THE INITIAL-BOUNDARY VALUE PROBLEM FOR SOME NONLOCAL NONLINEAR WAVE TYPE PROBLEM

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This work is dedicated to the memory of my late father Bikila Yadeta and my late mother Kore Belda.

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Abstract

In this doctoral thesis we study some nonlinear nonlocal wave type equations. We consider three related problems which have connections with the study of some physical models in the theory of nonlocal elasticity. The nonlocality term is introduced via a convolution type integral operator L_α with kernel α, defined on a bounded domain $\Omega \subset \mathbb{R}^n$ as

$$L_\alpha v(x) = \int_\Omega \alpha(x-y)v(y)dy.$$

The first problem is an initial value problem for the nonlocal nonlinear integro-partial differential equation given by,

$$u_{tt} = L_\alpha g(u) \quad x \in \Omega, \qquad t > 0.$$

For this problem firstly local well-posedness is studied. Further analysis of the solution, like global existence and finite time blow-up, are investigated by various approaches such as assumptions of various smoothness and growth conditions on the nonlinearity.

The second problem included in the thesis is the initial boundary value problem for some nonlocal nonlinear wave type equation given by the equation,

$$\begin{aligned} u_{tt} - \Delta u &= L_\alpha g(u) \quad x \in \Omega, \quad t > 0. \\ u &= 0 \qquad\qquad x \in \partial\Omega, \quad t > 0. \end{aligned} \tag{0.0.1}$$

Local well-posedness of this problem is studied in proper Banach space settings.

The third problem is variant form of the second problem and is given by the equation,

$$\begin{aligned} u_{tt} - \Delta u &= L_\alpha u + g(u) \quad x \in \Omega, \quad t > 0. \\ u &= 0 \qquad\qquad x \in \partial\Omega, \quad t > 0. \end{aligned} \tag{0.0.2}$$

Unlike the previous problem, here the nonlocality and nonlinearity are expressed with separate terms. While we have imposed general assumptions on $g(u)$ such as that it should be sufficiently smooth and $g(0) = 0$, in the last two problems we have used power type nonlinear function of the form $g(u) = |u|^{p-1}u, \quad p > 1$.

The symmetry of the integral operator involved in the last problem enables us to define an explicit energy, which is a conserved quantity. For further analysis of solutions of the third problem, we have used the method of *Nehari Manifold*. Functionals like the total energy, the potential energy and the *Nehari functional* associated to the equation are defined and the *potential well depth* is obtained in terms these functionals. The two subsets of the initial value space, namely the *stable set* and the *unstable set* that are invariant under the flow of the solution are obtained accordingly. Based on the initial energy and the sets where the initial data are located in, the blow-up or the global existence conditions for solutions is analysed.

Yerel ve Doğrusal Olmayan Dalga Tipi Bazı Problemler için
Başlangıç-Değer Problemi

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Anahtar Kelimeler: İyi konulmuşluk, Gronwall lemması, yerel olmayan problem, doğrusal olmayan problem, kuvvet tipi doğrusal olmayan terim, patlama koşulu, global varlık, enerji özdeşliği energy, Nehari manifoldu, potansiyel kuyusu yöntemi, Levine lemması, Banach sabit nokta teoremi, Sobolev gömülmesi

Bu doktora tezinde bazı yerel ve doğrusal olmayan dalga tipi denklemleri çalışıldı. Bu kapsamda, elastisite kuramının bazı fiziksel modelleriye bağıntılı üç problem ele alındı. Yerel olmayan terim, α çekirdeği ile belirlenen ve bir $\Omega \subset \mathbb{R}^n$ bölgesinde

$$L_\alpha v(x) = \int_\Omega \alpha(x-y)v(y)dy.$$

konvolüsyon tipi L_α operatörü vasıtasıyla tanımlanmıştır.

Birinci problem

$$u_{tt}(x,t) = L_\alpha g(u)(x,t) \quad x \in \Omega, \quad t > 0,$$

yerel olmayan integro-diferansiyel denklemine ait bir başlangıç değer problemidir. Bu problemde ilk olarak yerel iyi konulmuşluk çalışılmıştır. Çözümün

global varlık ve sonlu zamanda patlama gibi özellikleri değişik yaklaşımlarla ve doğrusal olmayan terimin düzgünlüğü ve büyüme koşulları gibi varsayımlarla araştırılmıştır.

Bu tezde yer alan ikinci problem

$$\begin{aligned} u_{tt} - \Delta u &= L_\alpha g(u) \quad x \in \Omega, \qquad t > 0, \\ u &= 0 \qquad\qquad\qquad x \in \partial\Omega, \quad t > 0. \end{aligned} \tag{0.0.3}$$

denklemiyle verilen yerel ve doğrusal olmayan dalga tipi denklem için başlangıç-sınır değer problemidir. Bu problemin uygun Banach uzayları üzerinde yerel iyi konulmuş olması çalışılmıştır.

Üçüncü problem, ikincinin bir benzeri olup,

$$\begin{aligned} u_{tt} - \Delta u &= L_\alpha u + g(u) \quad x \in \Omega, \qquad t > 0 \\ u &= 0 \qquad\qquad\qquad x \in \partial\Omega, \quad t > 0. \end{aligned} \tag{0.0.4}$$

denklemiyle tanımlanmıştır. Önceki problemden farklı olarak yerel olmama ve doğrusal olmama iki ayrı terimle temsil edilmektedir. İlk problemde doğrusal olmayan $g(u)$ terimi için yeterince düzgün olma ve $g(0) = 0$ gibi genel koşullar varsaymamıza karşın, son iki problemde daha özel olarak $g(u) = |u|^{p-1}u, \quad p > 1$, koşulu kullanıldı.

Son problemdeki integral operatörünün simetrik olması korunan bir büyüklük olan enerjiyi tanımlaya olanak sağlamaktadır. Üçüncü problemin çözümlerinin incelenmesinde *Nehari Manifold* yöntemini kulandık. Denklemin belirlediği toplam enerji, potansiyel enerji ve *Nehari fonksiyoneli* tanımlandı ve bu fonksiyoneller cinsinden *potansiyel kuyusunun derinliği* belirlendi. Başlangıç değerleri kümesinde, denklemin belirlediği akış altında invaryant olan, *kararlı* ve *karasız kümeler* elde edildi. Çözümlerin, başlangıç enerjisi ve başlangıç değerlerinin yer aldığı kümelere bağlı olarak patlama ya da global varlık koşulları incelendi.

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Contents

Chapter 1

Introduction and Preliminaries

This chapter is devoted to the preliminary information and results which we use throughout the thesis.

1.1 Introduction

A semi-linear wave equation is given by

$$u_{tt} - \Delta u + g(u) = 0,$$

where g is a function of u and not of its derivatives, which vanishes at more than first order. The linear case $g(u) = mu$, where $m \in \mathbb{R}$, corresponds to the classical Klien-Gordon equation in realistic particle physics; the constant m may be interpreted as mass and hence assumed as to be nonnegative. In attempt to model also nonlinear phenomenon like quantization, in the 1950s equations of the type with nonlinearities like

$$g(u) = mu + u^3, \quad m \geq 0$$

were proposed in relativistic quantum mechanics in local interactions, see[22] and [24] and the references therein.

Typically g grows like $|u|^p$ for some power p. If g is the gradient of some potential function G then we have the conserved energy as

$$\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla u\|^2 + \int G(u)dx,$$

where $\|.\|$ denotes the L^2- norm. For nonlinear Klein-Gordon there is an additional term of $\frac{1}{2}\|u\|^2$ in the energy, which is useful for controlling the low frequencies of u. We say that a semi-linear wave equation is *defocusing* if G is positive definite and hence $G(u)$ cooperates with the energy of the linear operator, making the whole energy to be positively defined. On the other hand if $G(u)$ is negative definite, we call the nonlinear wave equation *focusing*. The term "coercive" does not have a standard definition, but generally denotes a potential G which is positive for large values of u.

To analyze these equations in H^s, we need the non-linearity to be sufficiently smooth. More precisely, we will always assume either that g is smooth or that g is of p^{th} power-type non-linearity with

$$p > \lfloor s \rfloor + 1$$

where $\lfloor s \rfloor$ denotes the greatest integer not greater than s. On the other hand p must have some critical upper bound p_c which is dependent on the dimension n of the space. One such example is the Sobolev critical exponent which is the consequence of the Sobolev embedding theorem. A typical nonlinear wave equation with linear dissipation is given by

$$u_{tt} - \Delta u + \delta u_t = f(u)$$

and nonlinear wave equation with nonlinear dissipation is given as

$$u_{tt} - \Delta u + Q(t, x, u_t) = f(x, u)$$

Both of these equations were studied in several research works. In most cases the nonlinear source term is taken as

$$\sum_{i=1}^{m_1} a_i(x)|u|^{p_i-1}u$$

and nonlinear dissipative term is taken as

$$\sum_{i=1}^{m_2} b_i(x)|u_t|^{p_i-1}u_t.$$

The other form of nonlinear wave problems are those involving nonlocal terms which are introduced in various ways. One way is with terms involving integral expressions over a domain or time interval. Another way is with mixed type of nonlocalities which involve time and space nonlocalities.

Let us now explain how the nonlocal theory is different from the local or standard continuum theory. In standard elasticity it is assumed that the density of elastic energy stored per unit volume, w, depends only on the strain tensor, which is directly related to the deformation gradient, i.e., to the first gradient of the displacement field. The elastic energy stored by the entire body, W is then evaluated as the spatial integral of the elastic energy density. In the one-dimensional setting, one can write

$$W = \int_L w(u_x(x))dx$$

where $u = du/dx$ is the strain, further denoted as ε, and L is the interval representing geometrically the one-dimensional body. In linear elasticity, the elastic energy density is given by

$$w(\varepsilon) = \frac{1}{2}E\varepsilon^2,$$

which is a quadratic function of strain. In the standard continuum theory, propagation of waves in a homogeneous one dimensional linear elastic medium is described by the hyperbolic partial differential equation

$$\rho u_{tt} - E u_{xx} = 0, \tag{1.1.1}$$

where ρ is the mass density, E is the elastic modulus, $u(x,t)$ is the displacement. Since ρ and E are constant coefficients, equation (1.1.1) admits solutions of the form

$$u(x,t) = e^{i(kx-\omega t)}$$

where i is the imaginary unit, ω is the circular frequency, k is the wave number, and $c = \omega/k$ is the wave velocity. In the next few subsections, we will discuss how enrichments can be introduced to bring in certain scale parameters in the continuum equations.

1.2 Local vs nonlocal

Traditional partial differential equations are relations between the values of an unknown function and its derivatives of different orders. To calculate a partial derivative of a differentiable function at a point it suffice to have the function defined in an arbitrarily small neighborhood of the point . To check whether differential equation holds at a particular point, one needs to known only the values of the function in an arbitrarily small neighborhood, so that all derivatives can be computed.

A nonlocal equation is a relation for which the opposite holds. In order to check whether a nonlocal equation holds at a point, information about the values of the function far from that point is needed. Most of the times, this is because the equation involves integral operators acting on set of functions like

$$S = \{u| \quad u : \Omega \subset \mathbb{R}^n \to \mathbb{R}\}.$$

An operator T on a set S is nonlocal operator on S if $Tu(x), \quad x \in \Omega,$

is dependent on points in Ω that are far from x. However nonlocality can be of different type. For example, *spatial nonlocality*, *temporal nonlocality* (materials with memory) and *mixed nonlocality* can be listed. For further explanation on this, see [26].

As a specific example the transport equation,

$$u_t + cu_x = f(x,t),$$

may be considered as a local differential equation. Local problems may have solutions which is written as a nonlocal integral.

Some partial differential equations may be reduced to integral equations which are nonlocal wave equations For example, the generalized initial value problem for Boussinesq equation in one-dimensional space

$$u_{tt} = F(u)_{xx} + u_{xxtt} \tag{1.2.1}$$

was studied by Adrian Constantin and Luc Molinet [2] by reducing into convolution type integral equation. A more general class of convolution type integro partial differential equation,

$$u_{tt} = (\beta * (u + g(u)))_{xx} \tag{1.2.2}$$

was studied by N Duruk, H. A. Erbay and A. Erkip [1].

An initial value problem multidimensional generalized IMBq

$$u_{tt} - \Delta u_{tt} - \Delta u = \Delta f(u) \tag{1.2.3}$$

was studied in [3] by reducing into convolution type integral equation with the kernel as the free space Greens function for the operator $(1 - \Delta)$.

Cauchy problem of the generalized double dispersion equation

$$u_{tt} - u_{xx} - u_{xxtt} + u_{xxxx} - \alpha u_{xxt} = g(u)_{xx}, \quad r > 0 \tag{1.2.4}$$

was studied by Shubin Wang and Guowang Chen, ([27]) by reducing into an integro partial differential equations of the form

$$u_{tt} - u_{xx} = L[g(u)] + L[g(u_t)], \tag{1.2.5}$$

where $L := \partial_x^2(1-\partial_x^2)$ and $G(x) = \frac{1}{2}e^{-|x|}$.

My current research work is motivated by these and several other articles involving nonlocality and nonlinearity, includes three related problems. The first problem given in (2.1.1) is initial value problem for nonlinear nonlocal integro-partial differential equation of ordinary differential equation type with convolution type kernel on a bounded domain $\Omega \subset \mathbb{R}^n$. Here, there is no set boundary condition and the order of smoothness of the solution can not exceed that of the kernel even if we set initial data with more degrees of smoothness. The important aspect of this problem is the mapping property of the integral operator (2.1.2) involved in the problem. Following the local well-posedness of solution in L^2 space, we proceed to study solutions of higher regularity followed by smoother kernels. The second problem included in this thesis is an initial-boundary value problem which is a nonlocal nonlinear wave type problem. The nonlocality term is involved with same type of integral operator studied in problem (2.1.1) the nonlinearity is included by some nonlinear function g satisfying some desired properties. The third and the last problem (4.1.1) is similar to problem (3.1.1). However in this latter case the nonlinearity and the nonlocality are involved in separated terms. The integral operator is linear and symmetric. The symmetry allows us to calculate the energy identity corresponding to the problem. The rest of the chapter is the analysis of the solution via the so called potential well method. For Potential well method refer to [29],[30]. With the integral term involved and the problem (4.1.1) is different from the usual semi-linear wave equation we designed and introduced the corresponding definition for the Nehari functional, potential functional, the energy identity in some subsets of the Hilbert space H_0^1.

1.3 Sobolev Spaces

1.3.1 Distributions and Weak Derivatives

We denote by $L^1_{loc}(\mathbb{R})$ the **space of locally integrable functions** $f : \mathbb{R} \to \mathbb{R}$. These are the Lebesgue measurable functions which are integrable over every bounded interval. The **support** of a function ϕ, denoted by $supp\phi$ is the closure of the set $\{x : \quad \phi(x) \neq 0\}$ where ϕ does not vanish.

Every locally integrable function $f \in L^1_{loc}(\mathbb{R})$ determines a linear functional $L_f : C^\infty_c(\mathbb{R}) \to \mathbb{R}$ given by

$$L_f(\phi) := \int_{\mathbb{R}} f(x)\phi(x)dx.$$

The integral is well defined for all $\phi \in C^\infty_c(\mathbb{R})$, because ϕ vanishes outside of a compact set. Furthermore, if $Supp(\phi) \subset [a, b]$ we have the estimate

$$|L_f(\phi)| \leq \left(\int_a^b |f(x)|dx \right) \|\phi\|_{C^0}.$$

Definition 1.3.1. A function $u \in L^1_{loc}(\Omega)$ is weakly differentiable with respect to x_i if there exists a function, $v \in L^1_{loc}(\Omega)$ such that

$$\int_\Omega u\partial_{x_i}\phi = \int_\Omega v\phi \quad \text{for all} \quad \phi \in C^\infty_c(\Omega).$$

Definition 1.3.2. Suppose u, $v \in L^1_{loc}(\Omega)$ and $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multi-index. We say that v is the α^{th} weak derivative of u and write $v = D^\alpha u$ if

$$\int_\Omega u\, D^\alpha \phi\, dx = (-1)^{|\alpha|} \int_\Omega v\, \phi\, dx \qquad \forall \phi \in C^\infty_c(\Omega).$$

Note that the order of differentiation is irrelevant. For example, $u_{x_i x_j} = u_{x_j x_i}$ if one of them exists.

Example 1.3.3. Let $f(x) = |x|$, $x \in \mathbb{R}$. Then $f'(x) = 2H(x) - 1$, where

$H(\cdot)$ is the Heaviside function defined by

$$H(x) = \begin{cases} 1, & \text{if} \quad x > 0, \\ 0, & \text{if} \quad x < 0. \end{cases}$$

Example 1.3.4. The Heaviside function $H(\cdot)$ does not have a weak derivative. Indeed, $H' = \delta$ is the Dirac measure.

1.3.2 The Sobolev Spaces

1.3.3 Sobolev Spaces of Order of Non-negative Integers

Definition 1.3.5. Assume k is a non–negative integer and $p \in [1, \infty]$. The Sobolev space $W^{k,p}(\Omega)$ consists of those $L^p(\Omega)$ functions whose weak derivatives up to order k exist and are in $L^p(\Omega)$. Its norm is defined by

$$\|f\|_{W^{k,p}(\Omega)} = \sum_{|\alpha| \leq k} \|D^\alpha f\|_{L^p(\Omega)}.$$

When $p \in [1, \infty)$, the space $W_0^{k,p}(\Omega)$ is the completion of $C_c^\infty(\Omega)$ under the $\|\cdot\|_{W^{k,p}(\Omega)}$ norm.

When $p = 2$, $W^{k,2}$ and $W_0^{k,2}$ are often written as H^k and H_0^k respectively, which are Hilbert spaces.

Theorem 1.3.6. *The Sobolev space $W^{k,p}(\Omega)$ is a Banach space.*

1.3.4 Fractional Sobolev Spaces

Let $0 < \kappa < 1$. For any $p \in [0, \infty)$, the Sobolev space $W^{\kappa,p}(\Omega)$ of fractional order κ is defined as follows.

$$W^{\kappa,p}(\Omega) := \left\{ u \in L^p(\Omega) : \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{n+\kappa p}} dxdy < \infty \right\} \tag{1.3.1}$$

and

$$H^{\kappa}(\Omega) = W^{\kappa,2}(\Omega)$$

In the space $W^{\kappa,p}(\Omega)$ we define the semi-norm, $[.]_{\kappa,p,\Omega}$ given by,

$$[u]_{\kappa,p,\Omega} := \int_{\Omega}\int_{\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+\kappa p}} dxdy,$$

and a norm $\|.\|_{\kappa,p,\Omega}$ given by

$$\|u\|^p_{\kappa,p,\Omega} := \|u\|^p_{0,p,\Omega} + [u]^p_{\kappa,p,\Omega}.$$

For any $s \geq 0, s = k + \kappa$ where $0 < \kappa < 1$ and $k \geq 0$ an integer, we define

$$W^{s,p}(\Omega) := \left\{ u \in W^{k,p}(\Omega) : D^{\beta}u \in W^{\kappa,p}(\Omega), \quad |\beta| \leq k \right\}.$$

The following semi-norm $[u]_{\kappa,p,\Omega}$ and norm $\|u\|_{\kappa,p,\Omega}$ are defined in the space $W^{s,p}(\Omega)$ as

$$[u]_{s,p,\Omega} = \left(\sum_{|\beta|=k} [D^{\beta}u]_{\kappa,p,\Omega} \right)^{1/p}, \quad \|u\|_{s,p,\Omega} = \left(\sum_{|\beta|\leq k} |D^{\beta}u|_{\kappa,p,\Omega} \right)^{1/p}$$

$$[u]^2_{H^s(\Omega)} = \int_{\Omega}\int_{\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dydx < \infty \tag{1.3.2}$$

and

$$\|u\|_{H^s(\Omega)} = \left(\|u\|_{L^2(\Omega)} + [u]^2_{H^s(\Omega)} \right)^{\frac{1}{2}}. \tag{1.3.3}$$

1.3.5 Sobolev Embedding

Definition 1.3.7. Let X, Y be Banach spaces. We say that X is continuously imbedded in Y if X is the subset of Y and there exists a constant C such that

$$\|x\|_Y \leq C\|x\|_X \text{ for every } \quad x \in X,$$

We know that If $u \in W^{k,p}(\Omega)$, then $u \in L^p(\Omega)$ for $k \geq 0$. In Particular if $u \in W^{1,p}(\Omega)$ then $u \in L^p(\Omega)$.

Lemma 1.3.8. *[21]*

Let $m \geq 1$ be an integer. Let $1 \leq p < \infty$.

(i) If $\frac{1}{p} - \frac{m}{n} > 0$, then

$$W^{m,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n), \frac{1}{q} = \frac{1}{p} - \frac{m}{n} \tag{1.3.4}$$

(ii) If $\frac{1}{p} - \frac{m}{n} = 0$, then

$$W^{m,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n), q \in [p, \infty) \tag{1.3.5}$$

(iii) If $\frac{1}{p} - \frac{m}{n} < 0$, then

$$W^{m,p}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n) \tag{1.3.6}$$

In case (iii) set k to be the integral part and κ to be the fractional part of $m - \frac{n}{p}$ There exists $c > 0$ such that for all $u \in W^{m,p}(\mathbb{R}^n)$, we have

$$|D^\beta u|_{0,\infty,(\mathbb{R}^n)} \leq C\|u\|_{m,p,(\mathbb{R}^n)} \quad \forall |\beta| \leq k \tag{1.3.7}$$

and for almost all $x, y \in \mathbb{R}^n)$ and for all $|\beta| = k$,we have

$$|D^\beta u(x) - D^\beta u(y)| \leq C\|u\|_{m,p,(\mathbb{R}^n)}|x - y|^\kappa. \tag{1.3.8}$$

In particular we have the continuous inclusion

$$W^{m,p}(\mathbb{R}^n) \hookrightarrow C(\mathbb{R}^n), \quad \textit{for} \quad m > \frac{n}{p}. \tag{1.3.9}$$

The same result follows when $\mathbb{R}^n$ is replaced by $\mathbb{R}^n_+$ or by Ω of class C^m with bounded boundary and for the spaces $W_0^{m,p}(\Omega)$ for any open subset Ω of $\mathbb{R}^n$.

And thus if Ω is bounded and sufficiently smooth, we have

$$W^{m,p}(\Omega) \hookrightarrow \mathcal{C}^(\bar{\Omega}), \quad \textit{for} \quad m > \frac{n}{p}. \tag{1.3.10}$$

1.4 Eigenfunctions and Eigenvalues of Dirichlet Laplacian

1.4.1 Eigenfunction Expansion Methods

Let Ω be a bounded open set in $\mathbb{R}^n$. By $L^2(\Omega)$ we mean the usual real Hilbert space of square integrable functions which are defined on Ω. with inner product

$$\langle u, v \rangle = \int_\Omega uvdx,$$

and norm

$$\|u\| = \left(\int_\Omega u^2 dx\right)^{\frac{1}{2}}.$$

We know that the Laplace operator $-\Delta$ with Dirichlet boundary condition has a discrete spectrum consisting of increasing sequence

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \tag{1.4.1}$$

of eigenvalues satisfying the condition $\lim_n \lambda_n \to \infty$ as $n \to \infty$ and the corresponding sequence of eigenfunctions $\{\varphi_n\}$, which form a basis of $L^2(\Omega)$. We therefore have,

$$-\Delta\varphi_n(x) = \lambda_n\varphi_n(x), \quad \varphi_n(x) = 0 \quad x \in \partial\Omega \quad n = 1, 2, 3, \ldots \tag{1.4.2}$$

Furthermore, φ_n can be chosen to be an orthonormal set in $L^2(\Omega)$. That is,

$$\langle \varphi_i, \varphi_j \rangle = 0 \quad \text{if} \quad i \neq j \quad \text{and} \quad \|\varphi_i\| = 1.$$

By virtue of (1.4.2), we have for any smooth function g defined on $[0, \infty)$,

$$g(\sqrt{-\Delta})\varphi_n(x) = g(\sqrt{\lambda_n})\varphi_n(x) \tag{1.4.3}$$

By completeness, any $u \in L^2(\Omega)$ has eigenfunction expansion of the form,

$$u = \sum_{n=1}^{\infty} u_n \varphi_n \tag{1.4.4}$$

where u_n are the Fourier coefficients with respect to the basis φ_n are given by,

$$u_n = \int_{\Omega} u(x)\varphi_n(x)dx \quad n = 1, 2, 3, ... \tag{1.4.5}$$

From Parseval's theorem it follows,

$$\|u\|^2_{L^2(\Omega)} = \langle u, u\rangle = \sum_{n=1}^{\infty} u_n^2 \tag{1.4.6}$$

and the convergence is in the L_2 sense, that is,

$$\|S_N - u\|^2_{L^2(\Omega)} = \int_{\Omega} |u(x) - \sum_{n=1}^{N} u_n \varphi_n(x)|^2 dx \to 0 \quad \text{as} \quad N \to \infty. \tag{1.4.7}$$

$$\|\Delta u(x,t)\|^2_{L^2(\Omega)} = \sum_{n=1}^{\infty} \lambda_n^2 u_n^2(t) \tag{1.4.8}$$

Also from integration by parts and the Dirichlet boundary condition we have

$$\|\nabla u\|^2_{L^2(\Omega)} = \int_{\Omega} u(-\Delta u)dx = \sum_{n=1}^{\infty} \lambda_n u_n^2 \tag{1.4.9}$$

from (1.4.6) and (1.4.9) we see that, if $u \in H_0^1(\Omega)$,

$$\|u\|_{H^1(\Omega)} = \|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} = \sum_{n=1}^{\infty} (1 + \lambda_n) u_n^2 \tag{1.4.10}$$

If $u \in H^2(\Omega) \cap H_0^1(\Omega)$,

$$\|u\|_{H^2(\Omega)}^2 = \sum_{\alpha \leq 2} \|D^\alpha u\|_{L^2(\Omega)}^2 \approx \sum_{n=1}^{\infty} (1 + \lambda_n + \lambda_n^2) u_n^2 \tag{1.4.11}$$

For convenience and the use in latter chapters let us define a norm for a general $u \in H^s(\Omega), \quad s \geq 0$, by

$$\|u\|_{H^s(\Omega)}^2 := \sum_{n=1}^{\infty} \lambda_n^s u_n^2, \tag{1.4.12}$$

Otherwise we may have written (1.4.12) as

$$\|u\|_{H^s(\Omega)}^2 = C(s) \sum_{n=1}^{\infty} \lambda_n^s u_n^2,$$

where $C(s)$ is a constant that is dependent on s. This follows from (1.4.1) and the property that $\lim_n \lambda_n \to \infty$ as $n \to \infty$.

We wind up this section by giving an example of the Dirichlet Laplacian defined on an interval in the real line and the corresponding eigenvalues and eigenvectors. The structures of eigenfunctions of the Dirichlet Laplacian depend on the space dimension and the geometry of the domain. However they all posses some common features. For example, the eigenfunction of Dirichlet Lapalacian are infinitely differentiable. For further notes on this topic we refer to ([33]).

Example 1.4.1. Let $\Omega = (a, b)$ be an open interval in $\mathbb{R}$. For zero Dirichlet boundary value $u(a) = u(b) = 0$, the eigenfunctions $\varphi_m(x)$ and the corresponding eigenvalues λ_m of the Laplacian operator, $-\frac{d^2}{dx^2}$ are satisfying the boundary condition $\varphi_m(a) = \varphi_m(b) = 0, \quad m = 1, 2, 3, ...$ and the differential equation $\varphi_m''(x) + \lambda_m \varphi_m(x) = 0$ on (a, b). Any function $u \in H^2(a.b) \cap H_0^1(a, b)$ can be generated by $\{\varphi_m\}$, i.e. written in the form (1.4.4), which is the usual Fourier sine series of u. The eigenvalues and the eigenfunctions are given as follows:

$$\lambda_m = \left(\frac{m\pi}{b-a}\right)^2, \quad \varphi_m(x) = \sqrt{\frac{2}{b-a}}\sin\left(m\pi\left(\frac{x-a}{b-a}\right)\right) \quad m = 1, 2, 3...$$

Chapter 2

The Initial Value problem for Nonlinear Nonlocal Integro-Partial Differential Equation

2.1 Description of the problem

In this section we describe some problem with a non-local nonlinear integro-partial differential equation, and prove its local well-posedness. The problem is given as follows

$$\begin{cases} u_{tt} = L_\alpha g(u)(x,t) \quad x \in \Omega, t \geq 0 \\ u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x) \end{cases} \tag{2.1.1}$$

where the integral operator L_α is given by

$$L_\alpha u(x,t) := \int_\Omega \alpha(x-y)u(y,t)dy \tag{2.1.2}$$

where kernel α, is defined on a bounded domain $\Omega \subset \mathbb{R}^n$, and nonlinear function $g \in C^\infty(\mathbb{R})$ satisfies $g(0) = 0$. In this chapter we consider two cases for the kernel α. The case $\alpha \in W^{k,1}(\mathbb{R})$ and the case $\alpha \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$. The motivation for the current problem emerges from the possibility of some initial-boundary value problems to be written as integral equation with its kernel as Green's function of some differential operator. For example, a generalized Boussinesq equation in one-dimensional space is written as

$$\begin{cases} u_{tt} = [F(u)]_{xx} + u_{xxtt} \quad x \in \mathbb{R}, t \geq 0, \\ u(x,0) = u_0(x) \quad u_t(x,0) = u_1(x). \end{cases} \tag{2.1.3}$$

Problem (2.1.3) may also be rewritten as

$$\begin{cases} u_{tt} = (\beta * F(u))_{xx}, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \end{cases} \tag{2.1.4}$$

where

$$\beta(x) = \frac{1}{2}\exp(-|x|), \quad x \in \mathbb{R}, \tag{2.1.5}$$

is the free space Green's function for the operator $(1 - \partial_x^2)$ and "$*$ " is the convolution notation.

Problem of the form (2.1.4) has been generalized in the work of N. Duruk, H. A. Erbay and A. Erkip, in [1] by considering a wider class of kernel functions β that are not necessarily Green's function of some differential operator.

It is known that if $f \in W^{m,1}(\mathbb{R})$ and $g \in W^{n,1}(\mathbb{R})$, where m and n are positive integers, we have the property

$$D^{m+n}(f * g) = (D^m f) * (D^n g). \tag{2.1.6}$$

By aid of the property given in (2.1.6) problems of the form (2.1.4) may be written in the form

$$\begin{cases} u_{tt} = (\alpha * F(u)) \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \mathbb{R} \end{cases} \tag{2.1.7}$$

by carrying the derivatives only onto β. In doing so, it is assumed that $\beta_{xx} := \alpha$, and that β_{xx} is defined at least in some weak sense. For example, for β given as in (2.1.5), we have $\beta_{xx} = \beta - \delta$, where δ is the usual Dirac distribution which is not a regular function.

Equation (2.1.6) shows that convolution has some smoothing effect. That is, differentiation of a convolution of two integrable functions can be performed repeatedly as far as any one of them is differentiable. This property says that the order of smoothness of a convolution is the sum of orders of smoothness of the two functions. There are also some other important properties of convolution like the properties of Fourier transforms of convolutions.

The current problem (2.1.1) is essentially of the form (2.1.7) with convolution type integral operator. However rather than the whole space $\mathbb{R}^n$, our integral operator L_α given in (2.1.2) is defined on functions which are defined on a bounded domain $\Omega \subset \mathbb{R}^n$. The kernel α of the integral operator L_α is not necessarily a Green function of some differential operator. Also there is no boundary condition on the domain Ω imposed on unknown function u .

2.2 Local Well-posedness of the Problem

In this section we investigate the well-posedness of the initial value problem for the integro partial differential equation given in (2.1.1) with the kernel $\alpha \in L^1(\mathbb{R})$, initial data $\phi, \psi \in L^2(\Omega) \cap Ł^\infty(\Omega)$, and the nonlinear function g satisfying $g \in C^\infty(\mathbb{R})$ satisfying $\quad g(0) = 0$. Integrating with respect to t, we obtain first order integro-partial differential equation

$$u_t(x,t) = \psi(x) + \int_0^t L_\alpha g(u)(x,\tau)d\tau, \quad u(x,0) = \phi(x). \tag{2.2.1}$$

Integrating equation (2.2.1) with respect to t and applying the second initial condition, we obtain

$$u(x,t) = \int_0^t (t-\tau)L_\alpha g(u)(x,\tau)d\tau + t\psi(x) + \phi(x). \tag{2.2.2}$$

We analyse the solvability of the integral equation (2.2.2) which is equivalent to the original nonlocal problem (2.1.1).
Let the initial functions $\phi, \psi \in L^2(\Omega) \cap L^\infty(\Omega)$ satisfy

$$\|\phi\|_2 + \|\phi\|_\infty + \|\psi\|_2 + \|\psi\|_\infty =: M_0 \tag{2.2.3}$$

Define a function space,

$$Y_T = C\left([0,T]; L^2(\Omega) \cap L^\infty(\Omega)\right)$$

equipped with the norm,

$$\|u\|_{Y_T} = \max_{0 \le t \le T} \|u\|_2 + \max_{0 \le t \le T} \|u\|_\infty. \tag{2.2.4}$$

It is clear that Y_T is a Banach space with this norm. For the next task we need the following lemma.

Lemma 2.2.1. $L_\alpha : L^2(\Omega) \to L^2(\Omega)$ *is a continuous linear operator and,*

$$\|L_\alpha u\|_{L^2(\Omega)} \le \|\alpha\|_{L^1(\mathbb{R})}\|u\|_{L^2(\Omega)}. \tag{2.2.5}$$

Proof. By Cauchy-Schwarz inequality for integrals and Fubini's theorem we have,

$$\begin{aligned}
\|L_\alpha u\|^2_{L^2(\Omega)} &= \int_\Omega \left| \left(\int_\Omega \alpha(x-y)u(y)dy \right) \right|^2 dx \\
&\le \int_\Omega \left(\int_\Omega |\alpha(x-y)u(y)|dy \right)^2 dx \\
&= \int_\Omega \left(\int_\Omega \sqrt{|\alpha(x-y)|}\sqrt{|\alpha(x-y)|}|u(y)|dy \right)^2 dx \\
&\le \int_\Omega \left(\int_\Omega |\alpha(x-y)|dy \right) \left(\int_\Omega |\alpha(x-y)||u(y)|^2 dy \right) dx \\
&\le \|\alpha\|_{L^1(\mathbb{R})} \int_\Omega \int_\Omega |\alpha(x-y)||u(y)|^2 dydx \\
&= \|\alpha\|_{L^1(\mathbb{R})} \int_\Omega |u(y)|^2 \int_\Omega |\alpha(x-y)|dxdy \\
&\le \|\alpha\|^2_{L^1(\mathbb{R})} \int_\Omega |u(y)|^2 dy = \|\alpha\|^2_{L^1(\mathbb{R})} \|u(y)\|^2_{L^2(\Omega)}
\end{aligned}$$

This completes the proof. □

Lemma 2.2.2. *$L_\alpha : L^\infty(\Omega) \to L^\infty(\Omega)$ is a continuous linear operator and*

$$\|L_\alpha u\|_{L^\infty(\Omega)} \le \|\alpha\|_{L^1(\mathbb{R})} \|u\|_{L^\infty(\Omega)} \tag{2.2.6}$$

Proof. We have

$$\begin{aligned}
|L_\alpha u| = |\int_\Omega \alpha(x-y)u(y)dy| &\le \int_\Omega |\alpha(x-y)||u(y)|dy \\
&\le \|u\|_{L^\infty(\Omega)} \int_\Omega |\alpha(x-y)|dy \\
&\le \|\alpha\|_{L^1(\mathbb{R})} \|u\|_{L^\infty(\Omega)} \qquad (2.2.7)
\end{aligned}$$

which gives the result. □

Lemma 2.2.3. *Let $g \in C^1(\mathbb{R})$ be a function vanishing at zero. For all $u \in L^\infty(\Omega) \cap L^2(\Omega)$, the function $g(u)$ is also in $L^\infty(\Omega) \cap L^2(\Omega)$.*

Proof. We have

$$g(u) = \int_0^1 ug'(su)ds, \quad x \in \Omega,$$

so that if $u \in L^\infty(\Omega) \cap L^2(\Omega)$ then

$$|g(u)| \leq \sup_{|r| \leq \|u\|_{L^\infty}} \{g'(r)\} |u|.$$

Thus we have $g(u) \in L^\infty(\Omega) \cap L^2(\Omega)$. $\square$

If we have $u(x,t) \in C\left([0,T], L^\infty(\Omega) \cap L^2(\Omega)\right)$, we have

$$|u(x,t)| \leq \sup_{0 \leq \tau \leq T} \|u(.,\tau)\|_{L^\infty} \leq \|u\|_{Y_T}.$$

By Lemmas 2.2.1, 2.2.2 and 2.2.3, we know that $g(u) \in Y_T$ whenever $u \in Y_T$. In addition to that

$$\|g(u)\|_{Y_T} \leq M\|u\|_{Y_T} \tag{2.2.8}$$

where

$$M := M(T) = \sup_{|r| \leq \|u\|_{Y_T}} \{|g'(r)|\}. \tag{2.2.9}$$

Note that if $u(x,t) \in C\left([0,T], L^\infty(\Omega) \cap L^2(\Omega)\right)$, then $M(t)$ defined as in (2.2.9) is positive, continuous and nondecreasing on $[0,T]$.

Now we have,

$$\|L_\alpha g(u)\|_{Y_T} \leq \|\alpha\|_1 \|g(u)\|_{Y_T} \leq \|\alpha\|_1 M\|u\|_{Y_T} \tag{2.2.10}$$

$$\left\| \int_0^t (t-\tau) L_\alpha u(x,\tau) d\tau \right\|_{Y_T} \leq \int_0^t (t-\tau) \left\| L_\alpha g(u(x,\tau)) \right\|_{Y_T} d\tau$$
$$\leq \frac{1}{2} t^2 \|L_\alpha g(u)\|_{Y_T} \leq \frac{1}{2} t^2 \|\alpha\|_1 M \|u\|_{Y_T}$$

Define an operator $\mathcal{K} : Y_T \to Y_T$ as

$$\mathcal{K}(u)(x,t) = \int_0^t (t-\tau) L_\alpha g(u)(x,\tau) d\tau + t\psi(x) + \phi(x). \tag{2.2.11}$$

Clearly, $\mathcal{K}(u) \in Y_T$ and

$$\|\mathcal{K}(u)\|_{Y_T} \leq \frac{1}{2} T^2 \|\alpha\|_1 M \|u\|_{Y_T} + C M_0 \tag{2.2.12}$$

where $C := \max\{1, T\}$. Let us fix some bounded subset of Y_T which is the neighborhood of the initial functions ϕ, ψ. Then

$$Y_T(M_0) = \{u \in Y_T | \quad \|u\|_{Y_T} \leq M_0 + 1\}. \tag{2.2.13}$$

We can adjust T, so that $\mathcal{K}$ maps $Y_T(M_0)$ into itself on $[0, T]$. If $T \leq 1$, then $C := \max\{1, T\} = 1$ and choosing T in such a way that

$$\frac{1}{2} T^2 \|\alpha\|_1 M (M_0 + 1) < 1.$$

Consequently, by (2.2.12), we have

$$T \leq \min\left\{1, \frac{2}{\sqrt{2\|\alpha\|_1 M (M_0 + 1)}}\right\}. \tag{2.2.14}$$

So the operator $\mathcal{K}$ maps $Y_T(M_0)$ into itself on $[0, T]$.

Now we proceed to set $T > 0$ so that $\mathcal{K}$ is strictly contractive. Let u_1 and u_2 be in $Y_T(M_0)$. We have,

$$
\begin{aligned}
\|\mathcal{K}u_1 - \mathcal{K}u_2\|_{Y_T} &= \left\| \int_0^t (t-\tau)(L_\alpha(g(u_1) - g(u_2)))d\tau \right\|_{Y_T} \\
&\leq \int_0^t (t-\tau) \left\| L_\alpha(g(u_1) - g(u_2)) \right\|_{Y_T} d\tau \\
&\leq \frac{1}{2} T^2 \left\| L_\alpha(g(u_1) - g(u_2)) \right\|_{Y_T} \\
&\leq \frac{1}{2} T^2 \|\alpha\|_1 \left\| ((g(u_1) - g(u_2)) \right\|_{Y_T} \\
&\leq \frac{1}{2} T^2 \|\alpha\|_1 M \left\| u_1 - u_2 \right\|_{Y_T}. \qquad (2.2.15)
\end{aligned}
$$

From (2.2.14) and (2.2.15) if

$$
T \leq \min \left\{ 1, \frac{2}{\sqrt{2\|\alpha\|_1 M(M_0+1)}}, \frac{1}{\sqrt{M\|\alpha\|_1}} \right\}
$$

then,

$$
\|\mathcal{K}u_1 - \mathcal{K}u_2\|_{Y_T} \leq \frac{1}{2} \|u_1 - u_2\|_{Y_T}. \qquad (2.2.16)
$$

Now by Banach fixed point theorem there exists a unique $u \in Y_T(M_0)$ which is the solution of the integral equation (2.2.2), and equivalently the nonlocal problem (2.1.1). In conclusion, we have the following important theorem.

Theorem 2.2.4. *Let $\alpha \in L^1(\mathbb{R})$.For every initial data $\phi, \psi \in L^2(\Omega) \cap L^\infty(\Omega)$ and for every function $g \in C^\infty(\mathbb{R})$ with $g(0) = 0$, problem (2.1.1) has a unique local solution u such that*

$$
u(x,t) \in C^2\left([0,T), L^\infty(\Omega) \cap L^2(\Omega)\right).
$$

2.3 Global Existence of Solution.

2.3.1 The case of moderately growing nonlinearity

It can be shown that the linear version of problem (2.1.1) has global solution. In this section we focus on cases of nonlinearity that has mild growth so that it can be linearly bounded.

The first such case is when the function $g \in C^{\infty}(\mathbb{R})$ is uniformly bounded. This includes cases like $g(x) = A\sin(mx)$, $m \in \mathbb{R}\, g(x) = \arctan(x)$, $g(x) = \tanh(x)$ with infinitely many others. From the integral equation (3.2.1) we have

$$\begin{aligned} \|u(.,t)\|_{\infty} &\leq \|\phi\|_{\infty} + t\,\|\psi\| + t\int_0^t \|L_{\alpha} g(u)(.,\tau)\|_{\infty}\, d\tau \\ &\leq \|\phi\|_{\infty} + t\,\|\psi\| + t^2\,\|\alpha\|_1\,\|g(u)\|_{\infty} \\ &\leq \|\phi\|_{\infty} + t\,\|\psi\| + t^2\,\|\alpha\|_1\, M \end{aligned} \tag{2.3.1}$$

where $M = \sup_{x\in\mathbb{R}} |g(x)|$. Also from,

$$u_t(x,t) = \psi(x) + \int_0^t L_{\alpha} g(u)(x,\tau) d\tau, \quad u(x,0) = \phi(x), \tag{2.3.2}$$

we can obtain

$$\begin{aligned} \|u_t(.,t)\|_{\infty} &\leq \|\psi\|_{\infty} + \int_0^t \|L_{\alpha} g(u)(.,\tau)\|_{\infty}\, d\tau \\ &\leq \|\psi\|_{\infty} + \|\alpha\|_1\, Mt. \end{aligned} \tag{2.3.3}$$

Hence $\sup\limits_{t\uparrow T} \|u(.,t)\|_{\infty}$ and $\sup\limits_{t\uparrow T} \|u_t(.,t)\|_{\infty}$ are finite for every finite time $T > 0$. The second special case where we can obtain a global solution is the case where the function $g \in C^{\infty}(\mathbb{R})$ is bounded linearly. That is,

$$|g(x)| \leq A|x| + B, \quad x \in \mathbb{R}. \tag{2.3.4}$$

These includes all linear functions and sub-linear functions, i.e.,

$$|g(x)| \leq A|x|^{\rho} + B, \quad 0 < \rho < 1, \quad x \in \mathbb{R}, \tag{2.3.5}$$

as well as functions with sub-logarithmic growth , i.e.,

$$|g(x)| \leq A\ln(|x|+1)+B, \quad 0<\rho\leq 1, \quad x\in\mathbb{R}. \tag{2.3.6}$$

where A and B are positive constants.

Lemma 2.3.1 ([5]). *Let x and k be continuous, and a and b Riemann integrable functions on the interval $J=[t_0,t_1]$ with b and k non-negative on J.Then, we have the following;*

I. If

$$x(t) \leq a(t)+b(t)\int_{t_0}^{t} k(s)x(s)ds, \quad t\in J, \tag{2.3.7}$$

then

$$x(t) \leq a(t)+b(t)\int_{t_0}^{t} a(s)k(s)\exp\left(\int_{s}^{t} b(r)k(r)dr\right)ds, \quad t\in J. \tag{2.3.8}$$

Moreover, equality holds in (2.3.8) for a subinterval $J_1=[t_2,t_3]$ of J if equality holds in 2.3.7 for $t\in J_1$.

II. The result remains valid if $\leq$ is replaced by $\geq$ both in (2.3.7) and (2.3.8).

III. Both (I) and (II) remain valid if $\int_{t_0}^{t}$ is replaced by $\int_{t}^{t_1}$ and $\int_{s}^{t}$ by $\int_{t}^{s}$ throughout.

From our integral equation (2.2.2) and condition (2.3.4) on the function g, we have the norm inequalities

$$\begin{aligned}
\|u(.,t)\|_\infty &\leq \|\phi\|_\infty + t\|\psi\|_\infty + t\int_0^t \|L_\alpha g(u)(.,\tau)\|_\infty d\tau \\
&\leq \|\phi\|_\infty + t\|\psi\|_\infty + \|\alpha\|_1 t\int_0^t \|g(u)(.,\tau)\|_\infty d\tau \\
&\leq \|\phi\|_\infty + t\|\psi\|_\infty + \|\alpha\|_1 t\int_0^t (A\|u(.,\tau)\|_\infty + B)d\tau
\end{aligned}$$

Therefore,

$$\|u(.,t)\|_\infty \leq \|\phi\|_\infty + t\,\|\psi\|_\infty + B\,\|\alpha\|_1\, t^2 + A\,\|\alpha\|_1\, t \int_0^t \|u(.,\tau)\|_\infty \, d\tau. \quad (2.3.9)$$

The last inequality (2.3.8) satisfies the condition of the Lemma 2.3.8 with

$$\begin{aligned}
&a(t) = \|\phi\|_\infty + t\,\|\psi\|_\infty + B\,\|\alpha\|_1\, t^2,\\
&b(t) = A\,\|\alpha\|_1\,,\\
&k(\tau) = 1,\\
&J = [0,T], \quad t_0 = 0, t_1 = T,
\end{aligned}$$

Hence

$$\begin{aligned}
&\|u(.,t)\|_\infty \leq \|\phi\|_\infty + t\,\|\psi\|_\infty + B\,\|\alpha\|_1\, t^2\\
&+ A\,\|\alpha\|_1\, t \int_0^t (\|\phi\|_\infty + \tau\,\|\psi\|_\infty + B\,\|\alpha\|_1\, \tau^2) \exp(\frac{A}{2}\,\|\alpha\|_1\,(t^2 - \tau^2)) d\tau.
\end{aligned} \quad (2.3.10)$$

This means $\sup\limits_{t\uparrow T} \|u(.,t)\|_\infty < \infty$,for every finite time $T > 0$. From integro-differential equation (2.2.1) we get,

$$\begin{aligned}
\|u_t(.,t)\|_\infty &\leq \|\psi\|_\infty + \int_0^t \|L_\alpha g(u)(.,\tau)\|_\infty \, d\tau\\
&\leq \|\psi\|_\infty + \|\alpha\|_1 \int_0^t \|g(u)(.,\tau)\|_\infty \, d\tau\\
&\leq \|\psi\|_\infty + \|\alpha\|_1 \int_0^t (A\,\|u(.,\tau)\|_\infty + B) d\tau\\
&= \|\psi\|_\infty + B\,\|\alpha\|_1\, t + \|\alpha\|_1\, A \int_0^t \|u(.,\tau)\|_\infty \, d\tau\\
&\leq \|\psi\|_\infty + B\,\|\alpha\|_1\, t + \|\alpha\|_1\, AU(t), \qquad (2.3.11)
\end{aligned}$$

where

$$U(t) = t\left\|\phi\right\|_\infty + \frac{1}{2}t^2\left\|\psi\right\|_\infty + \frac{1}{3}Bt^3\left\|\alpha\right\|_1$$
$$+\int_0^t\int_0^s A\left\|\alpha\right\|_1 s(\left\|\phi\right\|_\infty + \tau\left\|\psi\right\|_\infty + B\left\|\alpha\right\|_1\tau^2)\exp(\frac{A}{2}\left\|\alpha\right\|_1(s^2-\tau^2))d\tau ds. \tag{2.3.12}$$

Therefore, $\limsup_{t\uparrow T}\left\|u_t(.,t)\right\|_\infty < \infty$. For every finite time $T > 0$. This proves the global existence of solutions under the conditions imposed on the function g. Note that, for bounded domain Ω like our case here, we have

$$\|u\|_{L^2(\Omega)} = \left(\int_\Omega |u|^2\right)^{1/2} \leq \sqrt{\mu(\Omega)}\|u\|_{L^\infty(\Omega)},$$

where $\mu(\Omega)$ is the measure of Ω. For this reason the L^2 norm in our work is inherently considered by the dominant L^∞ norm.

2.4 Solutions of Higher Regularity

We next show that the order smoothness of the solution can not exceed the order of smoothness of the kernel α of the integral operator L_α. As shown in the previous example, for $\alpha \in L^1(\mathbb{R})$ and initial data $\phi, \psi \in L^2(\Omega) \cap L^\infty(\Omega)$, we have seen that the problem is well-posed with a solution $u(x,t) \in C^2\left([0,T), L^2(\Omega) \cap L^\infty(\Omega)\right)$. However, we can not get a solution of better smoothness even if we take smoother initial data $\phi, \psi \in H^s(\Omega) \cap L^\infty(\Omega)$, $s > 0$, rather than $\phi, \psi \in L^2(\Omega) \cap L^\infty(\Omega)$. In the next lemma we see differentiation properties with respect to the spacial variable x under integral sign. Such differentiation is directly applied to the kernel α as the other variable under the integral is a dummy variable. We consider the following illustrative argument for case Ω an interval in $\mathbb{R}$.

Let $\Omega = [a,b] \subset \mathbb{R}$. If $u \in H^1(\Omega)$, by a simple change of variable, $y = x - z$, we may write the integral operator given by (2.1.2) as

$$L_\alpha u(x) = \int_{x-b}^{x-a} \alpha(z) u(x-z) dz, \quad x \in \Omega. \tag{2.4.1}$$

From (2.4.1) we get

$$\frac{\partial}{\partial x} L_\alpha u(x) = \Phi_0(x) + \int_\Omega \alpha(x-y) \frac{\partial}{\partial y} u(y) dy, \tag{2.4.2}$$

where

$$\Phi_0(x) = \alpha(x-a)u(a) - \alpha(x-b)u(b).$$

This implies that $L_\alpha u \in H^1(\Omega)$ if and only if $\Phi_0(x) \in L^2(\Omega)$. One possible case is when $u \in H_0^1(\Omega)$, where we have, $u(a) = u(b) = 0$, so that $\Phi_0 = 0 \in L^2(\Omega)$. From these arguments, we state the following theorem.

Theorem 2.4.1. *Let $\alpha \in L^1(\mathbb{R})$ and $u \in H^1(\Omega)$, then $L_\alpha u(x) \in H^1(\Omega)$ if and only if $\alpha(x-a)u(a) - \alpha(x-b)u(b) \in L^2(\Omega)$.*

Corollary 2.4.1. The operator $L_\alpha : H_0^1(\Omega) \to H^1(\Omega)$ is a bounded operator.

Proof. By Theorem 2.4.1 and differentiation under integral sign it follows that

$$\begin{aligned}
\|L_\alpha u\|^2_{H^1(\Omega)} &= \|\partial_x L_\alpha u\|^2_{L^2(\Omega)} + \|L_\alpha u\|^2_{L^2(\Omega)} \\
&\leq \|\alpha\|^2_{L^1(\mathbb{R})}(\|u_x\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)}) \\
&= \|\alpha\|^2_{L^1(\mathbb{R})}\|u\|^2_{H^1_0(\Omega)}.
\end{aligned}$$

Therefore we have,

$$\|L_\alpha u\|_{H^1(\Omega)} \leq \|\alpha\|_{L^1(\mathbb{R})}\|u\|_{H^1_0(\Omega)},$$

as required. □

If $\alpha \in W^m_1(\mathbb{R})$ and $u \in H^k(\Omega), m \geq k \geq 1$, by an argument that follows from (2.4.1) and (2.4.2) we may get

$$\partial^k_x L_\alpha u(x) = \int_\Omega \alpha(x-y)\partial^k_y u(y)dy + \sum_{i=1}^{k} \partial^{i-1}_x \Phi_{k-i}(x), \tag{2.4.3}$$

where,

$$\Phi_r(x) = \alpha(x-y)\partial^r_y u(y)|^{y=a}_{y=b}, \quad \text{for } 1 \leq r \leq k-1.$$

and

$$\sum_{i=1}^{k} \partial^{i-1}_x \Phi_{k-i}(x) \in L^2(\Omega) \tag{2.4.4}$$

In particular we have

$$\Phi_{k-i}(x) \in H^{i-1}(\Omega), \quad 1 \leq i \leq k. \tag{2.4.5}$$

The condition in (2.4.5) follows from definition of Sobolev space H^s. On the other hand we have

$$\partial^k_x L_\alpha u(x) = \int_\Omega \partial^k_x \alpha(x-y)u(y)dy = L_{D^k_x\alpha}u(x). \tag{2.4.6}$$

If $\alpha \in W_1^m(\mathbb{R}), m \geq k$ and $u \in H^k(\Omega), k \geq 1$ then $L_\alpha u(x,t) \in H^k(\Omega)$. In this case we have equivalent forms of writing $\partial_x^k L_\alpha u(x,t)$ given by (2.4.3) and (2.4.6).

Lemma 2.4.2. *Let $\alpha \in W^{k,1}(\mathbb{R})$, where is k is a positive integer. Then the operator*

$$L_\alpha : L^2(\Omega) \to H^k(\Omega)$$

is continuous.

Proof. By definition, $\alpha \in W^{k,1}(\mathbb{R})$ implies that

$$D_x^r \alpha \in L^1(\mathbb{R}), \quad 0 \leq r \leq k. \tag{2.4.7}$$

For $u \in L^2(\Omega) \cap L^\infty(\Omega)$, by differentiation under integral sign and (2.2.1) we have .

$$L_{D_x^r \alpha} u \in L^2(\Omega) \quad 0 \leq r \leq k \tag{2.4.8}$$

Applying partial differentiation under integral sign yields,

$$D_x^r L_\alpha u = L_{D_x^r \alpha} u \in L^2(\Omega), \quad 0 \leq r \leq k. \tag{2.4.9}$$

From (2.4.9) we get that $L_\alpha u \in H^k(\Omega)$. Next we set the norm estimate of $L_\alpha u$.

$$\begin{aligned} \|L_\alpha u\|_{H^k(\Omega)}^2 &= \sum_{r=0}^{k} \|\partial_x^r L_\alpha u\|_{L^2(\Omega)}^2 = \sum_{r=0}^{k} \|L_{(D_x^r \alpha)} u\|_{L^2(\Omega)}^2 \leq \sum_{r=0}^{k} \|D_x^r \alpha\|_{L^1(\mathbb{R})}^2 \|u\|_{L^2(\Omega)}^2 \\ &= \left(\sum_{r=0}^{k} \|D_x^r \alpha\|_{L^1(\mathbb{R})}^2 \right) \|u\|_{L^2(\Omega)}^2 \leq \left(\|\alpha\|_{W^{k,1}(\mathbb{R})} \right)^2 \|u\|_{L^2(\Omega)}^2 \end{aligned}$$

Therefore we have

$$\|L_\alpha u\|_{H^k(\Omega)} \leq \left(\|\alpha\|_{W^{k,1}(\mathbb{R})}\right)\|u\|_{L^2(\Omega)}. \tag{2.4.10}$$

$\square$

Theorem 2.4.3. *Let $k \geq 1$ be a positive integer. For the kernel $\alpha \in W^{k,1}(\mathbb{R})$ and initial data $\phi, \psi \in L^\infty \cap H^k(\Omega)$, we have a unique local solution of the initial problem (2.1.1) $u \in C^2\left([0,T], L^\infty \cap H^k(\Omega)\right)$.*

In the preceding lemmas and theorem we have seen that with smoother kernel of the integral operator we get a solution with order of smoothness provided that the initial data too are sufficiently smooth. However we have discussed only the case where solutions with order of smoothness a positive integer and not a fractional order. The next lemma and the theorem that follows are about how we can find a solution of any positive order s with a kernel $\alpha \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$.

Theorem 2.4.4. *Let $u \in L^2(\Omega)$ and $\alpha \in H^s(\mathbb{R}) \cap L^1(\mathbb{R}), \quad s \geq 0$. Then $L_\alpha \in H^s(\Omega)$ and the operator, $L_\alpha : L^2(\Omega) \to H^k(\Omega)$ is continuous.*

Proof.
- Case 1. For $s = 0$,That is, $\alpha \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, then $L_\alpha u \in L^2(\Omega)$. Indeed,

$$\begin{aligned}
\|L_\alpha u\|^2_{L^2(\Omega)} &= \int_\Omega \left|\left(\int_\Omega \alpha(x-y)u(y)dy\right)\right|^2 dx \\
&\leq \int_\Omega \left(\int_\Omega |\alpha(x-y)||u(y)|dy\right)^2 dx \\
&\leq \int_\Omega \left(\int_\Omega |\alpha(x-y)|^2 dy\right)\left(\int_\Omega |u(y)|^2 dy\right) dx \\
&\leq \|u\|^2_{L^2(\Omega)} \int_\Omega \int_\Omega |\alpha(x-y)|^2 dydx \\
&= \mu(\Omega)\|\alpha\|^2_{L^2(\mathbb{R})}\|u\|^2_{L^2(\Omega)}.
\end{aligned}$$

- Case 2. $s \in \mathbb{N}$.
  In this case $D^r \alpha \in L^2(\mathbb{R}), \quad \forall \quad 0 \leq r \leq s$. This implies that

$$D^r L_\alpha u = L_{D^r \alpha} u \in L^2(\Omega) \quad \forall \quad 0 \leq r \leq s.$$

  Hence $L_\alpha u \in H^s(\Omega)$ by Case 1 and definition of $H^s(\Omega)$.

- Case 3. For $0 < s < 1$.
  We have $\alpha \in L^2(\mathbb{R})$ and by Case 1 $L_\alpha u \in L^2(\Omega)$. Next we show that the Gagliardo semi-norm $[L_\alpha u]_{H^s(\Omega)} < \infty$. In fact;

$$\begin{aligned}
[L_\alpha u]^2_{H^s(\Omega)} &= \int_\Omega \int_\Omega \frac{|L_\alpha u(x) - L_\alpha u(y)|^2}{|x-y|^{1+2s}} dxdy \\
&\leq \int_\Omega \int_\Omega \frac{|\int_\Omega (\alpha(x-z) - \alpha(y-z))u(z)dz|^2}{|x-y|^{1+2s}} dxdy \\
&\leq \int_\Omega \int_\Omega \frac{(\int_\Omega |u(z)|^2 dz)(\int_\Omega |\alpha(x-z) - \alpha(y-z)|^2 dz)}{|x-y|^{1+2s}} dxdy \\
&\leq \|u\|^2_{L^2(\Omega)} \int_\Omega \left(\int_\mathbb{R} \int_\mathbb{R} \frac{|\alpha(x-z) - \alpha(y-z)|^2}{|x-y|^{1+2s}} dxdy \right) dz \\
&= \|u\|^2_{L^2(\Omega)} \mu(\Omega) [\alpha]^2_{H^s(\mathbb{R})}.
\end{aligned}$$

- Case 4. For arbitrary $s > 0$.
  In this case s can be written as $s = k + \kappa$, where k is a nonnegative integer and $0 < \kappa < 1$.Then,

$$\alpha \in H^s(\mathbb{R}) = H^{k+\kappa}(\mathbb{R}) \subset H^k(\mathbb{R}).$$

  Hence, $L_\alpha u \in H^k(\mathbb{R})$ by Case 2 and $D^k L_\alpha u \in H^\kappa(\mathbb{R})$ by Case 3. Consequently, we have $L_\alpha u \in H^s(\mathbb{R})$.

From Cases 1,2,3 and 4 we have the following norm estimate

$$
\begin{aligned}
\|L_\alpha u\|_{H^s(\Omega)} &= \left(\|L_\alpha u\|^2_{H^k(\Omega)} + [L_\alpha u]^2_{H^s(\Omega)}\right)^{1/2}\\
&= \left(\sum_{r=0}^{k} \|D^r L_\alpha u\|^2_{L^2(\Omega)} + [D^k L_\alpha u]^2_{H^\kappa(\Omega)}\right)^{1/2}\\
&= \left(\sum_{r=0}^{k} \|L_{D^r\alpha} u\|^2_{L^2(\Omega)} + [L_{D^k\alpha} u]^2_{H^\kappa(\Omega)}\right)^{1/2}\\
&\leq \left(\sum_{r=0}^{k} |\Omega|\|D^r\alpha\|^2_{L^2(\Omega)}\|u\|^2_{L^2(\Omega)} + |\Omega|[D^k\alpha]^2_{H^\kappa(\mathbb{R})}\|u\|^2_{L^2(\Omega)}\right)^{1/2}\\
&= \sqrt{|\Omega|}\|\alpha\|_{H^s(\mathbb{R})}\|u\|_{L^2(\Omega)}.
\end{aligned}
$$

We also have,

$$
\|L_\alpha u\|_{L^\infty(\Omega)} \leq \sqrt{\mu(\Omega)}\|\alpha\|_{L^2(\mathbb{R})}\|u\|_{L^\infty(\Omega)}
$$

□

Theorem 2.4.5. *For initial data $\phi, \psi \in H^s(\Omega) \cap L^\infty(\Omega), s \geq 0$ and $\alpha \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$, there exists a unique solution $u \in C^2\big([0,T); H^s(\Omega) \cap L^\infty(\Omega)\big)$ of problem (2.1.1) defined on some maximal interval of existence $[0,T)$. The solution depends on the initial data $u(x,0) = \phi(x), u_t(x,0) = \psi(x)$. If $T < \infty$, then*

$$
\limsup_{t\uparrow T} \left(\|u(.,t)\|_{H^s(\Omega)} + \|u_t(.,t)\|_{H^s(\Omega)} \|u(.,t)\|_{L^\infty(\Omega)} + \|u_t(.,t)\|_{L^\infty(\Omega)}\right) = \infty.
$$

Proof. We may write problem in (2.1.1) as a system of first order ODEs in the Banach space, $Y_s = H^s(\Omega) \cap L^\infty(\Omega)$. Let $u_t(x,t) := v(xt)$ .We can transform the problem (2.1.1) into a system of ODE

$$
\mathbf{U}_t(x,t) = \mathbf{F}(\mathbf{U}(x,t)), \mathbf{U}(x,0) = U_0,
$$

where,

$$\mathbf{U}(x,t) = \begin{bmatrix} u(x,t) \\ v(x,t) \end{bmatrix}$$

and

$$\mathbf{F}(\mathbf{U}) = \begin{bmatrix} f_1(\mathbf{U}(x,t)) \\ f_2(\mathbf{U}(x,t)) \end{bmatrix} = \begin{bmatrix} u(x,t) \\ L_\alpha g(u(x,t)) \end{bmatrix}$$

with the corresponding initial condition,

$$\mathbf{U}_0(x,t) = \begin{bmatrix} u(x,0) \\ v(x,0) \end{bmatrix} = \begin{bmatrix} \phi(x) \\ \psi(x) \end{bmatrix}.$$

By Theorem 2.4.4 we have

$$\begin{aligned} \|L_\alpha g(u_2) - L_\alpha g(u_2)\|_{Y_T(\Omega)} &= \|L_\alpha(g(u_2) - g(u_1))\|_{Y_T(\Omega)} \\ &\le \|\alpha\|_{H^s(\mathbb{R})}\|g(u_2) - g(u_1)\|_{Y_T(\Omega)} \\ &\le \|\alpha\|_{H^s(\mathbb{R})} M\|u_2 - u_1\|_{Y_T(\Omega)}, \end{aligned} \tag{2.4.11}$$

For all $u_1, u_2 \in Y_T(\Omega)$. We deduce that above ODE system is locally Lipschitz. Hence from the results from classical Picard theorem [38], we conclude the well-posedness of the problem. □

Theorem 2.4.6. *Let the initial data $\phi, \psi \in H^s(\Omega)$, $s > 1/2$ and $u(x,t)$ be the solution of (2.1.1) in the maximal interval of existence $[0,T)$. Then $T < \infty$ if and only if $\sup_{t \uparrow T} \|u_t(.,t)\|_{L^\infty(\Omega)} = \infty$.*

Proof. For $s > \frac{1}{2}, \Omega \subset \mathbb{R}$ we have the embedding,

$$H^s(\Omega) \hookrightarrow L^\infty(\Omega). \tag{2.4.12}$$

By composition theorem for Sobolev spaces we have,

$$\|g(u)\|_{H^s(\Omega)} \le C\|u\|_{H^s(\Omega)},$$

where C depends only on $\|u\|_{L^\infty(\Omega)}$. Now,

$$\begin{aligned}
\frac{d}{dt}(\|u\|_{H^s}^2 + \|v\|_{H^s}^2) &= 2\langle u, v\rangle_{H^s} + 2\langle v, v_t\rangle_{H^s} \\
&= 2\langle u, v\rangle_{H^s} + 2\langle v, L_\alpha g(u)\rangle_{H^s} \\
&= 2\|u\|_{H^s}\|v\|_{H^s} + 2\|v\|\|L_\alpha g(u)\|_{H^s} \\
&= 2\|u\|_{H^s}\|v\|_{H^s} + 2\|\alpha\|_{L^1(\mathbb{R})}\|v\|_{H^s}\|g(u)\|_{H^s} \\
&= 2\|u\|_{H^s}\|v\|_{H^s} + 2C\|\alpha\|_{L^1(\mathbb{R})}\|v\|_{H^s}\|g(u)\|_{H^s} \\
&\le (1 + C\|\alpha\|_{L^1(\mathbb{R})})(\|u\|_{H^s}^2 + \|v\|_{H^s}^2).
\end{aligned}$$

Therefore,

$$\frac{d}{dt}(\|u\|_{H^s}^2 + \|v\|_{H^s}^2) \le (1 + 2C\|\alpha\|_{L^1(\mathbb{R})})(\|u\|_{H^s}^2 + \|v\|_{H^s}^2).$$

Application of Gronwall's lemma verifies that the H^s- norms of u and v and consequently the L^∞-norms of u and v, do not blow up in finite time. By theorem — ,this implies that $T = \infty$ assuring global existence of solution. $\square$

We may summarize to a class of functions which includes the ones discussed above for which under certain conditions the global existence works. For initial data $\phi, \psi \in H^s(\Omega)$, $s \ge 1$, let us introduce the class

$$\mathcal{W} = \left\{ w : \mathbb{R}_+ \to (0, \infty), w \text{ is nondecreasing and } \int_1^\infty \frac{ds}{w(s)} = +\infty \right\}$$

Theorem 2.4.7. *Let $\alpha \in W_1^1(\mathbb{R})$ and function $g \in C^\infty(\mathbb{R})$ satisfy $g(0) = 0$ with*

$$|g(x)|^2 \le w(x^2),$$

for some $w \in \mathcal{W}$. Then for all $\phi, \psi \in H^1(\Omega)$ any local solution $u \in C^2([0,T); H^s(\Omega))$ of (2.1.1) is a global solution of (2.1.1), that is $T = \infty$.

Proof. We have

$$\|u\|^2_{H^1(\Omega)} = \int_\Omega (u^2 + u_x^2)dx, \quad \|v\|^2_{H^1(\Omega)} = \int_\Omega (v^2 + v_x^2)dx.$$

So,

$$\begin{aligned} \frac{d}{dt}\|u\|^2_{H^1(\Omega)} &= \int_\Omega (2uu_t + 2u_x u_{xt})dx = \int_\Omega (2uv + 2u_x v_x)dx \\ &\leq \int_\Omega (u^2 + v^2 + u_x^2 + v_x^2)dx \\ &= \int_\Omega (u^2 + u_x^2)dx + \int_\Omega (v^2 + v_x^2)dx \\ &= \|u\|^2_{H^1(\Omega)} + \|v\|^2_{H^1(\Omega)}, \end{aligned} \tag{2.4.13}$$

and

$$\begin{aligned} \frac{d}{dt}\|v\|^2_{H^1(\Omega)} &= \int_\Omega (2vv_t + 2v_x v_{xt})dx \\ &\leq \int_\Omega (v^2 + v_t^2 + v_x^2 + v_{xt}^2)dx \\ &= \int_\Omega (v^2 + v_x^2)dx + \int_\Omega (v_t^2 + v_{xt}^2)dx \\ &= \|v\|^2_{H^1(\Omega)} + \|v_t\|^2_{H^1(\Omega)}. \end{aligned} \tag{2.4.14}$$

However from (2.1.1) and Theorem 2.4.1 it follows that

$$\begin{aligned}\|v_t\|^2_{H^1(\Omega)} &= \|L_\alpha g(u)\|^2_{H^1(\Omega)} \leq C_\alpha^2 \|g(u)\|^2_{L^2(\Omega)} \\ &= C_\alpha^2 \int_\Omega |g(u)|^2 dx \leq C_\alpha^2 \int_\Omega w(u^2) dx \\ &\leq \mu(\Omega) C_\alpha^2 w(\|u\|^2_\infty) \leq \mu(\Omega) C_\alpha^2 w(\|u\|^2_{H^1(\Omega)}) \\ &\leq \mu(\Omega) C_\alpha^2 w(\|u\|^2_{H^1(\Omega)} + \|v\|^2_{H^1(\Omega)})\end{aligned} \tag{2.4.15}$$

Now from (2.4.14) and (2.4.15)we have,

$$\frac{d}{dt}\|v\|^2_{H^1(\Omega)} \leq \|v\|^2_{H^1(\Omega)} + \mu(\Omega) C_\alpha^2 w(\|u\|^2_{H^1(\Omega)} + \|v\|^2_{H^1(\Omega)}). \tag{2.4.16}$$

From (2.4.13) and (2.4.16) we get the differential inequality,

$$\frac{d}{dt} W(t) \leq \tilde{C}_\alpha (W(t) + w(W(t))) \tag{2.4.17}$$

where $W(t) := \|u(.,t)\|^2_{H^1(\Omega)} + \|v(.,t)\|^2_{H^1(\Omega)}$ and $\tilde{C}_\alpha := \max\{2, \mu(\Omega) C_\alpha^2\}$. Integrating from 0 to t both sides of the differential inequality (2.4.17), we get the integral inequality

$$W(t) \leq W(0) + \int_0^t \tilde{C}((W(s) + w(W(s)))ds, \tag{2.4.18}$$

where $W(0) := \|\phi\|^2_{H^1(\Omega)} + \|\psi\|^2_{H^1(\Omega)}$.
Applying Bihari-LaSalle inequality[5] we get,

$$W(t) \leq G^{-1}(G(W(0) + \tilde{C}_\alpha t),$$

or equivalently

$$\|u\|^2_{H^1(\Omega)} + \|v\|^2_{H^1(\Omega)} \leq G^{-1}\left(\tilde{C}_\alpha t + G\left(\|\phi\|^2_{H^1(\Omega)} + \|\psi\|^2_{H^1(\Omega)}\right)\right), \tag{2.4.19}$$

where
$$G(r) = \int_0^r \frac{ds}{s + w(s)}, \quad r \geq 0,$$
The last inequality guarantees the $H^1(\Omega)$-norms of u and u_t remain finite on every finite time interval $[0, T]$. Consequently,

$$\limsup_{t \uparrow T} \|u(., t)\|_\infty < \infty$$

and
$$\limsup_{t \uparrow T} \|u_t(., t)\|_\infty < \infty$$
for every finite time $T > 0$. $\square$

2.5 Energy Identity and Global Existence

We write equation (2.1.1) as

$$Pu_{tt} = -g(u) \tag{2.5.1}$$

where we assumed at this point that the integral operator L_α is negative symmetric and that
$$P := -L_\alpha^{-1}, \tag{2.5.2}$$
is a symmetric positive operator. By symmetry of P we mean that $\langle Pu, v \rangle = \langle u, Pv \rangle$ for all u, v in the range $R(L_\alpha)$ of the integral operator L_α. Basically, we assume that the kernel α is symmetric. That is,

$$\alpha(x - y) = \alpha(y - x). \tag{2.5.3}$$

Consequently the integral operator L_α is symmetric. By positivity of P we mean that
$$\langle Pu, u \rangle \geq 0, \quad \forall u \in R(L_\alpha). \tag{2.5.4}$$

By the conditions in (2.5.2) and (2.5.4), we have L_α is negative operator. That is,

$$\langle L_\alpha u, u\rangle \leq 0, \quad \forall u \in L^2(\Omega). \tag{2.5.5}$$

For $\alpha \in W^{1,1}(\mathbb{R})$, we have the operator P defined on the range $R(L_\alpha)$ of the integral operator L_α, that is ,

$$P : R(L_\alpha) \subset H^1(\Omega) \to L^2(\Omega). \tag{2.5.6}$$

We also have a norm $\|.\|_P$ defined on the range $R(L_\alpha)$ given by

$$u(t) \mapsto \langle Pu(t), u(t)\rangle^{\frac{1}{2}}_{L^2(\Omega)} := \|u(t)\|_P, \quad t \in [0, T). \tag{2.5.7}$$

Lemma 2.5.1. *Assume that,* $G(x) = \int_0^x g(s)ds, \quad Pu \in L^2(\Omega), \quad G \in L^1(\Omega)$ *and let* $u \in C^2([0, T); L^2(\Omega) \cap L^\infty(\Omega))$ *be the solution of problem (2.1.1). Then the quantity* $E(t)$ *given by*

$$E(t) = \frac{1}{2}\langle Pu_t, u_t\rangle_{L^2(\Omega)} + \int_\Omega G(u)dx \tag{2.5.8}$$

is a constant. Specifically we have

$$E(t) = \frac{1}{2}\langle P\psi, \psi\rangle_{L^2(\Omega)} + \int_\Omega G(\phi)dx = E(0), \quad t \in [0, T).$$

Proof. The proof follows by multiplying both sides of (2.5.1) by u_t and integrating over the domain Ω. $\square$

Sometimes it may be important to consider a nonlinear function g in problem (2.1.1) that begins with linear term u. For this case we may write $g(u) = u + f(u)$. To keep the characterization of g as smooth function with $g(0) = 0$, we set f to be smooth and that $f(0) = 0$ as well. This introduces a positive term $\frac{1}{2}\|u\|^2_{L^2(\Omega)}$ in the corresponding energy identity. So that new energy identity, which is a conserved quantity, may be written as

$$E(t) = \frac{1}{2}\langle Pu_t, u_t\rangle_{L^2(\Omega)} + \frac{1}{2}\|u\|^2_{L^2(\Omega)} + \int_\Omega F(u)dx. \tag{2.5.9}$$

In particular,

$$E(t) = E(0) = \frac{1}{2}\langle P\psi, \psi\rangle_{L^2(\Omega)} + \frac{1}{2}\|\phi\|^2_{L^2(\Omega)} + \int_\Omega F(\phi)dx, \quad t \in [0, T),$$

where, $F(x) = \int_0^x f(s)ds, \quad Pu_t \in L^2(\Omega), \quad F \in L^1(\Omega)$.

Theorem 2.5.2. *Let $\alpha \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$ and initial data $\phi, \psi \in H^s(\Omega) \cap L^\infty(\Omega), \quad s \geq 1$. Let u be a solution to problem (2.1.1) on the maximum interval of existence $[0, T)$. If $\int_\Omega G(u)dx \geq 0$, the quantity $\langle Pu(t), u(t)\rangle_{L^2(\Omega)}$ does not blow up in every finite time interval. In addition, if we have either the condition*

$$k \int_\Omega G(u(x,t))dx \geq \|u(.,t)\|_{L^2(\Omega)}, \quad t \in [0, T) \tag{2.5.10}$$

for some $k > 0$, or the condition

$$\|u(.,t)\|_{L^2(\Omega)} \leq \langle Pu(.,t), u(.,t)\rangle^{1/2}_{L^2(\Omega)}, \quad t \in [0, T) \tag{2.5.11}$$

then the solution is global in time, i.e. $T = \infty$.

Proof. From the energy identity (2.5.8) and the given additional condition that $\int_\Omega G(u)dx \geq 0$ on $[0, T)$, we have

$$\frac{1}{2}\langle Pu_t, u_t\rangle_{L^2(\Omega)} \leq E(0). \tag{2.5.12}$$

By the assumption that P is positive symmetric, using Cauchy-Schwartz inequality and equation (2.5.12), we have the inequality

$$\langle Pu, u_t\rangle \leq \langle Pu, u\rangle^{1/2}_{L^2(\Omega)}\langle Pu_t, u_t\rangle^{1/2}_{L^2(\Omega)} \leq \sqrt{2E(0)}\langle Pu, u\rangle^{1/2}_{L^2(\Omega)}$$

Rewriting the inequality as

$$\frac{d\langle Pu, u\rangle_{L^2(\Omega)}}{2\sqrt{\langle Pu, u\rangle_{L^2(\Omega)}}} \leq \sqrt{2E(0)}dt$$

and integrating over the interval $[0, t], \quad 0 < t < T$, we get

$$\langle Pu(t), u(t)\rangle_{L^2(\Omega)} \leq \left(\sqrt{\langle P\phi, \phi\rangle_{L^2(\Omega)}} + \sqrt{2E(0)}t\right)^2. \tag{2.5.13}$$

This shows that the quantity $\langle Pu(.,t), u(.,t)\rangle_{L^2(\Omega)}$ does not blow-up in a finite time interval. Let $u(x,t)$ a solution of (2.1.1). If we have additional condition given by inequality (2.5.10) then by Theorem 2.4.4, given condition (2.5.10) and energy identity (2.5.8) we have

$$\begin{aligned} \|L_\alpha g(u)\|_{H^1(\Omega)} &\leq \|\alpha\|_{H^s(\mathbb{R})}\|g(u)\|_{L^2(\Omega)} \\ &\leq \|\alpha\|_{H^s(\mathbb{R})}M(t)\|u\|_{L^2(\Omega)} \\ &\leq \|\alpha\|_{H^s(\mathbb{R})}M(t)k\int_\Omega G(u)dx \\ &\leq \|\alpha\|_{H^s(\mathbb{R})}M(t)kE(0) \end{aligned} \tag{2.5.14}$$

Now by virtue of the equations (2.2.2) and (2.2.1), we have the $\|u\|_{H^1(\Omega)}$ and $\|u_t\|_{H^1(\Omega)}$ are bounded over any finite interval of the form $[0, T)$. This assures the global existence of solution.
Now if we have the condition,

$$\|u(.,)\|_{L^2(\Omega)} \leq \langle Pu(.,t), u(.,t)\rangle^{1/2}_{L^2(\Omega)}, \tag{2.5.15}$$

By theorem (2.4.4) and given condition (2.5.15) we have,

$$
\begin{aligned}
\|L_\alpha g(u)\|_{H^s(\Omega)} &\leq \|\alpha\|_{H^s(\mathbb{R})}\|g(u)\|_{L^2(\Omega)} \\
&\leq \|\alpha\|_{H^s(\mathbb{R})}M(t)\|u\|_{L^2(\Omega)} \\
&\leq \|\alpha\|_{H^s(\mathbb{R})}M(t)\langle Pu(.,t),u(.,t)\rangle_{L^2(\Omega)}^{1/2} \\
&\leq \|\alpha\|_{H^s(\mathbb{R})}M(t)\left(\sqrt{\langle P\phi,\phi\rangle_{L^2(\Omega)}}+\sqrt{2E(0)}t\right). \qquad (2.5.16)
\end{aligned}
$$

Again by virtue of the equations (2.2.2) and (2.2.1) we have $\|u\|_{H^1(\Omega)}$ and $\|u_t\|_{H^1(\Omega)}$ are bounded over any finite interval of the form $[0,T)$. This assures the global existence of solution.

□

Particular examples of function g satisfying the condition in Theorem 2.5.2 are $g(x)=\sinh(x)$ and $g(x)=ax+f(x)$ where,

$$
a>0,\ f(0)=0,\ F(x)=\int_0^x f(r)dr,\ \int_\Omega F(u)dx \geq 0.
$$

2.6 Blow-up in Finite Time

In this section we discuss the case of non continuation of a solution beyond some finite time. We use an indirect argument by showing the non continuation of some positive quantity $\langle Pu,u\rangle$ defined on the range $R(L_\alpha)$ of the integral operator (2.1.2). The following classical result is of ultimate importance.

Lemma 2.6.1 (**Levine Lemma**). *Suppose that for $t\geq 0$, a positive, twice differentiable function $I(t)$ satisfies the inequality*

$$
I(t)I''(t)-\mu I'(t)^2 \geq 0,
$$

where $\mu>0$ is a constant. If $I(0)>0$ and $I'(0)>0$, then $I(t)\to\infty$ as $t\to t_1\leq \frac{I(0)}{\mu I'(0)}$, for some $0\leq t_1\leq \frac{I(0)}{\mu I'(0)}$.

Now we state a theorem on the non-continuation condition of solution u of (2.1.1) when the initial energy $E(0)$ is negative.

Theorem 2.6.2. *Let condition $ug(u) \leq qG(u)$ be satisfied for some $q > 2$. Then no solution u of problem (2.1.1) exist on some interval $J = [0, \infty)$ when the initial energy $E(0) < 0$.*

Proof. Let u be a solution on of (2.1.1) on the interval $J = [0, \infty)$. Define

$$I(t) = \frac{1}{2}\langle Pu, u\rangle + \beta(t), \tag{2.6.1}$$

where $\beta(t)$ is a positive twice differentiable function that we determine latter in the proof. So,

$$I'(t) = \langle Pu, u_t\rangle + \beta'(t). \tag{2.6.2}$$

From the given condition $ug(u) \leq qG(u)$ we have, $-\langle u, g(u)\rangle \geq -q\int_\Omega G(u)dx$. Therefore,

$$\begin{aligned} I''(t) &= \langle Pu_t, u_t\rangle + \langle Pu, u_{tt}\rangle + \beta''(t) \\ &= \langle Pu_t, u_t\rangle + \langle Pu, L_\alpha g(u)\rangle + \beta''(t) \\ &= \langle Pu_t, u_t\rangle - \langle u, g(u)\rangle + \beta''(t) \\ &= \langle Pu_t, u_t\rangle - q\int_\Omega G(u)dx + \beta''(t). \end{aligned} \tag{2.6.3}$$

But from the energy identity (2.5.8) we have,

$$-q\int_\Omega G(u)dx = \frac{q}{2}\langle Pu_t, u_t\rangle - qE(0).$$

We therefore have

$$I''(t) \geq (1 + \frac{q}{2})\langle Pu_t, u_t\rangle - qE(0) + \beta''(t). \tag{2.6.4}$$

Suppose that $E(0) < 0$, if we set,

$$\beta(t) = \beta_0(t + t_0)^2$$

where $\beta_0 = |E(0)|$ and $t_0 > 0$,then

$$-qE(0) + \beta''(t) = (q + 2)|E(0)| = 4\beta_0(1 + \mu).$$

We can observe that $I(t) > 0, \quad t \geq 0$, and consequently

$$I(0) = \frac{1}{2}\langle P\phi, \phi\rangle + \beta_0 t_0^2 > 0.$$

If the constant t_o is chosen sufficiently large, say for example,

$$t_0 > \frac{|\langle P\phi, \psi\rangle|}{2|E(0)|},$$

we have

$$I'(0) = \langle P\phi, \psi\rangle + 2\beta_0 t_0 > 0.$$

Up to this point we have shown that $I(t)$ satisfies two of the conditions in Levine lemma. Namely, $I(0) > 0$ and $I'(0) > 0$. It suffices to show that the quantity $I(t)I''(t) - (1+\mu)[I'(t)]^2 \geq 0$ to conclude that the quantity $I(t)$ and consequently, the quantity $\langle Pu, u\rangle$, blows up in finite time. We have

$$\begin{aligned}
H(t) \geq & \left(\frac{1}{2}\langle Pu, u\rangle + \beta(t)\right)\left((1 + \frac{q}{2})\langle Pu_t, u_t\rangle - qE(0) + \beta''(t)\right) \\
& - (1 + \mu)\left[\langle Pu, u_t\rangle + \beta'(t)\right]^2 \\
& = \frac{1}{2}(1 + \frac{q}{2})\langle Pu, u\rangle\langle Pu_t, u_t\rangle - (1 + \mu)\langle Pu, u_t\rangle^2 \\
& \left(-\frac{1}{2}qE(0) + \frac{1}{2}\beta''(t)\right)\langle Pu, u\rangle + (1 + \frac{q}{2})\beta(t)\langle Pu_t, u_t\rangle \\
& - 2(1 + \mu)\beta'(t)\langle Pu, u_t\rangle - qE(0)\beta(t) + \beta(t)\beta''(t) - (1 + \mu)[\beta'(t)]^2
\end{aligned}$$

If we let $\frac{1}{2}(1+\frac{q}{2}=(1+\mu)$, we get

$$H(t) \geq (1+\mu)[\langle Pu, u\rangle\langle Pu_t, u_t\rangle - \langle Pu, u_t\rangle^2] \\ + 2(1+\mu)\beta_0[\langle P(u-(t+t_0)u_t), u-(t+t_0)u_t\rangle] \geq 0$$

By Cauchy-Schwartz inequality the first term in bracket is nonnegative. And by the positive definiteness of the operator P, the expression in the second bracket is also non negative. Consequentiality the result follows by Levine lemma 2.6.1. $\square$

Chapter 3

Nonlinear Nonlocal Wave Type Problem

3.1 Description of the problem

In this chapter we consider a nonlinear nonlocal wave equation with nonlinearity and nonlocality terms are combined with an integral operator with kernel α. The problem is given as

$$\begin{cases} u_{tt} - \Delta u = L_\alpha g(u), \quad x \in \Omega, \quad t > 0, \\ u(x,0) = \phi(x), u_t(x,0) = \psi(x), \quad x \in \Omega, \\ u(x,t) = 0, \quad x \in \partial\Omega \quad t > 0, \end{cases} \tag{3.1.1}$$

where $\alpha \in L^1(\mathbb{R})$ is a known function, g is a given nonlinear function, the integral operator L_α defined as given by (2.1.2), and u is the unknown function.

For problem (3.1.1) and the current chapter, and the related problem (4.1.1) given in the next chapter, we set the nonlinearity term $g(u)$ to be power type nonlinearity given by

$$g(u) = |u|^{p-1}u, \quad p > 1. \tag{3.1.2}$$

This is due to the simplicity that we may discover later in the chapters. Some other functions with power type nonlinearities may take anyone of the forms

$$g(u) = \pm|u|^p, \quad g(u) = \pm u^p, \quad g(u) = \pm|u|^{p-1}u, \quad p > 1.$$

As desired by our solution, $g(u) = |u|^{p-1}u, \quad p > 1$, satisfies the following conditions.

i. $g \in C^1(\mathbb{R})$ and $g(0) = g'(0) = 0$.

ii. $G(u) := \int_0^u g(s)ds = \frac{1}{p+1}|u|^{p+1}$.

iii. $g(u)$ is monotone and convex for $u > 0$, and concave for $u < 0$.

iv. $(p+1)G(u) = ug(u)$.

3.2 Local well-posedness

Equation (3.1.1) may be written in an equivalent operator form which is an integral equation as

$$u(x,t) = \mathcal{A}(t)\phi(x) + \mathcal{B}(t)\psi(x) + \int_0^t \mathcal{B}(t-\tau)L_\alpha g(u(x,\tau))d\tau. \tag{3.2.1}$$

For notational simplicity, in (3.2.1) we have used the operator notations $\mathcal{A}$ and $\mathcal{B}$ meaning:

$$\mathcal{A}(t) := \cos(t\sqrt{-\Delta_D}), \quad \mathcal{B}(t) := \frac{\sin(t\sqrt{-\Delta_D})}{\sqrt{-\Delta}}. \tag{3.2.2}$$

According to (1.4.3), the operational definitions the operators given in (3.2.2), are given as follows. For any $u \in L^2(\Omega)$ with eigenfunction expansion given by (1.4.4),

$$\mathcal{A}(t)u = \sum_{n=1}^{\infty} \cos(t\sqrt{\lambda_n})u_n\varphi_n, \tag{3.2.3}$$

$$\mathcal{B}(t)u = \sum_{n=1}^{\infty} \frac{\sin(t\sqrt{\lambda_n})}{\sqrt{\lambda_n}}u_n\varphi_n. \tag{3.2.4}$$

We later see that definitions given in (3.2.3) and (3.2.4) are well defined. The integral representation for the first order time derivative u_t is given by

$$u_t(x,t) = (-\Delta_D)\mathcal{B}(t)\phi(x) + \mathcal{A}(t)\psi(x) + \int_0^t \mathcal{A}(t-\tau)L_\alpha g(u(x,\tau))d\tau. \tag{3.2.5}$$

In (3.2.5) we introduced the operational definition,

$$(-\Delta_D)\mathcal{B}(t) = -\sqrt{-\Delta_D}\sin(t\sqrt{-\Delta}). \tag{3.2.6}$$

Next, we make the analysis of the norm estimates of each of the terms involved in the integral equations (3.2.1) and (3.2.5).

The zero Dirichlet boundary conditions are satisfied by each of the eigenfunctions $\varphi_n(x)$ of the positive Dirichlet Laplace operator $-\Delta_D$. We have the following important mapping properties of the operators $\mathcal{A}(t)$ and $\mathcal{B}(t)$.

Theorem 3.2.1. *Each of the following mappings of the operator $\mathcal{A}(t)$ and $\mathcal{B}(t)$ between the indicated pair of spaces are continuous :*

$$\begin{aligned}
&\mathcal{A}(t) : L^2(\Omega) \to L^2(\Omega),\\
&\mathcal{A}(t) : H^1(\Omega) \to H_0^1(\Omega),\\
&\mathcal{A}(t) : H^k(\Omega) \to H_0^1(\Omega) \cap H^k(\Omega), k > 1,\\
&\mathcal{B}(t) : L^2(\Omega) \to H_0^1(\Omega),\\
&\mathcal{B}(t) : H^1(\Omega) \to H_0^1(\Omega) \cap H^2(\Omega),\\
&\mathcal{B}(t) : H^k(\Omega) \to H_0^1(\Omega) \cap H^{k+1}(\Omega), k > 0.
\end{aligned} \tag{3.2.7}$$

Proof. Let $u \in L^2(\Omega)$. By (1.4.4), (3.2.3) and (1.4.6) we have

$$\|\mathcal{A}(t)u\|^2_{L^2(\Omega)} = \sum_{i=1}^{\infty} \cos^2(t\sqrt{\lambda_i})v_i^2 \leq \sum_{i=1}^{\infty} u_i^2 = \|u\|^2_{L^2(\Omega)}$$

So that

$$\|\mathcal{A}(t)u\|_{L^2(\Omega)} \leq \|u\|_{L^2(\Omega)}. \tag{3.2.8}$$

If $u \in H^1(\Omega)$, then $\|u\|^2_{H^1(\Omega)} = \sum_{i=1}^{\infty} \lambda_i u_i^2 < \infty$. Hence,

$$\|\mathcal{A}(t)u\|^2_{H^1(\Omega)} = \sum_{i=1}^{\infty} \cos^2(t\sqrt{\lambda_i})\lambda_i u_i^2 \leq \sum_{i=1}^{\infty} \lambda_i u_i^2 = \|u\|^2_{H^1(\Omega)}.$$

Consequently,

$$\|\mathcal{A}(t)u\|_{H^s(\Omega)} \leq \|u\|_{H^s(\Omega)}. \tag{3.2.9}$$

If $u \in H^k(\Omega), k > 0$, then, $\|u\|^2_{H^k(\Omega)} = \sum_{i=1}^{\infty} \lambda_i^k u_i^2 < \infty$ and

$$\|\mathcal{A}(t)u\|^2_{H^k(\Omega)} = \sum_{i=1}^{\infty} \cos^2(t\sqrt{\lambda_i})\lambda_i^k u_i^2 \leq \sum_{i=1}^{\infty} \lambda_i^k u_i^2 = \|u\|^2_{H^k(\Omega)},$$

$$\|\mathcal{A}(t)u\|_{H^k(\Omega)} \leq \|u\|_{H^k(\Omega)}. \tag{3.2.10}$$

As a consequence of the operational definition given in (3.2.4) and the definition of H^1 norm given in (1.4.9), we have

$$\|\mathcal{B}(t)u\|^2_{H_0^1(\Omega)} = \sum_{i=1}^{\infty} \left(\frac{\sin(\sqrt{\lambda_i}t)}{\sqrt{\lambda_i}}\right)^2 \lambda_i u_i^2 \leq \sum_{i=1}^{\infty} u_i^2 = \|u\|^2_{L^2(\Omega)}$$

Consequently we have

$$\|\mathcal{B}(t)u\|_{H_0^1(\Omega)} \leq \|u\|_{L^2(\Omega)}. \tag{3.2.11}$$

By the operational definition given in (3.2.4) and the definition of the general

H^s norm given in (1.4.12), we have

$$\|\mathcal{B}(t)u\|^2_{H^2(\Omega)} = \sum_{i=1}^{\infty} \left(\frac{\sin(\sqrt{\lambda_i}t)}{\sqrt{\lambda_i}}\right)^2 \lambda_i^2 u_i^2 \leq \sum_{i=1}^{\infty} \lambda_i u_i^2 = \|u\|^2_{H^1(\Omega)}$$

and consequently

$$\|\mathcal{B}(t)u\|_{H^2(\Omega)} \leq \|u\|_{H^1(\Omega)}. \tag{3.2.12}$$

According to (1.4.12),

$$\|\mathcal{B}(t)u\|^2_{H^{k+1}(\Omega)} = \sum_{i=1}^{\infty} \left(\frac{\sin(t\sqrt{\lambda_i})}{\sqrt{\lambda_i}}\right)^2 \lambda_i^{k+1} v_i^2 = \sum_{i=1}^{\infty} \lambda_i^k u_i^2 = \|u\|^2_{H^k(\Omega)}.$$

Consequently

$$\|\mathcal{B}(t)u\|_{H^{k+1}(\Omega)} \leq \|u\|_{H^k(\Omega)}. \tag{3.2.13}$$

We observe that the operator $\mathcal{A}(t)$ and $\mathcal{B}(t)$ incorporate the Dirichlet boundary conditions while $\mathcal{B}(t)$, additionally, increases the order of smoothness by one. □

Theorem 3.2.2. *The mappings of the operator,*

$$\Delta_D \mathcal{B}(t) : H^{s+1}(\Omega) \to H^s(\Omega)$$

given as

$$u \mapsto \Delta_D \mathcal{B}(t)u = \sum_{n=1}^{\infty} -\sqrt{\lambda_n} \sin(t\sqrt{\lambda_n}) u_n \varphi_n(x)$$

is continuous.

Proof. If $u \in H^{s+1}(\Omega)$, then, $\|u\|_{H^{s+1}(\Omega)} = \sum_{i=1}^{\infty} \lambda_i^{s+1} u_i^2 < \infty$.

$$\Delta_D \mathcal{B}(t)u = \sum_{i=1}^{\infty} (-\sqrt{\lambda_i} \sin(t\sqrt{\lambda_i}) \varphi(x)$$

For $\Delta_D \mathcal{B}(t)u$, we have

$$
\begin{aligned}
\|\Delta_D \mathcal{B}(t)u\|^2_{H^s(\Omega)} &= \sum_{i=1}^{\infty} \lambda_i^s \left(-\sqrt{\lambda_i}\sin(t\sqrt{\lambda_i})u_i\right)^2 \\
&= \sum_{i=1}^{\infty} \lambda_i^{s+1} \sin^2(t\sqrt{\lambda_n})u_i^2 \\
&\leq \sum_{i=1}^{\infty} \lambda_i^{s+1} u_i^2 = \|u\|^2_{H^{s+1}(\Omega)}
\end{aligned}
$$

Therefore,

$$
\|\Delta_D \mathcal{B}(t)u\|_{H^s(\Omega)} \leq \|u\|_{H^{s+1}(\Omega)}, \tag{3.2.14}
$$

as required. □

Theorem 3.2.3. *The following mappings of the composition of operator $\mathcal{A}(t), \mathcal{B}(t)$ and L_α are continuous:*

$$
\begin{aligned}
&\mathcal{B}(t)L_\alpha : H_0^1(\Omega) \to H_0^1(\Omega), \\
&\mathcal{A}(t)L_\alpha : H_0^1(\Omega) \to H_0^1(\Omega), \\
&\mathcal{A}(t)L_\alpha : L^2(\Omega) \to L^2(\Omega)).
\end{aligned} \tag{3.2.15}
$$

Proof. From (2.4.1) we know that L_α maps $H_0^1(\Omega)$ into $H^1(\Omega)$ while from(3.2.7) we know that $\mathcal{B}(t)$ maps $H^1(\Omega)$ into $H_0^1(\Omega) \cap H^2(\Omega)$. Consequently, the composition of the continuous operators $\mathcal{B}(t)L_\alpha$ maps $H_0^1(\Omega)$ into itself. Similarly, the composition operator $\mathcal{A}(t)L_\alpha$ maps the space $H_0^1(\Omega)$ into itself. We also have the following norm estimates that follow from similar arguments.

$$
\begin{aligned}
&\|\mathcal{B}(t)L_\alpha u\|_{H_0^1(\Omega)} \leq \|\alpha\|_{L^1(\mathbb{R})} \|u\|_{H_0^1(\Omega}, \\
&\|\mathcal{A}(t)L_\alpha u\|_{H_0^1(\Omega)} \leq \|\alpha\|_{L^1(\mathbb{R})} \|u\|_{H_0^1(\Omega)}, \\
&\|\mathcal{A}(t)L_\alpha u\|_{L^2(\Omega)} \leq \|\alpha\|_{L^1(\mathbb{R})} \|u\|_{L^2(\Omega)}.
\end{aligned} \tag{3.2.16}
$$

This proves the claim. □

3.2.1 Weak Solutions

Definition 3.2.4. Let $\Omega_T := \Omega \times T$, where $\Omega \subset \mathbb{R}^n$, is an open connected set with smooth boundary $\partial\Omega$. A weak solution of the nonlocal nonlinear initial-boundary value problem (3.1.1) is any function u satisfying

$$u \in C([0,T), H_0^1(\Omega)) \cap C^1([0,T), L^2(\Omega) \cap L^\infty(\Omega)).$$

Moreover, it satisfies

$$\int_{\Omega_T} (u(x,s)v_{tt} - \nabla u(x,s)\nabla(x,s) - v(x,s)L_\alpha u(x,s) - g(x,s)v(x,s))dxds$$
$$= \int_\Omega (\phi(x)v(x,0) - \psi(x)v_t(x,s))\, dxds$$

for every $v \in C_c^\infty(\Omega_T \times (0,T))$.

We consider a power type nonlinearity for cases of space dimension $n \geq 2$ and a more general class of nonlinearity for space dimension $n = 1$. This is due to some embedding results that help to indicate where our desired solution should be. From Sobolev embedding theorem we have $H_0^1(\Omega) \subset L^{\frac{2n}{n-2}}(\Omega)$ if $n \geq 3$.However for $n = 2$, $\quad H_0^1(\Omega) \subset L^q(\Omega), \quad 1 \leq q < \infty$, and for $n = 1, \quad H_0^1(\Omega) \subset L^q(\Omega), \quad 1 \leq q \leq \infty$.

The case $n \geq 3$: Let $\phi \in H_0^1(\Omega), \quad \psi \in L^2(\Omega) \cap L^\infty(\Omega)$. We study the existence of a unique solution $u(x,t)$ of the nonlocal nonlinear initial-boundary value problem in the space

$$u \in C([0,T), H_0^1(\Omega)) \cap C^1([0,T), L^2(\Omega)).$$

We set the nonlinearity term $g(u)$ of power type as it is given by equation (3.1.2).

Lemma 3.2.5. *Let $\Omega \subset \mathbb{R}^n$ be abounded domain with measure $\mu(\Omega) < \infty$.*

We have the following embeddings,

$$H_0^1(\Omega) \hookrightarrow L^{2p}(\Omega) \hookrightarrow L^2(\Omega)$$

with

$$\|u\|_2 \leq \mu(\Omega)^{\frac{p-1}{2p}} \|u\|_{2p}, \quad \|u\|_{2p} \leq C_{2p} \|u\|_{H_0^1(\Omega)},$$

provided that

$$\begin{cases} 1 < p < \frac{n}{n-1} & \text{if} \quad n \geq 3, \\ 1 < p < \infty & \text{if} \quad n = 1, 2. \end{cases}$$

Proof. The first embedding follows from Sobolev embedding theorem. We have,

$$\|u\|_{L^{2p}(\Omega)} \leq C_{2p} \|u\|_{H_0^1(\Omega)}.$$

For the second embedding, since Ω is bounded domain, using Hölder's inequality, for any $u \in L^{2p}(\Omega)$ we get

$$\begin{aligned} \|u\|_2^2 = \int_\Omega |u(x)|^2 dx &\leq \left(\int_\Omega |u(x)|^{2p} dx \right)^{\frac{1}{p}} \left(\int_\Omega 1 dx \right)^{1-\frac{1}{p}} \\ &= \mu(\Omega)^{\frac{p-1}{p}} \|u\|_{2p}^2 \end{aligned} \tag{3.2.17}$$

which proves the second embedding. $\square$

Definition 3.2.6. Let X and Y be Banach spaces. A mapping $f : X \to Y$ is said to be locally Lipschitz if for each bounded subset U of X there exists a constant C_U with

$$\|f(u) - f(v)\|_Y \leq C_U \|u - v\|_X$$

for all $u, v \in U$.

Lemma 3.2.7. *Let $n \geq 3$. For $1 < p \leq \frac{n}{n-2}$, the mapping*

$$g : H_0^1(\Omega) \cap L^{2p}(\Omega) \to L^2(\Omega)$$

given by $u \mapsto g(u) = |u|^{p-1}u$ *is locally Lipschitz.*

Proof. Let $R > 0$. Define the set

$$B_R := \left\{ u \in H_0^1(\Omega) \cap L^{2p}(\Omega) : \|u\|_{H_0^1(\Omega)} \leq R \right\}$$

Then, we have

$$g(u) = |u|^{p-1}u = p \int_0^u |s|^{p-1} ds \tag{3.2.18}$$

For $u_1, u_2 \in B_R$ we have,

$$|g(u_2) - g(u_1)| = p \left| \int_{u_1}^{u_2} |s|^{p-1} ds \right| = p|u^*|^{p-1}|u_2 - u_1| \tag{3.2.19}$$

where, $u^* = \theta u_1 + (1-\theta)u_2$ for some $\theta \in (0,1)$, according to the mean value theorem. We also have $u^* \in H_0^1(\Omega) \cap L^{2p}(\Omega)$ and that

$$\begin{aligned}
\|u^*\|_{H_0^1(\Omega)} &= \|\theta u_1 + (1-\theta)u_2\|_{H_0^1(\Omega)} \\
&\leq \theta \|u_1\|_{H_0^1(\Omega)} + (1-\theta)\|u_2\|_{H_0^1(\Omega)} \\
&\leq \theta R + (1-\theta)R = R.
\end{aligned}$$

Consequently, $u^* \in B_R$, and from (3.2.19)

$$|g(u_2) - g(u_1)|^2 = p^2 |u^*|^{2(p-1)} |u_2 - u_1|^2$$

Finally by applying Hölder inequality with $\frac{1}{p}+\frac{1}{p'}=1$, we get

$$\begin{aligned}
\|g(u_2)-g(u_1)\|^2_{L^2(\Omega)} &= \int_\Omega |g(u_2(x))-g(u_1(x))|^2 dx \\
&= p^2\int_\Omega |u^*|^{2(p-1)}|u_1-u_2|^2 dx \\
&\le p^2\left(\int_\Omega |u^*|^{2(p-1)p'}dx\right)^{\frac{1}{p'}}\left(\int_\Omega |u_2-u_1|^{2p}dx\right)^{\frac{1}{p}} \\
&= p^2\|u^*\|^{2(p-1)}_{2p}\|u_2-u_1\|^2_{2p} \\
&\le p^2C^{2p}_{2p}\|u^*\|^{2(p-1)}_{H^1_0(\Omega)}\|u_2-u_2\|^2_{H^1_0(\Omega)} \\
&\le p^2C^{2p}_{2p}R^{2(p-1)}\|u_2-u_2\|^2_{H^1_0(\Omega)}
\end{aligned}$$

We therefore have

$$\|g(u_2)-g(u_1)\|_{L^2(\Omega)} \le pC^p_{2p}R^{p-1}\|u_2-u_2\|_{H^1_0(\Omega)}. \tag{3.2.20}$$

□

For the tasks that follow we define the following function space:

$$Y_T := C^1\left([0,T),H^1_0(\Omega)\right)\cap C\left([0,T),L^2(\Omega)\right) \tag{3.2.21}$$

with norm defined as

$$\|u\|_{Y_T} =: \max_{0\le\tau\le t}\|u(.,\tau)\|_{H^1_0(\Omega)} + \max_{0\le\tau\le t}\|u_t(.,\tau)\|_{L^2(\Omega)} \tag{3.2.22}$$

With the norm defined in (3.2.22), we notice that Y_T is a Banach Space. In connection to the integral equation (3.2.1), we define an integral operator $\mathcal{K}: Y_T \to Y_T$ given by

$$\mathcal{K}u(x,t) = \mathcal{A}(t)\phi(x) + \mathcal{B}(t)\psi(x) + \int_0^t \mathcal{B}(t-\tau)L_\alpha g(u(x,\tau))d\tau \tag{3.2.23}$$

For any initial data $\phi \in H_0^1(\Omega), \psi \in L^2(\Omega)$, let

$$M_0 := \|\phi\|_{H_0^1(\Omega)} + \|\nabla\phi\|_{L^2(\Omega)} + 2\|\psi\|_{L^2(\Omega)}.$$

Let us fix some bounded subset of Y_T which is a neighborhood of the initial data $\phi(x), \psi(x)$. Then we have

$$Y_T(M_0) = \{u| \quad u \in Y_T, \quad \|u\|_{Y_T} \leq M_0 + 1.\}$$

$Y_T(M_0)$ is a nonempty, bounded, closed and convex subset of Y_T. Our next task is to prove step by step that the integral operator $\mathcal{K}$ has a unique fixed point, which is also the unique solution of our problem.

Lemma 3.2.8. *Assume that $\phi \in H_0^1(\Omega), \psi \in L^2(\Omega)$. Then, $\mathcal{K}$ maps $Y_T(M_0)$ into itself and $\mathcal{K} : Y_T(M_0) \to Y_T(M_0)$ is strictly contractive if T is appropriately small relative to M_0.*

Proof. We calculate the $H_0^1(\Omega)$ norm estimates of all the terms involved in $\mathcal{K}u(x,t)$ as follows. From Theorem 3.2.1 we have the following norm estimates.

$$\|\mathcal{A}(t)\phi\|_{H_0^1(\Omega)} \leq \|\phi\|_{H_0^1}, \tag{3.2.24}$$

$$\|\mathcal{B}(t)\psi(x)\|_{H_0^1(\Omega)} \leq \|\psi\|_{L^2(\Omega)}. \tag{3.2.25}$$

Also we have; We also have;

$$\begin{aligned}
&\left\|\int_0^t \mathcal{B}(t-\tau)g(u(.,\tau))d\tau\right\|_{H_0^1(\Omega)} \leq \int_0^t \|\mathcal{B}(t-\tau)g(u(.,\tau))\|_{H_0^1(\Omega)}d\tau \\
&\leq \int_0^t \|g(u(.,\tau))\|_2 d\tau = \int_0^t \|u(.,\tau)\|_{2p}^p d\tau \leq C_{2p}^p \int_0^t \|u(.,\tau))\|_{H_0^1(\Omega)}^p d\tau \\
&\leq C_{2p}^p t \max_{0\leq\tau\leq t} \|u(.,\tau)\|_{H_0^1(\Omega)}^p \leq C_{2p}^p t \left(\|u\|_{Y_T}\right)^p \leq C_{2p}^p t \left(\|u\|_{Y_T}^{p-1}\right) \|u(.,t)\|_{Y_T}
\end{aligned} \tag{3.2.26}$$

We find the $H_0^1(\Omega)$ norm estimate for $\mathcal{K}u$ using the norm estimates of the terms of $\mathcal{K}u$ given in (3.2.24), 3.2.25 and (3.2.26) as

$$\max_{0\le\tau\le t}\|\mathcal{K}u(.,\tau)\|_{H_0^1(\Omega)} \le \|\phi\|_{H_0^1(\Omega)} + \|\psi\|_2 + \|\alpha\|_1 t\|u\|_{Y_T} + C_{2p}^p t\left(\|u\|_{Y_T}^{p-1}\right)\|u\|_{Y_T} \tag{3.2.27}$$

We calculate the L_2-norm estimates of each of the terms appearing in $\partial_t\mathcal{K}u$ as

$$\|\Delta_D\mathcal{B}(t)\phi\|_2 \le \|\nabla\phi\|_2, \tag{3.2.28}$$

$$\|\mathcal{A}(t)\psi\|_2 \le \|\psi\|_2. \tag{3.2.29}$$

$$\begin{aligned}&\left\|\int_0^t \mathcal{A}(t-\tau)g(u(.,\tau))d\tau\right\|_2 \le \int_0^t \|\mathcal{A}(t-\tau)g(u(.,\tau))\|_2 d\tau\\ &\le \int_0^t \|g(u(.,\tau))\|_2 d\tau \le \int_0^t \|u(.,\tau)\|_{2p}^p d\tau \le C_{2p}^p \int_0^t \|u(.,\tau))\|_{H_0^1(\Omega)}^p d\tau\\ &\le C_{2p}^p t \max_{0\le\tau\le t}\|u(.,\tau)\|_{H_0^1(\Omega)}^p \le C_{2p}^p t\left(\|u\|_{Y_T}^{p-1}\right)\|u\|_{Y_T}\end{aligned} \tag{3.2.30}$$

Therefore the L^2-norm estimate of $\partial_t\mathcal{K}u$ is calculated by summing up the estimates of terms of $\partial_t\mathcal{K}u$ given (3.2.28) to (**??**) so as to get,

$$\begin{aligned}\max_{0\le\tau\le t}\|\partial_t\mathcal{K}u\|_2 \quad &\le \|\nabla\phi\|_2 + \|\psi\|_2 + \|\alpha\|_1 t\|u\|_{Y_T} +\\ &+ C_{2p}^p t\left(\|u\|_{Y_T}^{p-1}\right)\|u\|_{Y_T}\end{aligned} \tag{3.2.31}$$

From (3.2.27) and (3.2.31) it follows that,

$$\begin{aligned}\|\mathcal{K}u\|_{Y_T} \le \|\nabla\phi\|_2 + 2\|\psi\|_2 + \|\phi\|_{H_0^1} + \|\alpha\|_1 t\|u\|_{Y_T} +\\ + C_{2p}^p t\left(\|u\|_{Y_T}^{p-1}\right)\|u\|_{Y_T}\end{aligned}$$

$$\begin{aligned}\|\mathcal{K}u\|_{Y_T} &\le M_0 + T\|\alpha\|_1(M_0+1) + C_{2p}^p T(M_0+1)^P\\ &\le M_0 + T(\|\alpha\|_1 + C_{2p}^p)\,(M_0+1)^p\end{aligned} \tag{3.2.32}$$

From (3.2.32), if T satisfies

$$T \le \frac{1}{(\|\alpha\|_1 + C_{2p}^p)\,(M_0+1)^p} \tag{3.2.33}$$

then $\|\mathcal{K}u\|_{Y_T} \le M_0 + 1$ whenever $u \in Y_T(M_0)$. Therefore $\mathcal{K}$ maps $Y_T(M_0)$ into itself.

Now we want to show that $\mathcal{K}$ is strictly contractive. Let $T > 0$ and $u_1, u_2 \in Y_T(M_0)$ be given.

$$\begin{aligned}
\|\mathcal{K}u_1 - \mathcal{K}u_1\|_{Y_T} &= \left\|\int_0^t \mathcal{B}(t-t)L_\alpha(g(u_1)-g(u_2))d\tau\right\|_{Y_T} \\
&\le \int_0^t \|\mathcal{B}(t-\tau)L_\alpha(g(u_1)-g(u_2))\|_{Y_T}\, d\tau \\
&\le \int_0^t \|L_\alpha(g(u_1)-g(u_2))\|_{Y_T}\, d\tau \\
&\le \|\alpha\|_1 \int_0^t \|g(u_1)-g(u_2)\|_{Y_T}\, d\tau \\
&\le \alpha\|_1 C \int_0^t \|u_1-u_2\|_{Y_T}\, d\tau \\
&\le \|\alpha\|_1 CT\,\|u_1-u_2\|_{Y_T}\,.
\end{aligned} \tag{3.2.34}$$

Now for

$$T < \min\left\{\frac{1}{(\|\alpha\|_1 + C_{2p}^p)\,(M_0+1)^p},\quad \frac{1}{2C\|\alpha\|_1}\right\}$$

From (3.2.33) and (3.2.34), we have

$$\|\mathcal{K}u_1 - \mathcal{K}u_2\|_{Y_T} \le \frac{1}{2}\|u_1-u_2\|_{Y_T}.$$

$\square$

Theorem 3.2.9. *Let $n \ge 3$. For any g given as in (3.1.2) with $1 < p \le \frac{n}{n-2}$ and for any initial data $\phi \in H_0^1(\Omega), \psi \in L^2(\Omega)$ problem (3.1.1) has a unique* *local weak solution* $u(x,t)$,

$$u(x,t) \in C^1([0,T), H_0^1(\Omega)) \cap C([0,T), L^2(\Omega))$$

Proof. From Lemma 3.2.8 and contraction mapping principle, it follows that for appropriately chosen $T > 0$, the integral operator $\mathcal{K}$ has a unique fixed point $u(x,t) \in Y_T(M_0) \subset C^1([0,T), H_0^1(\Omega)) \cap C([0,T), L^2(\Omega))$, which is the solution of integral equation (3.2.1) and equivalently weak solution of problem (3.1.1). □

The case of n=2:

The reason of considering the case $n = 2$ separately is that, while $p > 1$, in our power-type nonlinearity declared in (3.1.2), for case of $n = 2$ we can take p in the unbounded interval $1 < p < \infty$. Due to Sobolev embedding conditions given in lemma (3.2.5), we can not do the same for the case $n \geq 3$. For this reason we set the well-posedness of the problem (3.1.1) for $n = 2$ in the following theorem.

Theorem 3.2.10. *Suppose that* $n = 2, g(u) = |u|^{p-1}u$ *with* $p > 1$*. Then for any* $\phi \in H_0^1(\Omega), \psi \in L^2(\Omega)$*, problem (3.1.1) admits a unique weak solution*

$$u \in C([0,T), H_0^1(\Omega) \cap C^1([0,T), L^2(\Omega)$$

in some maximal interval of existence $[0,T)$*. Moreover, one of the following holds :*

(i) $T = +\infty$*, That is the solution is global in time.*

(ii) $T < \infty$*, and*

$$\lim_{t \uparrow T} \left(\|u(.,t)\|_{H_0^1(\Omega)} + \|u_t(.,t)\|_{L^2(\Omega)} \right) = +\infty$$

Proof. It suffices to notice that for $n = 2$, by Sobolev embedding theorem, $H_0^1(\Omega)$ is continuously embedded in $L^P(\Omega)$ for any $1 < p < \infty$. Hence, the function g is locally Lipschitz continuous. $\square$

The case of n=1:

While the case of $n = 1$ entertain the same condition of p given in (3.1.2) as the case of $n = 2$, we can set a wider class of nonlinear functions g other than the power type nonlinear functions. For this reason we set other theorem for the case of $n = 1$. We need the following lemma before stating the theorem.

Lemma 3.2.11. *For $g \in C^1(\mathbb{R})$ and $g(0) = 0$, the mapping $H_0^1(\Omega) \to H_0^1(\Omega)$ given by $u \mapsto g(u)$ is locally Lipschitz continuous.*

Proof. Let

$$B_R := \left\{ u \in H_0^1(\Omega) : \|u\|_{H_0^1(\Omega)} \leq R \right\}$$

and $u, v \in B_R$. Then by mean value theorem, we have

$$|g(u) - g(v)| = |g'(\theta u + (1-\theta)v||u - v|$$

and hence

$$\|g(u) - g(v)\|_{H_0^1(\Omega)} \leq \max_{|r| \leq R} |g'(r)| \|u - v\|_{H_0^1(\Omega)}$$

as required. $\square$

Theorem 3.2.12. *Suppose that $g \in C^1(\mathbb{R})$. Then for any $\phi \in H^1(\Omega)$, $\psi \in L^2(\Omega)$ problem (3.1.1) admits a unique solution*

$$u \in C\left([0,T), H_0^1(\Omega)\right) \cap C^1\left([0,T), L^2(\Omega)\right)$$

in some maximal interval of existence $[0, T)$. Moreover, one of the following holds :

(i) $T = +\infty$, That is the solution is global in time

(ii) $T < \infty$, *and*

$$\lim_{t \uparrow T} \left(\|u(.,t)\|_{H_0^1(\Omega)} + \|u_t(.,t)\|_{L^2(\Omega)} \right) = +\infty$$

Proof. For $n = 1$, the Sobolev embedding theorem guarantees that $H_0^1(\Omega)$ is continuously embedded into $C(\overline{\Omega})$. The further assumption given that $g \in C^1$ guarantees that g satisfies local Lipschitz condition. Thus local solvability follows. $\square$

3.2.2 Strong Solutions

Definition 3.2.13. Assume that $\phi \in H^2(\Omega) \cap H_0^1(\Omega)$ and $\psi \in H_0^1(\Omega) \cap L^{2p}(\Omega)$. A strong solution is of problem (3.1.1) is a weak solution with additional regularity,

$$u \in C^2\left([0,T), L^2(\Omega)\right) \cap C^1\left([0,T), H_0^1(\Omega)\right) \cap C\left([0,T), H^2(\Omega) \cap H_0^1\right).$$

In this case we consider the local well-posedness of problem (3.1.1) in the space where the domain of the Dirichlet Laplacian is normally defined.

Theorem 3.2.14. *Let* $n \geq 3$ *and* $g(u) = |u|^p u$ *with* $p > 1$ *satisfying the Sobolev embedding condition given in (3.2.5). Also assume that for the integral operator* L_α*, the kernel* $\alpha \in W^{2,1}(\mathbb{R})$*. For any initial conditions*

$$\phi \in H^2(\Omega) \cap H_0^1(\Omega), \quad \psi \in H_0^1(\Omega),$$

problem (3.1.1) admits a unique strong solution u *such that*

$$u \in C\left([0,T), H_0^1(\Omega) \cap H^2(\Omega)\right) \cap C^1\left([0,T), H_0^1(\Omega)\right) \cap C^2\left([0,T), L^2(\Omega)\right)$$

in some maximal interval of existence $[0,T)$*. Moreover, one of the following conditions hold:*

(i) $T = +\infty$, That is the solution is global in time.

(ii) $T < \infty$, and

$$\lim_{t \uparrow T} \left(\|u(.,t)\|_{H_0^1(\Omega)} + \|u_t(.,t)\|_{L^2(\Omega)} \right) = +\infty$$

Proof. Define a function space

$$X_T := C\left([0,T), H_0^1(\Omega) \cap H^2(\Omega)\right) C^1\left([0,T), H_0^1(\Omega)\right) \cap C^2\left([0,T), L^2(\Omega)\right). \tag{3.2.35}$$

with norm defined as,

$$\|u\|_{X_T} =: \max_{0 \le \tau \le t} \|u(.,\tau)\|_{H_0^1(\Omega)} + \max_{0 \le \tau \le t} \|u(.,\tau)\|_{H_0^1(\Omega)} + \max_{0 \le \tau \le t} \|u_t(.,\tau)\|_{L^2(\Omega)} \tag{3.2.36}$$

With the norm defined in (3.2.36), we notice that X_T is a Banach Space. By the result from Lemma 2.4.2 about the mapping property of the integral operator L_α and Lemma 3.2.1 about the mapping properties of the operators $\mathcal{A}$ and $\mathcal{B}$, for sufficiently small interval $[0, T_0]$, we have a continuous mapping of the operator,

$$\mathcal{K}u(x,t) := \mathcal{A}(t)\phi(x) + \mathcal{B}(t)\psi(x) + \int_0^t \mathcal{B}(t-\tau) L_\alpha g(u(x,\tau)) d\tau \tag{3.2.37}$$

in

$$C([0,T_0], H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0,T_0], H_0^1(\Omega)) \cap C^2([0,T_0]), L^2(\Omega)) \subset X_T$$

which is also strictly contractive. By Banach fixed point argument we have the fixed point of the integral operator (3.2.37) which is also the unique solution of the problem (3.1.1). $\square$

Chapter 4

Nonlinear Nonlocal Wave Type Problem with Separated Nonlinearity and Nonlocality

In this chapter we study a nonlinear nonlocal problem similar to problem (3.1.1) given in Chapter 3. However, here we have the nonlinear and nonlocal terms are given by two separate terms, with linear integral operator. Symmetry of the convolution type integral operator (2.1.2) involved in this problem has enabled us to further analyze the problem, beyond local well-posedness. We have defined the energy identity associated with (4.1.1) in (4.3.5). The condition of global well-posedness and finite time blow-up conditions were studied.

4.1 Description of the Problem

Let us analyse the well-posedness of a nonlocal wave-type equation given by

$$\begin{cases} u_{tt} - \Delta u = L_\alpha u + g(u), \quad x \in \Omega, \quad t > 0, \\ u(x,0) = \phi(x), u_t(x,0) = \psi(x), \quad x \in \Omega, \\ u(x,t) = 0, \quad x \in \partial\Omega, \quad t > 0, \end{cases} \tag{4.1.1}$$

where, $\alpha \in L^1(\mathbb{R})$ is a known function, $g(u) = |u|^{p-1}u$, and the integral operator L_α is defined in (2.1.2), u is a function which we are going to solve problem (4.1.1) on some maximum interval of existence [0,T).

4.2 Local Well-Posedness

4.2.1 Weak Solutions

Definition 4.2.1. Let $\Omega_T := \Omega \times T$, where $\Omega \subset \mathbb{R}^n$, is an open connected set with smooth boundary $\partial\Omega$. A weak solution of the nonlocal nonlinear initial-boundary value problem (4.1.1) is any function u Satisfying

$$u \in C([0,T), H_0^1(\Omega)) \cap C^1([0,T), L^2(\Omega) \cap L^\infty(\Omega)).$$

and

$$\begin{aligned} &\int_{\Omega_T} (u(x,s)v_{tt} - \nabla u(x,s)\nabla v(x,s) - v(x,s)L_\alpha u(x,s) - g(x,s)v(x,s))dxds \\ &\qquad = \int_\Omega (\phi(x)v(x,0) - \psi(x)v_t(x,s))\, dxds \end{aligned}$$

for every $v \in C_c^\infty(\Omega_T \times (0,T))$.

We consider a power type nonlinearity for cases of space dimension $n \geq 2$ and a more general class of nonlinearity for space dimension $n = 1$. This is due to some embedding results that help to indicate where our desired solution should be. From Sobolev embedding theorem we have $H_0^1(\Omega) \subset L^{\frac{2n}{n-2}}(\Omega)$ if $n \geq 3$. However for $n = 2, \quad H_0^1(\Omega) \subset L^q(\Omega), \quad 1 \leq q < \infty$, and for $n = 1, \quad H_0^1(\Omega) \subset L^q(\Omega), \quad 1 \leq q \leq \infty$.

Problem (4.1.1) may be written in an equivalent integral equation as

$$u(x,t) = \mathcal{A}(t)\phi(x) + \mathcal{B}\psi(x) + \int_0^t \mathcal{B}(t-\tau)L_\alpha u(x,\tau)d\tau + \\ + \int_0^t \mathcal{B}(t-\tau)g(u(x,\tau))d\tau. \qquad (4.2.1)$$

Define an integral operator $\mathcal{S}$ on the space $H_0^1(\Omega)$ as

$$\mathcal{S}(u) := \mathcal{A}(t)\phi(x) + \mathcal{B}\psi(x) + \int_0^t \mathcal{B}(t-\tau)L_\alpha u(x,\tau)d\tau \\ + \int_0^t \mathcal{B}(t-\tau)g(u(x,\tau))d\tau. \qquad (4.2.2)$$

The analysis of the solvability of nonlinear nonlocal problem (4.1.1) is essentially the same as that of problem (3.1.1). With the operational definitions and mapping properties of $\mathcal{A}$, $\mathcal{B}$ and L_α being the integral operator $\mathcal{S}$ given in (4.2.2) is well defined on the space $H_0^1(\Omega)$. The solution of problem (4.1.1) is the same as the solution of the equivalent integral equation (4.2.1). However, the solution of the integral equation (4.2.1) is the fixed point of the integral operator (4.2.2), with the function space Y_T defined as in (3.2.21) and the corresponding norm in (3.2.22). For sufficiently small time $T > 0$ we have a contractive mapping of the operator $\mathcal{S}$ from the space $C\left([0,T), H_0^1(\Omega)\right) \cap C^1\left([0,T)L^2(\Omega)\right)$. By Banach fixed point theorem, the integral equation (4.2.1) has a unique solution, which is also the solution of problem (4.1.1.

Given in Chapters 2 and 3, by applying the fixed point argument to integral equation (4.2.1) via the integral operator given by (4.2.2), we get problem (4.1.1) has a unique local solution

$$u(x,t) \in C\left([0,T)H_0^1(\Omega)\right) \cap C^1\left([0,T)L^2(\Omega)\right).$$

4.3 Energy Identity

Multiplying the equation in (4.1.1) by u_t and integrating over Ω we obtain

$$\int_\Omega u_t(x,t)u_{tt}(x,t)dx = \frac{1}{2}\frac{d}{dt}\int_\Omega u_t^2(x,t)dx = \frac{1}{2}\frac{d}{dt}\|u_t\|^2_{L^2(\Omega)} \tag{4.3.1}$$

$$\begin{aligned} -\int_\Omega u_t(x,t)\Delta u(x,t)dx &= \int_\Omega \nabla u_t(x,t)\nabla u(x,t)dx \\ &= \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2 dx = \frac{1}{2}\frac{d}{dt}\|\nabla u(.,t)\|^2_{L^2(\Omega)} \end{aligned} \tag{4.3.2}$$

From the assumption that the kernel α is symmetric, i.e, $\alpha(x-y) = \alpha(y-x)$ we have,

$$\begin{aligned} \int_\Omega u_t L_\alpha u dx &= \int_\Omega u_t \left(\int_\Omega \alpha(x-y)u(y,t)dy\right)dx \\ &= \int_\Omega\int_\Omega \alpha(x-y)u_t(x,t)u(y,t)dydx \\ &= \int_\Omega\int_\Omega \alpha(y-x)u_t(y,t)u(x,t)dxdy \end{aligned}$$

Consequently by taking the average,

$$\begin{aligned} \int_\Omega u_t(x,t)L_\alpha u(x,t)dx &= \frac{1}{2}\int_\Omega\int_\Omega \alpha(x-y)\left[u_t(x,t)u(y,t) + u_t(y,t)u(x,t)\right]dxdy \\ &= \frac{1}{2}\frac{d}{dt}\int_\Omega\int_\Omega \alpha(x-y)u(x,t)u(y,t)dxdy \end{aligned} \tag{4.3.3}$$

For the nonlinear term $g(u)$, we have

$$\int_{\Omega} g(u(x,t))u_t(x,t)dx = \frac{d}{dt}\int_{\Omega} G(u(x,t))dx \tag{4.3.4}$$

where $G(x) = \int_0^x g(s)ds$ so that $\quad G'(x) = g(x)$.

Combining the results from (4.3.1)-(4.3.4), we get quantity

$$E(t) := \frac{1}{2}\|u_t(.,t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla u(.,t)\|^2_{L^2(\Omega)} + \frac{1}{2}\Lambda(u)(t) - \mathcal{G}(u)(t), \tag{4.3.5}$$

where

$$\Lambda(u)(t) := -\int_{\Omega}\int_{\Omega} \alpha(x-y)u(x,t)u(y,t)dxdy = \langle L_\alpha u, u\rangle_{L^2(\Omega)} \tag{4.3.6}$$

$$\mathcal{G}(u)(t) := \int_{\Omega} G(u(x,t))dx = \frac{1}{p+1}\|u\|^{p+1}_{p+1} \tag{4.3.7}$$

which is actually independent of t. Specifically,

$$\begin{aligned} E(t) &= \frac{1}{2}\|\psi\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla\phi\|^2_{L^2(\Omega)} + \frac{1}{2}\Lambda(u)(0) - \mathcal{G}(u)(0) \\ &= E(0), \quad t \in [0,T), \end{aligned}$$

where,

$$\Lambda(0) = \int_{\Omega}\int_{\Omega} \alpha(x-y)\phi(x)\psi(y)dxdy, \quad \mathcal{G}(0) = \int_{\Omega} G(\phi(x))dx.$$

4.4 Finite Time Blow-up of solutions

In this the section we show that under some conditions the L^2- norm of the solution u of the problem (4.1.1) blows-up in finite time.

Theorem 4.4.1. *Suppose that there exists a constant $q > 2$ such that the* *conditions*

$$ug(u) \geq qG(u) \tag{4.4.1}$$

and

$$E(0) < 0 \tag{4.4.2}$$

hold true. Then the local solution of the problem (4.1.1) ceases to exist in finite time. Notice that for our problem involving power type nonlinearity, we have $ug(u) = (p+1)G(u)$*, which satisfies the condition (4.4.1) with* $q = p+1 > 2$.

Proof. Let

$$I(t) := \frac{1}{2}\langle u(.,t), u(.,t)\rangle + \beta(t) \tag{4.4.3}$$

where $\beta(t)$ is a positive, twice differentiable function which we will fix latter. Now,

$$I'(t) = \langle u(.,t), u_t(.,t)\rangle + \beta'(t) \tag{4.4.4}$$

and

$$\begin{aligned} I''(t) &= \langle u_t(.,t), u_t(.,t)\rangle + \langle u(.,t), u_{tt}(.,t)\rangle + \beta''(t) \\ &= \langle u_t(.,t), u_t(.,t)\rangle + \langle u(.,t), \Delta u(.,t)\rangle + \langle u(.,t), L_\alpha u(.,t)\rangle \\ &+ \langle u(.,t), g(u(.,t))\rangle + \beta''(t) \\ &= \|u_t\|^2 - \|\nabla u\|^2 + \Lambda u + \int_\Omega ug(u)dx + \beta''(t). \end{aligned} \tag{4.4.5}$$

From (4.4.1) and (4.3.7) we have the inequality

$$I''(t) \geq \|u_t\|^2 - \|\nabla u\|^2 + \Lambda u(t) + q\mathcal{G}(u)(t) + \beta''(t). \tag{4.4.6}$$

Now, from the energy identity given in (4.3.5) we have

$$q\mathcal{G}(u)(t) = \frac{q}{2}\|u_t\|^2 + \frac{q}{2}\|\nabla u\|^2 + \frac{q}{2}\Lambda u(t) - qE(0). \tag{4.4.7}$$

Substituting (4.4.7) into (4.4.6) we get

$$I''(t) \geq (1+\frac{q}{2})\|u_t\|^2 + (\frac{q}{2}-1)\|\nabla u\|^2 + (1-\frac{q}{2})\Lambda u(t) - qE(0) + \beta''(t). \quad (4.4.8)$$

By the assumption that $q > 2$, we have $\frac{q}{2} - 1 > 0$ and that the operator, $-L_\alpha$ is positive operator. Then both the terms $(\frac{q}{2} - 1)\|\nabla u\|^2$ and $\Lambda u(t)$ are positive. Consequently, we have the inequality

$$I''(t) \geq (1+\frac{q}{2})\|u_t\|^2 - qE(0) + \beta''(t). \quad (4.4.9)$$

Setting

$$\frac{1}{2}(1+\frac{q}{2}) = (1+\mu), \quad \mu > 0,$$

and

$$\beta(t) = \beta_0(t+t_0)^2, \quad \beta_0 = |E(0),|$$

from (4.4.3), (4.4.4) and (4.4.9) we have

$$\begin{aligned}
&I(t)I''(t) - (1+\mu)[I'(t)]^2 \\
&\geq \left[\frac{1}{2}\|u\|^2 + \beta(t)\right]\left[(1+\frac{q}{2})\|u_t\|^2 - qE(0) + \beta''(t)\right] - (1+\mu)\left[\langle u, u_t\rangle + \beta'(t)\right]^2 \\
&= (1+\mu)\left[\|u\|^2\|u_t\|^2 - |\langle u, u_t\rangle|^2\right] + 2\beta_0(1+\mu)\langle u-(t+t_0)u_t, u-(t+t_0)u_t\rangle \\
&\geq 0.
\end{aligned}$$

Hence, by Lemma 2.6.1, the quantity $I(t) = \|u(.,t)\|^2_{L^2(\Omega)} + \beta_0(t+t_0)^2$ and equivalently $\|u(.,t)\|^2_{L^2(\Omega)}$ ceases to exit in finite time. $\square$

4.5 Potential Well, Unstable Set and Blow-up of Solutions

In this section, we prove the finite time blow-up of solutions of the nonlinear nonlocal IBVP. We need the following definitions and lemmas for proof of the theorems that follows. The total energy identity of nonlinear nonlocal problem (4.1.1) is given by

$$E(t) := \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla u\|^2 + \frac{1}{2}\Lambda u - \frac{1}{p+1}\|u\|_{p+1}^{p+1}$$

We define potential energy functional for the nonlinear nonlocal initial-boundary value problem (4.1.1) as $J : H_0^1(\Omega) \cap L^{p+1}(\Omega) \to \mathbb{R}$ given by

$$J(u) := \frac{1}{2}\|\nabla u\|^2 + \frac{1}{2}\Lambda u - \frac{1}{p+1}\|u\|_{p+1}^{p+1}.$$

The Nehari functional, $\tilde{J} : H_0^1(\Omega) \cap L^{p+1}(\Omega) \to \mathbb{R}$, associated with nonlinear nonlocal initial-boundary value problem (4.1.1) is given by

$$\tilde{J}(u) := \|\nabla u\|^2 + \Lambda u - \|u\|_{p+1}^{p+1}.$$

The Nehari functional is introduced as follows. If $u \in H_0^1(\Omega)$ and $u \neq 0$, then there exists a unique $\lambda_c(u) > 0$ and that $\frac{d}{d\lambda}J(\lambda u)|_{\lambda=\lambda_c(u)}J(\lambda u) = 0$. This critical number $\lambda = \lambda_c(u)$ gives the maximum value of $J(\lambda u)$. The Nehari Manifold is the set defined by

$$\mathcal{N} := \left\{u \in H_0^1(\Omega) \mid \quad \tilde{J}(u) = 0, u \neq 0\right\}.$$

The potential well depth d is a positive number defined by

$$d := \inf_{u\in\mathcal{N}} J(u). \tag{4.5.1}$$

We define the stable set as

$$W := \left\{u \in H_0^1(\Omega), \mid \tilde{J}(u) > 0,\ J(u) < d\right\} \cap \{0\} \tag{4.5.2}$$

and the unstable set by,

$$\tilde{W} := \left\{ u \in H_0^1(\Omega), \mid \tilde{J}(u) < 0, \; J(u) < d \right\} \tag{4.5.3}$$

We observe that

$$W \cup \tilde{W} := \left\{ u \in H_0^1(\Omega) \mid \quad J(u) < d \right\}, \text{and} \quad W \cap \tilde{W} = \emptyset.$$

Lemma 4.5.1. *The potential well depth is given by*

$$d := \inf \left\{ \sup_{\lambda \geq 0} J(\lambda u) \quad u \in H_0^1(\Omega), \; \|\nabla u\| \neq 0 \right\}$$

is a positive quantity.

Proof. We have,

$$J(\lambda u) := \frac{1}{2}\lambda^2 \|\nabla u\|^2 + \frac{1}{2}\lambda^2 \Lambda u - \frac{\lambda^{p+1}}{p+1}\|u\|_{p+1}^{p+1} \tag{4.5.4}$$

From which we get,

$$\frac{d}{d\lambda} J(\lambda u) = \lambda \|\nabla u\|^2 + \lambda \Lambda u - \lambda^p \|u\|_{p+1}^{p+1} \tag{4.5.5}$$

From equation (4.5.5) setting $\frac{d}{d\lambda} J(\lambda u) = 0$, we get critical value of λ given by,

$$\lambda_c(u) = \left[\frac{\|\nabla u\|^2 + \Lambda u}{\|u\|_{p+1}^{p+1}} \right]^{\frac{1}{p-1}} \tag{4.5.6}$$

The second derivative at this critical point is,

$$\frac{d^2}{d\lambda^2} J(\lambda u)|_{\lambda = \lambda_c(u)} = (1-p)(\|\nabla u\|^2 + \Lambda u) < 0$$

This shows that the critical number $\lambda_c(u)$ indeed gives the maximum value.

$$\sup_{\lambda\in\mathbb{R}} J(\lambda u) = J(\lambda_c(u)u) = \frac{p-1}{2(p+1)}\left(\frac{\|\nabla u\|^2+\Lambda u}{\|u\|_{p+1}^2}\right)^{\frac{p+1}{p-1}}$$

$$\begin{aligned} d &:= \inf\left\{\sup_{\lambda\in\mathbb{R}} J(\lambda u),\ u\in H_0^1(\Omega),\ \|\nabla u\|\neq 0\right\} \\ &= \inf\left\{\frac{p-1}{2(p+1)}\left(\frac{\|\nabla u\|^2+\Lambda u}{\|u\|_{p+1}^2}\right)^{\frac{p+1}{p-1}} \quad u\in H_0^1(\Omega),\ \|\nabla u\|\neq 0\right\} \\ &\geq \frac{p-1}{2(p+1)} C_{p+1}^{\frac{2(p+1)}{1-p}} > 0 \end{aligned} \tag{4.5.7}$$

where the constant C_{p+1} is a constant of the Sobolev embedding.

$$C_{p+1} = \inf\left\{\frac{\|\nabla u\|}{\|u\|_{p+1}} \quad u\in H_0^1(\Omega),\ \|\nabla u\|\neq 0\right\}$$

This shows that the Nehari manifold is bounded away from zero by some positive number. $\square$

Lemma 4.5.2. *For each $u\in\tilde{W}$, we have the inequality*

$$\frac{p-1}{2(p+1)}\|u\|_{p+1}^{p+1} > d,$$

where d is the potential well depth.

Proof. We can write

$$\begin{aligned} J(u) &= \frac{1}{2}\|\nabla u\|^2 + \frac{1}{2}\Lambda u - \frac{1}{p+1}\|u\|_{p+1}^{p+1} \\ &= \frac{1}{2}\left(\|\nabla u\|^2 + \Lambda u - \|u\|_{p+1}^{p+1}\right) + \frac{p-1}{2(p+1)}\|u\|_{p+1}^{p+1} \\ &= \frac{1}{2}\tilde{J}(u) + \frac{p-1}{2(p+1)}\|u\|_{p+1}^{p+1} \end{aligned}$$

For $u\in\tilde{W}$, the unstable set, we have $\tilde{J}(u)<0$. However there exists a critical number $\lambda := \lambda_c(u) \in (0,1)$ such that $\tilde{J}(\lambda_c(u)u) = 0$. So

$$
\begin{aligned}
J(\lambda_c(u)u) &= \frac{1}{2}\tilde{J}(\lambda_c(u)u) + \lambda_c(u)^{p+1}\frac{p-1}{2(p+1)}\|u\|_{p+1}^{p+1} \\
&= \lambda_c(u)^{p+1}\frac{p-1}{2(p+1)}\|u\|_{p+1}^{p+1}.
\end{aligned}
$$

Therefore, for such choice of λ we obtain

$$
\frac{p-1}{2(p+1)}\|u\|_{p+1}^{p+1} > \lambda_c(u)^{p+1}\frac{p-1}{2(p+1)}\|u\|_{p+1}^{p+1} = J(\lambda_c(u)u) \geq d.
$$

□

Lemma 4.5.3 (**Invariance of Unstable Set**). *Let $\phi \in H_0^1(\Omega), \psi \in L^2(\Omega)$, and let $1 < p < \frac{n+2}{n-2}$ if $n \geq 3$, and $1 < p < \infty, \quad n = 1,2$. Let $u(x,t)$ be a local solution of the nonlocal nonlinear IBVP (4.1.1) on the maximum interval of existence $[0,T)$. If there exists $t_0 \in [0,T)$ such that $u(,t_0) \in \tilde{W}$(the unstable set) and $E(t_0) < d$, then $u(.,t)$ remains in the set $\tilde{W}$ for any $t \in [t_0,T)$.*

Proof. Suppose that there is $t_1 \in [t_0,T)$ such that $u(.,t) \in \tilde{W}$ for $[t_0,t_1)$ and $u(,t_1) \notin \tilde{W}$. From the definition of the stable set, $\tilde{W}$ and continuity in t of the functionals $J(u(.,t))$ and $\tilde{J}(u(.,t))$, we have either, $J(u(.,t_1)) = d$, or $\tilde{J}(u(.,t_1)) = 0$. If we assume that $J(u(.,t_1)) = d$ then from the energy identity, $E(0) = E(t_1) = \frac{1}{2}\|u_t(t_1)\|^2 + J(u(t_1)) \geq J(u(t_1)) \geq d$. This contradicts the energy condition $E(0) < d$ given in the Lemma. So the first condition is impossible. Now assume that $\tilde{J}(u(.,t_1)) = 0$. This implies that $u(t_1) \in \mathcal{N}$. Therefore, from the definition of d given in (4.5.1), $J(u(t_1)) \geq \inf_{u\in\mathcal{N}} J(u) = d$. This implies that $E(0) = E(t_1) \geq J(u(t_1)) \geq d$. Again this contradicts the energy condition given in the Lemma. Therefore, $u(t_1) \in W$ and that the stable set is invariant under the flow of the solution problem (4.1.1). □

Lemma 4.5.4. *Under the assumptions given in Lemma 4.5.2, the inequality*

$$(\|\nabla u\|^2 + \Lambda u) > \frac{2d(p+1)}{p-1}$$

is fulfilled for $t \in [t_0, T)$, where $u(x,t)$ is the local solution for nonlocal nonlinear IBVP (4.1.1).

Proof. We may write $J(u) = \frac{p-1}{2(p+1)} (\|\nabla u\|^2 + \Lambda u) + \frac{1}{p+1} \tilde{J}(u)$. For the critical coefficient $\lambda_c(u)$ given in (4.5.6) we have $\tilde{J}(\lambda_c(u)u) = 0$. In addition, for $u \in \tilde{W}$ we have $0 < \lambda_c(u) < 1$. Now we have

$$\begin{aligned} d \leq J(\lambda_c(u)u) &= (\lambda_c(u))^2 \frac{p-1}{2(p+1)} \left(\|\nabla u\|^2 + \Lambda u\right) + \frac{1}{p+1} \tilde{J}(\lambda_c(u)u) \\ &< \frac{p-1}{2(p+1)} \left(\|\nabla u\|^2 + \Lambda u\right) \end{aligned}$$

which completes the proof. □

Theorem 4.5.5. *Let $u(x,t)$ be the local solution of nonlocal nonlinear IBVP (4.1.1) on $[0,T)$ with initial data $\phi \in H_0^1(\Omega), \psi \in L^2(\Omega)$. If there exists a number $t_0 \in [0,T)$ such that $u(,t_0) \in \tilde{W}$ the unstable set and $E(t_0) < d$, then $T = \infty$.That is the solution of the nonlinear nonlocal IBVP doesn't exist globally in time. That is $T < \infty$.*

Proof. From Lemma () and energy identity we have,

$$\begin{aligned} d > E(t_0) = E(t) &= \frac{1}{2}\|u_t(.,t)\|^2 + J(u(.,t)) \\ &\geq C(\|\nabla u(.,t)\|^2 + \Lambda u(.,t) + \|u_t(.,t)\|^2 \end{aligned}$$

where, $c = \frac{p-1}{2(p+1)}$. This inequality and the principle of continuity lead to the global existence of solutions, i.e $T = \infty$. □

Theorem 4.5.6. *Let g satisfy the conditions (i)-(iv), and let W denote the corresponding potential well energy of (4.1.1). If $\phi(x) \in \tilde{W}$ and $E(0) < d$, then $\|u\|_{L^2(\Omega)} \to \infty$ in finite time.*

Proof. Let $I(t)$ be defined as

$$I(t) := \frac{1}{2}\|u\|^2_{L^2(\Omega)} \tag{4.5.8}$$

Then

$$I'(t) = \langle u, u_t \rangle_{L^2(\Omega)} \tag{4.5.9}$$

Since we know that $\tilde{J}(u) < 0$ for $u \in \tilde{W}$, we have

$$\begin{aligned} I''(t) &= \|u_t\|^2 + \langle u, u_{tt} \rangle_{L^2(\Omega)} \\ &= \|u_t\|^2 + \langle u, \Delta u + L_\alpha u + g(u) \rangle_{L^2(\Omega)} \\ &= \|u_t\|^2 + \langle u, \Delta u \rangle + \langle u, L_\alpha u \rangle + \langle u, g(u) \rangle \\ &= \|u_t\|^2 - \|\nabla u\|^2 - \Lambda u + \|u\|^{p+1}_{p+1} \\ &= \|u_t\|^2 - \tilde{J}(u) \geq 0 \end{aligned} \tag{4.5.10}$$

Where $\tilde{J}(u) = \|\nabla u\|^2 + \Lambda u - \langle u, g(u) \rangle_{L^2(\Omega)}$ is the Nehari functional. By the assumptions of the function $g(u) = |u|^{p-1}$ we have,

$$I''(t) \geq \|u_t\|^2 + (p+1)\int_\Omega G(u)dx - \|\nabla u\|^2 - \Lambda u. \tag{4.5.11}$$

Calculating the value of $\int_\Omega G(u)dx$ from the energy identity (4.3.5) and substituting into (4.5.11), we have

$$I''(t) \geq \frac{(p+3)}{2}\|u_t\|^2 + \frac{(p-1)}{2}\|\nabla u\|^2 + \frac{(p-1)}{2}\Lambda u - (p+1)E(0). \tag{4.5.12}$$

From the variational inequality given as

$$\|\nabla u\|^2 \geq \lambda_1 \|u\|^2 = 2\lambda_1 I(t), \tag{4.5.13}$$

where

$$\lambda_1 = \min_{u \in H^1_0(\Omega),\, \|\nabla u\| \neq 0} \left(\frac{\|\nabla u\|^2}{\|u\|^2} \right)$$

is the principal eigenvalue of the Dirichlet Laplacian, we get by substituting (4.5.13) into (4.5.12)

$$I''(t) \geq \frac{(p+3)}{2}\|u_t\|^2 + (p-1)\lambda_1\|u\|^2 + \frac{(p-1)}{2}\Lambda u - (p+1)E(0). \quad (4.5.14)$$

Since I is convex function of t ($I''(t) > 0$), it follows that, if there exists t_1 such that $I'(t_1) > 0$ then $I(t)$ is increasing $t > t_1$. Consequently the quantity

$$(p-1)\|u\|^2 + \frac{(p-1)}{2}\Lambda u - (p+1)E(0)$$

eventually became positive and will remain positive afterwards. Thus for sufficiently large t we would have

$$I'' \geq \frac{(p+3)}{2}\|u_t\|^2.$$

Now let us set $\mu = \frac{(p+3)}{4} > 1$. By applying Levine's Lemma 2.6.1 for $I(t)$ we have

$$\begin{aligned}
&I(t)I''(t) - \frac{(p+3)}{4}[I'(t)]^2 \\
&\geq \left(\frac{1}{2}\|u\|^2\right)\left(\frac{(p+3)}{2}\|u_t\|^2\right) - \frac{(p+3)}{4}\langle u, u_t\rangle^2 \\
&= \frac{(p+3)}{4}\left(\|u\|^2\|u_t\|^2 - \langle u, u_t\rangle^2\right) \geq 0.
\end{aligned}$$

Hence the L^2 norm of the solution u of problem (4.1.1) blows-up at some finite time T_0, that is,

$$\lim_{t \to T_0-} I(t) = \infty.$$

One more remaining condition required to complete the proof is to show that there exists some t_1 such that $I'(t_1) > 0$, which was assumed to be so.

Lemma 4.5.7. *Suppose that $I'(t) = \langle u(t), u_t(t)\rangle < 0$ for all $t \geq 0$. Then we have the following conditions.*

(i) $\|u(t)\| \leq \|\phi\|$ *for all* t *and that* $\|u(t)\| \downarrow A$ *for some constant* $A > 0$.
(ii) $I'(t) = \langle u(t), u_t(t) \rangle \uparrow 0$ *as* $t \to \infty$.
(iii) $I''(t) \downarrow 0$ *as* $t \to \infty$.

Proof. (i) $I'(t) = \langle u(t), u_t(t) \rangle < 0$ for all t implies that $I(t)$ is a decreasing function of t. This implies that

$$I(t) \leq I(0) \Rightarrow \frac{1}{2}\|u(t)\|^2 \leq \frac{1}{2}\|\phi\|^2 \Rightarrow \|u(t)\| \leq \|\phi\|$$

for all $t \geq 0$. Also as $\|u(t)\|$ is decreasing function of t, if we assume that $\|u(t)\| \downarrow 0$, then we have $u(t) \in W$ beyond some value of t. This contradicts with the stationary property of the unstable set $\tilde{W}$.

(ii) By the assumption that $I'(t) < 0$ and $I''(t) > 0$, we have $I'(t)$ is increasing. If we assume that $I'(t)$ is bounded above negatively, that is, if we assume that there exists some $\epsilon > 0$ such that, $I'(0) \leq I'(t) \leq -\epsilon$ for all $t \geq 0$, then

$$I(t) = I(0) + \int_0^t I'(s)ds \leq I(0) - \int_0^t \epsilon ds = I(0) - \epsilon t$$

This implies that $I(t) < 0$ for $t > \frac{I(0)}{\epsilon}$, and contradicts the fact that $I(t) \geq 0$ for all $t \geq 0$, as defined in (4.5.8).
(iii) Finally, if we assume that there exists $\delta > 0$ such that,$I''(t) \geq \delta > 0$, for all $t \geq 0$, then

$$I'(t) = I'(0) + \int_0^t I''(s)ds \geq I'(0) + \int_0^t \delta ds = I'(0) + \delta t$$

This implies that $I'(t) \geq 0$ for $t \geq \frac{|I'(0)|}{\delta}$ and contradicts with our assumption that $I'(t) < 0$ for all $t \geq 0$. Therefore we have, $I''(t) \downarrow 0$ as $t \to \infty$. $\square$

It turns out that I is bounded and decreasing (non increasing) and I'

is bounded and increasing with ,

$$I(0) \geq I(t) \geq A > 0, \quad I'(0) \leq I'(t) \leq 0.$$

We have $I(t)$ tend to a finite positive limit A where $0 < A \leq I(0)$. However $I(t)$ cannot tend to zero for this would place $u(t)$ inside the potential well. This cannot happen as was seen in Lemma 4.5.3. Therefore we have,

$$I \to A > 0, \quad I'(t) \to 0, \quad I''(t) \to 0 \tag{4.5.15}$$

We also have

$$\begin{aligned} I''(t) &= \|u_t\|^2 - \tilde{J}(u) \geq \|u_t\|^2 \geq 0, \\ I''(t) &= \|u_t\|^2 - \tilde{J}(u) \geq -\tilde{J}(u) \geq 0 \end{aligned} \tag{4.5.16}$$

From (4.5.16)we conclude that

$$\lim_{t\to\infty} \|u_t\|^2 = 0, \quad \lim_{t\to\infty} \tilde{J}(u) = 0. \tag{4.5.17}$$

□

We have to prove out that only one of the following two conditions holds true.

- The first condition is that $I'(t) < 0$, for all $t \geq 0$, and the maximum time of existence $T = \infty$. Let $u(t_0) \in \tilde{W}$. By lemma (4.5.3), we have $u(t) \in \tilde{W}$ for all $t \geq t_0$. Furthermore by lemma (4.5.2),

  $$J(u(t)) > d + \frac{1}{2}\tilde{J}(u(t)), \quad t \geq t_0.$$

  and,
  $$E(0) = \frac{1}{2}\|u_t\|^2 + J(u(t)) > \frac{1}{2}\|u_t\|^2 + d + \frac{1}{2}\tilde{J}(u(t))$$
  Applying limit as $t \to \infty$ and taking into account (4.5.17), we get

$E(0) \geq d > 0$. This contradicts the energy condition given in the Lemma and hence this condition can not happen. Therefore, we consider the second condition.

- The second condition, which is the negation of the first condition, states that either the maximum time of existence is $T < \infty$, or $I'(t_0) \geq 0$, for some $t_0 \in [0, T)$. However, the condition $T < \infty$, by the very definition, signifies the finite time blow-up condition of the L_2-norm $\|u\|$ of the solution u of problem (4.1.1). Alternatively, since $I''(t) \geq 0$, the condition $I'(t_0) \geq 0$, for some $t_0 \in [0, T)$ implies that $I'(t) \geq 0, \quad t \geq t_0$ so that the conditions of Levine Lemma 2.6.1 are satisfied and we resort back into our proof of finite time blow-up condition, assuming that $I'(t_0) \geq 0$, for some $t_0 \in [0, T)$.

4.6 Potential Well, Stable Set and Global Existence of Solution

Lemma 4.6.1 (**Invariance of Stable Set**). *Let $\phi \in H_0^1(\Omega), \psi \in L^2(\Omega)$ and let $1 < p < \frac{n}{n-2}$ if $n \geq 3$, and $1 < p < \infty, \quad n = 1, 2$. Let $u(x, t)$ be the solution of the nonlocal nonlinear IBVP (4.1.1) on the maximum interval of existence $[0, T)$. If there exists $t_0 \in [0, T)$ such that $u(, t_0) \in W$, the stable set, and $E(t_0) < d$, then $u(., t)$ remains in the set W for any $t \in [t_0, T)$.*

Proof. Suppose that there is $t_1 \in [t_0, T)$ such that $u(t) \in W$ for $[t_0, t_1)$ and $u(t_1) \notin W$. From the definition of the stable set W and the continuity in t of the functionals $J(u(., t))$ and $\tilde{J}(u(., t))$, we have either,

(I) $J(u(t_1)) = d$, or

(II) $\tilde{J}(u(t_1)) = 0$

However $J(u(t_1)) = d$ implies

$$E(0) = E(t_1) = \frac{1}{2}\|u_t(t_1)\|^2 + J(u(t_1)) \geq J(u(t_1)) = d$$

However this contradicts the energy condition $E(0) < d$ given in this lemma. Therefore condition (I) can not be true.

Assume that case (II) holds true. $\tilde{J}(u(t_1)) = 0$ implies that $u(t_1) \in \mathcal{N}$. Therefore by the definition of the potential well depth we have $J(u(t_1)) \geq d$. Consequently $E(0) = E(t_1) \geq J(u(t_1)) \geq d$. This contradicts the energy assumption given in the lemma. Therefore condition (II) as well cant not be true. $\square$

We now suppose that the solution starts inside a potential well, i,e,$\tilde{J}(\phi) > 0$ and with additional condition that $E(0) < d$. By Lemma 4.6.1 we have that the stable set W is invariant under the flow of the solution of (4.1.1). That is $\tilde{J}(u(t)) > 0, t \in [0, T)0$. We show that the solution is global in time.

Lemma 4.6.2. *For every $u(t_0) \in W$, i.e., $\tilde{J}(u(, t_0)) > 0$, We have the inequality,*

$$J(u(., t)) \geq \frac{p-1}{2(p+1)}(\|\nabla u(, t)\|^2 + \Lambda u(, t))$$

is fulfilled for $t \in [t_0, T)$, where $u(x, t)$ is the local solution for Nonlocal nonlinear IBVP and T is the maximum interval of existence.

Proof. For $t \in [t_0, T)$, by invariance property of the stable set W as was shown in Lemma 4.6.1, we have,

$$\tilde{J}(u(t)) = \|\nabla u(., t)\|^2 + \Lambda u(., t) - \|u(., t)\|_{p+1}^{p+1} > 0.$$

By the rearrangement, we obtain

$$
\begin{aligned}
J(u(t)) &= \frac{1}{2}\|\nabla u(.,t)\|^2 + \frac{1}{2}\Lambda u(.,t) - \frac{1}{p+1}\|u(.,t)\|_{p+1}^{p+1} \\
&= \frac{1}{(p+1)}\left(\|\nabla u(.,t)\|^2 + \Lambda u(.,t) - \|u(.,t)\|_{p+1}^{p+1}\right) \\
&\quad + \frac{p-1}{2(p+1)}\left(\|\nabla u(.,t)\|^2 + \Lambda u(.,t)\right) \\
&= \frac{1}{(p+1)}\tilde{J}(u(.,t)) + \frac{p-1}{2(p+1)}\left(\|\nabla u(.,t)\|^2 + \Lambda u(.,t)\right) \\
&\geq \frac{p-1}{2(p+1)}\left(\|\nabla u(,t)\|^2 + \Lambda u(.,t)\right).
\end{aligned}
$$

□

Theorem 4.6.3. *Let $u(x,t)$ be the local solution of nonlocal nonlinear IBVP (4.1.1) on $[0,T)$ with initial data $\phi \in H_0^1(\Omega), \psi \in L^2(\Omega)$. If there exists a number $t_0 \in [0,T)$ such that $u(,t_0) \in W$ and $E(t_0) < d$, then $T = \infty$. That is the solution is global in time.*

Proof. From Lemma 4.6.2 and energy identity (4.3.5) we have

$$
\begin{aligned}
d > E(t_0) = E(t) = \frac{1}{2}\|u_t(.,t)\|^2 + J(u(.,t)) &\geq J(u(x,t)) \\
\geq \left(\|\nabla u(.,t)\|^2 + \Lambda u(.,t)\right) &\geq \|\nabla u(.,t)\|^2.
\end{aligned}
$$

We therefore have the semi-norm $\|\nabla(.,t)\|$ is bounded and that

$$
\|\nabla u(.,t)\|^2 \leq \frac{2(p+1)}{p-1}d, \tag{4.6.1}
$$

for all $t \in [t_0,T)$. From the energy identity we have $\frac{1}{2}\|u_t(.,t)\|^2 \leq d$, for all $t \in [t_0,T)$ so that

$$
\|u_t(.,t)\| \leq \sqrt{2d}. \tag{4.6.2}
$$

Applying the differential inequality to equation (4.6.2) we get,

$$\frac{d}{dt}\|u(.,t)\| \leq \|u_t(.,t)\| \leq \sqrt{2d}.$$

Integrating over $[t_0, t]$, we get,

$$\|u(.,t)\| \leq \|u(.,t_0)\| + \sqrt{2d}(t - t_0), \tag{4.6.3}$$

for all $t \in [t_0, T)$. Hence the L_2- norm of u is bounded linearly. Alternatively we may proceed as follows. From the equation

$$J(u) = \frac{1}{2}\tilde{J}(u) + \frac{p-1}{2(p+1)}\|u\|_{p+1}^{p+1},$$

the given energy condition $E(0) < d$, and the property that $\tilde{J}(u) > 0$, $\forall u \in W$ leads to the inequality

$$\|u\|_{p+1}^{p+1} \leq \frac{2(p+1)}{p-1}d. \tag{4.6.4}$$

By the embedding $L^{p+1}(\Omega) \hookrightarrow L^2(\Omega)$ and inequality (4.6.4) we have,

$$\|u\| \leq |\Omega|^{\frac{p-1}{2(p+1)}}\|u\|_{p+1} \leq |\Omega|^{\frac{p-1}{2(p+1)}}\left(\frac{2(p+1)}{p-1}d\right)^{\frac{1}{p+1}} \tag{4.6.5}$$

From equations (74) and (75) we have,

$$\|u\| \leq |\Omega|^{\frac{p-1}{2(p+1)}}\left(\frac{2(p+1)}{p-1}d\right)^{\frac{1}{p+1}} \tag{4.6.6}$$

From inequalities given in (4.6.1), 4.6.2 and 4.6.5, we have a unique weak local solution

$$u(x,t) \in C([0,T), H_0^1(\Omega)) \cap C^1([0,T), L^2(\Omega)),$$

which is finite for every $T > 0$. By the principle of continuity this leads to the global existence of solutions that is $T = \infty$. $\square$

Chapter 5

Conclusions and Possible Future Work

With many other possible tasks the following problems may be considered.

- Let d be the potential well depth of a nonlinear nonlocal wave equation given in (4.5.7) and d' be the potential well depth of the usual nonlinear wave equation with nonlinear term $g(u) = |u|^{p-1}, \quad p > 1$. We have

$$d' := \inf \left\{ \frac{p-1}{2(p+1)} \left(\frac{\|\nabla u\|^2}{\|u\|_{p+1}^2} \right)^{\frac{p+1}{p-1}} \quad u \in H_0^1(\Omega),\ \|\nabla u\| \neq 0 \right\}.$$

Due to the positive term Λu involved in d, we have, $d \geq d'$. also from the inequality

$$\Lambda u := -\langle L_\alpha u, u \rangle \leq \|L_\alpha u\| \|u\| \leq \|\alpha\| \|u\|^2 \leq C_2 \|\alpha\| \|\nabla u\|^2$$

we have,

$$d \leq (1 + C_2\|\alpha\|)^{\frac{p+1}{p-1}} d'.$$

- Several variants of the above problems may be studied. For example, in my research I have considered the nonlinear function g to be of power type. That is, $g(u) = |u|^{p-1}u, \quad p > 1$. This is to apply some helpful Sobolev embedding theorems efficiently and easily. The corresponding term appearing in the energy identity is given by

$$\mathcal{G}(u)(t) := \int_\Omega G(u)dx = \frac{1}{p+1}\int_\Omega |u(.,t)|^{p+1}dx = \frac{1}{p+1}\|u\|_{p+1}^{p+1}$$

which is a positive definite quantity. This term appears as well in and the potential energy functional the Nehari functional up to a constant factor. On the other hand if we use the nonlinear term as $g(u) = |u|^p$, the corresponding term appearing in the energy identity would be

$$\mathcal{G}(u)(t) := \int_\Omega G(u)dx = \frac{1}{p+1}\int_\Omega |u(.,t)|^p u(x,t)dx$$

This term is sign changing depending on u and requires some more analysis in the study of existence of solutions and analysis of blow up conditions of solutions A finite linear combination of such power functions may also be considered or a more general nonlinear functions in the form

$$g(u) = \sum_{i=1}^{m} |u|^{p_i}u \quad p_i > 1, i = 1, ..., m$$

which does not include the nonlocality terms involved, such form nonlinearities were discussed some articles on nonlinear wave equations. For example, refer [39].

- A nonlocal problem involving dissipative term u_t or more generally nonlinear dissipative term $f(u_t)$ may be considered. Nonlinear dissipative problems with nonlocal term has been discussed in some articles.

- Nonlocality with respect to time variable t may be considered instead of the nonlocality in space variable x as the case of the current work.

- A problem with more general kernel of the form $k(x,t)$ may be considered in the integral operator involved in the problem. For example, integral operator of the form

$$L_k u(x,t) := \int_\Omega k(x,t)u(y,t)dy$$

can be considered.

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