# A KRONECKER'S LIMIT FORMULA FOR REAL QUADRATIC NUMBER FIELDS

by

SERHAN KALAYCI

Submitted to the Graduate School of Engineering and Natural Sciences
in partial fulfillment of
the requirements for the degree of
Master of Science

Sabancı University

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DATE OF APPROVAL: June 9th, 2005

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# Acknowledgments

I would like to express my gratitude and deepest regards to my supervisor Ebru Bekyel for her motivation, guidance and encouragement throughout this thesis.

Also I would like to thank my friends for their support and help.

A Kronecker's Limit Formula For Real Quadratic Number Fields

# Abstract

Let $K$ be a quadratic number field, the Dedekind zeta-function of $K$, $\zeta_K(s)$ can be written as a sum of partial zeta functions, $\zeta(s, A)$ where $A$ runs over the ideal class group of $K$ and $s$ a complex number. Then $\zeta(s, A)$ has an analytic continuation as a meromorphic function of $s$ with a simple pole at $s = 1$. Dirichlet proved that the residue of $\zeta(s, A)$ is independent of the ideal class $A$ chosen. For the constant in the Laurent expansion of partial zeta function around $s = 1$ we will examine Kronecker's and Zagier's results. Kronecker found the constant for the imaginary quadratic case. Working with imaginary quadratic fields is much easier because of the finiteness of unit group of the field. For real quadratic fields there are infinitely many units and Zagier computed the constant for this case. Also we will include continued fractions as Zagier used for the proof of the limit formula of zeta-function for real quadratic number fields.

Keywords: Quadratic Number Fields, Zeta Functions, Continued Fractions, Ideal Class.

# İKİNCİ DERECEDEN GERÇEL SAYILAR CİSİMLERİ İÇİN KRONECKER İN YAKINSAMA FORMÜLÜ


## Özet

$K$ ikinci dereceden sayılar cismi olsun. $s$ gerçel olmayan bir sayı ve $A$ da $K$ nın ideal sınıf grubundan bir ideal iken $K$' nın Dedekind zeta-fonksiyonu $\zeta_K(s)$ i kismı zeta- fonksiyonların, $\zeta_K(s, A)$ toplamıolarak yazabiliriz. O zaman $\zeta(s, A)$ analitik surekliliği olan $s = 1$ de tanımsız bir meromorphic fonksiyon olur. Zeta fonksiyonunun residu su Dirichlet tarafından hesaplanmış ve $A$ ya bağımlı olmadığı gösterilmiştir. Kronecker zeta fonksiyonunun Laurent açılımındaki sabit sayıyı gerçel olmayan ikinci dereceden sayılar cismi için bulmuş. Unit sayısı sonlu olduğundan bu cisimlerde hesap yapmak, unit sayısının sonsuz olduğu gerçel sayılar cisminde hesap yapmaktan daha kolaydır. Zagier sabit sayıyı gerçel sayılar cisminde hesaplamıştır. Biz de Kronecker ve Zagier in sonuçlarını inceleyeceğiz. Ayrıca Zagier in teoreminde kullandığı sürekli kesirlerden bahsedeceğiz.



Anahtar kelimeler: İkinci Dereceden Sayılar Cisimleri, Zeta Fonksiyonları, Sürekli Kesirler, İdeal Grublar.

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# Contents

# CHAPTER 1

## Quadratic number Fields

Many mathematicians were interested in finding the solutions to algebraic equations and this interest lied to different subjects in Mathematics, one of which is number theory. In the process of finding the zeros of a polynomial $f(x)$ over a field $k$, we sometimes need a larger field, so we pass to a larger field $K$, $K$ can be thought as the union of $k$ and the roots of the polynomial $f(x)$. $K$ is called a "*field of extension*" of $k$, denoted by $K/k$. For the special case $k = \mathbb{Q}$ and the irreducible polynomial $f(x)$ over $k$ has degree 2, we call the field of extension of $k$, *a quadratic number field*. We also give special names for the roots of $f(x)$. For the case $f(x) \in \mathbb{Q}[x]$ roots are called *algebraic numbers*, and when $f(x) \in \mathbb{Z}[x]$ and monic, then roots are called *algebraic integers*. It can be shown that for an algebraic number $\alpha$ over $K$, there exists a unique, monic, irreducible polynomial, $p$ of minimal degree subject to $p(\alpha) = 0$, which is called the *minimum polynomial* of $\alpha$ over $K$.

**Proposition 1.0.1.** *The quadratic fields are precisely those of the form $\mathbb{Q}(\sqrt{d})$ for $d$ a square free integer. If $d > 0$ we call it a real quadratic number field, if $d < 0$ we call it an imaginary quadratic number field.*

**Proof:** Firstly let us show that we can write $K = \mathbb{Q}(\theta)$ where $\theta$ is an algebraic integer.

Note that we have $[K : \mathbb{Q}] = 2$. Let $\theta \in K$ $\mathbb{Q}$

**Claim:** $\{1, \theta\}$ form a basis for $K$ over $\mathbb{Q}$, i.e $K = \mathbb{Q}(\theta)$.

**Proof:** It is clear that $\mathbb{Q}(\theta) \subseteq K$. Also it is enough to show that $\{1, \theta\}$ is linearly independent over $\mathbb{Q}$ since $[K : \mathbb{Q}] = 2$. Consider $a + b\theta = 0$ with $a, b \in \mathbb{Q}$ this implies $\theta \in \mathbb{Q}$.$\Box$

So there exists $\theta \in K/\mathbb{Q}$ such that $K = \mathbb{Q}(\theta)$, i.e $K = \{m + n\theta | m, n \in \mathbb{Q}\}$. Now $\theta^2 \in K$ so $\theta^2 = m + n\theta$ with $m, n \in \mathbb{Q}$. This implies $\theta$ is a root of the equation $t^2 - nt - m$, where $m, n \in \mathbb{Q}$. Multiplying with an appropriate integer we can clear the denominators and get an equation in integers $at^2 + bt + c$. Note that $\theta$ is still a root of this equation.

Hence $\theta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Let $b^2 - 4ac = r^2 d$ where $r, d \in \mathbb{Z}$ and $d$ square free.
So $\theta = \frac{-b \pm r\sqrt{d}}{2a}$ that is $K = \mathbb{Q}(\theta) = \mathbb{Q}(\sqrt{d})$ $\Box$ [7]

Now let us define the *norm* and *trace* functions of an algebraic number.

**Definition 1.0.1.** *Let $\alpha$ be an element of $K = \mathbb{Q}(\sqrt{d})$.*
*Consider the mapping $\mu_\alpha : K \to K$ such that $\mu_\alpha(\beta) = \alpha\beta$. $\mu_\alpha$ is a $\mathbb{Q}$-linear map. Choose a basis $\{w_1, w_2\}$ of $K/\mathbb{Q}$ and let $A$ be the matrix representing $\mu_\alpha$ with respect to $\{w_1, w_2\}$. $A$ is a $2 \times 2$ matrix over $\mathbb{Q}$. Then define*
$N(\alpha) := \det A \in \mathbb{Q}$ *and* $\mathrm{Tr}(\alpha) = \mathrm{tr}\ A \in \mathbb{Q}$.

Note that from the definition of norm it follows that norm function is multiplicative, i.e $\mathrm{N}(\alpha\beta) = \mathrm{N}(\alpha)\mathrm{N}(\beta)$, since $\mu_{(\alpha\beta)} = \mu_\alpha \circ \mu_\beta$ so $A_{(\alpha\beta)} = A_\alpha A_\beta$. Hence multiplicative property follows from $\det(A_{(\alpha\beta)}) = \det(A_\alpha)\det(A_\beta)$.

Another important fact is that norm and trace functions of an algebraic integer are integers.

**Theorem 1.0.1.** *Let $\alpha \in K$ be an algebraic integer and $p(x) = x^2 + ax + b$ be the minimal polynomial of $\alpha$ over $\mathbb{Z}$. Then*
*(i)*$\mathrm{N}(\alpha) = b$
*(ii)*$\mathrm{Tr}(\alpha) = -a$

**Proof:** We have $\{1, \alpha\}$ basis for $K(\alpha)/\mathbb{Q}$. Consider $\mu_\alpha(1) = \alpha$ and $\mu_\alpha(\alpha) = \alpha^2 = -a\alpha - b$. So the matrix for this map is $A = \begin{pmatrix} 0 & -b \\ 1 & -a \end{pmatrix}$. So $\mathrm{N}(\alpha) = \det A = b$

and Tr($\alpha$)=tr$A = -a$ $\square$

One can be interested in finding the algebraic integers of $\mathbb{Q}(\sqrt{d})$. The following theorem will enlighten this question in minds.

**Theorem 1.0.2.** *Let $d$ be a square free integer. Then the algebraic integers of $\mathbb{Q}(\sqrt{d})$ are*

*1) $\mathbb{Z}+\mathbb{Z}(\sqrt{d})$ for $d \neq 1 \pmod{4}$*

*2) $\mathbb{Z}+\mathbb{Z}(\frac{1+\sqrt{d}}{2})$ for $d \equiv 1 \pmod{4}$*

**Proof:**

Let $\alpha = a + b\sqrt{d} \ \epsilon \ \mathbb{Q}(\sqrt{d})$ be an algebraic integer. Then $\mathrm{N}(a+b\sqrt{d})$ and $\mathrm{Tr}(a+b\sqrt{d}) \in \mathbb{Z}$ by Theorem (1.0.1).

To find the norm and trace of $\alpha$ consider the minimal polynomial of $\alpha$ which is $f(x) = (x-((a+b\sqrt{d}))(x-(a-b\sqrt{d})) = x^2-2ax+(a^2-b^2d)$. So again by Theorem (1.0.1), $\mathrm{N}(a+b\sqrt{d}) = a^2-b^2d$ and $\mathrm{Tr}(a+b\sqrt{d}) = 2a$.

Consider $4(a^2-b^2d)-(2a)^2$ which is an integer. $2a \in \mathbb{Z}$ so $4b^2d \in \mathbb{Z}$. Hence $(2b)^2d \in \mathbb{Z}$ which implies $2b \in \mathbb{Z}$ since $d$ is square free and if $(2b)^2$ has denominator it will not be canceled by $d$, so it has no denominator. Now we have $2a$ and $2b$ are integers. Write $a = \frac{a'}{2}, b = \frac{b'}{2}$ where $a'$ and $b'$ are integers. So $\alpha = \frac{a'}{2}+\frac{b'}{2}\sqrt{d}$, $a', b' \in \mathbb{Z}$ and $a'^2 - b'^2d \equiv 0 \pmod{4}$ since $a' = 2a, b' = 2b$. We look at two cases:

Case1: $d \equiv 2,3 \pmod{4} \Longrightarrow a'^2 \equiv b'^2d \pmod{4}$. Note that for any integer $c \in \mathbb{Z}$ we have $c^2 \equiv 0,1 \pmod{4}$. So $a' \equiv b' \equiv 0 \pmod{2}$ Hence $\frac{a'}{2}, \frac{b'}{2} \in \mathbb{Z} \Longrightarrow \alpha \in \mathbb{Z}+\mathbb{Z}\sqrt{d}$. Conversely let $\alpha$ be an element of $\mathbb{Z}+\mathbb{Z}\sqrt{d}$. Then one sees that it satisfies the polynomial $f(x) = x^2-2ax+(a^2-b^2d)$ which is monic with integer coefficients, hence it is an algebraic integer.

Case2: $d \equiv 1 \pmod{4} \Longrightarrow a'^2 \equiv b'^2 \pmod{4} \Longrightarrow a' \equiv b' \pmod{2}$ Hence $\alpha = \frac{a'}{2}+\frac{b'}{2}\sqrt{d} = \frac{a'-b'}{2}+b'\frac{1+\sqrt{d}}{2} \in \mathbb{Z}+\mathbb{Z}\frac{1+\sqrt{d}}{2}$. Conversely let $\alpha$ be an element of $\mathbb{Z}+\mathbb{Z}\frac{1+\sqrt{d}}{2}$, i.e $\alpha = \frac{a+b\sqrt{d}}{2}$ where $a \equiv b \ (mod\ 2)$. Then one sees that it satisfies the polynomial $f(x) = x^2-ax+(\frac{a^2-b^2d}{4})$ which is monic with integer coefficients, hence it is an algebraic integer. $\square$

The set of algebraic integers of a quadratic number field form a ring. It is called *the ring of integers of* $K$. For a quadratic number field we have 2 embeddings, $\sigma_1, \sigma_2$ of $K$ into $\mathbb{C}$ as $\sigma_1$ is the identity map and $\sigma_2$ sends $a + b\sqrt{d}$ to $a - b\sqrt{d}$ where $a, b \in \mathbb{Q}$. Now let $\{\alpha_1, \alpha_2\}$ be a basis for $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$. Then we define *discriminant* of this basis as $d(\alpha_1, \alpha_2) = \det(\mathrm{Tr}(\alpha_i\alpha_j)) = \det(\sigma_i(\alpha_j))^2$. Also $\{\alpha_1, \alpha_2\}$ is called an *integral basis* for $\mathbb{Q}(\sqrt{d})$ if it is a $\mathbb{Z}$-basis for the ring of integers of $\mathbb{Q}(\sqrt{d})$, i.e $\vartheta = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2$, where $\vartheta$ is the ring of integers of $\mathbb{Q}(\sqrt{d})$. Then, *discriminant of* $\mathbb{Q}(\sqrt{d})$ is the discriminant of the integral basis of $\mathbb{Q}(\sqrt{d})$.

Now let us examine the discriminant of $\mathbb{Q}(\sqrt{d})$.

**Theorem 1.0.3.** *Let $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$ be a quadratic number field with $d$ a square free integer. Then*

*(a) If $d \neq 1 \pmod 4$ then $\mathbb{Q}(\sqrt{d})$ has an integral basis of the form $\{1, \sqrt{d}\}$ and discriminant $4d$.*

*(b) If $d \equiv 1 \pmod 4$ then $\mathbb{Q}(\sqrt{d})$ has an integral basis of the form $\{1, \frac{1+\sqrt{d}}{2}\}$ and discriminant $d$.*

**Proof:**

Case $d \equiv 2, 3 \pmod 4$:

$\{1, \sqrt{d}\}$ is an integral basis for $\mathbb{Q}$ by Theorem (1.0.2). Discriminant of this basis is

$$d(1, \sqrt{d}) = \det \begin{pmatrix} Tr(1) & Tr(\sqrt{d}) \\ Tr(\sqrt{d}) & Tr(d) \end{pmatrix} = \det \begin{pmatrix} 2 & 0 \\ 0 & 2d \end{pmatrix} = 4d$$

Case $d \equiv 1 \pmod 4$:

$\{1, \frac{1+\sqrt{d}}{2}\}$ is an integral basis for $\mathbb{Q}$. Discriminant of this basis is

$$d(\{1, \tfrac{1+\sqrt{d}}{2}\}) = \det \begin{pmatrix} Tr(1) & Tr(\frac{1+\sqrt{d}}{2}) \\ Tr(\frac{1+\sqrt{d}}{2}) & Tr((\frac{1+\sqrt{d}}{2})^2) \end{pmatrix} = \det \begin{pmatrix} 2 & 1 \\ 1 & \frac{1+\sqrt{d}}{2} \end{pmatrix} = d \, \square$$

## 1.1 Units in Quadratic Number Fields

After defining the ring of integers of a quadratic number field another question arises, finding the units of a quadratic number field, namely units of the ring of integers, which form a group. In this section we will explicitly find the units of an imaginary quadratic number field and also mention Dirichlet's Unit Theorem for a real quadratic number field.

**Definition 1.1.1.** *Let* $K = \mathbb{Q}(\sqrt{d})$ *and* $\vartheta_K$ *be the ring of integers of* $K$*. Define* $U_K = \{\epsilon \in \vartheta_K \mid \exists \xi \in K, \epsilon\xi = 1\}$ *to be the group of units of* $\vartheta_K$ *. Define* $W_K = \{\zeta \in K \mid \exists m \geq 1, \zeta^m = 1\}$ *to be the roots of unity in* $\vartheta_K$*. Note that* $W_K$ *is a subgroup of* $U_K$*.*

One can say whether an element of $K$ is a unit or not by looking at its norm.

**Proposition 1.1.1.** *For* $\alpha \in K$*, the following are equivalent*
*(i)*$\alpha \in U_K$
*(ii)*$\alpha \in \vartheta_K$ *and* $\mathrm{N}(\alpha) = \pm 1$*.*

**Proof:** Let us show $(i) \implies (ii)$. We have $\alpha \in U_K$, hence $\alpha \in \vartheta_K$. By the definition of a unit there exists $\alpha' \in \vartheta_K$ such that $\alpha\alpha' = 1$.
Consider $\mathrm{N}(\alpha\alpha')$=$\mathrm{N}(1) = 1$ =$\mathrm{N}(\alpha)\mathrm{N}(\alpha')$, note that $\mathrm{N}(\alpha)$,$\mathrm{N}(\alpha') \in \mathbb{Z}$ by Theorem (1.0.1). Hence $\mathrm{N}(\alpha) = \pm 1$.
Now let us show $(ii) \implies (i)$. Let $f(x) = x^2 + a_1x + a_0 \in \mathbb{Z}[x]$ be the minimal polynomial of $\alpha$. Then $\mathrm{N}(\alpha) = a_0$ by Theorem (1.1). We have $\mathrm{N}(\alpha) = \pm 1 = a_0$. Also $\alpha$ is a root of $f(x)$ so $0 = \alpha^2 + a_1\alpha + a_0 = \alpha(\alpha + a_1) + a_0$ Let $\alpha' = \alpha + a_1$ which is in $\vartheta_K$. Hence we have $\alpha\alpha' = \pm 1 \implies \alpha \in U_K$. □

**Theorem 1.1.1.** *Let* $\mathbb{Q}(\sqrt{d})$ *be an imaginary quadratic number field.*
*If* $d \neq -1, -3$*, then* $\mathbb{Q}(\sqrt{d})$ *has exactly two units* $\{\pm 1\}$*. Also* $\mathbb{Q}(\sqrt{-1})$ *has exactly four units* $\{\pm 1, \pm\sqrt{-1}\}$ *and* $\mathbb{Q}(\sqrt{-3})$ *has exactly six units*
$\{\pm 1, \pm(\frac{-1+\sqrt{-3}}{2}), \pm(\frac{-1-\sqrt{-3}}{2})\}$.

**Proof:** Let $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$ be an imaginary quadratic number field. Let $\alpha \in \mathbb{Q}(\sqrt{d})$ be a unit.
**Case 1:** For $d \neq 1 \pmod 4$ we write $\alpha = a + b\sqrt{d}$ with $a$ and $b$ are integers. We know that an algebraic number is a unit if and only if $\mathrm{N}(\alpha) = \pm 1$. Since $\alpha$ is a unit we have $\mathrm{N}(\alpha) = a^2 - b^2d = \pm 1$ and since we are considering the case of imaginary

quadratic number fields, $d < 0$ so $\mathrm{N}(\alpha) > 0$.

Hence we have $\mathrm{N}(\alpha) = 1 = a^2 + b^2(-d)$.

For $d < -2$ we have $\mathrm{N}(\alpha) \geq a^2 + 2b^2$ if $b \neq 0$ then $\mathrm{N}(\alpha) \geq 2$ so $\alpha$ will not be a unit. Hence $b$ must be zero and $a^2 = 1$ which implies $a = \pm 1$ the only units we have for the case $d \neq 1 \pmod 4$ and $d < -2$. Suppose $d = -1$. Then $\mathrm{N}(\alpha) = a^2 + b^2 = 1$. The only integer solutions are $a = \pm 1, b = 0$ or $a = 0, b = \pm 1$. So for $d = -1$ we have $\{\pm 1, \pm i\}$ as units.

**Case 2:** Now suppose $d \equiv 1 \pmod 4$. Let $\alpha$ be a unit in $\mathbb{Q}(\sqrt{d})$. Then $\alpha = \frac{a+b\sqrt{d}}{2}$ with $a \equiv b (\mathrm{mod}\ 2)$ and $\mathrm{N}(\alpha) = 1$. So $1 = \mathrm{N}(\alpha) = \frac{a^2+b^2(-d)}{4}$ this implies

$$a^2 + b^2(-d) = 4 \qquad (1.4)$$

For $d \leq -7$ and $b \neq 0$ we have $a^2 + b^2(-d) \geq a^2 + 7b^2 \geq a^2 + 7 \geq 7 > 4$. So we should have $b = 0$ and so $a^2 = 4 \Longrightarrow a = \pm 2$, so $\alpha = \pm 1$. Hence if $d \leq -7$ and $d \equiv 1 \pmod 4$ then units are $\pm 1$.

Now for $d > -7$ and $d \equiv 1 \pmod 4$, we have only $d = -3$. Hence from equation (1.4) we get $a^2 + 3b^2 = 4$ so we should have $b = \pm 1$ or $b = 0$. If $b = 0$ then $a = \pm 2$ and we have $\alpha = \pm 1$. If $b = 1$ then $a = \pm 1$ and we have $\alpha = \pm(\frac{-1+\sqrt{-3}}{2})$. If $b = -1$ then $a = \pm 1$ and we have $\alpha = \pm(\frac{-1-\sqrt{-3}}{2})$. $\square$

We found the units of an imaginary quadratic number field, how about real quadratic number fields? Now let $K$ be a real quadratic number field. Let $\sigma_1, \sigma_2 : K \hookrightarrow \mathbb{R}$ be the real embeddings, such that $\sigma_1$ is the identity and $\sigma_2$ is the map that send an element to its conjugate, i.e $\sigma_2(a + b\sqrt{d}) = a - b\sqrt{d}$. We have Dirichlet's Unit Theorem about units, using these embeddings. Before stating the theorem we will need the definition of a lattice.

In general a *lattice* in $\mathbb{R}^n$ is a subgroup $\Gamma \subseteq \mathbb{R}^n$ with respect to addition, with the following property: there exists $e_1, \ldots, e_r \in \Gamma$ which are linearly independent over $\mathbb{R}$, such that $\Gamma = \mathbb{Z}e_1 \bigoplus \ldots \bigoplus \mathbb{Z}e_r$. We call $(e_1, \ldots, e_r)$ a lattice basis of $\Gamma$ and $r = \dim(\Gamma)$.

**Theorem 1.1.2.** *(Dirichlet's Unit Theorem for real quadratic number fields) Let*

*$K = \mathbb{Q}(\sqrt{d})$ be a real quadratic number field, where $d$ is a square free positive integer. Then $U_K \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$. That is there is a fundamental unit $\epsilon \in U_K$ such that $U_K = \{\pm\epsilon^k : k \in \mathbb{Z}\}$*

**Proof:** Let $K = \mathbb{Q}(\sqrt{d})$, $d \in \mathbb{Z}$ square free and $d \geq 2$.
Consider the map

$$f : U_K \to \mathbb{R}$$

$$\varepsilon \to (\log|\sigma_1(\varepsilon)|)$$

We want to show (i) $\text{Ker}(f) = W_K$ and (ii) $\text{Im}(f) = \Gamma$ where $\Gamma$ is a lattice in $\mathbb{R}$ of dimension 1.

**(i)** Let us first show that $W_K = \{\pm 1\}$ for real quadratic number fields.

By the definition of $W_K$ there exists $m \geq 1$ such that $\xi^m = 1$. Consider $\xi^m = (a + b\sqrt{d})^m = 1$. Note that $\xi$ is a real number whose $m$th power is 1, the only real numbers that are root of unity are $\pm 1$. Hence $W_K = \{\pm 1\}$.
Clearly $W_K \subseteq \text{Ker}(f) = \{\varepsilon \in U_K : |\sigma_1(\varepsilon)| = 1\}$.
Let us show the other inclusion. So let $\varepsilon \in \text{Ker}(f)$. Then $|\sigma_1(\varepsilon)| = |\varepsilon| = 1$. Also note that $|\text{N}(\varepsilon)| = |\sigma_1(\varepsilon)||\sigma_2(\varepsilon)| = 1$. Hence $|\sigma_2(\varepsilon)| = \frac{1}{|\sigma_1(\varepsilon)|} = 1$ Consider the minimal polynomial of $\varepsilon$ $F_\varepsilon(x) = (x - \sigma_1(\varepsilon))(x - \sigma_2(\varepsilon)) = x^2 + a_1 x + a_0 \in \mathbb{Z}[x]$ where $a_1 = \sigma_1(\varepsilon) + \sigma_2(\varepsilon)$ and $a_0 = -\sigma_1(\varepsilon)\sigma_2(\varepsilon)$. So $|a_1| \leq 2$ and $|a_0| \leq 1$. Therefore there are only finitely $\varepsilon$ that has $|\sigma_i(\varepsilon)| = 1$ for $i = 1, 2$. That is $\varepsilon$ has finite order, so there exists an integer $n$ such that $\varepsilon^n = 1$. Hence $\varepsilon \in W_K$. So $\text{Ker}(f) \subseteq W_K$ therefore $\text{Ker} f = W_K$.

**(ii)** Let $\Gamma = \text{Im}(f)$ and consider the elements of $\Gamma$. Let $Y$ be any bounded subset of $\Gamma$. Then for every $y \in Y$ there exists a real number $M$ such that $|y| < M$ that is $-M < \log(|\sigma_1(\varepsilon)|) < M$, then $e^{-M} < |\sigma_1(\varepsilon)| < e^M$, i.e $e^{-M} < |\varepsilon| < e^M$. Also $|\text{N}(\epsilon)| = |\sigma_1(\epsilon)||\sigma_2(\epsilon)|$ so we have $|\sigma_2(\epsilon)| = \frac{1}{|\sigma_1(\epsilon)|} < e^M$. **Claim:** There are only finitely many $\alpha \in U_K$ for which $e^{-M} \leq |\sigma_1(\alpha)| = |\alpha| \leq e^M$ and $e^{-M} \leq |\sigma_2(\alpha)| \leq e^M$ for any positive real number $M$.

**Proof of the Claim:** Consider the characteristics polynomial of $\alpha$, which is by definition $F_\alpha(x) = (x-\sigma_1(\alpha))(x-\sigma_2(\alpha)) = x^2+a_1x+a_0 \in \mathbb{Z}[x]$ where $a_1 = \sigma_1(\alpha)+\sigma_2(\alpha)$ and $a_0 = -\sigma_1(\alpha)\sigma_2(\alpha)$.
Note that $|\mathrm{N}(\alpha)| = |\sigma_1(\alpha)||\sigma_2(\alpha)| = 1$, so $|\sigma_2(\alpha)| = \frac{1}{|\sigma_1(\alpha)|}$. Hence $|\sigma_i(\alpha)| \leq e^M$ for $i = 1, 2$, so we get $|a_1| \leq 2e^M$ and $|a_0| \leq e^M$. Hence there are only finitely many choices for $a_0$ and $a_1$, which implies there are only finitely many such $\alpha$ □
By our claim one can conclude that there are only finitely many such $\epsilon$ satisfying the above inequality. Hence any bounded subset of $\Gamma$ is finite. In fact we call such a set *discrete*. Now we need to show $\Gamma$ is a lattice in $\mathbb{R}$ of dimension 1.
**Claim:** Any discrete subgroup of $\mathbb{R}$ is a lattice.
**Proof of the claim:** Let $\alpha$ be a nonzero element of $\Gamma$ and let $A = \{\lambda \in \mathbb{R} : \lambda\alpha \in \Gamma\}$. Since $\Gamma$ is discrete the set $\{\gamma \in \Gamma : |\gamma| \leq |\alpha|\}$ is finite. Then $A \cap [-1, 1]$ is finite and contains a least positive element element $0 < \mu \leq 1$. Let $\beta = \mu\alpha$ and suppose that $\lambda\beta \in \Gamma$, with $\lambda \in \mathbb{R}$. Then

$$\lambda\beta - [\lambda]\beta = (\lambda - [\lambda])\beta = (\lambda - [\lambda])\mu\alpha \in \Gamma$$

by the minimality of $\mu$ we get $\lambda = [\lambda]$,i.e, $\lambda \in \mathbb{Z}$ which implies that $\Gamma = \mathbb{Z}\beta$ is a lattice of dimension 1. □
So we can write $U_K =\mathrm{Ker}(f)\oplus\mathrm{Im}(f) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$. This means that $\forall\varepsilon \in U_K$ can be expressed uniquely as $\varepsilon = \pm\epsilon^k$. One can choose $\epsilon$ in such a way that $1 < \epsilon$. By this condition $\epsilon$ is uniquely determined, and it is called the fundamental unit of $K$.□

## 1.2 Ideal Class Group

One may want to factorize elements of number fields in a unique way. However, this is not possible all the time. For example, $\mathbb{Q}(\sqrt{-5})$ is not a unique factorization domain. In that case we use different structures as our elements, namely, ideals of a quadratic number field, $K = \mathbb{Q}(\sqrt{d})$, to get the unique factorization. Furthermore one can define equivalence relation between ideals and obtain ideal classes which indeed form a group, as we will discuss below.

Firstly an *ideal* in a ring $R$ is an additive subgroup $I \subset R$ such that

$$ra \in I \text{ for all } r \in R,\ a \in I.$$

**Definition 1.2.1.** *Fractional ideal of $\vartheta_K$ is a finitely generated $\vartheta_K$-module, $0 \neq A \subseteq K$ such that $\alpha A \subseteq \vartheta_K$ for some $0 \neq \alpha \in \vartheta_K$, that is $I = \alpha A$ is an ideal in $\vartheta_K$. Denote $J_K$ be the group of fractional ideals of $K$ that is group of fractional ideals of $\vartheta_K$, where $\vartheta_K$ is the ring of integers of $K$.*
*Hence $J_K = \{I \mid I$ is a fractional ideal $\}$*
*Let $P_K = \{(a) \in J_K \mid 0 \neq a \in K\} \subseteq J_K$ be the group of principal ideals of $K$. $H_K := J_K/P_K$ is called the ideal class group of $K$.(class group of K).Let $h_K$ denote the number of element of $H_K$.*

**Theorem 1.2.1.** *There are finitely many ideal classes in a number field, that is $h_K < \infty$.*

The importance of class number is that as the class number approaches to 1, the number field gains the properties of a principal ideal domain. $h_K = 1$ for the number fields that are principal ideal domains.

Another definition we will encounter later will be narrow ideal class.

**Definition 1.2.2.** *Let $K$ be a real quadratic number field.*
*Define $P_K^+ = \{(\alpha) : \mathrm{N}(\alpha) > 0\}$ then the narrow ideal class is $J_K/P_K^+$*

For a real quadratic number field $K$ containing a unit of negative norm, narrow ideal class is the same as the ideal class we defined before, we can call it as *general ideal class*. However for the real quadratic number field $K$ with the property that $K$ contains no unit of negative norm, the narrow ideal class differs from the general ideal class. For example for the real quadratic number fields with $d \equiv 3 \pmod 4$, a unit has the form $\alpha = a + b\sqrt{d}$ where $a, b \in \mathbb{Z}$ and its norm is $a^2 - b^2 d$. If this unit has norm -1 then we have $a^2 - 3b^2 \equiv -1 \pmod 4$. But this can not be possible so we see that there is no unit of negative norm for the real number fields with $d \equiv 3$

(mod 4)

In that case each ordinary ideal class $A$ is the disjoint union of two narrow ideal classes $B$ and $B^*$ that is two ideals $a, \underline{b}$ are said to belong to the same narrow ideal class if $a = (\alpha)\underline{b}$ for some principal ideal $(\alpha)$ with $\mathrm{N}(\alpha) > 0$; clearly $B^* = \theta B$, where $\theta$ is the narrow ideal class of principal ideals $(\alpha)$ with $\mathrm{N}(\alpha) < 0$.

# CHAPTER 2

## Continued Fractions

In examining the Kronecker's limit formula for the real quadratic number fields we will need continued fractions, hence in this section we will define and examine some properties of continued fractions. For a quadratic irrational number $\alpha$, we will examine two continued fraction expressions. One with plus signs and the other with negative signs. [4]

**Definition 2.0.3.** *(i) A finite continued fraction is an expression of the form*

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ldots + \frac{1}{a_{n-1} + \frac{1}{a_m}}}}$$

*where each $a_i \in \mathbb{R}$ and $a_i \geq 0$ for $1 \leq i \leq m$. We use the notation $[a_0, a_1, ..., a_m]$ to denote the above expression.*

*(ii) $[a_0, a_1, ..., a_m]$ is called a simple continued fraction if $a_0, a_1, ..., a_m \in \mathbb{Z}$*

*(iii) The continued fraction $C_k = [a_0, a_1, ..., a_k], 0 \leq k \leq m$, is called the $k^{th}$ convergent of $[a_0, a_1, ..., a_m]$.*

**Proposition 2.0.1.** *Consider the continued fraction $[a_0, a_1, ..., a_m]$. Define the sequences $p_0, p_1, ..., p_m$ and $q_0, q_1, ..., q_m$ recursively as follows:*

$p_0 = a_0$ $\qquad q_0 = 1$

$p_1 = a_0 a_1 + 1$ $\qquad q_1 = a_1$

$p_k = a_k p_{k-1} + p_{k-2}$ $\qquad q_k = a_k q_{k-1} + q_{k-2}$ *for $k \geq 2$.*

*Then*

*(a) The* $k^{th}$ *convergent is* $C_k = \frac{p_k}{q_k}$

*(b)* $p_kq_{k-1} - p_{k-1}q_k = (-1)^{k-1}$, *for* $k \geq 1$.

*(c) We have the identities*

$C_k - C_{k-1} = \frac{(-1)^{k-1}}{q_kq_{k-1}}$ *for* $1 \leq k \leq m$ *and*

$C_k - C_{k-2} = \frac{a_k(-1)^k}{q_kq_{k-2}}$ *for* $2 \leq k \leq m$

*(d) We have* $C_1 > C_3 > C_5 > ...$, $C_0 < C_2 < C_4 < ...$

*and that every odd-numbered convergent* $C_{2k+1}$, $k \geq 0$,*is greater than every even-numbered convergent* $C_{2k}$, $k \geq 0$.

**Proof: (a)** We want to show $C_k = \frac{p_k}{q_k}$. Proof by induction:

for $k = 0$ $C_0 = a_0 = \frac{p_0}{q_0}$

for $k = 1$ $C_1 = a_0 + \frac{1}{a_1} = \frac{a_0a_1+1}{a_1} = \frac{p_1}{q_1}$

for $k = 2$ $C_2 = a_0 + \frac{1}{a_1+\frac{1}{a_2}} = a_0 + \frac{a_2}{a_1a_2+1} = \frac{a_0a_1a_2+a_0+a_2}{a_1a_2+1}$

Note that $p_2 = a_2(a_0q_1 + 1) + a_0$ and $q_2 = a_2a_1 + 1$. Hence $C_2 = \frac{p_2}{q_2}$

Assume it is true for $k$, i.e $C_k = \frac{p_k}{q_k}$ Using the recursive relations we get

$$
\begin{aligned}
(2.1)\frac{p_{k+1}}{q_{k+1}} &= \frac{a_{k+1}p_k + p_{k-1}}{a_{k+1}q_k + q_{k-1}} \\
&= \frac{a_{k+1}(a_kp_{k-1} + p_{k-2}) + p_{k-1}}{a_{k+1}(a_kq_{k-1} + q_{k-2}) + q_{k-1}} \\
&= \frac{(a_ka_{k+1} + 1)p_{k-1} + a_{k+1}p_{k-2}}{(a_ka_{k+1} + 1)q_{k-1} + a_{k+1}q_{k-2}}
\end{aligned}
$$

Note that $C_{k+1} = a_0 + \frac{1}{a_1+\frac{1}{a_2+...+\frac{1}{a_k+\frac{1}{a_{k+1}}}}}$ where $C_k = a_0 + \frac{1}{a_1+\frac{1}{a_2+...+\frac{1}{\frac{1}{a_k}}}}$ hence we have the term $a_k + \frac{1}{a_{k+1}}$ and $\frac{1}{a_k}$ in the same place respectively in $C_{k+1}$ and $C_k$. So if we write $a_k + \frac{1}{a_{k+1}} = \frac{a_ka_{k+1}+1}{a_{k+1}}$ in the place of $a_k$ in $C_k$ we will get $C_{k+1}$.

Hence

$$
\begin{aligned}
C_{k+1} &= \frac{(\frac{a_ka_{k+1}+1}{a_{k+1}})p_{k-1} + p_{k-2}}{(\frac{a_ka_{k+1}+1}{a_{k+1}})q_{k-1} + q_{k-2}} \\
&= \frac{(a_ka_{k+1} + 1)p_{k-1} + a_{k+1}p_{k-2}}{(a_ka_{k+1} + 1)p_{k-1} + a_{k+1}p_{k-2}}
\end{aligned}
$$

which is the same as equation (2.1) so we get $C_{k+1} = \frac{P_{k+1}}{q_{k+1}}$

**(b)** We compute

$$\begin{aligned} p_k q_{k-1} - p_{k-1} q_k &= (a_k p_{k-1} + p_{k-2}) q_{k-1} - p_{k-1}(a_k q_{k-1} + q_{k-2}) \\ &= p_{k-2} q_{k-1} - p_{k-1} q_{k-2} \\ &= -(p_{k-1} q_{k-2} - p_{k-2} q_{k-1}) \\ &= p_{k-2}(a_{k-1} q_{k-2} + q_{k-3}) - (a_{k-1} p_{k-2} + p_{k-3}) q_{k-2} \\ &= p_{k-2} q_{k-3} - p_{k-3} q_{k-2}. \end{aligned}$$

Repeating this process using the recursion after $k-1$ steps we will have

$$p_k q_{k-1} - p_{k-1} q_k = (-1)^{k-1}[p_1 q_0 - p_0 q_1] = (-1)^{k-1}[a_0 a_1 + 1 - a_0 a_1] = (-1)^{k-1}.$$

**(c)**From part (a) we have $C_k = \frac{p_k}{q_k}$.

Consider

$$C_k - C_{k-1} = \frac{p_k}{q_k} - \frac{p_{k-1}}{q_{k-1}} = \frac{p_k q_{k-1} - p_{k-1} q_k}{q_k q_{k-1}}$$

from part (b) we get

$$C_k - C_{k-1} = \frac{(-1)^{k-1}}{q_k q_{k-1}}$$

For $C_k - C_{k-2}$ we have

$$\begin{aligned} C_k - C_{k-2} &= \frac{p_k}{q_k} - \frac{p_{k-2}}{q_{k-2}} \\ &= \frac{p_k}{q_k} - \frac{p_k - a_k p_{k-1}}{q_k - a_k q_{k-1}} \\ &= \frac{-a_k(p_k q_{k-1} - p_{k-1} q_k)}{q_k q_{k-2}} \\ &= \frac{-a_k(-1)^{k-1}}{q_k q_{k-2}} \\ &= \frac{a_k(-1)^k}{q_k q_{k-2}}. \end{aligned}$$

**(d)**We want to show $C_0 < C_2 < C_4 < \dots$

In part (c) we had

$$C_k - C_{k-2} = \frac{a_k(-1)^k}{q_k q_{k-2}}, 2 \leq k \leq m$$

For $k = 2$, $C_2 - C_0 = \frac{a_2}{q_2 q_0} = \frac{a_2}{a_2 a_1 + 1} > 0$ which implies $C_2 > C_0$

For an even number $k$, say $k = 2n$, $n \in \mathbb{Z}$ we have

$$C_{2n} - C_{2n-2} = \frac{a_{2n}(-1)^{2n}}{q_{2n} q_{2n-2}} = \frac{a_{2n}}{q_{2n} q_{2n-2}}.$$

Since by the definition of continued fractions $a_i \geq 0$, $0 \leq i \leq m$, $q_k$ is also positive. Hence $C_{2n} - C_{2n-2}$ is positive and we get $C_0 < C_2 < C_4 < \ldots$

On the other hand, for an odd $k$, $k = 2n - 1, n \in \mathbb{Z}$ we have

$$C_{2n-1} - C_{2n-3} = \frac{a_{2n-1}(-1)^{2n-1}}{q_{2n-1} q_{2n-3}} = \frac{-a_{2n-1}}{q_{2n-1} q_{2n-3}}$$

Since $a_i$ and $q_i$ are positive for $0 \leq i \leq m$ we have $C_{2n-1} - C_{2n-3} \leq 0$. Hence we get $C_1 > C_3 > \ldots$.

In addition to these to show every $C_k$ with an odd index is greater that $C_k$ with an even index, consider $C_{2j+1} - C_{2j}$ for $j \geq 0$. From part (c) we had $C_{2j+1} - C_{2j} = \frac{(-1)^{2j}}{q_{2j+1} q_{2j}} = \frac{1}{q_{2j+1} q_{2j}} > 0$. Hence $C_{2j+1} > C_{2j}$ which implies $C_1 > C_3 > \ldots > C_{2k+1} > C_{2k} > \ldots > C_4 > C_2 > C_0$. □

**Definition 2.0.4.** *We define the continued fraction $[a_0, a_1, \ldots]$ to be the limit as $k \to \infty$ of its $k^{th}$ convergent $C_k$ and write*

$$[a_0, a_1, \ldots] = \lim_{k \to \infty} C_k.$$

**Proposition 2.0.2.** *Let $\{a_i\}_{i \geq 0}$ be an infinite sequence of integers with $a_i \geq 0$ for $i \geq 1$ and let $C_k = [a_0, \ldots, a_k]$. Then the sequence $\{C_k\}$ converges.*

**Proof:** From Proposition (2.0.1)(d) we know $\{C_{2j+1}\}_{j=0}^{\infty}$ is decreasing and bounded from below, for example by $C_0$. Then it is convergent, say $\lim_{j \to \infty} C_{2j+1} = \alpha$

Also, $C_1 > C_{2j}$ for $j = 1, \ldots$. So $\{C_{2j}\}_{j=1}^{\infty}$ is bounded above and it is increasing. So let $\lim_{j \to \infty} C_{2j} = \beta$. We want to show $\alpha = \beta$. In Proposition (2.0.1)(b) we had $C_k - C_{k-1} = \frac{(-1)^k}{q_k q_{k-1}}$ where $q_k q_{k-1} = a_k q_{k-1}^2 + q_{k-2} q_{k-1} \geq 0$

So it suffices to show that the denominator grows without bound. We will assume $a_i \geq 1$ for all $i \geq 1$. Then we have $q_0 = 1$, $q_1 = a_1 \geq 1$, then by recurrence relation we get $q_2 = a_2 a_1 + 1 \geq 2$. We claim that $q_k \geq k$ for $k \geq 2$. For $k = 2$

we see that this is true. Now assume this statement is true for $k$ and consider $q_{k+1} = a_{k+1}q_k + q_{k-1} \geq k + k - 1$ for every $k \geq 2$. Since $k \geq 2$ it follows that $2k - 1 \geq k + 1$. Therefore $q_k q_{k-1} \geq k(k-1)$ which implies

$$|C_{2j+1} - C_{2j}| = |\frac{1}{q_{2j+1}q_{2j}}| < \frac{1}{(2j+1)2j} \to 0 \mathrm{as} j \to \infty$$

□

The next proposition constructs a continued fraction expression for an irrational number $\alpha$, i.e it finds a sequence $\{a_i\}_{i=1}^{\infty}$ such that $[a_0, a_1, \ldots] = \alpha$.

**Proposition 2.0.3.** *Let $\alpha = \alpha_0$ be an irrational real number greater than zero. Define the sequence $\{a_i\}_{i\geq 0}$ recursively as follows, $a_k = \llcorner \alpha_k \lrcorner$, $\alpha_{k+1} = \frac{1}{\alpha_k - a_k}$. Then $\alpha = [a_0, \ldots]$ is a representation of $\alpha$ as a simple continued fraction.*

**Proof:** Using $\alpha_{k+1} = \frac{1}{\alpha_k - a_k}$ we have $\alpha_{k+1}\alpha_k - \alpha_{k+1}a_k = 1$ which gives $\alpha_k = a_k + \frac{1}{\alpha_{k+1}}$.

$$\alpha_0 = a_0 + \frac{1}{\alpha_1}$$

$$\alpha_0 = a_0 + \frac{1}{a_1 + \frac{1}{\alpha_2}}$$

$$\vdots$$

$$\alpha_0 = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ldots + a_k + \frac{1}{\alpha_{k+1}}}}.$$

Therefore $\alpha_0 = [a_0, \ldots, a_k, \alpha_{k+1}]$. We define $C_k = [a_0, \ldots, a_k]$ so we want to show that $\lim_{k\to\infty} C_k = \alpha = \alpha_0$

From Proposition (2.0.2) we know that $\{C_k\}_{k\geq 0}$ converges and we have $C_0 < C_2 < \ldots < C_5 < C_3 < C_1$. If we can show $C_{2k} \leq \alpha \leq C_{2k+1}$ for all $k$ we will be done.

We have $a_2 = \llcorner \alpha_2 \lrcorner$, so $a_2 \leq \alpha_2$ which gives $a_1 + \frac{1}{a_2} \geq a_1 + \frac{1}{\alpha_2}$ and $\frac{1}{a_1 + \frac{1}{a_2}} \leq \frac{1}{a_1 + \frac{1}{\alpha_2}}$

Hence $C_2 = a_0 + \frac{1}{a_1 + \frac{1}{a_2}} \leq a_0 + \frac{1}{a_1 + \frac{1}{\alpha_2}} = \alpha$.

By repeating the same argument one can show

$$C_{2k} \leq \alpha = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \ldots + a_{2k-1} + \frac{1}{\alpha_{2k}}}}$$

for all $k$.

Now let us show $C_{2k+1} \geq \alpha$. We have $a_k = \llcorner \alpha_k \lrcorner$ so we have $a_1 \leq \alpha_1$, this implies $C_1 = a_0 + \frac{1}{a_1} \leq a_0 + \frac{1}{\alpha_1} = \alpha$. Also if we start with $a_3 \leq \alpha_3$ we see that $a_2 + \frac{1}{a_3} \geq a_2 + \frac{1}{\alpha_3}$ and so $C_3 = a_0 + \frac{1}{a_2 + \frac{1}{a_3}} \leq a_0 + \frac{1}{a_2 + \frac{1}{\alpha_3}} = \alpha$. Applying the same argument one can show

$$C_{2k+1} \geq \alpha = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \ldots + a_{2k} + \frac{1}{\alpha_{2k+1}}}}$$

(Note that we started with $a_{2k+1} \leq \alpha_{2k+1}$).

We have $\lim_{k\to\infty} C_k = \alpha$ since $\lim_{k\to\infty} C_{2k} = \lim_{k\to\infty} C_{2k+1}$. $\square$

Next we look at continued fraction expressions for quadratic irrational numbers.

**Proposition 2.0.4.** *Let $\alpha$ be a positive quadratic irrational number. Then there are integers $P_0, Q_0, d$ such that $\alpha = \frac{P_0+\sqrt{d}}{Q_0}$ with $Q_0 | (d - P_0^2)$. Recursively define $\alpha_k = \frac{P_k+\sqrt{d}}{Q_k}$, $a_k = \llcorner \alpha_k \lrcorner$, $P_{k+1} = a_k Q_k - P_k$ and $Q_{k+1} = \frac{d-P_{k+1}^2}{Q_k}$. Then $[a_0, a_1, \ldots]$ is the simple continued fraction of $\alpha$.*

**Proof:** First let us show $\alpha$ can be written as $\frac{P_0+\sqrt{d}}{Q_0}$ where $P_0, Q_0, d$ are integers. So let $a, b, c, d'$ be integers where $d'$ is square free and $\alpha \in \mathbb{Q}(\sqrt{d})$. Then $\alpha = \frac{a+b\sqrt{d'}}{c} = \frac{a+\sqrt{b^2d'}}{c} = \frac{ac+\sqrt{c^2b^2d'}}{c^2}$. Let $P_0 = ac, Q_0 = c^2$ then $Q_0 | (d - P_0^2)$ where $d = c^2b^2d'$

Next we want to show $\alpha = [a_0, a_1, \ldots]$. In Proposition (2.0.3) we showed that if $a_k = \llcorner \alpha_k \lrcorner$ and $\alpha_{k+1} = \frac{1}{\alpha_k - a_k}$ then $[a_0, a_1, \ldots] = \alpha$, where $\alpha$ is an irrational real number. So it will be enough to show that $\alpha_{k+1} = \frac{1}{\alpha_k - a_k}$.

We have $\alpha_{k+1} = \frac{P_{k+1}+\sqrt{d}}{Q_{k+1}}$ and $P_{k+1} = a_k Q_k - P_k$ so $\alpha_{k+1} = \frac{Q_k(P_{k+1}+\sqrt{d})}{Q_k Q_{k+1}}$. By the definition of $Q_k$ we have

$$
\begin{aligned}
Q_{k+1} = \frac{d - P_{k+1}^2}{Q_k} \Longrightarrow \alpha_{k+1} &= \frac{Q_k(P_{k+1} + \sqrt{d})}{d - P_{k+1}^2} \\
&= \frac{Q_k}{\sqrt{d} - P_{k+1}} \\
&= \frac{Q_k}{\sqrt{d} - a_k Q_k + P_k} \\
&= \frac{1}{\frac{P_k + \sqrt{d}}{Q_k} - a_k} \\
&= \frac{1}{\alpha_k - a_k}
\end{aligned}
$$

Hence we are done by Proposition (2.0.3). □

**Definition 2.0.5.** *A simple continued fraction is called periodic with period $k$ if there exists positive integers $N, k$ such that $a_n = a_{n+k}$ for all $n \geq N$. We denote such a continued fraction by* $[a_0, \ldots, a_{N-1}, \overline{a_N, a_{N+1}, \ldots, a_N + k - 1}]$

**Proposition 2.0.5.** *The simple continued fraction expansion of a positive quadratic irrational is periodic.*

**Proof:** Let $\alpha$ be a positive quadratic irrational number. By Proposition (2.0.4), we can write $\alpha = \frac{P_0 + \sqrt{d}}{Q_0}$ where $Q_0 | (P_0^2 - d)$ and $Q_0, P_0, d \in \mathbb{Z}$ and $\alpha = [a_0, a_1, ...,]$ where $a_k = \llcorner \alpha_k \lrcorner$ as defined in Proposition (2.0.4). Now, from recursion definition in Proposition (2.0.1)(a) we get

$$
\begin{aligned}
\alpha = \frac{\alpha_k p_{k-1} + p_{k-2}}{\alpha_k q_{k-1} + q_{k-2}} &\Longrightarrow \alpha \alpha_k q_{k-1} + \alpha q_{k-2} = \alpha_k p_{k-1} + p_{k-2} \\
&\Longrightarrow \alpha_k = \frac{p_{k-2} - \alpha q_{k-2}}{\alpha q_{k-1} - p_{k-1}}
\end{aligned}
$$

Let $\alpha'$ be the $\mathbb{Q}$-conjugate of $\alpha$, i.e $\alpha' = \frac{P_0 - \sqrt{d'}}{Q_0}$. Then

$$
\alpha' = \frac{\alpha'_k p_{k-1} + p_{k-2}}{\alpha'_k q_{k-1} + q_{k-2}} \Longrightarrow \alpha'_k = -\frac{q_{k-2}}{q_{k-1}} \Big( \frac{\alpha' - C_{k-2}}{\alpha' - C_{k-1}} \Big)
$$

Since $C_{k-1}, C_{k-2} \to \alpha$ as $k \to \infty$ we have $\lim_{k \to \infty} \left( \frac{\alpha' - C_{k-2}}{\alpha' - C_{k-1}} \right) = 1 \Longrightarrow \alpha'_k < 0$. Also we know that $\alpha_k > 0$ so we get $\alpha_k - \alpha'_k = \frac{2\sqrt{d}}{Q_k} > 0$ for all sufficiently large $k$.

From the recursive relations defined in Proposition (2.0.4) we have $Q_k Q_{k+1} = d - P_{k+1}^2$ so $Q_k \leq Q_k Q_{k+1} = d - P_{k+1}^2 \leq d$ and $P_{k+1}^2 \leq d - Q_k \leq d$ for sufficiently large $k$. Thus there are only finitely many possible values for $P_k$ and we conclude that there exists integers $i < j$ such that $P_i = P_j$. Now let us show $Q_i = Q_j$.
We have $Q_i = \frac{d-P_i^2}{Q_{i-1}}$ and $Q_j = \frac{d-P_j^2}{Q_{j-1}}$ Since $P_i = P_j$ we get $Q_i Q_{i-1} = Q_j Q_{j-1}$ We also have $P_{k+1}^2 \leq d - Q_{k+1} \leq d$.
Now let us write this equation for $i$ and $j$.

$$P_i^2 \leq d - Q_i \leq d$$

$$P_j^2 \leq d - Q_j \leq d$$

Dividing these we get $1 \leq \frac{d-Q_i}{d-Q_j} \leq 1$ which gives $Q_i = Q_j$ implying that $a_i = a_j$, since $a_i$ are defined recursively depending on $Q_i$ and $P_i$. So we have $\alpha = [a_0, a_1, ..., a_{i-1}, \overline{a_i, ..., a_{j-1}}]$. $\square$

Now define the "minus" continued fraction for a quadratic irrational $\alpha$ with $\alpha > \alpha'$ as

$$\alpha = a_0 - \cfrac{1}{a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \vdots}}}$$

with $a_i \in \mathbb{Z}$, for all $i$ and $a_i \geq 2$ for $i = 1, 2, \ldots$. First of all let us show that one can have the "minus" continued fractions. First we need an analogy of Proposition (2.0.1).

**Proposition 2.0.6.** *Define* $p_0 = a_0 \qquad q_0 = 1$

$$p_1 = a_0 a_1 - 1 \qquad q_1 = a_1$$

$$p_k = a_k p_{k-1} - p_{k-2} \qquad q_k = a_k q_{k-1} - q_{k-2}$$

*and let*

$$C_k = a_0 - \cfrac{1}{a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots \frac{1}{a_k}}}}$$

*Then (a)* $C_k = \frac{p_k}{q_k}$

*(b)* $p_k q_{k-1} - p_{k-1} q_k = (-1)$ *for* $k \geq 1$

*(c)* $C_k - C_{k-1} = \frac{(-1)}{q_k q_{k-1}}$ *and* $C_k - C_{k-2} = \frac{a_k(-1)}{q_k q_{k-2}}$

*(d)* $C_0 > C_1 > C_2 > \ldots$

**Proof: (a)** Proof by induction:

For $k = 0$ $C_0 = a_0 = \frac{p_0}{q_0}$,

for $k = 1$ $C_1 = a_0 - \frac{1}{a_1} = \frac{a_0 a_1 - 1}{a_1} = \frac{p_1}{q_1}$,

for $k = 2$ $C_2 = a_0 - \frac{1}{a_1 - \frac{1}{a_2}} = a_0 - \frac{a_2}{a_1 a_2 - 1} = \frac{a_0 a_1 a_2 - a_0 - a_2}{a_1 a_2 - 1} = \frac{p_2}{q_2}$.

Assume it is true for $k$, i.e $C_k = \frac{p_k}{q_k}$ do we have $C_{k+1} = \frac{p_{k+1}}{q_{k+1}}$

Note that $C_{k+1} = a_0 - \cfrac{1}{a_1 - \cfrac{1}{a_2 - \vdots - \cfrac{1}{a_k - \cfrac{1}{a_{k+1}}}}}$ where $C_k = a_0 - \cfrac{1}{a_1 - \cfrac{1}{a_2 - \vdots - \cfrac{1}{\cfrac{1}{a_k}}}}$ hence we have the term $a_k - \frac{1}{a_{k+1}}$ and $\frac{1}{a_k}$ in the same place respectively in $C_{k+1}$ and $C_k$.

So if we write $a_k - \frac{1}{a_{k+1}} = \frac{a_k a_{k+1} - 1}{a_{k+1}}$ in the place of $a_k$ in $C_k$ we will get $C_{k+1}$.

Hence

$$\begin{aligned}
C_{k+1} &= \frac{(\frac{a_k a_{k+1}-1}{a_{k+1}})p_{k-1} - p_{k-2}}{(\frac{a_k a_{k+1}-1}{a_{k+1}})q_{k-1} - q_{k-2}} \\
&= \frac{(a_k a_{k+1} - 1)p_{k-1} - a_{k+1} p_{k-2}}{(a_k a_{k+1} - 1)p_{k-1} - a_{k+1} p_{k-2}} \\
&= \frac{a_{k+1}(a_k p_{k-1} - p_{k-2}) - p_{k-1}}{a_{k+1}(a_k q_{k-1} - q_{k-2}) - q_{k-1}} \\
&= \frac{p_{k+1}}{q_{k+1}}
\end{aligned}$$

**(b)** We have

$$\begin{aligned}
p_k q_{k-1} - p_{k-1} q_k &= (a_k p_{k-1} - p_{k-2}) q_{k-1} - p_{k-1}(a_k q_{k-1} - q_{k-2}) \\
&= -p_{k-2} q_{k-1} + p_{k-1} q_{k-2} \\
&= (p_{k-1} q_{k-2} - p_{k-2} q_{k-1}) \\
&= ((a_{k-1} p_{k-2} - p_{k-3}) q_{k-2} - p_{k-2}(a_{k-1} q_{k-2} - q_{k-3})) \\
&= (-p_{k-3} q_{k-2} + p_{k-2} q_{k-3}) \\
&= (p_{k-2} q_{k-3} - p_{k-3} q_{k-2})
\end{aligned}$$

Repeating this process using the recursion after $k-1$ steps we will have $p_1 q_0 - p_0 q_1 = a_0 a_1 - 1 - (a_0 a_1) = -1$

**(c)** we can compute

$$C_k - C_{k-1} = \frac{p_k}{q_k} - \frac{p_{k-1}}{q_{k-1}} = \frac{p_k q_{k-1} - q_k p_{k-1}}{q_k q_{k-1}} = \frac{(-1)}{q_k q_{k-1}}$$

and

$$\begin{aligned} C_k - C_{k-2} &= \frac{p_k}{q_k} - \frac{p_{k-2}}{q_{k-2}} \\ &= \frac{p_k q_{k-2} - q_k p_{k-2}}{q_k q_{k-2}} \\ &= \frac{(a_k p_{k-1} - p_{k-2}) q_{k-2} - (a_k q_{k-1} - q_{k-2}) p_{k-2}}{q_k q_{k-2}} \\ &= \frac{a_k (p_{k-1} q_{k-2} - q_{k-1} p_{k-2})}{q_k q_{k-2}} \\ &= \frac{-a_k}{q_k q_{k-2}} \end{aligned}$$

**(d)** We have $C_k - C_{k-1} = \frac{(-1)}{q_k q_{k-1}}$ also $q_k q_{k-1} > 0$ for any $k$, hence $C_k - C_{k-1} < 0$ therefore $C_k < C_{k-1}$ So we get what we want.□

The next proposition tells us how to construct a "minus" continued fraction for a quadratic irrational $\alpha$.

**Proposition 2.0.7.** *Let $\alpha = \alpha_0$ be an irrational real number greater than 0. Define $\{a_i\}_{i \geq 0}$ recursively as follows: $a_k = \ulcorner \alpha_k \urcorner$=least integer greater that $\alpha_k$, so we have $a_k \geq 2$. Define $\alpha_{k+1} = \frac{1}{a_k - \alpha_k}$. Then $\alpha = [a_0, \ldots]$ is a representation of $\alpha$ as a simple continued fraction.*

**Proof:** We have $\alpha_{k+1} = \frac{1}{a_k - \alpha_k} \implies \alpha_k = a_k - \frac{1}{\alpha_{k+1}} \implies \alpha_0 = a_0 - \frac{1}{\alpha_1} = a_0 - \frac{1}{a_1 - \frac{1}{\alpha_2}} \implies \alpha_0 = [a_0, a_1, \ldots, a_{k-1}, \alpha_k]$
We want to show $C_k \to \alpha$. Note that $a_k > \alpha_k$, so $a_{k-1} - \frac{1}{a_k} > a_{k-1} - \frac{1}{\alpha_k}$ which gives $a_{k-2} - \frac{1}{a_{k-1} - \frac{1}{a_k}} > a_{k-2} - \frac{1}{a_{k-1} - \frac{1}{\alpha_k}}$. By applying the same procedure we get

$$C_k = a_0 - \cfrac{1}{a_1 - \cfrac{1}{\vdots \frac{1}{a_k}}} > \alpha = a_0 - \cfrac{1}{a_1 - \cfrac{1}{\vdots \frac{1}{\alpha_k}}}.$$

Hence $C_k > \alpha$ for any $k$. So $\{C_k\}_{k=1}^{\infty}$ is a monotonicly decreasing and bounded below by $\alpha$, it is convergent. Now we have to show $\lim_{k \to \infty} C_k = \alpha$.

We have $C_k = \frac{p_k}{q_k} = \frac{a_k p_{k-1} - p_{k-2}}{a_k q_{k-1} - q_{k-2}}$ and if we write $\alpha_k$ instead of $a_k$ we will get $\alpha$. So

$$\begin{aligned} |C_k - \alpha| &= \Big|\frac{a_k p_{k-1} - p_{k-2}}{a_k q_{k-1} - q_{k-2}} - \frac{\alpha_k p_{k-1} - p_{k-2}}{\alpha_k q_{k-1} - q_{k-2}}\Big| \\ &= \Big|\frac{(a_k p_{k-1} - p_{k-2})(\alpha_k q_{k-1} - q_{k-2}) - (a_k q_{k-1} - q_{k-2})(\alpha_k p_{k-1} - p_{k-2})}{(a_k q_{k-1} - q_{k-2})(\alpha_k q_{k-1} - q_{k-2})}\Big| \\ &= \Big|\frac{a_k - \alpha_k}{q_k(\alpha_k q_{k-1} - q_{k-2})}\Big| \end{aligned}$$

Note that $0 \leq a_k - \alpha_k < 1$. We will show that the denominator grows without bound. First we will show $q_k > q_{k-1} + 1$

We have $q_0 = 1$ and $q_1 = a_1 \geq 2$ implies $q_1 \geq q_0 + 1$. For $k = 2$ since $a_1 \geq 2$ we have $a_2 \geq 1 + \frac{2}{a_1}$ so $a_1 a_2 \geq a_1 + 2$ implying $a_1 a_2 - 1 \geq a_1 + 1$. So $q_2 = a_2 a_1 - 1 \geq q_1 + 1 = a_1 + 1$.

Now let us use induction, assume for $k$ we have $q_k > q_{k-1} + 1$, consider $q_{k+1} = a_{k+1} q_k - q_{k-1}$ by induction hypothesis $q_{k-1} < q_k - 1$, so $q_{k+1} > a_{k+1} q_k - (q_k - 1) = (a_{k+1} - 1) q_k + 1$. Note that $(a_{k+1} - 1) \geq 1$ So $q_{k+1} > q_k + 1$. Further more we can show $q_k \geq k + 1$. This follows from the equation

$$q_k \geq q_{k-1} + 1 \geq q_{k-2} + 2 \geq \ldots \geq q_0 + k = 1 + k$$

Now consider $(\alpha_k q_{k-1} - q_{k-2})$ where $q_{k-1} > q_{k-2} + 1$. Also $\alpha_k > 1$ since $\alpha_k = \frac{1}{a_{k-1} - \alpha_{k-1}} > 1$. So $\alpha_k q_{k-1} > q_{k-2} + 1$,i.e $\alpha_k q_{k-1} - q_{k-2} > 1$. Hence

$$|C_k - \alpha| = \Big|\frac{a_k - \alpha_k}{q_k(\alpha_k q_{k-1} - q_{k-2})}\Big| < \Big|\frac{1}{q_k(\alpha_k q_{k-1} - q_{k-2})}\Big| < \Big|\frac{1}{k+1}\Big| \to 0 \text{as k} \to \infty$$

□

**Proposition 2.0.8.** *Let $\alpha$ be a positive quadratic irrational. Then there are integers $P_0, Q_0, d$ such that $\alpha = \frac{P_0 + \sqrt{d}}{Q_0}$ with $Q_0 | (d - P_0^2)$. Recursively define $\alpha_k = \frac{P_k + \sqrt{d}}{Q_k}$, $a_k = \ulcorner \alpha_k \urcorner$, $P_{k+1} = a_k Q_k - P_k$ ,$Q_{k+1} = \frac{P_{k+1}^2 - d}{Q_k}$. Then $[a_0, a_1, \ldots,]$ is the simple continued fraction of $\alpha$.*

**Proof:** The proof of the first part is the same as Proposition (2.0.4).

Next we want to show $\alpha = [a_0, a_1, \ldots]$. In Proposition (2.0.7) we showed that if

$a_k = \ulcorner \alpha_k \urcorner$ and $\alpha_{k+1} = \frac{1}{a_k - \alpha_k}$ then $[a_0, a_1, \ldots] = \alpha$, where $\alpha$ is an quadratic irrational. So it will be enough to show that $\alpha_{k+1} = \frac{1}{a_k - \alpha_k}$.
We have $\alpha = \frac{P_{k+1}+\sqrt{d}}{Q_{k+1}}$ and $P_{k+1} = a_k Q_k - P_k$ therefore $\alpha_{k+1} = \frac{Q_k(P_{k+1}+\sqrt{d})}{Q_k Q_{k+1}}$. By the definition of $Q_k$ we have

$$\begin{aligned} Q_{k+1} = \frac{P_{k+1}^2 - d}{Q_k} \Longrightarrow \alpha_{k+1} &= \frac{Q_k(P_{k+1} + \sqrt{d})}{P_{k+1}^2 - d} \\ &= \frac{Q_k}{P_{k+1} - \sqrt{d}} = \frac{Q_k}{a_k Q_k - P_k - \sqrt{d}} \\ &= \frac{1}{a_k - \frac{P_k + \sqrt{d}}{Q_k}} = \frac{1}{a_k - \alpha_k} \end{aligned}$$

Hence we are done by Proposition (2.0.7).□

**Proposition 2.0.9.** *$\alpha$ is a positive quadratic irratinal if and only if the simple continued fraction of $\alpha$ is periodic.*

**Proof:** Let $\alpha$ be a positive quadratic irrational number. Also we have $\alpha = [a_0, \ldots, a_k, \ldots] = [a_0, \ldots, a_{k-1}, \alpha_k]$. So we can write

$$\alpha_k = a_k - \cfrac{1}{a_{k+1} - \cfrac{1}{\vdots}}$$

By Proposition (2.8), we can write $\alpha = \frac{P_0+\sqrt{d}}{Q_0}$ where $Q_0|(P_0^2 - d)$ and $Q_0, P_0, d \in \mathbb{Z}$ and $\alpha = [a_0, a_1, ...,]$ where $a_k = \ulcorner \alpha_k \urcorner$ as defined in Proposition (2.0.8). Now, from recursion definition in Proposition (2.0.6) we get

$$\begin{aligned} \alpha = \frac{\alpha_k p_{k-1} - p_{k-2}}{\alpha_k q_{k-1} - q_{k-2}} &\Longrightarrow \alpha\alpha_k q_{k-1} - \alpha q_{k-2} = \alpha_k p_{k-1} - p_{k-2} \\ &\Longrightarrow \alpha_k = \frac{\alpha q_{k-2} - p_{k-2}}{\alpha q_{k-1} - p_{k-1}} \end{aligned}$$

Let $\alpha'$ be the $\mathbb{Q}$-conjugate of $\alpha$, i.e $\alpha' = \frac{P_0-\sqrt{d}}{Q_0}$. Then

$$\alpha' = \frac{\alpha'_k p_{k-1} - p_{k-2}}{\alpha'_k q_{k-1} - q_{k-2}} \Longrightarrow \alpha'_k = \frac{q_{k-2}}{q_{k-1}}\Big(\frac{\alpha' - C_{k-2}}{\alpha' - C_{k-1}}\Big)$$

Note that $q_k$ is increasing. One can show this by induction: we have $q_1 = a_1 \geq 2$ so assume for $k$ we have $1 = q_0 < q_1 < \ldots < q_k$. Then $q_{k+1} = a_{k+1}q_k - q_{k-1} \geq 2q_k > q_k$. So $\frac{q_{k-2}}{q_{k-1}} < 1$. Also $C_{k-1}, C_{k-2} \to \alpha$ as $k \to \infty$ and $C_k$ is monotone decreasing so $\frac{\alpha' - C_{k-2}}{\alpha' - C_{k-1}}$ tends to 1 from below. Thus we conclude that there exists $N > 1$ such that for all $n > N$ we have $0 < \alpha'_k < 1$. Also $\alpha_k = \frac{1}{a_{k-1} - \alpha_{k-1}} > 1$ since $a_{k-1} = \ulcorner \alpha_{k-1} \urcorner$. It follows that $0 < \frac{P_k - \sqrt{d}}{Q_k} < 1$ and $\frac{P_k + \sqrt{d}}{Q_k} > 1$. Since $\alpha_k - \alpha'_k = \frac{2\sqrt{d}}{Q_k} > 0$ implies $Q_k > 0$. So we get $|P_k - Q_k| < \sqrt{d}$ and hence can take only finitely many values for a given $d$. Thus we have $d - (Q_k - P_k)^2 > 0$ and this expression also can take only finitely many values.

Using the recursive relation $P_{k+1} = a_k Q_k - P_k$ and $Q_{k+1}Q_k = P_{k+1}^2 - d$ we can write

$$\begin{aligned} d - (P_k - Q_k)^2 &= d - P_k^2 - Q_k^2 + 2P_kQ_k = -Q_kQ_{k-1} - Q_k^2 + 2P_kQ_k \\ &= Q_k(-Q_{k-1} - Q_k + 2P_k) \end{aligned}$$

Hence $Q_k | (d - (P_k - Q_k)^2)$ this implies $Q_k$ takes only finitely many values and so does $P_k$. Therefore for some $j \neq k$ $\alpha_j = \alpha_k$ that is $a_j = a_k$. Hence the continued fraction of $\alpha$ is periodic. Conversely, it is easily seen that if $\alpha$ has a periodic continued fraction then it satisfies a polynomial of degree 2 with integer coefficients. □ [6]

**Proposition 2.0.10.** *Let $\alpha$ be a real quadratic irrational with $\alpha > 1$ and $0 < \alpha' < 1$. Such an $\alpha$ is called reduced if and only if the continued fraction of a reduced real quadratic irrational, $\alpha = [a_0, a_1, \ldots]$ is pure periodic, i.e $a_j = a_{j+r}$ for some period $r$ and for all $j \geq 0$.*

**Proof:** Note that in Proposition (2.0.9) we showed for $\alpha_k$ with $\alpha_k > 1$ and $0 < \alpha'_k < 1$ has a periodic continued fraction,i.e there exits $j \neq k$ with $a_j = a_k$, $a_{j+1} = a_{k+1}$ and so on. Now we will show it is in fact pure periodic, i.e we also have $a_{j-1} = a_{k-1}$. Now we have $\alpha_{i+1} = \frac{1}{a_i - \alpha_i}$ and $\alpha'_{i+1} = \frac{1}{a_i - \alpha'_i}$. We claim that $0 < \alpha'_i < 1$ for all $i \geq 0$. For $i = 0$ we have $0 < \alpha'_0 = \alpha' < 1$, assume this is true for $i$ so $0 < \alpha'_i < 1$, then we get $a_i - \alpha'_i > 1$ implying that $0 < \alpha'_{i+1} < 1$. We can write $a_{i-1} = \frac{1}{\alpha'_i} + \alpha'_{i-1}$ and since $\alpha'_{i-1} < 1$ we get $a_{i-1} = \ulcorner \frac{1}{\alpha'_i} \urcorner$. From this we can conclude that $\alpha_j = \alpha_k$ implies $a_{j-1} = a_{k-1}$. Conversely, if $\alpha$ is pure periodic, $\alpha = (\overline{a_1, \ldots, a_r})$

then $\alpha > 1$ since $a_1 \geq 2$. We continue this periodic sequence by setting $a_{i+r} = a_i$ for all $i \in \mathbb{N}$. Then we can define

$$x_i = \frac{1}{\overline{a_{i-1}, \ldots, a_{i-r}}}$$

Then $\frac{1}{x_{i+1}} = a_i - x_i$ or $x_i = a_i - \frac{1}{x_{i+1}}$. It follows that

$$x_1 = \frac{1}{\overline{a_r, \ldots, a_1}}$$

satisfies the equation

$$x_1 = (a_1, a_2, \ldots, a_{r-1}, x_1) = a_1 - \frac{1}{a_2 - \ldots - \frac{1}{a_{r-1} - \frac{1}{x_1}}}$$

which is the same as the quadratic equation that $\alpha$ satisfies, but $0 < x_1 < 1$ and so $x_1$ is different then $\alpha$. Therefore $x_1 = \alpha'$, so that $0 < \alpha' < 1$. $\square$.[6]

In this section we worked with quadratic number fields, which are the special type of number fields. We call a field $L$ a number field if $[L : \mathbb{Q}] = n < \infty$. In that case every element of $L$ will be an algebraic number, that is the elements will satisfy a corresponding polynomial of degree $n$ over $\mathbb{Q}$. For a number field of degree $n$ over $\mathbb{Q}$, one has Dirichlet's Unit Theorem which tells that the unit group is finitely generated. We also have ideal class groups of number fields and for a number field of degree $n$ the class number is finite. For further information about number fields, readers are suggested to read Marcus' Number Fields or Neukirch's Algebraic Number Theory. [1],[3]

# CHAPTER 3

## Zeta Functions

In this paper we will deal with zeta functions of quadratic number fields. Zeta functions among other things play an important role in the calculation of class number of a number field. However, we will not give formula for calculating class numbers but mention important formulas that are crucial for finding the class numbers. Now let us investigate the convergence of the Riemann's zeta function.

**Proposition 3.0.11.** *The Riemann's zeta function is defined by*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

*where s is a complex variable and Re(s)$> 1$. The series is absolutely and uniformly convergent in the domain Re(s)$\geq 1 + \delta$ for every $\delta > 0$. It therefore represents an analytic function in the half-plane Re(s)$> 1$. One has Euler's identity*

$$\zeta(s) = \prod_{p} \frac{1}{1 - p^{-s}}$$

*$Re(s) > 1$ where p runs through the prime numbers.*

**Proof:** To show that the Riemann's zeta function is absolutely convergent consider

$$\mid \zeta(s) \mid = \mid \sum_{n=1}^{\infty} \frac{1}{n^s} \mid \leq \sum_{n=1}^{\infty} \mid \frac{1}{n^s} \mid = \sum_{n=1}^{\infty} \frac{1}{\mid n^s \mid}$$

Note that $s = \sigma + it$ where $\sigma, t \in \mathbb{R}$ and $\mid n^s \mid = n^{Re(s)} = n^\sigma$.Then we have

$$\sum_{n=1}^{\infty} \frac{1}{\mid n^s \mid} = \sum_{n=1}^{\infty} \frac{1}{n^\sigma}$$

converges for $\text{Re}(s) = \sigma \geq 1 + \delta$ for every $\delta > 0$.

Now let us prove Euler's identity. An infinite product $\prod_{n=1}^{\infty} a_n$ of complex numbers $a_n$ is said to be convergent if the series $\sum_{n=1}^{\infty} \log(a_n)$ converges, where log denotes the principal branch of the logarithm ($\mid$ Image $\log z \mid < \pi$). The product is called absolutely convergent if the series converges absolutely, that is the product converges to the same limit after reordering of its terms. So let us examine the convergence of Euler's identity. Let us take the logarithm of $\prod_p \frac{1}{1-p^{-s}}$. We get

$$\sum_p \sum_{n=1}^{\infty} \frac{1}{np^{ns}}$$

Now take the absolute value and get,

$$\begin{aligned} \mid \sum_p \sum_{n=1}^{\infty} \frac{1}{np^{ns}} \mid &\leq \sum_p \sum_{n=1}^{\infty} \frac{1}{\mid np^{ns} \mid} \\ &\leq \sum_p \sum_{n=1}^{\infty} \frac{1}{\mid p^{ns} \mid} = \sum_p \sum_{n=1}^{\infty} (\frac{1}{p^\sigma})^n \\ &\leq \sum_p (\frac{1}{p^{1+\delta}})^n \\ &\leq 2 \sum_p \frac{1}{p^\sigma} \end{aligned}$$

This converges absolutely for $\text{Re}(s) \geq 1 + \delta$ for every $\delta \geq 0$. This implies the absolute convergence of $\prod_p \frac{1}{1-p^{-s}} = \exp(\sum_p \sum_{n=1}^{\infty} \frac{1}{np^{ns}})$ Now consider

$$\frac{1}{1-p^{-s}} = 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \ldots$$

for all primes $p_1, p_2, \ldots, p_r \leq N$, and obtain the equality

$$\prod_{p \leq N} \frac{1}{1-p^{-s}} = \sum_{v_1, v_2, \ldots, v_r = 0}^{\infty} \frac{1}{(p_1^{v_1} \ldots, p_r^{v_r})^s} = \sum_n{}' \frac{1}{n^s} \tag{3.1}$$

where $\sum'$ denotes the sum over all natural numbers which are divisible only by prime numbers $p \leq N$. Since the sum $\sum'$ contains in particular the terms corresponding to all $n \leq N$, we can write

$$\prod_{p \leq N} \frac{1}{1-p^{-s}} = \sum_{n \leq N} \frac{1}{n^s} + \sum_{n>N}{}' \frac{1}{n^s}$$

Now from equation (1) and the definition of $\zeta(s)$ we get

$$\begin{aligned} \left| \prod_{p \leq N} \frac{1}{1-p^{-s}} - \zeta(s) \right| &\leq \left| \sum_{\substack{n>N \\ p_i \nmid n}} \frac{1}{n^s} \right| \\ &\leq \sum_{n>N} \frac{1}{n^{1+\delta}} \end{aligned}$$

where the right hand side goes to zero as $N \to \infty$ because it is the remainder of a convergent series which completes the proof.□

The Riemann's zeta function is associated with the field $\mathbb{Q}$. It can be generalized to a quadratic number field $K$ as follows

**Definition 3.0.6.** *The Dedekind zeta function of the quadratic number field $K$ is defined by the series*

$$\zeta_K(s) = \sum_{a} \frac{1}{N(a)^s}$$

*where $a$ varies over the integral ideals of $K$, and $N(a) = \#(\vartheta_K : a)$ denotes their absolute norm.*

Remark: Dedekind's zeta function can be defined for any arbitrary number field. A good reference is Neukirch's Algebraic Number Theory book, Chapter 5.

Now we can break up $\zeta_K(s)$ into a finite sum

$$\zeta_K(s) = \sum_{A} \zeta(s, A)$$

where $A$ runs over the ideal class group of $K$ and

$$\zeta(s, A) = \sum_{a \in A} \frac{1}{N(a)^s} (Re(s) > 1)$$

We will need Euler's constant $\gamma$ later so let us examine it now. We have $\gamma = \lim_{n\to\infty}(\sum_{k=1}^{n} \frac{1}{k} - \ln n)$ by Whittaker and Watson. It is related to the Riemann zeta function by $\gamma = \lim_{s\to 1} \left[\zeta(s) - \frac{1}{s-1}\right]$

Also let us define here

$$Li_2(\alpha) = -\int_\alpha^1 \frac{\log(1-t)}{t} dt$$

which we will use latter in the proof of Zagier's theorem.

# CHAPTER 4

## A Kronecker Limit Formula for Real Quadratic Number Fields

The basic idea for studying $\zeta(s, A)$ is always the same, if one picks a fixed ideal $\underline{b}$ belonging to the class $A^{-1} = \{\alpha \in K | \alpha A \subseteq \vartheta_K\}$, then the correspondence

$$\psi : A \longrightarrow \text{set of principal ideals } (\lambda) \text{ divisible by } \underline{b}, \text{i.e } \lambda \in \underline{b}$$

$$a \longrightarrow a\underline{b} = (\lambda)$$

is a bijection.

We have $\psi$ is onto by construction also $\psi$ is one-to-one, since for $a_1, a_2 \in A$ with $a_1\underline{b} = a_2\underline{b}$. Fractional ideals forms a group [4], so we can cancel $\underline{b}$ from both sides of the equation, we get $a_1 = a_2$.

On the other hand, two numbers $\lambda_1, \lambda_2 \in \underline{b}$ define the same principal ideal if and only if $\lambda_1 = \epsilon\lambda_2$ for some unit $\epsilon$, i.e iff they have the same image in $\underline{b}/U_K$; where $U_K$ is the group of units in $K$.

If $(\lambda_1) = (\lambda_2)$ then $\lambda_1 = \lambda_2 d$ also $\lambda_2 = \lambda_1 c \Longrightarrow \lambda_1 = \lambda_1 cd \Longrightarrow cd = 1$ Hence $c, d$ are units.

Conversely, if $\lambda_1, \lambda_2 \in \underline{b}$ with $\lambda_1 = \epsilon\lambda_2 \Longrightarrow (\lambda_1) = (\lambda_2)$ since for $a \in (\lambda_1) \Longrightarrow a = \lambda_1 c = \epsilon\lambda_2 c \in (\lambda_2)$ and for $b \in (\lambda_2) \Longrightarrow b = \lambda_2 d = \epsilon\lambda_1 d \in (\lambda_1)$

Now consider

$$\zeta(s, A) = \sum_{a \in A} \frac{1}{N(a)^s} = \sum_{a \in A} \frac{N(\underline{b})^s}{N(a)^s N(\underline{b})^s} = \sum_{a \in A} \frac{N(\underline{b})^s}{N(a\underline{b})^s}$$

but $a\underline{b} = (\lambda)$ for $\lambda \in \underline{b}$ and $(\lambda_1) = (\lambda_2) \iff$ they have the same image in $\underline{b}/U_K$
Hence we get

$$(4.1)\zeta(s,A) = N(\underline{b})^s \sum_{\lambda \in \underline{b}/U_K} \frac{1}{N((\lambda))^s} = N(\underline{b})^s \sum_{\lambda \in \underline{b}/U_K} \frac{1}{\mid N(\lambda) \mid^s}$$

the second summation omits the value 0. From now on our sums will omit the value 0.

By the Theorem (1.0.4) if $K$ is an imaginary quadratic number field of discriminant $D < 0$, then $U_K$ is a finite group: of order 2,4, or 6. Hence

$$\zeta(s,A) = \frac{1}{\mid U_K \mid} N(\underline{b})^s \sum_{\lambda \in \underline{b}} \frac{1}{N(\lambda)^s}$$

(we can drop the absolute sign since $N(\lambda) = \lambda\overline{\lambda} = \mid \lambda \mid^2 > 0$.)

This formula is unchanged if we replace $\underline{b}$ by $\alpha\underline{b}$ for $\alpha \in K - 0$ so we can assume $\underline{b}$ has a basis of the form $\{1, w\}$. We also suppose that this basis is oriented, i.e $\mathrm{Im}(w) > 0$ (here we have fixed an embedding of $K$ in $\mathbb{C}$). Now let us examine N($\underline{b}$). By definition we have N($\underline{b}$)$= \#(\vartheta_K : \underline{b})$. Also $(\vartheta_K : \underline{b}) = C_{m_1} \oplus C_{m_2}$ where $C_{m_i}$ is a cyclic group of order $m_i$; by a well-known structure theorem for finite abelian groups.[1]

Say $C_{m_1} =< \alpha_1 >, C_{m_2} =< \alpha_2 >$, with $\alpha_i = \underline{b} + u_i$ where $\{u_1, u_2\}$ generates $\vartheta_K$ and $\{m_1u_1, m_2u_2\}$ generates $\underline{b}$.

Hence $\#(\vartheta_K : \underline{b}) = m_1m_2$, also $Vol(\underline{b}) = m_1m_2Vol(\vartheta_K)$ so $N(\underline{b}) = \frac{Vol(\underline{b})}{Vol(\vartheta_K)}$. We have $\{1, w\}$ basis for $\underline{b}$ where $w = x + y\sqrt{d}, x, y \in \mathbb{Q}$, so

$$\mathbf{Vol}(\underline{\mathbf{b}}) = \begin{pmatrix} i & j & k \\ x & y\sqrt{d} & 0 \\ 1 & 0 & 0 \end{pmatrix} = -y\sqrt{d} = -\frac{w - \overline{w}}{2i}$$

Also $\{1, \frac{1+\sqrt{D}}{2}\}$ is a basis for $\vartheta_K$,so

$$\mathbf{Vol}(\underline{\mathbf{b}}) = \begin{pmatrix} i & j & k \\ \frac{1}{2} & \frac{\sqrt{D}}{2} & 0 \\ 1 & 0 & 0 \end{pmatrix} = -\frac{\sqrt{D}}{2}$$

[1]Thomas W. Hungerford, Algebra, Theorem 2.1, page:76

and

$$N(\underline{b}) = -\frac{(w-\overline{w})}{2i}\frac{-2}{\sqrt{D}} = \frac{2Im(w)}{\sqrt{D}}$$

For an element $\alpha = mw + n$ of $\underline{b}$ we have

$$\begin{aligned} N(mw+n) &= (mw+n)(m\overline{w}+n) = n^2 + nm\overline{w} + nmw + m^2 w\overline{w} \\ &= n^2 + m^2 \mid w \mid^2 + nm(w+\overline{w}) \\ &= n^2 + m^2 \mid w \mid^2 + nm2Re(w) \end{aligned}$$

Back to Dirichlet's zeta function, we had $\zeta(s,A) = \frac{N(\underline{b})^s}{|U_K|}\sum_{\lambda\in\underline{b}}\frac{1}{N(\lambda)^s}$ with $\mathrm{N}(\underline{b})^s = 2^s \mid D \mid^{\frac{-s}{2}} (\mathrm{Im}(w))^s$.

For $\lambda \in \underline{b}$ we have $\lambda = n + mw, m, n \in \mathbb{Z}$ so $\mathrm{N}(\lambda) = \lambda\overline{\lambda} = \mid \lambda \mid^2 = \mid mw + n \mid^2$. Then

$$\begin{aligned} \zeta(s,A) &= \frac{\mid D \mid^{\frac{-s}{2}} 2^s Im(w)^s}{\mid U_K \mid} \sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(\mid mw+n \mid^2)^s} \\ &= \frac{\mid D \mid^{\frac{-s}{2}}}{\mid U_K \mid} \sum_{m,n\in\mathbb{Z}} \frac{1}{(\frac{|mw+n|^2}{2Im(w)})^s} \end{aligned}$$

Let

$$Q(m,n) = \frac{|mw+n|^2}{2Im(w)} \tag{4.2}$$

which is the binary form $\mathrm{N}(n+mw)$ and normalized to have discriminant -1. Because $Q(m,n) = \frac{m^2|w|^2+2mnRe(w)+n^2}{2Im(w)}$, $w = x + y\sqrt{d}$, $\mathrm{Re}(w) = x$, $\mathrm{Im}(w) = y\sqrt{\mid d \mid}$ discriminant of

$$\begin{aligned} Q(m,n) &= \left(\frac{2Re(w)}{2Im(w)}\right)^2 - 4\frac{\mid w \mid^2}{(2Im(w))^2} \\ &= \frac{4x^2 - 4(x^2 - y^2 d)}{4y^2 \mid d \mid} \\ &= \frac{4y^2 d}{4y^2 \mid d \mid} = -1 \end{aligned}$$

since $d < 0$.

Hence we have

$$\zeta(s,A) = \frac{\mid D \mid^{\frac{-s}{2}}}{\mid U_K \mid} \sum_{m,n\in\mathbb{Z}} \frac{1}{Q(m,n)^s}$$

**Theorem 4.0.1.** *(Kronecker)Let $Q(x,y) = ax^2 + bxy + cy^2, a, c > 0, b^2 - 4ac = -1$ be any positive definite quadratic form of discriminant -1, and $w$ be the solution with positive imaginary part of the quadratic equation $cw^2 - bw + a = 0$, so that $Q$ is given as in Equation (4.2). Then the zeta-function of $Q$, defined by*

$$\zeta_Q(s) = \sum_{m,n} \frac{1}{Q(m,n)^s}$$

*if $Re(s) > 1$, can be extended meromorphically to a neighbourhood of $s = 1$ and has there a Laurent expansion*

$$\zeta_Q(s) = \frac{2\pi}{s-1} + C + O(s-1)$$

*with residue independent of $Q$ and constant term given by*

$$C = 4\pi(\gamma + \frac{1}{2}logc - log \mid \eta(w) \mid^2)$$

*Here $\gamma$ denotes Euler's constant and*

$$\eta(w) = e^{\frac{\pi i w}{12}} \prod_{n=1}^{\infty}(1 - e^{2\pi i n w}),\ Im(w) > 0$$

*is Dedekind's eta-function.*

**Proof:** For $Q(x,y) = ax^2 + bxy + cy^2$ we can write

$$\mathbf{A} = \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix}$$

so that $Q(x,y) = X^tAX$. The matrix $A$ is said to be positive definite quadratic form, if $A$ is symmetric and $a > 0$, $ac - \frac{b^2}{4} > 0$. Now consider

$$\zeta_Q(s) = \sum_{m,n} \frac{1}{Q(m,n)^s} = \sum_{\substack{n=-\infty \\ m=0}}^{\infty} \frac{1}{Q(m,n)^s} + \sum_{\substack{n=-\infty \\ m\neq 0}}^{\infty} \frac{1}{Q(m,n)^s}$$

When $m = 0$ $Q(m,n) = cn^2$. So the first sum is

$$\begin{aligned}
\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{(cn^2)^s} &= c^{-s} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n^{2s}} \\
&= c^{-s}[\sum_{n=-\infty}^{1} \frac{1}{n^{2s}} + \sum_{n=1}^{\infty} \frac{1}{n^{2s}}] \\
&= c^{-s}[\zeta(2s) + \zeta(2s)].
\end{aligned}$$

Hence

$$\zeta_Q(s) = 2c^{-s}\zeta(2s) + \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{Q(m,n)^s}.$$

In order to evaluate the sum we write it as

$$= 2c^{-s}\zeta(2s) + \sum_{m=-\infty}^{1} \sum_{n=-\infty}^{\infty} \frac{1}{Q(m,n)^s} + \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{Q(m,n)^s}$$

and the first sum will be unchanged by replacing $m$ by $-m$.

Since $Q(m,n) = am^2 + bmn + cn^2$ we get

$$\zeta_Q(s) = 2c^{-s}\zeta(2s) + \sum_{m=1}^{\infty}[\sum_{n=-\infty}^{\infty} \frac{1}{(am^2 - bmn + cn^2)^s} + \sum_{n=-\infty}^{\infty} \frac{1}{(am^2 + bmn + cn^2)^s}]$$

$$= \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} (\frac{1}{(am^2 - bmn + cn^2)^s} + \frac{1}{(am^2 + bmn + cn^2)^s})$$

Note that for $n < 0$, $\frac{1}{(am^2+bm(-n)+cn^2)^s} = \frac{1}{(am^2+bmn+cn^2)^s}, n > 0$ therefore in the summation two parts give the same value. So

$$\begin{aligned} \zeta_Q(s) &= 2c^{-s}\zeta(2s) + \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \frac{2}{Q(m,n)^s} \\ \Longrightarrow & \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{Q(m,n)^s} = \frac{1}{2}\zeta_Q(s) - c^{-s}\zeta(2s) \end{aligned}$$

Let $I(s) = \int_{-\infty}^{\infty} \frac{dt}{Q(1,t)^s}$. This is clearly a holomorphic function of $s$ for Re(s)> $\frac{1}{2}$ Then

$$(4.3) \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{Q(m,n)^s} = \zeta(2s-1)I(s) + \sum_{m=1}^{\infty}( \sum_{n=-\infty}^{\infty} \frac{1}{Q(m,n)^s} - \frac{1}{m^{2s-1}}I(s)).$$

the last term in the above sum is

$$\frac{1}{m^{2s-1}}I(s) = \frac{m}{m^{2s}} \int_{-\infty}^{\infty} \frac{dt}{Q(1,t)^s} = \int_{\infty}^{\infty} \frac{mdt}{[m^2Q(1,t)]^s}.$$

Observing that $m^2Q(1,t) = am^2 + btm^2 + ct^2m^2 = Q(m,mt)$, we make the substitution $y = mt$ to get

$$\frac{1}{m^{2s-1}}I(s) = \int_{-\infty}^{\infty} \frac{dy}{Q(m,y)^s}$$

Writing

$$\int_{-\infty}^{\infty} \frac{dy}{Q(m,y)^s} = \sum_{n=-\infty}^{\infty} \int_n^{n+1} \frac{dy}{Q(m,y)^s}$$

we calculate

$$(4.4)\quad \sum_{n=-\infty}^{\infty} [\frac{1}{Q(m,n)^s} - \frac{1}{m^{2s-1}} I(s)] = \sum_{n=-\infty}^{\infty} [\frac{1}{Q(m,n)^s} - \int_n^{n+1} \frac{dy}{Q(m,y)^s}]$$
$$= \sum_{n=-\infty}^{\infty} \int_n^{n+1} (\frac{1}{Q(m,n)^s} - \frac{1}{Q(m,y)^s}) dy$$

By the mean value theorem for $n \leq y \leq n+1$ we get

$$\mid \frac{1}{Q(m,n)^s} - \frac{1}{Q(m,y)^s} \mid \leq max_{n \leq y \leq n+1} \mid \frac{d}{dy} \frac{1}{Q(m,y)^s} \mid = O\Big(\frac{max(m,n)}{(m^2+n^2)^{s+1}}\Big)$$

where the constant implied in O() depends only on $Q$ and on $s$ uniformly in latter for $\frac{1}{2} < s < 2$. Therefore Equation (4.3) is $O(\frac{1}{m^{2s}})$ from which it follows that the sum over $m$ in Equation (4.3) is absolutely convergent for $\text{Re}(s) > \frac{1}{2}$ and that

$$\lim_{s\to 1} (\sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{Q(m,n)^s} - \zeta(2s-1) I(s))$$

exists and equals $\sum_{m=1}^{\infty} (\sum_{n=-\infty}^{\infty} \frac{1}{Q(m,n)} - \frac{I(1)}{m})$.

The Taylor expansion of $I(s)$ around $s = 1$ is $I(s) = I(1) + (s-1)I'(1) + \frac{(s-1)^2}{2!} I''(1) + \ldots$. Also from the end of Chapter 3 we have $\gamma = \lim_{s\to 1} \Big[\zeta(2s-1) - \frac{\frac{1}{2}}{s-1}\Big] \implies \zeta(2s-1) = \frac{\frac{1}{2}}{s-1} + \gamma + O(s-1)$

Therefore

$$\zeta(2s-1) I(s) = \frac{\frac{1}{2} I(1)}{s-1} + \gamma I(1) + \frac{1}{2} I'(1) + (s-1)\gamma I'(1) + O(s-1)$$
$$= \frac{\frac{1}{2} I(1)}{s-1} + \Big[\gamma I(1) + \frac{1}{2} I'(1)\Big] + O(s-1) (4.5)$$

Now take limit as $s \to 1$ of

$$\sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{Q(m,n)^s} = \zeta(2s-1) I(s) + \sum_{m=1}^{\infty} \Big( \sum_{n=-\infty}^{\infty} \frac{1}{Q(m,n)^s} - \frac{1}{m^{2s-1} I(s)} \Big)$$

Consider (4.5) and we get

$$(4.6)\lim_{s\to 1}(\sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}\frac{1}{Q(m,n)^s}-\frac{\frac{1}{2}I(1)}{s-1})=\gamma I(1)+\frac{1}{2}I'(1)+\sum_{m=1}^{\infty}(\sum_{n=-\infty}^{\infty}\frac{1}{Q(m,n)}-\frac{I(1)}{m})$$

It remains to evaluate $I(1), I'(1)$ and the sum. We have from Equation (4.2)

$$Q(m,n)=\frac{\mid mw+n\mid^2}{2Im(w)}, w=x+y\sqrt{d}$$

Then

$$\begin{aligned}\frac{1}{Q(m,n)} &= \frac{2Im(w)}{\mid mw+n\mid^2}=\frac{1}{im}\frac{im2Im(w)}{\mid mw+n\mid^2}\\ &= \frac{1}{im}\Big(\frac{mw-m\overline{w}}{(m\overline{w}+n)(mw+n)}\Big)\\ &= \frac{1}{im}\Big(\frac{1}{m\overline{w}+n}-\frac{1}{mw+n}\Big)\end{aligned}$$

hence

$$I(1)=\int_{-\infty}^{\infty}\frac{dt}{Q(1,t)}=\frac{1}{i}\int_{-\infty}^{\infty}\Big(\frac{1}{t+\overline{w}}-\frac{1}{t+w}\Big)dt=\frac{1}{i}\log\frac{t+\overline{w}}{t+w}\Big|_{-\infty}^{\infty}$$

We have $\log(t+\overline{w})=\log\mid t+\overline{w}\mid+i\arg(t+\overline{w})$ and

$\log(t+w)=\log\mid t+w\mid+i\arg(t+w)$

$$\begin{aligned}\Longrightarrow \frac{1}{i}\Big[\log(t+\overline{w})-\log(t+w)\Big]\Big|_{-\infty}^{\infty} &= \frac{1}{i}\Big[i\arg(t+\overline{w})-i\arg(t+w)\Big]\Big|_{-\infty}^{\infty}\\ &= \arg(t+\overline{w})-\arg(t+w)\Big|_{-\infty}^{\infty}=2\pi\end{aligned}$$

Also

$$\sum_{n=-\infty}^{\infty}\frac{1}{Q(m,n)}=\frac{1}{im}\sum_{-\infty}^{\infty}\Big(\frac{1}{m\overline{w}+n}-\frac{1}{mw+n}\Big)=\frac{\pi}{i}\frac{1}{m}\Big(\cot\pi m\overline{w}-\cot\pi mw\Big)$$

where we used the standard expansion $\frac{\pi}{\tan\pi x}=\sum_n\frac{1}{n+x}$(principal value)

Therefore (4.6) becomes

$$\begin{aligned}(4.7)\lim_{s\to 1}\Big(\sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}\frac{1}{Q(m,n)^s}-\frac{\pi}{s-1}\Big) &= 2\pi\gamma+\frac{1}{2}I'(1)+\frac{\pi}{i}\sum_{m=1}^{\infty}\frac{1}{m}(\cot\pi m\overline{w}-i)\\ &- \frac{\pi}{i}\sum_{m=1}^{\infty}\frac{1}{m}(\cot\pi mw+i)\end{aligned}$$

the series converge because, for $\text{Im}(w) > 0, \cot \pi m w \to -i$ and $\cot \pi m \overline{w} \to i$ with exponential rapidity as $m \to \infty$.

Now

$$\frac{1}{i} \cot \pi m \overline{w} = \frac{e^{\pi i m \overline{w}} + e^{-\pi i m \overline{w}}}{e^{\pi i m \overline{w}} - e^{-\pi i m \overline{w}}} = 1 + 2e^{-2\pi i m \overline{w}} + 2e^{-4\pi i m \overline{w}} + \ldots$$

and so

$$\begin{aligned} \frac{1}{i} \sum_{m=1}^{\infty} \frac{1}{m}(\cot \pi m \overline{w} - i) &= 2 \sum_{m=1}^{\infty} \frac{1}{m} \sum_{n=1}^{\infty} e^{-2\pi i m n \overline{w}} \\ &= 2 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m} e^{-2\pi i m n \overline{w}} \\ &= -2 \sum_{n=1}^{\infty} \log[1 - e^{-2\pi i m n \overline{w}}] \\ &= -\frac{\pi i \overline{w}}{6} - 2 \log \eta(-\overline{w}) \end{aligned}$$

Note that Taylor series for log is $\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \ldots$, so

$$\sum_{m=1}^{\infty} \frac{1}{m}(e^{-2\pi i n \overline{w}})^m = -\log(1 - e^{-2\pi i n \overline{w}})$$

$$\sum_{n=1}^{\infty} \log(1 - e^{-2\pi i n \overline{w}}) = \log(\prod_{n=1}^{\infty}(1 - e^{2\pi i n \overline{w}}))$$

Similarly the second sum in Equation (4.7) equals $-i(\frac{\pi i w}{6} - 2 \log \eta(w))$

Now consider

$$\lim_{s \to 1} \Big( \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{Q(m,n)^s} - \frac{\pi}{s-1} \Big) = 2\pi\gamma + \frac{1}{2} I'(1) + \frac{\pi}{i} \sum_{m=1}^{\infty} (\cot \pi m \overline{w} - i)$$

$$-\frac{\pi}{i} \sum_{m=1}^{\infty} \frac{1}{m}(\cot \pi m w + i) = 2\pi\gamma + \frac{1}{2} I'(1) + \Big( -\frac{\pi^2 i \overline{w}}{6} - 2\pi \log \eta(-\overline{w}) + \frac{\pi^2 i w}{6} - 2\pi \log \eta(w) \Big)$$

Multiply by 2,

$$\begin{aligned} \lim_{s \to 1} \Big( \zeta_Q(s) - \frac{2\zeta(2s)}{e^s} - \frac{2\pi}{s-1} \Big) &= 4\pi\gamma + I'(1) - \frac{\pi^2 i \overline{w}}{3} - 4\pi \log \eta(-\overline{w}) \\ &- 4\pi \log \eta(w) + \frac{\pi^2 i w}{3} \\ &= 4\pi\gamma + I'(1) - \frac{\pi^2 i \overline{w}}{3} + \frac{\pi^2 i w}{3} + \frac{2}{c}\zeta(2) \\ &- 4\pi \log[\eta(-\overline{w})\eta(w)] \end{aligned}$$

Note that $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ so $\frac{2}{c}\zeta(2) = \frac{\pi^2}{3c}$. Also $\frac{1}{c} = \frac{w-\overline{w}}{i}$.

Hence we get

$$\lim_{s\to 1}\left(\zeta_Q(s) - \frac{2\pi}{s-1}\right) = 4\pi + I'(1) - \frac{\pi^2 wi}{3} + \frac{\pi^2 wi}{3} + \frac{\pi^2}{3c} - 4\pi\log[\eta(\overline{w})\eta(w)]$$

Notice that we could take $\lim_{s\to 1} \frac{2\zeta(2s)}{c^s}$ since $\zeta(2s)$ absolutely converges for $\mathrm{Re}(2s) > 1$, i.e $\mathrm{Re}(s) > \frac{1}{2}$. Also we have the assumption $\frac{1}{c} = \frac{w-\overline{w}}{i}$

$$\Longrightarrow \lim_{s\to 1}\left(\zeta_Q(s) - \frac{2\pi}{s-1}\right) = 4\pi + I'(1) - 4\pi\log[\eta(\overline{w})\eta(w)]$$

Now we have $\overline{e^{iz}} = e^{-i\overline{z}}$ and $\eta(w) = e^{\frac{\pi i w}{12}} \prod_{n=1}^{\infty}(1 - e^{2\pi i n w})$, ($Im$(w)$> 0$)

$\eta(-\overline{w}) = e^{\frac{-\pi i w}{12}} \prod_{n=1}^{\infty}(1 - e^{-2\pi i n w}) = \overline{\eta(w)} \Longrightarrow \eta(w)\eta(-\overline{w}) = \mid \eta(w) \mid^2$

Hence we get

$$\lim_{s\to 1}\left(\zeta_Q(s) - \frac{2\pi}{s-1}\right) = 4\pi\gamma + I'(1) - 4\pi\log \mid \eta(w) \mid^2$$

To complete the proof of the theorem we have to calculate the value of $I'(1)$. We have to show:

$$I'(1) = -\int_{-\infty}^{\infty} \frac{\log Q(1,t)}{Q(1,t)} dt$$

equals $2\pi\log c$.

This is easily checked by substituting $x = 2c(t+\mathrm{Re}w)$ which gives

$$I(s) = 2^{2s-1}c^{s-1}\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^s} = 2\pi\frac{\Gamma(2s-1)}{\Gamma(s)^2}c^{s-1}$$

Let us verify what we said above. First we will verify the expression for $I'(1)$.

We have $I(s) = \int_{-\infty}^{\infty} \frac{dt}{Q(1,t)^s}$.

$$\begin{aligned}
I'(1) &= \lim_{h\to 0} \frac{I(1+h) - I(1)}{h} \\
&= \lim_{h\to 0} \int_{-\infty}^{\infty} \frac{\frac{1}{Q(1,t)^{1+h}} - \frac{1}{Q(1,t)}}{h} dt \\
&= \int_{-\infty}^{\infty} \lim_{h\to 0} \frac{Q(1,t)^{-(1+h)} - Q(1,t)^{-1}}{h} dt \\
&= \int_{-\infty}^{\infty} \left[\frac{1}{Q(1,t)^s}\right]' \Big|_{s=1} dt \\
&= -\int_{-\infty}^{\infty} \frac{\ln Q(1,t)}{Q(1,t)} dt
\end{aligned}$$

Now we will do the substituting $x = 2c(t+\text{Re}(w))$ where $w$ is the root of $cw^2 - bw + a = 0$, so $w = \frac{b \pm \sqrt{b^2-4ac}}{2c}$, remember that we had the assumption $b^2 - 4ac = -1$. This gives $\text{Re}(w) = \frac{b}{2c}$. $x = 2ct + b$, $dx = 2cdt, t = \frac{x-b}{2c}$. Also

$$I(s) = \int_{-\infty}^{\infty} \frac{dt}{Q(1,t)^s} = \int_{-\infty}^{\infty} \frac{dt}{(a+bt+ct^2)^s}$$

Now put $t = \frac{x-b}{2c} \implies a + bt + ct^2 = \frac{1+x^2}{4c}$ So

$$I(s) = \int_{-\infty}^{\infty} \frac{(4c)^s}{(1+x^2)^s} \frac{dx}{2c} = 2^{2s-1} c^{s-1} \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^s} dx$$

**Claim:**

$$I(s) = 2\pi \frac{\Gamma(2s-1)}{\Gamma(s)^2} c^{s-1}$$

**Proof of Claim:** Since $(1+x^2)$ is an even function we can write

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^s} = 2\int_{0}^{\infty} \frac{dx}{(1+x^2)^s}$$

We will use the beta function which is defined as

$$\begin{aligned}
B(p,q) &= \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \\
&= \int_0^{\infty} \frac{\beta^{p-1}}{(1+\beta)^{p+q}} d\beta
\end{aligned}$$

where

$$\Gamma(s) = \int_0^{\infty} e^{-y} y^{s-1} dy$$

is the gamma function. Take $p = \frac{1}{2}, q = s - \frac{1}{2}$ and make the substitution $\beta = x^2$. Then the beta function gives

$$\begin{aligned} B(\frac{1}{2}, s - \frac{1}{2}) &= \frac{\Gamma(\frac{1}{2})\Gamma(s-\frac{1}{2})}{\Gamma(s)} \\ &= \int_0^\infty \frac{(x^2)^{\frac{-1}{2}} 2x dx}{(1+x^2)^s} \\ &= 2\int_0^\infty \frac{dx}{(1+x^2)^s} \end{aligned}$$

Hence we get

$$I(s) = 2^{s-1}c^{s-1}\frac{\Gamma(\frac{1}{2})\Gamma(s-\frac{1}{2})}{\Gamma(s)}$$

where $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ So

$$\begin{aligned} I(s) &= 2^{s-1}c^{s-1}\frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})}{\Gamma(s)} \\ &= 2^{s-1}c^{s-1}\sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})\Gamma(s)}{\Gamma(s)^2} \end{aligned}$$

and by the Legendre duplication formula which is stated as follows

$$\Gamma(s-\frac{1}{2})\Gamma(s) = \frac{2\sqrt{\pi}}{2^{2s-1}}\Gamma(2s-1)$$

$I(s)$ becomes

$$\begin{aligned} I(s) &= 2^{s-1}c^{s-1}\frac{2\pi\Gamma(2s-1)}{2^{2s-1}\Gamma(s)^2} \\ &= 2\pi\frac{\Gamma(2s-1)}{\Gamma(s)^2}c^{s-1} \end{aligned}$$

□ To found the value $I'(1)$ differentiate $I(s)$ we found above,

$$I'(s) = 2\pi\Big( \log(c)c^{s-1}\frac{\Gamma(2s-1)}{\Gamma(s)^2} + c^{s-1}\frac{2\Gamma(2s-1)'\Gamma(s)^2 - 2\Gamma(2s-1)\Gamma(s)\Gamma(s)'}{\Gamma(s)^4}\Big)$$

and put $s = 1$ then we get $I(1)' = 2\pi\log(c)$.

### 4.0.1 Hecke's Theorem

Now assume $K$ is a real quadratic field, $D > 0$ be its discriminant. The difficulty in working with real quadratic number fields is the existence of infinitely many units.

Let $\epsilon$ be a fundamental unit so that $U = \{\pm\epsilon^n; n \in \mathbb{Z}\}$. The same argument as in the beginning of Chapter 4 gives

$$\zeta(s, A) = \frac{1}{2} N(\underline{b}^s) \sum_{\lambda \in \frac{\underline{b}}{\epsilon}} \frac{1}{\mid N(\lambda) \mid^s}$$

Let $\Delta =$N$(\underline{b}) D^{\frac{1}{2}}$. Then

$$(4.8) \zeta(s, A) = \frac{1}{2D^{\frac{s}{2}}} \sum_{\lambda \in \frac{\underline{b}}{\epsilon}} \frac{N(\underline{b})^s D^{\frac{s}{2}}}{\mid N(\lambda) \mid^s} = \sum_{\lambda \in \frac{\underline{b}}{\epsilon}} \frac{\Delta^s}{\mid \lambda\lambda' \mid^s}$$

Consider the integral $c(s) = \int_{-\infty}^{\infty} \frac{dv}{(e^v + e^{-v})^s}$. Then by substituting $e^v = \frac{a}{b} e^w$ for non zero real numbers $a, b$ we get

$$\int_{-\infty}^{\infty} \frac{dv}{(a^2 e^v + b^2 e^{-v})^s} = \frac{c(s)}{\mid ab \mid^s}$$

indeed the integral depends only on the absolute values of $a$ and $b$ is homogenous of degree $-2s$, and only depends on the product $ab$. The substitution $a \to \lambda a, b \to \frac{b}{\lambda}$ corresponds $v \to v - 2\log\lambda$. Multiplying the Equation (4.8) by $2^s c(s)$, then we have

$$2^{s+1} c(s) D^{\frac{s}{2}} \zeta(s, A) = \sum_{\lambda \in \frac{\underline{b}}{\epsilon}} (2\Delta)^s \int_{-\infty}^{\infty} \frac{dv}{(\lambda^2 e^v + \lambda'^2 e^{-v})^s}$$

But replacing $\lambda$ by $\epsilon^n \lambda$ replaces $\lambda^2$ by $\epsilon^{2n}\lambda^2$, $\lambda'^2$ by $\epsilon^{-2n}\lambda'^2$ and this corresponds to $v \to v + 2n\log\epsilon$ that is, the action of $\epsilon$ by multiplication on $\lambda$ corresponds to an action on $v$ by translation through $2\log\epsilon$. Therefore the right hand side of the equation equals

$$\sum_{\lambda \in \underline{b}} \int_{-\log\epsilon}^{\log\epsilon} \frac{(2\Delta)^s dv}{(\lambda^2 \epsilon^v + \lambda'^2 e^{-v})^s}$$

But now the summation and the integration can be interchanged and the sum $\sum_{\lambda \in \underline{b}} \frac{(2\Delta)^s dv}{(\lambda^2 \epsilon^v + \lambda'^2 e^{-v})^s}$ is over the whole $\mathbb{Z}$-module $\underline{b}$ of a Dirichlet series with a definite quadratic form like the one in Kronecker's formula. If we again assume that $\underline{b}$ has a basis of the form $\{1, w\}$ which as before we assume to be oriented (this now means $w > w'$ where $w' = \sigma_2(w)$) then writing $\lambda = mw + n$ we find $\Delta = w - w' = \det(\sigma_i(w_j)) = \begin{pmatrix} 1 & w \\ 1 & w' \end{pmatrix}$ [$w'$ is the conjugate of w]. Consider

$$\lambda^2 e^v + \lambda'^2 e^{-v} = m^2[w^2 e^v + w'^2 e^{-v}] + 2mn[we^v + w'e^{-v}] + n^2[e^v + e^{-v}]$$

So for $Q_v(m,n) = am^2 + bmn + cn^2$ and $c = \frac{e^v + e^{-v}}{2\Delta}$ we have

$$Q_v(m,n) = \frac{(mw+n)^2 e^v + (mw'+n)^2 e^{-v}}{2\Delta}$$

has determinant -1, and we have

$$(4.9) 2^{s+1} c(s) D^{\frac{s}{2}} \zeta(s, A) = \int_{-\log \epsilon}^{\log \epsilon} \zeta_{Q_v}(s) dv$$

By the Kronecker's limit formula,

$$\zeta_{Q_v(s)} = \frac{2\pi}{s-1} + 4\pi \Big( \gamma + \frac{1}{2} \log \Big( \frac{e^v + e^{-v}}{2\delta} \Big) - \log \Big| \eta \Big( \frac{w + iw' e^{-v}}{1 + i\epsilon^{-v}} \Big) \Big|^2 \Big) + O(s-1)$$

Substituting this into Equation (4.9) and using easily calculated values $c(1) = \frac{\pi}{2}, c'(1) = -\pi \log 2$ we find $res_{s=1} \zeta(s, A) = \frac{2 \log \epsilon}{\sqrt{D}}$ and

$$\begin{aligned}
\varrho(A) &= \lim_{s \to 1} \Big( \zeta(s, A) - \frac{2 D^{\frac{-1}{2}} \log \epsilon}{s-1} \Big) \\
&= \frac{2 \log \epsilon}{\sqrt{D}} (-\frac{1}{2} \log D + 2\gamma) + \frac{1}{\sqrt{D}} \int_{-\log \epsilon}^{\log \epsilon} \Big( \log \Big( \frac{e^v + e^{-v}}{w - w'} \Big) \\
&- \log \Big| \eta \Big( \frac{w + iw' e^{-v}}{1 + ie^{-v}} \Big) \Big|^4 \Big) dv
\end{aligned}$$

This formula is not as nice as Kronecker's limit formula because it still involves an integral to be evaluated.

# CHAPTER 5

## Zagier's Theorem

If $\{1, w\}$ and $\{1, w_1\}$ are bases of fractional ideals in the same ideal class of an quadratic number field $K$, i.e $\underline{b} = \mathbb{Z} + \mathbb{Z}w$ and $\underline{b_1} = \mathbb{Z} + \mathbb{Z}w_1$, since they are in the same ideal class we have $\underline{b} = (\alpha)\underline{b_1}$ so $\mathbb{Z} + \mathbb{Z}w = \alpha\mathbb{Z} + \mathbb{Z}\alpha w$

$$\begin{pmatrix} \alpha \\ \alpha w_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w \\ 1 \end{pmatrix}$$

$$\Longrightarrow \alpha = aw + b, \alpha w_1 = cw + d \Longrightarrow w_1 = \frac{cw + d}{aw + b}$$

So $w$ and $w_1$ are related by a Mobius transformation $w_1 = \frac{aw+b}{cw+d}$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$

For the case of a real quadratic field, we call a number $w \in K$ reduced if it satisfies the inequalities

$$(5.1) w > 1, 0 < w' < 1$$

(always with respect to a fixed embedding $K \subseteq \mathbb{R}$, eg $\sqrt{D} > 0$)
Now the number $w \in K$ can (since we have fixed an embedding of $K$ in $\mathbb{R}$) be expanded in a unique way as a continued fraction.

$$w = a_0 - \cfrac{1}{a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots}}}$$

with $a_i \in \mathbb{Z}$(all i), $a_i \geq 2$ $(i = 1, 2, \ldots)$. By the standard theory of continued fractions, the fact that $w$ satisfies a quadratic equation over $\mathbb{Z}$ implies that the sequence $\{a_0, a_1, \ldots\}$ eventually becomes periodic, i.e $a_{i+r} = a_i$ for all $i \geq i_0$; the smallest such $r$ is called the period of $w$, and the corresponding $r$-tuple $((a_{i_0+1}, \ldots, a_{i_0+r}))$ the cycle associated to $w$.

If we choose a different number $w$ with $w > w'$ and $\underline{b} = \mathbb{Z}1 + \mathbb{Z}w$ then the period is unchanged, moreover this is also true if we replace $\underline{b}$ by another ideal in the same narrow ideal class $B$. Let us explain this statement. First consider the period of $w + 1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} w \\ 1 \end{pmatrix}$. Note that the continued fraction of $w + 1$ is the same as $w$ except the first entry. Hence they have the same period. Also consider $\frac{-1}{w} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} w \\ 1 \end{pmatrix}$, this has the continued fraction

$$-\cfrac{1}{a_0 - \cfrac{1}{a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots}}}}$$

, so this also have the same period with $w$. If we take any ideal $\underline{a}$ from the same narrow ideal class of $\underline{b} = \mathbb{Z} + \mathbb{Z}w$, then $\underline{a} = (\alpha)\underline{b} = \mathbb{Z}\alpha + \mathbb{Z}w\alpha = \mathbb{Z} + \mathbb{Z}\beta$ where $\beta = \frac{a+bw}{c+dw} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w \\ 1 \end{pmatrix}$ with $ad - bc = 1$. Since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ generates $SL_2(\mathbb{Z})$ we have $\beta$ has the same period with $w$.[5]

Thus to each such class $B$ we have associated an integer $r > 0$ and a cycle $((b_1, \ldots, b_r))$ of $r$ integers $\geq 2$, where the double parenthesis indicate that the order of the $b_i$ is only defined up to cyclic permutation. We denote the length $r$ of the cycle by $\ell(B)$ and call it the length of the ideal class $B$. The length of $B^*$ is the period of the continued fraction $\frac{-1}{w}$. The condition $w$ being reduced is equivalent to the condition that the continued fraction expression of $w$ is pure periodic, i.e satisfies $a_{i+r} = a_i$ for all $i \geq 0$. It follows that if $B$ is a narrow ideal class of $K$ with length $r = \ell(B)$ and cycle $((b_1, \ldots, b_r))$ $(b_i \in \mathbb{Z},\ b_i \geq 2)$, then there are exactly $r$ numbers $w \in K$ which are reduced and for which $\{1, w\}$ is a basis for some ideal in

$B$. These are the numbers defined by

$$(5.2) w_k = b_k - \cfrac{1}{b_{k+1} - \cfrac{1}{\vdots - \cfrac{1}{b_r - \cfrac{1}{b_1 - \vdots}}}}$$

for $k = 1, 2, \ldots, r$

Before giving the proof of the theorem we will need some preliminaries.

We define $w_k$ by Equation (5.2) for $k = 1, \ldots, r = \ell(B)$ and extend the definition to all $k \in \mathbb{Z}$ by requiring $w_k$ to depend only on $k (\text{mod } r)$. We also fix the ideal $\underline{b} = \mathbb{Z} + \mathbb{Z}w_0 \in B$.

Now, define a sequence of numbers

$$0 < \ldots < A_2 < A_1 < A_0 = 1 < A_{-1} < A_{-2} < \ldots$$

by

$$A_k = \frac{1}{w_1 \ldots w_k} \qquad (k \geq 1)$$

$$A_0 = 1$$

$$A_{-k} = w_0 w_{-1} \ldots w_{-k+1} \qquad (k \geq 1)$$

so that

$$(5.3) \qquad A_{k+1} = \frac{A_k}{w_{k+1}}$$

From the continued fraction expansion of $w_k$ in Equation (5.2) we have $w_k = b_k - \frac{1}{w_{k+1}}$, and from Equation (5.3) we have $w_{k+1} = \frac{A_k}{A_{k+1}}$.

Hence $w_k = b_k - \frac{A_{k+1}}{A_k}$ which implies $A_{k+1} = b_k A_k - w_k A_k$ and $w_k A_k = A_{k-1}$. So $A_{k+1} = b_k A_k - A_{k-1}$.

**Lemma 5.0.1.** *$\mathbb{Z}A_{k+1} + \mathbb{Z}A_k = \mathbb{Z}A_k + \mathbb{Z}A_{k-1}$, i.e $\{A_{k-1}, A_k\}$ form a basis for $\underline{b}$ for each integer $k$.*

**Proof:** For this let $aA_k + bA_{k-1} \in \mathbb{Z}A_k + \mathbb{Z}A_{k-1}$ be any element.Then

$$\begin{aligned} aA_k + bA_{k-1} &= aA_k + b(b_kA_k - A_{k+1}) \\ &= (a + bb_k)A_k - bA_{k+1} \in \mathbb{Z}A_k + \mathbb{Z}A_{k+1} \end{aligned}$$

So $\mathbb{Z}A_k + \mathbb{Z}A_{k-1} \subseteq \mathbb{Z}A_k + \mathbb{Z}A_{k+1}$

Now let $cA_{k+1} + dA_k \in \mathbb{Z}A_{k+1} + \mathbb{Z}A_k$ be any element. Then

$$\begin{aligned} cA_{k+1} + bA_k &= c(b_kA_k - A_{k-1}) + dA_k \\ &= (cb_k + d)A_k - cA_{k-1} \in \mathbb{Z}A_k + \mathbb{Z}A_{k-1} \end{aligned}$$

so $\mathbb{Z}A_k + \mathbb{Z}A_{k+1} \subseteq \mathbb{Z}A_k + \mathbb{Z}A_{k-1}$

Also note that we had fixed $\underline{b} = \mathbb{Z} + \mathbb{Z}w_0$.

For $k = -1$ we have $\mathbb{Z}A_{k+1} + \mathbb{Z}A_k = \mathbb{Z}A_0 + \mathbb{Z}A_{-1} = \mathbb{Z} + \mathbb{Z}w_0 = \underline{b}$.

so $\{A_{k-1}, A_k\}$ form a basis for $\underline{b}$ for any $k$.□

Moreover, the periodicity of the $w_k$ implies

**Lemma 5.0.2.** $A_rA_k = A_{k+r}$ *for all* $k \in \mathbb{Z}$.

**Proof:** We have

$$A_k = \tfrac{1}{w_1 \ldots w_k} \text{ and } A_r = \tfrac{1}{w_1 \ldots w_r}$$

$$\Longrightarrow A_rA_k = \frac{1}{(w_1 \ldots w_k)(w_1 \ldots w_r)} = \frac{1}{(w_1 \ldots w_r)(w_{r+1} \ldots w_{k+r})} = A_{k+r}$$

□

This implies the following lemma

**Lemma 5.0.3.** $A_r$ *is the fundamental unit of* $K$.

**Proof:** Consider

$$\psi : \underline{b} \to \underline{b}$$

$$\lambda \to A_r\lambda$$

Clearly $\psi$ respects addition and it is injective. Also since $\underline{b} = \mathbb{Z}A_{k+r-1} + \mathbb{Z}A_{k+r}$ from Lemma (5.1) and $A_r(aA_{k-1} + bA_k) = aA_{k+r-1} + bA_{k+r}$ by Lemma (5.2) we see that it is onto. Therefore, $\psi$ is an automorphism of $\underline{b}$. So there is an inverse mapping which implies $A_r$ has an inverse and hence $A_r$ is a unit. Also remember that $w$ was reduced that is $w > 1$ and $0 < w' < 1$ hence $A_r > 0$ and $A'_r = \frac{1}{w'_1 \ldots w'_r} > 0$. In a quadratic number field the units are of the form $\pm\epsilon^n$. Since $\epsilon > 1$ we have

$$A_r = \epsilon^{-n} \text{ for some } n \geq 1 \text{ since } 0 < A_r < 1. \ (5.4)$$

It can be shown that $n = 1$ □

**Proposition 5.0.12.** *There is a one-to-one correspondence between* $\{\lambda \in \underline{b} | \lambda >> 0\}$ *and* $\{(k, p, q) | k, p, q \in \mathbb{Z}, p \geq 1, q \geq 0\}$*. Moreover it is clear from (5.4)and (5.5) that if* $\lambda$ *corresponds to the triple (k,p,q) then the triple corresponding to* $\lambda\epsilon^n$ *is* $(k - nr, p, q)$*; hence there is a one-to-one correspondence between principal ideals* $(\lambda)$ *with* $\lambda >> 0, \lambda \in \underline{b}$ *and triples* $(k(mod\ r), p, q)$*.*

**Proof:** Now for any $k \in \mathbb{Z}$ any number $\lambda \in \underline{b}$ can be written in the form

$$(5.5)\lambda = pA_{k-1} + qA_k$$

[since $\underline{b} = \mathbb{Z}A_k + \mathbb{Z}A_{k-1}$] with $p, q \in \mathbb{Z}$, and it is clear that if $p, q \geq 0$ and not both are 0 then $\lambda$ is totally positive.(i.e $\lambda > 0, \lambda' > 0$;we write $\lambda >> 0$) Conversely; one can show that if $\lambda \in \underline{b}$ is totally positive then $\lambda$ can always be written as $pA_{k-1} + qA_k$ with $p, q \geq 0$ for some k.
Moreover this representation is unique unless $\lambda = nA_l$ ($n \in \mathbb{N}$) in which case we take $k = l$, $p = 0, q = n$ or $k = l + 1, p = n, q = 0$. Thus if we make the restriction $p \geq 1$, then to each $\lambda \in \underline{b}$ is associated a triple $(k, p, q)$ of integers with $p \geq 1, q \geq 0$ and $\lambda$ as in Equation (5.5). □

Now we are ready to prove

**Theorem 5.0.1.** *For* $Q(x, y) = ax^2 + bxy + cy^2$*,* $a, b, c > 0$*,* $b^2 - 4ac = 1$ *an indefinite* *binary quadratic form with positive real coefficients and discriminant 1, we define*

$$Z_Q(s) = \sum_{p=1}^{\infty}\sum_{q=0}^{\infty} \frac{1}{Q(p,q)^s}$$

*then for the zeta-function of a narrow ideal class $B$ of a real quadratic field of discriminant $D$, we have the decomposition*

$$D^{\frac{s}{2}}\zeta(s, B^{-1}) = \sum_{k=1}^{r} Z_{Q_k}(s)$$

*where $r = \ell(B)$ is the length of $B$ and the quadratic form $Q_k$ are defined as*

$$\text{(5.6)} \quad Q_k(p,q) = \tfrac{1}{w_k - w'_k}(q + pw_k)(q + pw'_k),$$

*$w_k$ being the elements of $K$ whose continued fractions correspond to the various cyclic permutations of the cycle $((b_1, b_2, \ldots, b_r))$ associated to $B$.*

**Proof:** Let $\lambda \in \underline{b}$ be totally positive. Then from Lemma (5.0.1) there are integers $k, p, q$ such that $\lambda = pA_{k-1} + qA_k$. We had $w_k = \frac{A_{k-1}}{A_k}$ so $\lambda = A_k(q + pw_k)$ and

$$\text{(5.7)} \ N(\lambda) = N(A_k)(q + pw_k)(q + pw'_k).$$

Also, from $w_k = b_k - \frac{1}{w_{k+1}}$ we obtain

$$w_k - w'_k = \frac{-1}{w_{k+1}} + \frac{1}{w'_{k+1}} = \frac{w_{k+1} - w'_{k+1}}{w_{k+1}w'_{k+1}}$$

or

$$(w_k - w'_k)A_kA'_k = (w_{k+1} - w'_{k+1})A_{k+1}A'_{k+1}$$

Now take $k = 0$ then we get

$$(w_0 - w'_0)A_0A'_0 = (w_1 - w'_1)A_1A'_1 = \ldots = (w_k - w'_k)A_kA'_k$$

$\Longrightarrow A_kA'_k = \frac{w_0 - w'_0}{w_k - w'_k}A_0A'_0$ Now using N($A_k$) $= A_kA'_k$ in Equation (5.7) we get

$$N(\lambda) = \frac{w_0 - w'_0}{w_k - w'_k}(q + pw_k)(q + pw'_k)$$

we have

$$(5.8)\ Q_k(p,q) = \frac{1}{w_k - w'_k}(q + pw_k)(q + pw'_k)$$

hence $N(\lambda)$ becomes

$$N(\lambda) = (w_0 - w'_0)Q_k(p,q)$$

Now, let $B$ be a narrow ideal class and $\underline{b} \in B$. Then there is a one-to-one correspondence between $a \in B^{-1}$ and $(\lambda) \in \underline{b}$ totally positive given by $a\underline{b} = (\lambda)$. Therefore,

$$\begin{aligned} D^{\frac{s}{2}}\zeta(s, B^{-1}) &= D^{\frac{s}{2}} \sum_{a \in B^{-1}} \frac{1}{N(a)^s} = D^{\frac{s}{2}} N(\underline{b})^s \sum_{\substack{\lambda \in \underline{b}/U \\ \lambda >> 0}} \frac{1}{N(\lambda)^s} \\ &= (w_0 - w'_0)^s \sum_{k=1}^{\infty} \sum_{p=1}^{\infty} \sum_{q=0}^{\infty} \frac{1}{N(pA_{k-1} + qA_k)^s} \\ &= \sum_{k=1}^{\infty} \sum_{p=1}^{\infty} \sum_{q=0}^{\infty} \frac{1}{Q_k(p,q)^s} \end{aligned}$$

Notice that $Q_k(p,q)$ is an indefinite binary quadratic form with positive coefficients, normalized to have discriminant 1. $\square$

Next we will prove a theorem about the residue and the constant term of the Laurent expansion of $Z_Q(s)$ at $s = 1$. But first we need the following lemma.

**Lemma 5.0.4.** *For $Q(x,y) = ax^2 + bxy + cy^2$, $a, b, c > 0$, $b^2 - 4ac = 1$ an indefinite binary quadratic form with positive real coefficients and discriminant 1, and $w, w'$ be the roots of the quadratic equation $cw^2 - bw + a = 0$, labelled so that $w > w' > 0$. Define*

$$I(s) = \int_0^\infty \frac{dt}{Q(1,t)^s}$$

*for $Re(s) > \frac{1}{2}$. Then $I(1) = \log(\frac{w}{w'})$ and*

$$I'(1) = (\log \frac{w'}{w})(\log \frac{w^2}{w - w'}) - g(\frac{w'}{w})$$

*with $g$ defined by $g(\alpha) = \int_0^\infty (\frac{1}{x+\alpha} - \frac{1}{x+1}) \log((x+1)(x+\alpha))dx$*

**Proof:**

We have $cw^2 - bw + a = 0$, factoring out c we get $c(w^2 - \frac{b}{c}w + \frac{a}{c}) = 0$ so $w$ and $w'$

satisfy the equation $t^2 - \frac{b}{c}t + \frac{a}{c} = 0$ so $ww' = \frac{a}{c}$ and $w + w' = \frac{b}{c}$. Now

$$\begin{aligned} Q(x,y) &= ax^2 + bxy + cy^2 = cww'x^2 + (cw + cw')xy + cy^2 \\ &= c(ww'x^2 + (w + w')xy + y^2) = c(y + xw)(y + xw') \end{aligned}$$

and

$$\frac{1}{Q(x,y)} = \frac{1}{c(y+xw)(y+xw')} = \frac{1}{c}\Big(\frac{1}{y+xw'} - \frac{1}{y+xw}\Big)\frac{1}{x(w-w')}$$

Also $b^2 - 4ac = 1$ which implies

$$[c(w + w')]^2 - 4c^2ww' = 1$$

$$c^2(w^2 + 2ww' + w'^2) - 4c^2ww' = 1$$

$$c^2w^2 + c^2w'^2 - 2c^2ww' = 1$$

$$c^2(w^2 - 2ww' + w'^2) = 1$$

$(w - w')^2 = \frac{1}{c^2}$ since $c > 0$ we get $\frac{1}{c} = (w - w')$.

So we have

$$\frac{1}{Q(x,y)} = \frac{1}{x}\Big(\frac{1}{y+xw'} - \frac{1}{y+xw}\Big)$$

Therefore,

$$\begin{aligned} I(1) &= \int_0^\infty \frac{dt}{Q(1,t)} \\ &= \int_0^\infty \Big(\frac{1}{t+w'} - \frac{1}{t+w}\Big)dt \\ &= \log \frac{t+w'}{t+w}\Big|_0^\infty \\ &= \log(\frac{w}{w'}) \end{aligned}$$

Now let us compute $I'(1)$.

$$\begin{aligned} I'(1) &= -\int_0^\infty \frac{\log Q(1,t)}{Q(1,t)} dt \\ &= -I(1) \log \frac{w^2}{w-w'} - \int_0^\infty \frac{\log(Q(1,t)\frac{w-w'}{w^2})}{Q(1,t)} dt \end{aligned}$$

We had

$$\begin{aligned} \frac{1}{Q(1,t)} &= (\frac{1}{t+w'} - \frac{1}{t+w}) \\ &= \frac{w-w'}{(t+w)(t+w')} \end{aligned}$$

so $Q(1,t)(w-w') = (t+w)(t+w')$

If we substitute $x = \frac{1}{w}t$ we get

$$Q(1,t)\frac{w-w'}{w^2} = \frac{1}{w^2}(xw+w)(xw+w') = (x+1)(x+\frac{w'}{w})$$

and

$$\begin{aligned} \frac{dt}{Q(1,t)} &= (\frac{1}{t+w'} - \frac{1}{t+w})dt \\ &= (\frac{1}{xw+w'} - \frac{1}{xw+w})wdx \\ &= (\frac{1}{x+\frac{w'}{w}} - \frac{1}{x+1})dx \end{aligned}$$

This yields

$$I'(1) = (\log \frac{w'}{w})(\log \frac{w^2}{w-w'}) - g(\frac{w'}{w})$$

with $g$ defined by

$$g(\alpha) = \int_0^\infty (\frac{1}{x+\alpha} - \frac{1}{x+1}) \log[(x+1)(x+\alpha)]dx.$$

□

Now we can state the main theorem.

**Theorem 5.0.2.** *For $Q(x,y) = ax^2+bxy+cy^2$, $a,b,c > 0$, $b^2-4ac = 1$ an indefinite binary quadratic form with positive real coefficients and discriminant 1, and $w, w'$ be the roots of the quadratic equation $cw^2 - bw + a = 0$, labelled so that $w > w' > 0$.*

*Then the function $Z_Q(s)$ has an analytic continuation to the half-plane $Re(s) > \frac{1}{2}$, with a single pole at $s = 1$, and its Laurent expansion there is*

$$Z_Q(s) = \frac{\frac{1}{2}\log\frac{w}{w'}}{s-1} + P(w, w') + O(s-1)$$

*where $P(x, y)$ is a universal function of two variables, given by*

$$P(x, y) = F(x) - F(y) + Li_2(\frac{y}{x}) - \frac{\pi^2}{6} + \log\frac{x}{y}(\gamma - \frac{1}{2}\log(x-y) + \frac{1}{4}\log(\frac{x}{y}))$$

*with $x > y > 0$ Here $\gamma$ is the Euler's constant and $Li_2(t)$ is the dilogarithm function, $\sum_{n=1}^{\infty}\frac{t^n}{n^2}$ $(0 < t < 1)$,and*

$$F(x) = \sum_{n=1}^{\infty}\frac{1}{n}\Big(\frac{\Gamma'(nx)}{\Gamma(nx)} - \log(nx)\Big) = \int_0^{\infty}\Big(\frac{1}{1-e^{-t}} - \frac{1}{t}\Big)\log(1-e^{-xt})dt$$

**Proof:**

Consider

$$Z_Q(s) - \zeta(2s-1)I(s) = \sum_{p=1}^{\infty}\Big[\sum_{q=0}^{\infty}\frac{1}{Q(p,q)^s} - \frac{1}{p^{2s-1}}I(s)\Big]$$

Let us investigate the convergence of this series.

$$\begin{aligned}\Big|Z_Q(s) - \zeta(2s-1)I(s)\Big| &\leq \sum_{p=1}^{\infty}\Big|\sum_{q=0}^{\infty}\frac{1}{Q(p,q)^s} - \frac{1}{p^{2s-1}}I(s)\Big| \\ &\leq \sum_{p=1}^{\infty}\sum_{q=0}^{\infty}\Big|\frac{1}{Q(p,q)^s}\Big| + \Big|\frac{I(s)}{p^{2s-1}}\Big| \\ &= \sum_{p=1}^{\infty}\sum_{q=0}^{\infty}\frac{1}{Q(p,q)^{\sigma}} + \frac{|I(s)|}{p^{2\sigma-1}}\end{aligned}$$

this absolutely converges for Re(s)> $\frac{1}{2}$ where $s = \sigma + it$. Since $\zeta(s)$ has analytic continuation to the complex plane and $I(s)$ is defined for Re$(s) > \frac{1}{2}$, $Z_Q(s)$ has analytic continuation to Re$(s) > \frac{1}{2}$. Also by Kronecker's limit formula we have

$$(5.9)\ \lim_{s\to 1}\Big[Z_Q(s) - \frac{\frac{1}{2}I(1)}{s-1}\Big] = \gamma I(1) + \frac{1}{2}I'(1) + \sum_{p=1}^{\infty}\Big(\sum_{q=0}^{\infty}\frac{1}{Q(p,q)} - \frac{I(1)}{p}\Big)$$

Let us evaluate $\sum_{q=0}^{\infty}\frac{1}{Q(p,q)}$. From Equation (5.8) we have

$$\sum_{q=0}^{\infty}\frac{1}{Q(p,q)} = \frac{1}{p}\sum_{q=0}^{\infty}\Big(\frac{1}{q+pw'} - \frac{1}{q+pw}\Big)$$

Let

$$\psi(x) = \lim_{N\to\infty}\Big(\log N - \sum_{q=0}^{N}\frac{1}{q+x}\Big) = \frac{\Gamma'(x)}{\Gamma(x)}$$

be the logarithmic derivative of the gamma-function. Then

$$\sum_{q=0}^{\infty}\frac{1}{Q(p,q)} = \frac{1}{p}(\psi(pw)-\psi(pw'))$$

therefore,

$$\begin{aligned}(5.9)\sum_{q=0}^{\infty}\frac{1}{Q(p,q)} - \frac{I(1)}{p} &= \frac{1}{p}\Big(\psi(pw)-\psi(pw')-\log(\frac{w}{w'})\Big)\\ &= \frac{1}{p}(\psi(pw)-\log(pw)) - \frac{1}{p}(\psi(pw')-\log(pw'))\end{aligned}$$

Now,

$$\begin{aligned}\sum_{q=0}^{N}\frac{1}{q+x} &= \frac{1}{x}+\frac{1}{x+1}+\ldots+\frac{1}{x+N}\\ &= (1+\ldots+\frac{1}{x+N}) - (1+\ldots+\frac{1}{x}) + \frac{1}{x}\\ &= (1+\ldots+\frac{1}{N}) + (\frac{1}{N+1}+\ldots+\frac{1}{x+N}) - (1+\ldots+\frac{1}{x}) + \frac{1}{x}\end{aligned}$$

Since $(1+\ldots+\frac{1}{N}) \approx \log N$ we have

$$\begin{aligned}\psi(x)-\log x &= \lim_{N\to\infty}\Big(\log N - \Big[\log N + (\frac{1}{N+1}+\ldots+\frac{1}{x+N} - \log x + \frac{1}{x}\Big]\Big) - \log x\\ &= \lim_{N\to\infty}(\frac{1}{N+1}+\ldots+\frac{1}{x+N}) - \frac{1}{x} = O(\frac{1}{x}).\end{aligned}$$

Therefore, both terms are $O(\frac{1}{p^2})$ for $p\to\infty$ So (5.9) can be summed over $p$. Putting (5.9) into (5.8) we obtain

$$\lim_{s\to 1}\Big(Z_Q(s) - \frac{\frac{1}{2}\log\frac{w}{w'}}{s-1}\Big) = \gamma\log\frac{w}{w'} + \frac{1}{2}I'(1) + F(w) - F(w')$$

where $F(x)$ is the function defined in the statement of the theorem. If we compare this with the equation for $P(x,y)$, we see that it remains to prove that

$$I'(1) = 2Li_2(\frac{w'}{w}) - \frac{\pi^2}{3} + \log\frac{w}{w'}\Big(\frac{1}{2}\log\frac{w}{w'} - \log(w-w')\Big)$$

Comparing this with the value of $I'(1)$ from Lemma (5.0.4)we need to prove

$$\begin{aligned} g(\frac{w'}{w}) &= -I'(1) + \log\frac{w'}{w}(\log\frac{w^2}{w-w'}) \\ &= -2Li_2(\frac{w'}{w}) + \frac{\pi^2}{3} + 2\log\frac{w}{w'}\log\frac{w-w'}{w} - \frac{1}{2}(\log\frac{w}{w'})^2 \end{aligned}$$

Letting $\alpha = \frac{w'}{w}$ we need to show

$$g(\alpha) = 2Li_2(\alpha) + \frac{\pi^2}{3} - 2\log\alpha\log(1-\alpha) - \frac{1}{2}(\log\alpha)^2$$

We have

$$Li_2(\alpha) + Li_2(1-\alpha) = \frac{\pi^2}{6} - \log\alpha\log(1-\alpha)$$

$(0 \leq \alpha \leq 1)$

so we need to prove

$$(5.10) g(\alpha) = 2Li_2(1-\alpha) - \frac{1}{2}(\log\alpha)^2$$

$(0 < \alpha \leq 1)$

Two sides of Equation (5.10) agree for $\alpha = 1$ (both plainly vanish) so it suffices to prove (5.10) after differentiation, i.e to show

$$g'(\alpha) = \frac{2\log\alpha}{1-\alpha} - \frac{\log\alpha}{\alpha}, (0 < \alpha < 1)$$

since $\frac{d}{dt}Li_2(t) = \sum_{n=1}^{\infty}\frac{t^{n-1}}{n} = -\frac{log(1-t)}{t}$.

We have

$$g'(\alpha) = -\int_0^\infty \frac{log((x+1)(x+\alpha))}{(x+\alpha)^2}dx + \int_0^\infty \Big(\frac{1}{x+\alpha} - \frac{1}{x+1}\Big)\frac{dx}{x+\alpha}$$

Let us evaluate the first integral by integration by parts.

Let $\log((x+1)(x+\alpha)) = u$ then $du = \frac{(x+\alpha)+(x+1)}{(x+1)(x+\alpha)}dx$ and let $\frac{-dx}{(x+\alpha)^2} = dx$ then $\frac{1}{x+\alpha} = v$. Therefore

$$\begin{aligned} -\int_0^\infty \frac{log[(x+1)(x+\alpha)]}{(x+\alpha)^2}dx &= \log((x+1)(x+\alpha))\frac{1}{x+\alpha}\Big|_0^\infty \\ &- \int_0^\infty \frac{1}{x+\alpha}\Big(\frac{1}{x+1} + \frac{1}{x+\alpha}\Big)dx \\ &= -\frac{\log\alpha}{\alpha} - \int_0^\infty \frac{1}{x+\alpha}\Big(\frac{1}{x+1} + \frac{1}{x+\alpha}\Big)dx \end{aligned}$$

Hence

$$g'(\alpha) = -\frac{\log \alpha}{\alpha} - 2\int_0^\infty \frac{dx}{(x+1)(x+\alpha)}$$

Evaluating the second integral will finish the proof. Consider

$$\frac{1}{(x+1)(x+\alpha)} = \frac{A}{x+1} + \frac{B}{x+\alpha} \implies Ax + A\alpha + Bx + B = 1$$
$$\implies A = -B, (1-\alpha)B = 1$$

Hence

$$\frac{1}{(x+1)(x+\alpha)} = \frac{-1}{1-\alpha}\frac{1}{(x+1)} + \frac{1}{1-\alpha}\frac{1}{(x+\alpha)}$$

so

$$\begin{aligned}\int_0^\infty \frac{1}{(x+1)(x+\alpha)}dx &= \frac{1}{1-\alpha}\int_0^\infty \Big(\frac{1}{x+\alpha} - \frac{1}{x+1}\Big)dx \\ &= \frac{1}{1-\alpha}\frac{\log(x+\alpha)}{\log(x+1)}\end{aligned}$$

we get

$$g'(\alpha) = -\frac{\log \alpha}{\alpha} - \frac{2}{1-\alpha}\log\frac{x+\alpha}{x+1}\Big|_0^\infty = \frac{-\log \alpha}{\alpha} + \frac{2\log\alpha}{1-\alpha}$$

This proves Theorem (5.0.2)□

Now we can state Zagier's theorem.

**Theorem 5.0.3.** *(Zagier) Let $B$ be a narrow ideal class in a real quadratic field of discriminant $D$, and $\epsilon > 1$ the smallest unit of $K$ of norm 1. Then*

$$\lim_{s\to 1}(D^{\frac{s}{2}}\zeta(s,B) - \frac{\log\epsilon}{s-1}) = \sum_{k=1}^{\ell(B)} P(w_k, w_k')$$

*where the summation is over all $w \in K$ satisfying Equation (5.1) for which $\{1, w\}$ is a basis for some fractional ideal of $B$ and $P(x,y)$ is as in Theorem (5.0.2).*

**Proof:**

Combining Theorems (5.0.1) and (5.0.2) we see that

$$D^{\frac{s}{2}}\zeta(s,B^{-1}) = \sum_{k=1}^{r} \mathbb{Z}_{Q_k}(s)$$

where $r = \ell(B)$ and

$$\mathbb{Z}_{Q_k} = \frac{\frac{1}{2}\log(\frac{w_k}{w'_k})}{s-1} + P(w_k, w'_k) + O(s-1)$$

. Then

$$D^{\frac{s}{2}}\zeta(s, B^{-1}) = \sum_{k=1}^{r}[\frac{\frac{1}{2}\log(\frac{w_k}{w'_k})}{s-1} + P(w_k, w'_k) + O(s-1)]$$

The stated formula for the residue is deduced by noting that,

$$\sum_{k=1}^{r}\log\frac{w_k}{w'_k} = \log\frac{w_1\dots w_r}{w'_1\dots w'_r} = \log\frac{A'_r}{A_r} = \log(\epsilon^2)$$

since $A_r = \epsilon^{-1}$. So

$$D^{\frac{s}{2}}\zeta(s, B^{-1}) = \frac{\frac{1}{2}2\log(\epsilon)}{s-1} + \sum_{k=1}^{r}P(w_k, w'_k) + O(s-1)$$

. Now taking the limit as $s \to 1$ we get the result:

$$\lim_{s\to 1}\left(D^{\frac{s}{2}}\zeta(s, B^{-1}) - \frac{\log(\epsilon)}{s-1}\right) = \sum_{k=1}^{r}P(w_k, w'_k)$$

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