UNIVERSALITY RESULTS FOR ZEROS OF RANDOM HOLOMORPHIC SECTIONS

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ABSTRACT. In this work we prove an universality result regarding the equidistribution of zeros of random holomorphic sections associated to a sequence of singular Hermitian holomorphic line bundles on a compact Kähler complex space X. Namely, under mild moment assumptions, we show that the asymptotic distribution of zeros of random holomorphic sections is independent of the choice of the probability measure on the space of holomorphic sections. In the case when X is a compact Kähler manifold, we also prove an off-diagonal exponential decay estimate for the Bergman kernels of a sequence of positive line bundles on X.

CONTENTS

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1. INTRODUCTION

In this paper we study the asymptotic distribution of zeros of random sequences of holomorphic sections of singular Hermitian holomorphic line bundles. We generalize our previous results from [CM1, CM2, CM3, CMM, Ba1, Ba3, Ba2] in several directions. We consider sequences (L_p, h_p), $p \geq 1$, of singular Hermitian holomorphic line bundles over Kähler spaces instead of the sequence of powers $(L^p, h^p) = (L^{\otimes p}, h^{\otimes p})$ of a fixed line bundle (L, h). Moreover, we endow the vector space of holomorphic sections with wide classes of probability measures (see condition (B) below and Section 4.2).

Recall that by the results of [T] (see also [MM1, Section 5.3]), if (X, ω) is a compact Kähler manifold and (L, h) is a line bundle such that the Chern curvature form $c_1(L, h)$ equals ω, then the normalized Fubini-Study currents $\frac{1}{p}\gamma_p$ associated to $H^0(X, L^p)$ (see (2.1)) are smooth for p sufficiently large and converge in the $\mathscr{C}^2$ topology to ω. This result can be applied to describe the asymptotic distribution of the zeros of sequences of Gaussian holomorphic sections. Indeed, it is shown in [SZ1] (see also [NV, DS, SZ2, S, DMS]) that for almost all sequences $\{s_p \in H^0(X, L^p)\}_{p\geq 1}$ the normalized zero-currents $\frac{1}{p}[s_p = 0]$ converge weakly to ω on X. Thus ω can be approximated by various algebraic or analytic objects in the semiclassical limit $p \to \infty$. Some important technical tools in higher dimensions were introduced in [FS]. Using these tools we generalized in [CM1, CM2, CM3, CMM, CMN1, CMN2, DMM] the above results to the case of singular positively curved Hermitian metrics h. We note that statistics of zeros of sections and hypersurfaces have been studied also in the context of real manifolds and real vector bundles, see e.g. [GW, NS].

In this paper we work in the following setting:

(A1) (X, ω) is a compact (reduced) normal Kähler space of pure dimension n, X_{reg} denotes the set of regular points of X, and X_{sing} denotes the set of singular points of X.

(A2) (L_p, h_p), $p \geq 1$, is a sequence of holomorphic line bundles on X with singular Hermitian metrics h_p whose curvature currents verify

$$c_1(L_p, h_p) \geq a_p\, \omega \text{ on } X, \text{ where } a_p > 0 \text{ and } \lim_{p\to\infty} a_p = \infty. \tag{1.1}$$

Let $A_p := \int_X c_1(L_p, h_p) \wedge \omega^{n-1}$. If $X_{\mathrm{sing}} \neq \emptyset$ we also assume that

$$\exists\, T_0 \in \mathscr{T}(X) \text{ such that } c_1(L_p, h_p) \leq A_p T_0\,,\ \forall\, p \geq 1\,. \tag{1.2}$$

Here $\mathscr{T}(X)$ denotes the space of positive closed currents of bidegree $(1,1)$ on X with local plurisubharmonic potentials (see Section 2.1). We let $H^0_{(2)}(X, L_p)$ be the Bergman space of L^2-holomorphic sections of L_p relative to the metric h_p and the volume form $\omega^n/n!$ on X,

$$H^0_{(2)}(X, L_p) = \left\{ S \in H^0(X, L_p) :\ \|S\|_p^2 := \int_{X_{\mathrm{reg}}} |S|^2_{h_p} \frac{\omega^n}{n!} < \infty \right\}, \tag{1.3}$$

endowed with the obvious inner product. For $p \geq 1$, let $d_p = \dim H^0_{(2)}(X, L_p)$ and let $S_1^p, \ldots, S_{d_p}^p$ be an orthonormal basis of $H^0_{(2)}(X, L_p)$.

Now, we describe the randomization on $H^0_{(2)}(X, L_p)$. Using the above orthonormal bases we identify the spaces $H^0_{(2)}(X, L_p) \simeq \mathbb{C}^{d_p}$ and endow them with probability measures σ_p verifying the following moment condition:

(B) There exist a constant $\nu \geq 1$ and for every $p \geq 1$ constants $C_p > 0$ such that

$$\int_{\mathbb{C}^{d_p}} \big| \log |\langle a, u\rangle| \big|^\nu \, d\sigma_p(a) \leq C_p \,, \;\; \forall \, u \in \mathbb{C}^{d_p} \text{ with } \|u\| = 1 \,.$$

We remark that the probability space $(H^0_{(2)}(X, L_p), \sigma_p)$ depends in general on the choice of the orthonormal basis (used for the identification $H^0_{(2)}(X, L_p) \simeq \mathbb{C}^{d_p}$). However, it follows from Theorem 1.1 below that the global distribution of zeros of random holomorphic sections does not depend on the choice of the orthonormal basis.

General classes of measures σ_p that satisfy condition (B) are given in Section 4.2. Important examples are provided by the Gaussians (see Section 4.2.1) and the Fubini-Study volumes (see Section 4.2.2), *which verify (B) for every $\nu \geq 1$ with a constant $C_p = \Gamma_\nu$ independent of p*. For such measures Theorem 1.1 below takes a particularly nice form. We note that for the measures σ_p from Sections 4.2.1, 4.2.2 and 4.2.3 (area measure of spheres), the probability space $(H^0_{(2)}(X, L_p), \sigma_p)$ does not depend on the choice of the orthonormal basis, since these measures are unitary invariant. In Section 4.2.4 we show that measures with *heavy tail probability* (see condition (B1) therein) and *small ball probability* (see condition (B2) therein) verify assumption (B). We also stress that random holomorphic sections with i.i.d. coefficients whose distribution has bounded density and logarithmically decaying tails arise as a special case (cf. Lemma 4.15). Moreover, locally moderate measures with compact support are also among the examples of such measures (cf. Lemma 4.16).

Given a section $s \in H^0(X, L_p)$ we denote by $[s = 0]$ the current of integration over the zero divisor of s. The expectation current $\mathbb{E}[s_p = 0]$ of the current-valued random variable $H^0_{(2)}(X, L_p) \ni s_p \mapsto [s_p = 0]$ is defined by

$$\big\langle \mathbb{E}[s_p = 0], \Phi \big\rangle = \int_{H^0_{(2)}(X, L_p)} \big\langle [s_p = 0], \Phi \big\rangle \, d\sigma_p(s_p),$$

where Φ is a $(n-1, n-1)$ test form on X. We consider the product probability space

$$(\mathcal{H}, \sigma) = \left(\prod_{p=1}^{\infty} H^0_{(2)}(X, L_p), \prod_{p=1}^{\infty} \sigma_p \right). \tag{1.4}$$

The following result gives the distribution of the zeros of random sequences of holomorphic sections of L_p, as well as the convergence in L^1 of the logarithms of their pointwise norms. Note that by the Poincaré-Lelong formula (see (2.4)) the latter are the potentials of the currents of integration on the zero sets, thus their convergence in L^1 implies the weak convergence of the zero-currents.

Theorem 1.1. *Assume that (X, ω), (L_p, h_p) and σ_p verify the assumptions (A1), (A2) and (B). Then the following hold:*

(i) If $\lim_{p\to\infty} C_p A_p^{-\nu} = 0$ *then* $\dfrac{1}{A_p}\big(\mathbb{E}[s_p = 0] - c_1(L_p, h_p)\big) \to 0$*, as* $p \to \infty$*, in the weak sense of currents on* X*.*

(ii) If $\liminf_{p\to\infty} C_p A_p^{-\nu} = 0$ *then there exists a sequence of natural numbers* $p_j \nearrow \infty$ *such that for* σ*-a. e. sequence* $\{s_p\} \in \mathcal{H}$ *we have*

$$\frac{1}{A_{p_j}} \log |s_{p_j}|_{h_{p_j}} \to 0\,, \quad \frac{1}{A_{p_j}}\big([s_{p_j} = 0] - c_1(L_{p_j}, h_{p_j})\big) \to 0\,, \ \text{ as } j \to \infty,$$

in $L^1(X, \omega^n)$*, respectively in the weak sense of currents on* X*.*

(iii) If $\sum_{p=1}^{\infty} C_p A_p^{-\nu} < \infty$ *then for* σ*-a. e. sequence* $\{s_p\} \in \mathcal{H}$ *we have*

$$\frac{1}{A_p} \log |s_p|_{h_p} \to 0\,, \quad \frac{1}{A_p}\big([s_p = 0] - c_1(L_p, h_p)\big) \to 0\,, \ \text{ as } p \to \infty,$$

in $L^1(X, \omega^n)$*, respectively in the weak sense of currents on* X*.*

Remark 1.2. If the measures σ_p verify condition (B) with constants $C_p = \Gamma_\nu$ independent of p then the hypothesis of (i) (and hence of (ii)), $\lim_{p\to\infty} \Gamma_\nu A_p^{-\nu} = 0$, is automatically verified since by (1.1),

$$A_p \geq a_p \int_X \omega^n\,, \ \text{ so } A_p \to \infty \text{ as } p \to \infty.$$

Moreover, the hypothesis of (iii) takes the simpler form $\sum_{p=1}^{\infty} A_p^{-\nu} < \infty$.

An important ingredient in the proof of Theorem 1.1 is the asymptotic behavior of the Bergman kernel functions P_p of the spaces $H^0_{(2)}(X, L_p)$ (see (2.1) for the definition) established in [CMM, Theorem 1.1]: namely, one has that

$$\frac{1}{A_p} \log P_p \to 0 \ \text{ as } p \to \infty \text{ in } L^1(X, \omega^n).$$

Theorem 1.1 will follow from this using Theorem 4.1, which shows, under very general assumptions, that the equidistribution of zeros of random holomorphic sections is a consequence of the asymptotic behavior of the Bergman kernel (see (4.1)). A similar approach was used in a different context in [CM1, Theorems 1.1 and 1.2].

If $(L_p, h_p) = (L^p, h^p)$, where (L, h) is a fixed singular Hermitian holomorphic line bundle, Theorem 1.1 gives analogues of the equidistribution results from [SZ1, CM1, CM2, CM3, CMM] for Gaussian ensembles and [DS, Ba1, Ba3, BL] for non-Gaussian ensembles on compact normal Kähler spaces. Note that in this case hypothesis (1.2) is automatically verified as $c_1(L^p, h^p) = p\, c_1(L, h)$, so we can take $T_0 = c_1(L, h)/\|c_1(L, h)\|$, where $\|c_1(L, h)\| := \int_X c_1(L, h) \wedge \omega^{n-1}$. We formulate here a corollary in this situation, for further variations of Theorem 1.1 see Section 4.

Corollary 1.3. *Let* (X, ω) *be a compact normal Kähler space and* (L, h) *be a singular Hermitian holomorphic line bundle on* X *such that* $c_1(L, h) \geq \varepsilon\omega$ *for some* $\varepsilon > 0$*. For* $p \geq 1$ *let* σ_p *be probability measures on* $H^0_{(2)}(X, L^p)$ *satisfying condition (B). Then the following hold:*

(i) If $\lim_{p\to\infty} C_p\, p^{-\nu} = 0$ *then* $\dfrac{1}{p}\,\mathbb{E}[s_p = 0] \to c_1(L,h)$*, as* $p \to \infty$*, weakly on* X.

(ii) If $\liminf_{p\to\infty} C_p\, p^{-\nu} = 0$ *then there exists a sequence of natural numbers* $p_j \nearrow \infty$ *such that for* σ*-a. e. sequence* $\{s_p\} \in \mathcal{H}$ *we have as* $j \to \infty$*,*

$$\frac{1}{p_j}\,\log|s_{p_j}|_{h^{p_j}} \to 0 \ \textit{ in } L^1(X,\omega^n)\,, \ \ \frac{1}{p_j}\,[s_{p_j} = 0] \to c_1(L,h)\,, \textit{ weakly on } X.$$

(iii) If $\sum_{p=1}^{\infty} C_p\, p^{-\nu} < \infty$ *then for* σ*-a. e. sequence* $\{s_p\} \in \mathcal{H}$ *we have as* $p \to \infty$*,*

$$\frac{1}{p}\,\log|s_p|_{h^p} \to 0 \ \textit{ in } L^1(X,\omega^n)\,, \ \ \frac{1}{p}\,[s_p = 0] \to c_1(L,h)\,, \textit{ weakly on } X.$$

It is by now well established that the off-diagonal decay of the Bergman/Szegő kernel for powers L^p of a line bundle L implies the asymptotics of the variance current and variance number for zeros of random holomorphic sections of L^p, cf. [Ba2, ST, SZ2]. Note also that the Bergman kernel provides the 2-point correlation function for the determinantal random point process defined by the Bergman projection [Ber, §6.1].

We wish to consider here the off-diagonal decay for Bergman kernels of a sequence L_p satisfying (1.1). We expect that this will have applications in obtaining a Central Limit Theorem for smooth linear statistics of zero divisors. To state our result, let us introduce the relevant definitions. We consider the situation where X is smooth and the Hermitian metrics h_p on L_p are also smooth. Let $L^2(X,L_p)$ be the space of L^2 integrable sections of L_p with respect to the metric h_p and the volume form $\omega^n/n!$. We assume now that h_p is smooth, hence $H^0_{(2)}(X,L_p) = H^0(X,L_p)$. Let $P_p : L^2(X,L_p) \to H^0(X,L_p)$ be the orthogonal projection. The Bergman kernel $P_p(x,y)$ is defined as the integral kernel of this projection, see [MM1, Definition 1.4.2]. Let $d_p = \dim H^0(X,L_p)$ and $(S^p_j)_{j=1}^{d_p}$ be an orthonormal basis of $H^0(X,L_p)$. We have

$$P_p(x,y) = \sum_{j=1}^{d_p} S^p_j(x) \otimes S^p_j(y)^* \in L_{p,x} \otimes L^*_{p,y}\,,$$

where $S^p_j(y)^* = \langle\,\cdot\,, S^p_j(y)\rangle_{h_p} \in L^*_{p,y}$. We set $P_p(x) := P_p(x,x)$.

The next result provides the exponential off-diagonal decay of the Bergman kernels $P_p(x,y)$ for sequences of positive line bundles (L_p,h_p). Adapting methods from [L, Be] we prove the following:

Theorem 1.4. *Let* (X,ω) *be a compact Kähler manifold of dimension* n *and* (L_p,h_p)*,* $p \geq 1$*, be a sequence of holomorphic line bundles on* X *with Hermitian metrics* h_p *of class* $\mathscr{C}^3$ *whose curvature forms verify* (1.1)*. Assume that*

$$\varepsilon_p := \|h_p\|_3^{1/3}\, a_p^{-1/2} \to 0 \ \textit{ as } p \to \infty\,. \tag{1.5}$$

Then there exist constants $C, T > 0$*,* $p_0 \geq 1$*, such that for every* $x,y \in X$ *and* $p > p_0$ *we have*

$$\big|P_p(x,y)\big|^2_{h_p} \leq C\exp\big(-T\sqrt{a_p}\,d(x,y)\big)\,\frac{c_1(L_p,h_p)^n_x}{\omega^n_x}\,\frac{c_1(L_p,h_p)^n_y}{\omega^n_y}\,. \tag{1.6}$$

Here $\|h_p\|_3$ denotes the sup-norm of the derivatives of h_p of order at most three with respect to a reference cover of X as defined in Section 2.3, and $d(x, y)$ denotes the distance on X induced by the Kähler metric ω. We also recall that, in the hypotheses of Theorem 1.4, the first order asymptotics of the Bergman kernel function $P_p(x) = P_p(x, x)$ was obtained in [CMM, Theorem 1.3] (see Theorem 3.3 below).

The situation when $(L_p, h_p) = (L^p, h^p)$ was intensively studied. Let $(L_p, h_p) = (L^p, h^p)$, such that there exists a constant $\varepsilon > 0$ with

$$c_1(L, h) \geqslant \varepsilon\omega\,. \tag{1.7}$$

Then $a_p = p\varepsilon$ and $\|h_p\|_3 \lesssim p$ so (1.1) and (1.5) are satisfied, thus (1.6) holds in this case, and is a particular case of (1.8) below. Namely, by [MM2, Theorem 1], there exist $T > 0$, $p_0 > 0$ so that for any $k \in \mathbb{N}$, there exists $C_k > 0$ such that for any $p \geqslant p_0$, $x, y \in X$, we have

$$|P_p(x, y)|_{\mathscr{C}^k} \leqslant C_k\, p^{n+\frac{k}{2}} \exp(-T\sqrt{p}\, d(x, y))\,. \tag{1.8}$$

In [DLM, Theorem 4.18], [MM1, Theorem 4.2.9], a refined version of (1.8) was obtained, i.e., the asymptotic expansion of $P_p(x, y)$ for $p \to +\infty$ with an exponential estimate of the remainder. The estimate (1.8) holds actually for complete Kähler manifolds with bounded geometry and for the Bergman kernel of the bundle $L^p \otimes E$, where E is a fixed holomorphic Hermitian vector bundle.

Assume that $X = \mathbb{C}^n$ with the Euclidean metric, $L = \mathbb{C}^{n+1}$ and $h = e^{-\varphi}$ where $\varphi : X \to \mathbb{R}$ is a smooth plurisubharmonic function such that (1.7) holds. Then the estimate (1.8) with $k = 0$ was obtained by [Ch1] for $n = 1$ and [De], [L] for $n \geqslant 1$ (cf. also [Be]). In [Ba2, Theorem 2.4] the exponential decay was obtained for a family of weights having super logarithmic growth at infinity.

Assume that X is a compact Kähler manifold, $c_1(L, h) = \omega$ and take $k = 0$ and $d(x, y) > \delta > 0$. Then (1.8) was obtained in [LZ, Theorem 2.1] (see also [Ber]) and a sharper estimate than (1.8) is due to Christ [Ch2].

The paper is organized as follows. After introducing necessary notions in Section 2, we prove Theorem 1.4 in Section 3. In Section 4 we prove Theorem 1.1 and we provide examples of measures satisfying condition (B) showing how Theorem 1.1 transforms in these cases.

2. Preliminaries

2.1. **Plurisubharmonic functions and currents on analytic spaces.** Let X be a complex space. A chart (U, τ, V) on X is a triple consisting of an open set $U \subset X$, a closed complex space $V \subset G \subset \mathbb{C}^N$ in an open set G of $\mathbb{C}^N$ and a biholomorphic map $\tau : U \to V$ (in the category of complex spaces). The map $\tau : U \to G \subset \mathbb{C}^N$ is called a local embedding of the complex space X. We write

$$X = X_{\rm reg} \cup X_{\rm sing}\,,$$

where $X_{\rm reg}$ (resp. $X_{\rm sing}$) is the set of regular (resp. singular) points of X. Recall that a reduced complex space $(X, \mathscr{O})$ is called normal if for every $x \in X$ the local ring $\mathscr{O}_x$ is integrally closed in its quotient field $\mathscr{M}_x$. Every normal complex space is locally irreducible and locally pure dimensional, cf. [GR2, p. 125], $X_{\rm sing}$ is a closed complex subspace of X with $\operatorname{codim} X_{\rm sing} \geq 2$. Moreover, Riemann's second extension theorem holds on normal complex spaces [GR2, p. 143]. In particular, every holomorphic function on $X_{\rm reg}$ extends uniquely to a holomorphic function on X.

Let X be a complex space. A continuous (resp. smooth) function on X is a function $\varphi : X \to \mathbb{C}$ such that for every $x \in X$ there exists a local embedding $\tau : U \to G \subset \mathbb{C}^N$ with $x \in U$ and a continuous (resp. smooth) function $\widetilde{\varphi} : G \to \mathbb{C}$ such that $\varphi|_U = \widetilde{\varphi} \circ \tau$.

A *(strictly) plurisubharmonic (psh)* function on X is a function $\varphi : X \to [-\infty, \infty)$ such that for every $x \in X$ there exists a local embedding $\tau : U \to G \subset \mathbb{C}^N$ with $x \in U$ and a (strictly) psh function $\widetilde{\varphi} : G \to [-\infty, \infty)$ such that $\varphi|_U = \widetilde{\varphi} \circ \tau$. If $\widetilde{\varphi}$ can be chosen continuous (resp. smooth), then φ is called a continuous (resp. smooth) psh function. The definition is independent of the chart, as is seen from [N, Lemma 4]. The analogue of Riemann's second extension theorem for psh functions holds on normal complex spaces [GR1, Satz 4]. In particular, every psh function on $X_{\rm reg}$ extends uniquely to a psh function on X. We let $PSH(X)$ denote the set of psh functions on X, and refer to [GR1], [N], [FN], [D2] for the properties of psh functions on X. We recall here that psh functions on X are locally integrable with respect to the area measure on X given by any local embedding $\tau : U \to G \subset \mathbb{C}^N$ [D2, Proposition 1.8].

Let X be a complex space of pure dimension n. We consider currents on X as defined in [D2] and we denote by $\mathcal{D}'_{p,q}(X)$ the space of currents of bidimension (p, q), or bidegree $(n-p, n-q)$ on X. In particular, if $v \in PSH(X)$ then $dd^c v \in \mathcal{D}'_{n-1,n-1}(X)$ is positive and closed. Let $\mathscr{T}(X)$ be the space of positive closed currents of bidegree $(1, 1)$ on X which have local psh potentials: $T \in \mathscr{T}(X)$ if every $x \in X$ has a neighborhood U (depending on T) such that there exists a psh function v on U with $T = dd^c v$ on $U \cap X_{\rm reg}$. Most of the currents considered here, such as the curvature currents $c_1(L_p, h_p)$ and the Fubini-Study currents γ_p, belong to $\mathscr{T}(X)$. A Kähler form on X is a current $\omega \in \mathscr{T}(X)$ whose local potentials extend to smooth strictly psh functions in local embeddings of X to Euclidean spaces. We call X a Kähler space if X admits a Kähler form (see also [G, p. 346], [O], [EGZ, Sec. 5]).

2.2. **Singular Hermitian holomorphic line bundles on analytic spaces.** Let L be a holomorphic line bundle on a normal Kähler space (X, ω). The notion of singular Hermitian metric h on L is defined exactly as in the smooth case (see [D3], [MM1, p. 97])): if e_α is a holomorphic frame of L over an open set $U_\alpha \subset X$ then $|e_\alpha|_h^2 = e^{-2\varphi_\alpha}$ where $\varphi_\alpha \in L^1_{loc}(U_\alpha, \omega^n)$. If $g_{\alpha\beta} = e_\beta/e_\alpha \in \mathcal{O}^*_X(U_\alpha \cap U_\beta)$ are the transition functions of L then $\varphi_\alpha = \varphi_\beta + \log|g_{\alpha\beta}|$. The curvature current $c_1(L, h) \in \mathcal{D}'_{n-1,n-1}(X)$ of h is defined by $c_1(L, h) = dd^c \varphi_\alpha$ on $U_\alpha \cap X_{\rm reg}$. We will denote by h^p the singular Hermitian metric induced by h on $L^p := L^{\otimes p}$. If $c_1(L, h) \geq 0$ then the weight φ_α is psh on $U_\alpha \cap X_{\rm reg}$ and since X is normal it extends to a psh function on U_α [GR1, Satz 4], hence $c_1(L, h) \in \mathscr{T}(X)$.

Let L be a holomorphic line bundle on a compact normal Kähler space (X, ω). Then the space $H^0(X, L)$ of holomorphic sections of L is finite dimensional (see e.g. [A, Théorème 1, p.27]). The space $H^0_{(2)}(X, L)$ defined as in (1.3) is therefore also finite dimensional.

For $p \geq 1$, we consider the space $H^0_{(2)}(X, L_p)$ defined in (1.3). Recall that $d_p = \dim H^0_{(2)}(X, L_p)$ and $S^p_1, \dots, S^p_{d_p}$ is an orthonormal basis of $H^0_{(2)}(X, L_p)$. If $x \in X$ and

e_p is a local holomorphic frame of L_p in a neighborhood U_p of x we write $S_j^p = s_j^p e_p$, where $s_j^p \in \mathcal{O}_X(U_p)$. Then the Bergman kernel functions and the Fubini-Study currents of the spaces $H^0_{(2)}(X, L_p)$ are defined as follows:

$$P_p(x) = \sum_{j=1}^{d_p} |S_j^p(x)|^2_{h_p}\,, \quad \gamma_p|_{U_p} = \frac{1}{2}\, dd^c \log\left(\sum_{j=1}^{d_p} |s_j^p|^2\right), \tag{2.1}$$

where $d = \partial + \overline{\partial}$ and $d^c = \frac{1}{2\pi i}(\partial - \overline{\partial})$. Note that P_p, γ_p are independent of the choice of basis $S_1^p, \ldots, S_{d_p}^p$. It follows from (2.1) that $\log P_p \in L^1(X, \omega^n)$ and

$$\gamma_p - c_1(L_p, h_p) = \frac{1}{2}\, dd^c \log P_p\,. \tag{2.2}$$

Moreover, as in [CM1, CM2], one has that

$$P_p(x) = \max\left\{|S(x)|^2_{h_p} : S \in H^0_{(2)}(X, L_p),\ \|S\|_p = 1\right\}, \tag{2.3}$$

for all $x \in X$ where $|e_p(x)|_{h_p} < \infty$.

We recall that if $S \in H^0(X, L_p)$ the Lelong-Poincaré formula shows that

$$[S = 0] = c_1(L_p, h_p) + dd^c \log |S|_{h_p}\,. \tag{2.4}$$

This follows exactly as in the case when X is smooth (see [MM1, Theorem 2.3.3]). Indeed, if X is a compact (reduced) analytic space of pure dimension and $S \in H^0(X, L_p)$, the current of integration $[S = 0] \in \mathscr{T}(X)$ is defined as the current with local psh potentials of the form $\log |s|$, where $S = se_p$, $s \in \mathcal{O}_X(U_p)$, and e_p is a holomorphic frame of L_p on the open set $U_p \subset X$. If $|e_p|_{h_p} = e^{-\varphi}$, then $\log |S|_{h_p} = \log |s| - \varphi$, which gives (2.4).

2.3. **Special weights of Hermitian metrics on reference covers.** Let (X, ω) be a compact Kähler manifold of dimension n. Let (U, z), $z = (z_1, \ldots, z_n)$, be local coordinates centered at a point $x \in X$. For $r > 0$ and $y \in U$ we denote by

$$\Delta^n(y, r) = \{z \in U : |z_j - y_j| \leq r,\ j = 1, \ldots, n\}$$

the (closed) polydisk of polyradius $(r, \ldots, r)$ centered at y. The coordinates (U, z) are called Kähler at $y \in U$ if

$$\omega_z = \frac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\overline{z}_j + O(|z - y|^2) \text{ on } U. \tag{2.5}$$

Definition 2.1 ([CMM, Definition 2.6]). A *reference cover* of X consists of the following data: for $j = 1, \ldots, N$, a set of points $x_j \in X$ and

(1) Stein open simply connected coordinate neighborhoods $(U_j, w^{(j)})$ centered at $x_j \equiv 0$,
(2) $R_j > 0$ such that $\Delta^n(x_j, 2R_j) \Subset U_j$ and for every $y \in \Delta^n(x_j, 2R_j)$ there exist coordinates on U_j which are Kähler at y,
(3) $X = \bigcup_{j=1}^N \Delta^n(x_j, R_j)$.

Given the reference cover as above we set $R = \min R_j$.

We can construct a reference cover as in [CMM, Section 2.5]. On U_j we consider the differential operators D_w^α, $\alpha \in \mathbb{N}^{2n}$, corresponding to the real coordinates associated to $w = w^{(j)}$. For a function $\varphi \in \mathscr{C}^k(U_j)$ we set

$$\|\varphi\|_k = \|\varphi\|_{k,w} = \sup\big\{|D_w^\alpha \varphi(w)| : \, w \in \Delta^n(x_j, 2R_j), |\alpha| \leq k\big\}. \tag{2.6}$$

Let (L, h) be a Hermitian holomorphic line bundle on X, where the metric h is of class $\mathscr{C}^\ell$. Note that $L|_{U_j}$ is trivial. For $k \leq \ell$ set

$$\begin{aligned} \|h\|_{k,U_j} &= \inf\big\{\|\varphi_j\|_k : \, \varphi_j \in \mathscr{C}^\ell(U_j) \text{ is a weight of } h \text{ on } U_j\big\}, \\ \|h\|_k &= \max\big\{1, \|h\|_{k,U_j} : \, 1 \leq j \leq N\big\}. \end{aligned} \tag{2.7}$$

Recall that φ_j is a weight of h on U_j if there exists a holomorphic frame e_j of L on U_j such that $|e_j|_h = e^{-\varphi_j}$. We have the following:

Lemma 2.2 ([CMM, Lemma 2.7]). *There exists a constant $C > 1$ (depending on the reference cover) with the following property: Given any Hermitian line bundle (L, h) on X, any $j \in \{1, \ldots, N\}$ and any $x \in \Delta^n(x_j, R_j)$ there exist coordinates $z = (z_1, \ldots, z_n)$ on $\Delta^n(x, R)$ which are centered at $x \equiv 0$ and Kähler coordinates for x such that*

(i) $n!\, dm \leq (1 + Cr^2)\omega^n$ and $\omega^n \leq (1 + Cr^2) n!\, dm$ hold on $\Delta^n(x, r)$ for any $r < R$ where $dm = dm(z)$ is the Euclidean volume relative to the coordinates z,

(ii) (L, h) has a weight φ on $\Delta^n(x, R)$ with $\varphi(z) = \sum_{j=1}^n \lambda_j |z_j|^2 + \widetilde{\varphi}(z)$, where $\lambda_j \in \mathbb{R}$ and $|\widetilde{\varphi}(z)| \leq C\|h\|_3 |z|^3$ for $z \in \Delta^n(x, R)$.

3. Bergman kernel asymptotics

We prove in Section 3.1 an L^2-estimate for the solution of the $\overline{\partial}$-equation in the spirit of Donnelly-Fefferman, which is used in Section 3.2 to prove Theorem 1.4.

3.1. L^2-estimates for $\overline{\partial}$.

Let us recall the following version of Demailly's estimates for the $\overline{\partial}$ operator [D1, Théorème 5.1].

Theorem 3.1 ([CMM, Theorem 2.5]). *Let Y, $\dim Y = n$, be a complete Kähler manifold and let Ω be a Kähler form on Y (not necessarily complete) such that its Ricci form $\operatorname{Ric}_\Omega \geq -2\pi B\Omega$ on Y, for some constant $B > 0$. Let (L_p, h_p) be singular Hermitian holomorphic line bundles on Y such that $c_1(L_p, h_p) \geq 2a_p\Omega$, where $a_p \to \infty$ as $p \to \infty$, and fix p_0 such that $a_p \geq B$ for all $p > p_0$. If $p > p_0$ and $g \in L^2_{0,1}(Y, L_p, loc)$ verifies $\overline{\partial} g = 0$ and $\int_Y |g|^2_{h_p} \Omega^n < \infty$ then there exists $u \in L^2_{0,0}(Y, L_p, loc)$ such that $\overline{\partial} u = g$ and $\int_Y |u|^2_{h_p} \Omega^n \leq \frac{1}{a_p} \int_Y |g|^2_{h_p} \Omega^n$.*

The next result gives a weighted estimate for the solution of $\overline{\partial}$-equation which goes back to Donnelly-Fefferman [DF]. The idea is to twist with a non-necessarily plurisubharmonic weight whose gradient is however controlled in terms of its complex Hessian. We follow here [Ber, Theorem 4.3], similar estimates were used for $\mathbb{C}^n$ in [De, L].

Theorem 3.2. *Let (X, ω) be a compact Kähler manifold, $\dim X = n$, and (L_p, h_p) be singular Hermitian holomorphic line bundles on X such that h_p have locally bounded weights and*

$c_1(L_p,h_p)\geq a_p\omega$*, where* $a_p\to\infty$ *as* $p\to\infty$*. Then there exists* $p_0\in\mathbb{N}$ *with the following property: If* v_p *are real valued functions of class* $\mathscr{C}^2$ *on* X *such that*

$$\|\overline{\partial}v_p\|_{L^\infty(X)}\leq\frac{\sqrt{a_p}}{8}\,,\;\;dd^cv_p\geq-\frac{a_p}{2}\,\omega\,, \tag{3.1}$$

then

$$\int_X|u|^2_{h_p}e^{2v_p}\,\omega^n\leq\frac{16}{a_p}\int_X|\overline{\partial}u|^2_{h_p}e^{2v_p}\,\omega^n$$

holds for $p>p_0$ *and for every* $\mathscr{C}^1$*-smooth section* u *of* L_p *which is orthogonal to* $H^0(X,L_p)$ *with respect to the inner product induced by* h_p *and* ω^n*.*

Proof. We fix a constant $B>0$ such that $\mathrm{Ric}_\omega\geq-2\pi B\omega$ on X and p_0 such that $a_p\geq 4B$ if $p>p_0$. Consider the metric $g_p=h_pe^{-2v_p}$ on L_p. Then by (3.1),

$$c_1(L_p,g_p)=c_1(L_p,h_p)+dd^cv_p\geq\frac{a_p}{2}\,\omega.$$

Moreover

$$\big(e^{2v_p}u,S\big)_{g_p}:=\int_X\langle e^{2v_p}u,S\rangle_{g_p}\,\frac{\omega^n}{n!}=\int_X\langle u,S\rangle_{h_p}\,\frac{\omega^n}{n!}=0\,,\;\;\forall\,S\in H^0(X,L_p).$$

Let $\alpha=\overline{\partial}\big(e^{2v_p}u\big)=e^{2v_p}(2\overline{\partial}v_p\wedge u+\overline{\partial}u)$. By Theorem 3.1 there exists a section $\widetilde{u}\in L^2_{0,0}(X,L_p)$ such that $\overline{\partial}\widetilde{u}=\alpha$ and

$$\int_X\big|e^{2v_p}u\big|^2_{g_p}\,\omega^n\leq\int_X|\widetilde{u}|^2_{g_p}\,\omega^n\leq\frac{4}{a_p}\int_X|\alpha|^2_{g_p}\,\omega^n,$$

where the first inequality follows since $e^{2v_p}u$ is orthogonal to $H^0(X,L_p)$ with respect to the inner product $(\cdot,\cdot)_{g_p}$. Using (3.1) we obtain

$$|\alpha|^2_{g_p}=e^{2v_p}|2\overline{\partial}v_p\wedge u+\overline{\partial}u|^2_{h_p}\leq 2e^{2v_p}(4|\overline{\partial}v_p\wedge u|^2_{h_p}+|\overline{\partial}u|^2_{h_p})\leq 2e^{2v_p}\big(\frac{a_p}{16}\,|u|^2_{h_p}+|\overline{\partial}u|^2_{h_p}\big).$$

It follows that

$$\int_X|u|^2_{h_p}e^{2v_p}\,\omega^n\leq\frac{1}{2}\int_X|u|^2_{h_p}e^{2v_p}\,\omega^n+\frac{8}{a_p}\int_X|\overline{\partial}u|^2_{h_p}e^{2v_p}\,\omega^n,$$

which implies the conclusion. $\square$

3.2. **Proof of Theorem 1.4.** We recall the following result about the first term asymptotic expansion of the Bergman kernel function $P_p(x)=P_p(x,x)$ (see (2.1)):

Theorem 3.3 ([CMM, Theorem 1.3])**.** *Let* (X,ω) *be a compact Kähler manifold of dimension* n*. Let* (L_p,h_p)*,* $p\geq1$*, be a sequence of holomorphic line bundles on* X *with Hermitian metrics* h_p *of class* $\mathscr{C}^3$ *whose curvature forms verify* (1.1) *and such that* (1.5) *holds. Then there exist* $C>0$ *depending only on* (X,ω) *and* $p_0\in\mathbb{N}$ *such that*

$$\left|P_p(x)\,\frac{\omega_x^n}{c_1(L_p,h_p)^n_x}-1\right|\leq C\varepsilon_p^{2/3} \tag{3.2}$$

holds for every $x\in X$ *and* $p>p_0$*.*

Recall that $d(x,y)$, $x,y\in X$, denotes the distance induced by the Kähler metric ω.

Proof of Theorem 1.4. We use ideas from the proof of [L, Proposition 9] together with methods from [Be, Section 2] and [CMM, Theorem 1.3]. Let us consider a reference cover of X as in Definition 2.1. Let $p_0 \in \mathbb{N}$ be sufficiently large such that

$$r_p := a_p^{-1/2} < R/2$$

and the conclusions of Theorems 3.2 and 3.3 hold for $p > p_0$. If $y \in X$ and $r > 0$ we let $B(y,r) := \{\zeta \in X : d(y,\zeta) < r\}$ and we fix a constant $\tau > 1$ such that, for every $y \in X$, $\Delta^n(y,r_p) \subset B(y,\tau r_p)$, where $\Delta^n(y,r_p)$ is the (closed) polydisc centered at y defined using the coordinates centered at y given by Lemma 2.2.

We show first that there exists a constant $C' > 1$ with the following property: If $y \in X$, so $y \in \Delta^n(x_j,R_j)$ for some j, and z are coordinates centered at y as in Lemma 2.2, then

$$|S(y)|^2_{h_p} \leq C' \, \frac{c_1(L_p,h_p)^n_y}{\omega^n_y} \int_{\Delta^n(y,r_p)} |S|^2_{h_p} \, \frac{\omega^n}{n!} \, , \tag{3.3}$$

where $\Delta^n(y,r_p)$ is the (closed) polydisc centered at $y=0$ in the coordinates z and S is any continuous section of L_p on X which is holomorphic on $\Delta^n(y,r_p)$. Indeed, let

$$\varphi_p(z) = \varphi'_p(z) + \widetilde{\varphi}_p(z) \, , \;\; \varphi'_p(z) = \sum_{l=1}^{n} \lambda^p_l |z_l|^2 \, ,$$

be a weight of h_p on $\Delta^n(y,R)$ so that $\widetilde{\varphi}_p$ verifies (ii) in Lemma 2.2 and let e_p be a frame of L_p on U_j with $|e_p|_{h_p} = e^{-\varphi_p}$. Writing $S = se_p$, where $s \in \mathcal{O}(\Delta^n(y,r_p))$, and using the sub-averaging inequality for psh functions we get

$$|S(y)|^2_{h_p} = |s(0)|^2 \leq \frac{\int_{\Delta^n(0,r_p)} |s|^2 e^{-2\varphi'_p} \, dm}{\int_{\Delta^n(0,r_p)} e^{-2\varphi'_p} \, dm} \, .$$

If $C > 1$ is the constant from Lemma 2.2 then

$$\begin{aligned} \int_{\Delta^n(0,r_p)} |s|^2 e^{-2\varphi'_p} \, dm &\leq (1+Cr_p^2) \exp\big(2 \max_{\Delta^n(0,r_p)} \widetilde{\varphi}_p\big) \int_{\Delta^n(0,r_p)} |s|^2 e^{-2\varphi_p} \frac{\omega^n}{n!} \\ &\leq (1+Cr_p^2) \exp\big(2C\|h_p\|_3 \, r_p^3\big) \int_{\Delta^n(0,r_p)} |S|^2_{h_p} \, \frac{\omega^n}{n!} \, . \end{aligned}$$

Set

$$E(r) := \int_{|\xi|\leq r} e^{-2|\xi|^2} \, dm(\xi) = \frac{\pi}{2} \, \left(1 - e^{-2r^2}\right),$$

where dm is the Lebesgue measure on $\mathbb{C}$. Since $\lambda^p_j \geq a_p$ and $E(1) > 1$ we have

$$\int_{\Delta^n(0,r_p)} e^{-2\varphi'_p} \, dm \geq \frac{E(r_p\sqrt{a_p}\,)^n}{\lambda^p_1 \ldots \lambda^p_n} \geq \frac{1}{\lambda^p_1 \ldots \lambda^p_n} \, .$$

Hence

$$|S(y)|^2_{h_p} \leq (1+Cr_p^2) \exp\big(2C\|h_p\|_3 \, r_p^3\big) \, \lambda^p_1 \ldots \lambda^p_n \int_{\Delta^n(0,r_p)} |S|^2_{h_p} \, \frac{\omega^n}{n!} \, .$$

Note that at y, $\omega_y = \frac{i}{2}\sum_{j=1}^n dz_j\wedge d\bar z_j$, $c_1(L_p,h_p)_y = dd^c\varphi_p(0) = \frac{i}{\pi}\sum_{j=1}^n \lambda_j^p dz_j\wedge d\bar z_j$, thus

$$\lambda_1^p\ldots\lambda_n^p = \Big(\frac{\pi}{2}\Big)^n\, \frac{c_1(L_p,h_p)_y^n}{\omega_y^n}\,.$$

Since $r_p\to 0$ and, by (1.5), $\|h_p\|_3\, r_p^3 = \varepsilon_p^3 \to 0$, there exists a constant $C'>1$ such that

$$\Big(\frac{\pi}{2}\Big)^n\,(1+Cr_p^2)\exp\big(2C\|h_p\|_3\, r_p^3\big)\leq C'$$

for all $p\geq 1$. This yields (3.3).

We continue now with the proof of the theorem. Fix $x\in X$. There exists a section $S_p=S_{p,x}\in H^0(X,L_p)$ such that

$$|S_p(y)|^2_{h_p} = |P_p(x,y)|^2_{h_p}\,,\ \ \forall\, y\in X.$$

Then

$$\|S_p\|_p^2 = \int_X |S_p(y)|^2_{h_p}\,\frac{\omega_y^n}{n!} = \int_X |P_p(x,y)|^2_{h_p}\,\frac{\omega_y^n}{n!} = P_p(x).$$

By Theorem 3.3 there exists a constant $C''>1$ such that for all $p\geq 1$ and $y\in X$,

$$P_p(y)\leq C''\,\frac{c_1(L_p,h_p)_y^n}{\omega_y^n}\,. \tag{3.4}$$

Assume first that $y\in X$ and $d(x,y)\leq 4\tau r_p = 4\tau a_p^{-1/2}$. Using (2.3) and (3.4) we obtain

$$\begin{aligned}|P_p(x,y)|^2_{h_p} &= |S_p(y)|^2_{h_p}\leq P_p(y)\|S_p\|_p^2 = P_p(x)P_p(y)\leq (C'')^2\,\frac{c_1(L_p,h_p)_x^n}{\omega_x^n}\,\frac{c_1(L_p,h_p)_y^n}{\omega_y^n}\\ &\leq e^{4\tau}(C'')^2\,\frac{c_1(L_p,h_p)_x^n}{\omega_x^n}\,\frac{c_1(L_p,h_p)_y^n}{\omega_y^n}\,e^{-\sqrt{a_p}\,d(x,y)}\,.\end{aligned}$$

We treat now the case when $y\in X$ and $\delta:=d(x,y)>4\tau r_p=4\tau a_p^{-1/2}$. By (3.3) and the definition of S_p we have

$$|P_p(x,y)|^2_{h_p} = |S_p(y)|^2_{h_p}\leq C'\,\frac{c_1(L_p,h_p)_y^n}{\omega_y^n}\int_{\Delta^n(y,r_p)}|P_p(x,\zeta)|^2_{h_p}\,\frac{\omega_\zeta^n}{n!}\,. \tag{3.5}$$

Note that

$$\Delta^n(x,r_p)\subset B(x,\delta/4)\,,\ \ \Delta^n(y,r_p)\subset\{\zeta\in X: d(x,\zeta)>3\delta/4\}.$$

Let χ be a non-negative smooth function on X such that

$$\chi(\zeta)=1 \text{ if } d(x,\zeta)\geq 3\delta/4,\ \ \chi(\zeta)=0 \text{ if } d(x,\zeta)\leq\delta/2,\ \text{ and } |\overline\partial\chi(\zeta)|^2\leq\frac{c}{\delta^2}\,\chi(\zeta) \tag{3.6}$$

for some constant $c>0$. Then we have

$$\begin{aligned}\int_{\Delta^n(y,r_p)}|P_p(x,\zeta)|^2_{h_p}\,\frac{\omega_\zeta^n}{n!} &\leq \int_X |P_p(x,\zeta)|^2_{h_p}\chi(\zeta)\,\frac{\omega_\zeta^n}{n!}\\ &= \max\left\{|P_p(\chi S)(x)|^2_{h_p}:\ S\in H^0(X,L_p),\ \int_X |S|^2_{h_p}\chi\,\frac{\omega^n}{n!}=1\right\},\end{aligned}$$

where

$$P_p(\chi S)(x) = \int_X P_p(x,\zeta)(\chi(\zeta)S(\zeta))\,\frac{\omega_\zeta^n}{n!}$$

is the Bergman projection of the smooth section χS to $H^0(X, L_p)$.

It remains to estimate $|P_p(\chi S)(x)|^2_{h_p}$, where $S \in H^0(X, L_p)$ and $\int_X |S|^2_{h_p}\chi\,\frac{\omega^n}{n!} = 1$. To this end we consider the smooth section u of L_p given by

$$u := \chi S - P_p(\chi S).$$

Note that u is orthogonal to $H^0(X, L_p)$ with respect to the inner product $(\cdot,\cdot)_p$ induced by h_p and $\omega^n/n!$. Moreover, since $\chi(x) = 0$, and since u is holomorphic in the polydisc $\Delta^n(x, r_p)$ centered at x and defined using the coordinates centered at x given by Lemma 2.2, it follows by (3.3) that

$$|P_p(\chi S)(x)|^2_{h_p} = |u(x)|^2_{h_p} \leq C'\,\frac{c_1(L_p,h_p)^n_x}{\omega^n_x}\int_{\Delta^n(x,r_p)} |u|^2_{h_p}\,\frac{\omega^n}{n!}\,. \tag{3.7}$$

We will estimate the latter integral using Theorem 3.2. Let $f : [0,\infty) \to (-\infty, 0]$ be a smooth function such that $f(x) = 0$ for $x \leq 1/4$, $f(x) = -x$ for $x \geq 1/2$, and set $g_\delta(x) := \delta f(x/\delta)$. There exists a constant $M > 0$ such that $|g'_\delta(x)| \leq M$ and $|g''_\delta(x)| \leq M/\delta$ for all $x \geq 0$. We define the function

$$v_p(\zeta) := \varepsilon\sqrt{a_p}\,g_\delta(d(x,\zeta))\,,\ \ \zeta \in X.$$

Then there exists a constant $M' > 0$ such that

$$\|\overline{\partial}v_p\|_{L^\infty(X)} \leq M'\varepsilon\sqrt{a_p}\ ,\ \ dd^c v_p \geq -\frac{M'\varepsilon}{\delta}\,\sqrt{a_p}\,\omega \geq -\frac{M'\varepsilon a_p}{4\tau}\,\omega,$$

since $\delta > 4\tau {a_p}^{-1/2}$. So v_p satisfies (3.1) if we take $\varepsilon = 1/(8M')$. We have that $v_p = 0$ in $B(x,\delta/4) \supset \Delta^n(x, r_p)$. Moreover

$$\overline{\partial}u = \overline{\partial}(\chi S) = \overline{\partial}\chi \wedge S$$

is supported in the set $V_\delta := \{\zeta \in X :\ \delta/2 \leq d(x,\zeta) \leq 3\delta/4\}$, and $v_p(\zeta) = -\varepsilon\sqrt{a_p}\,d(x,\zeta) \leq -\varepsilon\sqrt{a_p}\,\delta/2$ on this set. By Theorem 3.2 and (3.6) we get

$$\begin{aligned}\int_{\Delta^n(x,r_p)} |u|^2_{h_p}\,\frac{\omega^n}{n!} &\leq \int_X |u|^2_{h_p}e^{2v_p}\,\frac{\omega^n}{n!} \leq \frac{16}{a_p}\int_{V_\delta} |\overline{\partial}(\chi S)|^2_{h_p}e^{2v_p}\,\frac{\omega^n}{n!}\\ &\leq \frac{16c}{a_p\delta^2}\,e^{-\varepsilon\sqrt{a_p}\,\delta}\int_{V_\delta} |S|^2_{h_p}\chi\,\frac{\omega^n}{n!} \leq c\,e^{-\varepsilon\sqrt{a_p}\,\delta},\end{aligned}$$

since $a_p\delta^2 > 16\tau^2 > 16$. Hence (3.7) implies that

$$|P_p(\chi S)(x)|^2_{h_p} \leq C'c\,\frac{c_1(L_p,h_p)^n_x}{\omega^n_x}\,e^{-\varepsilon\sqrt{a_p}\,\delta}\,.$$

It follows that

$$\int_{\Delta^n(y,r_p)} |P_p(x,\zeta)|^2_{h_p}\,\frac{\omega^n_\zeta}{n!} \leq C'c\,\frac{c_1(L_p,h_p)^n_x}{\omega^n_x}\,e^{-\varepsilon\sqrt{a_p}\,d(x,y)}.$$

Combined with (3.5) this gives

$$|P_p(x,y)|^2_{h_p} \leq c(C')^2 \, \frac{c_1(L_p,h_p)^n_x}{\omega^n_x} \, \frac{c_1(L_p,h_p)^n_y}{\omega^n_y} \, e^{-\varepsilon\sqrt{a_p}\, d(x,y)},$$

and the proof is complete. □

4. Equidistribution for zeros of random holomorphic sections

In Section 4.1 we prove Theorem 1.1. We provide examples of measures satisfying condition (B) and give applications of Theorem 1.1 in Section 4.2.

4.1. **Proof of Theorem 1.1.**

We prove first the following general equidistribution result which combined with [CMM, Theorem 1.1] will yield Theorem 1.1.

Theorem 4.1. *Let X be a compact (reduced) analytic space of pure dimension n and ω be a Hermitian form on X. Let (L_p,h_p), $p\geq 1$, be singular Hermitian holomorphic line bundles on X and let $H^0_{(2)}(X,L_p)$ be the corresponding Bergman spaces defined in* (1.3) *endowed with probability measures σ_p that verify assumption (B). Let $(\mathcal{H},\sigma)$ be the product probability space defined in* (1.4). *Assume that there exist constants $\alpha_p>0$ such that*

$$\frac{1}{\alpha_p}\log P_p \to 0 \text{ as } p\to\infty, \text{ in } L^1(X,\omega^n), \tag{4.1}$$

where P_p is the Bergman kernel function of $H^0_{(2)}(X,L_p)$ defined in (2.1). *Then the following hold:*

(i) If $\lim_{p\to\infty} C_p\alpha_p^{-\nu}=0$ then $\frac{1}{\alpha_p}\big(\mathbb{E}[s_p=0]-c_1(L_p,h_p)\big)\to 0$, as $p\to\infty$, in the weak sense of currents on X.

(ii) If $\liminf_{p\to\infty} C_p\alpha_p^{-\nu}=0$ then there exists a sequence of natural numbers $p_j\nearrow\infty$ such that for σ-a. e. sequence $\{s_p\}\in\mathcal{H}$ we have

$$\frac{1}{\alpha_{p_j}}\log|s_{p_j}|_{h_{p_j}}\to 0\,,\;\; \frac{1}{\alpha_{p_j}}\big([s_{p_j}=0]-c_1(L_{p_j},h_{p_j})\big)\to 0\,,\;\text{ as } j\to\infty,$$

in $L^1(X,\omega^n)$, respectively in the weak sense of currents on X.

(iii) If $\sum_{p=1}^{\infty} C_p\alpha_p^{-\nu}<\infty$ then for σ-a. e. sequence $\{s_p\}\in\mathcal{H}$ we have

$$\frac{1}{\alpha_p}\log|s_p|_{h_p}\to 0\,,\;\; \frac{1}{\alpha_p}\big([s_p=0]-c_1(L_p,h_p)\big)\to 0\,,\;\text{ as } p\to\infty,$$

in $L^1(X,\omega^n)$, respectively in the weak sense of currents on X.

Proof. Note that if $H^0_{(2)}(X,L_p)\neq\{0\}$ then $\log P_p\in L^1(X,\omega^n)$, since it is locally the difference of a psh and an integrable function. Let γ_p be the Fubini-Study currents of the spaces $H^0_{(2)}(X,L_p)$ defined in (2.1).

(i) Let Φ be a smooth real valued $(n-1,n-1)$ form on X. By (2.2) and hypothesis (4.1) we have

$$\frac{1}{\alpha_p}\langle\gamma_p - c_1(L_p,h_p),\Phi\rangle = \frac{1}{\alpha_p}\int_X \log P_p\, dd^c\Phi \to 0\,,$$

so for the first assertion of (i) it suffices to show that

$$\frac{1}{\alpha_p}\langle\mathbb{E}[s_p=0]-\gamma_p,\Phi\rangle \to 0\,,\ \text{ as } p\to\infty. \tag{4.2}$$

Note that there exists a constant $c>0$ such that for every smooth real valued $(n-1,n-1)$ form Φ on X,

$$-c\|\Phi\|_{\mathscr{C}^2}\,\omega^n \leq dd^c\Phi \leq c\|\Phi\|_{\mathscr{C}^2}\,\omega^n.$$

Hence the total variation of $dd^c\Phi$ satisfies $|dd^c\Phi|\leq c\|\Phi\|_{\mathscr{C}^2}\,\omega^n$. Indeed, let $\tau: U\hookrightarrow G\subset\mathbb{C}^N$ be a local embedding of X, where $U\subset X$ and $G\subset\mathbb{C}^N$ are open, such that there exist a smooth real valued $(N-1,N-1)$ form $\widetilde{\Phi}$ and a Hermitian form Ω on G with $\Phi\mid_{U_{reg}}=\tau^\star\widetilde{\Phi}$ and $\omega\mid_{U_{reg}}=\tau^\star\Omega$. There exists a constant $c'>0$ such that for any smooth real valued $(N-1,N-1)$ form φ on G and any open set $G_0\Subset G$, we have

$$-c'\|\varphi\|_{\mathscr{C}^2(G_0)}\Omega^n \leq dd^c\varphi\mid_{G_0}\leq c'\|\varphi\|_{\mathscr{C}^2(G_0)}\Omega^n.$$

Our claim follows by taking a finite cover of X with sets of form $U_0=\tau^{-1}(G_0)$.

If $s_p\in H^0_{(2)}(X,L_p)$, using (2.4) and (2.2), we see that

$$\big\langle[s_p=0],\Phi\big\rangle = \big\langle c_1(L_p,h_p),\Phi\big\rangle + \int_X \log|s_p|_{h_p}dd^c\Phi = \big\langle\gamma_p,\Phi\big\rangle + \int_X \log\frac{|s_p|_{h_p}}{\sqrt{P_p}}\,dd^c\Phi. \tag{4.3}$$

Note that $\log\frac{|s_p|_{h_p}}{\sqrt{P_p}}\in L^1(X,\omega^n)$ as it is locally the difference of two psh functions.

We write

$$s_p=\sum_{j=1}^{d_p}a_jS_j^p.$$

Moreover, for $x\in X$ we let e_p be a holomorphic frame of L_p on a neighborhood U of x and we write $S_j^p=s_j^pe_p$, where $s_j^p\in\mathscr{O}_X(U)$. Let $\langle a,u^p\rangle=a_1u_1+\ldots+a_{d_p}u_{d_p}$, where

$$u^p(x):=(u_1(x),\ldots,u_{d_p}(x))\,,\ \ u_j(x)=\frac{s_j^p(x)}{\sqrt{|s_1^p(x)|^2+\ldots+|s_{d_p}^p(x)|^2}}\,. \tag{4.4}$$

Using Hölder's inequality and assumption (B) it follows that

$$\int_{H^0_{(2)}(X,L_p)}\Big|\log\frac{|s_p(x)|_{h_p}}{\sqrt{P_p(x)}}\Big|d\sigma_p(s_p)=\int_{\mathbb{C}^{d_p}}\big|\log|\langle a,u^p(x)\rangle|\big|d\sigma_p(a)\leq C_p^{1/\nu}.$$

Hence by Tonelli's theorem

$$\int_{H^0_{(2)}(X,L_p)}\int_X\Big|\log\frac{|s_p|_{h_p}}{\sqrt{P_p}}\Big||dd^c\Phi|\,d\sigma_p(s_p)\leq C_p^{1/\nu}\int_X|dd^c\Phi|\leq c\,C_p^{1/\nu}\|\Phi\|_{\mathscr{C}^2}\int_X\omega^n.$$

By (4.3) we conclude that

$$\big\langle \mathbb{E}[s_p=0],\Phi\big\rangle=\int_{H^0_{(2)}(X,L_p)}\big\langle[s_p=0],\Phi\big\rangle\,d\sigma_p(s_p)$$

is a well-defined positive closed current which satisfies

$$\big|\big\langle \mathbb{E}[s_p=0]-\gamma_p,\Phi\big\rangle\big|\leq c\,C_p^{1/\nu}\|\Phi\|_{\mathscr{C}^2}\int_X\omega^n.$$

Thus (4.2) holds since $C_p^{1/\nu}/\alpha_p\to 0$.

For the proof of assertion (ii), since $\liminf_{p\to\infty}C_p\alpha_p^{-\nu}=0$ we can find a sequence of natural numbers $p_j\nearrow\infty$ such that $\sum_{j=1}^{\infty}C_{p_j}\alpha_{p_j}^{-\nu}<\infty$. Then we proceed as in the proof of assertion (iii) given below, working with $\{p_j\}$ instead of $\{p\}$.

(iii) We define

$$Y_p,\,Z_p:\mathcal{H}\to[0,\infty)\,,\;\;Y_p(s)=\frac{1}{\alpha_p}\int_X\big|\log|s_p|_{h_p}\big|\,\omega^n\,,\;\;Z_p(s)=\frac{1}{\alpha_p}\int_X\Big|\log\frac{|s_p|_{h_p}}{\sqrt{P_p}}\Big|\,\omega^n\,,$$

where $s=\{s_p\}$. So

$$0\leq Y_p(s)\leq Z_p(s)+m_p\,,\;\;\text{where } m_p:=\frac{1}{2\alpha_p}\int_X|\log P_p|\,\omega^n.$$

Hypothesis (4.1) shows that $m_p\to 0$ as $p\to\infty$. By Hölder's inequality

$$0\leq Z_p(s)^\nu\leq\frac{1}{\alpha_p^\nu}\left(\int_X\omega^n\right)^{\nu-1}\int_X\Big|\log\frac{|s_p|_{h_p}}{\sqrt{P_p}}\Big|^\nu\omega^n.$$

For $x\in X$ and $u^p(x)$ as in (4.4) we obtain using (B) that

$$\int_{H^0_{(2)}(X,L_p)}\Big|\log\frac{|s_p(x)|_{h_p}}{\sqrt{P_p(x)}}\Big|^\nu d\sigma_p(s_p)=\int_{\mathbb{C}^{d_p}}\big|\log|\langle a,u^p(x)\rangle|\big|^\nu d\sigma_p(a)\leq C_p.$$

Hence by Tonelli's theorem

$$\int_{\mathcal{H}}Z_p(s)^\nu d\sigma(s)\leq\frac{1}{\alpha_p^\nu}\left(\int_X\omega^n\right)^{\nu-1}\int_X\int_{H^0_{(2)}(X,L_p)}\Big|\log\frac{|s_p|_{h_p}}{\sqrt{P_p}}\Big|^\nu d\sigma_p(s_p)\,\omega^n\leq\frac{C_p}{\alpha_p^\nu}\left(\int_X\omega^n\right)^\nu.$$

Therefore

$$\sum_{p=1}^{\infty}\int_{\mathcal{H}}Z_p(s)^\nu d\sigma(s)\leq\left(\int_X\omega^n\right)^\nu\sum_{p=1}^{\infty}\frac{C_p}{\alpha_p^\nu}<\infty\,.$$

It follows that $Z_p(s)\to 0$, and hence $Y_p(s)\to 0$ as $p\to\infty$ for σ-a. e. $s\in\mathcal{H}$. This means that $\frac{1}{\alpha_p}\log|s_p|_{h_p}\to 0$ in $L^1(X,\omega^n)$, hence by (2.4), $\frac{1}{\alpha_p}\big([s_p=0]-c_1(L_p,h_p)\big)\to 0$ weakly on X, for σ-a. e. sequence $\{s_p\}\in\mathcal{H}$. The proof of Theorem 4.1 is finished. $\square$

Proof of Theorem 1.1. By [CMM, Theorem 1.1] we have that

$$\frac{1}{A_p}\log P_p \to 0 \text{ as } p\to\infty\,,\text{ in } L^1(X,\omega^n).$$

Hence Theorem 1.1 follows at once from Theorem 4.1 with $\alpha_p := A_p$. $\square$

Let us give now a variation of Theorem 1.1 modeled on [CMM, Corollary 5.6]. It allows to approximate arbitrary ω-psh functions by logarithms of absolute values of holomorphic sections. Let (X,ω) be a Kähler manifold with a positive line bundle (L,h_0), where h_0 is a smooth Hermitian metric such that $c_1(L,h_0)=\omega$. The set of singular Hermitian metrics h on L with $c_1(L,h)\geq 0$ is in one-to-one correspondence to the set $PSH(X,\omega)$ of ω-plurisubharmonic (ω-psh) functions on X, by associating to $\psi\in PSH(X,\omega)$ the metric $h_\psi = h_0 e^{-2\psi}$ (see e.g., [D3, GZ]). Note that $c_1(L,h_\psi)=\omega+dd^c\psi$.

Corollary 4.2. *Let (X,ω) be a compact Kähler manifold and (L,h_0) be a positive line bundle on X such that $c_1(L,h_0)=\omega$. Let h be a singular Hermitian metric on L with $c_1(L,h)\geq 0$ and let $\psi\in PSH(X,\omega)$ be its global weight such that $h=h_0e^{-2\psi}$. Let $\{n_p\}_{p\geq 1}$ be a sequence of natural numbers such that*

$$n_p\to\infty \text{ and } n_p/p\to 0 \text{ as } p\to\infty. \tag{4.5}$$

Let h_p be the metric on L^p given by

$$h_p = h^{p-n_p}\otimes h_0^{n_p} = h_0^p\, e^{-2(p-n_p)\psi}. \tag{4.6}$$

For $p\geq 1$ let σ_p be probability measures on $H^0_{(2)}(X,L_p)=H^0_{(2)}(X,L^p,h_p)$ satisfying condition (B). Then the following hold:

(i) If $\lim\limits_{p\to\infty} C_p\,p^{-\nu}=0$ then $\frac{1}{p}\mathbb{E}[s_p=0]\to c_1(L,h)$, as $p\to\infty$, weakly on X.

(ii) If $\liminf\limits_{p\to\infty} C_p\,p^{-\nu}=0$ then there exists a sequence of natural numbers $p_j\nearrow\infty$ such that for σ-a. e. sequence $\{s_p\}\in\mathcal{H}$ we have as $j\to\infty$,

$$\frac{1}{p_j}\log|s_{p_j}|_{h_0^{p_j}}\to\psi \text{ in } L^1(X,\omega^n)\,,\quad \frac{1}{p_j}[s_{p_j}=0]\to c_1(L,h)\,,\text{ weakly on } X.$$

(iii) If $\sum\limits_{p=1}^{\infty} C_p\,p^{-\nu}<\infty$ then for σ-a. e. sequence $\{s_p\}\in\mathcal{H}$ we have as $p\to\infty$,

$$\frac{1}{p}\log|s_p|_{h_0^p}\to\psi \text{ in } L^1(X,\omega^n)\,,\quad \frac{1}{p}[s_p=0]\to c_1(L,h)\,,\text{ weakly on } X.$$

Proof. Note that $\log|s_p|_{h_p}=\log|s_p|_{h_0^p}-(p-n_p)\psi$. The corollary follows from Theorem 1.1 and the proofs of Corollaries 5.2 and 5.6 from [CMM]. $\square$

Corollary 4.2 is an extension of [BL, Theorem 5.2] which deals with the special case when $\psi=\mathcal{V}^*_{K,q}$ is the weighted ω-psh global extremal function of a compact $K\subset X$. Note that we use here a different scalar product than in [BL].

Remark 4.3. Let us give a local version of Theorem 1.1. Note that when X is smooth any holomorphic line bundle on X is trivial on any contractible Stein open subset $U \subset X$. Assume that (X, ω), (L_p, h_p) and σ_p verify the assumptions (A1), (A2) and (B). Let $U \subset X$ such that for every $p \geq 1$, $L_p|_U$ is trivial and let $e_p : U \to L_p$ be a holomorphic frame with $|e_p|_{h_p} = e^{-\varphi_p}$, where $\varphi_p \in PSH(U)$. For a section $s \in H^0(X, L_p)$ write $s = \widetilde{s}\, e_p$, with $\widetilde{s} \in \mathcal{O}(U)$. If $\sum_{p=1}^{\infty} C_p A_p^{-\nu} < \infty$, then for σ-a. e. sequence $\{s_p\} \in \mathcal{H}$ we have as $p \to \infty$,

$$\frac{1}{A_p}\Big(\log|\widetilde{s}_p| - \varphi_p\Big) \to 0 \text{ in } L^1(U, \omega^n), \quad \frac{1}{A_p}\Big([\,\widetilde{s}_p = 0] - dd^c\varphi_p\Big) \to 0, \text{ weakly on } U.$$

In particular, let $(L_p, h_p) = (L^p, h^p)$, where (L, h) is a fixed singular Hermitian holomorphic line bundle on X such that $c_1(L, h) \geq \varepsilon\omega$ for some $\varepsilon > 0$. Let $U \subset X$ such that $L|_U$ is trivial, let $e : U \to L$ be a holomorphic frame and with $|e|_h = e^{-\varphi}$, where $\varphi \in PSH(U)$. Consider the holomorphic frames $e_p = e^{\otimes p}$ of $L^p|_U$. If $\sum_{p=1}^{\infty} C_p\, p^{-\nu} < \infty$, then for σ-a. e. sequence $\{s_p\} \in \mathcal{H}$ we have as $p \to \infty$,

$$\frac{1}{p}\log|\widetilde{s}_p| \to \varphi \text{ in } L^1_{loc}(U), \quad \frac{1}{p}[\,\widetilde{s}_p = 0] \to dd^c\varphi, \text{ weakly on } U.$$

Example 4.4. We formulate now some of the previous results in the case of polynomials in $\mathbb{C}^n$. Consider $X = \mathbb{P}^n$ and $L_p = \mathcal{O}(p)$, $p \geq 1$, where $\mathcal{O}(1) \to \mathbb{P}^n$ is the hyperplane line bundle. Let $\mathbb{C}^n \hookrightarrow \mathbb{P}^n$, $\zeta \mapsto [1 : \zeta]$, be the standard embedding. The global holomorphic sections $H^0(\mathbb{P}^n, \mathcal{O}(p))$ of $\mathcal{O}(p)$ are given by homogeneous polynomials of degree p in the homogeneous coordinates $z_0, \ldots, z_n$ on $\mathbb{C}^{n+1}$. For any $\alpha \in \mathbb{N}^{n+1}$ the map $\mathbb{C}^{n+1} \ni z \mapsto z^\alpha$ is identified to a section $s_\alpha \in H^0(\mathbb{P}^n, \mathcal{O}(p))$.

On $U_0 = \{[1 : \zeta] \in \mathbb{P}^n : \zeta \in \mathbb{C}^n\} \cong \mathbb{C}^n$ we consider the holomorphic frame $e_p = s_{(p,0,\ldots,0)}$ of $\mathcal{O}(p)$, corresponding to z_0^p. The trivialization of $\mathcal{O}(p)$ using this frame gives an identification

$$H^0(\mathbb{P}^n, \mathcal{O}(p)) \to \mathbb{C}_p[\zeta], \;\; s \mapsto s/z_0^p, \tag{4.7}$$

with the space of polynomials of total degree at most p,

$$\mathbb{C}_p[\zeta] = \mathbb{C}_p[\zeta_1, \ldots, \zeta_n] := \{f \in \mathbb{C}[\zeta_1, \ldots, \zeta_n] : \deg(f) \leq p\}.$$

Let $\omega_{_{\rm FS}}$ denote the Fubini-Study Kähler form on $\mathbb{P}^n$ and $h_{_{\rm FS}}$ be the Fubini-Study metric on $\mathcal{O}(1)$, so $c_1(\mathcal{O}(1), h_{_{\rm FS}}) = \omega_{_{\rm FS}}$. The set $PSH(\mathbb{P}^n, p\,\omega_{_{\rm FS}})$ is in one-to-one correspondence to the set $p\mathcal{L}(\mathbb{C}^n)$, where $\mathcal{L}(\mathbb{C}^n)$ is the Lelong class of entire psh functions with logarithmic growth (cf. [GZ, Section 2]):

$$\mathcal{L}(\mathbb{C}^n) = \big\{\varphi \in PSH(\mathbb{C}^n) : \exists\, C_\varphi \in \mathbb{R} \text{ such that } \varphi(z) \leq \log^+\|z\| + C_\varphi \text{ on } \mathbb{C}^n\big\}.$$

The map $\mathcal{L}(\mathbb{C}^n) \to PSH(\mathbb{P}^n, \omega_{_{\rm FS}})$ is given by $\varphi \mapsto \widetilde{\varphi}$ where

$$\widetilde{\varphi}(w) = \begin{cases} \varphi(w) - \frac{1}{2}\log(1 + |w|^2), & w \in \mathbb{C}^n, \\ \limsup\limits_{z \to w, z \in \mathbb{C}^n} \widetilde{\varphi}(z), & w \in \mathbb{P}^n \setminus \mathbb{C}^n. \end{cases}$$

The one-to-one correspondence between singular Hermitian metrics h_p on $\mathcal{O}(p)$ with $c_1(\mathcal{O}(p), h_p) \geq 0$ and $p\mathcal{L}(\mathbb{C}^n)$ is given by sending a metric h_p to its weight φ_p on U_0 with respect to the standard frame e_p. Define the L^2-space

$$H^0_{(2)}(\mathbb{P}^n,\mathcal{O}(p),h_p)=\left\{s\in H^0(\mathbb{P}^n,\mathcal{O}(p)):\int_{\mathbb{P}^n}|s|^2_{h_p}\frac{\omega^n_{\mathrm{FS}}}{n!}<\infty\right\},$$

with the obvious scalar product. The map (4.7) induces an isometry between this space and the L^2-space of polynomials

$$\mathbb{C}_{p,(2)}[\zeta]=\left\{f\in\mathbb{C}_p[\zeta]:\int_{\mathbb{C}^n}|f|^2e^{-2\varphi_p}\frac{\omega^n_{\mathrm{FS}}}{n!}<\infty\right\}. \tag{4.8}$$

If σ_p are probability measures on $\mathbb{C}_{p,(2)}[\zeta]$ we denote by $\mathcal{H}$ the corresponding product probability space $(\mathcal{H},\sigma)=\Big(\prod_{p=1}^\infty\mathbb{C}_{p,(2)}[\zeta],\prod_{p=1}^\infty\sigma_p\Big)$.

Corollary 4.5. *Consider a sequence of functions $\varphi_p\in p\mathcal{L}(\mathbb{C}^n)$ such that $dd^c\varphi_p\geq a_p\,\omega_{\mathrm{FS}}$ on $\mathbb{C}^n$, where $a_p>0$ and $a_p\to\infty$ as $p\to\infty$. For $p\geq 1$ let σ_p be probability measures on $\mathbb{C}_{p,(2)}[\zeta]$ satisfying condition (B). Assume that $\sum_{p=1}^\infty C_pp^{-\nu}<\infty$. Then for σ-a. e. sequence $\{f_p\}\in\mathcal{H}$ we have as $p\to\infty$,*

$$\frac{1}{p}\Big(\log|f_p|-\varphi_p\Big)\to 0\ \text{ in } L^1(\mathbb{C}^n,\omega^n_{\mathrm{FS}})\,,\ \text{ hence in } L^1_{loc}(\mathbb{C}^n)\,,$$
$$\frac{1}{p}\Big([f_p=0]-dd^c\varphi_p\Big)\to 0\,,\ \text{ weakly on } \mathbb{C}^n\,.$$

Proof. If h_p is the singular Hermitian metric on $\mathcal{O}(p)$ corresponding to φ_p then

$$A_p=\int_{\mathbb{P}^n}c_1(\mathcal{O}(p),h_p)\wedge\omega^{n-1}_{\mathrm{FS}}=p,\ \text{and}\ c_1(\mathcal{O}(p),h_p)\mid_{\mathbb{C}^n}=dd^c\varphi_p\geq a_p\,\omega_{\mathrm{FS}}.$$

If T denotes the trivial extension of $dd^c\varphi_p$ to $\mathbb{P}^n$ then $T\geq a_p\,\omega_{\mathrm{FS}}$ on $\mathbb{P}^n$. By Siu's decomposition theorem, $c_1(\mathcal{O}(p),h_p)=T+b[z_0=0]$, where $b\geq 0$. Hence $c_1(\mathcal{O}(p),h_p)\geq T\geq a_p\,\omega_{\mathrm{FS}}$ on $\mathbb{P}^n$. The corollary now follows directly from Theorem 1.1. □

In particular, we obtain:

Corollary 4.6. *Let $\varphi\in\mathcal{L}(\mathbb{C}^n)$ such that $dd^c\varphi\geq\varepsilon\,\omega_{\mathrm{FS}}$ on $\mathbb{C}^n$ for some constant $\varepsilon>0$. For $p\geq 1$ construct the spaces $\mathbb{C}_{p,(2)}[\zeta]$ by setting of $\varphi_p=p\varphi$ in (4.8) and let σ_p be probability measures on $\mathbb{C}_{p,(2)}[\zeta]$ satisfying condition (B). If $\sum_{p=1}^\infty C_p\,p^{-\nu}<\infty$, then for σ-a. e. sequence $\{f_p\}\in\mathcal{H}$ we have as $p\to\infty$,*

$$\frac{1}{p}\log|f_p|\to\varphi\ \text{ in } L^1(\mathbb{C}^n,\omega^n_{\mathrm{FS}})\,,\quad \frac{1}{p}[f_p=0]\to dd^c\varphi\,,\ \text{ weakly on } \mathbb{C}^n\,. \tag{4.9}$$

We can also apply Corollary 4.2 to the setting of polynomials in $\mathbb{C}^n$ and obtain a version of Corollary 4.6 for arbitrary $\varphi\in\mathcal{L}(\mathbb{C}^n)$.

Corollary 4.7. *Let $\varphi\in\mathcal{L}(\mathbb{C}^n)$ and let h be the singular Hermitian metric on $\mathcal{O}(1)$ corresponding to φ. Let $\{n_p\}_{p\geq1}$ be a sequence of natural numbers such that (4.5) is satisfied. Consider the metric h_p on $\mathcal{O}(p)$ given by $h_p=h^{p-n_p}\otimes h^{n_p}_{\mathrm{FS}}$ (cf. (4.6)). For $p\geq 1$ let σ_p be probability measures on $H^0_{(2)}(\mathbb{P}^n,\mathcal{O}(p),h_p)\cong\mathbb{C}_{p,(2)}[\zeta]$ satisfying condition (B). If $\sum_{p=1}^\infty C_p\,p^{-\nu}<\infty$, then for σ-a. e. sequence $\{f_p\}\in\mathcal{H}$ we have (4.9) as $p\to\infty$.*

This is an extension (with a different scalar product) of [BL, Theorem 4.2] which deals with the special case when $\psi = V^*_{K,Q}$ is the weighted pluricomplex Green function of a nonpluripolar compact $K \subset \mathbb{C}^n$ [BL, (3.2)].

4.2. **Classes of measures verifying assumption (B).** In this section we give important examples of measures that verify condition (B) and we specialize Theorem 1.1 to these measures.

4.2.1. *Gaussians.* We consider here the measures σ_k on $\mathbb{C}^k$ that have Gaussian density,

$$d\sigma_k(a) = \frac{1}{\pi^k} e^{-\|a\|^2} dV_k(a)\,, \tag{4.10}$$

where $a = (a_1, \dots, a_k) \in \mathbb{C}^k$ and V_k is the Lebesgue measure on $\mathbb{C}^k$.

Lemma 4.8. *For every integer $k \geq 1$ and every $\nu \geq 1$,*

$$\int_{\mathbb{C}^k} |\log|\langle a, u\rangle||^\nu \, d\sigma_k(a) = \Gamma_\nu := 2\int_0^\infty r|\log r|^\nu e^{-r^2}\, dr\,,\ \ \forall\, u \in \mathbb{C}^k,\ \|u\| = 1\,.$$

Proof. Since σ_k is unitary invariant we have

$$\int_{\mathbb{C}^k} |\log|\langle a, u\rangle||^\nu \, d\sigma_k(a) = \int_{\mathbb{C}^k} |\log|a_1||^\nu \, d\sigma_k(a) = \frac{1}{\pi}\int_{\mathbb{C}} |\log|a_1||^\nu e^{-|a_1|^2}\, dV_1(a_1)\,.$$

□

Lemma 4.8 implies at once that in this case Theorem 1.1 takes the following simpler form:

Theorem 4.9. *Assume that (X, ω), (L_p, h_p) verify the assumptions (A1), (A2), and $\sigma_p := \sigma_{d_p}$ is the measure given by* (4.10) *on $H^0_{(2)}(X, L_p) \simeq \mathbb{C}^{d_p}$. Then the following hold:*

(i) $\dfrac{1}{A_p}\big(\mathbb{E}[s_p = 0] - c_1(L_p, h_p)\big) \to 0$, as $p \to \infty$, in the weak sense of currents on X. Moreover, there exists a sequence $p_j \nearrow \infty$ such that for σ-a. e. sequence $\{s_p\} \in \mathcal{H}$ we have

$$\frac{1}{A_{p_j}} \log|s_{p_j}|_{h_{p_j}} \to 0\,,\ \ \frac{1}{A_{p_j}}\big([s_{p_j} = 0] - c_1(L_{p_j}, h_{p_j})\big) \to 0\,,\ \ \textit{as } j \to \infty,$$

in $L^1(X, \omega^n)$, respectively in the weak sense of currents on X.

(ii) If $\sum_{p=1}^\infty A_p^{-\nu} < \infty$ for some $\nu \geq 1$, then for σ-a. e. sequence $\{s_p\} \in \mathcal{H}$ we have

$$\frac{1}{A_p} \log|s_p|_{h_p} \to 0\,,\ \ \frac{1}{A_p}\big([s_p = 0] - c_1(L_p, h_p)\big) \to 0\,,\ \ \textit{as } p \to \infty,$$

in $L^1(X, \omega^n)$, respectively in the weak sense of currents on X.

4.2.2. *Fubini-Study volumes.* The Fubini-Study volume on the projective space $\mathbb{P}^k \supset \mathbb{C}^k$ is given by the measure σ_k on $\mathbb{C}^k$ with density

$$d\sigma_k(a) = \frac{k!}{\pi^k}\,\frac{1}{(1+\|a\|^2)^{k+1}}\,dV_k(a)\,, \tag{4.11}$$

where $a = (a_1, \ldots, a_k) \in \mathbb{C}^k$ and V_k is the Lebesgue measure on $\mathbb{C}^k$.

Lemma 4.10. *For every integer $k \geq 1$ and every $\nu \geq 1$,*

$$\int_{\mathbb{C}^k} |\log|\langle a, u\rangle||^\nu\, d\sigma_k(a) = \Gamma_\nu := 2\int_0^\infty \frac{r|\log r|^\nu}{(1+r^2)^2}\,dr\,,\ \ \forall\, u \in \mathbb{C}^k,\ \|u\| = 1\,.$$

Proof. Recall that the area of the unit sphere in $\mathbb{C}^k$ is $s_{2k} = 2\pi^k/(k-1)!$. Since σ_k is unitary invariant we have

$$\int_{\mathbb{C}^k} |\log|\langle a, u\rangle||^\nu\, d\sigma_k(a) = \int_{\mathbb{C}^k} |\log|a_1||^\nu\, d\sigma_k(a) = 4k(k-1)\int_0^\infty\int_0^\infty \frac{r|\log r|^\nu \rho^{2k-3}}{(1+r^2+\rho^2)^{k+1}}\,d\rho\,dr,$$

where we used polar coordinates for a_1 and spherical coordinates for $(a_2, \ldots, a_k) \in \mathbb{C}^{k-1}$. Changing variables $\rho^2 = (1+r^2)x(1-x)^{-1}$, $2\rho\,d\rho = (1+r^2)(1-x)^{-2}\,dx$, in the inner integral we obtain

$$\int_0^\infty \frac{\rho^{2k-3}}{(1+r^2+\rho^2)^{k+1}}\,d\rho = \frac{1}{2(1+r^2)^2}\int_0^1 x^{k-2}(1-x)\,dx = \frac{1}{2k(k-1)(1+r^2)^2}\,,$$

and the lemma follows. $\square$

Lemma 4.10 shows that the conclusions of Theorem 4.9 hold for the measures $\sigma_p := \sigma_{d_p}$ given by (4.11) on $H^0_{(2)}(X, L_p) \simeq \mathbb{C}^{d_p}$.

More generally, one can consider radial probability measures on $\mathbb{C}^k$ with density

$$d\sigma_{k,\alpha}(a) = \frac{\Gamma(k+\alpha)}{\Gamma(\alpha)\pi^k}\,\frac{1}{(1+\|a\|^2)^{k+\alpha}}\,dV_k(a)\,, \tag{4.12}$$

where $\alpha > 0$ and Γ is the Gamma function. As in the proof of Lemma 4.10 one can show that for every integer $k \geq 1$ and every $\nu \geq 1$,

$$\int_{\mathbb{C}^k} |\log|\langle a, u\rangle||^\nu\, d\sigma_{k,\alpha}(a) = \Gamma_{\nu,\alpha} := 2\alpha\int_0^\infty \frac{r|\log r|^\nu}{(1+r^2)^{1+\alpha}}\,dr\,,\ \ \forall\, u \in \mathbb{C}^k,\ \|u\| = 1\,.$$

4.2.3. *Area measure of spheres.* Let $\mathcal{A}_k$ be the surface measure on the unit sphere $\mathbf{S}^{2k-1}$ in $\mathbb{C}^k$, so $\mathcal{A}_k\big(\mathbf{S}^{2k-1}\big) = 2\pi^k/(k-1)!$, and let

$$\sigma_k = \frac{1}{\mathcal{A}_k\big(\mathbf{S}^{2k-1}\big)}\,\mathcal{A}_k\,. \tag{4.13}$$

Lemma 4.11. *If $\nu \geq 1$ there exists a constant $M_\nu > 0$ such that for every integer $k \geq 2$,*

$$\int_{\mathbf{S}^{2k-1}} |\log|\langle a, u\rangle||^\nu\, d\sigma_k(a) \leq M_\nu\,(\log k)^\nu\,,\ \ \forall\, u \in \mathbb{C}^k,\ \|u\| = 1\,.$$

Proof. We use spherical coordinates $(\theta_1, \ldots, \theta_{2k-2}, \varphi) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]^{2k-2} \times [0, 2\pi]$ on $\mathbf{S}^{2k-1}$ such that

$$a_k = \sin\theta_{2k-3}\cos\theta_{2k-2} + i\sin\theta_{2k-2}\,, \quad d\mathcal{A}_k = \cos\theta_1\cos^2\theta_2\ldots\cos^{2k-2}\theta_{2k-2}\; d\theta_1\ldots d\theta_{2k-2}d\varphi\,.$$

Since σ_k is unitary invariant we argue in the proof of [CMM, Lemma 4.3] and obtain that there exists a constant $c > 0$ such that for every k and ν,

$$\begin{aligned}\int_{\mathbf{S}^{2k-1}} |\log|\langle a, u\rangle||^\nu\, d\sigma_k(a) &= \int_{\mathbf{S}^{2k-1}} |\log|a_k||^\nu\, d\sigma_k(a) \\ &\leq \frac{ck}{2^\nu}\int_0^1\int_0^1 (1-x^2)^{k-3/2}(1-y^2)^{k-2}|\log(x^2+y^2-x^2y^2)|^\nu\, dxdy \\ &\leq \frac{\pi ck}{2^{\nu+1}}\int_0^1 (1-t)^{k-2}|\log t|^\nu\, dt\,.\end{aligned}$$

Note that

$$f(t) := t^{1/2}|\log t|^\nu \leq f\big(e^{-2\nu}\big) = (2\nu/e)^\nu\,, \text{ for } 0 < t \leq 1.$$

It follows that

$$\begin{aligned}\int_0^1 (1-t)^{k-2}|\log t|^\nu\, dt &\leq \left(\frac{2\nu}{e}\right)^\nu \int_0^{1/k^2} (1-t)^{k-2}t^{-1/2}\, dt + \int_{1/k^2}^1 (1-t)^{k-2}|\log t|^\nu\, dt \\ &\leq \left(\frac{2\nu}{e}\right)^\nu \int_0^{1/k^2} t^{-1/2}\, dt + 2^\nu(\log k)^\nu \int_{1/k^2}^1 (1-t)^{k-2}\, dt \\ &\leq \left(\frac{2\nu}{e}\right)^\nu \frac{2}{k} + \frac{2^\nu(\log k)^\nu}{k-1}\,,\end{aligned}$$

which implies the conclusion of the lemma. □

Lemma 4.11 implies that in this case Theorem 1.1 takes the following simpler form:

Theorem 4.12. *Assume that (X, ω), (L_p, h_p) verify the assumptions (A1), (A2), and $\sigma_p := \sigma_{d_p}$ is the measure given by* (4.13) *on the unit sphere of $H^0_{(2)}(X, L_p) \simeq \mathbb{C}^{d_p}$. Then the following hold:*

(i) If $\lim_{p\to\infty} \frac{\log d_p}{A_p} = 0$ then $\frac{1}{A_p}\big(\mathbb{E}[s_p = 0] - c_1(L_p, h_p)\big) \to 0$, as $p \to \infty$, in the weak sense of currents on X.

(ii) If $\liminf_{p\to\infty} \frac{\log d_p}{A_p} = 0$ then there exists a sequence $p_j \nearrow \infty$ such that for σ-a. e. sequence $\{s_p\} \in \mathcal{H}$ we have

$$\frac{1}{A_{p_j}}\log|s_{p_j}|_{h_{p_j}} \to 0\,, \quad \frac{1}{A_{p_j}}\big([s_{p_j} = 0] - c_1(L_{p_j}, h_{p_j})\big) \to 0\,, \text{ as } j \to \infty,$$

in $L^1(X, \omega^n)$, respectively in the weak sense of currents on X.

(iii) If $\sum_{p=1}^{\infty}\left(\frac{\log d_p}{A_p}\right)^{\nu} < \infty$ *for some* $\nu \geq 1$*, then for* σ*-a. e. sequence* $\{s_p\} \in \mathcal{H}$ *we have*

$$\frac{1}{A_p}\log|s_p|_{h_p} \to 0\,,\;\; \frac{1}{A_p}\big([s_p=0]-c_1(L_p,h_p)\big) \to 0\,,\;\; \textit{as } p\to\infty,$$

in $L^1(X,\omega^n)$*, respectively in the weak sense of currents on* X*.*

We remark that the assertion (ii) of Theorem 4.12 was proved in [CMM, Theorem 4.2]. That paper also gives two general examples of sequences of line bundles L_p for which

$$\lim_{p\to 0}\frac{\log\dim H^0(X,L_p)}{A_p}=0,$$

see [CMM, Proposition 4.4] and [CMM, Proposition 4.5]. In particular, if X is smooth and each L_p is semiample then it is shown in [CMM, Proposition 4.5] that

$$\dim H^0(X,L_p)=O(A_p^N).$$

Therefore $\lim_{p\to\infty}(\log d_p)/A_p = 0$. Moreover, since $\log d_p < \sqrt{A_p}$ for p sufficiently large, the hypothesis that $\sum_{p=1}^{\infty}\left(\frac{\log d_p}{A_p}\right)^{\nu} < \infty$, for some $\nu \geq 1$, in Theorem 4.12 (iii), can be replaced by the condition that $\sum_{p=1}^{\infty} A_p^{-\nu} < \infty$ for some $\nu \geq 1$.

Remark 4.13. We note that for unitary invariant measures σ_p, like those from Sections 4.2.1-4.2.3, the probability space $(H^0_{(2)}(X,L_p),\sigma_p)$ does not depend on the choice of orthonormal basis. Other important classes of probability measures which do not depend on the choice of orthonormal basis and are not unitary invariant are given in [FZ] (see formulas (5), (6) and (7) therein). These measures γ_N are easily seen to be dominated by measures σ_N on the space $\mathcal{P}_N \simeq \mathbb{C}^{N+1}$ of polynomials in $\mathbb{C}$ of degree at most N, with Gaussian type density of the form

$$d\sigma_N(a)=e^{C-\varepsilon\|a\|^2}\,dV_{N+1}(a)\,.$$

Indeed, the polynomial $P(x)$ from [FZ, (7)] is bounded from below on $[0,+\infty)$, hence $P(x)\geq \varepsilon x - C$ for all $x \geq 0$, with some constants $\varepsilon, C > 0$. An argument analogous to that in the proof of Lemma 4.8 shows that the measures γ_N verify assumption (B) for every $\nu \geq 1$ with constants $C_N = \Gamma_\nu$ independent of N. In particular, if the metric h and the measure ν in the definition of γ_N [FZ, (5)] is positively curved, respectively a Kähler form on $\mathbb{P}^1$, then our Theorem 1.1 holds in the setting of [FZ] for the measures γ_N.

4.2.4. *Measures with heavy tail and small ball probability.* Let σ_p be probability measures on $H^0_{(2)}(X,L_p)\simeq\mathbb{C}^{d_p}$ verifying the following: There exist a constant $\rho > 1$ and for every $p \geq 1$ constants $C'_p > 0$ such that:

(B1) For all $R \geq 1$ the *tail probability* satisfies

$$\sigma_p\big(\{a\in\mathbb{C}^{d_p}:\log\|a\|>R\}\big)\leq\frac{C'_p}{R^\rho}\,;$$

(B2) For all $R \geq 1$ and for each unit vector $u \in \mathbb{C}^{d_p}$, the *small ball probability* satisfies

$$\sigma_p\big(\{a \in \mathbb{C}^{d_p} : \log|\langle a, u\rangle| < -R\}\big) \leq \frac{C'_p}{R^\rho} \cdot$$

Lemma 4.14. *If σ_p are probability measures on $\mathbb{C}^{d_p}$ verifying (B1) and (B2) with some constant $\rho > 1$, then σ_p verify (B) for any constant $1 \leq \nu < \rho$.*

Proof. Let $\nu < \rho$ and $u \in \mathbb{C}^{d_p}$ be a unit vector. By (B1), (B2) we have

$$\sigma_p\big(\{a \in \mathbb{C}^{d_p} : |\log|\langle a, u\rangle|| > R\}\big) \leq \frac{2C'_p}{R^\rho}\,,\ \ \forall\, R \geq 1\,.$$

Hence

$$\begin{aligned}\int_{\mathbb{C}^{d_p}} |\log|\langle a, u\rangle||^\nu \, d\sigma_p(a) &= \nu \int_0^\infty R^{\nu-1}\sigma_p\big(\{a \in \mathbb{C}^{d_p} : |\log|\langle a, u\rangle|| > R\}\big)\, dR \\ &\leq \nu \int_0^1 R^{\nu-1}\, dR + 2\nu C'_p \int_1^\infty R^{\nu-\rho-1}\, dR = 1 + \frac{2\nu C'_p}{\rho - \nu} =: C_p\,.\end{aligned}$$

□

4.2.5. *Random holomorphic sections with i.i.d. coefficients.* Next, we consider random linear combinations of the orthonormal basis $(S^p_j)_{j=1}^{d_p}$ with independent identically distributed (i.i.d.) coefficients. More precisely, let $\{a^p_j\}_{j=1}^{d_p}$ be an array of i.i.d. complex random variables whose distribution law is denoted by $\boldsymbol{P}$. Then a random holomorphic section is of the form

$$s_p = \sum_{j=1}^{d_p} a^p_j S^p_j.$$

We endow the space $H^0_{(2)}(X, L_p)$ with the d_p-fold product measure σ_p induced by $\boldsymbol{P}$.

Lemma 4.15. *Assume that a^p_j are i.i.d. complex valued random variables whose distribution law $\boldsymbol{P}$ has density ϕ, such that $\phi : \mathbb{C} \to [0, M]$ is a bounded function and there exist $c > 0$, $\rho > 1$ with*

$$\boldsymbol{P}(\{z \in \mathbb{C} : \log|z| > R\}) \leq \frac{c}{R^\rho}\,,\ \ \forall\, R \geq 1. \tag{4.14}$$

Then the product measures σ_p on $\mathbb{C}^{d_p}$ satisfy condition (B) for any $1 \leq \nu < \rho$, with constants $C_p = \Gamma d_p^{\nu/\rho}$, where $\Gamma = \Gamma(M, c, \rho, \nu) > 0$. In particular, if $d_p = O(A_p^N)$ for some $N \in \mathbb{N}$ and $\rho > N$, then σ_p satisfy condition (B) for any $1 \leq \nu < \rho$ with $C_p = O(A_p^{N\nu/\rho}) = o(A_p^\nu)$.

Proof. Let $u = (u_1, \dots, u_{d_p}) \in \mathbb{C}^{d_p}$ be a unit vector. For $R \geq \log d_p$ we have

$$\{a \in \mathbb{C}^{d_p} : \log|\langle a, u\rangle| > R\} \subset \bigcup_{j=1}^{d_p} \big\{a_j : |a_j| > e^{R - \frac{1}{2}\log d_p}\big\},$$

so by (4.14),

$$\sigma_p\big(\{a \in \mathbb{C}^{d_p} : \log|\langle a, u\rangle| > R\}\big) \leq d_p\, \boldsymbol{P}\big(\{a^p_j \in \mathbb{C} : |a^p_j| > e^{R-\frac{1}{2}\log d_p}\}\big) \leq \frac{2^\rho c d_p}{R^\rho} \cdot \tag{4.15}$$

On the other hand, we have $|u_j| \geq d_p^{-1/2}$ for some $j \in \{1, \dots, d_p\}$. We may assume $j = 1$ for simplicity and apply the change of variables

$$\alpha_1 = \sum_{j=1}^{d_p} a_j^p u_j,\ \alpha_2 = a_2^p,\ \dots, \alpha_{d_p} = a_{d_p}^p.$$

Then, using the assumption $\phi \leq M$,

$$\begin{aligned}
&\sigma_p\big(\{a \in \mathbb{C}^{d_p} : \log|\langle a, u\rangle| < -R\}\big) \\
&= \int_{\mathbb{C}^{d_p-1}} \int_{|\alpha_1|<e^{-R}} \phi\left(\frac{\alpha_1 - \sum_{j=2}^{d_p} \alpha_j u_j}{u_1}\right) \phi(\alpha_2)\dots\phi(\alpha_{d_p})\,\frac{d\alpha_1\dots d\alpha_{d_p}}{|u_1|^2} \qquad (4.16)\\
&\leq M\pi d_p e^{-2R}.
\end{aligned}$$

For $R_0 \geq \log d_p$ we obtain using (4.15) and (4.16)

$$\begin{aligned}
\int_{\mathbb{C}^{d_p}} |\log|\langle a, u\rangle||^\nu\, d\sigma_p(a) &= \nu \int_0^\infty R^{\nu-1}\sigma_p\big(\{a \in \mathbb{C}^{d_p} : |\log|\langle a, u\rangle|| > R\}\big)\, dR \\
&\leq \nu \int_0^{R_0} R^{\nu-1} dR + \nu \int_{R_0}^\infty R^{\nu-1}\sigma_p\big(\{a \in \mathbb{C}^{d_p} : |\log|\langle a, u\rangle|| > R\}\big)\, dR \\
&\leq R_0^\nu + \nu \int_{R_0}^\infty R^{\nu-1}\left(\frac{2^\rho c d_p}{R^\rho} + M\pi d_p e^{-2R}\right) dR\,.
\end{aligned}$$

Since $R^{\nu-1}e^{-R} \leq ((\nu-1)/e)^{\nu-1}$ for $R > 0$, and since $R_0 \geq \log d_p$, we get

$$\begin{aligned}
\int_{\mathbb{C}^{d_p}} |\log|\langle a, u\rangle||^\nu\, d\sigma_p(a) &\leq R_0^\nu + \frac{2^\rho \nu c d_p R_0^{\nu-\rho}}{\rho - \nu} + M\pi\nu d_p \left(\frac{\nu-1}{e}\right)^{\nu-1} \int_{R_0}^\infty e^{-R} dR \\
&\leq R_0^\nu \left(1 + \frac{2^\rho \nu c d_p}{(\rho-\nu)R_0^\rho}\right) + M\pi\nu \left(\frac{\nu-1}{e}\right)^{\nu-1}.
\end{aligned}$$

Choosing $R_0^\rho = d_p$ this implies that

$$\int_{\mathbb{C}^{d_p}} |\log|\langle a, u\rangle||^\nu\, d\sigma_p(a) \leq \Gamma d_p^{\nu/\rho},$$

where $\Gamma > 0$ is a constant that depends on M, c, ρ and ν. □

We remark that if X is smooth and each L_p is semiample then $d_p = O(A_p^N)$ (see [CMM, Proposition 4.5]) and Lemma 4.15 applies.

4.2.6. *Locally moderate measures.* Let X be a complex manifold and σ be a positive measure on X. Following [DNS], we say that σ is locally moderate if for any open set $U \subset X$, any compact set $K \subset U$, and any compact family $\mathscr{F}$ of psh functions on U, there exist constants $c, \alpha > 0$ such that

$$\int_K e^{-\alpha\psi} d\sigma \leq c\,,\ \ \forall\, \psi \in \mathscr{F}. \qquad (4.17)$$

Note that a locally moderate measure σ does not put any mass on pluripolar sets. The existence of c, α in (4.17) is equivalent to existence of $c', \alpha' > 0$ satisfying

$$\sigma(\{z \in K : \psi(z) < -t\}) \leq c' e^{-\alpha' t},$$

for any $t \geq 0$ and $\psi \in \mathscr{F}$. Important examples are provided by the Monge-Ampère measures of Hölder continuous psh functions [DNS, Theorem 1.1, Corollary 1.2].

Lemma 4.16. *If σ_p, $p \geq 1$, is a locally moderate probability measure with compact support in $\mathbb{C}^{d_p} \simeq H^0_{(2)}(X, L_p)$, then σ_p satisfies condition (B) for every $\nu \geq 1$.*

Proof. Consider the compact family of psh functions $\mathscr{F} = \{\psi_u : u \in \mathbf{S}^{2d_p-1}\}$, where $\psi_u : \mathbb{C}^{d_p} \to [-\infty, \infty)$, $\psi_u(a) = \log|\langle a, u\rangle|$. Let $R_p \geq 1$ be such that $\|a\| \leq R_p$ for all $a \in \operatorname{supp} \sigma_p$. Then

$$|\psi_u(a)| = -\psi_u(a) + \max\{0, 2\psi_u(a)\} \leq -\psi_u(a) + 2\log R_p$$

holds for all $a \in \operatorname{supp} \sigma_p$ and $\psi_u \in \mathcal{F}$. Since σ_p is locally moderate and with compact support, there exist constants $c_p, \alpha_p > 0$ such that (4.17) holds for every $\psi_u \in \mathcal{F}$ and with the integral over $\mathbb{C}^{d_p}$. Fix $\nu \geq 1$. As $x^\nu \leq c' e^{\alpha_p x}$ for all $x \geq 0$, with some constant $c' > 0$ depending on p, ν, we conclude that

$$\int_{\mathbb{C}^{d_p}} |\psi_u(a)|^\nu \, d\sigma_p(a) \leq c' \int_{\mathbb{C}^{d_p}} e^{\alpha_p |\psi_u(a)|} \, d\sigma_p(a) \leq c' R_p^{2\alpha_p} \int_{\mathbb{C}^{d_p}} e^{-\alpha_p \psi_u(a)} \, d\sigma_p(a) \leq c' c_p R_p^{2\alpha_p}.$$

$\square$

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