (ii) applies. Thus, this will lead to either no change in outcome or it will change the outcome to a^j. In the former, $g(s) = a$ and the claim trivially holds. The latter case is possible only if $a^j \in L(a, R_i)$, which is worse for agent j. Therefore, player j does not have any incetive to deviate.i.e for all $i \in N$, $(g(s))R_i(g(s_i', s_{-i}))$, for all $s_i' \in S_i$ holds and hence, s is a Nash equilibrium with respect to R. ■

With this claim, we established the requirement (1) –that there is a Nash equilibrium of μ corresponding to each F optimal alternative– of Nash implementability. To establish the requirement (2) –that every Nash equilibrium of μ is F optimal–we propose the second claim below.

Claim 2 *Let $s \in S$ be a Nash equilibrium of μ with respect to real preference profile R^* of the society. Then $g(s) \in F(R^*)$.*

Proof. We will divide the proof into subcases:

Case 1: Assume $s_i \neq s_j$ for some $i, j \in N$ then either (ii) or (iii) applies. However, in both cases, it is possible for $n-1$ agents to obtain any alternative in A by a unilateral deviation. (To do so, they should just increase their proposed natural number to a higher number than the current proposed highest number.) But since s is a Nash Equilibrium then, $g(s)R^*g(s'_i, s_{-i})$ for all $i \in N$, and for all $s'_i \in S_i$. Hence, it must be that there is $j \in N$ such that for all $j \neq i$ $g(s)R^*b$ for all $b \in A$. Hence, by no veto power property $g(s) \in F(R^*)$.

Case 2: Assume $s_i = (R, a, m)$ for all $i \in N$. Now we have additional subcases:

Subcase 1: If $a \notin F(R)$ then (iii) applies and as above $g(s) \in F(R^*)$.

Subcase 2: If $a \in F(R)$ then $g(s) = a$ by (i). Now we need to show that $a \in F(R^*)$. In this case, we have for all $i \in N$ and for all $b \in A$ with aR_ib implies aR^*_ib. To see why, assume it does not hold, i.e for some $i \in N$ and some $b \in A$ aR_ib holds but $aR^*_ib)$ does not hold. Then, by (ii) agent i can change the outcome to b by just changing his strategy to $s'_i = (R^*, b, r^i)$. But this means $g(s'_i, s_{-i})R^*g(s)$ This is a contradiction of

s being Nash equilibrium with respect to R^*. Thus, aR_ib implies aR_i^*b and since F is monotonic; this implies $a \in F(R^*)$ i.e $g(s) \in F(R^*)$. ■

Hence, we established requirement (2) in the definition of Nash implementability as well. By Claim 1 and Claim 2 we conclude that any monotonic social choice rule which satisfy no veto power condition, is implementable in Nash equilibrium when there are at least three agents in the society. ■

In the proof above, a very abstract mechanism is used. Nevertheless, its complexity derives from its generality. One can think this theorem as a kind of existence theorem for the implementation in Nash equilibrium. Practical mechanisms to be used in real world examples are another subject to be considered after these kind of characterization theorems.

A point we should mention before ending this chapter is the case where there are 2 agents in the society. This may seem awkward to the reader, a society with two agents only, but it is of obvious importance since there are many bilateral interactions that one would want the theory to explain. It is interesting that there are non-trivial differences between the case of $n = 2$ and $n \leqq 3$. Note that the no veto power condition is vague in case of $n = 2$. A sufficient condition called "non-empty lower intersection condition" appears in Dutta and Sen (1991b).Interested reader can find the characterization for the case $n = 2$ in Dutta and Sen (1991b) and Moore and Repullo (1990).

Chapter 4

Continuity of Maskin's Implementation Result

4.1 Introduction

In this chapter, we will consider a continuity measure on implementability of social choice rules on a restricted domain of preferences. We restrict ourselves to a domain where each preference ordering can be represented by a cardinal utility function. The continuity measure we will consider, will employ those cardinal utilities in order to approximate payoffs for both equilibrium considerations and monotonicity and no veto power properties.

4.2 The Domain

For a special class of preferences, it is possible to use utility functions to denote the preference orderings of agents in the society, similarly the preference profile of the society can be identified by the use of the utility functions for those special class of preferences. (Recall: Debreu's representation theorem[1].)

We restrict ourselves to the domain of preferences where preference orderings of the agents are representable by cardinal utility functions on the non-empty compact alternative set A. That is, we consider a society $N = \{1, ..., n\}$ where every individual $i \in N$ has a cardinal utility function $u_i : A \to \mathbb{R}$ representing his/her preference orderings on the compact alternative set A. Hence the preference profile of the society is represented by a vector of functions $u = \{u_1, ..., u_n\}$. An example would be preferences that are represented by von-Neumann Morgenstern utility functions.

Let U_A represent the set of all cardinal utility functions on A. For every agent $i \in N$, let $U_i \subseteq U_A$ be the set of all possible utility functions denoting agent i's possible preferences. Then, the set of all possible utility profiles of the society is represented by $U = \times_{i \in N} U_i$.

Now, we give the definition of a social choice rule on the domain we restrict our attention. Note that, this definition presents nothing new, it is just the restriction of the canonical definition to our domain.

Definition 15 *A social choice rule $F : U \twoheadrightarrow A$ is a correspondence, which selects a feasible subset $F(u) \subseteq A$ for all possible utility profile $u \in U$ of the*

[1]Debreu's representation theorem basically says that, any continuous complete preorder on an arbitrary set is representable by a continuous utility function, in fact it is possible to narrow down the assumptions of this theorem, for any arbitrary set and a complete preorder R on this set, there exists a utility function representing R if and only if there exist a subset of this arbitrary set which is countable and R-order-dense. See Debreu (1959), Kreps (1988), and Fishburn (1970) for more details.

society.

4.3 The continuity measure

4.3.1 Equilibrium Concept

We start with the equilibrium concept, for which we will characterize the implementable social choice rules on our domain. Let us first define what epsilon-equilibrium of a mechanism μ means:

Definition 16 *Let $\mu = (S, g)$ be a normal form mechanism. A strategy profile $s^* \in \times_{i \in N} S_i$ is called an* ***epsilon-equilibrium of*** μ ***at*** u*, if, given any $\varepsilon \in \mathbb{R}$, for all $i \in N$, and for all $s_i \in S_i$*

$$u_i(g(s^*)) \geqq u_i(g(s_i, s^*_{-i})) - \varepsilon.$$

Note that, ε in the definition of epsilon-equilibrium is allowed to be a negative real number. The notion of epsilon-equilibrium is aimed to generalize the notion of Nash equilibrium where during game play an agent requires a payoff at least as much as ε to deviate from an outcome. That is exactly why we restrict attention to cardinal utility functions because otherwise the particular value of ε does not have any meaning.

According to the particular values of $\varepsilon \in \mathbb{R}$ we can interpret the epsilon-equilibrium in three phases: If $\varepsilon > 0$ epsilon-equilibrium is equivalent to the epsilon-Nash equilibrium introduced by Radner (1980), it coincides with Nash equilibrium when $\varepsilon = 0$ and when $\varepsilon < 0$ it prescribes another equilibrium concept, which we will refer to it as epsilon-strict Nash equilibrium.

Another important point is that; these three phases of ε, define a subset-superset relation between epsilon-equilibrium and the Nash equilibrium; that is, when $\varepsilon > 0$ epsilon-equilibrium is a superset of Nash equilibrium, when $\varepsilon = 0$ epsilon-equilibrium is exactly equivalent to Nash equilibrium, and when $\varepsilon < 0$ epsilon-equilibrium is a refinement, that is a subset of Nash equilibrium[2].

We continue with the definition of implementability of a social choice rule in epsilon-equilibrium.

Definition 17 (ε-implementability) *Let $\varepsilon \in \mathbb{R}$. A social choice rule $F : U \twoheadrightarrow A$ is **ε-implementable** if there exists a mechanism $\mu = (S, g)$ such that the following conditions hold:*

1. *For all $u \in U$, and for all $a \in F(u)$, there exists $s \in S$ such that $g(s) = a$ and for all $i \in N$*

$$u_i(g(s)) \geqslant u_i(g(s'_i, s_{-i})) - \varepsilon, \quad \text{for all } s'_i \in S_i,$$

   and;

2. *For any $s \in S$ which satisfies, for all $i \in N$*

$$u_i(g(s)) \geqslant u_i(g(s'_i, s_{-i})) - \varepsilon, \quad \text{for all } s'_i \in S_i,$$

   $g(s)$ must be in $F(u)$.

Requirement (1) in the definition of epsilon-Nash implementability of a social choice rule F says that, there is an epsilon-equilibrium of μ corresponding to each F-optimal alternative. On the other hand requirement (2) says,

[2]Note that; when $\varepsilon < 0$, the set of epsilon equilibria of a game can be empty. An example is zero-sum games.

every epsilon-equilibrium of μ is F-optimal. Together, they imply that if F is epsilon-implementable by a mechanism μ then the epsilon-equilibria of μ and F-optimal alternatives coincide as in the case of the Nash implementation.[3]

When the utility profile and the game form to be used are fixed, the size of ε may be useful to compare two social choice rules in terms of implementability. Obviously, 0-Nash Implementability coincides with Nash implementability. In the next section below, we continue with the necessity and sufficiency conditions for epsilon-implementability.

4.3.2 Epsilon-Monotonicity

We start with the definition of epsilon-monotonicity, which will be turned out to be a necessary condition for epsilon-implementability.

Definition 18 (ε-Monotonicity) *Let $\varepsilon \in \mathbb{R}$. A social choice rule $F : U \twoheadrightarrow A$ is **ε-monotonic** if for all $u, u' \in U$, and for all $a \in F(u)$, the following is true:*

$$\textit{for all } b \in A \textit{ such that } u_i(a) \geq u_i(b) - \varepsilon \quad \textit{implies} \quad u'_i(a) \geq u'_i(b) - \varepsilon$$

implies $a \in F(u')$.

The notion of ε-monotonicity is aimed to generalize Maskin monotonicity, with which it coincides when ε is set to equal 0. In words it tells us that if a is chosen when the preference profile is given by u, then a must also be chosen with the preference profile altered to u' such that under u' none of

[3]Note that, we restrict ourselves to pure strategies only.

the alternatives that have a utility figure lower than $u_i(a)+\varepsilon$ receive a utility figure higher than $u_i'(a)+\varepsilon$.

Indeed, for each $\varepsilon > 0$, the notion of ε-monotonicity is stronger than monotonicity. To observe this, note that given a utility profile $u \in U$, if $\varepsilon > 0$ is chosen sufficiently high, e.g. $\varepsilon > \max_{a,b\in A, i\in N} |u_i(a) - u_i(b)|$, only constant social choice rules satisfy ε-monotonicity condition. On the other hand, for each $\varepsilon < 0$, the notion of ε-monotonicity is also stronger than monotonicity condition since, if $\varepsilon < 0$ is chosen sufficiently small, e.g. $\varepsilon < -\max_{a,b\in A, i\in N} |u_i(a) - u_i(b)|$, again only constant social choice rules satisfy ε-monotonicity condition.

Now, we present the necessity theorem for epsilon-implementation which states that epsilon-Monotonicity is inescapable for epsilon-implementability.

Theorem 3 *Let $\varepsilon \in \mathbb{R}$. If a social choice rule $F : U \twoheadrightarrow A$ is ε-implementable, then it is ε-monotonic.*

Proof. Let $F : U \twoheadrightarrow A$ be ε-implementable by the mechanism $\mu = (S, g)$. Consider two utility profiles $u, u' \in U$ with for all $b \in A$ such that $u_i(a) \geq u_i(b) - \varepsilon$ implies $u_i'(a) \geq u_i'(b) - \varepsilon$ for all $i \in N$. Let $a \in F(u)$ What we need to show is $a \in F(u')$.

Since F is implementable by $\mu = (S, g)$ in epsilon-equilibrium, there exists $s \in S$ such that $g(s) = a$ where $u_i(g(s)) \geq u_i(g(s_i', s_{-i})) - \varepsilon$, for all $i \in N$, for all $s_i' \in S_i$. Then by assumption $u_i'(g(s)) \geq u_i'(g(s_i', s_{-i})) - \varepsilon$ for all $s_i' \in S_i$, for all $i \in N$ but this means s satisfies the condition of (2) in the definition of ε-implementability, then by (2) $g(s) \in F(u')$ *i.e.* $a \in F(u')$ therefore F is ε-monotonic. ∎

4.3.3 Epsilon-No Veto Power

We continue with the definition of another condition called epsilon-no-veto power which is a variant of no veto power condition and which will be turned out to be a sufficient condition for epsilon-implementability when combined with epsilon-monotonicity.

Definition 19 (ε-NVP) *Let $\varepsilon \in \mathbb{R}$. A social choice rule is said to satisfy* ***ε-no veto power*** *condition if there exists a player $j \in N$ such that for all the other players $i \neq j$, and for all $b \in A$, $u_i(a) \geq u_i(b) - \varepsilon$ implies $a \in F(u)$.*

The ε-no veto power condition is a generalization of the standard no veto power condition to which this new notion equals when $\varepsilon = 0$. In words, ε-no veto power condition can be interpreted in two different ways according to the particular value of ε: When $\varepsilon > 0$, it implies that, if all the players but one were to think that an alternative provides a return figure that is not less than ε from the utility level of their highest ranked alternative, then it must be chosen. On the other hand, when $\varepsilon < 0$ it basically says that if there exists an alternative which provides a return figure which is more than $|\,\varepsilon\,|$ from the utility level of all the other alternatives for all but one player, then it must be chosen.

When $\varepsilon > 0$ is sufficiently high, e.g. $\varepsilon > \max_{a,b\in A, i\in N} |u_i(a) - u_i(b)|$, we point out that no social choice rule but only the one which chooses A, i.e identity correspondence satisfy ε-no veto power property. When $\varepsilon < 0$ ε-no veto power property is weaker than the no veto power property. To see this consider $\bar{\varepsilon} < -\max_{a,b\in A, i\in N} |u_i(a) - u_i(b)|$. Then, the restriction put in the definition of ε-no veto power will not bind, hence, any social choice rule will be $\bar{\varepsilon}$-NVP.

We, now present the sufficiency theorem for epsilon-implementation which basically says that epsilon-monotonicity coupled with epsilon-no veto power turns out to be sufficient for epsilon-Nash implementation where there are at least three agents in the society.

Theorem 4 *($\#N \geq 3$)Let $\varepsilon \in \mathbb{R}$. If a social choice rule $F : U \twoheadrightarrow A$ satisfies ε-monotonicity and ε-no veto power, then it is ε-implementable.*

Proof. The proof is by construction, we will construct a mechanism $\mu = (S, g)$ which implements F in epsilon-Nash equilibrium. Consider the mechanism $\mu = (S, g)$ such that $g : S \to A$ is the outcome function and the strategy spaces are defined as $S_i = (U, A, (0, 1))$ for all $i \in N$. That is, every agent proposes a utility profile for the society, an alternative and a real number in the open interval (0,1). The outcome function g is given by the following:

(i) If $s_i = (u, a, r)$ for all $i \in N$ and $a \in F(u)$, then $g(s) = a$.

(ii) If there exists a player $j \in N$ such that for all the other players $i \neq j$ $s_i = (u, a, r)$ and $s_j = (u^j, a^j, r^j)$ and $a \in F(u)$ then $g(s) = a^j$ if $u_j(a) \geq u_j(a^j) - \varepsilon$, and $g(s) = a$ otherwise.

(iii) If neither (i) nor (ii) applies, then $g(s) = a^k$ where $k \in \text{argmax}_i(r^i)$.

Now the rest of the proof is to show that the mechanism $\mu = (S, g)$ defined above ε-implements F.

Claim 3 *For all $u \in U$ and $a \in A$, $s_i = (u, a, 1/2)$ for all $i \in N$, is an ε-equilibrium, i.e for all $i \in N$, $u_i(g(s)) \geq u_i(g(s'_i, s_{-i})) - \varepsilon$, for all $s'_i \in S_i$.*

Proof. To understand why consider a unilateral deviation of agent j. Then (*ii*) applies. So, this will lead to either no changes in outcome or it will change the outcome to a^j. In the former, $g(s) = a$ and the claim trivially holds. The latter case is possible only if $u_j(a) \geqslant u_j(a^j) - \varepsilon$ i.e $u_j(g(s)) \geqslant u_j(g(s'_j, s_{-j})) - \varepsilon$. Therefore, player j does not have any deviation opportunities. Hence, (1) in the definition of ε-implementability holds. ■

Claim 4 *Consider any ε-equilibrium $s \in S$ with respect to the real utility profile u^* of the society. Then $g(s) \in F(u^*)$.*

Proof. We will work with subcases:

Case 1:$s_i \neq s_j$ for some $i, j \in N$. Then either (*ii*) or (*iii*) applies. But in both cases, it is possible for $n - 1$ agents to get any alternative in A by a unilateral deviation. (To do so, they should just increase their proposed real number to a higher number than the current proposed highest number.) But since s is an ε-equilibrium then, $u_i^*(g(s)) \geq u_i^*(g(s'_i, s_{-i})) - \varepsilon$, for all $i \in N$ for all $s'_i \in S_i$. Hence, it must be that there exists $j \in N$ such that for all $j \neq i$ $u_i^*(g(s)) \geq u_i^*(b) - \varepsilon$ for all $b \in A$. Hence, by ε-no veto power condition $g(s) \in F(u^*)$.

Case 2: $s_i = (u, a, r)$ for all $i \in N$. Now we have additional subcases:

Subcase 1: $a \notin F(u)$. Then (*iii*) applies and as above $g(s) \in F(u')$.

Subcase 2: $a \in F(u)$. Then $g(s) = a$ by (*i*). Now we need to show that $a \in F(u^*)$. In this case we have for all $i \in N$ and for all $b \in A$ with $u_i(a) \geq u_i(b) - \varepsilon$ implies $u_i^*(a) \geq u_i^*(b) - \varepsilon$. To see why assume not, i.e for some $i \in N$ and some $b \in A$ $u_i(a) \geq u_i(b) - \varepsilon$ holds but $u_i^*(a) < u_i^*(b) - \varepsilon$. Then, agent i can change the outcome to b by just changing his strategy to $s'_i = (u^i, b, r^i)$

by (ii). But then $u_i^*(g(s)) < u_i^*(g(s_i'), s_{-i}) - \varepsilon$ This is a contradiction of s being an ε-equilibrium. Thus, by ε-monotonicity, $a \in F(u^*)$ i.e $g(s) \in F(u')$. Hence the requirement (2) in the definition of epsilon-implementability is satisfied. ■

By the two claims above we established both requirement (1) and (2) in the definition of epsilon-implementability. Therefore, every social choice rule which satisfies epsilon-monotonicity and epsilon-no veto power conditions is implementable in epsilon-equilibrium. ■

Chapter 5

Conclusion

In this thesis, we briefly surveyed Nash implementation and after concentrating on a restricted domain of cardinal preferences we defined a continuity measure by using an equilibrium concept, which we call epsilon-equilibrium.

Although, we characterized the implementable social choice rules in epsilon-equilibrium, we have not yet discovered a relevant example on our domain to examine the regularities of our conditions.

We believe that, our characterization of epsilon-implementation may lead us to a distance notion for social choice rules in terms of implementability in Nash equilibrium. In turn, one may use these to construct a notion which in some sense measures the sacrifice of a society when a non-Nash implementable social choice rule is to be implemented.

Finally, the analysis of some of the prominent social choice rules and some other intuitive examples of social choice rules on our domain obviously constitute a future avenue for research.

Bibliography

Allen, B. (1997): "Implementation theory with incomplete information," in *Cooperation: Game Theoretic Approaches*, ed. by S. Hart, and A. Mas-Colell. Springer, Heidelberg.

Arrow, K. (1951): *Social Choice and Individual Values*. John Wiley and Sons, New York.

Bernheim, B., and M. Whinston (1987): "Coalition Proof Nash equilibrium, II: Applications," *Journal of Economic Theory*, 42, 13–29.

Corchón (1996): *The Theory of Implementation of Socially Optimal Decisions in Economics*. St. Martin's Press, New York.

Danilov, V. (1992): "Implementation via Nash Equilibria," *Econometrica*, 60(1), 43–56.

Debreu, G. (1959): *Theory of Value*. Wiley, New York.

Dutta, B., and A. Sen (1991a): "Implementation Under Strong Equilibria: A Complete Characterization," *Journal of Mathematical Economics*, 20, 49–68.

——— (1991b): "Necessary and sufficient conditions for 2-person Nash implementation," *Review of Economic Studies*, 58, 121–129.

FISHBURN, P. (1970): *Utility Theory for Decision Making*. John Wiley and Sons, New York.

GROVES, T., AND J. LEDYARD (1977): "Optimal allocation of public goods: a solution to the 'free rider' dilemma," *Econometrica*, 45, 783–811.

——— (1987): "Incentive Compatibility since 1972," in *Information, Incentives and Economic Mechanisms*, ed. by T. Groves, R. Radner, and S. Reiter. University of Minnesota Press, Minneapolis.

HURWICZ, L., AND A. P. E. MASKIN (1995): "Feasible Nash Implementation of social choice rules when the designer does not know endowments or production sets," in *The Economics of Informationla Decentralization: Complexity, Efficiency and Stability*, ed. by J. Ledyard. Kluwer Academic Publishers, Amsterdam.

HURWICZ, L., AND D. SCHMEIDLER (1978): "Construction of outcome functions guaranteeing existence and Pareto-optimality of Nash equilibria," *Econometrica*, 46, 1447–1474.

JACKSON, M. (2001): "A crash course in implementation theory," *Social Choice and Welfare*, 18, 655–708.

KREPS, D. (1988): *Notes on the Theory of Choice*. Westview Press, Boulder and London.

MASKIN, E. (1977): "Nash Equilibrium and Welfare Optimality," *mimeo.*

——— (1985): "The theory of implementation in Nash Equilibrium: a survey," in *Social Goals and Social Organization*, ed. by L. Hurwicz, D. Schmeidler, and H. Sonnenschein. Cambridge university Press, Cambridge.

——— (1999): "Nash Equilibrium and Welfare Optimality," *The Review of Economic Studies*, 66(1), Special Issue: Contracts. 23–38.

Maskin, E., and T. Sjöström (2002): "Implementation theory," in *Handbook of Social Choice and Welfare Vol. 1*, ed. by K. Arrow, A. Sen, and K. Suzumura. North-Holland, Amsterdam.

McKelvey, R. (1989): "Game forms for Nash implementation of general general choice correspondences," *Social Choice and Welfare*, 6, 139–156.

Moore, J. (1992): "Implementation, contracts and renegotiation in environments with complete information," in *Advances in Economic Theory Vol. 1*, ed. by J.-J. Laffont. Cambridge University Press, Cambridge.

Moore, J., and R. Repullo (1990): "Nash Implementation: a full characterization," *Econometrica*, 58(5), 1083–1099.

Moulin, H. (1982): "Non-Cooperative Implementation: A Survey of Recent Results," *Mathematical Social Sciences*, 3, 243–257.

Muller, E., and M. Satterthwaite (1977): "The equivalence of strong positive association and strategy proofness," *Journal of Economic Theory*, 14, 412–418.

PALFREY, T. (1992): "Implementation in Bayesian Equilibrium: the multiple equilibrium problem in mechanism design," in *Advances in Economic Theory Vol. 1*, ed. by J.-J. Laffont. Cambridge University Press, Cambridge.

——— (2001): "Implementation theory," in *Handbook of Game Theory Vol. 3*, ed. by R. Aumann, and S. Hart. North-Holland, Amsterdam.

PALFREY, T., AND S. SRIVASTAVA (1993): *Bayesian Implementation.* Harwood Academic Publishers, Reading.

POSTLEWAITE, A. (1985): "Incentive Compatibility since 1972," in *Social goals and Organization: Essays in memory of Eliza Pazner*, ed. by L. Hurwicz, D. Schmeidler, and H. Sonnenschein. Cambridge University Press, Cambridge.

RADNER, R. (1980): "Collusive Behaviour in Oligopolies with Long but Finite Lives," *Journal of Economic Theory*, 22, 136–156.

REPULLO, R. (1987): "A simple Proof of Maskin's Theorem on Nash Implementation," *Social Choice and Welfare*, 4, 39–41.

SAIJO, T. (1988): "Strategy space reductions in Maskin's Theorem: Sufficient conditions for Nash implementation," *Econometrica*, 56, 693–700.

WILLIAMS, S. (1986): "Realization and Nash Implementation: Two aspects of mechanism design," *Econometrica*, 54, 139–151.