The Highest Dynamical Frequency in the Inner Region of an Accretion Disc

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ABSTRACT
In the inner regions of accretion discs around compact objects, the orbital frequency of the gas deviates from the local Keplerian value. For long-wavelength modes in this region, the radial epicyclic frequency $\kappa$ is higher than the azimuthal frequency $\Omega$. This has significant implications for models of the twin kHz QPOs observed in many neutron-star sources that traditionally identify the frequencies of the two kHz QPOs with dynamical frequencies in the accretion disc. The recognition that the highest frequency in the transition or boundary region of the disc is actually the epicyclic frequency also modifies significantly the constraints imposed by the observation of a high-frequency QPOs on the mass and radius of the compact objects.

Keywords: black hole physics – accretion, accretion discs – MHD – instability – turbulence.

1 INTRODUCTION
The X-ray brightness of an accreting compact object is often observed to be modulated quasi-periodically at different characteristic frequencies comparable to the dynamical frequency of the central object (see, e.g., van der Klis 2006). The physical origin of the various types of quasi-periodic oscillations (QPOs) in accreting black holes and neutron stars is still a matter of debate (see, e.g., Psaltis 2004; van der Klis 2006). Indeed, different variability models attribute the observed QPOs to different dynamical frequencies at a particular radius in the accretion disc (e.g., Alpar & Shaham 1985; Miller, Lamb, & Psaltis 1998; Stella, Vietri & Morsink 1999; Abramowicz et al. 2003) or at the frequencies of wave modes in the disc, which are also related to the dynamical frequencies (e.g., Alpar et al. 1992; Alpar & Yilmaz 1997; Wagener 1999; Kato 2001). Most proposed models agree that the highest frequency of a large amplitude QPO cannot be larger than the frequency of a stable dynamical oscillation in the accretion disc. Around a non-rotating black hole or a relatively compact neutron star of mass $M$, the highest stable dynamical frequency is the azimuthal orbital frequency at the radius of the innermost stable circular orbit

$$f_{\text{isco}} = \frac{c^3}{12\pi \sqrt{6} GM}.$$  (1)

Requiring this frequency to be larger than the highest observed QPO frequency imposes an upper bound on the mass of the compact object (e.g., Miller, Lamb, & Psaltis 1998) or even provides evidence that the compact object is rapidly spinning (e.g., Strohmayer 2001).

In a realistic accretion disc, the dynamical frequencies of oscillations of fluid elements are approximately equal to the dynamical frequencies of test particles (such as eq. (11)) only away from the boundaries. For example, in the accretion disc around a neutron star with a dynamically important magnetic field, the orbital frequency of a fluid element near the Alfvén radius (see, e.g., Ghosh & Lamb 1991) deviates from the local Keplerian frequency because of magnetic and viscous stresses (Erkut & Alpar 2005). In accretion flows without large scale magnetic fields, such as those around a non-magnetic neutron star or a black hole, the Maxwell stresses due to turbulent small-scale magnetic fields can also affect the dynamical frequencies in the inner disc (see, e.g., Hawley & Krolik 2001). Even in the absence of any magnetic fields, radiation drag forces can alter significantly the dynamical frequencies at the inner regions of accretion discs (Miller & Lamb 1996).

Non-gravitational forces alter the dynamical frequencies predominantly near the inner regions of the disc, where the observed QPOs are expected to originate. In this paper, we argue that in realistic accretion discs, in which the azimuthal orbital frequency has a maximum at some radius outside the surface or horizon of a compact object, the radial epicyclic frequency at a comparable radius is the highest dynamical frequency in the system. This has significant implications for models of QPOs in accreting compact objects and for constraints imposed on the masses and spins of compact objects by the observation of high-frequency QPOs.

2 THE HIGHEST DYNAMICAL FREQUENCY
In order to discuss the relative order of different dynamical frequencies in an accretion disc independently of the details of the
additional forces that affect the motion of fluid elements, we use the simple but transparent derivation of the equations of motion discussed by Hill (1878). These equations are valid for the motion of test particles in a frame rotating with an angular velocity \( \Omega \). Balbus & Hawley (1992) have used the Hill equations to elucidate the physics behind the magnetorotational instability in MHD accretion discs (see Pessah & Psaltis 2005 for a more general case of MHD instabilities).

Consider particles in a stable circular orbit at a radius \( r_0 \) and with an angular velocity \( \Omega_0 = \Omega(r_0) \), i.e., \( r = r_0 \), \( \phi = \Omega_0 t \), where \( \Omega(r) \) is not necessarily the Keplerian angular velocity. Small perturbations around the circular orbit, i.e., \( r = r_0 + x \) and \( \phi = \Omega_0 t + y/r_0 \), give, to first order in \( x \) and \( y \), the Hill equations

\[
\begin{align*}
\ddot{x} - 2\Omega\dot{y} &= -x \left( \frac{d\Omega^2}{dr} \right)_{r_0} + f_x, \\
\ddot{y} + 2\Omega_0 \dot{x} &= f_y,
\end{align*}
\]

where \( f_x \) and \( f_y \) represent the \( x \)- and \( y \)-components of the sum of the perturbations of non-gravitational forces acting on the particles. For small displacements away from the stable equilibrium orbit, \( f_x \) and \( f_y \) are linear in \( x \) and \( y \) with negative first derivatives,

\[
\begin{align*}
\ddot{x} - 2\Omega\dot{y} &= -x \left( \frac{d\Omega^2}{dr} \right)_{r_0} - \frac{\partial f_x}{\partial x} \left( \frac{\partial f_x}{\partial y} \right) + y \\
\ddot{y} + 2\Omega_0 \dot{x} &= - \frac{\partial f_x}{\partial x} x - \frac{\partial f_y}{\partial y} y.
\end{align*}
\]

Looking for solutions of the form \( e^{-i\omega t} \), with axial symmetry, \( \partial f_x/\partial y = \partial f_y/\partial y = 0 \), the dispersion relation

\[
\omega^4 - \left( \kappa^2 + \left( \frac{d\Omega^2}{d\ln r} \right)_{r_0} \right) \omega^2 - 2\Omega_0 \left( \frac{d\Omega}{d\phi} \right)_{r_0} \omega = 0.
\]

is obtained, where

\[
\kappa^2 = 4\Omega^2_0 + \left( \frac{d\Omega^2}{d\ln r} \right)_{r_0}
\]

When non-gravitational forces are absent, the solutions are stable modes with \( \omega = 0 \) and \( \omega = \kappa \). As can be verified easily, \( \omega = \kappa \) is the frequency associated with the mode of radial oscillations. As azimuthal perturbations simply shift the phase angle \( \phi \) among equivalent phases on the stable circular orbit (“spontaneous symmetry breaking”), there is no restoring force, and the solution corresponding to azimuthal oscillations is simply \( \omega = 0 \). Indeed, after an azimuthal perturbation the particle would proceed from its shifted phase on the same circular orbit, at the original orbital frequency \( \Omega_0 \). This is an instance of the Goldstone theorem that zero frequency modes are associated with spontaneous symmetry breaking. For radial perturbations the long range correlations introduced by the presence of rotation, i.e., the Coriolis force and the gradient of the centrifugal potential provide the restoring force in the Hill equations (4), leading to the nonzero frequency \( \omega = \kappa \). Modes affected by long range forces always have nonzero frequencies, as exemplified by the plasma frequency due to the Coulomb force, the Jeans mass associated with gravitational instabilities, and the finite mass of Higgs bosons.

The effect of short-range forces can be traced in the realistic limit

\[
\frac{\partial f_x}{\partial x} = \frac{\partial f_x}{\partial y} = \frac{\partial f_y}{\partial x} = \frac{\partial f_y}{\partial y} \ll \kappa^2.
\]

The \( \omega \approx \kappa \) modes are now quasi-stable with \( \mathcal{R}(\omega) \approx \kappa \) and \( \mathcal{T}(\omega) \sim \mathcal{O}(\partial f_{x,u}/\partial v_{r_0}/\kappa) \ll \mathcal{R}(\omega) \). There is still an exact \( \omega = 0 \) mode, due to the axial symmetry and a decaying mode with \( \mathcal{R}(\omega) = 0 \) and \( \mathcal{T}(\omega) \sim -\mathcal{O}(\kappa) \). This decay corresponds to the effect of the shear on the initial perturbations.

When the disc is treated as a continuous medium and its hydrodynamic or magnetohydrodynamic wave modes are studied, the analogues of restoring forces on perturbations of single particle orbits in the Hill equations are spatial derivatives of the forces on a fluid element. These restoring force terms contain \( (V_A \cdot \dot{k})^2 \) or \((c_s k)^2\), where \( k \) is the wave vector, and \( V_A \) and \( c_s \) are the Alfven velocity and sound speed, respectively (see, e.g., Alpar et al. 1992; Balbus & Hawley 1992; Alpar & Yılmaz 1997; Kato 2001; Pessah & Psaltis 2005). Conditions analogous to (7) describe global modes of long wavelength. These modes are expected to be the ones that produce the largest amplitude of brightness (accretion rate) modulation. In the limit of small wavenumber \( k \), the frequencies of oscillations have values close to \( \kappa \) and \( \Omega \). For modes with \( m \) azimuthal nodes, frequencies comparable to \( \kappa \) will appear; the dominant global modes will be those with \( m = 0 \) and \( m = 1 \), which represents the beat between the radial epicyclic frequency and the disc rotation. This fundamental effect of rotation can be seen in all dispersion relations of waves in rotating fluids and accretion discs, in models of varying degrees of complexity (see, e.g., Chandrasekhar 1961; Papaloizou & Pringle 1984). We emphasize here that not all oscillations described by the Hill equations are stable. This is of particular relevance to MHD discs that are subject to the magnetorotational instability, especially at the limit of small wavenumbers.

Where magnetic or radiation forces significantly affect the orbital motion of gas elements, the azimuthal frequency in the disc differs from the local Keplerian value. As an example, for accretion onto a rotating and magnetic neutron star, the local azimuthal frequency \( \Omega(r) \) is lower than the Keplerian frequency inside some characteristic radius \( r_\ast \) comparable to the Alfven radius. It has a maximum \( \Omega_{\text{max}} \) at \( r_\ast \) and then decreases to match the rotation frequency of the star \( \Omega_{\ast} \) at a radius \( r_\ast \), which marks the disc-magnetosphere boundary. The simplest mathematical model of such a transition employs a quadratic form for \( \Omega(r) \),

\[
\Omega(r) = \begin{cases} 
\Omega_{\text{max}} - (\Omega_{\text{max}} - \Omega_{\ast}) \left( \frac{r - r_\ast}{r_\ast - r_2} \right)^2, & r < r_\ast \\
\Omega_{\ast}, & r > r_2
\end{cases}
\]

where \( M \) is the mass of the star. Matching these two expressions, as well as their derivatives, at \( r_\ast \) relates the maximum frequency and its location to the transition region parameters \( r_\ast, r_2 \); and the neutron-star rotation rate \( \Omega_{\ast} \).

\[
\begin{align*}
\Omega_{\text{max}} &= \frac{\Omega_{\ast}}{2} - \frac{9(16/15)(r_2/r_1-2/3)}{\Omega_{\ast}/2 + (3/2)(r_2/r_1-5/3)} \\
&= \frac{7}{4} \left(1 - \frac{3}{7} \frac{r}{r_2} \right) \Omega_{\ast}.
\end{align*}
\]

This model quite accurately represents the \( \Omega(r) \) curves that meet a boundary condition at the stellar magnetosphere and asymptotically join the Keplerian curve within a transition region (Erkut & Alpar 2005). The model of the azimuthal frequency profile described by equation (8) is shown in Figure 1, for \( \Omega_2/\Omega_\ast = 1.1, \Omega_{\text{max}}/\Omega_\ast = 1.2, \) and \( r_1/r_2 = 0.84 \), together with the radial epicyclic frequency calculated according to equation (6).

At radius \( r < r_\ast \), where the azimuthal frequency is lower than the Keplerian frequency, the radial epicyclic frequency is the larger of the two dynamical frequencies. At the radius \( r_\ast \), where the azimuthal frequency has a maximum, \( \kappa = 2\Omega_{\text{max}} \). The ratio \( \kappa/\Omega \) increases to even larger values in the region \( r_1 < r < r_\ast \).
the azimuthal frequency is decreasing with decreasing radius. For a wide range of radii inside \( r_2 \), the radial epicyclic frequency is larger than the local Keplerian frequency. As a result, if low-\( k \)-modes in this transition region are responsible for the observed QPOs in accreting neutron stars and black holes, as envisioned by most models, then an upper bound of the QPO frequencies may be set not by the maximum azimuthal frequency but by the maximum radial epicyclic frequency in the region.

3 DISCUSSION

In \( \S 2 \), we argued that, in the inner regions of accretion discs around compact objects, where magnetic, radiation, pressure, and viscous forces become comparable to the gravitational force, the orbital frequencies of fluid elements are altered, leading to radial epicyclic frequencies in excess of the orbital frequencies. Here, we discuss the implications for models of the kHz QPOs observed from many accreting neutron-star sources (for a review see van der Klis 2000). These are pairs of QPOs with frequencies comparable to a kHz that vary on timescales longer than any of the dynamical timescales in the vicinity of the neutron stars.

As the frequencies of the kHz QPOs vary, they follow a number of intriguing patterns. In all sources for which the spin frequency of the neutron star is known, either via observations of X-ray pulses or burst oscillations, the difference frequency between the two kHz QPOs was shown to be comparable to the neutron star spin frequency or to half its value. This property gave rise to the beat-frequency models of kHz QPOs (see Strohmayer et al. 1996; Miller et al. 1998; Chakrabarty et al. 2003). In sources for which the frequencies of the two kHz QPOs were measured with high accuracy, such as Sco X-1, it was shown that the two frequencies followed a quadratic correlation (Psaltis et al. 1998; Psaltis, Belloni, & van der Klis 1999). Together with the correlation between the frequencies of the kHz QPOs and other low-frequency QPOs observed simultaneously in the same sources, this provided support for the relativistic precession models (Stella & Vietri 1999; Stella, Vietri, & Morsink 1999; Psaltis & Norman 2000). Finally, the ratios of the frequencies of the two kHz QPOs are roughly comparable to the ratio 2/3 and this gave rise to models that rely on resonances between modes (Abramowicz et al. 2003). Mathematically, not all three of the above patterns can characterize simultaneously the frequency correlations of the kHz QPOs and, in fact, none of them is valid to within the measurement errors of the QPO frequencies. However, all three of them describe the data roughly, typically to within 30% for the first and last alternative and to within 5% for the quadratic correlation (see Psaltis et al. 1998 for a detailed discussion of the statistical significance of various correlations).

In all these models one or both of the observed QPO frequencies are interpreted as dynamical frequencies (azimuthal, radial, or vertical) at different characteristic radii in the accretion flows. These dynamical frequencies have been calculated so far assuming that the only force that affects the motion of fluid elements is gravity, even though there is always an implicit assumption that some additional physical mechanism picks the characteristic radius where these QPOs originate. As we have shown, the relative ordering of the frequencies for long-wavelength modes depends on the radial profile of the azimuthal frequency. Therefore, the identification of observed QPOs with particular dynamical frequencies may not be self consistent in any of the above models.

For most realistic models of the azimuthal frequency, e.g., for the quadratic model discussed in \( \S 2 \) (eq. [8]), the largest of the dynamical frequencies is \( \kappa \) and the smallest is \( \Omega \), in the inner part of the transition region where \( d\Omega/dr > 0 \) (see also Fig. 1). Wavepackets in the disc can modulate the accretion flow onto the compact object at the radial epicyclic frequency band \( \sim \kappa \), at the orbital frequency band \( \sim \Omega \), or at the beat \( \kappa - \Omega \) of the radial and orbital motions. It is clear that \( \kappa > 2\Omega \) and, therefore, \( \kappa > \Omega \) in the region where \( d\Omega/dr > 0 \); the order of the frequencies is \( \kappa > \kappa - \Omega \).

How does one identify the observed upper and lower kHz QPO frequencies, hereafter \( r_2 \) and \( \nu_1 \), respectively, with two of these three possible frequencies? There are two simple trends that all kHz QPOs are observed to obey: (i) the two kHz QPO frequencies always increase or decrease together, and (ii) their difference \( \Delta \nu \equiv \nu_2 - \nu_1 \) decreases as both frequencies increase. The only pair of frequencies that satisfies these two conditions are \( \kappa = \Omega \). Indeed, if we associate these two frequencies with the upper and lower kilohertz QPO frequencies, i.e., \( \kappa = 2\pi\nu_2 \) and \( \kappa = \Omega = 2\pi\nu_1 \), they will satisfy the trend (ii) if \( d\Omega/dr < 0 \), which is equivalent to

\[
\frac{d\log(\Omega/d\Omega/dr)}{d\log r} < -5.
\]

This condition holds as long as the transition region size is about a fifth of the disc inner radius or less.

Perhaps the most interesting application of kilohertz QPO models is the constraints they provide on the mass-radius relation of neutron stars (Miller et al. 1998). The interpretation of the highest QPO frequency as the epicyclic rather than the Keplerian azimuthal frequency modifies the bounds obtained from the upper kilohertz frequency. To estimate the effect of the phenomena discussed here on the bounds in the mass-radius plane, we use the simple model (8). In this model, the maximum epicyclic frequency, to a first approximation, is equal to the radial epicyclic frequency at radius \( \tilde{r} \) and hence

\[
2\pi\nu_{2,\text{max}} \simeq \kappa(\tilde{r}) = 2\Omega_{\text{max}}.
\]

The constraints the kHz QPOs impose on the mass and the radius of a neutron star are obtained from two requirements. First, the neutron star radius \( r_{\text{NS}} \) must be less than a representative inner disc.
radius that is associated with upper kilohertz QPO frequency, which was interpreted as a Keplerian frequency in earlier applications (see van der Klis 2006). Recognition of the epicyclic frequency as the highest frequency modifies this constraint to

\[ r_{\text{NS}} < \frac{(GM)^{1/3}}{(2\pi v_2)^{2/3}} \int_0^f f_{1/3}, \]

where the factor

\[ f = \frac{\kappa(\tilde{r})}{\Omega_K(\tilde{r})} \approx \frac{7}{2} \left( \frac{\tilde{r}}{r_2} \right)^{3/2} \left( 1 - \frac{3}{7} \frac{\tilde{r}}{r_2} \right) \]

depends on the width of the transition region. The second constraint simply states that the radius of the last stable orbit is less than the disc radius associated with the upper kilohertz QPO, \( r_{\text{ISCO}} = 6GM/c^2 < r \). This leads to

\[ M < \frac{c^3}{6\sqrt{6G}} \frac{f}{2\pi v_2}, \]

if the upper kilohertz QPO is the radial epicyclic frequency rather than the Keplerian frequency. For an infinitely narrow boundary layer, \( f = 2 \), while for \( r/r_2 = 0.9 \) and 0.8, \( f = 1.84 \) and 1.65 respectively. The resulting modified constraints are shown in Figure 2. We conclude that a careful interpretation of the kHz QPO frequencies makes the constraints on the masses and radii of neutron stars much less stringent.

Most sources that show kHz QPOs also exhibit thermonuclear X-ray bursts. This allows for the possibility that the masses of the neutron stars in these sources can be measured via modeling of their X-ray spectral properties during such bursts (as in Özel 2006). Measurement of the mass of a neutron star in excess of the maximum mass allowed when interpreting the upper kHz QPO as an orbital frequency (e.g., as in Miller et al. 1998) will be very strong evidence that the QPOs are hydrodynamic modes and that are excited in a region of the disc that is sub-Keplerian.

In general relativity the radial epicyclic frequency for test particle orbits deviates from the Keplerian form at radii close to \( r_{\text{ISCO}} \). Unlike the Newtonian case, radial and vertical epicyclic frequencies do not coincide with the azimuthal frequency. Simple derivations of epicyclic oscillation frequencies in the (Newtonian, as well as) general relativistic cases are provided by Abramowicz & Kluzniak (2004), who give expressions for the dynamical frequencies in terms of the metric. General relativistic expressions for the radial epicyclic frequency incorporating fluid effects and appropriate formulations of the metric around the neutron star will be the subject of future work. By continuity from the Newtonian treatment here, we anticipate that the highest frequency will turn out to be the radial epicyclic frequency at radii \( r_f, f > r_{\text{ISCO}} \).

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