# COMPARISON OF SOLUTIONS OF SOME PAIRS OF NONLINEAR WAVE EQUATIONS

by

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**Abstract**

In this thesis, we compare solutions of the Camassa-Holm equation with solutions of the Double Dispersion equation and the Hunter-Saxton equation. In the first part of this thesis work, we determine a class of Boussinesq-type equations from which can be asymptotically derived. We use an expansion determined by two small positive parameters measuring nonlinear and dispersive effects. We then rigorously show that solutions of the Camassa-Holm equation are well approximated by corresponding solutions of a certain class of the Double Dispersion equation over a long time scale. Finally we show that any solution of the Double Dispersion equation can be written as the sum of solutions of the two decoupled Camassa-Holm equations moving in opposite directions up to a small error. We observe that the approximation error for the decoupled problem is greater than the approximation error characterized by single Camassa-Holm approximation. We also obtain similar results for Benjamin-Bona-Mahony approximation to the Double Dispersion equation in the long wave limit. In the literature, Hunter-Saxton equation arises as high frequency limit of the Camassa-Holm equation. In the second part of this thesis work, we establish convergence results between the solutions of the Hunter-Saxton equations and the solutions of the Camassa-Holm equation in periodic setting providing a precise estimate for the approximation error.

DOĞRUSAL OLMAYAN BAZI DALGA DENKLEMLERİNİN ÇÖZÜMLERİNİN KARŞILAŞTIRILMASI

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## Özet

Bu tezde Camassa-Holm denkleminin çözümleri ile İkili Dispersif ve Hunter-Saxton denklemlerinin çözümlerini kıyasladık. Tezin ilk bölümünde, Camassa-Holm denkleminin elde edilebileceği bir Boussinesq sınıfı belirledik. Açılımda doğrusal olmayan terimi ve saçılmayı ölçen iki küçük pozitif parametre kullandık. Daha sonra Camassa-Holm ve İkili Dispersif denklemlerin çözümlerinin uzun zaman aralığında birbirine yakın kaldığını ispatladık. Buna ek olarak, İkili Dispersif denklemin bir çözümünün zıt yönde giden iki Camassa-Holm denklemin çözümlerine ayrışabileceğini ve bu aşamada ortaya çıkan hatanın tek yönlü Camassa-Holm yaklaşımı ile elde edilen hatadan büyük olduğunu gözlemledik. İkili Dispersif ve Benjamin-Bona-Mahony denklemlerinin çözümleri için de benzer sonuçlar elde ettik. Tezin ikinci bölümünde, Camassa-Holm denkleminin yüksek frekans limiti olan Hunter-Saxton denkleminin periyodik çözümleri ile ona karşılık gelen Camassa-Holm denkleminin periyodik çözümlerinin uzun zaman aralığında birbirine yakın kaldığını gösterdik. Yaklaşımdan elde edilen hatayı net bir şekilde hesapladık.

*to whom supported my study*

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# Chapter 1

## Introduction and Preliminaries

In this thesis, we will present some recent results regarding comparisons of solutions of some pairs of nonlinear wave equations in asymptotic regimes.

In the literature, there are many work about rigourously relating the solutions of asymptotic equations with the equations of the physical problem. For example, the Korteweg-DeVries (KdV), the Benjamin-Bona-Mahony (BBM) and the Camassa-Holm (CH) equations are derived as long wave limits of water wave equations in the scope of fluid dynamics [6], [19] and references therein. Moreover, there are some other work showing that bidirectional, small amplitude long wave solutions of the water wave problem are well-approximated first by combinations of the two uncoupled KdV equations [26] and later by the CH equations by [9]. These equations are generally obtained from either the Euler equation or the Green Naghdi equations. Then, researchers consider the problem within the scope of elasticity and present the rigorous derivation of the CH equation from the Improved Boussinesq equation (IBq) in the long wave limit [10]. Then, they prove that solutions of the CH equations are well approximated by the corresponding solutions of the IBq equation [11]. They also show that any solution of the IBq equation can be written as the sum of solutions of right and left going CH equations up to a small error [12].

In the first part of this thesis work, we also consider the problem within the scope of elasticity. We derive the CH equation as the long wave, small amplitude limit of a certain class of the Double Dispersion (DD) equation. We use an expansion determined by two small positive parameters measuring nonlinear and dispersive effects. We then prove that solutions of the CH equations are well approximated by the corresponding solutions of the DD equation. Finally, we show that any solution of the DD equation can be written as the sum of solutions of CH equations moving in opposite directions up to a small error. We also obtain similar results for the Benjamin-Bona-Mahony approximation to the DD equation. All the results we obtain so far are the extensions of the results obtained in [10], [11] and [12] to "Improved Boussinesq-like DD equations".

In the second part of this thesis, we consider similar problems between the Hunter-Saxton (HS) equation and CH equation. In the literature, the Hunter-Saxton equation arises as high frequency limit of the Camassa-Holm equation [7], [17] and [22]. Existence-uniqueness of the solutions of Cauchy-problem for the HS equation is also studied in different domains. In the second part of my thesis work, we approach the problem within the periodic setting and work on the conditions under which the solutions of the CH and the HS equation remain close to each other.

The thesis is organized as follows. In the rest of this chapter, we present the main tools and notations that are going to be used throughout the thesis. In Chapter 2, we present the asymptotic derivation of the CH from the DD equation in the long wave limit. We then examine the problem for the Good Boussinesq (Good Bq) and Bad Boussinesq (Bad Bq) equations and show that the CH equation can be derived from Bad Boussinesq equation whereas it cannot be derived from the Good Boussinesq equation in the long wave limit. We then consider BBM and KdV approximations to the DD equation. In Chapter 3, we first present the main convergence result between the solutions of the DD and the CH equation. To this aim, we briefly explain the general methodology for the comparison. Then we recall the well-posedness theorems for Cauchy problems for both DD and CH equations before going through the details of the proof. In Chapter 4, following the same methodology in [12], we verify that any solution of the DD equation can be written as the sum of solutions of the two CH equations moving in opposite directions up to a small error. Here, we mainly invoke the theorems and their proofs obtained in Chapter 3 since they are almost parallel. As in [12], we show that the approximation error for the decoupling problem is greater than the approximation error characterized by the single CH equation. In Chapter 5, we first present the derivation of the HS equation from the CH equation provided in [22]. Then, we state the convergence result between the periodic solutions of the HS and CH equation since most of the well-posedness results on the initial problem for the HS equation rely on the periodicity. For that reason, we recall existence uniqueness results for Cauchy problems for both the HS and CH equation in the periodic setting before giving the proof.

Now we present the main tools and theorems that will be used in this study.

### 1.1 Sobolev Spaces

In this section, we recall the definitions of Sobolev spaces and related concepts [14]. Let $\Omega \subseteq \mathbb{R}^n$ be open set and $\alpha$ be multiindex. By $D_x^\alpha u$ we define the $\alpha$-th weak derivative of $u$. Then for each $k = 1, 2, ...$, $H^k(\Omega)$ is a Banach space with the norm

$$||u||^2_{H^k(\Omega)} = \sum_{|\alpha|\leq k} ||D_x^\alpha u||^2_{L^2(\Omega)}.$$

We will closely look at the two cases where $\Omega = \mathbb{R}$ and $\Omega = \mathbb{T} = [0, 2\pi) \subseteq \mathbb{R}$, repectively. Let $\Omega = \mathbb{R}$. Assume $u$ is an integrable function. Then define Fourier Transform and inverse Fourier Transform by

$$\hat{u}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} u(x)dx,$$
$$u(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ix\xi} u(\xi)d\xi.$$

Using the Fourier Transform and Plancherel's identity, norm on $H^s(\mathbb{R})$ for all real numbers $s \geq 0$ can be equivalently defined by

$$||u||^2_{H^s(\mathbb{R})} = \int_{\mathbb{R}} (1+\xi^2)^s |\hat{u}|^2 \, d\xi$$

where

$$\langle u, v \rangle_{H^s} = \langle \Lambda^s u, \Lambda^s v \rangle_{L^2} = \int_{\mathbb{R}} (1+\xi^2)^s \hat{u}\hat{v} \, d\xi$$

is the inner product on $H^s$ and $\Lambda^s = (1 - D_x^2)^{s/2}$.

### 1.1.1 Fourier series representation for periodic functions and Sobolev spaces on $\mathbb{T}$

Let $u$ be periodic function with period $2\pi$ and integrable on $[0, 2\pi)$. Then Fourier series of $u$ is given by

$$u(x) = \sum_{n=-\infty}^{\infty} u_n e^{inx}.$$

We can find the Fourier coefficients precisely using the Fourier Transform:

$$\begin{aligned}
\int_0^{2\pi} u(x)e^{-imx} dx &= \int_0^{2\pi} \sum_{n=-\infty}^{\infty} u_n e^{inx} e^{-imx} dx \\
&= \sum_{n=-\infty}^{\infty} \int_0^{2\pi} u_n e^{inx} e^{-imx} dx \\
&= \sum_{n=-\infty}^{\infty} 2\pi u_n \delta_{mn} dx \\
&= 2\pi u_m
\end{aligned}$$

where $\delta_{mn}$ is the Kronecker delta defined as $\delta_{mn} = 1$ for $m = n$ and $\delta_{mn} = 0$ for $m \neq n$. Thus we arrive at

$$u_n = \hat{u}(n) = \frac{1}{2\pi} \int_0^{2\pi} u(x) e^{-inx} dx. \tag{1.1}$$

Assume $D_x^j u \in L^2(\mathbb{T})$ for $j = 0, 1, 2, ...., k$. Then

$$||u||^2_{L^2(\mathbb{T})} = \sum_{n=-\infty}^{\infty} |u_n|^2$$

and

$$D_x^j u(x) = \sum_{n=-\infty}^{\infty} u_n (in)^j e^{inx}.$$

It follows that

$$||D_x^j u||_{L^2(\mathbb{T})} = \sum_{n=-\infty}^{\infty} n^{2j}|u_n|^2$$

and

$$\begin{aligned} ||u||^2_{H^k(\mathbb{T})} &= \sum_{j=0}^{k} \sum_{n=-\infty}^{\infty} n^{2j}|u_n|^2 \\ &\approx \sum_{n=-\infty}^{\infty} (1+n^2)^k |u_n|^2. \end{aligned}$$

Similarly, for all real numbers $s \geq 0$, we define

$$||u||^2_{H^s(\mathbb{T})} = \sum_{n=-\infty}^{\infty} (1+n^2)^s |u_n|^2.$$

### 1.1.2 Some Useful Sobolev Inequalities

- For any $0 < s_1 \leq s_2 < \infty$ there holds

  $$H^{s_1}(\mathbb{R}) \supset H^{s_2}(\mathbb{R})$$

  and

  $$||h||_{H^{s_1}(\mathbb{R})} \leq ||h||_{H^{s_2}(\mathbb{R})}.$$

- Let $f, g \in H^s(\mathbb{R})$ and $s \geq 0$. Then

  1. $||fg||_{H^s(\mathbb{R})} \leq C\left(||f||_{H^s(\mathbb{R})}||g||_{L^\infty(\mathbb{R})} + ||f||_{L^\infty(\mathbb{R})}||g||_{H^s(\mathbb{R})}\right)$,
  2. $||f||_\infty \leq C||f||_s$ if $s > 1/2$.

## 1.2 Homogeneous Sobolev spaces and properties of the inverse operator $D_x^{-1}$ on $\mathbb{T}$

Assume $u$ is the antiderivative of $2\pi$-periodic function $w$. That is

$$u(x) = D_x^{-1}w(x) = \int_0^x w(y)dy.$$

Assume moreover that $w$ has mean zero:

$$w_0 = \hat{w}(0) = \frac{1}{2\pi}\int_0^{2\pi} w(x)\,dx = 0.$$

Then $u$ is also $2\pi$ periodic function and we have the following relation between the Fourier series of $D_x u$ and $w$:

$$D_x u(x) = \sum_{n=-\infty}^{\infty} u_n ine^{inx} = \sum_{n=-\infty}^{\infty} w_n e^{inx} = w(x)$$

with $w_n = inu_n$, for $n \neq 0$ .

All the observations above imply that in order for $D_x^{-1} w = u$ to exist $w$ should have mean zero.

Let $k$ be a positive integer. The Homogenous Sobolev Space $\dot{H}^k(\mathbb{T})$, the subspace of Sobolev space $H^k(\mathbb{T})$, is defined by

$$\dot{H}^k(\mathbb{T}) = \{u : u \in H^k(\mathbb{T});\ \hat{u}(0) = u_0 = 0\}.$$

This time norm is defined by

$$||u||^2_{\dot{H}^k(\mathbb{T})} = \sum_{n\neq 0} n^{2k}|u_n|^2$$

since $u_0 = 0$.

**Lemma 1.2.1** *Let $w \in \dot{H}^k(\mathbb{T})$ and $D_x^{-1}w = u$ as above. Then*

1. $||D_x^{-1}w||_{\dot{H}^k(\mathbb{T})} \leq ||w||_{\dot{H}^{k-1}(\mathbb{T})} \leq ||w||_{H^{k-1}(\mathbb{T})}$
2. $||D_x^{-1}w||_{H^k(\mathbb{T})} \leq 2^{k/2}||D_x^{-1}w||_{\dot{H}^k(\mathbb{T})}$
3. $||D_x^{-1}w||_{H^k(\mathbb{T})} \leq C||w||_{H^{k-1}(\mathbb{T})}$.

**Proof**: Firstly, note that

$$||D_x^{-1}w||^2_{\dot{H}^k(\mathbb{T})} = ||u||^2_{\dot{H}^k(\mathbb{T})} = \sum_{n\neq 0} n^{2k}|u_n|^2 = \sum_{n\neq 0} n^{2k}\frac{|w_n|^2}{(in)^2} = ||w||^2_{\dot{H}^{k-1}(\mathbb{T})}$$

and $n^{2k} \leq 1 + n^{2k} \leq (1+n^2)^k$. Hence

$$||w||^2_{\dot{H}^{k-1}(\mathbb{T})} = \sum_{n\neq 0} n^{2(k-1)}|w_n|^2 \leq \sum_{n=-\infty}^{\infty}(1+n^2)^{k-1}|w_n|^2 = ||w||^2_{H^{k-1}(\mathbb{T})}.$$

We also have

$$
\begin{aligned}
||D_x^{-1}w||^2_{H^k(\mathbb{T})} &= \sum_{n=-\infty}^{\infty}(1+n^2)^k|\widehat{D_x^{-1}w}|^2 \\
&= \sum_{n\neq 0}(1+n^2)^k|\frac{\hat{w}_n}{in}|^2 \\
&= \sum_{n\neq 0}\left[1+\binom{k}{1}n^2+\binom{k}{2}n^4+..+\binom{k}{k}n^{2k}\right]|\frac{\widehat{w}_n}{in}|^2 \\
&\leq \sum_{n\neq 0}2^k n^{2k}|\frac{\widehat{w}_n}{in}|^2 \\
&= 2^k||D_x^{-1}w||^2_{\dot{H}^k(\mathbb{T})}.
\end{aligned}
$$

Combining the above estimates, we obtain part 3 where $C = 2^{k/2}$. □

Note that The Homogenous Sobolev Spaces can also be defined for all real numbers $s \geq 0$. Thus all the inequalities above are also valid for real numbers $s \geq 0$.

In the rest of the thesis we will use $||.||$,$||.||_s$ and $||.||_{\dot{s}}$ for the $L^2$, $H^s$ and $\dot{H}^s$ norms respectively and $C$ is a generic constant.

### 1.3 Commutator estimates

The commutator of two operators $K$ and $L$ is defined as $[K, L] = KL - LK$. In this section, we present some commutator estimates listed from [19] for the completeness of the work.

**Proposition 1.3.2** *Let $q_0 \geq 1/2$, $s \geq 0$ and $\Lambda^s = (1 - D_x^2)^{s/2}$. If $-q_0 < r \leq q_0 + 1 - s$ and $w \in H^{q_0+1}$, then for all $g \in w \in H^{r+s-1}$ one has*

$$||[\Lambda^s, w]g||_r \leq C||w||_{q_0+1}||g||_{r+s-1}.$$

1. Assume $q_0 = s > 1/2$. Then $-s < r \leq 1$ and
   - $||[\Lambda^s, w]g|| \leq C||w||_{s+1}||g||_{s-1}$
   - $||[\Lambda^s, w]g||_1 \leq C||w||_{s+1}||g||_s$.
2. Assume $s > 3/2$ and $q_0 = s - 1 > 1/2$. Then $-(s - 1) < r \leq 0$ and
   - $||[\Lambda^s, w]g|| \leq C||w||_{H^s}||g||_{H^{s-1}}$.

**Lemma 1.3.3** *Assume $u$ and $h$ are smooth enough, then*

*1.* $\langle h\Lambda^s u_x, \Lambda^s u\rangle = -\frac{1}{2}\langle h_x\Lambda^s u, \Lambda^s u\rangle$,

2. $\langle \Lambda^s(hu_x), \Lambda^s u\rangle = \langle [\Lambda^s, h]u_x, \Lambda^s u\rangle - \frac{1}{2}\langle h_x \Lambda^s u, \Lambda^s u\rangle$.

**Proof**: Using the fact that the operator $D_x$ is skew-symmetric, we obtain

$$\begin{aligned}\langle h\Lambda^s u_x, \Lambda^s u\rangle = \langle h\Lambda^s u, \Lambda^s u_x\rangle &= -\langle D_x(h\Lambda^s u), \Lambda^s u\rangle \\ &= -\langle h_x\Lambda^s u, \Lambda^s u\rangle - \langle h\Lambda^s u_x, \Lambda^s u\rangle.\end{aligned}$$

Thus $2\langle h\Lambda^s u_x, \Lambda^s u\rangle = -\langle h_x \Lambda^s u, \Lambda^s u\rangle$ and the result follows.

Note also that

$$\begin{aligned}\langle \Lambda^s(hu_x), \Lambda^s u\rangle &= \langle [\Lambda^s, h]u_x, \Lambda^s u\rangle + \langle h\Lambda^s u_x, \Lambda^s u\rangle \\ &= \langle [\Lambda^s, h]u_x, \Lambda^s u\rangle - \frac{1}{2}\langle h_x\Lambda^s u, \Lambda^s u\rangle\end{aligned}$$

where we use part 1 for the second term. □

### 1.4 Asymptotic expansion

In this section, we want to summarize some results about asymptotic expansions. We mainly refer to [13], [18] and [30] for the definitions, examples and notations.

Let $R$ be a set. The sequence of functions $\{\phi_n\}$ is called an asymptotic sequence for $x \to x_0$ in $R$ if for each $n$, $\phi_n$ is defined in $R$ and $\phi_{n+1} = o(\phi_n)$ as $x \to x_0$. In other words,

$$\lim_{x\to x_0} [\phi_{n+1}(x)/\phi_n(x)] = 0 \quad \forall n = 0, 1, 2...$$

For example $\{\phi_n\}(x) = x^n$ is an asymptotic sequence for $x \to 0$.

The series $\sum_{n=0}^{N} a_n\phi_n(x)$ is said to be asymptotic expansion of $f(x)$ if

$$f(x) = \sum_{n=0}^{N} a_n\phi_n(x) + o(\phi_N) \quad \text{as } x \to x_0.$$

It is usually written

$$f(x) \sim \sum_{n=0}^{N} a_n\phi_n(x) \quad \text{as } x \to x_0. \tag{1.2}$$

Following example illustrates that an asymptotic expansion of a function can be different from its Taylor expansion.

**Example 1.4.1** *Let $c$ be an arbitrary constant. Then*

$$\frac{1}{1-x} + ce^{-1/x} \sim 1 + x + x^2 + ....... \quad \text{as } x \to 0^+$$

*since* $e^{-1/x} = o(x^n)$. *However,*

$$\frac{1}{1-x} + ce^{-1/x} \neq 1 + x + x^2 + ....... \quad \text{as } x \to 0^+.$$

Following example demonstrates a useful property of asymptotic expansions.

**Example 1.4.2** *Consider the error function* $\operatorname{erf}(x) : \mathbb{R} \to \mathbb{R}$ *defined by*

$$\operatorname{erf}(x) = \frac{1}{2\pi} \int_0^x e^{-t^2} dt.$$

*Its power series expansion*

$$\operatorname{erf}(x) = \frac{1}{2\pi} \left\{ x - \frac{x^3}{2} + ... + \frac{(-1)^n}{(2n+1)!} x^{2n+1} + .... \right\}$$

*obtained by integrating the power series expansion of* $e^{-t^2}$ *is convergent for every* $x \in \mathbb{R}$. *For large values of $x$, however, the convergence is very slow for the Taylor series of the error function at $x = 0$. Instead, we can use the following divergent asymptotic expansion to obtain accurate approximations of* $\operatorname{erf}(x)$ *for large $x$:*

$$\operatorname{erf}(x) \sim 1 - \frac{e^{-x^2}}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^{n+1} \frac{1 \cdot 3 \cdot \ldots \cdot (2n-1)!}{2^n} \frac{1}{x^{n+1}}$$

*is divergent as $x \to \infty$. For example, when $x = 3$, we need 31 terms in the Taylor series at $x = 0$ to approximate* $\operatorname{erf}(3)$ *to an accuracy of* $10^{-5}$ *, whereas we only need 2 terms in the asymptotic expansion.*

# Chapter 2

# Asymptotic Derivations

In the literature, there are many work on asymptotic approximations to the Euler equation. One of the most typical model equation is the Camassa-Holm equation (CH) which is given by

$$v_\tau + \kappa_1 v_\xi + 3vv_\xi - v_{\xi\xi\tau} = \kappa_2(2v_\xi v_{\xi\xi} + vv_{\xi\xi\xi}) \tag{2.1}$$

derived for the unidirectional propagation of long water waves in the context of a shallow water approximation to the Euler equation [6]. Also, the CH equation has been derived as long wave limit of the Improved Boussinesq equation (IBq) over a long time scale in [10]. In the present chapter, we consider the problem within the scope of elasticity as in [10] and give the class of Bq-type equations from which the CH equation can be formally derived.

## 2.1 Derivation of the Camassa-Holm equation from the Double Dispersion equation

We consider the Double Dispersive equation (DD)

$$u_{tt} - u_{xx} + au_{xxxx} - bu_{xxtt} = (u^2)_{xx}, \tag{2.2}$$

where $u(x,t)$ is a real-valued function, the subscripts $x$ and $t$ denote partial differentiations and $a$ and $b$ are positive constants. Now, we will provide formal derivation of the CH equation from the DD equation in the long wave limit. In other words, we are going to show that right-going, small-but-finite amplitude, long wave solutions of DD equation satisfy CH equation asymptotically. For this purpose, we introduce the scaling transformation

$$u(x,t) = \epsilon U(\delta(x-t), \delta t) = \epsilon U(Y,S)$$

where $\epsilon$ and $\delta$ are positive small parameters measuring the effects of nonlinearity and dispersion, respectively. Then we are going to plug this solution into (2.2). We first find all the derivatives appearing there as follows:

$$
\begin{aligned}
u_t &= [-U_Y + U_S] \\
u_{tt} &= \epsilon\delta^2 \left[U_{YY} - 2U_{YS} + U_{SS}\right] \\
D_x^i u &= \epsilon\delta^i U_Y \quad \text{for} \quad i = 1,2,3,4 \\
u_{xxt} &= \epsilon\delta^3 \left[-U_{YYY} + U_{YYS}\right] \\
u_{xxtt} &= \epsilon\delta^4 \left[U_{YYYY} - 2U_{YYYS} + U_{YYSS}\right] \\
(u^2)_{xx} &= \epsilon^2\delta^2 (U^2)_{YY}.
\end{aligned}
$$

Now we plug them all in equation (2.2) to obtain

$$
U_{SS} - 2U_{YS} + \delta^2 \left[(a-b)U_{YYYY} + 2bU_{YYYS} - bU_{YYSS}\right] = \epsilon(U^2)_{YY}. \tag{2.3}
$$

Our aim is to seek asymptotic solution of (2.3) in the form

$$
U(Y,S;\epsilon,\delta) = U_0(Y,S) + \epsilon U_1(Y,S) + \delta^2 U_2(Y,S) + \epsilon\delta^2 U_3(Y,S) + \mathcal{O}(\epsilon^2,\delta^4). \tag{2.4}
$$

We note that only even powers of $\delta$ appear in this form since there are only even order spatial derivatives in the DD equation.

We plug (2.4) in (2.3). Then we obtain

$$
\begin{aligned}
&U_{0SS} - 2U_{0YS} + \delta^2 \left((a-b)U_{0YYYY} + 2bU_{0YYYS} - bU_{0YYSS}\right) \\
&+\epsilon \left[U_{1SS} - 2U_{1YS} + \delta^2 \left((a-b)U_{1YYYY} + 2bU_{1YYYS} - bU_{1YYSS}\right)\right] \\
&+\delta^2 \left[U_{2SS} - 2U_{2YS} + \delta^2 \left((a-b)U_{2YYYY} + 2bU_{2YYYS} - bU_{2YYSS}\right)\right] \\
&\epsilon\delta^2 \left[U_{3SS} - 2U_{3YS} + \delta^2 \left((a-b)U_{3YYYY} + 2bU_{2YYYS} - bU_{3YYSS}\right)\right] \\
&-\epsilon \left[U_0^2 + \epsilon^2 U_1^2 + \delta^4 U_2^2 + \epsilon^2\delta^4 U_3^2\right]_{YY} \\
&-2\epsilon \left[\epsilon U_0 U_1 + \delta^2 U_0 U_2 + \epsilon\delta^2 U_0 U_3 + \epsilon\delta^2 U_1 U_2 + \epsilon^2\delta^2 U_1 U_3 + \epsilon\delta^4 U_2 U_3)\right]_{YY} + \ldots = 0.
\end{aligned}
$$

We can rewrite the equation at all orders in the following way:

$$
\mathcal{O}(1): \quad U_{0SS} - 2U_{0YS} = 0 \tag{2.5}
$$

$$
\mathcal{O}(\epsilon): \quad U_{1SS} - 2U_{1YS} - (U_0^2)_{YY} = 0 \tag{2.6}
$$

$$
\mathcal{O}(\delta^2): \quad (a-b)U_{0YYYY} + 2bU_{0YYYS} - bU_{0YYSS} + U_{2SS} - 2U_{2YS} = 0 \tag{2.7}
$$

$$
\mathcal{O}(\epsilon\delta^2): \quad (a-b)U_{1YYYY} + 2bU_{1YYYS} - bU_{1YYSS} + U_{3SS} - 2U_{3YS} - 2(U_0U_2)_{YY} = 0. \tag{2.8}
$$

Now, we are going to solve these equations iteratively and find $U_i$ for $i = 1,2,3$. We assume that all unknowns $U_i$ and their derivatives decay to zero as $|Y|$ tends to infinity.

Equation (2.5) implies $(D_S - 2D_Y)U_{0S} = 0$. Then

$$U_{0S} = 0 \text{ and } U_0 = U_0(Y). \tag{2.9}$$

If we rewrite equation (2.6), we get

$$(D_S - 2D_Y)U_{1S} - (U_0^2)_{YY} = 0. \tag{2.10}$$

Using $U_0 = U_0(Y)$, we differentiate (2.10) with respect to $S$ to obtain

$$(D_S - 2D_Y)U_{1SS} = 0,$$

which implies that

$$U_{1SS} = 0 \text{ and } U_{1S} = U_{1S}(Y). \tag{2.11}$$

From equation (2.10) and (2.11) we have $-2U_{1SY} = (U_0^2)_{YY}$. This implies

$$U_{1S} = -\frac{1}{2}(U_0^2)_Y. \tag{2.12}$$

Rewriting equation (2.7), we get

$$(D_S - 2D_Y)U_{2S} + (a-b)U_{0YYYY} + 2bU_{0YYYS} - bU_{0YYSS} = 0.$$

By (2.9), we have

$$(D_S - 2D_Y)U_{2S} + (a-b)U_{0YYYY} = 0. \tag{2.13}$$

Now we differentiate this equation with respect to $S$ and use (2.9) to obtain

$$(D_S - 2D_Y)U_{2SS} = 0$$

and

$$U_{2SS} = 0 \text{ and } U_{2S} = U_{2S}(Y). \tag{2.14}$$

From equation (2.13) and (2.14) we have $-2U_{2SY} + (a-b)U_{0YYYY} = 0$ and

$$U_{2S} = \frac{(a-b)}{2}U_{0YYY}. \tag{2.15}$$

We rewrite equation (2.8), we get

$$(D_S - 2D_Y)U_{3S} + (a-b)U_{1YYYY} + 2bU_{1YYYS} - bU_{1YYSS} - 2(U_0U_2)_{YY} = 0.$$

Using (2.9), latter equation reduces to

$$(D_S - 2D_Y)U_{3S} + (a-b)U_{1YYYY} + 2bU_{1YYYS} - 2(U_0U_2)_{YY} = 0. \tag{2.16}$$

We differentiate equation (2.16) with respect to $S$ and use (2.11) to get

$$(D_S - 2D_Y)U_{3SS} + (a-b)U_{1YYYYS} - 2(U_0U_2)_{YYS} = 0. \tag{2.17}$$

If we differentiate equation (2.17) with respect to $S$ once again and use (2.11), we obtain

$$(D_S - 2D_Y)U_{3SSS} - 2(U_0U_2)_{YYSS} = 0.$$

However,

$$(U_0U_2)_{YYSS} = (U_{0S}U_2 + U_0U_{2S})_{YYS} = (U_0U_{2S})_{YYS} = (U_{0S}U_{2S} + U_0U_{2SS})_{YY} = 0$$

by (2.9) and (2.14). Then it follows that

$$(D_S - 2D_Y)U_{3SSS} = 0.$$

Therefore,

$$U_{3SSS} = 0 \text{ and } U_{3SS} = U_{3SS}(Y).$$

Then equation (2.17) reduces to

$$-2U_{3SSY} + (a-b)U_{1YYYYS} - 2(U_0U_2)_{YYS} = 0$$

and

$$U_{3SS} = \frac{(a-b)}{2}U_{1YYYS} - (U_0U_2)_{YS}. \tag{2.18}$$

However,

$$\begin{aligned}(U_0U_2)_{YS} &= (U_{0Y}U_2 + U_0U_{2Y})_S \\ &= (U_{0YS}U_2 + U_{0Y}U_{2S} + U_{0S}U_{2Y} + U_0U_{2YS}) \\ &= (U_{0Y}U_{2S} + U_0U_{2YS}) \\ &= (U_0U_{2S})_Y.\end{aligned}$$

From (2.12) and (2.15), equation (2.18) becomes

$$U_{3SS} = -\frac{(a-b)}{4}(U_0^2)_{YYYY} - \frac{(a-b)}{2}(U_0U_{0YYY})_Y. \tag{2.19}$$

We now plug equation (2.19) in equation (2.16), and solve for $U_{3SY}$ and obtain that

$$\begin{aligned}U_{3SY} = &-\frac{(a-b)}{8}(U_0^2)_{YYYY} - \frac{(a-b)}{4}(U_0U_{0YYY})_Y \\ &+ \frac{(a-b)}{2}U_{1YYYY} + bU_{1YYYS} - (U_0U_2)_{YY}.\end{aligned} \tag{2.20}$$

Integrating equation (2.20) we get

$$U_{3S} = -\frac{(a-b)}{8}(U_0^2)_{YYY} - \frac{(a-b)}{4}(U_0U_{0YYY}) + \frac{(a-b)}{2}U_{1YYY} + bU_{1YYS} - (U_0U_2)_Y. \tag{2.21}$$

Finally we insert all results (2.9),(2.12),(2.15),and (2.21) in equation (2.4) and obtain

$$\begin{aligned}U_S =&U_{0S} + \epsilon U_{1S} + \delta^2 U_{2S} + \epsilon\delta^2 U_{3S} + \mathcal{O}(\epsilon^2, \delta^4) \\ =&0 - \epsilon\frac{1}{2}(U_0^2)_Y + \delta^2\frac{(a-b)}{2}U_{0YYY} \\ &+ \epsilon\delta^2\left[-\frac{(a-b)}{8}(U_0^2)_{YYY} - \frac{(a-b)}{4}(U_0U_{0YYY})\right] \\ &+ \epsilon\delta^2\left[\frac{(a-b)}{2}U_{1YYY} + bU_{1YYS} - (U_0U_2)_Y\right] \\ &+ \mathcal{O}(\epsilon^2, \delta^4).\end{aligned} \tag{2.22}$$

On the other hand,

$$\frac{1}{2}(U_0^2)_Y = U_0 U_{0Y}$$
$$(U_0 U_{0Y})_{YY} = 3U_{0Y}U_{0YY} + U_0 U_{0YYY}$$
$$(U_0 U_2)_Y = U_{0Y}U_2 + U_0 U_{2Y}$$
$$U_{1S} = -U_0 U_{0Y}.$$

If we plug all these in (2.22), we get

$$\begin{aligned}
U_S &= -\epsilon U_0 U_{0Y} + \frac{\delta^2(a-b)}{2}U_{0YYY} \\
&\quad + \epsilon\delta^2\left[-\frac{3(a-b)}{4}U_{0Y}U_{0YY} - \frac{(a-b)}{4}U_0U_{0YYY} - \frac{(a-b)}{4}U_0U_{0YYY}\right] \\
&\quad + \epsilon\delta^2\left[\frac{(a-b)}{2}U_{1YYY} - b(U_0U_{0Y})_{YY} - U_{0Y}U_2 - U_0U_{2Y}\right] + \mathcal{O}(\epsilon^2,\delta^4) \\
&= -\epsilon U_0 U_{0Y} + \frac{\delta^2(a-b)}{2}U_{0YYY} \\
&\quad + \epsilon\delta^2\left[-\frac{3(a-b)}{4}U_{0Y}U_{0YY} - \frac{(a-b)}{2}U_0U_{0YYY} + \frac{(a-b)}{2}U_{1YYY}\right] \\
&\quad + \epsilon\delta^2\left[-U_{0Y}U_2 - U_0U_{2Y} - 3bU_{0Y}U_{0YY} - bU_0U_{0YYY}\right] + \mathcal{O}(\epsilon^2,\delta^4) \\
&= -\epsilon U_0 U_{0Y} + \frac{\delta^2(a-b)}{2}U_{0YYY} \\
&\quad + \epsilon\delta^2\left[-\frac{(3a+9b)}{4}U_{0Y}U_{0YY} - \frac{(a+b)}{2}U_0U_{0YYY} + \frac{(a-b)}{2}U_{1YYY}\right] \\
&\quad + \epsilon\delta^2\left[-U_{0Y}U_2 - U_0U_{2Y}\right] + \mathcal{O}(\epsilon^2,\delta^4) \\
&= -\epsilon U_0 U_{0Y} + \frac{\delta^2(a-b)}{2}U_{0YYY} \\
&\quad + \epsilon\delta^2\left[-\frac{(3a+9b)}{4}U_{0Y}U_{0YY} - \frac{(a+b)}{2}U_0U_{0YYY} + \frac{(a-b)}{2}U_{1YYY}\right] \\
&\quad + \epsilon\delta^2\left[-U_{0Y}U_2 - U_0U_{2Y}\right] + \mathcal{O}(\epsilon^2,\delta^4).
\end{aligned} \tag{2.23}$$

We observe the following

| $UU_Y$ | $U_{0Y}$ | $\epsilon U_{1Y}$ | $\delta^2 U_{2Y}$ | $\epsilon\delta^2 U_{3Y}$ |
|---|---|---|---|---|
| $U_0$ | $U_0U_{0Y}$ | $\epsilon U_0U_{1Y}$ | $\delta^2U_0U_{2Y}$ | $\epsilon\delta^2U_0U_{3Y}$ |
| $\epsilon U_1$ | $\epsilon U_1U_{0Y}$ | $\epsilon^2U_1U_{1Y}$ | $\epsilon\delta^2U_1U_{2Y}$ | $\epsilon^2\delta^2U_1U_{3Y}$ |
| $\delta^2U_2$ | $\delta^2U_2U_{0Y}$ | $\epsilon\delta^2U_2U_{1Y}$ | $\delta^4U_2U_{2Y}$ | $\epsilon\delta^4U_2U_{3Y}$ |
| $\epsilon\delta^2U_3$ | $\epsilon\delta^2U_3U_{0Y}$ | $\epsilon^2\delta^2U_3U_{1Y}$ | $\epsilon\delta^3U_3U_{2Y}$ | $\epsilon^2\delta^3U_3U_{3Y}$ |

Multiplying $UU_Y$ by $\epsilon$ requires multiplication of each entry by $\epsilon$ individually. At $\mathcal{O}(\epsilon^2, \delta^4)$, all the entries above become

| $\epsilon UU_Y$ | $U_{0Y}$ | $\epsilon U_{1Y}$ | $\delta^2 U_{2Y}$ | $\epsilon\delta^2 U_{3Y}$ |
|---|---|---|---|---|
| $\epsilon U_0$ | $\epsilon U_0 U_{0Y}$ | 0 | $\epsilon\delta^2 U_0 U_{2Y}$ | 0 |
| $\epsilon^2 U_1$ | 0 | 0 | 0 | 0 |
| $\epsilon\delta^2 U_2$ | $\epsilon\delta^2 U_2 U_{0Y}$ | 0 | 0 | 0 |
| $\epsilon^2\delta^2 U_3$ | 0 | 0 | 0 | 0 |

Thus,

$$\epsilon UU_Y = \epsilon U_0 U_{0Y} + \epsilon\delta^2 \left[U_0 U_{2Y} + U_2 U_{0Y}\right] + \mathcal{O}(\epsilon^2, \delta^4).$$

On the other hand,

$$\begin{aligned}\frac{\delta^2}{2} U_{YYY} &= \frac{\delta^2}{2} \left[U_0 + \epsilon U_1 + \delta^2 U_2 + \epsilon\delta^2 U_3\right]_{YYY} \\ &= \frac{\delta^2}{2} \left[U_{0YYY} + \epsilon U_{1YYY}\right] + \mathcal{O}(\epsilon^2, \delta^4).\end{aligned}$$

Similar argument gives that

$$\begin{aligned}\epsilon\delta^2 U_Y U_{YY} &= \epsilon\delta^2 U_{0Y} U_{0YY} \\ \epsilon\delta^2 UU_{YYY} &= \epsilon\delta^2 U_0 U_{0YYY}.\end{aligned}$$

Now we rewrite equation (2.23) as

$$\begin{aligned}0 &= U_S + \epsilon U_0 U_{0Y} + \epsilon\delta^2 \left[U_{0Y} U_2 + U_0 U_{2Y}\right] - \frac{\delta^2(a-b)}{2} \left[U_{0YYY} + \epsilon U_{1YYY}\right] \\ &\quad + \frac{\epsilon\delta^2}{4} \left[(3a+9b) U_{0Y} U_{0YY} + 2(a+b) U_0 U_{0YYY}\right] + \mathcal{O}(\epsilon^2, \delta^4) \\ &= U_S + \epsilon UU_Y - \frac{\delta^2(a-b)}{2} U_{YYY} + \frac{\epsilon\delta^2}{4} \left[(3a+9b) U_Y U_{YY} + 2(a+b) UU_{YYY}\right].\end{aligned}$$

**Remark 2.1.1** *Solution of* (2.2) *of the form* (2.4) *satisfies*

$$U_S + \epsilon UU_Y - \frac{\delta^2(a-b)}{2} U_{YYY} + \frac{\epsilon\delta^2}{4} \left[(3a+9b) U_Y U_{YY} + 2(a+b) UU_{YYY}\right] = 0 \tag{2.24}$$

*asymptotically.*

*Note that we require $U_Y, U_{YSS}$ terms in the equation* (2.24) *in the corresponding coordinate system. On the other hand, we need scaled free form of this equation and we need right coefficients of the terms and correct ratio for $U_Y U_{YY}$ and $UU_{YYY}$ as in equation* (2.1).

**Step 1** We let $X = mY + nS$ and $T = rS$ where $m, n, r$ are positive constants to obtain the term $U_Y$. Then

$$U(Y,S) = V(mY + nS, rS) = V(X,T)$$
$$U_S = V_X X_S + V_T T_S = nV_X + rV_T$$
$$D_Y^i U D_Y^j U = m^{i+j} D_X^i V D_X^j V.$$

In other words, whenever we take derivative with respect to $Y$ we gain $m$ as a product.

We plug all these in (2.24), then

$$\begin{aligned} rV_T + nV_X + \epsilon m V V_X - \frac{\delta^2(a-b)}{2} m^3 V_{XXX} \\ + \frac{\epsilon\delta^2 m^3}{4}\left[(3a+9b)V_X V_{XX} + 2(a+b)VV_{XXX}\right] = 0. \end{aligned} \tag{2.25}$$

**Step 2** Now, we need $V_{TXX}$ term in the equation. At level $\mathcal{O}(1)$, equation (2.25) becomes $rV_T + nV_X + \epsilon m V V_X = 0$. We solve for $V_X$ and differentiate with respect to $X$ twice to obtain

$$\begin{aligned} V_X &= -\frac{r}{n}V_T - \frac{\epsilon m}{n} V V_X + \mathcal{O}(\delta^2, \epsilon\delta^2) \\ V_{XXX} &= -\frac{r}{n} V_{TXX} - \frac{\epsilon m}{n}(VV_X)_{XX} + \mathcal{O}(\delta^2, \epsilon\delta^2). \end{aligned} \tag{2.26}$$

We plug (2.26) in (2.25) and keep the terms up to $\mathcal{O}(\delta^2, \epsilon\delta^2)$,

$$\begin{aligned} rV_T + nV_X + \epsilon m V V_X + \frac{\delta^2(a-b)}{2} m^3 \left[\frac{r}{n} V_{TXX} + \frac{\epsilon m}{n}(VV_X)_{XX}\right] \\ + \frac{\epsilon\delta^2 m^3}{4}\left[(3a+9b)V_X V_{XX} + 2(a+b)VV_{XXX}\right] = 0. \end{aligned} \tag{2.27}$$

However,

$$(VV_X)_{XX} = (V_X^2 + VV_{XX})_X = 2V_X V_{XX} + V_X V_{XX} + VV_{XXX} = 3V_X V_{XX} + VV_{XXX}.$$

We plug this in (2.27) and divide by $r$ to obtain

$$\begin{aligned} V_T + \frac{n}{r}V_X + \frac{\epsilon m}{r} V V_X + \frac{\delta^2 m^3 (a-b)}{2n} V_{TXX} \\ + \frac{\epsilon\delta^2 m^3}{2r}\left[\left(\frac{3(a-b)m}{n} + \frac{(3a+9b)}{2}\right) V_X V_{XX}\right] \\ + \frac{\epsilon\delta^2 m^3}{2r}\left[\left(\frac{(a-b)m}{n} + a + b\right) VV_{XXX}\right] = 0. \end{aligned} \tag{2.28}$$

**Step 3** We need parameters-free form of (2.28). We let $v = \epsilon V$, $X = \delta\xi$, and $T = \delta\tau$. Then we multiply equation (2.28) by $\epsilon\delta$ to get

$$\begin{aligned} \epsilon\delta V_T + \epsilon\delta\frac{n}{r}V_X + \epsilon^2\delta\frac{m}{r} V V_X + \epsilon\delta^3 \frac{m^3 (a-b)}{2n} V_{TXX} \\ + \epsilon^2\delta^3\frac{m^3}{2r}\left[\left(\frac{3(a-b)m}{n} + \frac{(3a+9b)}{2}\right) V_X V_{XX}\right] \\ + \epsilon^2\delta^3\frac{m^3}{2r}\left[\left(\frac{(a-b)m}{n} + a + b\right) VV_{XXX}\right] = 0. \end{aligned} \tag{2.29}$$

However, we have

$$v = \epsilon V(X,T) = \epsilon V(\delta\xi, \delta\tau)$$
$$v_\tau = \epsilon\delta V_T$$
$$v_\xi = \epsilon\delta V_X.$$

Note that we gain $\epsilon$ from nonlinearity and $\delta$ as a product from each derivative. If we plug all these in (2.29), we obtain

$$\begin{aligned} v_\tau + \frac{n}{r}v_\xi + \frac{m}{r}vv_\xi + \frac{m^3(a-b)}{2n}v_{\tau\xi\xi} \\ + \frac{m^3}{2r}\left[\left(\frac{3(a-b)m}{n} + \frac{(3a+9b)}{2}\right)v_\xi v_{\xi\xi} + \left(\frac{(a-b)m}{n} + a + b\right)vv_{\xi\xi\xi}\right] = 0. \quad (2.30)\end{aligned}$$

**Step 4** Now, we need to determine the coefficients in equation (2.30). Sign of $v_{\tau\xi\xi}$ should be negative. This requires that $b > a > 0$. Moreover, we have to choose $m, n, r$ so that the following hold.

$$\frac{m}{r} = 3 \tag{2.31}$$
$$\frac{m^3(a-b)}{2n} = -1 \tag{2.32}$$
$$\frac{3(a-b)m}{n} + \frac{3a+9b}{2} = 2\left(\frac{(a-b)m}{n} + a + b\right). \tag{2.33}$$

From (2.33), we get

$$\frac{(b-a)m}{n} = \frac{3a+9b}{2} - 2(a+b) = \frac{5b-a}{2} \tag{2.34}$$

which is

$$\frac{m}{n} = \frac{5b-a}{2(b-a)}. \tag{2.35}$$

Conditions (2.32) and (2.34) imply

$$\frac{(b-a)m}{n} = \frac{2}{m^2} = \frac{5b-a}{2} \quad \text{and} \quad m^2 = \frac{4}{5b-a}. \tag{2.36}$$

Moreover (2.31) and (2.36) imply

$$\frac{n}{r} = \frac{n}{m}\frac{m}{r} = 3\frac{n}{m} = \frac{6(b-a)}{5b-a} \tag{2.37}$$
$$\frac{m^3}{2r} = \frac{m^2}{2}\frac{m}{r} = \frac{3}{2}m^2 = \frac{6}{5b-a}. \tag{2.38}$$

From (2.35) we have

$$\frac{3(a-b)m}{n} + \frac{(3a+9b)}{2} = \frac{3(a-b)(5b-a)}{2(b-a)} + \frac{(3a+9b)}{2} = 3(a-b). \tag{2.39}$$

If we plug (2.31),(2.32), (2.37),(2.38) and (2.39) in equation (2.30), we get

$$v_\tau + \frac{6(b-a)}{5b-a}v_\xi + 3vv_\xi - v_{\tau\xi\xi} + \frac{6}{5b-a}\left[3(a-b)v_\xi v_{\xi\xi} + \frac{3(a-b)}{2}vv_{\xi\xi\xi}\right] = 0$$

which is equivalent to

$$v_\tau + \frac{6(b-a)}{5b-a}v_\xi + 3vv_\xi - v_{\tau\xi\xi} = \frac{9(b-a)}{5b-a}\left(2v_\xi v_{\xi\xi} + vv_{\xi\xi\xi}\right). \tag{2.40}$$

**Remark 2.1.2** *Equation* (2.40) *is of CH-type with*

$$\kappa_1 = \frac{6(b-a)}{5b-a}, \quad \kappa_2 = \frac{9(b-a)}{5b-a}$$

*if and only if* $b > a$.

**Step 5** We want to find the coordinate transformation between $(\xi, \tau)$ and $(x, t)$ as follows: We know that $v = v(\xi, \tau)$. On the other hand we have

$$X = \delta\xi, \quad Y = \delta(x-t), \quad S = \delta t.$$

Therefore

$$\begin{aligned}
X &= mY + nS \\
\delta\xi &= m\delta(x-t) + n\delta t \\
\xi &= m(x-t) + nt = m\left(x - \left(1 - \frac{n}{m}\right)t\right) \\
T &= rS \\
\delta\tau &= r\delta t \\
\tau &= rt
\end{aligned}$$

So we need $m$, $1-\frac{n}{m}$, and $r$. We observe that $m = \frac{2}{\sqrt{5b-a}}$. Then

$$\begin{aligned}
1 - \frac{n}{m} &= \frac{3b+a}{5b-a} \\
r &= \frac{m}{3} = \frac{2}{3\sqrt{5b-a}}.
\end{aligned}$$

All these imply that

$$\xi = \frac{2}{\sqrt{5b-a}}\left(x - \frac{3b+a}{5b-a}t\right); \quad \tau = \frac{2}{3\sqrt{5b-a}}t \tag{2.41}$$

$$v(\xi,\tau) = v\left(\frac{2}{\sqrt{5b-a}}x - \frac{2(3b+a)}{(5b-a)\sqrt{5b-a}}t, \frac{2}{3\sqrt{5b-a}}t\right)$$

$$
\begin{aligned}
v_x &= \frac{2}{\sqrt{5b-a}} v_\xi \\
v_t &= -\frac{2(3b+a)}{(5b-a)\sqrt{5b-a}} v_\xi + \frac{2}{3\sqrt{5b-a}} v_\tau \\
v_{xx} &= \frac{4}{5b-a} v_{\xi\xi} \\
v_{xxx} &= \frac{8}{(5b-a)\sqrt{5b-a}} v_{\xi\xi\xi} \\
v_{txx} &= -\frac{2(3b+a)}{(5b-a)\sqrt{5b-a}} \frac{4}{5b-a} v_{\xi\xi\xi} + \frac{2}{3\sqrt{5b-a}} \frac{4}{5b-a} v_{\tau\xi\xi} \\
&= \frac{4}{5b-a} \left( -\frac{2(3b+a)}{(5b-a)\sqrt{5b-a}} v_{\xi\xi\xi} + \frac{2}{3\sqrt{5b-a}} v_{\tau\xi\xi} \right).
\end{aligned}
$$

Now,

$$
\begin{aligned}
v_\xi &= \frac{\sqrt{5b-a}}{2} v_x \\
\frac{2}{3\sqrt{5b-a}} v_\tau &= v_t + \frac{2(3b+a)}{(5b-a)\sqrt{5b-a}} v_\xi \\
&= v_t + \frac{2(3b+a)}{(5b-a)\sqrt{5b-a}} \frac{\sqrt{5b-a}}{2} v_x \\
&= v_t + \frac{3b+a}{5b-a} v_x.
\end{aligned}
$$

Then

$$
v_\tau = \frac{3\sqrt{5b-a}}{2} v_t + \frac{3\sqrt{5b-a}}{2} \frac{3b+a}{5b-a} v_x.
$$

Moreover

$$
\frac{5b-a}{4} v_{txx} = -\frac{2(3b+a)}{(5b-a)\sqrt{5b-a}} v_{\xi\xi\xi} + \frac{2}{3\sqrt{5b-a}} v_{\tau\xi\xi}.
$$

We solve for $v_{\tau\xi\xi}$:

$$
\begin{aligned}
v_{\tau\xi\xi} &= \frac{3\sqrt{5b-a}}{2} \left( 3\sqrt{5b-a} \frac{5b-a}{4} v_{txx} + \frac{2(3b+a)}{(5b-a)\sqrt{5b-a}} v_{\xi\xi\xi} \right) \\
&= \frac{3\sqrt{5b-a}}{2} \left( 3\sqrt{5b-a} \frac{5b-a}{4} v_{txx} + \frac{2(3b+a)}{(5b-a)\sqrt{5b-a}} \frac{(5b-a)\sqrt{5b-a}}{8} v_{xxx} \right) \\
&= \frac{3\sqrt{5b-a}}{2} \left( 3\sqrt{5b-a} \frac{5b-a}{4} v_{txx} + \frac{3b+a}{4} v_{xxx} \right).
\end{aligned}
$$

We put everything in (2.40) to obtain

$$
\begin{aligned}
\frac{3\sqrt{5b-a}}{2} v_t &+ \frac{3\sqrt{5b-a}}{2} \frac{3b+a}{5b-a} v_x + \frac{6(b-a)}{5b-a} \frac{\sqrt{5b-a}}{2} v_x + 3v \frac{\sqrt{5b-a}}{2} v_x \\
&- \frac{3\sqrt{5b-a}}{2} \left( 3\sqrt{5b-a} \frac{5b-a}{4} v_{txx} + \frac{3b+a}{4} v_{xxx} \right) \\
&= \frac{9(b-a)}{5b-a} \frac{(5b-a)\sqrt{5b-a}}{8} \left( 2v_x v_{xx} + v v_{xxx} \right). \qquad (2.42)
\end{aligned}
$$

We simplify the coefficient of $v_x$ as

$$\frac{3\sqrt{5b-a}}{2}\frac{3b+a}{5b-a}+\frac{6(b-a)}{5b-a}\frac{\sqrt{5b-a}}{2}=\frac{3\sqrt{5b-a}}{2}\left(\frac{3b+a}{5b-a}+\frac{2(b-a)}{5b-a}\right)=\frac{3\sqrt{5b-a}}{2}.$$

We divide equation (2.42) by $\frac{3\sqrt{5b-a}}{2}$ to get

$$v_t+v_x+vv_x-\frac{5b-a}{4}v_{txx}-\frac{3b+a}{4}v_{xxx}=\frac{3(b-a)}{4}\left(2v_xv_{xx}+vv_{xxx}\right). \tag{2.43}$$

Thus, we obtain the following result.

**Corollary 2.1.1** *Solutions of the form* (2.4) *of the DD equation* (2.2) *with* $b > a$ *satisfy CH equation* (2.43) *asymptotically.*

### 2.1.1 The case of the Improved Boussinesq equation

We consider the Improved Boussinesq equation (IBq) given as

$$u_{tt}-u_{xx}-u_{ttxx}=(u^2)_{xx}. \tag{2.44}$$

Note that (2.44) is not DD-type since $a=0, b=1$, yet the approximation above holds. Thus the right going solutions of equation (2.44) satisfy the CH equation (2.43) asymptotically. This coincides with the result obtained in [10].

### 2.1.2 The case of the Bad Boussinesq equation

We consider the Bad Boussinesq equation (Bad Bq) given as

$$u_{tt}-u_{xx}-u_{xxxx}=(u^2)_{xx}. \tag{2.45}$$

This equation is not DD-type equation for $a=-1, b=0$. However, asymptotic expansion holds also here. Thus, the right going solutions of (2.45) of the form (2.4) satisfy CH equation asymptotically.

### 2.1.3 The case of the Good Boussinesq equation

We consider Good Boussinesq equation (Good Bq) given as

$$u_{tt} - u_{xx} + u_{xxxx} = (u^2)_{xx}. \tag{2.46}$$

This is the special case of the DD equation since $a = 1, b = 0$. Solutions of (2.46) of the form (2.4) satisfy

$$\begin{aligned} v_\tau + \frac{n}{r}v_\xi + \frac{m}{r}vv_\xi + \frac{m^3}{2n}v_{\tau\xi\xi} \\ + \frac{m^3}{2r}\left[\left(\frac{3m}{n} + \frac{3}{2}\right)v_\xi v_{\xi\xi} + \left(\frac{m}{n} + 1 + 0\right)vv_{\xi\xi\xi}\right] = 0. \end{aligned} \tag{2.47}$$

However, this is not a CH like equation since the sign of $v_{\tau\xi\xi}$ is positive. Thus we conclude that right-going solutions of the Good Bq equation do not satisfy any CH equation.

**Remark 2.1.3** *We will call the Double Dispersion equation with $b > a > 0$ as "Improved Boussinesq-like DD equation".*

## 2.2 Derivation of the Benjamin-Bona-Mahony and Korteweg-De Vries equations

Now we can provide a result for lower order approximations:

**Corollary 2.2.2** *Right-going solutions of the DD equation with $b > a > 0$ satisfy the KdV equation.*

**Proof**: We consider asymptotic solution of (2.3) in the form

$$U(Y, S; \epsilon, \delta) = U_0(Y, S) + \epsilon U_1(Y, S) + \delta^2 U_2(Y, S) + \mathcal{O}(\epsilon, \epsilon\delta^2, \delta^4). \tag{2.48}$$

Then equation (2.24) reduces to

$$U_S + \epsilon UU_Y + \frac{\delta^2(b-a)}{2}U_{YYY} = 0. \tag{2.49}$$

We use the same coordinate transformations in step 1, step 3 and step 5 and (2.25) becomes

$$V_T + \frac{n}{r}V_X + \epsilon\frac{m}{r}VV_X + \frac{\delta^2(b-a)}{2r}m^3V_{XXX} = 0. \tag{2.50}$$

Letting $v = \epsilon V$, $X = \delta\xi$, and $T = \delta\tau$ we get

$$v_\tau + \frac{n}{r}v_\xi + \frac{m}{r}vv_\xi + \frac{(b-a)}{2r}m^3 v_{\xi\xi\xi} = 0. \tag{2.51}$$

This is the KdV equation. Thus we conclude that the right-going solutions of the DD equation of the form (2.48) satisfies the KdV equation (2.51) asymptotically. □

**Corollary 2.2.3** *The right-going solutions of the DD equation with $b > a > 0$ satisfy the BBM equation asymptotically.*

**Proof**: At the order $\mathcal{O}(1)$, equation (2.50) becomes $rV_T + nV_X = 0$. However, we need the term $V_{TXX}$. So we solve for $V_X$ and differentiate with respect to $X$ twice to obtain

$$\begin{aligned} V_X &= -\frac{r}{n}V_T + \mathcal{O}(\epsilon, \delta^2, \epsilon\delta^2) \\ V_{XXX} &= -\frac{r}{n}V_{TXX} + \mathcal{O}(\epsilon, \delta^2, \epsilon\delta^2). \end{aligned} \tag{2.52}$$

However, $V_{XXX} = -\frac{r}{n}V_{TXX}$ at $\mathcal{O}(1)$ . Then

$$V_T + \frac{n}{r}V_X + \epsilon\frac{m}{r}VV_X - \frac{\delta^2(b-a)}{2}\frac{m^3}{n}V_{TXX} = 0. \tag{2.53}$$

Afterwards we use the same coordinate transformations in step 1, step 3 and step 5 and we obtain the BBM equation:

$$v_t + v_x + vv_x - \frac{5b-a}{4}v_{txx} - \frac{3b+a}{4}v_{xxx} = 0. \tag{2.54}$$

□

**Corollary 2.2.4** 1. *The right-going solutions of the Improved Boussinesq equation satisfy the BBM equation asymptotically:*

$$v_t + v_x + vv_x - \frac{3}{4}v_{txx} + \frac{5}{4}v_{xxx} = 0. \tag{2.55}$$

2. *The right-going solutions of the Bad Bq equation satisfy the BBM equation asymptotically:*

$$v_t + v_x + vv_x + \frac{1}{4}v_{txx} - \frac{1}{4}v_{xxx} = 0. \tag{2.56}$$

# Chapter 3

# The Camassa-Holm equation as the long-wave limit of the Double Dispersion equation

## 3.1 Problem setting

We consider following Cauchy problems for the Double Dispersion equation and the Camassa-Holm equation in scaled forms respectively in the same coordinate system:

$$u_{tt} - u_{xx} + a\delta^2 u_{xxxx} - b\delta^2 u_{xxtt} - \epsilon(u^2)_{xx} = 0 \qquad x \in \mathbb{R}, t > 0 \tag{3.1}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad x \in \mathbb{R} \tag{3.2}$$

$$w_t + w_x + \epsilon w w_x - \frac{5b-a}{4}\delta^2 w_{txx} - \frac{3b+a}{4}\delta^2 w_{xxx} = \frac{3}{4}(b-a)\epsilon\delta^2(2w_x w_{xx} + w w_{xxx}) \tag{3.3}$$

$$w(x,0) = w_0(x), \quad x \in \mathbb{R},\ t > 0 \tag{3.4}$$

where $b > a > 0$, $\epsilon > 0$ measures nonlinearity and $\delta > 0$ dispersive effects. We will show that solutions of CH equations are well approximated by solutions of DD equation with $b > a > 0$ over a long time scale. In other words, we are going to show that it is always possible to find a solution of (3.1) whenever we are given the solution of (3.3) that remains close to it over a long time:

**Theorem 3.1.1** *Let $w_0 \in H^{s+6}(\mathbb{R})$, $s > 3/2$ and suppose $w^{\epsilon,\delta}$ is the solution of Cauchy problem* (3.3)-(3.4)*. Then there exists $T > 0$ and $\delta_0 \leq 1$ such that the solution $u^{\epsilon,\delta}$ of the Cauchy problem* (3.1)-(3.2) *with the same initial values*

$$u_0(x) = w_0(x) \quad u_1(x) = w_t(x,0)$$

*and $b > a > 0$ that satisfy*

$$||u^{\epsilon,\delta}(t) - w^{\epsilon,\delta}(t)||_s \leq C(\epsilon^2 + \delta^4)t$$

*for all $t \in \left[0, \frac{T}{\epsilon}\right]$ and all $0 < \epsilon \leq \delta \leq \delta_0$.*

There is a general methodology that is used to compare the solutions of the model equation and parent equation in the literature [25], [2] and [11]. Firstly, we show that model equation is derived from parent equation asymptotically. Secondly, we need existence-uniqueness results for Cauchy problems for both equations. In addition to that, solutions of the model equation should be uniformly bounded over a long time for the estimation of the residual term coming from the approximation. Lastly, using energy methods, we show that difference between solutions of the parent and model equations remain small in some appropriate function spaces on a relevant time interval. One can also use the scaled-free form of the equations above. However, scaled forms are more appropriate to deal with long waves with small amplitude. Note that derivation part of the methodology has already been discussed in the previous chapter.

There are many work on well-posedness of the Cauchy problem for the Double Dispersion equation. One of them is stated in [28]. The researchers in [1] also obtained well-posedness of the solutions of the Cauchy problem for a generalized form of the Double Dispersion equation.

On the other hand, there are many work on wellposedness of Cauchy problem for different forms of Camassa-Holm equation in both perodic and non-periodic cases. Most of them are on Cauchy problem for (2.1) with $\kappa_1 = 0$. In 1997, Constantin [3] showed the local well-posedness in the Sobolev spaces $H^s(\mathbb{T})$ for $s > 4$. Then, Constantin and Escher [5] improved the result in 1998 with $s > 3$. Then, Danchin [8] considered the same problem with initial conditions in Besov spaces in 2001. In 2002, Misiolek [23] proved local well-posedness in the space of continuously differentiable functions again in periodic setting by viewing the equation as an ODE in a Banach space using the geometric interpretation. On the other hand, Li and Olver [21] obtained results for $s > 3/2$ in non-periodic setting using regularization technique in 2000. Then, in 2001, Rodriguez-Blanco obtained the same result by using Kato semigroup theory for the quasilinear differential equations [24]. In 2016, Lee and Preston [20] obtained well-posedness results in the space of continuously differentiable functions by using group diffeomorphisms.

The equation (3.3) involves the term $w_{xxx}$. Besides this, we will need uniform bounds for the solutions of Cauchy problem for (3.3) on the real line. For that reason, we consider the work (Proposition 4 in [6]) on well-posedness of the Cauchy problem for a more generalized form of the Camassa-Holm equation which covers (3.3) as well. It not only provides existence-uniqueness results but also provides information about uniform bounds of the solutions. As in [11], we will use rephrased form of the result to adapt it to our problem:

**Theorem 3.1.2 (Corollary 1 in [11])** *Let $w_0 \in H^{s+k+1}(\mathbb{R})$, $s > 1/2$, $k \geq 1$. Then there exists $T > 0$, $C > 0$ and a unique family of solutions*

$$w^{\epsilon,\delta} \in C\left([0, \frac{T}{\epsilon}], H^{s+k}(\mathbb{R})\right) \cap C^1\left([0, \frac{T}{\epsilon}], H^{s+k-1}(\mathbb{R})\right)$$

*to (3.3) with the initial value $w(x,0) = w_0(x)$, satisfying*

$$||w^{\epsilon,\delta}(t)||_{s+k} + ||w_t^{\epsilon,\delta}(t)||_{s+k-1} \leq C,$$

*for all $0 < \epsilon \leq \delta \leq 1$ and $t \in [0, \frac{T}{\epsilon}]$.*

In the next section, we are going to concentrate on the remaining steps of the methodology.

### 3.2 Energy for the Double Dispersion Equation

In other to make the rest of the steps more clear, we are going to first consider

$$r_{tt} - r_{xx} + a\delta^2 r_{xxxx} - b\delta^2 r_{xxtt} - \epsilon(r^2 + 2wr)_{xx} = -F_x, \tag{3.5}$$

$$r(x,0) = 0, \quad r_t(x,0) = q_x(x), \tag{3.6}$$

where $r, w \in C([0,\bar{T}], H^{s+1}(\mathbb{R})) \cap C^1([0,\bar{T}], H^s(\mathbb{R})),\ F \in C([0,\bar{T}], H^s(\mathbb{R}))$ for $s > 3/2$.

We are going to find the energy for equation (3.5) and an estimate for it. Note that $r_t(x,0)$ is derivative of some function and nonhomogenous part of (3.5) is of the form $F_x$. Then we can take $r = \rho_x$ for some function $\rho(x,t)$ and equation (3.5) becomes

$$\rho_{ttx} - \rho_{xxx} + a\delta^2 \rho_{xxxxx} - b\delta^2 \rho_{xxxtt} - \epsilon((r^2 + 2wr)\rho_x)_{xx} = -F_x. \tag{3.7}$$

We integrate over $x$

$$\rho_{tt} - \rho_{xx} + a\delta^2 \rho_{xxxx} - b\delta^2 \rho_{xxtt} - \epsilon(r^2 + 2wr)_x = -F. \tag{3.8}$$

We multiply (3.8) by $\Lambda^s \rho_t$ and integrate the equation over $\mathbb{R}$:

$$\begin{aligned}\langle \Lambda^{s/2}\rho_t, \Lambda^{s/2}\left(\rho_{tt} - \rho_{xx} + a\delta^2\rho_{xxxx} - b\delta^2\rho_{xxtt}\right)\rangle - \epsilon\langle \Lambda^{s/2}\rho_t, \Lambda^{s/2}(r^2+2wr)_x\rangle \\ = -\langle \Lambda^{s/2}\rho_t, \Lambda^{s/2}F\rangle.\end{aligned}$$

Now we use integration by parts to obtain

$$\begin{aligned}\langle \Lambda^{s/2}\rho_t, \Lambda^{s/2}\rho_{tt}\rangle + \langle \Lambda^{s/2}\rho_{tx}, \Lambda^{s/2}\rho_x\rangle + a\delta^2\langle \Lambda^{s/2}\rho_{txx}, \Lambda^{s/2}\rho_{xx}\rangle + b\delta^2\langle \Lambda^{s/2}\rho_{tx}, \Lambda^{s/2}\rho_{xtt}\rangle \\ + \epsilon\langle \Lambda^{s/2}\rho_{tx}, \Lambda^{s/2}(r^2+2wr)\rangle = -\langle \Lambda^{s/2}\rho_t, \Lambda^{s/2}F\rangle.\end{aligned}$$

Use the fact that $\rho_x = r$ to get

$$\begin{aligned}\langle \Lambda^{s/2}\rho_t, \Lambda^{s/2}\rho_{tt}\rangle + \langle \Lambda^{s/2}r_t, \Lambda^{s/2}r\rangle + a\delta^2\langle \Lambda^{s/2}r_{tx}, \Lambda^{s/2}r_x\rangle + b\delta^2\langle \Lambda^{s/2}r_t, \Lambda^{s/2}r_{tt}\rangle \\ + \epsilon\langle \Lambda^{s/2}r_t, \Lambda^{s/2}(r^2+2wr)\rangle = -\langle \Lambda^{s/2}\rho_t, \Lambda^{s/2}F\rangle.\end{aligned}$$

Finally we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(||\rho_t(t)||_s^2 + ||r(t)||_s^2 + a\delta^2||r_x(t)||_s^2 + b\delta^2||r_t(t)||_s^2\right) \\ &+\epsilon\langle \Lambda^s(r^2+2wr)(t), \Lambda^s r_t(t)\rangle + \langle \Lambda^s F(t), \Lambda^s \rho_t(t)\rangle = 0.\end{aligned} \tag{3.9}$$

Thus, we define the modified energy as follows:

$$E_s^2(t) = \frac{1}{2}\left(||\rho_t(t)||_s^2 + ||r(t)||_s^2 + a\delta^2||r_x(t)||_s^2 + b\delta^2||r_t(t)||_s^2\right) + \frac{\epsilon}{2}\langle \Lambda^s(r^2+2wr)(t), \Lambda^s r(t)\rangle. \tag{3.10}$$

### 3.3 Energy Estimate for the Double Dispersion Equation

**Lemma 3.3.3** *Assume $s > 3/2$ and $r, w \in C([0,\bar{T}], H^s(\mathbb{R}))$. Let $||r(t)||_s < 1$ for $0 \le t \le \bar{T}$ and $\epsilon < \frac{1}{2\sup\limits_{0\le t\le \bar{T}}(1+2||w(t)||_s)}$, then the energy (3.10) for the Cauchy problem (3.5)-(3.6) is equivalent to*

$$E_s(t) \approx ||\rho_t(t)||_s + ||r(t)||_s + \sqrt{a}\delta||r_x(t)||_s + \sqrt{b}\delta||r_t(t)||_s$$

*for $0 \le t \le \bar{T}$.*

**Proof**: Note that

$$|\langle \Lambda^s(r^2+2wr)(t), \Lambda^s r(t)\rangle| \le ||r(t)||_s^3 + 2||w(t)||_s||r(t)||_s^2 \le (1+2||w(t)||_s)||r(t)||_s^2$$

Then

$$\begin{aligned} E_s^2(t) &\ge \frac{1}{2}\left(||\rho_t(t)||_s^2 + ||r(t)||_s^2 + a\delta^2||r_x(t)||_s^2 + b\delta^2||r_t(t)||_s^2\right) - \frac{\epsilon}{2}\langle \Lambda^s(r^2+2wr)(t), \Lambda^s r(t)\rangle \\ &\ge \left(\frac{1}{2} - \frac{\epsilon}{2}\sup_{0\le t\le \bar{T}}(1+2||w(t)||_s)\right)\left(||\rho_t(t)||_s^2 + ||r(t)||_s^2 + a\delta^2||r_x(t)||_s^2 + b\delta^2||r_t(t)||_s^2\right). \end{aligned}$$

Since $\epsilon < \frac{1}{2\sup\limits_{0\le t\le \bar{T}}(1+2||w(t)||_s)}$, we have

$$E_s^2(t) \ge \frac{1}{4}\left(||\rho_t(t)||_s^2 + ||r(t)||_s^2 + a\delta^2||r_x(t)||_s^2 + b\delta^2||r_t(t)||_s^2\right).$$

Also

$$\begin{aligned} E_s^2(t) &\le \frac{1}{2}\left(||\rho_t(t)||_s^2 + ||r(t)||_s^2 + a\delta^2||r_x(t)||_s^2 + b\delta^2||r_t(t)||_s^2\right) + \frac{\epsilon}{2}\langle \Lambda^s(r^2+2wr)(t), \Lambda^s r(t)\rangle \\ &\le \left(\frac{1}{2} + \epsilon\sup_{0\le t\le \bar{T}}(1+2||w(t)||_s)\right)\left(||\rho_t(t)||_s^2 + ||r(t)||_s^2 + a\delta^2||r_x(t)||_s^2 + b\delta^2||r_t(t)||_s^2\right) \\ &\le ||\rho_t(t)||_s^2 + ||r(t)||_s^2 + a\delta^2||r_x(t)||_s^2 + b\delta^2||r_t(t)||_s^2. \end{aligned}$$

Taking square roots of the expressions we complete the proof. □

**Lemma 3.3.4** *Assume $s > 3/2$ and $r, w \in C([0,\bar{T}], H^s(\mathbb{R})) \cap C'([0,\bar{T}], H^s(\mathbb{R}))$ and $F \in C([0,\bar{T}], H^s(\mathbb{R}))$. Let $||r(t)||_s < 1$ for $t \leq \bar{T}$ and $\epsilon < \frac{1}{2\sup\limits_{0\leq t\leq \bar{T}}(1+2||w(t)||_s)}$, then there exists some $C$ such that the energy (3.10) for the Cauchy Problem (3.5)-(3.6) satisfies*

$$E_s(t) \leq C\left(E_s(0) + t\sup_{0\leq t\leq \bar{T}}||F(t)||_s\right)$$

*for $0 \leq t \leq \bar{T} \leq T/\epsilon$.*

**Proof**: We differentiate the energy defined in (3.10):

$$\begin{aligned}\frac{d}{dt}E_s^2(t) =&\langle\Lambda^{s/2}\rho_t, \Lambda^{s/2}\rho_{tt}\rangle + \langle\Lambda^{s/2}r_t, \Lambda^{s/2}r\rangle + a\delta^2\langle\Lambda^{s/2}r_{tx}, \Lambda^{s/2}r_x\rangle + b\delta^2\langle\Lambda^{s/2}r_t, \Lambda^{s/2}r_{tt}\rangle\\ &+ \frac{d}{dt}\frac{\epsilon}{2}\langle\Lambda^s(r^2+2wr), \Lambda^s r\rangle.\end{aligned}$$

We eliminate the term $\rho_{tt}$ by using (3.8) as follows

$$\begin{aligned}\frac{d}{dt}E_s^2(t) =&\langle\Lambda^{s/2}\rho_t, \Lambda^{s/2}\left(\rho_{xx} - a\delta^2\rho_{xxxx} + b\delta^2\rho_{xxtt} + \epsilon(r^2+2wr)_x - F\right)\rangle\\ &+ \langle\Lambda^{s/2}r_t, \Lambda^{s/2}r\rangle + a\delta^2\langle\Lambda^{s/2}r_{tx}, \Lambda^{s/2}r_x\rangle + b\delta^2\langle\Lambda^{s/2}r_t, \Lambda^{s/2}r_{tt}\rangle\\ &+ \frac{d}{dt}\frac{\epsilon}{2}\langle\Lambda^s(r^2+2wr), \Lambda^s r\rangle\\ =&-\langle\Lambda^{s/2}r_t, \Lambda^{s/2}r\rangle - a\delta^2\langle\Lambda^{s/2}r_{tx}, \Lambda^{s/2}r_x\rangle - b\delta^2\langle\Lambda^{s/2}r_t, \Lambda^{s/2}r_{tt}\rangle\\ &+ \langle\Lambda^{s/2}r_t, \Lambda^{s/2}r\rangle + a\delta^2\langle\Lambda^{s/2}r_{tx}, \Lambda^{s/2}r_x\rangle + b\delta^2\langle\Lambda^{s/2}r_t, \Lambda^{s/2}r_{tt}\rangle\\ &- \epsilon\langle\Lambda^s(r^2+2wr), \Lambda^s r_t\rangle - \langle\Lambda^s F, \Lambda^s\rho_t\rangle + \frac{d}{dt}\frac{\epsilon}{2}\langle\Lambda^s(r^2+2wr), \Lambda^s r\rangle\\ =&\frac{d}{dt}\frac{\epsilon}{2}\langle\Lambda^s(r^2+2wr), \Lambda^s r\rangle - \epsilon\langle\Lambda^s(r^2+2wr), \Lambda^s r_t\rangle - \langle\Lambda^s F, \Lambda^s\rho_t\rangle\\ =&\frac{d}{dt}\left[\frac{\epsilon}{2}\langle\Lambda^s r^2), \Lambda^s r\rangle + \epsilon\langle\Lambda^s wr, \Lambda^s r\rangle\right] - \epsilon\langle\Lambda^s(r^2+2wr), \Lambda^s r_t\rangle - \langle\Lambda^s F, \Lambda^s\rho_t\rangle\\ =&\frac{\epsilon}{2}\langle\Lambda^s r^2, \Lambda^s r\rangle_t + \epsilon\langle\Lambda^s wr, \Lambda^s r\rangle_t - \epsilon\langle\Lambda^s(r^2+2wr), \Lambda^s r_t\rangle - \langle\Lambda^s F, \Lambda^s\rho_t\rangle.\end{aligned}$$

Thus we obtain that

$$\begin{aligned}\frac{d}{dt}E_s^2(t) =&\epsilon\langle\Lambda^s rr_t, \Lambda^s r\rangle + \frac{\epsilon}{2}\langle\Lambda^s r^2, \Lambda^s r_t\rangle + \epsilon\langle\Lambda^s(wr)_t, \Lambda^s r\rangle + \epsilon\langle\Lambda^s wr, \Lambda^s r_t\rangle\\ &- \epsilon\langle\Lambda^s r^2, \Lambda^s r_t\rangle - 2\epsilon\langle\Lambda^s wr, \Lambda^s r_t\rangle - \langle\Lambda^s F, \Lambda^s\rho_t\rangle\\ =&\epsilon\langle\Lambda^s rr_t, \Lambda^s r\rangle - \frac{\epsilon}{2}\langle\Lambda^s r^2, \Lambda^s r_t\rangle + \epsilon\langle\Lambda^s(wr)_t, \Lambda^s r\rangle - \epsilon\langle\Lambda^s wr, \Lambda^s r_t\rangle - \langle\Lambda^s F, \Lambda^s\rho_t\rangle\\ =&\epsilon\langle\Lambda^s rr_t, \Lambda^s r\rangle - \frac{\epsilon}{2}\langle\Lambda^s r^2, \Lambda^s r_t\rangle - \epsilon\langle\Lambda^s wr, \Lambda^s r_t\rangle\\ &+ \epsilon\langle\Lambda^s w_t r, \Lambda^s r\rangle + \epsilon\langle\Lambda^s wr_t, \Lambda^s r\rangle - \langle\Lambda^s F, \Lambda^s\rho_t\rangle.\end{aligned}$$

Hence we get

$$|\langle\Lambda^s w_t r, \Lambda^s r\rangle| \leq ||w_t||_\infty||\Lambda^s r||^2 \leq ||w_t||_{s-1}||\Lambda^s r||^2 \leq ||w_t||_s||\Lambda^s r||^2 \leq C||r||_s^2 \leq CE_s^2(t)$$

$$|\langle\Lambda^s F, \Lambda^s\rho_t\rangle| \leq ||\Lambda^s F||||\Lambda^s\rho_t|| = ||F||_s||\rho_t||_s \leq ||F||_s E_s(t)$$

since $s-1>1/2$. We observe that

$$\langle \Lambda^s w r_t, \Lambda^s r\rangle = \langle [\Lambda^s, w] r_t, \Lambda^s r\rangle - \langle w\Lambda^s r, \Lambda^s r_t\rangle$$

$$-\langle \Lambda^s w r, \Lambda^s r_t\rangle = -\langle [\Lambda^s, w] r, \Lambda^s r_t\rangle + \langle w\Lambda^s r_t, \Lambda^s r\rangle$$

and $\langle w\Lambda^s r, \Lambda^s r_t\rangle = \langle w\Lambda^s r_t, \Lambda^s r\rangle$. Moreover,

$$|\langle [\Lambda^s, w] r_t, \Lambda^s r\rangle| \le ||w||_s ||r_t||_{s-1} ||r||_s \le CE_s^2(t)$$

$$\begin{aligned}\langle [\Lambda^s, w] r, \Lambda^s r_t\rangle &= |\langle \Lambda[\Lambda^s, w] r, \Lambda^{s-1} r_t\rangle| \\ &\le ||w||_s ||r||_s ||r_t||_{s-1} \le CE_s^2(t)\end{aligned}$$

by Proposition 1.3.2. On the other hand,

$$\begin{aligned}\langle \Lambda^s r r_t, \Lambda^s r\rangle &= \langle \Lambda^{s-1}\Lambda^2 r, \Lambda^{s-1} r r_t\rangle \\ &= \langle \Lambda^{s-1}(1-D_x^2) r, \Lambda^{s-1} r r_t\rangle \\ &= \langle \Lambda^{s-1} r, \Lambda^{s-1} r r_t\rangle - \langle \Lambda^{s-1} r_{xx}, \Lambda^{s-1} r r_t\rangle \\ \frac{1}{2}\langle \Lambda^s r^2, \Lambda^s r_t\rangle &= \langle \Lambda^{s-1}\Lambda^2 r^2, \Lambda^{s-1} r_t\rangle \\ &= \frac{1}{2}\langle \Lambda^{s-1}(1-D_x^2) r^2, \Lambda^{s-1} r_t\rangle \\ &= \frac{1}{2}\langle \Lambda^{s-1} r^2, \Lambda^{s-1} r_t\rangle - \frac{1}{2}\langle \Lambda^{s-1}(r^2)_{xx}, \Lambda^{s-1} r_t\rangle \\ &= \frac{1}{2}\langle \Lambda^{s-1} r^2, \Lambda^{s-1} r_t\rangle - \langle \Lambda^{s-1}(r r_x)_x, \Lambda^{s-1} r_t\rangle \\ &= \frac{1}{2}\langle \Lambda^{s-1} r^2, \Lambda^{s-1} r_t\rangle - \langle \Lambda^{s-1} r_x^2, \Lambda^{s-1} r_t\rangle - \langle \Lambda^{s-1} r r_{xx}, \Lambda^{s-1} r_t\rangle.\end{aligned}$$

We now find estimates for all of them in terms of $||r||_s$ and $||r_t||_{s-1}$ using Proposition 1.3.2:

$$|\langle \Lambda^{s-1} r, \Lambda^{s-1} r r_t\rangle| \le ||r||_{s-1} ||r r_t||_{s-1} \le C||r||_{s-1}^2 ||r_t||_{s-1} \le C||r||_s^2 ||r_t||_{s-1}$$

$$|\langle \Lambda^{s-1} r^2, \Lambda^{s-1} r_t\rangle| \le ||r^2||_{s-1} ||r_t||_{s-1} \le C||r||_s^2 ||r_t||_{s-1}$$

$$|\langle \Lambda^{s-1} r_x^2, \Lambda^{s-1} r_t\rangle| \le ||r_x^2||_{s-1} ||r_t||_{s-1} \le C||r_x||_{s-1}^2 ||r_t||_{s-1} \le C||r||_s^2 ||r_t||_{s-1}.$$

For the terms with $r_{xx}$ we see that

$$\langle \Lambda^{s-1} r r_{xx}, \Lambda^{s-1} r_t\rangle = \langle [\Lambda^{s-1}, r] r_{xx}, \Lambda^{s-1} r_t\rangle - \langle r\Lambda^{s-1} r_{xx}, \Lambda^{s-1} r_t\rangle$$

$$-\langle \Lambda^{s-1} r_{xx}, \Lambda^{s-1} r r_t\rangle = -\langle [\Lambda^{s-1}, r] r_t, \Lambda^{s-1} r_{xx}\rangle + \langle r\Lambda^{s-1} r_t, \Lambda^{s-1} r_{xx}\rangle$$

and $\langle r\Lambda^{s-1} r_{xx}, \Lambda^{s-1} r_t\rangle = \langle r\Lambda^{s-1} r_t, \Lambda^{s-1} r_{xx}\rangle$. Moreover using commutator estimates in Proposition 1.3.2 we get

$$|\langle [\Lambda^{s-1}, r] r_{xx}, \Lambda^{s-1} r_t\rangle| \le ||r||_{s-1} ||r_{xx}||_{s-2} ||r_t||_{s-1} \le C||r||_s^2 ||r_t||_{s-1}$$

$$\begin{aligned}|\langle [\Lambda^{s-1}, r] r_t, \Lambda^{s-1} r_{xx}\rangle| &= |\langle \Lambda[\Lambda^{s-1}, r] r_t, \Lambda^{s-2} r_{xx}\rangle| \\ &\le C||r||_s ||r_t||_{s-1} ||r_{xx}||_{s-2} \le C||r||_s^2 ||r_t||_{s-1}.\end{aligned}$$

Thus,

$$||r||_s^2||r_t||_{s-1} \leq C||r||_s||r_t||_{s-1} \leq CE_s^2(t)$$

since $||r||_s < 1$ for $t \leq \bar{T}$ and

$$||r_t||_{s-1} = ||\rho_{xt}||_{s-1} \leq C|||\rho_t||_s \leq E_s(t).$$

We invoke Grönwall's Lemma and all these calculations above imply that

$$\frac{d}{dt}E_s^2(t) \leq C\epsilon E_s^2(t) + \sup ||F(t)||_s E_s(t)$$

$$\begin{aligned}
E_s(t) &\leq e^{C\epsilon t}\left(E_s(0) + \sup ||F(t)||_s \frac{1}{\epsilon}\left(1 - e^{-C\epsilon t}\right)\right) \\
&\leq e^{C\epsilon t}E_s(0) + \sup ||F(t)||_s \frac{1}{\epsilon}\left(e^{C\epsilon t} - 1\right) \\
&\leq C\left(E_s(0) + t\sup ||F(t)||_s\right) \quad \text{for} \quad t \leq \bar{T} \leq \frac{T}{\epsilon}.
\end{aligned}$$

### 3.4 Residual term corresponding to Camassa-Holm approximation

Let $w^{\epsilon,\delta}$ be the family of solutions for the Cauchy problem of the CH equation (3.3) with $w(x,0) = w_0(x)$ and $u^{\epsilon,\delta}$ be the family of solution of the Cauchy problem for DD equation (3.1) such that $u(x,0) = w(x,0) = w_0(x)$ and $u_t(x,0) = w_t(x,0)$ where we dropped the indices for simplicity and $w_t(x,0)$ is function obtained from

$$w_t = \mathcal{Q}\left(-w_x - \epsilon w w_x + \frac{3b+a}{4}\delta^2 w_{xxx} + \frac{3(b-a)}{4}\epsilon\delta^2(2w_x w_{xx} + w w_{xxx})\right)$$

at $t = 0$ and it is a function in terms of $w_0$ where $\mathcal{Q} = \left(1 - \frac{5b-a}{4}\delta^2 D_x^2\right)^{-1}$.

Let $r = u - w$. Then $r(x,0) = r_t(x,0) = 0$. We substitute the function $r$ into the equation (2.2) and observe that

$$\begin{aligned}
0 =& u_{tt} - u_{xx} + a\delta^2 u_{xxxx} - b\delta^2 u_{xxtt} - \epsilon(u^2)_{xx} \\
=& (r+w)_{tt} - (r+w)_{xx} + a\delta^2(r+w)_{xxxx} - b\delta^2(r+w)_{xxtt} - \epsilon(r+w)^2_{xx} \\
=& r_{tt} - r_{xx} + a\delta^2 r_{xxxx} - b\delta^2 r_{xxtt} \\
&+ \left(w_{tt} - w_{xx} + a\delta^2 w_{xxxx} - b\delta^2 w_{xxtt}\right) - \epsilon(r^2 + 2wr)_{xx} - \epsilon(w^2)_{xx}.
\end{aligned}$$

Then we have Cauchy problem for function $r$

$$r_{tt} - r_{xx} + a\delta^2 r_{xxxx} - b\delta^2 r_{xxtt} - \epsilon(r^2 + 2wr)_{xx} = -f \tag{3.11}$$

$$r(x,0) = 0, \quad r_t(x,0) = 0, \tag{3.12}$$

where

$$f = w_{tt} - w_{xx} + a\delta^2 w_{xxxx} - b\delta^2 w_{xxtt} - \epsilon(w^2)_{xx}. \tag{3.13}$$

We will show that $f = F_x$ for some $F \in C([0, \frac{T}{\epsilon}], H^s(\mathbb{R}))$ under some reasonable conditions.

We rewrite equation (3.3) as follows

$$w_t + w_x = -\epsilon w w_x + \frac{5b-a}{4}\delta^2 w_{txx} + \frac{3b+a}{4}\delta^2 w_{xxx} + \frac{3}{4}(b-a)\epsilon\delta^2(2w_x w_{xx} + w w_{xxx}). \tag{3.14}$$

We continue with inserting (3.14) in (3.13):

$$\begin{aligned}
f =&(D_t - D_x)(w_t + w_x) + a\delta^2 w_{xxxx} - b\delta^2 w_{xxtt} - \epsilon(w^2)_{xx} \\
=&(D_t - D_x)\left(-\epsilon w w_x + \frac{5b-a}{4}\delta^2 w_{txx} + \frac{3b+a}{4}\delta^2 w_{xxx} + \frac{3}{4}(b-a)\epsilon\delta^2(2w_x w_{xx} + w w_{xxx})\right) \\
&+a\delta^2 w_{xxxx} - b\delta^2 w_{xxtt} - \epsilon(w^2)_{xx} \\
=&-\epsilon\left[(D_t - D_x) w w_x + (w^2)_{xx}\right] + (D_t - D_x)\left(\frac{5b-a}{4}\delta^2 w_{txx} + \frac{3b+a}{4}\delta^2 w_{xxx}\right) \\
&+ a\delta^2 w_{xxxx} - b\delta^2 w_{xxtt} - \epsilon(w^2)_{xx} + (D_t - D_x)\frac{3}{4}(b-a)\epsilon\delta^2(2w_x w_{xx} + w w_{xxx}) \\
=&-\epsilon\left[(w w_x)_t - (w w_x)_x + 2(w w_x)_x\right] + \frac{5b-a}{4}\delta^2 w_{ttxx} - \frac{5b-a}{4}\delta^2 w_{txxx} \\
&+ \frac{3b+a}{4}\delta^2 w_{txxx} - \frac{3b+a}{4}\delta^2 w_{xxxx} + a\delta^2 w_{xxxx} - b\delta^2 w_{xxtt} \\
&+ (D_t - D_x)\frac{3}{4}(b-a)\epsilon\delta^2(2w_x w_{xx} + w w_{xxx}) \\
=&-\epsilon\left[w_t w_x + w w_{xt} + w_x w_x + w w_{xx}\right] + \left(\frac{5b-a}{4} - b\right)\delta^2 w_{ttxx} \\
&+ \left(\frac{3b+a}{4} - \frac{5b-a}{4}\right)\delta^2 w_{txxx} + \left(a - \frac{3b+a}{4}\right)\delta^2 w_{xxxx} \\
&+ (D_t - D_x)\frac{3}{4}(b-a)\epsilon\delta^2(2w_x w_{xx} + w w_{xxx}) \\
=&-\epsilon\left[w_x(w_t + w_x) + w(w_x + w_t)_x\right] - \frac{a-b}{4}\delta^2 w_{ttxx} + \frac{a-b}{2}\delta^2 w_{txxx} + \frac{3(a-b)}{4}\delta^2 w_{xxxx} \\
&+ (D_t - D_x)\frac{3}{4}(b-a)\epsilon\delta^2(2w_x w_{xx} + w w_{xxx}) \\
=&-\epsilon\left[w(w_x + w_t)\right]_x + \frac{a-b}{2}\delta^2\left(w_{txxx} + w_{xxxx}\right) + \frac{a-b}{4}\delta^2\left(w_{xxxx} - w_{ttxx}\right) \\
&+ (D_t - D_x)\frac{3}{4}(b-a)\epsilon\delta^2(2w_x w_{xx} + w w_{xxx}) \\
=&-\epsilon D_x\left[w(w_x + w_t)\right] + \frac{a-b}{2}\delta^2 D_x^3(w_x + w_t) + \frac{a-b}{4}\delta^2\left[D_x^2(D_x - D_t)(w_x + w_t)\right] \\
&+ (D_t - D_x)\frac{3}{4}(b-a)\epsilon\delta^2(2w_x w_{xx} + w w_{xxx}) \\
=&-\epsilon D_x\left[w(w_x + w_t)\right] + \frac{a-b}{4}\delta^2\left[3D_x^3 - D_x^2 D_t\right]\left[w_x + w_t\right] \qquad (3.15) \\
&+ (D_t - D_x)\frac{3}{4}(b-a)\epsilon\delta^2(2w_x w_{xx} + w w_{xxx}).
\end{aligned}$$

We continue by using (3.14) one more time:

$$
\begin{aligned}
f =& - \epsilon D_x \left[ w \left( -\epsilon w w_x + \frac{5b-a}{4}\delta^2 w_{txx} + \frac{3b+a}{4}\delta^2 w_{xxx} \right) \right] \\
& - \epsilon D_x \left[ w \frac{3}{4}(b-a)\epsilon\delta^2(2w_x w_{xx} + w w_{xxx}) \right] \\
& + \frac{a-b}{4}\delta^2 \left[ 3D_x^3 - D_x^2 D_t \right] \left( -\epsilon w w_x + \frac{5b-a}{4}\delta^2 w_{txx} + \frac{3b+a}{4}\delta^2 w_{xxx} \right) \\
& + \frac{a-b}{4}\delta^2 \left[ 3D_x^3 - D_x^2 D_t \right] \frac{3}{4}(b-a)\epsilon\delta^2(2w_x w_{xx} + w w_{xxx}) \\
& + (D_t - D_x)\frac{3}{4}(b-a)\epsilon\delta^2(2w_x w_{xx} + w w_{xxx}) \\
=& \epsilon^2 (w^2 w_x)_x - \frac{\epsilon\delta^2}{4} D_x \left[ (5b-a) w w_{txx} + (3b+a) w w_{xxx} \right] \\
& - \epsilon D_x \left[ w \frac{3}{4}(b-a)\epsilon\delta^2(2w_x w_{xx} + w w_{xxx}) \right] \\
& + \frac{b-a}{4}\epsilon\delta^2 D_x \left[ 3D_x^2 - D_x D_t) w w_x \right] \\
& + \frac{b-a}{16}\delta^4 \left[ (D_x^2 D_t - 3D_x^3) \left[ (5b-a) w_{xxt} + (3b+a) w_{xxx} \right] \right] \\
& + \frac{a-b}{4}\delta^2 \left[ 3D_x^3 - D_x^2 D_t \right] \frac{3}{4}(b-a)\epsilon\delta^2(2w_x w_{xx} + w w_{xxx}) \\
& + (D_t - D_x)\frac{3}{4}(b-a)\epsilon\delta^2(2w_x w_{xx} + w w_{xxx}) \\
=& \epsilon^2 D_x^2 \frac{w^3}{3} + \frac{\epsilon\delta^2}{4} D_x \left[ (b-a)(3D_x^2 - D_t D_x) w w_x - (5b-a) w w_{txx} - (3b+a) w w_{xxx} \right] \\
& - \epsilon D_x \left[ w \frac{3}{4}(b-a)\epsilon\delta^2(2w_x w_{xx} + w w_{xxx}) \right] \\
& + \frac{b-a}{16}\delta^4 \left[ (D_x^2 D_t - 3D_x^3) \left[ (5b-a) w_{xxt} + (3b+a) w_{xxx} \right] \right] \\
& + \frac{a-b}{4}\delta^2 \left[ 3D_x^3 - D_x^2 D_t \right] \frac{3}{4}(b-a)\epsilon\delta^2(2w_x w_{xx} + w w_{xxx}) \\
& + (D_t - D_x)\frac{3}{4}(b-a)\epsilon\delta^2(2w_x w_{xx} + w w_{xxx}).
\end{aligned}
$$

Now we continue as

$$
\begin{aligned}
f =& \epsilon^2 D_x^2 \left( \frac{w^3}{3} \right) + \frac{b-a}{16}\delta^4 \left[ (D_x^2 D_t - 3D_x^3) \left[ (5b-a) w_{xxt} + (3b+a) w_{xxx} \right] \right] \\
& + \frac{\epsilon\delta^2}{4} D_x \left[ (b-a)(3D_x^2 - D_t D_x) w w_x - (5b-a) w w_{txx} - (3b+a) w w_{xxx} \right] \\
& + \frac{3}{4}\epsilon^2\delta^2 (a-b) D_x \left[ w(2w_x w_{xx} + w w_{xxx}) \right] \\
& + \frac{3}{16}\epsilon\delta^4 (b-a)^2 \left( D_x^2 D_t - 3D_x^3 \right) (2w_x w_{xx} + w w_{xxx}) \\
& + \frac{3}{4}\epsilon\delta^2 (b-a)(D_t - D_x)(2w_x w_{xx} + w w_{xxx}).
\end{aligned}
$$

Now, we use the fact that $D_x\left(w_x^2+2ww_{xx}\right)=2(2w_xw_{xx}+ww_{xxx})$.

$$
\begin{aligned}
f =&\epsilon^2 D_x^2\left(\frac{w^3}{3}\right)+\frac{b-a}{16}\delta^4\left[(D_x^2D_t-3D_x^3)\left[(5b-a)w_{xxt}+(3b+a)w_{xxx}\right]\right]\\
&+\frac{\epsilon\delta^2}{4}D_x\left[(b-a)(3D_x^2-D_tD_x)ww_x-(5b-a)ww_{txx}-(3b+a)ww_{xxx}\right]\\
&+\frac{3}{8}\epsilon^2\delta^2(a-b)D_x\left[wD_x\left(w_x^2+2ww_{xx}\right)\right]\\
&+\frac{3}{32}\epsilon\delta^4(b-a)^2D_x\left(D_x^2D_t-3D_x^3\right)\left(w_x^2+2ww_{xx}\right)\\
&+\frac{3}{8}\epsilon\delta^2(b-a)D_x(D_t-D_x)\left(w_x^2+2ww_{xx}\right).
\end{aligned}
\tag{3.16}
$$

At $\mathcal{O}(\epsilon\delta^2)$:

$$
\begin{aligned}
&\frac{\epsilon\delta^2}{8}D_x\left[2(b-a)(3D_x-D_t)D_x(ww_x)+3(b-a)(D_t-D_x)\left(w_x^2+2ww_{xx}\right)\right]\\
&+\frac{\epsilon\delta^2}{4}D_x\left[-(5b-a)ww_{txx}-(3b+a)ww_{xxx}\right]\\
=&\frac{\epsilon\delta^2}{8}D_x\left[2(b-a)(3D_x-D_t)(w_x^2+ww_{xx})+3(b-a)(D_t-D_x)\left(w_x^2+2ww_{xx}\right)\right]\\
&-\frac{\epsilon\delta^2}{4}D_x\left[-(5b-a)ww_{txx}-(3b+a)ww_{xxx}\right]\\
=&\frac{\epsilon\delta^2}{8}D_x\left[(b-a)(3D_x+D_t)w_x^2+4(b-a)D_t\left(ww_{xx}\right)\right]\\
&-\frac{\epsilon\delta^2}{4}D_x\left[(5b-a)ww_{txx}+(3b+a)ww_{xxx}\right]\\
=&\frac{\epsilon\delta^2}{4}D_x\left[3(b-a)w_xw_{xx}+(b-a)w_xw_{xt}+2(b-a)w_tw_{xx}+2(b-a)ww_{xxt}\right]\\
&-\frac{\epsilon\delta^2}{4}D_x\left[(5b-a)ww_{txx}+(3b+a)ww_{xxx}\right]\\
=&\frac{\epsilon\delta^2}{4}D_x\left[(-3b-a)w(w_x+w_t)_{xx}+(2b-2a)w_{xx}(w_x+w_t)+(b-a)w_x(w_x+w_t)_x\right].
\end{aligned}
\tag{3.17}
$$

We then obtain

$$
\begin{aligned}
f =&\epsilon^2 D_x^2\left(\frac{w^3}{3}\right)+\frac{b-a}{16}\delta^4\left[(D_x^2D_t-3D_x^3)\left[(5b-a)w_{xxt}+(3b+a)w_{xxx}\right]\right]\\
&+\frac{3}{8}\epsilon^2\delta^2(a-b)D_x\left[wD_x\left(w_x^2+2ww_{xx}\right)\right]\\
&+\frac{3}{32}\epsilon\delta^4(b-a)^2D_x\left(D_x^2D_t-3D_x^3\right)\left(w_x^2+2ww_{xx}\right)\\
&+\frac{\epsilon\delta^2}{4}D_x\left[(-3b-a)wD_x^2+(2b-2a)w_{xx}+(b-a)w_xD_x\right](w_x+w_t)
\end{aligned}
$$

$$
\begin{aligned}
=&\epsilon^2 D_x^2\left(\frac{w^3}{3}\right)+\frac{b-a}{16}\delta^4\left[(D_x^2D_t-3D_x^3)\left[(5b-a)w_{xxt}+(3b+a)w_{xxx}\right]\right]\\
&+\frac{3}{8}\epsilon^2\delta^2(a-b)D_x\left[w\left(w_x^2+ww_{xx}\right)_x\right]\\
&+\frac{3}{32}\epsilon\delta^4(b-a)^2D_x\left(D_x^2D_t-3D_x^3\right)\left(w_x^2+2ww_{xx}\right)\\
&-\frac{\epsilon^2\delta^2}{4}D_x\left[(-3b-a)wD_x^2+(2b-2a)w_{xx}+(b-a)w_xD_x\right]ww_x\\
&+\frac{\epsilon\delta^4}{16}D_x\left[(-3b-a)wD_x^2+(2b-2a)w_{xx}+(b-a)w_xD_x\right]\left[(5b-a)w_{xxt}\right]\\
&+\frac{\epsilon\delta^4}{16}D_x\left[(-3b-a)wD_x^2+(2b-2a)w_{xx}+(b-a)w_xD_x\right]\left[(3b+a)w_{xxx}\right]\\
&+\frac{3\epsilon^2\delta^4}{16}D_x\left[(-3b-a)wD_x^2+(2b-2a)w_{xx}+(b-a)w_xD_x\right]\left[(b-a)\left(2w_xw_{xx}\right)\right]\\
&+\frac{3\epsilon^2\delta^4}{16}D_x\left[(-3b-a)wD_x^2+(2b-2a)w_{xx}+(b-a)w_xD_x\right]\left[(b-a)\left(ww_{xxx}\right)\right].
\end{aligned}
$$

All the terms in the above expression contains $D_x$ as a multiplication. Thus, we can write $f = F_x$:

$$
\begin{aligned}
F=&\epsilon^2 D_x\left(\frac{w^3}{3}\right)+\frac{b-a}{16}\delta^4\left[(D_xD_t-3D_x^2)\left[(5b-a)w_{xxt}+(3b+a)w_{xxx}\right]\right]\\
&-\frac{\epsilon^2\delta^2}{8}\left[3(b-a)w\left(w_x^2+ww_{xx}\right)_x\right]\\
&-\frac{\epsilon^2\delta^2}{8}\left[(-3b-a)w(w^2)_{xxx}+(2b-2a)w_{xx}(w^2)_x+(b-a)w_xD(w^2)_{xx}\right]\\
&+\frac{\epsilon\delta^4}{32}\left[3(b-a)^2\left(D_x^2D_t-3D_x^3\right)\left(w_x^2+2ww_{xx}\right)\right]\\
&+\frac{\epsilon\delta^4}{32}\left[2(-3b-a)wD_x^2+(2b-2a)w_{xx}+(b-a)w_xD_x\right]\left[(5b-a)w_{xxt}+(3b+a)w_{xxx}\right]\\
&+\frac{\epsilon^2\delta^4}{16}\left[(b-a)(-9b-3a)wD_x^2+6(b-a)^2w_{xx}+3(b-a)^2w_xD_x\right]\left[(2w_xw_{xx}+ww_{xxx})\right].
\end{aligned}
\tag{3.18}
$$

### 3.5 Estimate for the residual term corresponding to Camassa-Holm approximation

**Lemma 3.5.5** *Let $w_0 \in H^{s+6}(\mathbb{R})$, $s > 3/2$. Then there is some $C > 0$ so that the family of solutions $w^{\epsilon,\delta}$ to the CH* (3.3) *equation with initial value $w(x,0) = w_0(x)$, satisfy*

$$w_{tt} - w_{xx} + a\delta^2 w_{xxxx} - b\delta^2 w_{xxtt} - \epsilon(w^2)_{xx} = F_x$$

*with*

$$||F(t)||_s \le C(\epsilon^2 + \delta^4)$$

*for all* $0 < \epsilon \le \delta \le 1$ *and* $t \in \left[0, \frac{T}{\epsilon}\right]$.

**Proof**: We observe that $F$ in (3.18) is a combination of terms of the form $D_x^j w$ with $j \le 5$ or $D_x^l w_t$ with $l \le 4$. We only need to check the term $w_{xxxtt}$. We first recall that we rewrite the CH equation as

$$w_t = \mathcal{Q}\left(-w_x - \epsilon w w_x + \frac{3b+a}{4}\delta^2 w_{xxx} + \frac{3(b-a)}{4}\epsilon\delta^2(2w_x w_{xx} + w w_{xxx})\right) \tag{3.19}$$

where $\mathcal{Q} = \left(1 - \frac{5b-a}{4}\delta^2 D_x^2\right)^{-1}$. We then apply $D_x^3 D_t$ to (3.19) and get

$$\begin{aligned} w_{ttxxx} &= D_x^3 D_t Q\left(-w_x - \epsilon w w_x + \frac{3b+a}{4}\delta^2 w_{xxx} + \frac{3(b-a)}{4}\epsilon\delta^2(2w_x w_{xx} + w w_{xxx})\right) \\ &= D_t Q\left(-w_{xxxx} - \epsilon(w w_x)_{xxx}\right) \\ &+ D_t Q\left(\frac{3b+a}{4}\delta^2 D_x^2 w_{xxxx} + \epsilon\delta^2 D_x^2 \frac{3(b-a)}{4}(2w_x w_{xx} + w w_{xxx})_x\right), \end{aligned} \tag{3.20}$$

which implies that

$$||w_{xxxtt}||_s \le C\left(||Q w_{xxxxt}||_s + ||\delta^2 Q D_x^2 w_{xxxxt}||_s\right). \tag{3.21}$$

Note that the operator $Q$ and $\delta^2 Q D_x^2$ are bounded on the $H^s(\mathbb{R})$:

$$\begin{aligned} ||Qv||_s^2 &= \int_{\mathbb{R}} (1+\xi^2)^s (\widehat{Qv})^2 d\xi \\ &\le \int_{\mathbb{R}} (1+\xi^2)^s \left(1 + \frac{5b-a}{4}\delta^2\xi^2\right)^{-2} \hat{v}^2 d\xi \\ &\le \int_{\mathbb{R}} (1+\xi^2)^s \hat{v}^2 d\xi = ||v||_s^2. \end{aligned} \tag{3.22}$$

On the other hand,

$$\begin{aligned} ||\delta^2 Q D_x^2 v||_s^2 &= \int_{\mathbb{R}} (1+\xi^2)^s \left[(\widehat{\delta^2 Q D_x^2 v})\right]^2 d\xi \\ &\le \int_{\mathbb{R}} (1+\xi^2)^s \delta^4\xi^4 \left(1 + \frac{5b-a}{4}\delta^2\xi^2\right)^{-2} \hat{v}^2 d\xi \\ &\le \left(\frac{4}{5b-a}\right)^2 \int_{\mathbb{R}} (1+\xi^2)^s \hat{v}^2 d\xi = \left(\frac{4}{5b-a}\right)^2 ||v||_s^2. \end{aligned} \tag{3.23}$$

Now (3.22) and (3.23) both imply that

$$||Q||_s \le 1 \quad \text{and} \quad ||\delta^2 Q D_x^2||_s \le \frac{4}{5b-a}.$$

Thus

$$\begin{aligned}
||w_{xxxtt}||_s &\leq C\left(||Q||_s||w_{xxxxt}||_s + ||\delta^2 Q D_x^2||_s||w_{xxxxt}||_s\right) \\
&\leq C||w_{xxxxt}||_s + \left(\frac{4}{5b-a}\right)||w_{xxxxt}||_s \\
&\leq C||w_{xxxxt}||_s \\
&\leq C||w_t||_{s+4}, \qquad (3.24)
\end{aligned}$$

where C is generic constant.

Since all the terms of $F$ have coefficients $\epsilon^2, \epsilon\delta^2, \delta^4$, we get the following estimate:

$$||F(t)||_s \leq C(\epsilon^2 + \delta^4)\left(||w(t)||_{s+5} + ||w_t(t)||_{s+4}\right), \qquad (3.25)$$

for $0 \leq t \leq T/\epsilon$. By Theorem 3.1.2, the solutions $w$ of Cauchy problem for (3.3) are bounded with $k = 5$ and this completes the proof. □

### 3.6 Convergence result with Camassa-Holm approximation

Now we are ready to prove Theorem 3.1.1.

#### 3.6.1 Proof of the Theorem 3.1.1

Let $w^{\epsilon,\delta}$ be the solution of CH equation with $w(x,0) = w_0$. So we consider the Cauchy problem for the DD equation with $u^{\epsilon,\delta}(x,0) = w^{\epsilon,\delta}(x,0) = w_0(x)$ and $u_t(x,0) = w_t(x,0)$. Note that solution $w^{\epsilon,\delta}$ of CH equation exists for all times $t \leq T/\epsilon$ by Theorem 3.1.2. Therefore $r = u - w$ will exist over the same interval as long as the solution $u$ of DD does not blow up in a shorter time. Note that we have $r(x,0) = 0$. Therefore, by continuity there exists some $\bar{t}$ such that $||r(t)||_s \leq 1$ for all $0 \leq t \leq \bar{t} \leq T/\epsilon$. We define

$$T_0^{\epsilon,\delta} = \sup\{t \leq \frac{T}{\epsilon} : ||r(t)||_s \leq 1 \quad \text{for all} \quad t \in [0,\bar{t}]\}. \qquad (3.26)$$

Note that difference $r$ satisfies (3.11)-(3.12) with $F$ in (3.18). Consider the energy (3.10). We observe that $E_s(0) = 0$. Choose $\epsilon \leq \frac{1}{2\sup\limits_{0\leq t\leq \bar{t}}(1+2||w(t)||_s)} = \epsilon_0$. Then by Lemma 3.3.4 the energy satisfies

$$E_s(t) \leq C(\epsilon^2 + \delta^4)t \quad for \quad t \leq \bar{t} \leq T/\epsilon$$

for some generic constant $C$ and $0 \leq \epsilon \leq \delta \leq 1$ and $\epsilon \leq \epsilon_0$. We now choose $\delta$ so that $0 \leq \epsilon \leq \delta \leq \delta_0 \leq \epsilon_0 \leq 1$. Then $C(\epsilon^2 + \delta^4)T_0^{\epsilon,\delta} \leq C(\epsilon_0^2 + \delta_0^4)T_0^{\epsilon,\delta} = \epsilon' << 1$. Then $||u^{\epsilon,\delta}(t) - v^{\epsilon,\delta}(t)||_s = ||r(t)||_s \leq CE_s(t) \leq C(\epsilon_0^2 + \delta_0^4)T_0^{\epsilon,\delta} = \epsilon' << 1$ remains very small for $0 \leq t \leq T/\epsilon$. □

**Remark 3.6.1** *Initially we know that $u^{\epsilon,\delta}$ exists locally for some $t \leq T^{\epsilon,\delta}$. However, the estimate above shows that $u^{\epsilon,\delta}$ stays bounded and so exists for $[0, T/\epsilon]$.*

### 3.7 Convergence result with Benjamin-Bona-Mahony approximation

**Theorem 3.7.6** *Let $w_0 \in H^{s+6}(\mathbb{R})$, $s > 3/2$ and suppose $w^{\epsilon,\delta}$ is the solution of the BBM equation with initial value $w(x,0) = w_0(x)$. Then there exists $T > 0$ and $\delta_0 \leq 1$ such that the solution $u^{\epsilon,\delta}$ of the Cauchy problem for the DD equation, with the same initial values, satisfies*

$$||u^{\epsilon,\delta} - w^{\epsilon,\delta}||_s \leq C(\epsilon^2 + \delta^4)t$$

*for all $t \in \left[0, \frac{T}{\epsilon}\right]$ and all $0 < \epsilon \leq \delta \leq \delta_0$.*

**Proof**: Note that the proof will be the same with proof of Theorem 3.1.1. But we need to find the residual term arising when we plug the solution of BBM equation into DD equation. Method is the same but we are going to obtain less terms in $f$ since we do not have higher powers of $\epsilon, \delta$. The residual term $f$ for the DD is

$$f = w_{tt} - w_{xx} + a\delta^2 w_{xxxx} - b\delta^2 w_{xxtt} - \epsilon(w^2)_{xx}, \tag{3.27}$$

where we dropped the indices $\epsilon, \delta$ for simplicity. We rewrite the BBM equation as follows:

$$w_t + w_x = -\epsilon w w_x + \frac{5b-a}{4}\delta^2 w_{txx} + \frac{3b+a}{4}\delta^2 w_{xxx}. \tag{3.28}$$

Now we plug (3.28) in (3.27) and get (3.15)

$$\begin{aligned} f =&(D_t - D_x)(w_t + w_x) + a\delta^2 w_{xxxx} - b\delta^2 w_{xxtt} - \epsilon(w^2)_{xx} \\ =& -\epsilon D_x \left[w(w_x + w_t)\right] + \frac{a-b}{4}\delta^2 \left[3D_x^3 - D_x^2 D_t\right]\left[w_x + w_t\right]. \end{aligned} \tag{3.29}$$

We plug (3.28) in (3.29) and obtain

$$f = -\epsilon D_x \left[w(w_x + w_t)\right] + \frac{a-b}{4}\delta^2 \left[3D_x^3 - D_x^2 D_t\right]\left[w_x + w_t\right] \tag{3.30}$$

$$\begin{aligned} =& -\epsilon D_x \left[w\left(-\epsilon w w_x + \frac{5b-a}{4}\delta^2 w_{txx} + \frac{3b+a}{4}\delta^2 w_{xxx}\right)\right] \\ &+\frac{a-b}{4}\delta^2\left(3D_x^3 - D_x^2 D_t\right)\left(-\epsilon w w_x + \frac{5b-a}{4}\delta^2 w_{txx} + \frac{3b+a}{4}\delta^2 w_{xxx}\right) \\ =&\epsilon^2 (w^2 w_x)_x - \frac{\epsilon\delta^2}{4} D_x\left[(5b-a) w w_{txx} + (3b+a) w w_{xxx}\right] + \frac{b-a}{4}\epsilon\delta^2 D_x\left[3D_x^2 - D_x D_t) w w_x\right] \\ &+ \frac{b-a}{16}\delta^4\left[(D_x^2 D_t - 3D_x^3)\left[(5b-a)w_{xxt} + (3b+a)w_{xxx}\right]\right] \\ =&\epsilon^2 D_x^2 \frac{w^3}{3} + \frac{\epsilon\delta^2}{4} D_x\left[(b-a)(3D_x^2 - D_t D_x) w w_x - (5b-a) w w_{txx} - (3b+a) w w_{xxx}\right] \\ &+ \frac{b-a}{16}\delta^4\left[(D_x^2 D_t - 3D_x^3)\left[(5b-a)w_{xxt} + (3b+a)w_{xxx}\right]\right] \end{aligned} \tag{3.31}$$

and we see that $f = F_x$:

$$\begin{aligned} F =& \epsilon^2 D_x \frac{w^3}{3} + \frac{\epsilon\delta^2}{4} \left[ (b-a)(3D_x^2 - D_t D_x) w w_x - (5b-a) w w_{txx} - (3b+a) w w_{xxx} \right] \\ &+ \frac{b-a}{16} \delta^4 \left[ (D_x D_t - 3D_x^2) \left[ (5b-a) w_{xxt} + (3b+a) w_{xxx} \right] \right]. \end{aligned} \tag{3.32}$$

We observe that $F$ is a combination of terms of the form $D_x^j w$ with $j \leq 5$ or $D_x^l w_t$ with $l \leq 4$. Thus all the terms of $F$ are uniformly bounded according to Constantin and Lannes [6]. We only need to check the term $w_{xxxtt}$: We first rewrite BBM equation as

$$w_t - \frac{5b-a}{4} \delta^2 w_{txx} = -w_x - \epsilon w w_x + \frac{3b+a}{4} \delta^2 w_{xxx}$$

$$w_t = \left( 1 - \frac{5b-a}{4} \delta^2 D_x^2 \right)^{-1} \left( -w_x - \epsilon w w_x + \frac{3b+a}{4} \delta^2 w_{xxx} \right) \tag{3.33}$$

$$\begin{aligned} w_{ttxxx} &= D_x^3 D_t Q \left( -w_x - \epsilon w w_x + \frac{3b+a}{4} \delta^2 w_{xxx} \right) \\ &= D_t Q \left( -w_{xxxx} - \epsilon (w w_x)_{xxx} + \frac{3b+a}{4} \delta^2 D_x^2 w_{xxxx} \right). \end{aligned} \tag{3.34}$$

Thus,

$$||F(t)||_s \leq C(\epsilon^2 + \delta^4) \left( ||w(t)||_{s+5} + ||w_t(t)||_{s+4} \right), \tag{3.35}$$

for $0 \leq t \leq T/\epsilon$. By Theorem 3.1.2, the solutions $w$ of Cauchy problem for (3.3) are bounded with $k = 5$ and this completes the proof. $\square$

# Chapter 4

## Decoupling of the Double Dispersion equation into two uncoupled Camassa-Holm equations

In the previous chapter, we have shown that solutions of

$$w_t^+ + w_x^+ + \epsilon w^+ w_x^+ - \frac{5b-a}{4}\delta^2 w_{txx}^+ - \frac{3b+a}{4}\delta^2 w_{xxx}^+ = \frac{3}{4}(b-a)\epsilon\delta^2(2w_x^+ w_{xx}^+ + w^+ w_{xxx}^+) \tag{4.1}$$

are well-approximated by associated solutions of Cauchy problem for the DD equation

$$u_{tt} - u_{xx} + a\delta^2 u_{xxxx} - b\delta^2 u_{xxtt} - \epsilon(u^2)_{xx} = 0 \tag{4.2}$$

$$u(x,0) = 0, \quad u_t(x,0) = u_1(x) \tag{4.3}$$

with a proper choice of initial data where we placed $w$ by $w^+$ to emphasize the direction of the wave. Now, we are going to concentrate on the solutions of the Cauchy Problem for the DD equation traveling both sides. For this aim, we replace $t$ by $-t$ to obtain CH equations for left going waves:

$$\begin{aligned} w_t^- - w_x^- - \epsilon w^- w_x^- - \frac{5b-a}{4}\delta^2 w_{txx}^- + \frac{3b+a}{4}\delta^2 w_{xxx}^- \\ = -\frac{3}{4}(b-a)\epsilon\delta^2(2w^- w_{xx}^- + w^- w_{xxx}^-). \end{aligned} \tag{4.4}$$

We are going to show that any solution $u$ of Cauchy Problem (4.2)-(4.3) can be approximated by the sum of solutions of CH equations (4.1) and (4.4). In other words, we are going to establish the conditions for the existence of solutions $w^+$ and $w^-$ of Cauchy Problems (4.1) and (4.4) with initial values $w^+(x,0) = w_0^+$ and $w^-(x,0) = w_0^-$ satisfying $u = w^+ + w^-$ up to a small error.

Naturally, we would like to select $w_0^+$ and $w_0^-$ so that

$$u_0(x) = w^+(x,0) + w^-(x,0) = w_0^+ + w_0^- \tag{4.5}$$

$$u_1(x) = w_t^+(x,0) + w_t^-(x,0). \tag{4.6}$$

hold. We see from (4.1)-(4.4) that

$$
\begin{aligned}
w_t^+ + w_t^- =& - w_x^+ + w_x^- - \epsilon(w^+ w_x^+ - w^- w_x^-) \\
&+ \frac{5b-a}{4}\delta^2 \left(w_{txx}^+ + w_{txx}^-\right) + \frac{3b+a}{4}\delta^2 (w_{xxx}^+ - w_{xxx}^-) \\
&+ \frac{3}{4}(b-a)\epsilon\delta^2 \left(2w_x^+ w_{xx}^+ + w^+ w_{xxx}^+ - 2w^- w_{xx}^- + w^- w_{xxx}^-\right)
\end{aligned} \tag{4.7}
$$

and $w_t^+ + w_t^- = -w_x^+ + w_x^- + \mathcal{O}(\epsilon, \delta^2, \epsilon\delta^2)$.

We take $u_1(x) = -w_x^+(x,0) + w_x^-(x,0)$ and assume that $u_1 = (v_0)_x$. Then

$$
u_0 = w_0^+ + w_0^-, \quad v_0 = -w_0^+ + w_0^-. \tag{4.8}
$$

Solving for $w_0^+$ and $w_0^-$ yields

$$
w_0^+ = \frac{1}{2}(u_0 - v_0), \quad w_0^- = \frac{1}{2}(u_0 + v_0). \tag{4.9}
$$

Assume that $r = u - (w^+ + w^-)$. This gives $r(x,0) = 0$ but $r_t(x,0) \neq 0$ because of the approximation. Let us calculate $r_t(x,0)$. Recall that $\mathcal{Q} = \left(1 - \frac{5b-a}{4}\delta^2 D_x^2\right)^{-1}$. Then

$$
\begin{aligned}
w_t^+ + w_t^- =& \mathcal{Q}\left(-w_x^+ + w_x^- - \epsilon(w^+ w_x^+ - w^- w_x^-) + \frac{3b+a}{4}\delta^2 (w_{xxx}^+ + w_{xxx}^-)\right) \\
&+ \mathcal{Q}\frac{3}{4}(b-a)\epsilon\delta^2 \left(2w_x^+ w_{xx}^+ + w^+ w_{xxx}^+ - 2w_x^- w_{xx}^- - w^- w_{xxx}^-\right) \\
=& \mathcal{Q}\left(-w_x^+ + w_x^- - \frac{\epsilon}{2}D_x((w^+)^2 - (w^-)^2) + \frac{3b+a}{4}\delta^2 D_x^3 (w^+ - w^-)\right) \\
&+ \mathcal{Q}\frac{3}{4}(b-a)\epsilon\delta^2 D_x \left(\frac{1}{2}((w_x^+)^2 - (w_x^-)^2)2 + w^+ w_{xx}^+ - w^- w_{xx}^-\right).
\end{aligned} \tag{4.10}
$$

We write (4.10) in terms of $u_0, v_0$ by using (4.8) and obtain:

$$
\begin{aligned}
r_t(x,0) =& u_t(x,0) - (w_t^+(x,0) + w_t^-(x,0)) \\
=& - w_x^+(x,0) + w_x^-(x,0) - (w_t^+(x,0) + w_t^-(x,0)) \\
=& D_x v_0 - \mathcal{Q}\left(D_x v_0 + \frac{\epsilon}{2}D_x u_0 v_0 - \frac{3b+a}{4}\delta^2 D_x^3 v_0\right) \\
&- \frac{3}{4}(b-a)\epsilon\delta^2 \mathcal{Q} D_x \left(\frac{(u_0)_x(-v_0)_x}{2} - \frac{(u_0+v_0)(u_0+v_0)_{xx}}{4}\right) \\
&- \frac{3}{4}(b-a)\epsilon\delta^2 \mathcal{Q} D_x \left(\frac{(u_0-v_0)(u_0-v_0)_{xx}}{4}\right).
\end{aligned}
$$

We write $v_0 = \mathcal{Q}\mathcal{Q}^{-1}v_0$, then $\mathcal{Q}^{-1}v_0 = v_0 - \frac{5b-a}{4}\delta^2 (v_0)_{xx}$ and

$$\begin{aligned}
r_t(x,0) =& D_x\mathcal{Q}\left(v_0 - \frac{5b-a}{4}\delta^2(v_0)_{xx} - v_0 - \frac{\epsilon}{2}u_0v_0 + \frac{3b+a}{4}\delta^2(v_0)_{xx}\right) \\
&+ D_x\mathcal{Q}\frac{3}{8}(b-a)\epsilon\delta^2(u_0)_x(v_0)_x + \mathcal{Q}\frac{3}{8}(b-a)\epsilon\delta^2\left(-u_0(v_0)_{xx} - v_0(u_0)_{xx}\right) \\
=& D_x\mathcal{Q}\left(-\frac{b-a}{2}\delta^2(v_0)_{xx} - \frac{\epsilon}{2}u_0v_0\right) \\
&+ D_x\mathcal{Q}\left(\frac{3}{8}(b-a)\epsilon\delta^2\left((u_0)_x(v_0)_x + u_0(v_0)_{xx} + v_0(u_0)_{xx}\right)\right) \\
=& q_x(x).
\end{aligned} \tag{4.11}$$

### 4.1 Estimate for the residual term corresponding to two uncoupled Camassa-Holm approximation

**Lemma 4.1.1** *Let $w_0^+, w_0^- \in H^{s+6}(\mathbb{R}), s > 1/2$. Then there is some $C > 0$ so that the family of solutions $(w^+)^{\epsilon,\delta}$, $(w^-)^{\epsilon,\delta}$ to the CH equations (4.1),(4.4) with initial values $w^+(x,0) = w_0^+(x)$, and $w^-(x,0) = w_0^-(x)$ satisfy*

$$w^+_{tt} - w^+_{xx} + a\delta^2 w^+_{xxxx} - b\delta^2 w^+_{xxtt} - \epsilon(w^+)^2_{xx} = F^+_x$$

$$w^-_{tt} - w^-_{xx} + a\delta^2 w^-_{xxxx} - b\delta^2 w^-_{xxtt} - \epsilon(w^-)^2_{xx} = F^-_x.$$

*Moreover*

$$||\tilde{F}(t)||_s \le C(\epsilon + \delta^4)$$

*for all $0 < \epsilon \le \delta \le 1$ and $t \in \left[0, \frac{T}{\epsilon}\right]$ where $\tilde{F} = F^+ + F^- - 2\epsilon(w^+w^-)_x$.*

**Proof**: We know that

$$w^+_{tt} - w^+_{xx} + a\delta^2 w^+_{xxxx} - b\delta^2 w^+_{xxtt} - \epsilon(w^+)^2_{xx} = F^+_x$$

with $F^+$ as in (3.18).

Replace $t$ by $-t$, then we have

$$w^-_{tt} - w^-_{xx} + a\delta^2 w^-_{xxxx} - b\delta^2 w^-_{xxtt} - \epsilon(w^-)^2_{xx} = F^-_x,$$

where

$$
\begin{aligned}
F^- =&\epsilon^2 D_x\left(\frac{w^3}{3}\right)+\frac{b-a}{16}\delta^4\left[(-D_xD_t-3D_x^2)\left[-(5b-a)w_{xxt}+(3b+a)w_{xxx}\right]\right]\\
&-\frac{\epsilon^2\delta^2}{8}\left[3(b-a)w\left(w_x^2+ww_{xx}\right)_x\right]\\
&-\frac{\epsilon^2\delta^2}{8}\left[(-3b-a)w(w^2)_{xxx}+(2b-2a)w_{xx}(w^2)_x+(b-a)w_x(w^2)_{xx}\right]\\
&+\frac{\epsilon\delta^4}{32}\left[3(b-a)^2\left(-D_x^2D_t-3D_x^3\right)\left(w_x^2+2ww_{xx}\right)\right]\\
&+\frac{\epsilon\delta^4}{32}\left[2(-3b-a)wD_x^2+(2b-2a)w_{xx}+(b-a)w_xD_x\right]\left[-(5b-a)w_{xxt}\right]\\
&+\frac{\epsilon\delta^4}{32}\left[2(-3b-a)wD_x^2+(2b-2a)w_{xx}+(b-a)w_xD_x\right]\left[(3b+a)w_{xxx}\right]\\
&+\frac{\epsilon^2\delta^4}{16}\left[(b-a)(-9b-3a)wD_x^2+6(b-a)^2w_{xx}\right]\left[D_x\left(2w_xw_{xx}+ww_{xxx}\right)\right]\\
&+\frac{\epsilon^2\delta^4}{16}\left[3(b-a)^2w_xD_x\right]\left[D_x\left(2w_xw_{xx}+ww_{xxx}\right)\right]
\end{aligned}
\tag{4.12}
$$

Then by Lemma 3.5.5 there exist some $C_1, C_2 > 0$ so that

$$
\begin{aligned}
||\tilde{F}||_s &= ||F^+ + F^- - 2(w^+w^-)_x||_s\\
&\le ||F^+(t)||_s + ||F^-(t)||_s + 2\epsilon C_0||w^+||_{s+1}||w^-||_{s+1}\\
&\le C_1(\epsilon^2+\delta^4)+C_2(\epsilon^2+\delta^4)+C_3(\epsilon+\delta^4)\\
&\le C(\epsilon+\delta^4)
\end{aligned}
$$

for all $0 < \epsilon \le \delta \le 1$ and $t \in \left[0, \frac{T}{\epsilon}\right]$. □

## 4.2 Convergence result for the Double Dispersion equation with uncoupled Camassa-Holm equations

Plugging $u = r + w^+ + w^-$ into equation (3.1) gives

$$
\begin{aligned}
0 =&u_{tt}-u_{xx}+a\delta^2u_{xxxx}-b\delta^2u_{xxtt}-\epsilon(u^2)_{xx}\\
=&(r+w^++w^-)_{tt}-(r+w^++w^-)_{xx}+a\delta^2(r+w^++w^-)_{xxxx}\\
&-b\delta^2(r+w^++w^-)_{xxtt}-\epsilon(r+w^++w^-)^2_{xx}\\
=&r_{tt}-r_{xx}+a\delta^2r_{xxxx}-b\delta^2r_{xxtt}-\epsilon(r^2+2(w^++w^-)r)_{xx}\\
&+w^+_{tt}-w^+_{xx}+a\delta^2w^+_{xxxx}-b\delta^2w^+_{xxtt}-\epsilon(w^+)^2_{xx}\\
&+w^-_{tt}-w^-_{xx}+a\delta^2w^-_{xxxx}-b\delta^2w^-_{xxtt}-\epsilon(w^-)^2_{xx}-2\epsilon(w^+w^-)_{xx}.
\end{aligned}
$$

It follows that the function $r$ is the solution of the Cauchy Problem

$$r_{tt} - r_{xx} + a\delta^2 r_{xxxx} - b\delta^2 r_{xxtt} - \epsilon(r^2 + 2(w^+ + w^-)r)_{xx} = -\tilde{F}_x \tag{4.13}$$

$$r(x,0) = 0, \quad r_t(x,0) = q_x(x), \tag{4.14}$$

where $\tilde{F} = F^+ + F^- - 2\epsilon(w^+w^-)_x$ and $q_x(x)$ in (4.11).

**Theorem 4.2.2** *Let $u_0 \in H^{s+6}(\mathbb{R})$, and $v_0 \in H^{s+7}(\mathbb{R})$, $s > 3/2$. Assume $u^{\epsilon,\delta}$ be the solution of DD equation (4.2)-(4.3). Let*

$$w_0^+ = \frac{1}{2}(u_0 - v_0), \quad w_0^- = \frac{1}{2}(u_0 + v_0).$$

*Then for any given $t_0$ there exists $\delta_0 \leq 1$ so that the solutions $(w^+)^{\epsilon,\delta}$, $(w^-)^{\epsilon,\delta}$ of the CH equations (4.1)-(4.4) with initial values $w^+(x,0) = w_0^+(x)$, and $w^-(x,0) = w_0^-(x)$ satisfy*

$$||u^{\epsilon,\delta} - (w^+)^{\epsilon,\delta} - (w^-)^{\epsilon,\delta}||_s \leq ((\epsilon + \delta^2) + (\epsilon + \delta^4)t)$$

*for all $t \in [0, t_0]$ and all $0 < \epsilon \leq \delta \leq \delta_0$.*

**Proof**: Note that $u_0 \in H^{s+6}(\mathbb{R})$, and $u_1 \in H^{s+7}(\mathbb{R})$. Let $r = u - w^+ - w^-$. Then $w^+, w^-$ and hence $r, \tilde{w}$ are in $H^{s+6}(\mathbb{R})$ since $||\tilde{w}||_{s+6} \leq (||w^+||_{s+6} + ||w^-||_{s+6})$. Moreover, $r(x,0) = 0$ and $r_t(x,0) = (q(x))_x$ for $q(x)$ described in (4.11). Then $r$ satisfies the Dispersive Equation (4.13)-(4.14). We can consider the energy as in (3.10) for $w = w^+ + w^-$. We know from Lemma 4.1.1 that

$$||\tilde{F}(t)||_s \leq C(\epsilon + \delta^4)$$

for all $0 < \epsilon \leq \delta \leq 1$ and $t \in \left[0, \frac{T}{\epsilon}\right]$. Using the same argument in the proof of Theorem 3.1.1, we use the same set (3.26). Note that $\sup\limits_{0\leq t\leq \bar{t}}(1 + 2||w(t)||_s) = \sup\limits_{0\leq t\leq \bar{t}}\left(1 + 2||(w^+ + w^-)(t)||_{H^s(\mathbb{R})}\right) \leq 1 + 2(M_1 + M_2)$ where $M_1 = \sup ||w^+(t)||_s, M_2 = \sup ||w^-(t)||_s$. Assume that $\epsilon \leq \frac{1}{1+2(M_1+M_2)} = \epsilon_0$, then by Lemma 3.3.4 the modified energy in (3.10) satisfies

$$E_s(t) \leq C\left(E_s(0) + t\sup_{0\leq t\leq \bar{t}}||\tilde{F}(t)||_s\right)$$

for $0 \leq t \leq \bar{t} \leq T/\epsilon$. Let us find an estimate for $E_s(0)$. We know that

$$\begin{aligned} E_s^2(0) =& \frac{1}{2}\left(||\rho_t(0)||_s^2 + ||r(0)||_s^2 + a\delta^2||r_x(0)||_s^2 + b\delta^2||r_t(0)||_s^2\right) \\ &+ \frac{\epsilon}{2}\langle\Lambda^s(r^2 + 2wr)(0), \Lambda^s r(0)\rangle \end{aligned}$$

for some $\rho$ with $r = \rho_x$. Note that $r_x(x,0) = 0$. Then

$$E_s^2(0) = \frac{1}{2}\left(||\rho_t(0)||_s^2 + b\delta^2||r_t(0)||_s^2\right).$$

We know from (4.11) that

$$||r_t(0)||_s \leq ||\mathcal{Q}||_s||u_0||_{H^{s+3}}||v_0||_{H^{s+3}}(\epsilon + \delta^2)$$

and

$$||\rho_t(0)||_s \leq C||\rho_{xt}(0)||_{H^{s-1}} = ||r_t(0)||_{H^{s-1}}||\mathcal{Q}||_s||u_0||_{H^{s+2}}||v_0||_{H^{s+2}}(\epsilon + \delta^2).$$

Since $\mathcal{Q}$ is bounded operator on $H^s$, it follows that

$$E_s(0) \leq C_1(\epsilon + \delta^2) + \delta C_2(\epsilon + \delta^2) \leq C(\epsilon + \delta^2 + \epsilon\delta + \delta^3) \leq C(\epsilon + \delta^2). \tag{4.15}$$

Then energy satisfies

$$E_s(t) \leq C\left((\epsilon + \delta^2) + t(\epsilon + \delta^4)\right) \leq C\left((\epsilon + \delta^2) + T_0^{\epsilon,\delta}(\epsilon + \delta^4)\right)$$

by (4.15) and Lemma 4.1.1. Choose $\delta_0$ so that $0 < \epsilon \leq \delta \leq \delta_0 \leq \epsilon_0 \leq 1$, then

$$||u^{\epsilon,\delta} - w^{\epsilon,\delta}||_s = ||r(t)||_s \leq CE_s(t) \leq C\left((\delta_0 + \delta_0^2) + T_0^{\epsilon,\delta}(\delta_0 + \delta_0^4)\right) = \epsilon' << 1$$

for $t \leq T/\epsilon$.

## 4.3 Convergence result for the Double Dispersion equation with uncoupled Benjamin-Bona-Mahony equations

We consider right-going and left-going BBM equations

$$w_t^+ + w_x^+ + \epsilon w^+ w_x^+ - \frac{5b-a}{4}\delta^2 w_{txx}^+ - \frac{3b+a}{4}\delta^2 w_{xxx}^+ = 0, \tag{4.16}$$

and

$$w_t^- - w_x^- - \epsilon w^- w_x^- + \frac{5b-a}{4}\delta^2 w_{txx}^- + \frac{3b+a}{4}\delta^2 w_{xxx}^- = 0. \tag{4.17}$$

**Theorem 4.3.3** *Let $u_0 \in H^{s+6}(\mathbb{R})$, and $u_1 \in H^{s+7}(\mathbb{R})$, $s > 3/2$. Assume $u^{\epsilon,\delta}$ be the solution of DD equation (4.2) with initial data*

$$u(x,0) = u_0(x), \quad u_t(x,0) = (v_0(x))_x.$$

*Let*

$$w_0^+ = \frac{1}{2}(u_0 - v_0), \quad w_0^- = \frac{1}{2}(u_0 + v_0).$$

*Then for any given $t_0$ there exists $\delta_0 \leq 1$ so that the solutions $(w^+)^{\epsilon,\delta}$, $(w^-)^{\epsilon,\delta}$ of the BBM equations (4.16)-(4.17) with initial values $w^+(x,0) = w_0^+(x)$, and $w^-(x,0) = w_0^-(x)$ satisfy*

$$||u^{\epsilon,\delta} - (w^+)^{\epsilon,\delta} - (w^-)^{\epsilon,\delta}||_s \leq ((\epsilon + \delta^2) + (\epsilon + \delta^4)t)$$

*for all $t \in [0, t_0]$ and all $0 < \epsilon \leq \delta \leq \delta_0$.*

**Proof**: We will follow the same argument as in the proof of Theorem 4.2.2. We just take the residues $F^+, F^-$ in $\tilde{F}$ as

$$\begin{aligned}
F^+ =&\epsilon^2 D_x \frac{{w^+}^3}{3} + \frac{\epsilon\delta^2}{4}\left[(b-a)(3D_x^2 - D_t D_x)w^+ {w^+}_x\right] \\
&+ \frac{\epsilon\delta^2}{4}\left[-(5b-a)w^+ {w^+}_{txx} - (3b+a)w^+ {w^+}_{xxx}\right] \\
&+ \frac{b-a}{16}\delta^4\left[(D_x D_t - 3D_x^2)\left[(5b-a){w^+}_{xxt} + (3b+a){w^+}_{xxx}\right]\right] \\
F^- =&\epsilon^2 D_x \frac{{w^-}^3}{3} + \frac{\epsilon\delta^2}{4}\left[(b-a)(3D_x^2 + D_t D_x)w^- {w^-}_x\right] \\
&+ \frac{\epsilon\delta^2}{4}\left[(5b-a)w^- {w^-}_{txx} - (3b+a)w^- {w^-}_{xxx}\right] \\
&+ \frac{b-a}{16}\delta^4\left[(-D_x D_t - 3D_x^2)\left[-(5b-a){w^-}_{xxt} + (3b+a){w^-}_{xxx}\right]\right]
\end{aligned} \tag{4.18}$$

where we find $F^-$ by just replacing $t$ by $-t$ in (3.32). $\square$

# Chapter 5

# The Hunter-Saxton equation as high frequency limit of the Camassa-Holm equation

The Hunter-Saxton (HS) equation is given by

$$u_{txx} + 2u_x u_{xx} + u u_{xxx} = 0 \tag{5.1}$$

where $u(x,t)$ is a real-valued function and subscripts $x$ and $t$ denote partial differentiations. The equation (5.1) arises as high frequency (or short wave) limit of the CH equation in the case of water waves [17] and elastic waves [7]. This asymptotic relation between the CH equation and the HS equation also provided by Matsuno in [22].

In Section 5.1, we present the derivation of the HS equation from the CH equation given in [22] for the convenience of the reader. Then, in Section 5.2, we first state the main result, Theorem 5.2.1, for convergence of the solutions of the HS and the CH equations. To this end, we state local well-posedness results for the CH equation [23] and the HS equation [29], estimate the residual term, and finally show that the error term in the approximation remains small and complete the proof.

## 5.1 Derivation of the Hunter-Saxton equation from the Camassa-Holm equation

Consider the CH equation

$$V_T + 3VV_X - V_{TXX} = 2V_X V_{XX} + VV_{XXX}. \tag{5.2}$$

To obtain the high frequency limit of the CH equation, we introduce the short wave scaling [22]

$$V(X,T) = \gamma^2 W(\xi,\tau) = \gamma^2 W(\frac{X}{\gamma}, \gamma T)$$

where $\gamma$ is a positive small parameter. Plug this into the CH equation to obtain

$$\gamma^2 W_\tau + 3\gamma^2 W W_\xi - W_{\tau\xi\xi} = 2W_\xi W_{\xi\xi} + WW_{\xi\xi\xi}.$$

Thus for $W = W_0 + \gamma W_1 + ...$ , we get at $\mathcal{O}(1)$

$$W_{\tau\xi\xi} + 2W_\xi W_{\xi\xi} + WW_{\xi\xi\xi} = 0, \tag{5.3}$$

where we replace $W_0$ by $W$ afterwards. Thus, high frequency limit of the CH equation satisfies the HS equation asymptotically. By this scaling, equation (5.1) may be considered as short wave limit of the CH equation. This equation, called as the HS equation, was already obtained by Hunter and Saxton in [16] as an asymptotic equation for weakly nonlinear waves.

Since we will compare the solutions of the CH and HS equations rigorously, we carry both equations into the same coordinate system. The solution $V(X, T)$ for the CH equation (5.2)

$$v^{\epsilon,\delta}(x,t) = \frac{1}{\epsilon}V(X,T) = \frac{1}{\epsilon}V(\frac{x}{\delta},\frac{t}{\delta}), \tag{5.4}$$

takes the form

$$v_t + 3\epsilon v v_x - \delta^2 v_{txx} = \epsilon\delta^2(2v_x v_{xx} + v v_{xxx}). \tag{5.5}$$

Similarly, with the transformation

$$W(\xi,\tau) = W(\frac{x}{\delta\gamma},\frac{\gamma}{\delta}t) = w^{\gamma,\delta}(x,t),$$

the HS equation (5.3) takes the form

$$w_{txx} + \gamma^2(2w_x w_{xx} + w w_{xxx}) = 0. \tag{5.6}$$

Note that we drop the superscripts in (5.5) and (5.6) afterwards.

## 5.2 Convergence result in periodic setting

In the present section, we compare periodic solutions of the CH and HS equations, rigorously, and provide an estimate for the error term in the HS approximation.
We consider the Cauchy problem for the CH equation

$$\begin{aligned} &v_t + 3\epsilon v v_x - \delta^2 v_{txx} = \epsilon\delta^2(2v_x v_{xx} + v v_{xxx}) && t > 0,\ x \in \mathbb{R} \\ &v(x,0) = v_0(x) && x \in \mathbb{R} \\ &v(x,t) = v(x + 2\pi, t) && t \geq 0,\ x \in \mathbb{R} \end{aligned} \tag{5.7}$$

and the Cauchy problem for the HS equation

$$\begin{aligned} &w_{txx} + \gamma^2(2w_x w_{xx} + w w_{xxx}) = 0 && t > 0,\ x \in \mathbb{R} \\ &w(x,0) = w_0(x) && x \in \mathbb{R} \\ &w(x,t) = w(x + 2\pi, t) && t \geq 0,\ x \in \mathbb{R} \end{aligned} \tag{5.8}$$

where $v_0^{\epsilon,\delta}(x) = \frac{\gamma^2}{\epsilon} w_0^{\gamma,\delta}(x)$ and dependence of the parameter $\gamma$ on $\epsilon$ and $\delta$ are to be determined later.

Given the solution $w^{\gamma,\delta}$ of the Cauchy problem for the HS equation (5.8), we prove that it is possible to find a solution $v^{\epsilon,\delta}$ of the Cauchy problem for the CH (5.7) such that $||v^{\epsilon,\delta} - \frac{\gamma^2}{\epsilon} w^{\gamma,\delta}||$ is small over a long time in suitable function spaces. The main result of this section is the following.

**Theorem 5.2.1** *Let $s > 3/2$. Assume $w_0 \in H^{s+2}(\mathbb{T})$. Suppose $w^{\gamma,\delta}$ is the solution of the Cauchy problem for the periodic HS equation* (5.8)*. Then there exists $T > 0$ and $\delta_1$ so that the solution $v^{\epsilon,\delta}$ of the Cauchy problem for the periodic CH equation* (5.7) *satisfies*

$$||v^{\epsilon,\delta}(t) - \frac{\gamma^2}{\epsilon} w^{\gamma,\delta}(t)||_s \leq C\frac{\gamma^4}{\epsilon} t$$

*for all $t \in [0, \frac{T}{\epsilon}]$ and sufficiently small positive parameters $\epsilon$, $\delta$ and $\gamma$.*

As in the previous chapters, we will follow the same methodology for the convergence. To this end, we first recall well-posedness results for both Cauchy problems in parameters-free forms in Section 5.2.1. In the same section, we rewrite the theorems for the Cauchy problems (5.7) and (5.8). In Section 5.2.2, we find an estimate for the residual term in suitable Sobolev Spaces, and finally we find an estimate for the energy of the equation satisfied by the error term, and complete the proof.

### 5.2.1 Well-posedness results for the periodic Hunter Saxton and Camassa-Holm equations

The initial value problem for the Hunter-Saxton equation

$$(u_t + uu_x)_x = \frac{u_x^2}{2} \qquad t > 0,\, x \in \mathbb{R} \tag{5.9}$$

$$u(x,0) = u_0 \qquad x \in \mathbb{R} \tag{5.10}$$

was studied over the real line by Hunter and Saxton in [16]. Using the method of characteristics, a formula for the solution is provided. However, it is not possible to work with $H^s(\mathbb{R})$ spaces, as the formula involves a term that is not an $L^2(\mathbb{R})$ function. Thus, most of the problems for the HS equation are considered in periodic setting.
The Cauchy problem for the periodic Hunter-Saxton equation was first studied by Yin in 2004 [29]:

$$\begin{aligned} & u_t + uu_x = D_x^{-1}\left(\frac{u_x^2}{2} + d(t)\right) + h(t), && t > 0,\, x \in \mathbb{R} \\ & u(x,0) = u_0 && x \in \mathbb{R} \\ & u(x,t) = u(x+2\pi, t) && t \geq 0,\, x \in \mathbb{R} \end{aligned} \tag{5.11}$$

where $d$ and $h$ are continuous functions. Using the Kato semigroup method, it is shown that strong solutions to the periodic HS equation exist locally in time in $H^s$ for $s > 3/2$. Following is the well-posedness result for the Cauchy problem of the periodic HS equation (5.11) given in [29]:

**Theorem 5.2.2 ( [29])** *Given $h$, a continuous function, and $u_0 \in H^s(\mathbb{T})$, $s > 3/2$ . Then there exists a maximal $T = T(d, h(t), u_0) > 0$, and a unique solution $u$ to (5.11), such that $u \in C([0,T), H^s(\mathbb{T})) \cap C^1([0,T), H^{s-1}(\mathbb{T}))$. Moreover, the solution depends continuously on the initial data.*

If we take $t = \frac{1}{\gamma^2}\tau$, then (5.8) reduces to

$$\bar{u}_{\tau xx} + 2\bar{u}_x\bar{u}_{xx} + \bar{u}\bar{u}_{xxx} = 0 \tag{5.12}$$

with a relation

$$w^{\gamma,\delta}(x,t) = \bar{u}(x,\tau) = \bar{u}(x,\gamma^2 t)$$

between the solution $w^{\gamma,\delta}(x,t)$ of (5.8) and the solution $\bar{u}(x,\tau)$ of (5.12).

According to Theorem 2.12 in [29] and the discussion in [15], solutions to the Cauchy problem for the HS equation (5.11) are also solutions to the twice differentiated form of the HS equation (5.12). Thus, solution $\bar{u}$ exists and lives in $C([0,T), H^{s+2}(\mathbb{T}))$ for $0 < \tau \le T$ and we can rephrase Theorem 5.2.2 for (5.8) as follows:

**Theorem 5.2.3** *Let $w_0 \in H^{s+2}(\mathbb{T})$, $s > 3/2$ . Then there exists $T = T(w_0) > 0$ such that the Cauchy problem (5.6) has a family of solutions $w^{\gamma,\delta} \in C([0, \frac{T}{\gamma^2}), H^s(\mathbb{T})) \cap C^1([0, \frac{T}{\gamma^2}), H^{s-1}(\mathbb{T}))$. Moreover, the solution $w^{\gamma,\delta}$ is uniformly bounded in $C([0, \frac{T}{\gamma^2}), H^s(\mathbb{T}))$.*

Thus we have not only long time existence-uniquness of the solutions (5.8) but also uniform bounds over a long time interval.

As we are working on periodic problems, we need the well-posedness result for the periodic CH equation (5.7) as well. For this reason, we will recall well-posedness result for the Cauchy problem for periodic Camassa-Holm equation

$$\begin{aligned}
&v_t + 3\epsilon v v_x - \delta^2 v_{txx} - \epsilon\delta^2(2v_x v_{xx} + v v_{xxx}) = 0, && x \in \mathbb{R},\ t > 0 \\
&v(x,0) = v_0(x) && x \in \mathbb{R} \\
&v(x,t) = v(x+2\pi, t) && t > 0
\end{aligned} \tag{5.13}$$

provided in [23].

**Theorem 5.2.4 ( [23])** *If $s > 3/2$, then given any $v_0 \in H^s(\mathbb{T})$ there exists a $T^{\epsilon,\delta} > 0$ and a unique solution $v^{\epsilon,\delta}$ to the Cauchy problem (5.13) such that $v^{\epsilon,\delta} \in C([0, T^{\epsilon,\delta}), H^s(\mathbb{T})) \cap C^1([0, T^{\epsilon,\delta}), H^{s-1}(\mathbb{T}))$ and which depends continuously on the initial data $v_0$.*

**Remark 5.2.1** *The proof in (5.13) is actually parameter-free form of (5.7). However, coefficients do not affect the proof of well-posedness. We note that the existence time $T^{\epsilon,\delta}$ may be different for each value of the parameter. Moreover Theorem 5.2.4 does not say anything about uniform bounds for $v^{\epsilon,\delta}$. However, this is not crucial since uniform bounds for the solutions are necessary only for the model equation, namely for equation (5.8).*

### 5.2.2 Estimate for the residual term

Assume $v^{\epsilon,\delta}$ be the solution of (5.7) and $w^{\gamma,\delta}$ be the solution of (5.8). Let $r = v^{\epsilon,\delta} - \frac{\gamma^2}{\epsilon} w^{\gamma,\delta}$. Then we have

$$\begin{aligned}(r + \frac{\gamma^2}{\epsilon}w)_t + &3\epsilon(r + \frac{\gamma^2}{\epsilon}w)(r + \frac{\gamma^2}{\epsilon}w)_x - \delta^2(r + \frac{\gamma^2}{\epsilon}w)_{txx} \\ &- \epsilon\delta^2\left(2(r + \frac{\gamma^2}{\epsilon}w)_x(r + \frac{\gamma^2}{\epsilon}w)_{xx} + (r + \frac{\gamma^2}{\epsilon}w)(r + \frac{\gamma^2}{\epsilon}w)_{xxx}\right) = 0.\end{aligned}$$

Straightforward calculations imply that the error term $r$ satisfies the differential equation

$$\begin{aligned}r_t + 3\epsilon r r_x - \delta^2 r_{txx} - &\epsilon\delta^2(2r_x r_{xx} + r r_{xxx}) + 3\gamma^2(rw)_x \\ &- \delta^2\gamma^2(2r_x w_{xx} + 2w_x r_{xx} + r w_{xxx} + w r_{xxx}) = -f,\end{aligned}$$

where

$$\begin{aligned}f &= \frac{\gamma^2}{\epsilon}w_t + 3\frac{\gamma^4}{\epsilon}ww_x - \delta^2\frac{\gamma^2}{\epsilon}w_{txx} - \frac{\gamma^4}{\epsilon}\delta^2(2w_x w_{xx} + ww_{xxx}) \\ &= \frac{\gamma^2}{\epsilon}w_t + 3\frac{\gamma^4}{\epsilon}ww_x\end{aligned}$$

is the residual term. Note that the last two terms in the above expression disappear as $w$ is the solution of (5.8):

$$w_{txx} + \gamma^2(2w_x w_{xxx} + ww_{xxx}) = 0.$$

The following lemma gives an estimate for $H^s$ norm of the residual term.

**Lemma 5.2.5** *Let $w_0 \in H^{s+2}(\mathbb{R})$, $s > 3/2$. Then there is some $C > 0$ so that the family of solutions $w^{\gamma,\delta}$ to the periodic HS equation* (5.6) *with initial value $w^{\gamma,\delta}(x,0) = w_0(x)$, satisfy*

$$f = \frac{\gamma^2}{\epsilon}w_t + 3\frac{\gamma^4}{\epsilon}ww_x$$

*with*

$$||f(t)||_{H^s(\mathbb{T})} \leq C\frac{\gamma^4}{\epsilon}$$

*for $t \in \left[0, \frac{T}{\gamma^2}\right]$ and sufficiently small positive parameters $\epsilon$, $\delta$ and $\gamma$.*

**Proof**: We need an expression for $w_t$. We observe that

$$(w_{tx})_x = -\gamma^2 \left( \frac{w_x^2}{2} + w w_{xx} \right)_x .$$

We integrate with respect to $x$

$$\begin{aligned} w_{tx} &= -\gamma^2 \left( \frac{w_x^2}{2} + w w_{xx} + d(t) \right) \\ &= -\gamma^2 \left( \frac{w_x^2}{2} + (w w_x)_x - w_x^2 + d(t) \right). \end{aligned}$$

where use the fact that $w w_{xx} = (w w_x)_x - w_x^2$ and $d$ is to be determined later. Thus,

$$(w_t + \gamma^2 w w_x)_x = \gamma^2 \left( \frac{w_x^2}{2} - d(t) \right).$$

We integrate once more

$$w_t + \gamma^2 w w_x = \gamma^2 \left( D_x^{-1} \left( \frac{w_x^2}{2} - d(t) \right) + h(t) \right)$$

where $h(t)$ is a continuous function. Then residual term $f$ can be expressed as

$$\begin{aligned} f &= \frac{\gamma^2}{\epsilon} \left( -\gamma^2 w w_x + \gamma^2 \left( D_x^{-1} \left( \frac{w_x^2}{2} - d(t) \right) + h(t) \right) \right) + 3\frac{\gamma^4}{\epsilon} w w_x \\ &= \frac{\gamma^4}{\epsilon} D_x^{-1} \left( \frac{w_x^2}{2} - d(t) \right) + \frac{\gamma^4}{\epsilon} h(t) + 2\frac{\gamma^4}{\epsilon} w w_x. \end{aligned} \tag{5.14}$$

In order for $D_x^{-1}\left(\frac{w_x^2}{2} - d(t)\right)$ to be defined $\frac{w_x^2}{2} - d(t)$ must have mean zero. To do so, we choose $d(t) = \frac{1}{4\pi}\int_0^{2\pi} w_x^2\, dx$. As shown in Lemma 3.2 of Yin [29], when $w_0 \in H^s$, $s \geq 3$, then $w(x,t)$ satisfies

$$\int_0^{2\pi} w_x^2\, dx = \int_0^{2\pi} (w_0)_x^2\, dx$$

and $d(t) = \frac{1}{4\pi}\int_0^{2\pi} (w_0)_x^2\, dx = d$ becomes a constant. Thus $\frac{w_x^2}{2} - d$ has mean zero:

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{w_x^2}{2} - d \right) dx &= \frac{1}{2\pi} \int_0^{2\pi} \frac{w_x^2}{2}\, dx - d \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{w_x^2}{2}\, dx - \frac{1}{4\pi} \int_0^{2\pi} w_0^2\, dx = 0. \end{aligned}$$

Now, we are ready to find an estimate for the residual term:

$$\begin{aligned} ||f(t)||_s &\leq \frac{\gamma^4}{\epsilon} ||D_x^{-1} \left( \frac{w_x^2}{2} - d \right) ||_s + \frac{\gamma^4}{\epsilon} ||h(t)||_s + 2\frac{\gamma^4}{\epsilon} ||w w_x||_s \\ &\leq C\frac{\gamma^4}{\epsilon} \left( ||\frac{w_x^2}{2} - d||_{s-1} + ||w||_s ||w||_{s+1} \right) \\ &\leq C\frac{\gamma^4}{\epsilon} \left( ||w||_s^2 + ||w||_s ||w||_{s+1} \right) \\ &\leq C\frac{\gamma^4}{\epsilon}, \end{aligned}$$

where we use the fact that $H^s$ norms of $w$ are uniformly bounded by Theorem 5.2.3.
□

### 5.2.3 Energy Estimate

The error term $r$ satisfies

$$r_t + 3\epsilon r r_x - \delta^2 r_{txx} - \epsilon\delta^2(2r_x r_{xx} + r r_{xxx}) + 3\gamma^2(rw)_x$$
$$- \delta^2\gamma^2(2r_x w_{xx} + 2w_x r_{xx} + r w_{xxx} + w r_{xxx}) = -f \tag{5.15}$$
$$r(x,0) = 0 \tag{5.16}$$

Define the energy as in [6]:

$$E_s^2(t) = \frac{1}{2}\left(||r(t)||_s^2 + \delta^2||r_x(t)||_s^2\right). \tag{5.17}$$

Then we have the following estimate for the energy:

**Lemma 5.2.6** *Assume $s > 3/2$ and let $w^{\gamma,\delta}$ and $v^{\epsilon,\delta}$ be solutions of (5.8) and (5.7), respectively and let $r = v^{\epsilon,\delta} - \frac{\gamma^2}{\epsilon}w^{\gamma,\delta}$. Assume that $||r(t)||_{H^s} < 1$ for $t \leq \bar{T}$. Then there exists some $C$ such that the energy (3.10) for the Cauchy Problem (5.15)-(5.16) satisfies*

$$E_s(t) \leq C\left(E_s(0) + t\sup_{0\leq t\leq\bar{T}}||f(t)||_s\right)$$

*for $0 \leq t \leq \bar{T} \leq \min\{\frac{T}{\epsilon}, \frac{T}{\gamma^2}\}$ and sufficiently small positive parameters $\epsilon$, $\delta$ and $\gamma$.*

**Proof**: Take derivative of the energy with respect to $t$

$$\begin{aligned}
\frac{d}{dt}E_s^2(t) =& \langle\Lambda^s r_t, \Lambda^s r\rangle + \delta^2\langle\Lambda^s r_x, \Lambda^s r_{xt}\rangle \\
=& -3\epsilon\langle\Lambda^s(r r_x), \Lambda^s r\rangle + \epsilon\delta^2\langle\Lambda^s(2r_x r_{xx} + r r_{xxx}), \Lambda^s r\rangle \\
& - 3\gamma^2\langle\Lambda^s(rw)_x, \Lambda^s r\rangle + \gamma^2\delta^2\langle\Lambda^s(2r_x w_{xx} + 2w_x r_{xx}, \Lambda^s r\rangle \\
& + \gamma^2\delta^2\langle\Lambda^s(r w_{xxx} + w r_{xxx}), \Lambda^s r\rangle - \langle\Lambda^s f, \Lambda^s r\rangle.
\end{aligned} \tag{5.18}$$

Now, we are going to find an estimate for $H^s$ norm of each term in (5.18). Note that whenever we have $||r_x||_s$ , we need $\delta$ as a multiplier. This is as same as to say we can use $||r_x||_s^k$, if we have $\delta^k$ as a coefficient.

- We start with the last term of equation (5.18) and we obtained that

$$|\langle\Lambda^s f, \Lambda^s r\rangle| \leq ||f||_s||r||_s. \tag{5.19}$$

- Now we estimate the first term of equation (5.18).

We use first property of Lemma 1.3.3 for the second term with $h = r$ and $u = r$.

$$\begin{aligned}\langle \Lambda^s r r_x, \Lambda^s r\rangle &= \langle [\Lambda^s, r] r_x, \Lambda^s r\rangle + \langle r\Lambda^s r_x, \Lambda^s r\rangle \\ &= \langle [\Lambda^s, r] r_x, \Lambda^s r\rangle - \frac{1}{2}\langle r_x \Lambda^s r, \Lambda^s r\rangle.\end{aligned}$$

Thus the condition $s - 1 > 1/2$ leads:

$$\begin{aligned}|\langle \Lambda^s(r r_x), \Lambda^s r\rangle| &\le C\left(||r||_s ||r_x||_{s-1}||r||_s + ||r_x||_\infty ||r||_s^2\right) \\ &\le C\left(||r||_s^3 + ||r_x||_{s-1}||r||_s^2\right) \\ &\le C||r||_s^3. \qquad (5.20)\end{aligned}$$

- Now we estimate third term of equation (5.18). We see that

$$\langle \Lambda^s (rw)_x, \Lambda^s r\rangle = \langle \Lambda^s r w_x, \Lambda^s r\rangle + \langle \Lambda^s r_x w, \Lambda^s r\rangle.$$

Note that

$$\langle \Lambda^s r w_x, \Lambda^s r\rangle = \langle [\Lambda^s, w_x] r, \Lambda^s r\rangle + \langle w_x \Lambda^s r, \Lambda^s r\rangle.$$

On the other hand

$$\langle \Lambda^s w r_x, \Lambda^s r\rangle = -\frac{1}{2}\langle w_x \Lambda^s r, \Lambda^s r\rangle$$

with $h = w$ and $u = r$ in Lemma 1.3.3. Thus

$$\begin{aligned}|\langle \Lambda^s (rw)_x, \Lambda^s r\rangle| &\le C\left(||w_x||_s ||r||_{s-1}||r||_s + ||w_x||_\infty ||r||_s^2\right) \\ &\le C\left(||w||_{s+1}||r||_s^2 + ||w_x||_{s-1}||r||_s^2\right) \\ &\le C(||w||_{s+1} + ||w||_s)||r||_s^2 \qquad (5.21)\end{aligned}$$

since $s - 1 > 1/2 > 0$.

- We estimate the second term of equation (5.18). Note that

$$\langle \Lambda^s (2r_x r_{xx} + r r_{xxx}), \Lambda^s r\rangle = \langle \Lambda^s (\frac{r_x^2}{2} + r r_{xx})_x, \Lambda^s r\rangle = -\langle \Lambda^s (\frac{r_x^2}{2} + r r_{xx}), \Lambda^s r_x\rangle.$$

We consider the sum separately. We see that

$$\langle \Lambda^s r_x r_x, \Lambda^s r_x\rangle = \langle [\Lambda^s, r_x] r_x\rangle + \langle r_x \Lambda^s r_x, \Lambda^s r_x\rangle.$$

Thus

$$\begin{aligned}|\langle \Lambda^s (r_x^2), \Lambda^s r_x\rangle| &\le C\left(||r_x||_s ||r_x||_{s-1}||r_x||s + ||r_x||_\infty ||r_x||_s ||r_x||_s\right) \\ &\le C\left(||r||_s ||r_x||_s^2 + ||r_x||_{s-1}||r_x||_s^2\right) \\ &\le C||r||_s ||r_x||_s^2 \qquad (5.22)\end{aligned}$$

where we use $s-1>1/2$. For the second term, we use the second property of Lemma 1.3.3 with $h=r$ and $u=r_x$. It follows that

$$\langle \Lambda^s r r_{xx}, \Lambda^s r\rangle = \langle [\Lambda^s, r] r_{xx}, \Lambda^s r_x\rangle - \frac{1}{2}\langle r_x \Lambda^s r_x, \Lambda^s r_x\rangle.$$

Thus we get

$$\begin{aligned} |\langle \Lambda^s r r_{xx}, \Lambda^s r\rangle| &\le C\left(||r||_s ||r_{xx}||_{s-1}||r_x||_s + ||r_x||_\infty ||r_x||_s^2\right)\\ &\le C\left(||r||_s||r_x||_s^2 + ||r_x||_{s-1}||r_x||_s^2\right)\\ &\le C||r||_s||r_x||_s^2. \end{aligned} \tag{5.23}$$

- Lastly we estimate the forth and fifth term of equation (5.18) together. We observe that

$$2r_x w_{xx} + 2w_x r_{xx} + r w_{xxx} + w r_{xxx} = (r w_{xx} + w r_{xx} + r_x w_x)_x.$$

Therefore

$$\langle \Lambda^s(2r_x w_{xx} + 2w_x r_{xx} + r w_{xxx} + w r_{xxx}), \Lambda^s r\rangle = -\langle \Lambda^s(r w_{xx} + w r_{xx} + r_x w_x), \Lambda^s r_x\rangle.$$

Now we estimate the three sum separately. For the first term we have

$$\langle \Lambda^s r w_{xx}, \Lambda^s r_x\rangle = \langle [\Lambda^s, w_{xx}] r, \Lambda^s r_x\rangle + \langle w_{xx}\Lambda^s r, \Lambda^s r_x\rangle$$

and therefore

$$\begin{aligned} |\langle \Lambda^s r w_{xx}, \Lambda^s r_x\rangle| &\le C\left(||w_{xx}||_s ||r||_{s-1}||r_x||_s + ||w_{xx}||_\infty ||r||_s||r_x||_s\right)\\ &\le C\left(||w||_{s+2}||r||_s||r_x||_s + ||w_{xx}||_{s-1}||r||_s||r_x||_s\right)\\ &\le C\left(||w||_{s+2} + ||w||_{s+1}\right)||r||_s||r_x||_s. \end{aligned} \tag{5.24}$$

For the second term we use second property of Lemma 1.3.3 with $h=w$ and $u=r_x$:

$$\langle \Lambda^s w r_{xx}, \Lambda^s r_x\rangle = \langle [\Lambda^s, w] r_{xx}, \Lambda^s r_x\rangle - \frac{1}{2}\langle w_x \Lambda^s r_x, \Lambda^s r_x\rangle.$$

Thus

$$\begin{aligned} |\langle \Lambda^s w r_{xx}, \Lambda^s r_x\rangle| &\le C\left(||w||_s ||r_{xx}||_{s-1}||r_x||_s + ||w_x||_\infty ||r_x||_s^2\right)\\ &\le C\left(||w||_s||r_x||_s^2 + ||w_x||_{s-1}||r_x||_s^2\right)\\ &\le C||w||_s||r_x||_s^2 \end{aligned} \tag{5.25}$$

where we use the fact $s-1>1/2$ once again. For the last term we have

$$\langle \Lambda^s r_x w_x, \Lambda^s r_x\rangle = \langle [\Lambda^s, w_x] r_x, \Lambda^s r_x\rangle + \langle w_x\Lambda^s r_x, \Lambda^s r_x\rangle$$

and hence

$$\begin{aligned} |\langle \Lambda^s r_x w_x, \Lambda^s r_x\rangle| &\le C\left(||w_x||_s ||r_x||_{s-1}||r_x||_s + ||w_x||_\infty ||r_x||_s^2\right)\\ &\le C\left(||w||_{s+1}||r||_s||r_x||_s + ||w_x||_{s-1}||r_x||_s^2\right)\\ &\le C||w||_{s+1}\left(||r||_s||r_x||_s + ||r_x||_s^2\right). \end{aligned} \tag{5.26}$$

Using inequalities (5.19),(5.20),(5.21),(5.22),(5.23),(5.24),(5.25) and (5.26) altogether we get

$$\begin{aligned}\frac{d}{dt}E_s^2(t) =&C\left(\epsilon||r||_s^3 + \epsilon\delta^2||r||_s||r_x||_s^2 + \gamma^2(||w||_{s+1} + ||w||_s)||r||_s^2\right)\\&+ C\gamma^2\delta^2\left(||w||_{s+2} + ||w||_{s+1}\right)||r||_s||r_x||_s\\&+ C\gamma^2\delta^2||w||_s||r_x||_s^2\\&+ C\gamma^2\delta^2||w||_{s+1}\left(||r||_s||r_x||_s + ||r_x||_s^2\right)\\&+ ||f||_s||r||_s. \end{aligned} \tag{5.27}$$

We are given that $||r(t)||_s < 1$ for $t \leq \bar{T}w \in H^s(\mathbb{T})$. Using the fact that $||r||_s \leq CE_s(t)$ and $\delta||r_x||_s \leq CE_s(t)$ for $t \leq \bar{T}$, we get

$$\begin{aligned}\frac{d}{dt}E_s^2(t) &= C\left(\epsilon + \epsilon + \gamma^2 + \gamma^2\delta + \gamma^2 + \gamma^2\delta + \gamma^2\right)E_s^2(t) + ||f||_sE_s(t)\\&C\left(\epsilon + \gamma^2 + \gamma^2\delta\right)E_s^2(t) + ||f(t)||_sE_s(t)\end{aligned} \tag{5.28}$$

Thus

$$\begin{aligned}\frac{d}{dt}E_s(t) &\leq C\left(\epsilon + \gamma^2 + \gamma^2\delta\right)E_s(t) + ||f(t)||_s\\&\leq C\left(\epsilon + \gamma^2\right)E_s(t) + ||f(t)||_s.\end{aligned}$$

We use Grönwall's inequality to obtain

$$\begin{aligned}\frac{d}{dt}\left(E_s(t)e^{-C(\epsilon+\gamma^2)t}\right) &\leq e^{-C(\epsilon+\gamma^2)t)}\sup||f(t)||_s\\E_s(t) &\leq e^{-C(\epsilon+\gamma^2)t}\left[E_s(0) + \sup||f(t)||_s\int_0^t e^{-C(\epsilon+\gamma^2)s}ds\right]\end{aligned}$$

$E_s(0) = 0$ since $r(x,0) = 0$. It follows that

$$\begin{aligned}E_s(t) &\leq \sup\left[\frac{e^{C(\epsilon+\gamma^2)t} - 1}{\epsilon + \gamma^2}\right]\\&\leq t\sup||f(t)||_s\end{aligned}$$

for $t \leq \bar{T} \leq \min\{\frac{T}{\epsilon}, \frac{T}{\gamma^2}\}$.

### 5.2.4 Proof of Theorem 5.2.1

Let $w_0 \in H^{s+2}(\mathbb{T})$. Note that solution $w$ of the HS equation (5.8) exists for all times $t \leq T/\gamma^2$ by the discussion above. We consider the Cauchy problem for periodic CH equation with $v(x,0) = aw(x,0) = v_0(x)$. Therefore $r = v - aw$ will exist over the same interval as long as the solution $v$ of CH equation does not blow up in shorter time. Moreover, we have $r(x,0) = 0$. Therefore, by continuity there exists some $\bar{t}$ such that $||r(t)||_{H^s} \leq 1$ for all $0 \leq t \leq \bar{t} \leq T/\gamma^2$. We define

$$T_0^{\gamma,\delta} = \sup\{t \leq \frac{T}{\gamma^2} : ||r(t)||_{H^s} \leq 1 \quad \text{for all} \quad t \in [0,\bar{t}]\} \tag{5.29}$$

Note that the error $r$ satisfies (5.15)-(5.16) with $f$ in Lemma 5.2.5. Consider the energy (5.17). We observe that $E_s(0) = 0$. Then by Lemma 5.2.6 the energy satisfies

$$E_s(t) \leq C\frac{\gamma^4}{\epsilon}t \quad for \quad t \leq \bar{t} \leq \min\{\frac{T}{\epsilon}, \frac{T}{\gamma^2}\}$$

for some generic constant $C$. We choose parameters small enough then

$$||v^{\epsilon,\delta}(t) - \frac{\gamma^2}{\epsilon}w^{\gamma,\delta}(t)||_s = ||r(t)||_s \leq CE_s(t) \leq C\frac{\gamma^4}{\epsilon}t << 1.$$

Therefore existence time $T^{\epsilon,\delta}$ of $v$ becomes $\min\{\frac{T}{\epsilon}, \frac{T}{\gamma^2}\}$.

**Remark 5.2.2** *Initially we know that $v^{\epsilon,\delta}$ exists locally in time for some $t \leq T^{\epsilon,\delta}$. However, the estimate above shows that $v^{\epsilon,\delta}$ stays bounded and so exists for $t \leq \min\{\frac{T}{\epsilon}, \frac{T}{\gamma^2}\}$.*

**Remark 5.2.3** *Assume $\gamma = \epsilon$ and consider Cauchy problem for the periodic CH equation*

$$\begin{aligned} v_t + 3\epsilon v v_x - \delta^2 v_{txx} &= \epsilon\delta^2(2v_x v_{xx} + v v_{xxx}) && t > 0,\ x \in \mathbb{R} \\ v(x,0) &= \epsilon v_0(x), && x \in \mathbb{R} \\ v(x,t) &= v(x+2\pi,t) && t \geq 0,\ x \in \mathbb{R} \end{aligned} \tag{5.30}$$

*and Cauchy problem for the periodic HS equation*

$$\begin{aligned} w_{txx} &= \epsilon^2(2w_x w_{xx} + w w_{xxx}) && t > 0,\ x \in \mathbb{R} \\ w(x,0) &= w_0(x) && x \in \mathbb{R} \\ w(x,t) &= w(x+2\pi,t) && t \geq 0,\ x \in \mathbb{R} \end{aligned} \tag{5.31}$$

*and*

$$||v^{\epsilon,\delta}(t) - \epsilon w^{\epsilon,\delta}(t)||_s \leq C\epsilon^3 t.$$

*Note that the order of the error is related to the order of the residual term and the residual term is the combination of the solutions of the model equation* (5.31)*. We observe that this estimation is good since the order of the error term is greater than the order of the parameters appearing in the equation* (5.31)*.*

**Remark 5.2.4** *Assume $\gamma = \sqrt{\epsilon}$ and consider Cauchy problem for the periodic CH equation*

$$\begin{aligned} v_t + 3\epsilon v v_x - \delta^2 v_{txx} &= \epsilon\delta^2(2v_x v_{xx} + v v_{xxx}) && t > 0,\ x \in \mathbb{R} \\ v(x,0) &= \epsilon v_0(x), && x \in \mathbb{R} \\ v(x,t) &= v(x+2\pi,t) && t \geq 0,\ x \in \mathbb{R} \end{aligned} \tag{5.32}$$

*and Cauchy problem for the periodic HS equation*

$$\begin{aligned} & w_{txx} = \epsilon(2w_x w_{xx} + w w_{xxx}) && t > 0,\ x \in \mathbb{R} \\ & w(x,0) = w_0(x) && x \in \mathbb{R} \\ & w(x,t) = w(x + 2\pi, t) && t \geq 0,\ x \in \mathbb{R} \end{aligned} \tag{5.33}$$

$$||v^{\epsilon,\delta}(t) - w^{\gamma,\delta}(t)||_s \leq C\epsilon t.$$

*We observe that the order of parameters in the equation* (5.33) *and the order of the error are the same. However, we need higher orders in the error. However, it is not possible to use the HS iteratively in derivation of the residual term* (5.14)*. Thus this estimation is not good for* $\gamma = \sqrt{\epsilon}$*.*

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