Ramanujan's Congruences for the Partition Function modulo 5, 7, 11

by
AYŞEGÜL YAVUZ

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APPROVED BY

Assoc. Prof. Dr.	Kağan Kurşungöz
(Thesis Supervisor)	
Asst. Prof. Dr.	Ayesha Asloob Qureshi
Asst. Prof. Dr.	Zafeirakis Zafeirakopoulos (Gebze Technical University)

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to all my struggles

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Ayşegül Yavuz

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Thesis Supervisor:
Kağan Kurşungöz

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Abstract

In 1919, Ramanujan introduced three congruences satisfied by the partition function $p(n)$, namely $p(5n+4) \equiv 0 \pmod 5$, $p(7n+5) \equiv 0 \pmod 7$ and $p(11n+6) \equiv 0 \pmod{11}$. In this thesis, our goal is to present different proofs of each of these congruences. For the congruence $p(5n+4) \equiv 0 \pmod 5$, we present three types of elementary to non-elementary proofs. In these proofs, we observe that the elementary proofs of the congruences $p(5n+4)$ and $p(7n+5)$ are analogues with minor variations. The second proof that we introduce can be regarded as non-elementary proof. Even though their non-elementary proofs are similar to each other, the proof of the congruence in modulo 7 involves considerably more computations on identities, inevitably, than the proof of the congruence in modulo 5. We further present three worth-stressing proofs for the congruence $p(11n+6) \equiv 0 \pmod{11}$; The first is proved by Winquist using a representation of $(q;q)_\infty^{10}$ as a double series and a two parameter identity is utilized for this double sum. Then Hirschhorn proves this congruence using a four parameter generalization of Winquist's identity and modifies the representation of $(q;q)_\infty^{10}$. Lastly, owing to Ramanujan, B. Berndt, et al., prove the congruence $p(11n+6) \equiv 0 \pmod{11}$ directly using a new representation for $(q;q)_\infty^{10}$. We complete the thesis by presenting a more direct and a uniform proof, given by Hirschhorn in 1994, that can be applicable to all three congruences. This proof is partially based on linear algebra, which makes it reasonably different from Winquist's and Hirschhorn's earlier proofs.

Ramanujan'ın Parçalanış Fonksiyonu için (mod 5), (mod 7) ve (mod 11)' deki Denklikleri

Ayşegül Yavuz

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Tez Danışmanı:
Kağan Kurşungöz

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Özet

1919'da Ramanujan parçalanış fonksiyonu için $p(5n+4) \equiv 0 \pmod 5$, $p(7n+5) \equiv 0 \pmod 7$ ve $p(11n+6) \equiv 0 \pmod{11}$ denkliklerini ortaya attı. Bu tezde amacımız bu denkliklerin farklı kanıtlarını sunmak. $p(5n+4) \equiv 0 \pmod 5$ denkliği için temelden zora doğru üç farklı kanıt gösterildi. Burada temel olan kanıtlar aynı doğrultuda ilerliyor. Temel olmayan kanıtlar da birbirlerine benzer olmasına rağmen mod 7'deki denklik kanıtı, mod 5'teki denklik kanıtından daha karmaşık özdeşlikler kullanyor. Son denklik $p(11n+6) \equiv 0 \pmod{11}$ için, üzerinde durulması gereken üç kanıt sunduk. İlki, Winquist'in iki parametreli özdeşlik ve q serisi $(q;q)_\infty^{10}$ sonsuz çarpımını çift toplam olarak yazdığı kanıt. Diğeri, Hirschhorn tarafından verilen, iki parametreli Winquist özdeşliğinin dört parametreye genişletip ve $(q;q)_\infty^{10}$ serisini Winquist' den farklı kullanarak yaptığı kanıt. Son olarak, Berndt ve diğerlerinin Ramanujan özdeşliklerini kullanarak tekrar ifade ettikleri $(q;q)_\infty^{10}$ serisini kullanan bir kanıt. Tezi 1994'te Hirschhorn tarafından verilen, bütün denkliklerin kanıtını aynı anda çıkarabileceğimizi gösteren makaleyle tamamladık. Bu kanıtı diğerlerinden farklı yapan tarafı, kısmen lineer cebir kullanıyor olmasıdır.

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Table of Contents

CHAPTER 1

Introduction

In 1919, Ramanujan introduced three congruences satisfied by the partition function $p(n)$:

$$p(5n+4) \equiv 0 \pmod 5 \tag{1.1}$$

$$p(7n+5) \equiv 0 \pmod 7 \tag{1.2}$$

$$p(11n+6) \equiv 0 \pmod{11}. \tag{1.3}$$

He proved (1.1) and (1.2) in [17] and later in [16] announced that he had also established a proof of (1.3). Later on, G. H. Hardy extracted different proofs of these congruences from unpublished works of Ramanujan in [18]. Ramanujan offered a general conjecture of these congruences in [17]:

" Let $\delta = 5^a 7^b 11^c$ and λ be an integer such that $24\lambda \equiv 1 \pmod \delta$. Then $p(n\delta+\lambda) \equiv 0 \pmod \delta$."

He started to prove his conjecture for some a, b, c, but he did not complete. After Ramanujan died, S. Chowla in [9] realized that $p(243)$ is not divisible by 7^3 even if $24.243 \equiv 1 \pmod{7^3}$. Berndt points out in [5] that Watson prove the correct version of Ramanujan's conjecture for $a = c = 0$ in [20], that is:

"Define $\delta' = 5^a 7^{b'} 11^c$ where $b' = b$ if $b = 0, 1, 2$ and $b' = \left\lceil \frac{b+2}{2} \right\rceil$ if $b > 2$. Then

$$p(n\delta+\lambda) \equiv 0 \pmod{\delta'}.\text{"} \tag{1.4}$$

In this thesis we have collected different proofs for (1.1), (1.2) and (1.3).

The second chapter of this study is devoted to basic facts about q series. We start with essential definitions and theorems that are used in the later chapters. All of these can be found in Chapter 1 of Number Theory In the Spirit of Ramanujan, [5].

Chapter 3 focuses on three different proofs for the partition function congruence in modulo 5. These proofs are taken from [5]. The first proof provides an undemanding approach using only principal theorems: Euler's Pentagonal number theorem and Jacobi's identity. This proof is taken from [17] in [5]. The second proof was chosen to highlight some observations on q-series. The extensive generalizations and results can be found in [3]. The third proof is non-elementary, but easy to follow. Understanding this proof also assists in overcoming the non-elementary proof of Ramanujan congruence in modulo 7.

In chapter 4, we introduce two different types of proof of the Ramanujan congruence in modulo 7. These proofs were selected from [5], and for the omitted calculations in the book we consult [7]. This choice enables us to observe that the elementary proof of Ramanujan congruences in modulos 5 and 7 are very similar with some variation. However the non-elementary proof of Ramanujan congruence for modulo 7 requires heavy calculations on q-series, and some complicated identities compared to Ramanujan's congruence in modulo 5.

In chapter 5, we present three proofs of Ramanujan's conjecture modulo 11, which appeared in [17] in 1919, that prove the congruence $p(11n + 6) \equiv 0 \pmod{11}$. The first, by Winquist, [21] uses a two parameter identity whose construction is a little bit tricky but not hard to prove. By proving this identity we were able to facilitate a proof of Ramanujan's congruence modulo 11. The second, from Hirschhorn [13], improves the two parameter identity to a four parameter identity regarded as a generalization of Winquist's identity. Then Hirschhorn uses a slightly modified representation of $(q; q)_\infty^{10}$ to conclude the proof. Later Berndt, et al. introduce a new representation of $(q; q)_\infty^{10}$ only using some identities in respect of Ramanujan and then they prove the congruence $p(11n + 6) \equiv 0 \pmod{11}$.

In chapter 6, we added a proof by Hirschhorn [14], present a uniform approach that applicable to all the congruences for the partition function modulo 5, 7 and 11. He begins by Jacobi's triple product identity which is essential and used in almost all proofs we introduce. Then he presents different matrices for each case and uses some facts from linear algebra to prove the congruences. This approach is quite different from the earlier ones we include.

In addition, for each congruence for the partition function we add a nice illustration that $p(25n+24) \equiv 0 \pmod{25}$ and $p(49n+47) \equiv 0 \pmod{49}$. Hirschhorn asserts that $p(121n + 116) \equiv 0 \pmod{121}$ from the more general formula.

CHAPTER 2

Preliminaries

In this chapter we begin with definition of an integer partition and exhibit the established connection with q-series. We proceed to essential theorems that will be used throughout the thesis. All of these can be found in [5] and [4] including their detailed proofs.

Definition 2.1 *An integer partition of n is a finite unordered sequence of positive integers $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that $\lambda_1 + \lambda_2 + \ldots + \lambda_k = n$. The λ_k's are called the parts of the partition.*

Examples: For $n = 0$ the only partition of 0 is the empty partition.

$$\begin{aligned} \text{For } n = 3: \qquad 3&, \\ 2+1&, \\ 1+1+1&. \end{aligned}$$

$$\begin{aligned} \text{For } n = 5: \qquad 5&, \\ 4+1&, \\ 3+2&, \\ 3+1+1&, \\ 2+2+1&, \\ 2+1+1+1&, \\ 1+1+1+1+1&. \end{aligned}$$

$$\begin{aligned}
\text{For } n = 7: \quad & 7, \\
& 6+1, \\
& 5+2, \\
& 5+1+1, \\
& 4+3, \\
& 4+2+1, \\
& 4+1+1+1, \\
& 3+3+1, \\
& 3+2+2, \\
& 3+2+1+1, \\
& 3+1+1+1+1, \\
& 2+2+2+1, \\
& 2+2+1+1+1, \\
& 2+1+1+1+1+1, \\
& 1+1+1+1+1+1+1.
\end{aligned}$$

The total number of unrestricted partitions of positive integer n is denoted by $p(n)$. We call $p(n)$ the partition function. One observes that $p(3) = 3$, $p(5) = 7$, $p(7) = 15$ in the above example. The partition function increases rapidly with n. For instance, $p(10) = 42$, $p(15) =$, $p(50) = 204226$.

Definition 2.2 *The generating function of an infinite sequence $a_0, a_1, a_2, a_3 \dots$ is*

$$f(q) = a_0 + a_1 q + a_2 q^2 + a_3 q^3 + \ldots = \sum_{n=0}^{\infty} a_n q^n \tag{2.1}$$

where q is an indeterminate. Then we say the series

$$f(q) = \sum_{n \geq 0} p(n) q^n$$

is a generating function for the partition function $p(n)$, due to Euler, and counts all partitions of n as the coefficient of q^n.

$$\sum_{n=0}^{\infty} p(n) q^n = \frac{1}{(q;q)_\infty}, \qquad p(0) = 1. \tag{2.2}$$

Definition 2.3 *For $|q| < 1$, $(a;q)_0 := 1$ and*

$$(a;q)_n := (1-a)(1-aq)(1-aq^2)\ldots(1-aq^{n-1}) = \prod_{j=1}^{n} (1-aq^{j-1})$$

$$(a;q)_\infty := (1-a)(1-aq)(1-aq^2)\ldots = \prod_{j=1}^{\infty}(1-aq^{j-1}) \tag{2.3}$$

We call q the base, a the parameter.

If we take $a = q$ in (2.3):

$$(q;q)_\infty = (1-q)(1-q^2)(1-q^3)\ldots = \prod_{n=1}^{\infty}(1-q^n) \tag{2.4}$$

The following theorems are taken from [5].

Theorem 2.1 *(q-Series Version of Euler's Pentagonal Number Theorem)*

$$(q;q)_\infty = \prod_{n\geq 1}(1-q^n) = \sum_{r=-\infty}^{\infty}(-1)^r q^{\frac{r(3r+1)}{2}} = \sum_{r=-\infty}^{\infty}(-1)^r q^{\frac{r(3r-1)}{2}} \tag{2.5}$$

Notice that the second summation arises from the first summation replacing $-r$ for r and vice versa.

Theorem 2.2 *(Jacobi's Identity) For $|q| < 1$,*

$$(q;q)_\infty^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{\frac{n(n+1)}{2}} \tag{2.6}$$

We use two versions of Jacobi's Triple Product Identity:

Theorem 2.3 *(Jacobi's Triple Product Identity 1)*

For $z \neq 0$ and $|q| < 1$

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = (-zq;q^2)_\infty(-z^{-1}q;q^2)_\infty(q^2;q^2)_\infty \tag{2.7}$$

Theorem 2.4 *(Jacobi's Triple Product Identity 2)*

For $z \neq 0$ and $|q| < 1$

$$\sum_{m=-\infty}^{\infty}(-1)^m z^m q^{\binom{m}{2}} = (z;q)_\infty(z^{-1}q;q)_\infty(q;q)_\infty \tag{2.8}$$

Definition 2.4 *Ramanujan's general theta function $f(a,b)$ is defined by*

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \qquad |ab| < 1.$$

In particular, there are special cases defined by

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{\frac{n(n+1)}{2}} q^{\frac{n(n-1)}{2}} = \sum_{n=-\infty}^{\infty} q^{n^2}, \tag{2.9}$$

$$\psi(q) := f(q,q^3) = \sum_{n=-\infty}^{\infty} q^{\frac{n(n+1)}{2}} q^{3\frac{n(n-1)}{2}} = \sum_{n=-\infty}^{\infty} q^{\frac{n(n+1)}{2}}.$$

Corollary 2.5 *From [5, Corollary 1.3.4], we have*

$$\varphi(q) = (-q;q^2)^2_\infty (q^2;q^2)_\infty \tag{2.10}$$

$$f(-q) = (q;q)_\infty. \tag{2.11}$$

Theorem 2.6 *(the Binomial expansion)*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

Observe that, by this theorem, we have

$$(1-q^j)^n = \sum_{k=0}^{n} \binom{n}{k} (-q)^{jk}$$

Remark 2.1

$$(1-q^j)^n = \sum_{k=0}^{n} \binom{n}{k} (-q)^{jk} \equiv 1 - q^{jk} \pmod{d} \text{ where } d \text{ is a prime divisor of } n. \tag{2.12}$$

Example: For $n = 8$,

$$\begin{aligned} (1-q^j)^8 &= \binom{8}{0}(-q^j)^0 + \binom{8}{1}(-q^j)^1 + \binom{8}{2}(-q^j)^2 + \ldots + \binom{8}{7}(-q^j)^7 + \binom{8}{8}(-q^j)^8 \\ &\equiv 1 - q^{8j} \pmod{2} \end{aligned}$$

Using the remark above, we observe that:

$$\begin{aligned} (q;q)^8_\infty = \prod_{k=1}^{\infty}(1-q^k)^8 &= (1-q)^8(1-q^2)^8(1-q^3)^8 \ldots \\ &\equiv (1-q^{8\cdot1})(1-q^{8\cdot2})(1-q^{8\cdot3})\ldots \qquad \pmod{2} \\ &= \prod_{k=1}^{\infty}(1-q^{8k}) = (q^8;q^8)_\infty \end{aligned}$$

Hence in general

$$(q;q)^l_\infty \equiv (q^l;q^l)_\infty \pmod{d} \text{ where } d \text{ is a prime divisor of } l. \tag{2.13}$$

We use this fact many times in the proofs that follow.

CHAPTER 3

Ramanujan's congruence mod 5

Theorem 3.1 *For each $n \in \mathbb{N}$,*

$$p(5n+4) \equiv 0 \pmod 5.$$

We present three ways to prove this congruence.

3.1. An elementary proof

This proof is taken from [17] and [12] that was reproduced by Hardy, in [5].

Proof of Theorem 3.1 : Begin by writing

$$q(q;q)_\infty^4 \frac{(q^5;q^5)_\infty}{(q;q)_\infty^5} = q\frac{(q^5;q^5)_\infty}{(q;q)_\infty} = q(q^5;q^5)_\infty \sum_{m=0}^{\infty} p(m)q^m.$$

In the first equation we simplify the fraction, in the second we use the definition of the generating function of $p(n)$, (2.2).

Observe that using (2.13) where $l = d = 5$, we have $(q^5;q^5)_\infty \equiv (q;q)_\infty^5 \pmod 5$, equivalently this means $\frac{(q^5;q^5)_\infty}{(q;q)_\infty^5} \equiv 1 \pmod 5$.

Hence, if we reduce modulo 5, the left hand side of the equation becomes $q(q;q)_\infty^4$. Then,

$$q(q;q)_\infty^4 \equiv q(q^5;q^5)_\infty \sum_{m=0}^{\infty} p(m)q^m = (q^5;q^5)_\infty \sum_{m=0}^{\infty} p(m)q^{m+1} \pmod 5. \tag{3.1}$$

When we write $m = 5n+4$ on the right hand side, we look for the coefficient of q^{5n+5} that gives zero in modulo 5. Similarly, we consider the coefficient of q^{5n+5} on the left hand side $q(q;q)_\infty^4$. By the Pentagonal number theorem (2.5) and Jacobi‘s identity (2.7):

$$q(q;q)_\infty^4 = q(q;q)_\infty(q;q)_\infty^3 = q\sum_{j=-\infty}^{\infty}(-1)^j q^{\frac{j(3j+1)}{2}}\sum_{k=0}^{\infty}(-1)^k(2k+1)q^{\frac{k(k+1)}{2}}$$
$$= \sum_{j=-\infty}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k}(2k+1)q^{1+\frac{j(3j+1)}{2}+\frac{k(k+1)}{2}}. \quad (3.2)$$

Then the following question arises: When are the exponents of q multiples of $5n+5$? Writing the exponent of q as

$$\begin{aligned}
8\left(1+\frac{j(3j+1)}{2}+\frac{k(k+1)}{2}\right) &= 8+12j^2+4j+4k^2+4k \\
&= 5+2j^2+4j+2+4k^2+4k+10j^2+1 \\
&= 2(j^2+2j+1)+4k^2+4k+1+10j^2+5 \\
&= 2(j+1)^2+(2k+1)^2+10j^2+5.
\end{aligned}$$

Then

$$8\left(1+\frac{j(3j+1)}{2}+\frac{k(k+1)}{2}\right)-10j^2-5 = 2(j+1)^2+(2k+1)^2.$$

Hence

$$1+\frac{j(3j+1)}{2}+\frac{k(k+1)}{2} \equiv 0 \pmod 5$$

if and only if

$$2(j+1)^2+(2k+1)^2 \equiv 0 \pmod 5.$$

Observe that when we plug in the numbers from the residue classes in modulo 5, $2(j+1)^2$ gives $0,2,3$ in modulo 5 and similarly $(2k+1)^2$ gives $0,1,4$ in modulo 5. Therefore to obtain the sum zero both $2(j+1)^2$ and $(2k+1)^2$ must be 0 in modulo 5. It follows that $j+1 \equiv 0 \pmod 5$ and $2k+1 \equiv 0 \pmod 5$.

To sum up, the exponents of q are multiples of $5n+5$ if and only if $2k+1 \equiv 0 \pmod 5$. By (3.2), this implies that the coefficient of q^{5n+5} in $q(q;q)_\infty^4$ is a multiple of 5 and then the coefficient of q^{5n+5} on the right side of (3.1) is a multiple of 5. Therefore $p(5n+4) \equiv 0 \pmod 5$.

□

3.2. A less elementary proof

This proof is due to Andrews, [1] and generalization of the following lemma is in [3].

Proof of Theorem 3.1 : The second proof of Ramanujan's congruence in modulo 5 is a consequence of this lemma:

Lemma: Let $\{a_n\}$ be any sequence of integers and $n \geq 0$.

Then the coefficient of q^{5n+3} in

$$L(q) := \frac{1}{(q;q)_\infty^2} \sum_{n=0}^{\infty} a_n q^{n^2}$$

is divisible by 5.

Proof: Write

$$\begin{aligned}
\frac{1}{(q;q)_\infty^2} \sum_{m=0}^{\infty} a_m q^{m^2} &= (q;q)_\infty^3 \frac{\sum_{m=0}^{\infty} a_m q^{m^2}}{(q;q)_\infty^5} \\
&\equiv (q;q)_\infty^3 \frac{\sum_{m=0}^{\infty} a_m q^{m^2}}{(q^5;q^5)_\infty} \pmod 5 \\
&= \sum_{j=0}^{\infty} (-1)^j (2j+1) q^{\frac{j(j+1)}{2}} \frac{\sum_{m=0}^{\infty} a_m q^{m^2}}{(q^5;q^5)_\infty} \pmod 5.
\end{aligned}$$

We used (2.13) in the denominator for the infinite product in the second step and the Jacobi's identity, (2.6), for the last equality. By this, it is enough to consider only the series

$$(q;q)_\infty^3 \sum_{m=0}^{\infty} a_m q^{m^2} = \sum_{j=0}^{\infty} (-1)^j (2j+1) q^{\frac{j(j+1)}{2}} \sum_{m=0}^{\infty} a_m q^{m^2} = \sum_{j,m=0}^{\infty} (-1)^j (2j+1) a_m q^{\frac{j(j+1)}{2}+m^2}.$$

So when do we have $\frac{j(j+1)}{2} + m^2 = 5n+3$ for $n \geq 0$? This is equivalent to ask when we have $\frac{j^2+j}{2} + m^2 - 3 \equiv 0 \pmod 5$. Multiply this congruence by 8:

$$\begin{aligned}
0 \equiv 4j^2 + 4j + 8m^2 - 24 &= (2j+1)^2 + 8m^2 - 25 \\
&\equiv (2j+1)^2 + 3m^2 \pmod 5.
\end{aligned}$$

When we plug in the numbers from the residue classes in modulo 5, $(2j+1)^2$ gives us $0, \pm 1 \pmod 5$ and $3m^2$ gives us $0, 2, 3 \pmod 5$. Thus the sum is zero if and only if both $(2j+1)^2$ and $3m^2$ are zero in modulo 5. This implies that $(2j+1) \equiv 0 \pmod 5$ and $m \equiv 0 \pmod 5$. Since the coefficients of q^{5n+3} in $L(q)$ corresponds to the coefficients $(-1)^j(2j+1)a_m$, we conclude that the coefficients of q^{5n+3} in $L(q)$ are divisible by 5.

$\square$

Lets make some observations:

- Using definition of $(q;q)_\infty$, we have $(q;q)_\infty = (q;q^2)_\infty (q^2;q^2)_\infty$ and $(q^2;q^2)_\infty = (q;q)_\infty(-q;q)_\infty$.

- Recall that $\varphi(q) = (-q;q^2)^2_\infty (q^2;q^2)_\infty$ from (2.10) and by the definition in the special case of Ramanujan's general theta function (2.9):

$$\begin{aligned}\varphi(-q) = \sum_{n=-\infty}^{\infty} (-q)^{n^2} = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} &= \sum_{n=-\infty}^{-1} (-1)^n q^{n^2} + 1 + \sum_{n=1}^{\infty} (-1)^n q^{n^2} \\ &= 1 + 2\sum_{n=1}^{\infty} (-1)^n q^{n^2}.\end{aligned}$$

Using these facts, it follows that $\varphi(-q) = \frac{(q;q)_\infty}{(-q;q)_\infty}$. Because

$$\begin{aligned}\frac{(q;q)_\infty}{(-q;q)_\infty} = \frac{(q;q^2)_\infty (q^2;q^2)_\infty}{(-q;q)_\infty} = \frac{(q;q^2)_\infty (q;q)_\infty (-q;q)_\infty}{(-q;q)_\infty} &= (q;q^2)_\infty (q;q^2)_\infty (q^2;q^2)_\infty \\ &= (q;q^2)^2_\infty (q^2;q^2)_\infty \\ &= \varphi(-q).\end{aligned}$$

Then

$$\begin{aligned}\sum_{k=0}^{\infty} p(k) q^{2k} = \frac{1}{(q^2;q^2)_\infty} = \frac{1}{(q;q)_\infty (-q;q)_\infty} &= \frac{(q;q)_\infty}{(q;q)^2_\infty (-q;q)_\infty} \\ &= \frac{1}{(q;q)^2_\infty} \left(1 + 2\sum_{m=1}^{\infty} (-1)^m q^{m^2}\right).\end{aligned}$$

In the third equality multiply both the numerator and the denominator by $(q;q)_\infty$ and then use the last observation. By the previous lemma the coefficients of q^{5n+3} are divisible by 5. Then on the left hand side the coefficient of q^{2k} are multiples of 5 when $2k \equiv 5n+3$. It follows that $k \equiv 5n+4 \pmod 5$. Hence $p(5n+4) \equiv 0 \pmod 5$.

□

3.3. A non-elementary proof

Proof of Theorem 3.1 : The third proof is based on the following theorem taken from [5].

Theorem 3.2 *We have*

$$\sum_{n=0}^{\infty} p(5n+4) q^n = 5\frac{(q^5;q^5)^5_\infty}{(q;q)^6_\infty}. \tag{3.3}$$

Proof : By the pentagonal number theorem (2.5),

$$(q^{1/5};q^{1/5})_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{5}\frac{n(3n-1)}{2}}. \tag{3.4}$$

Our aim is to write this series as a combination of power series with integral coefficients and integral powers. If n is the index of the summation, divide the terms into residue classes modulo 5. The outcomes are as follows.

(mod 5)		
n	$3n-1$	$(n(3n-1))/2$
0	4	0
1	2	1
2	0	0
3	3	2
4	1	2

The exponents of q are only $0/5, 1/5, 2/5$ in modulo 1. Let's observe that the exponent is $0/5 \pmod 1$, when $n \equiv 0 \pmod 5$ or $n \equiv 2 \pmod 5$. We find a power series with integral coefficients and integral powers in both cases. Therefore to illustrate, we'll show it for the case when $n \equiv 2 \pmod 5$. Let n be of the form $5k+2$. When the indices in (3.4) are in this form, we have

$$\begin{aligned}\sum_{k=-\infty}^{\infty} (-1)^{5k+2} q^{\frac{1}{5}\frac{(5k+2)(15k+5)}{2}} = \sum_{k=-\infty}^{\infty} (-1)^{5k+2} q^{\frac{15k^2+11k+2}{2}} &= \sum_{k=-\infty}^{\infty} (-1)^{5k+2} q^{\frac{15k^2+5k}{2}+3k+1} \\ &= \sum_{k=-\infty}^{\infty} (-1)^{5k} q^{5\frac{k(3k+1)}{2}} q^{3k+1}.\end{aligned}$$

Using the Pentagonal number theorem, (2.5), and doing some operations on the sum, this sum can be written as $(q^5;q^5)_\infty J_1(q)$, where $J_1(q)$ is a power series with integral coefficients and integral powers.

The exponent is $1/5 \pmod 1$, when $n \equiv 1 \pmod 5$.

$$\begin{aligned}\sum_{k=-\infty}^{\infty} (-1)^{5k+1} q^{\frac{1}{5}\frac{(5k+1)(15k+2)}{2}} &= \sum_{k=-\infty}^{\infty} (-1)^{5k+1} q^{\frac{15k^2+5k}{2}+\frac{1}{5}} = \sum_{k=-\infty}^{\infty} (-1)^{5k+1} q^{\frac{5k(3k+1)}{2}} q^{\frac{1}{5}} \\ &= -q^{\frac{1}{5}} \sum_{k=-\infty}^{\infty} (-1)^{5k} q^{5\frac{k(3k+1)}{2}} = -q^{\frac{1}{5}} (q^5;q^5)_\infty.\end{aligned}$$

Finally the exponent is $2/5 \pmod 1$ when $n \equiv 3 \pmod 5$ or $n \equiv 4 \pmod 5$. To illustrate we'll show it for the case $n \equiv 3 \pmod 5$, because we obtain a power series with integral coefficients and integral powers in both cases. Then the indices of the form $n = 5k+3$ gives

$$\begin{aligned}\sum_{k=-\infty}^{\infty} (-1)^{5k+3} q^{\frac{1}{5}\frac{(5k+3)(15k+8)}{2}} = \sum_{k=-\infty}^{\infty} (-1)^{5k+3} q^{\frac{5(3k+1)k+12k}{2}+\frac{12}{5}} &= \sum_{k=-\infty}^{\infty} (-1)^{5k+3} q^{\frac{5k(3k+1)}{2}+6k+2+\frac{2}{5}} \\ &= q^{\frac{2}{5}} \sum_{k=-\infty}^{\infty} (-1)^{5k} q^{5\frac{k(3k+1)}{2}} q^{6k+2}.\end{aligned}$$

Again after some operations, this sum can be written as $q^{2/5}(q^5;q^5)_\infty J_2(q)$, where $J_2(q)$ is a power series with integral coefficients and integral powers. Therefore

$$(q^{1/5};q^{1/5})_\infty = (q^5;q^5)_\infty J_1(q) + (-q)^{1/5}(q^5;q^5)_\infty + q^{2/5}(q^5;q^5)_\infty J_2(q).$$

Divide both sides by $(q^5;q^5)_\infty$:

$$\frac{(q^{1/5};q^{1/5})_\infty}{(q^5;q^5)_\infty} = J_1(q) - q^{1/5} + J_2(q)q^{2/5}. \tag{3.5}$$

Take the cube of both sides:

$$\begin{aligned}
\frac{(q^{1/5};q^{1/5})^3_\infty}{(q^5;q^5)^3_\infty} &= (J_1(q) - q^{1/5} + J_2(q)q^{2/5})^3 \\
&= -q^{3/5} + 3q^{2/5}J_1 + 3q^{2/5}J_2q^{2/5} - 3q^{1/5}J_1^2 - 6q^{1/5}J_1J_2q^{2/5} \\
&- 3q^{1/5}J_2^2q^{4/5} + J_1^3 + 3J_1^2J_2q^{2/5} + 3J_1J_2^2q^{4/5} + J_2^3q^{6/5} \\
&= (J_1^3 - 3J_2^2q) + q^{1/5}(-3J_1^2 + J_2^3q) + q^{2/5}(3J_1 + 3J_1J_2) \\
&+ q^{3/5}(-1 - 6J_1J_2) + q^{4/5}(3J_2 + 3J_1J_2^2).
\end{aligned} \tag{3.6}$$

By Jacobi's identity, $(q^{1/5};q^{1/5})^3_\infty = \sum_{n=0}^\infty (-1)^n(2n+1)q^{\frac{1}{5}\frac{n(n+1)}{2}}$. Similar to the previous argument, divide the index of the summation into residue classes modulo 5.

(mod 5)		
n	$n+1$	$(n(n+1))/2$
0	1	0
1	2	1
2	3	3
3	4	1
4	5	0

The exponents of q are only $0/5, 1/5$ and $3/5$ on modulo 1. Then we can write $(q^{1/5};q^{1/5})^3_\infty$ as a sum of power series with integral coefficients and integral powers. We demonstrate the case $n \equiv 2 \pmod 5$. Let's observe what happens when the index is of the form $n = 5k+2$:

$$\begin{aligned}
\sum_{k=0}^\infty (-1)^{5k+2}(2(5k+2)+1)q^{\frac{1}{5}\frac{(5k+2)(5k+3)}{2}} &= \sum_{k=0}^\infty (-1)^{5k}5(2k+1)q^{\frac{5k(k+1)}{2}+\frac{3}{5}} \\
&= 5q^{3/5}\sum_{k=0}^\infty (-1)^{5k}(2k+1)q^{5\frac{k(k+1)}{2}} \\
&= 5q^{3/5}(q^5;q^5)^3_\infty
\end{aligned}$$

It was shown that the terms with the exponents $n \equiv 2 \pmod 5$ are equal to $5q^{3/5}(q^5;q^5)^3_\infty$. Similarly do this for $q^{0/5}$ and $q^{1/5}$ to obtain the power series $G_1(q)$

and $G_2(q)$ with integral powers and integral coefficients. Then

$$\begin{aligned}\frac{(q^{1/5};q^{1/5})^3_\infty}{(q^5;q^5)^3_\infty} &= \frac{G_1(q)(q^5;q^5)^3_\infty + G_2(q)q^{1/5}(q^5;q^5)^3_\infty + 5q^{3/5}(q^5;q^5)^3_\infty}{(q^5;q^5)^3_\infty}\\ &= G_1(q)+G_2(q)q^{1/5}+5q^{3/5}.\end{aligned}$$

Then equating the coefficient of this with (3.6), the coefficients of $q^{2/5}$ and $q^{4/5}$ are 0, the coefficient of $q^{3/5}$ is 5. This gives

$$3J_1+3J_1^2J_2=0, \qquad 3J_2+3J_1J_2^2=0, \qquad -1-6J_1J_2=5 \implies J_1J_2=-1. \quad (3.7)$$

Now in (3.5), replace $q^{1/5}$ by $\omega q^{1/5}$ where ω is a fifth root of unity.

$$\frac{(\omega q^{1/5};\omega q^{1/5})_\infty}{(q^5;q^5)_\infty} = J_1(q)-\omega q^{1/5}+J_2(q)\omega^2q^{2/5}. \quad (3.8)$$

Let ω run through all five fifth roots of unity and multiply all such equalities to obtain

$$\prod_\omega \frac{(\omega q^{1/5};\omega q^{1/5})_\infty}{(q^5;q^5)_\infty} = \prod_\omega \{J_1(q)-\omega q^{1/5}+J_2(q)\omega^2q^{2/5}\}.$$

First examine the left hand side of the equality:

$$\begin{aligned}(\omega q^{1/5};\omega q^{1/5})_\infty = \prod_{k=0}^{\infty}(1-\omega^{1+k}q^{\frac{k+1}{5}}) &= (1-\omega q^{1/5})(1-\omega^2q^{2/5})(1-\omega^3q^{3/5})(1-\omega^4q^{4/5})\\ &\quad\times(1-q)(1-\omega q^{6/5})(1-\omega^2q^{7/5})(1-\omega^3q^{8/5})\ldots\end{aligned}$$

If the exponent is not a multiple of 5 then

$$(1-q^n)=(1-q^{n/5})(1-\omega q^{n/5})(1-\omega^2q^{n/5})(1-\omega^3q^{n/5})(1-\omega^4q^{n/5})$$

If $n=5m$ then $(1-q^m)(1-q^m)(1-q^m)(1-q^m)(1-q^m)=(1-q^m)^5$. Using these take the product over ω :

$$\begin{aligned}\prod_\omega(\omega q^{1/5};\omega q^{1/5})_\infty &= \prod_\omega\left(\prod_{k=0}^{\infty}1-\omega^{1+k}q^{\frac{k+1}{5}}\right)\\
&= (1-\omega q^{1/5})(1-\omega^2q^{1/5})(1-\omega^3q^{1/5})(1-\omega^4q^{1/5})(1-q^{1/5})\\
&\quad\times(1-\omega^2q^{2/5})(1-\omega^4q^{2/5})(1-\omega q^{2/5})(1-\omega^3q^{2/5})(1-q^{2/5})\\
&\quad\times\ldots\times(1-\omega^4q^{4/5})\ldots\ldots(1-q^{4/5})(1-q)(1-q)(1-q)\\
&\quad\times(1-q)(1-q)(1-\omega q^{6/5})(1-\omega^2q^{6/5})(1-\omega^3q^{6/5})\\
&\quad\times\cdots\times(1-q^2)(1-q^2)(1-q^2)(1-q^2)(1-q^2)(1-\omega q^{11/5})\\
&\quad\times(1-\omega^2q^{11/5})\ldots\times(1-q^5)(1-q^5)(1-q^5)(1-q^5)(1-q^5)\ldots\\
&\quad\times(1-q^{26/5})\ldots\\
&= (1-q)(1-q^2)(1-q^3)(1-q^4)(1-q)^5(1-q^6)(1-q^7)\ldots(1-q^2)^5\\
&\quad\times(1-q^{11})\ldots(1-q^3)^5(1-q^{16})\ldots(1-q^4)^5\ldots(1-q^5)^5\ldots\\
&\quad\times(1-q^{10})^5\ldots\\
&= (1-q)^6(1-q^2)^6(1-q^3)^6(1-q^4)^6(1-q^5)^5(1-q^6)^6(1-q^7)^6\ldots\\
&\quad\times(1-q^{10})^5(1-q^{11})^6\ldots(1-q^{15})^5\ldots\end{aligned}$$

Multiplying and dividing by $(q^5;q^5)_\infty$, we have

$$\prod_\omega (\omega q^{1/5};\omega q^{1/5})_\infty = \frac{(q;q)_\infty^6}{(q^5;q^5)_\infty}. \tag{3.9}$$

Hence

$$\prod_\omega \frac{(\omega q^{1/5};\omega q^{1/5})_\infty}{(q^5;q^5)_\infty} = \frac{(q;q)_\infty^6}{(q^5;q^5)_\infty^5(q^5;q^5)_\infty} = \frac{(q;q)_\infty^6}{(q^5;q^5)_\infty^6}. \tag{3.10}$$

Second, examine the right hand side of the equality: Since there are no fractional powers of q on the left hand side, there are none on the right hand side as well. Thus it is enough to examine only those terms in the product $\prod_\omega\{J_1(q)-\omega q^{1/5}+J_2(q)\omega^2q^{2/5}\}$ that gives rise to integral powers of q. In accordance with this purpose, we consider on $\{J_1(q)-\omega q^{1/5}+J_2(q)\omega^2q^{2/5}$ as a quadratic expression and we factorize it in terms of q and power series A, B, C and D with integral powers and integral coefficients in q.

$$\begin{aligned}\prod_\omega\{J_1(q)-\omega q^{1/5}+J_2(q)\omega^2q^{2/5}\} &= \prod_\omega (A-B\omega q^{1/5})(C-D\omega q^{1/5})\\ &= (A^5-B^5q)(C^5-D^5q)\\ &= (AC)^5-(A^5D^5+B^5C^5)q+(BD)^5q^2\end{aligned}$$

where $AC=J_1$, $BD=J_2$, $AD+BC=1$ with $J_1J_2=-1$, then $ACBD=J_1J_2=-1$. Using these,

$$\begin{aligned}1=(AD+BC)^5 &= (AD)^5+5(AD)^4BC+10(AD)^3(BC)^2+10(AD)^2(BC)^3\\ &\quad +5(AD)(BC)^4+(BC)^5\\ &= (AD)^5-5(AD)^3+10AD+10BC-5(BC)^3+(BC)^5\\ &= (AD)^5-5((AD)^3+(BC)^3)+(BC)^5+10. \end{aligned}\tag{3.11}$$

By a similar argument,

$$\begin{aligned}1=(AD+BC)^3 &= (AD)^3+3(AD)^2(BC)+3(AD)(BC)^2+(BC)^3\\ &= (AD)^3-3(AD)-3(BC)+(BC)^3.\end{aligned}$$

Then $(AD)^3+(BC)^3=4$. Substituting this values in (3.11), it follows that $(AD)^5+(BC)^5=11$. Hence

$$\prod_\omega\{J_1(q)-\omega q^{1/5}+J_2(q)\omega^2q^{2/5}\} = J_1^5(q)-11q+J_2^5(q)q^2.$$

It has been shown that

$$\begin{aligned}\frac{1}{J_1(q)-q^{1/5}+J_2(q)q^{2/5}} &= \frac{\prod_{\omega\neq1}J_1(q)-\omega q^{1/5}+J_2(q)\omega^2q^{2/5}}{\prod_\omega J_1(q)-\omega q^{1/5}+J_2(q)\omega^2q^{2/5}}\\ &= \frac{\prod_{\omega\neq1}J_1(q)-\omega q^{1/5}+J_2(q)\omega^2q^{2/5}}{J_1^5(q)-11q+J_2^5(q)q^2}\\ &= \frac{F(q)}{J_1^5(q)-11q+J_2^5(q)q^2},\end{aligned}$$

where
$$F(q) = \frac{J_1^5(q) - 11q + J_2^5(q)q^2}{J_1(q) - q^{1/5} + J_2(q)q^{2/5}}.$$
After long polynomial division in $q^{1/5}$, $F(q)$ was found as
$$(J_1^4 + 3J_2q) + q^{1/5}(J_1^3 + 2J_2^2q) + q^{2/5}(2J_1^2 + J_2^3q) + q^{3/5}(3J_1 + J_2^4q) + 5q^{4/5}.$$

Then (3.5) follows as
$$\begin{aligned}\frac{(q^5;q^5)_\infty}{(q^{1/5};q^{1/5})_\infty} &= \frac{1}{J_1(q) - q^{1/5} + J_2(q)q^{2/5}} \\ &= \frac{(J_1^4 + 3J_2q) + q^{1/5}(J_1^3 + 2J_2^2q) + q^{2/5}(2J_1^2 + J_2^3q) + q^{3/5}(3J_1 + J_2^4q) + 5q^{4/5}}{J_1^5(q) - 11q + J_2^5(q)q^2}.\end{aligned} \tag{3.12}$$

By definition of the generating function of $p(n)$,(2.2),
$$\frac{1}{(q^{1/5};q^{1/5})_\infty} = \sum_{n=0}^{\infty} p(n)q^{n/5}.$$

We select the terms that the exponents are congruent to $4/5 \pmod 1$ on both sides of (3.12):
$$\frac{(q^5;q^5)_\infty}{(q^{1/5};q^{1/5})_\infty} = (q^5;q^5)_\infty \sum_{n=0}^{\infty} p(5n+4)q^{\frac{5n+4}{5}} = \frac{5q^{4/5}}{J_1^5(q) - 11q + J_2^5(q)q^2}.$$

Divide by $q^{4/5}$,
$$(q^5;q^5)_\infty \sum_{n=0}^{\infty} p(5n+4)q^n = \frac{5}{J_1^5(q) - 11q + J_2^5(q)q^2}.$$

Then
$$\begin{aligned}\sum_{n=0}^{\infty} p(5n+4)q^n &= \frac{5}{(q^5;q^5)_\infty(J_1^5(q) - 11q + J_2^5(q)q^2)} \\ &= \frac{5}{(q^5;q^5)_\infty \prod_\omega (J_1(q) - \omega q^{1/5} + J_2(q)\omega^2 q^{2/5})} \\ &= \frac{5}{(q^5;q^5)_\infty \prod_\omega \frac{(\omega q^{1/5};\omega q^{1/5})}{(q^5;q^5)_\infty}} \\ &= \frac{5}{(q^5;q^5)_\infty \frac{(q;q)_\infty^6}{(q^5;q^5)_\infty^6}} \\ &= \frac{5(q^5;q^5)_\infty^5}{(q;q)_\infty^6}.\end{aligned}$$

□

If we reduce modulo 5,
$$\sum_{n=0}^{\infty} p(5n+4)q^n \equiv 0.$$

It follows that $p(5n+4) \equiv 0$.

$\square$

The following theorem can be seen as an application of Ramanujan conjecture (1.4) for $a = 2$, $b' = c = 0$ and where $\delta = 25$, $\lambda = 24$ with $24\lambda \equiv 1 \pmod{25}$.

Theorem 3.3 *For any $n \in \mathbb{N}$,*

$$p(25n+24) \equiv 0 \pmod{25}.$$

Proof : By Theorem 3.2 we know

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6}. \tag{3.13}$$

Using the fact (2.13) when $l = 5$, (mod 5) and definition of generating function (2.2):

$$\begin{aligned} 5\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6} &= 5\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty (q;q)_\infty^5} = 5\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty (q^5;q^5)_\infty} \\ &= 5\frac{(q^5;q^5)_\infty^4}{(q;q)_\infty} = 5(q^5;q^5)^4 \sum_{m=0}^{\infty} p(n)q^n. \end{aligned} \tag{3.14}$$

From Theorem 3.1 we know the coefficients of q^{5n+4} are multiples of 5. If we consider the exponents of the form $5n+4$ in (3.13) and (3.14),

$$\sum_{n=0}^{\infty} p(25n+24)q^{5n+4} = 5(q^5;q^5)^4 \sum_{m=0}^{\infty} p(5n+4)q^{5n+4}.$$

Hence the coefficients of q^{5n+4} are multiples of 25. This implies that

$$p(25n+24) \equiv 0 \pmod{25}.$$

$\square$

CHAPTER 4

Ramanujan's congruence mod 7

Theorem 4.1 *For each $n \in \mathbb{N}$,*

$$p(7n+5) \equiv 0 \pmod 7.$$

4.1. A straightforward proof

This proof is taken from [17] and [12] in [5]. Since this proof is pretty similar to the elementary proof of Ramanujan congruence in modulo 5 we can consider as a straightforward proof.

Proof of Theorem 4.1 : Using the definition of the generating function, (2.2), and binomial equivalence (2.13) with $l = d = 7$:

$$\begin{aligned}
(q^7;q^7)_\infty \sum_{n=0}^{\infty} p(n) q^{n+2} = q^2 (q^7;q^7)_\infty \sum_{n=0}^{\infty} p(n) q^n &= q^2 \frac{(q^7;q^7)_\infty}{(q;q)_\infty} \\
&= q^2 (q;q)_\infty^6 \frac{(q^7;q^7)_\infty}{(q;q)_\infty^7} \\
&\equiv q^2 (q;q)_\infty^6 \pmod 7. \quad (4.1)
\end{aligned}$$

If we show that the coefficient of the terms q^{7n+7} in $q^2(q;q)_\infty^6$ is a multiple of 7, it will follow that the coefficient of q^{7n+7} on the left side is congruent to zero on modulo 7, i.e $p(7n+5) \pmod 7$. Apply Jacobi's identity, (2.6), on (4.1)

$$\begin{aligned}
q^2(q;q)_\infty^6 = q^2((q;q)_\infty^3)^2 &= q^2 \left(\sum_{j=0}^{\infty} (-1)^j (2j+1) q^{\frac{j(j+1)}{2}} \right) \left(\sum_{k=0}^{\infty} (-1)^k (2k+1) q^{\frac{k(k+1)}{2}} \right) \\
&= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{j+k} (2j+1)(2k+1) q^{\frac{j(j+1)}{2} + \frac{k(k+1)}{2} + 2}. \quad (4.2)
\end{aligned}$$

We need to know when the exponent $\frac{j(j+1)}{2}+\frac{k(k+1)}{2}+2$ is a multiple of 7. Observe that

$$\begin{aligned}8\left(\frac{j(j+1)}{2}+\frac{k(k+1)}{2}+2\right) &= 4j(j+1)+4k(k+1)+16\\ &= 4j^2+4j+1+4k^2+4k+1+14\\ &= (2j+1)^2+(2k+1)^2+14.\end{aligned}$$

Then

$$\frac{j(j+1)}{2}+\frac{k(k+1)}{2}+2\equiv 0 \pmod 7$$

if and only if

$$(2j+1)^2+(2k+1)^2\equiv 0 \pmod 7.$$

Observe that $(2j+1)^2\equiv 0,1,2,4 \pmod 7$ and the same result holds for $(2k+1)^2$ as well. Hence the sum is congruent to zero if and only if both $(2j+1)$ and $(2k+1)$ are congruent to zero in modulo 7. Therefore the coefficients in (4.2) are multiples of 7 when the exponent of q is a multiple of 7.

□

4.2. A non-straightforward proof

Proof of Theorem 4.1 : We demonstrate an analogue of the non-elementary proof of the Ramanujan congruence in modulo 5 for the Ramanujan congruence in modulo 7. The following theorem was stated without proof by Ramanujan in his paper [17] and gives a sketch of its proof in [18]. The complete proof was acquired from [7] and we worked on the article [7] to give the detailed proof. We also attached the omitted calculations from [5] by proving in a required order with the intention of giving an accurate proof to the reader.

Theorem 4.2

$$\sum_{n=0}^{\infty}p(7n+5)q^n=7\frac{(q^7;q^7)^3_\infty}{(q;q)^4_\infty}+49q\frac{(q^7;q^7)^7_\infty}{(q;q)^8_\infty} \tag{4.3}$$

Since the right hand side is congruent to zero in modulo 7, it follows that $p(7n+5)\equiv 0 \pmod 7$. Thus Theorem 4.1 can be seen as a corollary of this theorem.

Proof : First apply the Euler's Pentagonal Number Theorem,(2.5),

$$(q^{1/7};q^{1/7})_\infty=\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{1}{7}\frac{n(3n+1)}{2}} \tag{4.4}$$

and then divide the index of the summation into the residue classes in modulo 7.

		(mod 7)
n	$3n+1$	$(n(3n+1))/2$
0	1	0
1	4	2
2	7	0
3	10	1
4	13	2
5	16	5
6	19	2

The exponents of q are only $0/7, 1/7, 2/7, 5/7$ in modulo 1. As one sees from the table, the exponent is $0/5 \pmod 1$, when $n \equiv 0 \pmod 7$ or $n \equiv 2 \pmod 7$. The exponent is $1/7 \pmod 1$, when $n \equiv 1 \pmod 7$. We find a power series with integral coefficients and integral powers in all cases. To illustrate, we'll show it for the cases when $n \equiv 1 \pmod 7$ and when $n \equiv 3 \pmod 7$.

Let n be of the form $7k+1$. When the indices in (4.4) are in this form, we have

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} (-1)^{7k+1} q^{\frac{1}{7}\frac{(7k+1)(21k+4)}{2}} &= -\sum_{k=-\infty}^{\infty} (-1)^{7k} q^{\frac{7^2.3.k^2+7^2k+4}{2.7}} = -\sum_{k=-\infty}^{\infty} (-1)^{7k} q^{\frac{7.k(3k+1)}{2}+\frac{2}{7}} \\
&= -\sum_{k=-\infty}^{\infty} (-1)^{7k} q^{7\frac{k(3k+1)}{2}} q^{\frac{2}{7}} = -q^{2/7}(q^7;q^7)_\infty.
\end{aligned}$$

Let n be of the form $7k+3$. When the indices in (4.4) are in this form, we have

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} (-1)^{7k+3} q^{\frac{1}{7}\frac{(7k+3)(21k+10)}{2}} &= -\sum_{k=-\infty}^{\infty} (-1)^{7k} q^{\frac{7^2.3.k^2+133kk+30}{2.7}} \\
&= -\sum_{k=-\infty}^{\infty} (-1)^{7k} q^{\frac{7k(21k+19)+30}{2.7}} \\
&= -\sum_{k=-\infty}^{\infty} (-1)^{7k} q^{\frac{7k(3k+1)+12k}{2}\frac{15}{7}} \\
&= -\sum_{k=-\infty}^{\infty} (-1)^{7k} q^{7\frac{k(3k+1)}{2}} q^{6k} q^2 q^{\frac{1}{7}} \\
&= q^{\frac{1}{7}}(q^7;q^7)_\infty J_2(q),
\end{aligned}$$

where $J_2(q)$ is a power series with integral powers and integral coefficients. Hence (4.4) can be written as

$$(q^{1/7};q^{1/7})_\infty = q^{0/7}(q^7;q^7)_\infty J_1(q) + q^{1/7}(q^7;q^7)_\infty J_2(q) - q^{2/7}(q^7;q^7)_\infty + q^{5/7}(q^7;q^7)_\infty J_3(q),$$

where J_1, J_2 and J_3 are power series in q with integral coefficients and integral powers. Divide both sides by $(q^7;q^7)_\infty$,

$$\frac{(q^{1/7};q^{1/7})_\infty}{(q^7;q^7)_\infty} = J_1(q) + q^{1/7}J_2(q) - q^{2/7} + q^{5/7}J_3(q). \tag{4.5}$$

Take the cube of both sides:

$$\begin{aligned}
\frac{(q^{1/7};q^{1/7})^3_\infty}{(q^7;q^7)^3_\infty} &= (J_1^3 + 3J_2^2J_3q - 6J_1J_3q) + q^{1/7}(3J_1J_2 - 6J_2J_3q + J + 3^2q^2) \\
&+ 3q^{2/7}(J_1J_2^2 - J_1^2 + J_3q) + q^{3/7}(J_2^3 - 6J_1J_2 + 3J_1J_3^2q) \\
&+ 3q^{4/7}(J_1 - J_2^2 + J_2J_3^2q) + 3q^{5/7}(J_2 + J_1^2J_3 - J_3^2q) \\
&+ q^{6/7}(6J_1J_2J_3 - 1).
\end{aligned} \tag{4.6}$$

Now apply the Jacobi's identity to $(q^{1/7};q^{1/7})^3_\infty$,

$$(q^{1/7};q^{1/7})^3_\infty = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{\frac{1}{7}\frac{n(n+1)}{2}}. \tag{4.7}$$

Separate the index of summation into the residue classes modulo 7.

		(mod 7)
n	$n+1$	$(n(n+1))/2$
0	1	0
1	2	1
2	3	3
3	4	6
4	5	3
5	6	1
6	7	0

The exponents are $0/7, 1/7, 3/7, 6/7$ in modulo 1. A similar argument to the previous ones works here. To exemplify, we'll show it for the cases when $n \equiv 5 \pmod 7$ and when $n \equiv 3 \pmod 7$. Let n be of the form $7k+5$. Then by Jacobi's identity, in the sum (4.7), we obtain these terms:

$$\begin{aligned}
\sum_{k=0}^{\infty}(-1)^{7k+5}(2(7k+5)+1)q^{\frac{1}{7}\frac{(7k+5)(7k+6)}{2}} &= -\sum_{k=0}^{\infty}(-1)^{7k}(14k+11)q^{\frac{7^2k^2+77k+30}{7.2}} \\
&= -\sum_{k=0}^{\infty}(-1)^{7k}(14k+11)q^{\frac{7k(k+1)+4k}{2}+\frac{15}{7}} \\
&= -\sum_{k=0}^{\infty}(-1)^{7k}(14k+11)q^{7\frac{k(k+1)}{2}}q^{2k+2}q^{\frac{1}{7}} \\
&= q^{\frac{1}{7}}G_2(q)(q^7;q^7)^3_\infty.
\end{aligned}$$

The exponent is $6/7 \pmod 1$ when $n \equiv 3 \pmod 7$. Let n be of the form $7k+3$,

$$\begin{aligned}\sum_{k=0}^{\infty}(-1)^{7k+3}(2(7k+3)+1)q^{\frac{1}{7}\frac{(7k+3)(7k+4)}{2}} &= -\sum_{k=0}^{\infty}(-1)^{7k}7(2k+1)q^{\frac{49k^2+49k+12}{7.2}} \\ &= -7\sum_{k=0}^{\infty}(-1)^{7k}(2k+1)q^{7\frac{k(k+1)}{2}}q^{\frac{6}{7}} \\ &= -7q^{6/7}(q^7;q^7)_\infty^3.\end{aligned}$$

One can repeat the same procedure for other exponents and then (4.7) can be written as

$$(q^{1/7};q^{1/7})_\infty^3 = G_1(q)(q^7;q^7)_\infty^3 + q^{1/7}G_2(q)(q^7;q^7)_\infty^3 + q^{3/7}(q^7;q^7)_\infty^3 G_3(q) - 7q^{6/7}(q^7;q^7)_\infty^3$$

where G_i's are power series in q with integral powers and exponents, $i = 1,2,3$. Divide both sides by $(q^7;q^7)_\infty^3$,

$$\frac{(q^{1/7};q^{1/7})_\infty^3}{(q^7;q^7)_\infty^3} = G_1(q) + q^{1/7}G_2(q) + q^{3/7}G_3(q) - 7q^{6/7}.$$

Comparing this with (4.5), we see in that equation there is no q with the powers $2/7, 4/7, 5/7$. Then equating them gives

$$\begin{aligned}&J_1J_2^2 - J_1^2 + J_3q = 0\\ &J_1 - J_2^2 + J_2J_3^2q = 0\\ &J_2 + J_1^2J_3 - J_3^2q = 0\\ &6J_1J_2J_3 - 1 = -7\end{aligned}$$

and these imply

$$\begin{aligned}J_1^2 &= J_1J_2^2 + J_3q\\ J_2^2 &= J_1 + J_2J_3^2q\\ J_3^2q &= J_2 + J_1^2J_3\\ J_1J_2J_3 &= -1.\end{aligned} \tag{4.8}$$

Replace $q^{1/7}$ by $\omega q^{1/7}$ in (4.4) where ω is a seventh root of unity,

$$\frac{(\omega q^{1/7};\omega q^{1/7})_\infty}{(q^7;q^7)_\infty} = J_1(q) + \omega q^{1/7}J_2(q) - \omega^2q^{2/7} + \omega^5q^{5/7}J_3(q). \tag{4.9}$$

Taking the products of both sides over all seventh roots of unity, The left hand side is transformed as follows:

$$\begin{aligned}(\omega q^{1/7};\omega q^{1/7})_\infty = \prod_{k=0}^{\infty}(1-\omega^{1+k}q^{\frac{k+1}{7}}) &= (1-\omega q^{1/7})(1-\omega^2q^{2/7})(1-\omega^3q^{3/7})(1-\omega^4q^{4/7})\\ &\times(1-\omega^5q^{5/7})(1-\omega^6q^{6/7})(1-q)(1-\omega^1q^{8/7})\\ &\times(1-\omega^2q^{9/7})(1-\omega^3q^{10/7})\dots\end{aligned}$$

$$\begin{aligned}
\prod_{\omega}(\omega q^{1/7};\omega q^{1/7})_\infty &= \prod_{\omega}\left(\prod_{k=0}^{\infty} 1-\omega^{1+k}q^{\frac{k+1}{5}}\right)\\
&= \prod_{\omega}(1-\omega q^{1/7})(1-\omega^2q^{2/7})(1-\omega^3q^{3/7})(1-\omega^4q^{4/7})\\
&\quad \times(1-\omega^5q^{5/7})(1-\omega^6q^{6/7})(1-q)(1-\omega^1q^{8/7})(1-\omega^2q^{9/7})\dots\\
&= (1-\omega q^{1/7})(1-\omega^2q^{1/7})(1-\omega^3q^{1/7})\dots(1-\omega^6q^{1/7})(1-q^{1/7})\\
&\quad \times(1-\omega^2q^{2/7})(1-\omega^4q^{2/7})(1-\omega^6q^{2/7})(1-\omega q^{2/7})\dots(1-q^{2/7})\\
&\quad \times\dots\times(1-\omega^6q^{6/7})\dots\dots(1-q^{6/7})(1-q)(1-q)(1-q)(1-q)\\
&\quad \times(1-q)(1-q)(1-q)(1-\omega q^{8/7})(1-\omega^2q^{8/7})(1-\omega^3q^{8/7})\\
&\quad \times\dots\times(1-q^2)(1-q^2)(1-q^2)(1-q^2)(1-q^2)(1-q^2)(1-q^2)\\
&\quad \times(1-\omega q^{15/7})(1-\omega^2q^{15/7})\dots\\
&\quad \times(1-q^3)(1-q^3)(1-q^3)(1-q^3)(1-q^3)(1-q^3)(1-q^3)\dots\\
&= (1-q)(1-q^2)(1-q^3)(1-q^4)(1-q^5)(1-q^6)(1-q^1)^7(1-q^8)\dots\\
&\quad \times(1-q^2)^7\dots(1-q^3)^7\dots(1-q^4)^7\dots(1-q^7)^7\dots(1-q^8)^7\dots\\
&= (1-q)^8(1-q^2)^8(1-q^3)^8(1-q^4)^8(1-q^5)^8(1-q^6)^8(1-q^7)^7\\
&\quad \times(1-q^8)^8(1-q^9)^8\dots(1-q^{14})^7(1-q^{15})^8\dots(1-q^{21})^7\dots
\end{aligned}$$

Multiply and divide both sides by $(q^7;q^7)_\infty$. Then,

$$\prod_{\omega}(\omega q^{1/7};\omega q^{1/7})_\infty = \frac{(q;q)_\infty^8}{(q^7;q^7)_\infty}. \tag{4.10}$$

Hence

$$\prod_{\omega}\frac{(\omega q^{1/7};\omega q^{1/7})_\infty}{(q^7;q^7)_\infty} = \frac{(q;q)_\infty^8}{(q^7;q^7)_\infty^7(q^7;q^7)_\infty} = \frac{(q;q)_\infty^8}{(q^7;q^7)_\infty^8}.$$

Then the equation (4.9) becomes

$$\frac{(q;q)_\infty^8}{(q^7;q^7)_\infty^8} = \prod_{\omega}\{J_1(q)+\omega q^{1/7}J_2(q)-\omega^2q^{2/7}+\omega^5q^{5/7}J_3(q)\}. \tag{4.11}$$

Using the generating function for $p(n)$, (2.2), multiplying both the numerator and the denominator by $(q^{49};q^{49})_\infty^8$ and $(q^7;q^7)_\infty^8$:

$$\begin{aligned}
\sum_{n=0}^{\infty}p(n)q^n &= \frac{1}{(q;q)_\infty}\\
&= \frac{(q^{49};q^{49})_\infty^7}{(q^7;q^7)_\infty^8}\frac{(q^7;q^7)_\infty^8}{(q^{49};q^{49})_\infty^8}\frac{(q^{49};q^{49})_\infty}{(q;q)_\infty}\\
&= \frac{(q^{49};q^{49})_\infty^7}{(q^7;q^7)_\infty^8}\frac{\prod_\omega\{J_1(q^7)+\omega qJ_2(q^7)-\omega^2q+\omega^5qJ_3(q^7)\}}{J_1(q^7)+qJ_2(q^7)-q+qJ_3(q^7)}\\
&= \frac{(q^{49};q^{49})_\infty^7}{(q^7;q^7)_\infty^8}\prod_{\omega\neq 1}\{J_1(q^7)+\omega qJ_2(q^7)-\omega q^2+\omega q^5J_3(q^7)\}. \qquad (4.12)
\end{aligned}$$

We need to compute the terms in $\prod_{\omega \neq 1}\{J_1(q^7)+\omega q J_2(q^7)-\omega q^2+\omega q^5 J_3(q^7)\}$ where the powers of q are of the form $7n+5$. Expanding the product in (4.11) we have

$$
\begin{aligned}
\frac{(q;q)_\infty^8}{(q^7;q^7)_\infty^8} = J_1^7 + J_2^7 q + J_3^7 q^5 - q^2 \quad &+ \quad 7(J_1 J_2^5 q + J_3 J_1^5 q + J_2 J_3^5 q^4) \\
&+ \quad 7(J_1^4 J_2^2 J_3 q + J_1 J_2^4 J_3^2 q^2 + J_2 J_3^4 J_1^2 q^3) \\
&+ \quad 7(J_1^3 J_2 q + J_2^3 J_3 q^2 + J_3^3 J_1 q^3) \\
&+ \quad 14(J_1^2 J_2^3 q + J_3^2 J_1^3 q^2 + J_2^2 J_3^3 q^3) \\
&+ \quad 7J_1^2 J_2^2 J_3^2 q^2 + 14 J_1 J_2 J_3 q^2. \qquad (4.13)
\end{aligned}
$$

Rewriting (4.13), using the fact $J_1 J_2 J_3 = -1$ in the second and last rows, the equation is further transformed:

$$
\begin{aligned}
= J_1^7 + J_2^7 q + J_3^7 q^5 - q^2 \quad &+ \quad 7q(J_1 J_2^5 + J_3 J_1^5 + J_2 J_3^5 q^3) \\
&+ \quad 7(-J_1^3 J_2^1 q - J_2^3 J_3 q^2 - J_3^3 J_1 q^3) \\
&+ \quad 7(J_1^3 J_2 q + J_2^3 J_3 q^2 + J_3^3 J_1 q^3) \\
&+ \quad 14q(J_1^2 J_2^3 + J_3^2 J_1^3 q + J_2^2 J_3^3 q^2) \\
&+ \quad 7q^2 - 14q^2. \qquad (4.14)
\end{aligned}
$$

Second and third rows are opposites of each other. Now we will examine the factors $(J_1 J_2^5 + J_3 J_1^5 + J_2 J_3^5 q^3)$ and $(J_1^2 J_2^3 + J_3^2 J_1^3 q + J_2^2 J_3^3 q^2)$, using the identities in (4.8).

$$
\begin{aligned}
J_1^2 J_2^3 + J_3^2 J_1^3 q + J_2^2 J_3^3 q^2 \quad &= \quad J_1^2 J_2^3 + J_3^2 J_1^3 q + J_2^2 J_3^3 q^2 \\
&= \quad J_1^2 J_2 J_2^2 + J_3^2 J_1 q J_1^2 + J_2^2 J_3 q J_3^2 q \\
&= \quad J_1^2 J_2 (J_1 + J_2 J_3^2 q) + J_3^2 J_1 q (J_1 J_2^2 + J_3 q) + J_2^2 J_3 q (J_2 + J_1^2 J_3) \\
&= \quad J_1^3 J_2 + J_1^2 J_2^2 J_3^2 q + J_1^2 J_2^2 J_3^2 q + J_3^3 J_1 q^2 + J_2^3 J_3 q + J_1^2 J_2^2 J_3^2 q \\
&= \quad J_1^3 J_2 + J_2^3 J_3 q + J_3^3 J_1 q^2 + 3q. \qquad (4.15)
\end{aligned}
$$

Again by using the identities in (4.8) and (4.15)

$$
\begin{aligned}
J_1 J_2^5 + J_3 J_1^5 + J_2 J_3^5 q^3 \quad &= \quad J_1 J_2^5 + J_3 J_1^5 + J_2 J_3^5 q^3 \\
&= \quad J_1 J_2 J_2^4 + J_3 J_1^4 + J_2 J_3 q (J_3^2 q)^2 \\
&= \quad J_1 J_2 (J_1 + J_2 J_3^2 q)^2 + J_3 (J_1 J_2^2 + J_3 q)^2 + J_2 J_3 q (J_2 + J_1^2 J_3)^2 \\
&= \quad J_1^3 J_2 + 2 J_1^2 J_2^2 J_3^2 q + J_1 J_2^3 J_3^4 q^2 + J_1^3 J_2^4 J_3 + 2 J_1^2 J_2^2 J_3^2 q + J_3^3 J_1 q^2 \\
&\quad\; + J_2^3 J_3 q + J_1^2 J_2^2 J_3^2 q + J_1^4 J_2 J_3^3 q \\
&= \quad J_1^3 J_2 + J_2^3 J_3 q + J_3^3 J_1 q^2 - (J_1^2 J_2^3 + J_1^3 J_3^2 q + J_2^2 J_3^3 q^2) + 6q \\
&= \quad -3q + 6q = 3q. \qquad (4.16)
\end{aligned}
$$

Then (4.14) continues to be transformed as follows, by substitutions (4.15) and (4.16) in the corresponding places,

$$\begin{aligned}
&= J_1^7 + J_2^7 q + J_3^7 q^5 - q^2 + 7q.3q \\
&\quad +14q(J_1^3 J_2 + J_2^3 J_3 q + J_3^3 J_1 q^2 + 3q) + 7q^2 - 14q^2 \\
&= J_1^7 + J_2^7 q + J_3^7 q^5 - q^2 + 21q^2 \\
&\quad +14q(J_1^3 J_2 + J_2^3 J_3 q + J_3^3 J_1 q^2) + 42q^2 + 7q^2 - 14q^2 \\
&= J_1^7 + J_2^7 q + J_3^7 q^5 + 14q(J_1^3 J_2 + J_2^3 J_3 q + J_3^3 J_1 q^2) + 55q^2. \qquad (4.17)
\end{aligned}$$

Let's make some observations. Let $z = J_1^3 J_2 + J_2^3 J_3 q + J_3^3 J_1 q^2$. Then, take the square of z and using the identities in (4.8) and for the fifth equality using (4.15) and (4.16):

$$\begin{aligned}
(J_1^3 J_2 + J_2^3 J_3 q + J_3^3 J_1 q^2)^2 &= J_1^6 J_2^2 + J_2^6 J_3^2 q^2 + J_3^6 J_1^2 q^4 \\
&\quad +2(J_1^3 J_2^4 J_3 q + J_1 J_2^3 J_3^4 q^3 + J_1^4 J_2 j_3^3 q^2) \\
&= J_1^6 (J_1 + J_2 J_3^2 q) + J_2^6 q (J_2 + J_1^2 J_3) + J_3^6 q^4 (J_1 J_2^2 + J_3 q) \\
&\quad +2(-J_1^2 J_2^3 q - J_2^2 J_3^3 q^3 - J_1^3 j_3^2 q^2) \\
&= J_1^7 + J_1^6 J_2 J_3^2 q + J_2^7 q + J_2^6 J_1^2 J_3 q + J_3^6 J_1 J_2^2 q^4 + J_3^7 q^5 \\
&\quad -2(J_1^2 J_2^3 q + J_1^3 j_3^2 q^2 + J_2^2 J_3^3 q^3) \\
&= J_1^7 + J_2^7 q + J_3^7 q^5 - (J_1^5 J_3 q + J_2^5 J_1 q + J_3^5 J_2 q^4) \\
&\quad -2q(J_1^2 J_2^3 + J_1^3 j_3^2 q + J_2^2 J_3^3 q^2) \\
&= J_1^7 + J_2^7 q + J_3^7 q^5 - q3q - 2q(J_1^3 J_2 + J_2^3 J_3 q + J_3^3 J_1 q^2 + 3q) \\
&= J_1^7 + J_2^7 q + J_3^7 q^5 - 3q^2 - 2q(J_1^3 J_2 + J_2^3 J_3 q + J_3^3 J_1 q^2) - 6q^2 \\
&= J_1^7 + J_2^7 q + J_3^7 q^5 - 2q(J_1^3 J_2 + J_2^3 J_3 q + J_3^3 J_1 q^2) - 9q^2.
\end{aligned}$$

Then

$$z^2 = J_1^7 + J_2^7 q + J_3^7 q^5 - 2qz - 9q^2.$$

It follows that

$$J_1^7 + J_2^7 q + J_3^7 q^5 = (z+q)^2 + 8q^2. \qquad (4.18)$$

To continue to transform (4.17) , substitute (4.18) in the last line of (4.17):

$$\begin{aligned}
J_1^7 + J_2^7 q + J_3^7 q^5 + 14q(J_1^3 J_2 + J_2^3 J_3 q + J_3^3 J_1 q^2) + 55q^2 &= (z+q)^2 + 8q^2 + 14qz + 55q^2 \\
&= (z+8q)^2.
\end{aligned}$$

Eventually we have

$$\frac{(q;q)_\infty^8}{(q^7;q^7)_\infty^8} = (z+8q)^2.$$

Observe that from the left hand side of (4.5), J_2 is negative for sufficiently small q. Because when we consider on infinite products in (4.5), we obtain

$$\begin{aligned}\frac{(1-q^{1/7})}{\prod_{n\geq 1}(1-q^7)^n} = \frac{(1-q^{1/7})}{\sum_{n=0}^{\infty} q^{7n}} &= (1-q^{1/7}) + (q^7 - q^{50/7}) + \ldots \\ &= (1-q^{1/7}) + O(q^{2/7})\end{aligned}$$

where $O(q^{2/7})$ respesent the remaining parts of the series with exponents greater than or equal to 2/7. Then multiply this with the remaining factors of $(q^{1/7};q^{1/7})_\infty$, again we have

$$(1-q^{1/7}) + O(q^{2/7}).$$

Here one sees that when we group them according the exponent of $q^{1/7}$, besides the other exponents, we have $q^{1/7}(-1+O(q))$ where $O(q)$ respesent the remaining parts of the series with integral exponents greater than or equal to 1. On the other hand we have expressed this series as J_2 with integral exponents and with integral coefficients in (4.5). Hence J_2 contains -1 and by this reason for sufficiently small q, z is negative. Taking this into account, we take the square root of both sides and solve for z:

$$z = -\frac{(q;q)_\infty^4}{(q^7;q^7)_\infty^4} - 8q$$

which is

$$J_1^3 J_2 + J_2^3 J_3 q + J_3^3 J_1 q^2 = -\frac{(q;q)_\infty^4}{(q^7;q^7)_\infty^4} - 8q. \tag{4.19}$$

Now we claim that

$$J_1^2 J_2^3 + J_3^2 J_1^3 q + J_2^2 J_3^3 q^2 = -\frac{(q;q)_\infty^4}{(q^7;q^7)_\infty^4} - 5q. \tag{4.20}$$

Because from (4.15) we know that

$$J_1^2 J_2^3 + J_3^2 J_1^3 q + J_2^2 J_3^3 q^2 = J_1^3 J_2 + J_2^3 J_3 q + J_3^3 J_1 q^2 + 3q$$

and substitution (4.19) in corresponding place gives

$$J_1^2 J_2^3 + J_3^2 J_1^3 q + J_2^2 J_3^3 q^2 = -\frac{(q;q)_\infty^4}{(q^7;q^7)_\infty^4} - 8q + 3q = -\frac{(q;q)_\infty^4}{(q^7;q^7)_\infty^4} - 5q.$$

Returning to (4.12), by using the computer algebra system MAXIMA we compute the terms in $\prod_{\omega\neq 1}\{J_1(q^7) + \omega q J_2(q^7) - \omega q^2 + \omega q^5 J_3(q^7)\}$ where the powers of q are of the form $7n+5$, we find:

$$\begin{aligned}-(J_1(q^7)J_2^5(q^7) + J_3(q^7)J_1^5(q^7) + 3J_1^3(q^7)J_2(q^7) + 4J_1^2(q^7)J_2^3(q^7))q^5 \\ -(3J_2^3(q^7)J_3(q^7) + 4J_3^2(q^7)J_1^3(q^7) - 8)q^{12} \\ -(4J_2^2(q^7)J_3^3(q^7) + 3J_3^3(q^7)J_1(q^7))q^{19} \\ -J_2(q^7)J_3^5(q^7)q^{26}.\end{aligned}$$

Group according to the coefficients of powers of q:

$$-(J_1J_2^5 + J_3J_1^5 + J_2J_3^5q^{21})q^5 - 3(J_1^3J_2 + J_2^3J_3q^7 + J_3^3J_1q^{14})q^5$$
$$-4(J_1^2J_2^3 + J_3^2J_1^3q^7 + J_2^2J_3^3q^{14})q^5 + 8q^{12}, \tag{4.21}$$

where J_1,J_2 and J_3 are power series in q^7 with integral powers and integral coefficients.

Observe that when we substitute q^7 in for q in (4.19), we obtain

$$J_1^3J_2 + J_2^3J_3q^7 + J_3^3J_1q^{14} = -\frac{(q^7;q^7)_\infty^4}{(q^{49};q^{49})_\infty^4} - 8q^7 \tag{4.22}$$

Similarly if we substitute q^7 in for q in (4.16), we have

$$J_1J_2^5 + J_3J_1^5 + J_2J_3^5q^{21} = 3q^7 \tag{4.23}$$

and substituting q^7 in for q in (4.20),

$$J_1^2J_2^3 + J_3^2J_1^3q^7 + J_2^2J_3^3q^{14} = -\frac{(q^7;q^7)_\infty^4}{(q^{49};q^{49})_\infty^4} - 5q^7 \tag{4.24}$$

where J_1,J_2,J_3 are power series in q^7 with integral powers and integral coefficients.Then substituting the outcomes in (4.22), (4.23), (4.24) to (4.21) we have the desired result

$$7q^5\frac{(q^7;q^7)_\infty^4}{(q^{49};q^{49})_\infty^4} + 49q^{12}.$$

Therefore if we consider the terms with the exponent of the form $7n+5$ on both sides of (4.12):

$$\begin{aligned}\sum_{n=0}^{\infty} p(7n+5)q^{7n+5} &= \frac{(q^{49};q^{49})_\infty^7}{(q^7;q^7)_\infty^8}\left[7q^5\frac{(q^7;q^7)_\infty^4}{(q^{49};q^{49})_\infty^4} + 49q^{12}\right] \\ &= 7q^5\frac{(q^{49};q^{49})_\infty^3}{(q^7;q^7)_\infty^4} + 49q^{12}\frac{(q^{49};q^{49})_\infty^7}{(q^7;q^7)_\infty^8}.\end{aligned}$$

Simplify q^5 from both sides and then replace q^7 by q:

$$\sum_{n=0}^{\infty} p(7n+5)q^n = 7\frac{(q^7;q^7)_\infty^3}{(q;q)_\infty^4} + 49q\frac{(q^7;q^7)_\infty^7}{(q;q)_\infty^8}.$$

□

If we reduce modulo 7

$$\sum_{n=0}^{\infty} p(7n+5)q^n \equiv 0 \pmod 7.$$

This implies that $p(7n+5) \equiv 0 \pmod 7$. □

Now we will illustrate (1.4) when $b = b' = 2$ and $a = c = 0$. Since $b = b'$ we have $\delta = \delta' = 49$ and $24.\lambda \equiv 1 \pmod{49}$ is satisfied when $\lambda = 47$.

Theorem 4.3 *For any $n \in \mathbb{N}$,*

$$p(49n+47) \equiv 0 \pmod{49}.$$

Proof : By Theorem 4.2 we know that

$$\sum_{n=0}^{\infty} p(7n+5)q^n = 7\frac{(q^7;q^7)_\infty^3}{(q;q)_\infty^4} + 49q\frac{(q^7;q^7)_\infty^7}{(q;q)_\infty^8}.$$

Rewrite this applying (2.13) to the denominator when $l = d = 7$ and then apply Jacobi's identity (2.6):

$$\begin{aligned} 7\frac{(q^7;q^7)_\infty^3(q;q)_\infty^3}{(q;q)_\infty^7} + 49q\frac{(q^7;q^7)_\infty^7}{(q;q)_\infty^8} &= 7(q^7;q^7)_\infty^2 \sum_{m=0}^{\infty}(-1)^m(2m+1)q^{\frac{m(m+1)}{2}} + 49q\frac{(q^7;q^7)_\infty^7}{(q;q)_\infty^8} \\ &\equiv 7(q^7;q^7)_\infty^2 \sum_{m=0}^{\infty}(-1)^m(2m+1)q^{\frac{m(m+1)}{2}} \pmod{49}. \end{aligned}$$

Then,

$$\sum_{n=0}^{\infty} p(7n+5)q^n \equiv 7(q^7;q^7)_\infty^2 \sum_{m=0}^{\infty}(-1)^m(2m+1)q^{\frac{m(m+1)}{2}} \pmod{49}. \quad (4.25)$$

Now examine the terms on the right hand side where the powers are of the form $7n+6$:

		(mod 7)
m	$m+1$	$(m(m+1))/2$
0	1	0
1	2	1
2	3	3
3	4	6
4	5	3
5	6	1
6	7	0

By this observation we see that the exponents are of the form $7n+6$ if and only if $m \equiv 3 \pmod{7}$. But in this case the coefficient $(2m+1)$ in (4.25) is congruent to zero modulo 7. Then $7(2m+1)$ is a multiple of 49. It follows that the coefficients of q^{7n+6} are multiple of 49 on the right hand side of (4.25). Then the coefficients of q^{7n+6} are multiple of 49 on the left hand side of (4.25), i.e $p(7(7n+6)+5) = p(49n+47)$ must be a multiple of 49.

□

CHAPTER 5

Ramanujan's congruence mod 11

We'll present Ramanujan's congruence for the partition function modulo 11 in two different ways such that the second uses an evolved identity compared to the first. Sure, there are other proofs using different ideas both easy and hard. However we include a more elegant one, written by Winquist in 1969; An Elementary Proof of $p(11n+6) \equiv 0 \pmod{11}$, [21], than others. Winquist answers the question: Is it possible to prove the modulo 11 congruence along the same line with the proofs for modulo 5 and 7 that written by Hardy, [12], using $\prod_{n\geq 0}(1-x^n)^4$ and $\prod_{n\geq 0}(1-x^n)^6$ as a double series? In this direction, we also expect to write $\prod_{n\geq 0}(1-x^n)^{10}$ as a double series, and then prove the congruence modulo 11. Winquist did this. Later on, we examine the article written by Hirschhorn, [13]. He improved the two parameter identity, used by Winquist, to a four parameter identity. Then he provided a proof for the congruence in modulo 11.

Theorem 5.1 *For each $n \in \mathbb{N}$,*

$$p(11n+6) \equiv 0 \pmod{11}.$$

5.1. Proof by a two-parameter identity

We begin by introducing the following theorem for a two parameter identity and then the proof for Theorem 5.1 will follow.

Theorem 5.2

$$\begin{aligned} F(a,b,x) &= \prod_{n\geq 1}(1-ax^{n-1})(1-a^{-1}x^n)(1-bx^{n-1})(1-b^{-1}x^n)(1-ab^{-1}x^{n-1}) \\ &\quad \times(1-a^{-1}bx^n)(1-abx^{n-1})(1-a^{-1}b^{-1}x^n)(1-x^n)^2 \\ &= \sum_{j=-\infty}^{\infty}(-1)^j(b^{-3j}-b^{-3j+1})x^{j(3j+1)/2}\sum_{i\geq 0}(-1)^i(a^{-3i}-a^{3i+3})x^{3i(i+1)/2} \\ &\quad +\sum_{i\geq 0}(-1)^i(b^{3i+2}-b^{-3i-1})x^{3i(i+1)/2}\sum_{j=-\infty}^{\infty}(-1)^j(a^{-3j+1}-a^{3j+2})x^{j(3j+1)/2}. \end{aligned} \tag{5.1}$$

Proof : Observe that $F(ax,b,x) = -\frac{1}{a^3}F(a,b,x)$, since

$$\begin{aligned} F(ax,b,x) &= \prod_{n\geq 1}(1-ax^n)(1-a^{-1}x^{n-1})(1-bx^{n-1})(1-b^{-1}x^n)(1-ab^{-1}x^n) \\ &\quad \times(1-a^{-1}bx^{n-1})(1-abx^n)(1-a^{-1}b^{-1}x^{n-1})(1-x^n)^2 \\ &= \prod_{n\geq 1}(1-a)^{-1}(1-ax^{n-1})(1-a^{-1})(1-a^{-1}x^n)(1-bx^{n-1})(1-b^{-1}x^n) \\ &\quad \times(1-ab^{-1})^{-1}(1-ab^{-1}x^{n-1})(1-a^{-1}b)(1-a^{-1}bx^n)(1-ab)^{-1}(1-abx^{n-1}) \\ &\quad \times(1-(ab)^{-1})(1-a^{-1}b^{-1}x^n)(1-x^n)^2 \\ &= \frac{(1-a^{-1})(1-a^{-1}b)(1-(ab)^{-1})}{(1-a)(1-ab^{-1})(1-ab)}\prod_{n\geq 1}(1-ax^{n-1})(1-a^{-1}x^n)(1-bx^{n-1}) \\ &\qquad \times(1-b^{-1}x^n)(1-ab^{-1}x^{n-1})(1-a^{-1}bx^n)(1-abx^{n-1})(1-a^{-1}b^{-1}x^n) \\ &\qquad \times(1-x^n)^2 \\ &= -\frac{1}{a^3}F(a,b,x). \end{aligned}$$

By similar computations one can show that

$$F(\frac{1}{a},b,x) = -\frac{1}{a^3}F(a,b,x) \tag{5.2}$$

Let

$$F(a,b,x) = \sum_{r=-\infty}^{\infty} C_r(b,x)a^r. \tag{5.3}$$

Then

$$F(ax,b,x) = \sum_{r=-\infty}^{\infty} C_r(b,x)x^r a^r. \tag{5.4}$$

Consider

$$\begin{aligned} -\frac{1}{a^3}F(a,b,x) &\overset{(5.3)}{=} -\frac{1}{a^3}\sum_{r=-\infty}^{\infty} C_r(b,x)a^r \\ &= \sum_{r=-\infty}^{\infty} -C_r(b,x)a^{r-3} \\ &\overset{r\mapsto r+3}{=} \sum_{r=-\infty}^{\infty} -C_{r+3}(b,x)a^r. \end{aligned} \tag{5.5}$$

Then since

$$F(ax,b,x) = -\frac{1}{a^3}F(a,b,x),$$

combining (5.4) and (5.5) gives

$$\sum_{r=-\infty}^{\infty} C_r(b,x)x^r a^r = \sum_{r=-\infty}^{\infty} -C_{r+3}(b,x)a^r.$$

Hence

$$C_r(b,x)x^r = -C_{r+3}(b,x). \tag{5.6}$$

Again using (5.3)

$$\begin{aligned} F\left(\frac{1}{a},b,x\right) &= \sum_{r=-\infty}^{\infty} C_r(b,x)a^{-r} \\ &\overset{r\mapsto -r}{=} \sum_{r=-\infty}^{\infty} C_{-r}(b,x)a^r \end{aligned} \tag{5.7}$$

By the second observation (5.2) and (5.5) we have

$$\sum_{r=-\infty}^{\infty} C_{-r}(b,x)a^r = \sum_{r=-\infty}^{\infty} -C_{r+3}(b,x)a^r$$

Then

$$-C_{r+3}(b,x) = C_{-r}(b,x). \tag{5.8}$$

Put $r = -1$ in (5.8) This gives $-C_2 = C_1$. Therefore if we know C_0 and C_1 then by using (5.6) and (5.8) we know all C_r's.

$$\begin{aligned} F(a,b,x) &= C_0(b,x)\sum_{i\geq 0}(-1)^i(a^{-3i} - a^{3i+3})x^{3i(i+1)/2} \\ &+ C_1(b,x)\sum_{j=-\infty}^{\infty}(-1)^j(a^{-3j+1} - a^{3j+2})x^{j(3j+1)/2}. \end{aligned} \tag{5.9}$$

Put x^3 for x and x for a, in this order, then

$$\begin{aligned} F(x,b,x^3) &= C_0(b,x^3)\sum_{i\geq 0}(-1)^i(x^{-3i} - x^{3i+3})x^{3\frac{3i(3i+1)}{2}} \\ &+ C_1(b,x^3)\sum_{j=-\infty}^{\infty}(-1)^j(x^{-3j+1} - x^{3j+2})x^{3\frac{j(3j+1)}{2}}. \end{aligned} \tag{5.10}$$

Consider the first summation:

$$\begin{aligned} C_0(b,x^3)\sum_{i\geq 0}(-1)^i x^{-3i}x^{3\frac{3i(3i+1)}{2}} &= C_0(b,x^3)\sum_{i\geq 0}(-1)^i x^{\frac{9i^2+3i}{2}} \\ &= C_0(b,x^3)\sum_{i\geq 0}(-1)^i x^{3\frac{i(3i+1)}{2}} \end{aligned} \tag{5.11}$$

$$
\begin{aligned}
C_0(b,x^3)\sum_{i\geq 0}(-1)^i(-x^{3i+3})x^{3\frac{3i(3i+1)}{2}} &= C_0(b,x^3)\sum_{i\geq 0}(-1)^{i+1}x^{\frac{(3i+2)(3i+3)}{2}}\\
&\overset{i\mapsto i-1}{=} C_0(b,x^3)\sum_{i\geq 1}(-1)^i x^{\frac{(3i-1)3i}{2}}\\
&\overset{i\mapsto -i}{=} C_0(b,x^3)\sum_{i\leq -1}(-1)^i x^{3\frac{i(3i+1)}{2}} \qquad (5.12)
\end{aligned}
$$

Combining the sums in (5.11) and in (5.12) we have

$$
C_0(b,x^3)\sum_{i=-\infty}^{\infty}(-1)^i x^{\frac{3i(3i+1)}{2}}. \qquad (5.13)
$$

Now consider the second summation in $F(x,b,x^3)$:

$$
\begin{aligned}
C_1(b,x^3)\sum_{j=-\infty}^{\infty}(-1)^j x^{-3j+1}x^{3\frac{j(3j+1)}{2}} &= C_1(b,x^3)\sum_{j=-\infty}^{\infty}(-1)^j x^{\frac{9j^2-3j+2}{2}}\\
&= C_1(b,x^3)\sum_{j=-\infty}^{\infty}(-1)^j x(x^3)^{\frac{j(3j-1)}{2}} \qquad (5.14)
\end{aligned}
$$

$$
\begin{aligned}
C_1(b,x^3)\sum_{j=-\infty}^{\infty}(-1)^j(-x^{3j+2})x^{3\frac{j(3j+1)}{2}} &= \sum_{j=-\infty}^{\infty}(-1)^j x^2 x^{9\frac{j(j+1)}{2}}\\
&= x^2\left[\sum_{j<0}^{-\infty}(-1)^j(x^9)^{\frac{j(j+1)}{2}}+\sum_{j\geq 0}^{\infty}(-1)^j(x^9)^{\frac{j(j+1)}{2}}\right]\\
&= x^2\left[\sum_{j<1}^{-\infty}(-1)^{j-1}(x^9)^{\frac{(j-1)j}{2}}+\sum_{j\geq 0}^{\infty}(-1)^j(x^9)^{\frac{j(j+1)}{2}}\right]\\
&= x^2\left[-\sum_{j>-1}^{\infty}(-1)^j(x^9)^{\frac{(j+1)j}{2}}+\sum_{j\geq 0}^{\infty}(-1)^j(x^9)^{\frac{j(j+1)}{2}}\right]\\
&= 0.
\end{aligned}
$$

Consequently combining (5.13) and (5.14) gives this result:

$$
C_0(b,x^3)\sum_{i=-\infty}^{\infty}(-1)^i x^{\frac{3i(3i+1)}{2}}+C_1(b,x^3)x\sum_{j=-\infty}^{\infty}(-1)^j(x^3)^{\frac{j(3j-1)}{2}}.
$$

By the Pentagonal number theorem,(2.5), this equals to

$$
C_0(b,x^3)\prod_{n\geq 1}(1-x^{3n})+C_1(b,x^3)x\prod_{n\geq 1}(1-x^{3n}). \qquad (5.15)
$$

Now make the same substitutions, x^3 for x and x for a, in this order, for $F(a,b,x)$:

$$\begin{aligned} F(x,b,x^3) &= \prod_{n\geq 1}(1-xx^{3n-3})(1-x^{-1}x^{3n})(1-bx^{3n-3})(1-b^{-1}x^{3n})(1-xb^{-1}x^{3n-3}) \\ &\quad \times(1-x^{-1}bx^{3n})(1-xbx^{3n-3})(1-x^{-1}b^{-1}x^{3n})(1-x^{3n})^2 \\ &= \prod_{n\geq 1}(1-x^{3n-2})(1-x^{3n-1})(1-b)(1-bx^{3n})(1-b^{-1}x^{3n})(1-b^{-1}x^{3n-2}) \\ &\quad \times(1-bx^{3n-1})(1-bx^{3n-2})(1-b^{-1}x^{3n-1})(1-x^{3n})(1-x^{3n}) \\ &= \prod_{n\geq 1}(1-x^n)(1-bx^n)(1-b^{-1}x^n)(1-x^{3n})(1-b) \\ &= \prod_{n\geq 1}(1-x^n)(1-bx^{n-1})(1-b^{-1}x^n)(1-x^{3n}). \end{aligned} \tag{5.16}$$

Then (5.10) is transformed to the following, using (5.15) and (5.16):

$$\begin{aligned} & C_0(b,x^3)\prod_{n\geq 1}(1-x^{3n}) + C_1(b,x^3)x\prod_{n\geq 1}(1-x^{3n}) \\ &= \prod_{n\geq 1}(1-x^n)(1-bx^{n-1})(1-b^{-1}x^n)(1-x^{3n}). \end{aligned}$$

Divide both sides by

$$\prod_{n\geq 1}(1-x^{3n})$$

.

$$C_0(b,x^3) + C_1(b,x^3)x = \prod_{n\geq 1}(1-x^n)(1-bx^{n-1})(1-b^{-1}x^n). \tag{5.17}$$

The right hand side can be written as $(x;x)_\infty(b;x)_\infty(xb^{-1};x)_\infty$. Then apply Jacobi's triple product identity 2, (2.8):

$$\begin{aligned} (x;x)_\infty(b;x)_\infty(xb^{-1};x)_\infty &= \sum_{n=-\infty}^{\infty}(-1)^n b^n x^{\binom{n}{2}} \\ &\overset{n\mapsto n+1}{=} \sum_{n=-\infty}^{\infty}(-1)^{n+1}b^{n+1}x^{\binom{n+1}{2}} \\ &= \sum_{n=-\infty}^{-1}(-1)^{n+1}b^{n+1}x^{\binom{n+1}{2}} + \sum_{n=0}^{\infty}(-1)^{n+1}b^{n+1}x^{\binom{n+1}{2}} \\ &\overset{n\mapsto -n}{=} \sum_{n=1}^{\infty}(-1)^{-n+1}b^{-n+1}x^{\binom{n}{2}} + \sum_{n=0}^{\infty}(-1)^{n+1}b^{n+1}x^{\binom{n+1}{2}} \\ &\overset{n\mapsto n+1}{=} \sum_{n=0}^{\infty}(-1)^{-n}b^{-n}x^{\binom{n+1}{2}} - \sum_{n=0}^{\infty}(-1)^{n+1}b^{n+1}x^{\binom{n+1}{2}} \\ &= \sum_{n=0}^{\infty}(-1)^n(b^{-n}-b^{n+1})x^{\binom{n+1}{2}}. \end{aligned}$$

Hence (5.17) is transformed to the following

$$C_0(b,x^3)+C_1(b,x^3)x=\sum_{n=0}^{\infty}(-1)^n(b^{-n}-b^{n+1})x^{\binom{n+1}{2}}.$$

Observe that in the series $C_0(b,x^3)$, exponents of x are congruent to 0 modulo 3 and in the series $C_1(b,x^3)x$, exponents of x are congruent to 0 modulo 3. Now we are looking for the exponents that are congruent to 1 modulo 3 and congruent to 0 modulo 3 for the right hand side.

	(mod 3)	
n	$n+1$	$n(n+1)/2$
0	1	0
1	2	1
2	3	0

As one sees in the table, the exponent is congruent to 0 when $n\equiv 0 \pmod 3$. For $n=3j$ for some $j\in N$, the corresponding terms for $C_0(b,x^3)$ is

$$\sum_{j=0}^{\infty}(-1)^{3j}(b^{-3j}-b^{3j+1})x^{\binom{3j+1}{2}}. \tag{5.18}$$

For $n=3j+2$ for some $j\in N$, the corresponding residue for $C_0(b,x^3)$ is

$$\begin{aligned}
\sum_{j=0}^{\infty}(-1)^{3j+2}(b^{-(3j+2)}-b^{3j+3})x^{\binom{3j+3}{2}} &\overset{j\mapsto -j}{=} \sum_{j=0}^{-\infty}(-1)^{-3j}(b^{3j-2}-b^{-3j+3})x^{\binom{-3j+3}{2}}\\
&\overset{j\mapsto j+1}{=} \sum_{j=-1}^{-\infty}(-1)^{-3j-3}(b^{3j+1}-b^{-3j})x^{\frac{(-3j)(-3j-1)}{2}}\\
&= -\sum_{j=-1}^{-\infty}(-1)^{-3j}(b^{3j+1}-b^{-3j})x^{\frac{(3j)(3j+1)}{2}}\\
&= \sum_{j=-1}^{-\infty}(-1)^{j}(b^{-3j}-b^{3j+1})x^{\binom{3j+1}{2}}. \qquad (5.19)
\end{aligned}$$

Combining (5.18) and (5.19) gives

$$\begin{aligned}
\sum_{j=0}^{\infty}(-1)^{3j}(b^{-3j}-b^{3j+1})x^{\binom{3j+1}{2}} &+ \sum_{j=-1}^{-\infty}(-1)^{j}(b^{-3j}-b^{3j+1})x^{\binom{3j+1}{2}}\\
&= \sum_{j=-\infty}^{-\infty}(-1)^{j}(b^{-3j}-b^{3j+1})x^{\binom{3j+1}{2}}\\
&= \sum_{j=-\infty}^{-\infty}(-1)^{j}(b^{-3j}-b^{3j+1})x^{\frac{(3j+1)3j}{2}}.
\end{aligned}$$

Writing x for x^3, the corresponding residue for $C_0(b,x)$ is

$$\sum_{j=-\infty}^{-\infty}(-1)^j(b^{-3j}-b^{3j+1})x^{\frac{(3j+1)j}{2}}. \tag{5.20}$$

In the table, the exponent is congruent to 1 modulo 3 when $n \equiv 1 \pmod 1$. Say $n=3i+1$ for some $i \in N$. Then the corresponding residue for $C_1(b,x^3)x$ is

$$\begin{aligned}\sum_{i=0}^{\infty}(-1)^{3i+1}(b^{-(3i+1)}-b^{3i+2})x^{\binom{3i+2}{2}} &= -\sum_{i=0}^{\infty}(-1)^i(b^{-3i-1}-b^{3i+2})x^{\frac{(3i+2)(3i+1)}{2}}\\ &= \sum_{i=0}^{\infty}(-1)^i(b^{3i+2}-b^{-3i-1})x^{\frac{9i^2+9i+2}{2}}\\ &= x\sum_{i=0}^{\infty}(-1)^i(b^{3i+2}-b^{-3i-1})(x^9)^{\frac{i(i+1)}{2}}.\end{aligned}$$

Writing x for x^3, we obtain

$$C_1(b,x)=\sum_{i=0}^{\infty}(-1)^i(b^{3i+2}-b^{-3i-1})(x^3)^{\frac{i(i+1)}{2}}. \tag{5.21}$$

Therefore when we substitute (5.21) and (5.20) in (5.9), we obtain (5.1).

□

Now we will derive $\prod_{n\geq 1}(1-x^n)^{10}$ as a double series. Then the proof will follow. Factor out the term $(1-b)$ in theorem 5.2 and divide both sides by $(1-b)$:

$$\begin{aligned}F(a,b,x) &= \prod_{n\geq 1}(1-ax^n)(1-a^{-1}x^n)(1-bx^n)(1-b^{-1}x^n)(1-ab^{-1}x^{n-1})\\ &\quad \times(1-a^{-1}bx^n)(1-abx^{n-1})(1-a^{-1}b^{-1}x^n)(1-x^n)^2\\ &= \sum_{j=-\infty}^{\infty}(-1)^j\frac{b^{-3j}-b^{-3j+1}}{1-b}x^{j(3j+1)/2}\sum_{i\geq 0}(-1)^i(a^{-3i}-a^{3i+3})x^{3i(i+1)/2}\\ &\quad +\sum_{i\geq 0}(-1)^i\frac{b^{3i+2}-b^{-3i-1}}{1-b}x^{3i(i+1)/2}\sum_{j=-\infty}^{\infty}(-1)^j(a^{-3j+1}-a^{3j+2})x^{j(3j+1)/2}.\end{aligned}$$

Let $b \to 1$ on both sides. To do this we apply L'hospital rule to the right hand side. Then

$$\begin{aligned}&\prod_{n\geq 1}(1-ax^{n-1})^3(1-a^{-1}x^n)^3(1-x^n)^4\\ &= \sum_{j=-\infty}^{\infty}(-1)^j(6j+1)x^{j(3j+1)/2}\sum_{i\geq 0}(-1)^i a^{-3i}-a^{3i+3}x^{3i(i+1)/2}\\ &+ \sum_{i\geq 0}(-1)^i(-6i-3)x^{3i(i+1)/2}\sum_{j=-\infty}^{\infty}(-1)^j(a^{-3j+1}-a^{3j+2})x^{j(3j+1)/2}.\end{aligned}$$

This time factor the term $(1-a)^3$ out from the left hand side and then divide both sides by $(1-a)^3$

$$\begin{aligned}
&\prod_{n\geq 1}(1-ax^n)^3(1-a^{-1}x^n)^3(1-x^n)^4 \\
&= \sum_{j=-\infty}^{\infty}(-1)^j(6j+1)x^{j(3j+1)/2}\sum_{i\geq 0}(-1)^i\frac{a^{-3i}-a^{3i+3}}{(1-a)^3}x^{3i(i+1)/2} \\
&+ \quad -3\sum_{i\geq 0}(-1)^i(2i+1)x^{3i(i+1)/2}\sum_{j=-\infty}^{\infty}(-1)^j\frac{a^{-3j+1}-a^{3j+2}}{(1-a)^3}x^{j(3j+1)/2}.
\end{aligned}$$

Now let $a \to 1$, the right hand side becomes

$$\prod_{n\geq 1}(1-x^n)^{10}. \tag{5.22}$$

Again for the right hand side, we need to use L'hospital rule as we differentiate three times. Hence

$$\begin{aligned}
&\lim_{a\to 1}\sum_{j=-\infty}^{\infty}(-1)^j(6j+1)x^{j(3j+1)/2}\sum_{i\geq 0}(-1)^i\frac{a^{-3i}-a^{3i+3}}{(1-a)^3}x^{3i(i+1)/2} \\
&\quad -3\sum_{i\geq 0}(-1)^i(2i+1)x^{3i(i+1)/2}\sum_{j=-\infty}^{\infty}(-1)^j\frac{a^{-3j+1}-a^{3j+2}}{(1-a)^3}x^{j(3j+1)/2} \\
&= \lim_{a\to 1}\left[\sum_{j=-\infty}^{\infty}(-1)^j(6j+1)x^{j(3j+1)/2}\right. \\
&\quad \sum_{i\geq 0}(-1)^i\frac{(-3i)(-3i-1)(-3i-2)a^{-3i-3}-(3i+3)(3i+2)(3i+1)a^{3i}}{3.2.(-1)(-1)(-1)} \\
&\quad \left.\times x^{3i(i+1)/2}\right] - 3\left[\sum_{i\geq 0}(-1)^i(2i+1)x^{3i(i+1)/2}\right. \\
&\quad \left.\sum_{j=-\infty}^{\infty}(-1)^j\frac{(-3j+1)(-3j)(-3j-1)a^{-3j-2}-(3j+2)(3j+1)(3j)a^{3j-1}}{3.2.(-1)(-1)(-1)}x^{j(3j+1)/2}\right] \\
&= \sum_{j=-\infty}^{\infty}(-1)^j(6j+1)x^{j(3j+1)/2}\sum_{i\geq 0}(-1)^i\frac{(3i+1)(3i+2)(6i+3)}{6}x^{3i(i+1)/2} \\
&\quad -3\sum_{i\geq 0}(-1)^i(2i+1)x^{3i(i+1)/2}\sum_{j=-\infty}^{\infty}(-1)^j\frac{(3j)(3j+1)(6j+1)}{6}x^{j(3j+1)/2} \\
&= \sum_{j=-\infty}^{\infty}(-1)^{i+j}\frac{(6j+1)(3i+1)(3i+2)(2i+1)}{2}x^{\frac{3i(i+1)}{2}+\frac{j(3j+1)}{2}} \\
&\quad -\sum_{i\geq 0}(-1)^{i+j}\frac{(2i+1)(3j)(3j+1)(6j+1)}{2}x^{\frac{3i(i+1)}{2}+\frac{j(3j+1)}{2}}.
\end{aligned} \tag{5.23}$$

Combining (5.22) and (5.23), we have

$$\prod_{n\geq 1}(1-x^n)^{10} = \sum(-1)^{i+j}(2i+1)(6j+1)\left[\frac{(3i+1)(3i+2)}{2} - \frac{3j(3j+1)}{2}\right] \times x^{\frac{3i(i+1)}{2}+\frac{j(3j+1)}{2}}$$

summed over i,j integers, $i \geq 0$, $-\infty \leq j \leq \infty$. Multiply both sides by x^5:

$$x^5\prod_{n\geq 1}(1-x^n)^{10} = \sum(-1)^{i+j}(2i+1)(6j+1)\left[\frac{(3i+1)(3i+2)}{2} - \frac{3j(3j+1)}{2}\right] \times x^{\frac{3i(i+1)}{2}+\frac{j(3j+1)}{2}+5}. \quad (5.24)$$

We investigate in what circumstances the exponent $\frac{3i(i+1)}{2} + \frac{j(3j+1)}{2} + 5$ is divisible by 11.

i	$i+1$	(mod 11) $\frac{3}{2}i(i+1)$
0	1	0
1	2	3
2	3	9
3	4	7
4	5	8
5	6	1
6	7	8
7	8	7
8	9	9
9	10	3
10	11	0
11	12	0

j	$3j+1$	(mod 11) $\frac{j(3j+1)}{2}$
0	1	0
1	4	2
2	7	7
3	10	4
4	13	8
5	16	3
6	19	4
7	22	0
8	25	1
9	28	5
10	31	1
11	34	0

The possibilities to obtain 6 for the sum $\frac{j(3j+1)}{2} + \frac{3}{2}i(i+1)$ are the followings and in each case the coefficient $(2i+1)(6j+1)$ gives a different result in modulo 11:

(i) $i = 2$, $j = 4$ implies the coefficient $(2i+1)(6j+1)$ is congruent to $5.25 \equiv 4 \pmod{11}$

(ii) $i = 5$, $j = 9$ implies that $(2.5+1)(6.9+1) = 11.55 \equiv 0 \pmod{11}$

(iii) $i = 8$, $j = 4$ implies that $(2.8+1)(6.4+1) \equiv 7 \pmod{11}$

(iv) $i = 9$, $j = 5$ implies that $(2.9+1)(6.5+1) \equiv 6 \pmod{11}$.

This shows that the coefficient is not congruent to zero whenever the exponent of x is congruent to zero. However the exponent is congruent to zero if and only if the second case holds. In other words the exponent is congruent to zero if and only if the factors of the coefficient is congruent to zero in modulo 11 separately. Then from (5.24) we deduce that the coefficients of x with exponent divisible by 11 in $x^5 \prod_{n\geq 1}(1-x^n)^{10}$ are divisible by 11.

Since $1 - x^{11} \equiv (1-x)^{11} \pmod{11}$ as in the remark 2.12. Then by (2.13) when $d = l = 11$,

$$\frac{(x^{11};x^{11})_\infty}{(x;x)_\infty^{11}} \equiv 1 \pmod{11}.$$

Therefore

$$\begin{aligned} x^5(x;x)_\infty^{10} &\equiv x^5(x;x)_\infty^{10}\frac{(x^{11};x^{11})_\infty}{(x;x)_\infty^{11}} \qquad \pmod{11} \\ &= x^5\frac{(x^{11};x^{11})_\infty}{(x;x)_\infty} \\ &= x^5\frac{1}{(x;x)_\infty}(1-x^{11})(1-x^{22})(1-x^{33})\dots \\ &= x^5\left(\sum p(n)x^n\right)P(x^{11}) = \sum p(n)x^{n+5}P(x^{11}). \end{aligned} \tag{5.25}$$

Here p(11n+6) leads to x with exponents divisible by 11. But we know in the series $x^5(x;x)_\infty^{10}$ the coefficients of x with these exponents are divisible by 11 Hence on the right hand side the coefficients of x with exponent $11n + 11$ are divisible by 11, i.e $p(11n + 6) \equiv 0 \pmod{11}$.

□

5.2. Proof by a four parameter identity

We introduce a four parameter identity, due to Hirschhorn [13]. First there are successive substitutions yield us a compact identity to apply Jacobi's triple product identity, (2.8). Then the proof of Ramanujan's congruence for the partition function $p(11n + 6)$ modulo 11 will follow.

$$
\begin{aligned}
&\prod_{n\geq 1}(1+aq^{2n-1})(1+a^{-1}q^{2n-1})(1+bq^{2n-1})(1+b^{-1}q^{2n-1})\\
&\qquad\times(1+cq^{2n-1})(1+c^{-1}q^{2n-1})(1+dq^{2n-1})(1+d^{-1}q^{2n-1})(1-q^{2n})^4\\
&= \Bigg\{\prod_{n\geq 1}(1+acdq^{6n-3})(1+(acd)^{-1}q^{6n-3})(1+bcd^{-1}q^{6n-3})(1+(bc)^{-1}dq^{6n-3})\\
&\qquad\times(1+abc^{-1}q^{6n-3})(1+(ab)^{-1}cq^{6n-3})(1+a(bd)^{-1}q^{6n-3})(1+a^{-1}bdq^{6n-3}\\
&+\ aq\prod_{n\geq 1}(1+acdq^{6n-1})(1+(acd)^{-1}q^{6n-5})(1+bcd^{-1}q^{6n-3})(1+(bc)^{-1}dq^{6n-3})\\
&\qquad\times(1+abc^{-1}q^{6n-1})(1+(ab)^{-1}cq^{6n-5})(1+a^{-1}bdq^{6n-5})(1+a(bd)^{-1}q^{6n-1})\\
&+\ a^{-1}q\prod_{n\geq 1}(1+acdq^{6n-5})(1+(acd)^{-1}q^{6n-1})(1+bcd^{-1}q^{6n-3})(1+(bc)^{-1}dq^{6n-3})\\
&\qquad\times(1+(ab)^{-1}cq^{6n-1})(1+abc^{-1}q^{6n-5})(1+a^{-1}bdq^{6n-1})(1+a(bd)^{-1}q^{6n-5})\\
&+\ bq\prod_{n\geq 1}(1+acdq^{6n-3})(1+(acd)^{-1}q^{6n-3})(1+bcd^{-1}q^{6n-1})(1+(bc)^{-1}dq^{6n-5})\\
&\qquad\times(1+(ab)^{-1}cq^{6n-5})(1+abc^{-1}q^{6n-1})(1+a^{-1}bdq^{6n-1})(1+a(bd)^{-1}q^{6n-5})\\
&+\ b^{-1}q\prod_{n\geq 1}(1+acdq^{6n-3})(1+(acd)^{-1}q^{6n-3})(1+bcd^{-1}q^{6n-5})(1+(bc)^{-1}dq^{6n-1})\\
&\qquad\times(1+(ab)^{-1}cq^{6n-1})(1+abc^{-1}q^{6n-5})(1+a^{-1}bdq^{6n-5})(1+a(bd)^{-1}q^{6n-1})\\
&+\ cq\prod_{n\geq 1}(1+acdq^{6n-1})(1+(acd)^{-1}q^{6n-5})(1+bcd^{-1}q^{6n-1})(1+(bc)^{-1}dq^{6n-5})\\
&\qquad\times(1+(ab)^{-1}cq^{6n-1})(1+abc^{-1}q^{6n-5})(1+a^{-1}bdq^{6n-3})(1+a(bd)^{-1}q^{6n-3})\\
&+\ c^{-1}q\prod_{n\geq 1}(1+acdq^{6n-5})(1+(acd)^{-1}q^{6n-1})(1+bcd^{-1}q^{6n-5})(1+(bc)^{-1}dq^{6n-1})\\
&\qquad\times(1+(ab)^{-1}cq^{6n-5})(1+abc^{-1}q^{6n-1})(1+a^{-1}bdq^{6n-3})(1+a(bd)^{-1}q^{6n-3})\\
&+\ dq\prod_{n\geq 1}(1+acdq^{6n-1})(1+(acd)^{-1}q^{6n-5})(1+bcd^{-1}q^{6n-5})(1+(bc)^{-1}dq^{6n-1})\\
&\qquad\times(1+(ab)^{-1}cq^{6n-3})(1+abc^{-1}q^{6n-3})(1+a^{-1}bdq^{6n-1})(1+a(bd)^{-1}q^{6n-5})\\
&+\ d^{-1}q\prod_{n\geq 1}(1+acdq^{6n-5})(1+(acd)^{-1}q^{6n-1})(1+bcd^{-1}q^{6n-1})(1+(bc)^{-1}dq^{6n-5})\\
&\qquad\times(1+(ab)^{-1}cq^{6n-3})(1+abc^{-1}q^{6n-3})(1+a^{-1}bdq^{6n-5})(1+a(bd)^{-1}q^{6n-1})\Bigg\}\\
&\qquad\times\prod_{n\geq 1}(1-q^{6n})^4.
\end{aligned}
\tag{5.26}
$$

In this identity we‘ll change the parameters: $-aq^{-1}$ for a, $-bq^{-1}$ for b, $-abq^{-1}$ for c, $-ab^{-1}q^{-1}$ for d and q for q^2, in the given order. First substitute the new parameters on the left hand side:

$$\prod_{n\geq 1}(1-aq^{2n-2})(1-a^{-1}q^{2n-2})(1-bq^{2n-2})(1-b^{-1}q^{2n-2})(1-abq^{2n-2})$$
$$\times(1-(ab)^{-1}q^{2n-2})(1-ab^{-1}q^{2n-2})(1-a^{-1}bq^{2n-2})(1-(q^2)^n)^4.$$

Write q for q^2,

$$\prod_{n\geq 1}(1-aq^{n-1})(1-a^{-1}q^{n-1})(1-bq^{n-1})(1-b^{-1}q^{n-1})(1-abq^{n-1})$$
$$\times(1-(ab)^{-1}q^{n-1})(1-ab^{-1}q^{n-1})(1-a^{-1}bq^{n-1})(1-q^n)^4,$$

Divide by $\prod_{n\geq 1}(1-q^n)^2 = \prod_{n\geq 1}(1-q^{3n-2})^2(1-q^{3n-1})^2(1-q^{3n})^2$ then the left hand side becomes

$$\prod_{n\geq 1}(1-aq^{n-1})(1-a^{-1}q^{n-1})(1-bq^{n-1})(1-b^{-1}q^{n-1})(1-abq^{n-1})$$
$$\times(1-(ab)^{-1}q^{n-1})(1-ab^{-1}q^{n-1})(1-a^{-1}bq^{n-1})(1-q^n)^2. \tag{5.27}$$

Second, substitute the new parameters on the right hand side:

$$
\begin{aligned}
&\Bigg\{\prod_{n\geq 1}(1-a^3q^{3n-3})(1-a^{-3}q^{3n})(1-b^3q^{3n-2})(1-b^{-3}q^{3n-1})(1-q^{3n-2})(1-q^{3n-1})\\
&\qquad\times(1-q^{3n-1})(1-q^{3n-2})\\
&\quad-\ a\prod_{n\geq 1}(1-a^3q^{3n-2})(1-a^{-3}q^{3n-1})(1-b^3q^{3n-2})(1-b^{-3}q^{3n-1})(1-q^{3n-1})(1-q^{3n-2})\\
&\qquad\times(1-q^{3n-3})(1-q^{3n})\\
&\quad-\ a^{-1}q\prod_{n\geq 1}(1-a^3q^{3n-4})(1-a^{-3}q^{3n+1})(1-b^3q^{3n-2})(1-b^{-3}q^{3n-1})(1-q^{3n})(1-q^{3n-3})\\
&\qquad\times(1-q^{3n-1})(1-q^{3n-2})\\
&\quad-\ b\prod_{n\geq 1}(1-a^3q^{3n-3})(1-a^{-3}q^{3n})(1-b^3q^{3n-1})(1-b^{-3}q^{3n-2})(1-q^{3n-2})(1-q^{3n-1})\\
&\qquad\times(1-q^{3n-1})(1-q^{3n-2})\\
&\quad-\ b^{-1}q\prod_{n\geq 1}(1-a^3q^{3n-3})(1-a^{-3}q^{3n})(1-b^3q^{3n-3})(1-b^{-3}q^{3n})(1-q^{3n})(1-q^{3n-3})\\
&\qquad\times(1-q^{3n-3})(1-q^{3n})\\
&\quad-\ ab\prod_{n\geq 1}(1-a^3q^{3n-2})(1-a^{-3}q^{3n-1})(1-b^3q^{3n-1})(1-b^{-3}q^{3n-2})(1-q^{3n})(1-q^{3n-3})\\
&\qquad\times(1-q^{3n-2})(1-q^{3n-1})\\
&\quad-\ (ab)^{-1}q\prod_{n\geq 1}(1-a^3q^{3n-4})(1-a^{-3}q^{3n+1})(1-b^3q^{3n-3})(1-b^{-3}q^{3n})(1-q^{3n-2})(1-q^{3n-1})\\
&\qquad\times(1-q^{3n-2})(1-q^{3n-1})\\
&\quad-\ ab^{-1}\prod_{n\geq 1}(1-a^3q^{3n-2})(1-a^{-3}q^{3n-1})(1-b^3q^{3n-3})(1-b^{-3}q^{3n})(1-q^{3n-1})(1-q^{3n-2})\\
&\qquad\times(1-q^{3n-1})(1-q^{3n-2})\\
&\quad-\ a^{-1}bq\prod_{n\geq 1}(1-a^3q^{3n-4})(1-a^{-3}q^{3n+1})(1-b^3q^{3n-1})(1-b^{-3}q^{3n-2})(1-q^{3n-1})(1-q^{3n-2})\\
&\qquad\times(1-q^{3n-3})(1-q^{3n})\Bigg\}\times\prod_{n\geq 1}(1-q^{3n})^4.
\end{aligned}
\tag{5.28}
$$

Observe that the parts that begin with the parameters a, $a^{-1}q$, $b^{-1}q$, ab, $a^{-1}bq$ vanish since those parts contain the term $(1-q^{3n-3})$, which is zero for $n=1$, in the product. When we distribute the product $\prod_{n\geq 1}(1-q^{3n})^4$ inside,

$$
\begin{aligned}
&\prod_{n\geq 1}(1-a^3q^{3n-3})(1-a^{-3}q^{3n})(1-b^3q^{3n-2})(1-b^{-3}q^{3n-1})\\
&\qquad\times(1-q^{3n-2})^2(1-q^{3n-1})^2(1-q^{3n})^4\\
&\quad-\quad b\prod_{n\geq 1}(1-a^3q^{3n-3})(1-a^{-3}q^{3n})(1-b^3q^{3n-1})(1-b^{-3}q^{3n-2})\\
&\qquad\times(1-q^{3n-2})^2(1-q^{3n-1})^2(1-q^{3n})^4\\
&\quad-\quad (ab)^{-1}q\prod_{n\geq 1}(1-a^3q^{3n-4})(1-a^{-3}q^{3n+1})(1-b^3q^{3n-3})(1-b^{-3}q^{3n})\\
&\qquad\times(1-q^{3n-2})^2(1-q^{3n-1})^2(1-q^{3n})^4\\
&\quad-\quad ab^{-1}\prod_{n\geq 1}(1-a^3q^{3n-2})(1-a^{-3}q^{3n-1})(1-b^3q^{3n-3})(1-b^{-3}q^{3n})\\
&\qquad\times(1-q^{3n-1})^2(1-q^{3n-2})^2(1-q^{3n})^4.
\end{aligned}
$$

Dividing by $\prod_{n\geq 1}(1-q^{3n-2})^2(1-q^{3n-1})^2)(1-q^{3n})^2$, we obtain

$$
\begin{aligned}
&\prod_{n\geq 1}(1-a^3q^{3n-3})(1-a^{-3}q^{3n})(1-b^3q^{3n-2})(1-b^{-3}q^{3n-1})(1-q^{3n})^2\\
&\quad-\quad b\prod_{n\geq 1}(1-a^3q^{3n-3})(1-a^{-3}q^{3n})(1-b^3q^{3n-1})(1-b^{-3}q^{3n-2})(1-q^{3n})^2\\
&\quad-\quad (ab)^{-1}q\prod_{n\geq 1}(1-a^3q^{3n-4})(1-a^{-3}q^{3n+1})(1-b^3q^{3n-3})(1-b^{-3}q^{3n})(1-q^{3n})^2\\
&\quad-\quad ab^{-1}\prod_{n\geq 1}(1-a^3q^{3n-2})(1-a^{-3}q^{3n-1})(1-b^3q^{3n-3})(1-b^{-3}q^{3n})(1-q^{3n})^2.
\end{aligned}
$$

Write the first two rows and the last two rows in a common parenthesis :

$$
\begin{aligned}
&\prod_{n\geq 1}(1-a^3q^{3n-3})(1-a^{-3}q^{3n})(1-q^{3n})\\
&\qquad\times\left[\prod_{n\geq 1}(1-b^3q^{3n-2})(1-b^{-3}q^{3n-1})(1-q^{3n})-b\prod_{n\geq 1}(1-b^3q^{3n-1})(1-b^{-3}q^{3n-2})(1-q^{3n})\right]\\
&-\quad ab^{-1}\prod_{n\geq 1}(1-b^3q^{3n-3})(1-b^{-3}q^{3n})(1-q^{3n})\\
&\qquad\left[a^{-2}q\prod_{n\geq 1}(1-a^3q^{3n-4})(1-a^{-3}q^{3n+1})(1-q^{3n})\right.\\
&\qquad\left.+\prod_{n\geq 1}(1-a^3q^{3n-2})(1-a^{-3}q^{3n-1})(1-q^{3n})\right]\\
&=\quad\prod_{n\geq 1}(1-a^3q^{3n-3})(1-a^{-3}q^{3n})(1-q^{3n})\\
&\qquad\times\left[\prod_{n\geq 1}(1-b^3q^{3n-2})(1-b^{-3}q^{3n-1})(1-q^{3n})-b\prod_{n\geq 1}(1-b^3q^{3n-1})(1-b^{-3}q^{3n-2})(1-q^{3n})\right]\\
&-\quad ab^{-1}\prod_{n\geq 1}(1-b^3q^{3n-3})(1-b^{-3}q^{3n})(1-q^{3n})\\
&\qquad\times\left[(a^{-2}q)\frac{1-a^3/q}{1-q/a^3}\prod_{n\geq 1}(1-a^3q^{3n-1})(1-a^{-3}q^{3n-2})(1-q^{3n})\right.\\
&\qquad\left.+\prod_{n\geq 1}(1-a^3q^{3n-2})(1-a^{-3}q^{3n-1})(1-q^{3n})\right]\\
&=\quad\prod_{n\geq 1}(1-a^3q^{3n-3})(1-a^{-3}q^{3n})(1-q^{3n})\\
&\qquad\times\left[\prod_{n\geq 1}(1-b^3q^{3n-2})(1-b^{-3}q^{3n-1})(1-q^{3n})-b\prod_{n\geq 1}(1-b^3q^{3n-1})(1-b^{-3}q^{3n-2})(1-q^{3n})\right]\\
&-\quad ab^{-1}\prod_{n\geq 1}(1-b^3q^{3n-3})(1-b^{-3}q^{3n})(1-q^{3n})\\
&\qquad\times\left[-a\prod_{n\geq 1}(1-a^3q^{3n-1})(1-a^{-3}q^{3n-2})(1-q^{3n})+\prod_{n\geq 1}(1-a^3q^{3n-2})(1-a^{-3}q^{3n-1})(1-q^{3n})\right].
\end{aligned}
$$

With the left hand side (5.27) the four parameter identity becomes

$$
\begin{aligned}
&\prod_{n\geq 1}(1-aq^{n-1})(1-a^{-1}q^n)(1-bq^{n-1})(1-b^{-1}q^n)(1-abq^{n-1})\\
&\qquad \times(1-(ab)^{-1}q^n)(1-ab^{-1}q^{n-1})(1-a^{-1}bq^n)(1-q^n)^2\\
&= \prod_{n\geq 1}(1-a^3q^{3n-3})(1-a^{-3}q^{3n})(1-q^{3n})\\
&\qquad \times\left[\prod_{n\geq 1}(1-b^3q^{3n-2})(1-b^{-3}q^{3n-1})(1-q^{3n})-b\prod_{n\geq 1}(1-b^3q^{3n-1})(1-b^{-3}q^{3n-2})(1-q^{3n})\right]\\
&- ab^{-1}\prod_{n\geq 1}(1-b^3q^{3n-3})(1-b^{-3}q^{3n})(1-q^{3n})\\
&\qquad \times\left[-a\prod_{n\geq 1}(1-a^3q^{3n-1})(1-a^{-3}q^{3n-2})(1-q^{3n})+\prod_{n\geq 1}(1-a^3q^{3n-2})(1-a^{-3}q^{3n-1})(1-q^{3n})\right].
\end{aligned}
$$

By Jacobi's triple product identity, (2.8), the right hand side can be written as

$$
\begin{aligned}
&(a^3;q^3)_\infty(a^{-3}q^3;q^3)_\infty(q^3;q^3)_\infty(b^3q;q^3)_\infty(b^{-3}q^2;q^3)_\infty(q^3;q^3)_\infty\\
&\quad -b(a^3q^3;q^3)_\infty(a^{-3}q^3;q^3)_\infty(q^3;q^3)_\infty(b^3q^2;q^3)_\infty(b^{-3}q;q^3)_\infty(q^3;q^3)_\infty\\
&\qquad +a^2b^{-1}(b^3;q^3)_\infty(b^{-3}q^3;q^3)_\infty(q^3;q^3)_\infty(a^3q^2;q^3)_\infty(a^{-3}q;q^3)_\infty(q^3;q^3)_\infty\\
&\qquad\quad -ab^{-1}(b^3;q^3)_\infty(b^{-3}q^3;q^3)_\infty(q^3;q^3)_\infty(a^3q;q^3)_\infty(a^{-3}q^2;q^3)_\infty(q^3;q^3)_\infty\\
&= \sum_{m=-\infty}^{\infty}(-1)^m(a^{-3}q^3)^m(q^3)^{\binom{m}{2}}\sum_{n=-\infty}^{\infty}(-1)^n(b^{-3}q^2)^n(q^3)^{\binom{n}{2}}\\
&\quad -b\sum_{m=-\infty}^{\infty}(-1)^m(a^{-3}q^3)^m(q^3)^{\binom{m}{2}}\sum_{n=-\infty}^{\infty}(-1)^n(b^3q^2)^n(q^3)^{\binom{n}{2}}\\
&\qquad +a^2b^{-1}\sum_{m=-\infty}^{\infty}(-1)^m(b^{-3}q^3)^m(q^3)^{\binom{m}{2}}\sum_{n=-\infty}^{\infty}(-1)^n(a^3q^2)^n(q^3)^{\binom{n}{2}}\\
&\qquad\quad -ab^{-1}\sum_{m=-\infty}^{\infty}(-1)^m(b^{-3}q^3)^m(q^3)^{\binom{m}{2}}\sum_{n=-\infty}^{\infty}(-1)^n(a^{-3}q^2)^n(q^3)^{\binom{n}{2}}\\
&= \sum_{m,n=-\infty}^{\infty}(-1)^{m+n}(a^{-3}b^{-3n}-a^{-3m}b^{3n+1}-b^{-3m-1}a^{3n+2}-b^{-3m-1}a^{-3n+1})\\
&\qquad \times q^{\frac{3m^2+3m+3n^2+n}{2}}.
\end{aligned}
$$

By this, the four parameter identity becomes

$$
\begin{aligned}
&\prod_{n\geq 1}(1-aq^{n-1})(1-a^{-1}q^n)(1-bq^{n-1})(1-b^{-1}q^n)(1-abq^{n-1})\\
&\qquad \times(1-(ab)^{-1}q^n)(1-ab^{-1}q^{n-1})(1-a^{-1}bq^n)(1-q^n)^2\\
&= \sum_{m,n=-\infty}^{\infty}(-1)^{m+n}(a^{-3}b^{-3n}-a^{-3m}b^{3n+1}-b^{-3m-1}a^{3n+2}-b^{-3m-1}a^{-3n+1})\\
&\qquad \times q^{\frac{3m^2+3m+3n^2+n}{2}}.
\end{aligned}
$$

Write a^2 for a, b^2 for b and then multiply by $1/ab$:

$$\frac{(1-a^2)(1-b^2)}{ab}\prod_{n\geq 1}(1-a^2q^n)(1-a^{-2}q^n)(1-b^2q^n)(1-b^{-2}q^n)(1-a^2b^2q^{n-1})$$
$$\times(1-(ab)^{-2}q^n)(1-a^2b^{-2}q^{n-1})(1-a^{-2}b^2q^n)(1-q^n)^2$$
$$= \sum_{m,n=-\infty}^{\infty}(-1)^{m+n}(a^{-6m-1}b^{-6n-1}-a^{-6m-1}b^{6n+1}-b^{-6m-3}a^{-6n+1}+b^{-6m-3}a^{6n+3})$$
$$\times q^{\frac{1}{2}(3m^2+3m+3n^2+n)}. \tag{5.29}$$

Now take the derivative with respect to b and substitute $b=1$. Then the left hand side becomes

$$\frac{(1-a^2)(1^2+1)}{a1^2}\prod_{n\geq 1}(1-a^2q^n)(1-a^{-2}q^n)(1-1^2q^n)(1-1^{-2}q^n)(1-a^21^2q^{n-1})$$
$$\times(1-(a1)^{-2}q^n)(1-a^21^{-2}q^{n-1})(1-a^{-2}1^2q^n)(1-q^n)^2,$$

because of the product rule, the coefficient $\frac{(1-a^2)(1-b^2)}{ab^2}$ remains the same in each derivative of the other factors and when we substitute $b=1$, it annihilates the whole term. Multiply by a^{-2} and equate the base to q^n in the product, then some terms factor out

$$\frac{2(1-a^2)(1-a^2)(1-a^2)}{a^3}\prod_{n\geq 1}(1-a^2q^n)(1-a^{-2}q^n)(1-q^n)(1-q^n)(1-a^2q^n)$$
$$\times(1-a^{-2}q^n)(1-a^2q^n)(1-a^{-2}q^n)(1-q^n)^2$$
$$= 2\left(\frac{1-a^2}{a}\right)^3\prod_{n\geq 1}(1-a^2q^n)^3(1-a^{-2}q^n)^3(1-q^n)^4.$$

For the right hand side of (5.29), after taking the derivative with respect to b, substituting $b=1$, and multiplying by a^{-2}, we have

$$\sum_{m,n=-\infty}^{\infty}(-1)^{m+n}[a^{-6m-3}(-6n-1)-a^{-6m-3}(6n+1)+(6m+3)a^{-6n-1}+(-6m-3)a^{6n+1}]$$
$$\times q^{\frac{1}{2}(3m^2+3m+3n^2+n)}$$
$$= \sum_{m,n=-\infty}^{\infty}(-1)^{m+n}[-2(6n+1)a^{-6m-3}+(6m+3)a^{-6n-1}-(6m+3)a^{6n+1}]$$
$$\times q^{\frac{1}{2}(3m^2+3m+3n^2+n)}.$$

Hence

$$2\left(\frac{1-a^2}{a}\right)^3\prod_{n\geq 1}(1-a^2q^n)^3(1-a^{-2}q^n)^3(1-q^n)^4$$
$$= \sum_{m,n=-\infty}^{\infty}(-1)^{m+n}[-2(6n+1)a^{-6m-3}+(6m+3)a^{-6n-1}-(6m+3)a^{6n+1}]$$
$$\times q^{\frac{1}{2}(3m^2+3m+3n^2+n)}.$$

Next equations show the process that we apply to both sides three times: take the derivative with respect to a and multiply by a. For the right hand side this allows us to collect coefficients without reducing the exponents.

$$\sum_{m,n=-\infty}^{\infty}(-1)^{m+n}[-2(6n+1)(-6m-3)(-6m-3)(-6m-3)a^{-6m-3}$$
$$+(6m+3)(-6n-1)(-6n-1)(-6n-1)a^{-6n-1}$$
$$-(6m+3)(6n+1)(6n+1)(6n+1)a^{6n+1}]$$
$$\times q^{\frac{1}{2}(3m^2+3m+3n^2+n)}.$$

Write $a=1$ and divide by 2:

$$\sum_{m,n=-\infty}^{\infty}(-1)^{m+n}[(6n+1)(6m+3)^3-(6m+3)(6n+1)^3]q^{\frac{1}{2}(3m^2+3m+3n^2+n)}.$$

For the left hand side we apply the same argument. Similarly in each derivative the only remaining part after substituting $a=1$ is the first part. Then the product becomes

$$6\left(9a^3-a-\frac{1}{a}+\frac{9}{a^3}\right)\prod_{n\geq 1}(1-a^2q^n)^3(1-a^{-2}q^n)^3(1-q^n)^4.$$

Write $a=1$ and divide by 2, then the equality becomes

$$48\prod_{n\geq 1}(1-q^n)^{10}=\sum_{m,n=-\infty}^{\infty}(-1)^{m+n}[(6n+1)(6m+3)^3-(6m+3)(6n+1)^3]$$
$$\times q^{\frac{1}{2}(3m^2+3m+3n^2+n)}. \quad (5.30)$$

Let $\alpha,\beta\in\mathbb{Z}$ such that $\alpha=6m+3$, $\beta=6n+1$. Then rewrite (5.30):

$$48\prod_{n\geq 1}(1-q^n)^{10}=\sum_{\substack{\alpha\equiv 3 \pmod 6\\ \beta\equiv 1 \pmod 6}}(-1)^{(\alpha+\beta-4)/6}[\beta\alpha^3-\alpha\beta^3]q^{(\alpha^2+\beta^2-10)/24}.$$

If we write

$$(q;q)_\infty^{10}=\prod_{n\geq 1}(1-q^n)^{10}=\sum_{n=0}^{\infty}a(n)q^n,$$

then equating the coefficients of q^n we find that

$$a(n)=\frac{1}{48}\sum_{\substack{\alpha\equiv 3 \pmod 6\\ \beta\equiv 1 \pmod 6\\ (\alpha^2+\beta^2-10)/24=n}}(-1)^{(\alpha+\beta-4)/6}\alpha\beta(\alpha-\beta)(\alpha+\beta).$$

If $n\equiv 6 \pmod{11}$ then the exponent becomes

$$\frac{1}{24}(\alpha^2+\beta^2-10)\equiv 6 \pmod{11},$$

$$\alpha^2+\beta^2-10 \equiv 1 \pmod{11},$$
$$\alpha^2+\beta^2 \equiv 0 \pmod{11}.$$

The only solution of this is $\alpha \equiv 0 \pmod{11}$, $\beta \equiv 0 \pmod{11}$ then $(\alpha-\beta) \equiv 0 \pmod{11}$, $(\alpha+\beta) \equiv 0 \pmod{11}$. Hence $a(11n+6) \equiv 0 \pmod{11^4}$. By the definition of the generating function,(2.2),

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n\geq 1}(1-q^n) = (q;q)_\infty = \frac{(q;q)_\infty^{10}}{(q;q)_\infty^{11}} \overset{(\text{mod } 11)}{\equiv} \frac{(q;q)_\infty^{10}}{(q^{11};q^{11})_\infty} = \frac{\sum_{n=0}^{\infty} a(n)q^n}{(q^{11};q^{11})_\infty}.$$

Therefore

$$\begin{aligned} \sum p(11n+6)q^{11n+6} &\equiv \frac{\sum_{n=0}^{\infty} a(11n+6)q^{11n+6}}{(q^{11};q^{11})_\infty} \pmod{11} \\ &\equiv 0 \pmod{11}. \end{aligned}$$

We have

$$p(11n+6) \equiv 0 \pmod{11}.$$

□

5.2.1 Proof of the four parameter identity

Consider

$$\begin{aligned} S &= \prod_{n\geq 1}(1+a^3q^{6n-3})(1+a^{-3}q^{6n-3})(1+b^3q^{6n-3})(1+b^{-3}q^{6n-3}) \\ &\qquad \times(1+c^3q^{6n-3})(1+c^{-3}q^{6n-3})(1+d^3q^{6n-3})(1+d^{-3}q^{6n-3})(1-q^{6n})^4. \end{aligned} \tag{5.31}$$

Using Jacobi's Triple Product identity it follows that

$$\begin{aligned} S &= \prod_{n\geq 1}(1+a^3q^3q^{6n-6})(1+a^{-3}q^3q^{6n-6})(1+b^3q^3q^{6n-6})(1+b^{-3}q^3q^{6n-6})(1+c^3q^3q^{6n-6}) \\ &\qquad \times(1+c^{-3}q^3q^{6n-6})(1+d^3q^3q^{6n-6})(1+d^{-3}q^3q^{6n-6})(1-q^{6n})^4 \\ &= (-a^3q^3;(q^3)^2)_\infty(-a^{-3}q^3;(q^3)^2)_\infty((q^3)^2;(q^3)^2)_\infty \\ &\qquad \times(-b^3q^3;(q^3)^2)_\infty(-b^{-3}q^3;(q^3)^2)_\infty((q^3)^2;(q^3)^2) \\ &\qquad\quad \times(-c^3q^3;(q^3)^2)_\infty(-c^{-3}q^3;(q^3)^2)_\infty((q^3)^2;(q^3)^2)_\infty \\ &\qquad\qquad \times(-d^3q^3;(q^3)^2)_\infty(-d^{-3}q^3;(q^3)^2)_\infty((q^3)^2;(q^3)^2)_\infty \\ &= \sum_{r=-\infty}^{\infty} a^{3r}q^{3r^2} \sum_{s=-\infty}^{\infty} b^{3s}q^{3s^2} \sum_{t=-\infty}^{\infty} c^{3t}q^{3t^2} \sum_{u=-\infty}^{\infty} d^{3u}q^{3u^2} \\ &= \sum_{r,s,t,u=-\infty}^{\infty} a^{3r}b^{3s}c^{3t}d^{3u}q^{3r^2+3s^2+3t^2+3u^2}. \end{aligned} \tag{5.32}$$

The exponent of q can be written as a sum of squares

$$3r^2 + 3s^2 + 3t^2 + 3u^2 = l^2 + m^2 + n^2 + p^2,$$

where

(i) $l = r + t + u$,

(ii) $m = s + t - u$,

(iii) $n = r + s - t$,

(iv) $p = r - s - u$.

Observe that sum of the equations (i), (iii), (iv) gives

$$l + n + p = 3r \qquad \text{which means} \qquad l + n + p \equiv 0 \pmod 3, \tag{5.33}$$

the sum of the equations (i), (ii), (iii) gives

$$l + m - n = 3t \qquad \text{which means} \qquad l + m - n \equiv 0 \pmod 3, \tag{5.34}$$

the sum of the equations (ii), (iii), (iv) gives

$$m + n - p = 3s \qquad \text{which means} \qquad m + n - p \equiv 0 \pmod 3, \tag{5.35}$$

the sum of (i), (ii), (iv) gives

$$l - m - p = 3u \qquad \text{which means} \qquad l - m - p \equiv 0 \pmod 3. \tag{5.36}$$

Then (5.32) can be written as

$$\begin{aligned} S &= \sum_{l,m,n,p=-\infty}^{\infty} a^{l+n+p} b^{m+n-p} c^{l+m-n} d^{l-m-p} q^{l^2+m^2+n^2+p^2} \\ &= \sum_{l,m,n,p=-\infty}^{\infty} (acd)^l (bcd^{-1})^m (abc^{-1})^n (ab^{-1}d^{-1})^p q^{l^2+m^2+n^2+p^2}. \end{aligned}$$

The indices are restricted to satisfy the equivalences (5.33), (5.34), (5.35) and (5.36), which have nine solutions in modulo 3.

l	m	n	p
0	0	0	0
1	0	1	1
-1	0	-1	-1
0	1	1	-1
0	-1	-1	1
1	1	-1	0
-1	-1	1	0
1	-1	0	-1
-1	1	0	1

Lets see what happens to S when the indices satisfy first solution. In the third equality, use Jacobi‘s triple product identity, (2.8), and then write it as an infinite product:

$$\begin{aligned}
&\sum_{l,m,n,p=-\infty}^{\infty}(acd)^{3l}(bcd^{-1})^{3m}(abc^{-1})^{3n}(ab^{-1}d^{-1})^{3p}q^{(3l)^2+(3m)^2+(3n)^2+(3p)^2}\\
&= \sum_{l=-\infty}^{\infty}((acd)^3)^l(q^9)^{l^2}\sum_{m=-\infty}^{\infty}((bcd^{-1})^3)^m(q^9)^{m^2}\sum_{n=-\infty}^{\infty}((abc^{-1})^3)^n(q^9)^{n^2}\sum_{p=-\infty}^{\infty}((ab^{-1}d^{-1})^3)^p(q^9)^{p^2}\\
&= (-(acd)^3q^9;q^{18})_\infty(-(acd)^{-3}q^9;q^{18})_\infty(q^{18};q^{18})_\infty\\
&\qquad\times(-(bcd^{-1})^3q^9;q^{18})_\infty(-(bcd^{-1})^{-3}q^9;q^{18})_\infty(q^{18};q^{18})_\infty\\
&\qquad\quad\times(-(abc^{-1})^3q^9;q^{18})_\infty(-(abc^{-1})^{-3}q^9;q^{18})_\infty(q^{18};q^{18})_\infty\\
&\qquad\qquad\times(-(ab^{-1}d^{-1})^3q^9;q^{18})_\infty(-(ab^{-1}d^{-1})^{-3}q^9;q^{18})_\infty(q^{18};q^{18})_\infty\\
&= \prod_{n\geq 1}(1+(acd)^3q^{18n-18+9})(1+((acd)^3)^{-1}q^{18n-18+9})(1-q^{18n-18+18})\\
&\qquad\times(1+(bcd^{-1})^3q^{18n-18+9})(1+((bcd^{-1})^3)^{-1}q^{18n-18+9})(1-q^{18n-18+18})\\
&\qquad\quad\times(1+(abc^{-1})^3q^{18n-18+9})(1+((abc^{-1})^3)^{-1}q^{18n-18+9})(1-q^{18n-18+18})\\
&\qquad\qquad\times(1+(ab^{-1}d^{-1})^3q^{18n-18+9})(1+((ab^{-1}d^{-1})^3)^{-1}q^{18n-18+9})(1-q^{18n-18+18}).
\end{aligned}$$

Now do the simplifications and replace a^3 by a, b^3 by b, c^3 by c, d^3 by d and q^3 by q. We have

$$\begin{aligned}
&\prod_{n\geq 1}(1+acdq^{6n-3})(1+(acd)^{-1}q^{6n-3})(1-q^{6n})\\
&\qquad\times(1+bcd^{-1}q^{6n-3})(1+(bcd^{-1})^{-1}q^{6n-3})(1-q^{6n})\\
&\qquad\quad\times(1+abc^{-1}q^{6n-3})(1+(abc^{-1})^{-1}q^{6n-3})(1-q^{6n})\\
&\qquad\qquad\times(1+ab^{-1}d^{-1}q^{6n-3})(1+(ab^{-1}d^{-1})^{-1}q^{6n-3})(1-q^{6n}).
\end{aligned}$$

This is the first row of the right hand side of the four parameter identity. Notice that the factor $(1-q^{6n})^4$ which is out of the parenthesis in (5.26) was distributed through the terms here. Let's do the same operations for the terms satisfying the second solution.

$$\begin{aligned}
&\sum_{l,m,n,p=-\infty}^{\infty} (acd)^{3l+1}(bcd^{-1})^{3m}(abc^{-1})^{3n+1}(ab^{-1}d^{-1})^{3p+1}q^{(3l+1)^2+(3m)^2+(3n+1)^2+(3p+1)^2} \\
&= \sum_{l=-\infty}^{\infty}((acd)^3)^l(acd)q^{9l^2+6l+1}\sum_{m=-\infty}^{\infty}((bcd^{-1})^3)^m(q^9)^{m^2} \\
&\qquad \sum_{n=-\infty}^{\infty}((abc^{-1})^3)^n(abc^{-1})q^{9n^2+6n+1}\sum_{p=-\infty}^{\infty}((ab^{-1}d^{-1})^3)^p(ab^{-1}d^{-1})q^{9p^2+6p+1} \\
&= \left(acdq\sum_{l=-\infty}^{\infty}((acdq^2)^3)^l(q^9)^{l^2}\right)\left(\sum_{m=-\infty}^{\infty}((bcd^{-1})^3)^m(q^9)^{m^2}\right) \\
&\qquad \left(abc^{-1}q\sum_{n=-\infty}^{\infty}((abc^{-1}q^2)^3)^n(q^9)^{n^2}\right)\left(ab^{-1}d^{-1}q\sum_{p=-\infty}^{\infty}((ab^{-1}d^{-1}q^2)^3)^p(q^9)^{p^2}\right) \\
&= a^3q^3(-(acdq^2)^3q^9;q^{18})_\infty(-(acdq^2)^{-3}q^9;q^{18})_\infty(q^{18};q^{18})_\infty \\
&\quad \times(-(bcd^{-1})^3q^9;q^{18})_\infty(-(bcd^{-1})^{-3}q^9;q^{18})_\infty(q^{18};q^{18})_\infty \\
&\qquad \times(-(abc^{-1}q^2)^3q^9;q^{18})_\infty(-(abc^{-1}q^2)^{-3}q^9;q^{18})_\infty(q^{18};q^{18})_\infty \\
&\qquad\quad \times(-(ab^{-1}d^{-1}q^2)^3q^9;q^{18})_\infty(-(ab^{-1}d^{-1}q^2)^{-3}q^9;q^{18})_\infty(q^{18};q^{18})_\infty \\
&= a^3q^3\prod_{n\geq 1}(1+(acd)^3q^{18n-18+9+6})(1+((acd)^3)^{-1}q^{18n-18+9-6})(1-q^{18n-18+18}) \\
&\qquad \times(1+(bcd^{-1})^3q^{18n-18+9})(1+((bcd^{-1})^3)^{-1}q^{18n-18+9})(1-q^{18n-18+18}) \\
&\qquad\quad \times(1+(abc^{-1})^3q^{18n-18+9+6})(1+((abc^{-1})^3)^{-1}q^{18n-18+9-6})(1-q^{18n-18+18}) \\
&\qquad\qquad \times(1+(ab^{-1}d^{-1})^3q^{18n-18+9+6})(1+((ab^{-1}d^{-1})^3)^{-1}q^{18n-18+9-6}) \\
&\qquad\qquad\quad \times(1-q^{18n-18+18}).
\end{aligned}$$

Again do the simplifications and replace a^3 by a, b^3 by b, c^3 by c, d^3 by d and q^3 by q. We have

$$\begin{aligned}
&aq\prod_{n\geq 1}(1+acdq^{6n-1})(1+(acd)^{-1}q^{6n-5})(1-q^{6n}) \\
&\quad \times(1+bcd^{-1}q^{6n-3})(1+(bcd^{-1})^{-1}q^{6n-3})(1-q^{6n}) \\
&\qquad \times(1+abc^{-1}q^{6n-1})(1+(abc^{-1})^{-1}q^{6n-5})(1-q^{6n}) \\
&\qquad\quad \times(1+ab^{-1}d^{-1}q^{6n-1})(1+(ab^{-1}d^{-1})^{-1}q^{6n-5})(1-q^{6n}),
\end{aligned}$$

which is the second row of the right hand side of the four parameter identity. If we do the same operations for all terms satisfying the solutions in the table, each of them gives rise to products as above. Then S can be expressed as a sum of these nine products. Notice that as we do the replacements of a^3 by a, b^3 by b, c^3 by c, d^3 by d and q^3 by q, we do the same also for S in (5.31). Then, we obtain (5.26). $\square$

5.3. Proof by only using an identity for $(q;q)_\infty^{10}$

Besides the previous two proofs there is a new representation for $(q;q)_\infty^{10}$ given by B.Berndt, et al. [8]. This is used to prove the congruence $p(11n+6) \equiv 0 \pmod{11}$ without any need as Winquist's identity, proceed with the help of some Ramanujan's identities. On the way to express the new representation we need a lemma:

Lemma 5.3 *We have*

$$1+3\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}-27\sum_{n=1}^{\infty}\frac{nq^{9n}}{1-q^{9n}}=\frac{(q^3;q^3)_\infty^{10}}{(q;q)_\infty^3(q^9;q^9)_\infty^3} \tag{5.37}$$

Proof : Recall the following fact from Ramanujan's notebooks [6, p.475, Entry7 (i)],

$$\begin{aligned}1&+3\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}-27\sum_{n=1}^{\infty}\frac{nq^{9n}}{1-q^{9n}}\\ &=\frac{f^6(-q^3)}{f^2(-q)f^2(-q^9)}\{f^6(-q)+9qf^3(-q)f^3(-q^9)+27q^2f^6(-q^9)\}^{1/3}.\end{aligned} \tag{5.38}$$

With the help of (2.11), observe that $f^6(-q^3)=(q^3;q^3)_\infty^6$, $f^6(-q)=(q;q)_\infty^6$, $f^3(-q^9)=(q^9;q^9)_\infty^3$ and so on. Then the right hand side of (5.38) becomes

$$\frac{(q^3;q^3)_\infty^6}{(q;q)_\infty^2(q^9;q^9)_\infty^2}\{(q;q)_\infty^6+9q(q;q)_\infty^3(q^9;q^9)_\infty^3+27q^2(q^9;q^9)_\infty^6\}^{1/3}. \tag{5.39}$$

Then it is enough to show that

$$\{(q;q)_\infty^6+9q(q;q)_\infty^3(q^9;q^9)_\infty^3+27q^2(q^9;q^9)_\infty^6\}^{1/3}=\frac{(q^3;q^3)_\infty^4}{(q;q)_\infty(q^9;q^9)_\infty}. \tag{5.40}$$

Take cube of the right hand side, that is

$$\frac{(q^3;q^3)_\infty^{12}}{(q;q)_\infty^3(q^9;q^9)_\infty^3}. \tag{5.41}$$

From Ramanujan' notebooks [6, p.345, Entry1 (iv)], we know that

$$3+\frac{f^3(-q^{1/3})}{q^{1/3}f^3(-q^3)}=\left(27+\frac{f^{12}(-q)}{qf^{12}(-q^3)}\right)^{1/3}.$$

By (2.11), this is

$$3+\frac{(q^{1/3};q^{1/3})_\infty^3}{q^{1/3}(q^3;q^3)_\infty^3}=\left(27+\frac{(q;q)_\infty^{12}}{q(q^3;q^3)_\infty^{12}}\right)^{1/3}.$$

Replace $q^{1/3}$ by q and take cube of both sides:

$$\left(3+\frac{(q;q)_\infty^3}{q(q^9;q^9)_\infty^3}\right)^3 = 27+\frac{(q^3;q^3)_\infty^{12}}{q^3(q^9;q^9)_\infty^{12}}$$

Then solve the equation for

$$\frac{(q^3;q^3)_\infty^{12}}{(q^9;q^9)_\infty^3},$$

which is very similar to (5.41).

$$\begin{aligned}
\frac{(q^3;q^3)_\infty^{12}}{(q^9;q^9)_\infty^3} &= \left(\left(3+\frac{(q;q)_\infty^3}{q(q^9;q^9)_\infty^3}\right)^3-27\right)q^3(q^9;q^9)_\infty^9 \\
&= \left(27\frac{(q;q)_\infty^3}{q(q^9;q^9)_\infty^3}+9\frac{(q;q)_\infty^6}{q^2(q^9;q^9)_\infty^6}+\frac{(q;q)_\infty^9}{q^3(q^9;q^9)_\infty^9}\right)q^3(q^9;q^9)_\infty^9 \\
&= 27q^2(q;q)_\infty^3(q^9;q^9)_\infty^6+9q(q;q)_\infty^6(q^9;q^9)_\infty^3+(q;q)_\infty^9
\end{aligned}$$

Divide both sides by $(q;q)_\infty^3$, we obtain

$$\frac{(q^3;q^3)_\infty^{12}}{(q;q)_\infty^3(q^9;q^9)_\infty^3} = 27q^2(q^9;q^9)_\infty^6+9q(q;q)_\infty^3(q^9;q^9)_\infty^3+(q;q)_\infty^6.$$

Then taking cube root of both sides proves the equation in (5.40) Replacing

$$\frac{(q^3;q^3)_\infty^4}{(q;q)_\infty(q^9;q^9)_\infty}$$

in (5.39) yields the desired result (5.37).

□

Theorem 5.4 *For $|q|<1$,*

$$\begin{aligned}
32(q;q)_\infty^{10} = 9&\left(\sum_{n=-\infty}^{\infty}(-1)^n(2n+1)^3q^{3n(n+1)/2}\right)\left(\sum_{n=-\infty}^{\infty}(-1)^n(2n+1)q^{n(n+1)/6}\right) \\
&-\left(\sum_{n=-\infty}^{\infty}(-1)^n(2n+1)q^{3n(n+1)/2}\right)\left(\sum_{n=-\infty}^{\infty}(-1)^n(2n+1)^3q^{n(n+1)/6}\right).
\end{aligned} \tag{5.42}$$

Proof : Recall Jacobi's identity from 2.6, then

$$(q;q)_\infty^3 = \frac{1}{2}\sum_{n=-\infty}^{\infty}(-1)^n(2n+1)q^{\frac{n(n+1)}{2}}. \tag{5.43}$$

Take the derivative with respect to q. To do this we use logarithmic differentiation. For the left hand side let

$$F(q)=\prod_{n=1}^{\infty}(1-q^n)^3=(q;q)_\infty^3.$$

Take the logarithm

$$\log F(q) = \sum_{n=1}^{\infty} \log(1-q^n)^3 = 3\sum_{n=1}^{\infty} \log(1-q^n) = 3(\log(1-q) + \log(1-q^2) + \ldots)$$

Take the derivative with respect to q:

$$\frac{F'(q)}{F(q)} = 3\left(\frac{-1}{1-q} + \frac{-2q}{1-q^2} + \frac{-3q^2}{1-q^3} \ldots\right) = 3\sum_{n=1}^{\infty} \frac{-nq^{n-1}}{1-q^n}$$

Then

$$F'(q) = -3(q;q)_\infty^3 \sum_{n=1}^{\infty} \frac{nq^{n-1}}{1-q^n}.$$

The equation (5.43) transforms to

$$\begin{aligned}
-3(q;q)_\infty^3 \sum_{n=1}^{\infty} \frac{nq^{n-1}}{1-q^n} &= \frac{1}{4} \sum_{n=-\infty}^{\infty} (-1)^n (2n+1)(n^2+n) q^{n(n+1)/2-1} \\
&= \frac{1}{16} \sum_{n=-\infty}^{\infty} (-1)^n (2n+1)(4n^2+4n+1-1) q^{n(n+1)/2-1} \\
&= \frac{1}{16} \sum_{n=-\infty}^{\infty} (-1)^n (2n+1)((2n+1)^2-1) q^{n(n+1)/2-1} \\
&= \frac{1}{16} \sum_{n=-\infty}^{\infty} (-1)^n ((2n+1)^3-(2n+1)) q^{n(n+1)/2-1}
\end{aligned}$$

Multiply both sides by $16q$:

$$\begin{aligned}
-48(q;q)_\infty^3 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} &= \sum_{n=-\infty}^{\infty} (-1)^n (2n+1)^3 q^{n(n+1)/2} - \sum_{n=-\infty}^{\infty} (-1)^n (2n+1) q^{n(n+1)/2} \\
&\overset{(5.43)}{=} \sum_{n=-\infty}^{\infty} (-1)^n (2n+1)^3 q^{n(n+1)/2} - 2(q;q)_\infty^3
\end{aligned}$$

Then

$$2(q;q)_\infty^3 \left(1 - 24\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}\right) = \sum_{n=-\infty}^{\infty} (-1)^n (2n+1)^3 q^{n(n+1)/2}. \tag{5.44}$$

Using (5.43) and (5.44)

$$\begin{aligned}
&9\left(\sum_{n=-\infty}^{\infty}(-1)^n(2n+1)^3q^{3n(n+1)/2}\right)\left(\sum_{n=-\infty}^{\infty}(-1)^n(2n+1)q^{n(n+1)/6}\right)\\
&\qquad-\left(\sum_{n=-\infty}^{\infty}(-1)^n(2n+1)q^{3n(n+1)/2}\right)\left(\sum_{n=-\infty}^{\infty}(-1)^n(2n+1)^3q^{n(n+1)/6}\right)\\
&= \quad 9\cdot 2(q^3;q^3)_\infty^3\left(1-24\sum_{n=1}^{\infty}\frac{nq^{3n}}{1-q^{3n}}\right)\cdot 2(q^{1/3};q^{1/3})_\infty^3\\
&\quad\quad -2(q^3;q^3)_\infty^3\cdot 2(q^{1/3};q^{1/3})_\infty^3\left(1-24\sum_{n=1}^{\infty}\frac{nq^{n/3}}{1-q^{n/3}}\right)\\
&= \quad 4(q^3;q^3)_\infty^3(q^{1/3};q^{1/3})_\infty^3\left(9-9\cdot 24\frac{nq^{3n}}{1-q^{3n}}-1+24\frac{nq^{n/3}}{1-q^{n/3}}\right)\\
&= \quad 4(q^3;q^3)_\infty^3(q^{1/3};q^{1/3})_\infty^3\left(8\left(1+3\frac{nq^{n/3}}{1-q^{n/3}}-27\frac{nq^{3n}}{1-q^{3n}}\right)\right)\\
&\overset{(5.37)}{=} \quad 32(q^3;q^3)_\infty^3(q^{1/3};q^{1/3})_\infty^3\frac{(q;q)_\infty^{10}}{(q^{1/3};q^{1/3})_\infty^3(q^3;q^3)_\infty^3}\\
&= \quad 32(q;q)_\infty^{10}.
\end{aligned}$$

For the step marked with (5.37), we use Lemma 5.3 replacing q by $q^{1/3}$.

□

Back to the proof of Theorem 5.1. Rewrite the Theorem 5.4, (5.42)

$$32(q;q)_\infty^{10}=\sum_{m,n=-\infty}^{\infty}(-1)^{m+n}(9(2m+1)^3(2n+1)-(2m+1)(2n+1)^3)q^{\frac{9m^2+9m+n^2+n}{6}} \tag{5.45}$$

Let $u=2m+1$ and $v=2n+1$, then $m=\frac{u-1}{2}$, $n=\frac{v-1}{2}$ and (5.45) transforms to

$$32(q;q)_\infty^{10}=\sum_{\substack{u,v=-\infty\\ u,v\equiv 1 \pmod 2}}^{\infty}(-1)^{(u+v-2)/2}(9u^3v-uv^3)q^{(9u^2+v^2-10)/24}. \tag{5.46}$$

If we write

$$(q;q)_\infty^{10}=\sum_{n=0}^{\infty}\alpha(n)q^n \tag{5.47}$$

then the coefficients of q's on both sides of the equation in 5.46 are equal. It follows that

$$\alpha(n)=\frac{1}{32}\sum_{\substack{u,v=-\infty\\ u,v\equiv 1 \pmod 2}}^{\infty}(-1)^{(u+v-2)/2}uv(9u^2-v^2) \tag{5.48}$$

for which $9u^2+v^2-10=24n$. If $n\equiv 6 \pmod{11}$ then $9u^2+v^2-10\equiv 1 \pmod{11}$ or equivalently $9u^2+v^2\equiv 0 \pmod{11}$. However this congruence holds if and only if both u and v are congruent to 0 modulo 11. Then since $(3u - v) \equiv 0 \pmod{11}$ and $(3u + v) \equiv 0 \pmod{11}$ we have

$$\alpha(11n + 6) \equiv 0 \pmod{11^4}. \tag{5.49}$$

Begin by writing the definition of generating function (2.2)

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty} = \frac{(q;q)_\infty^{10}}{(q;q)_\infty^{11}} \overset{(\text{mod } 11)}{\equiv} \frac{(q;q)_\infty^{10}}{(q^{11};q^{11})_\infty} \overset{(5.47)}{=} \frac{\sum_{n=0}^{\infty} \alpha(n)q^n}{(q^{11};q^{11})_\infty}$$

Then

$$\sum_{n=0}^{\infty} p(11n+6)q^{11n+6} = \frac{\sum_{n=0}^{\infty} \alpha(11n+6)q^{11n+6}}{(q^{11};q^{11})_\infty} \overset{(5.49)}{\equiv} 0 \pmod{11}$$

Hence $p(11n + 6) \equiv 0 \pmod{11}$.

$\square$

CHAPTER 6

A uniform proof for Ramanujan's congruences

In this section we present Hirschhorn's proof, [14], comprehend all congruences for the partition function modulo 5, 7 and 11. Hirschhorn uses the papers [11] and [10] written by Garvan, Stanton and Kim who perform improvements on the partition function. Hirschhorn's aim is to present a more direct proof besides their contributions giving a uniform proof of all congruences for the partition function. Hirschhorn deduce the congruence $p(5n+4) \equiv 0 \pmod{5}$ by introducing a 5×5 matrix. The proof follows with the help of linear algebra. For the other two congruences he provides guidance on how to proceed with the given matrices. With this guidance we present the proof of the congruence $p(7n+5) \equiv 0 \pmod{7}$.

We begin recalling Jacobi's triple product identity (2.7)

$$\sum_{n=-\infty}^{\infty} a^n q^{n^2} = (-aq;q^2)_\infty(-a^{-1}q;q^2)_\infty(q^2;q^2)_\infty$$

Then

$$\frac{\sum_{n=-\infty}^{\infty} a^n q^{n^2}}{(q^2;q^2)_\infty} = (-aq;q^2)_\infty(-a^{-1}q;q^2)_\infty. \tag{6.1}$$

Let's observe that

$$
\begin{aligned}
(-aq;q^2)_\infty &= \prod_{i\geq 1}(1+aq^{2i-2}) = (1+aq)(1+aq^3)(1+aq^5)(1+aq^7)(1+aq^9)\\
&\quad \times(1+aq^{11})(1+aq^{13})(1+aq^{15})(1+aq^{17})(1+aq^{19})\\
&\quad \times(1+aq^{21})(1+aq^{23})(1+aq^{25})\dots\\
&\quad \times(1+aq^{31})(1+aq^{33})(1+aq^{35})\dots\\
&= (1+aq)(1+aqq^{10})(1+aqq^{20})(1+aqq^{30})(1+aqq^{40})\dots\\
&\quad \times(1+aq^3)(1+aq^3q^{10})(1+aq^3q^{20})(1+aq^3q^{30})(1+aq^3q^{40})\dots\\
&\quad \times(1+aq^5)(1+aq^5q^{10})(1+aq^5q^{20})(1+aq^5q^{30})(1+aq^5q^{40})\dots\\
&\quad \times(1+aq^7)(1+aq^7q^{10})(1+aq^7q^{20})(1+aq^7q^{30})(1+aq^7q^{40})\dots\\
&\quad \times(1+aq^9)(1+aq^9q^{10})(1+aq^9q^{20})(1+aq^9q^{30})(1+aq^9q^{40})\dots\\
&= \prod_{i\geq 1}(1+aqq^{10i-10})(1+aq^3q^{10i-10})(1+aq^5q^{10i-10})\\
&\qquad \times(1+aq^7q^{10i-10})(1+aq^9q^{10i-10})\\
&= (-aq;q^{10})_\infty(-aq^3;q^{10})_\infty(-aq^5;q^{10})_\infty(-aq^7;q^{10})_\infty(-aq^9;q^{10})_\infty.
\end{aligned}
$$

Similarly one can obtain

$$
(-a^{-1}q;q^2)_\infty = (-a^{-1}q;q^{10})_\infty(-a^{-1}q^3;q^{10})_\infty(-a^{-1}q^5;q^{10})_\infty(-a^{-1}q^7;q^{10})_\infty(-a^{-1}q^9;q^{10})_\infty.
$$

Combining these results yields the product in (6.1) to continue as follows:

$$
\begin{aligned}
(-aq;q^2)_\infty(-a^{-1}q;q^2)_\infty &= (-aq;q^{10})_\infty(-a^{-1}q;q^{10})_\infty(-aq^3;q^{10})_\infty(-a^{-1}q^3;q^{10})_\infty\\
&\quad \times(-aq^5;q^{10})_\infty(-a^{-1}q^5;q^{10})_\infty(-aq^7;q^{10})_\infty(-a^{-1}q^7;q^{10})_\infty\\
&\quad \times(-aq^9;q^{10})_\infty(-a^{-1}q^9;q^{10})_\infty\\
&= \frac{1}{(q^{10};q^{10})_\infty^5}(-aq;q^{10})_\infty(-a^{-1}q;q^{10})_\infty(q^{10};q^{10})_\infty\\
&\quad \times(-aq^3;q^{10})_\infty(-a^{-1}q^3;q^{10})_\infty(q^{10};q^{10})_\infty\\
&\quad \times(-aq^5;q^{10})_\infty(-a^{-1}q^5;q^{10})_\infty(q^{10};q^{10})_\infty\\
&\quad \times(-aq^7;q^{10})_\infty(-a^{-1}q^7;q^{10})_\infty(q^{10};q^{10})_\infty\\
&\quad \times(-aq^9;q^{10})_\infty(-a^{-1}q^9;q^{10})_\infty(q^{10};q^{10})_\infty\\
&= \frac{1}{(q^{10};q^{10})_\infty^5}\sum_{m_1=-\infty}^{\infty}(aq^{-4})^{m_1}(q^5)^{m_1^2}\sum_{m_2=-\infty}^{\infty}(aq^{-2})^{m_2}(q^5)^{m_2^2}\\
&\quad \times\sum_{m_3=-\infty}^{\infty}(a^{m_3}(q^5)^{m_3^2}\sum_{m_4=-\infty}^{\infty}(aq^2)^{m_4}(q^5)^{m_4^2}\sum_{m_5=-\infty}^{\infty}(aq^4)^{m_5}(q^5)^{m_5^2}\\
&= \frac{1}{(q^{10};q^{10})_\infty^5}\sum_{m_1=-\infty}^{\infty}a^{m_1}q^{5m_1^2-4m_1}\sum_{m_2=-\infty}^{\infty}a^{m_2}q^{5m_2^2-2m_2}\\
&\quad \times\sum_{m_3=-\infty}^{\infty}a^{m_3}q^{5m_3^2}\sum_{m_4=-\infty}^{\infty}a^{m_4}q^{5m_4^2+2m_4}\sum_{m_5=-\infty}^{\infty}a^{m_5}q^{5m_5^2+4m_5}.
\end{aligned}
$$

If we consider the constant term, when $n = 0$, we obtain

$$\frac{1}{(q^2;q^2)_\infty} = \frac{1}{(q^{10};q^{10})_\infty^5} \sum_{\hat{m}\in\mathbb{Z}^5} a^{m_1+m_2+m_3+m_4+m_5} q^{R(\hat{m})}$$

where $R(\hat{m}) = (5m_1^2 - 4m_1 + 5m_2^2 - 2m_2 + 5m_3^2 + 5m_4^2 + 2m_4 + 5m_5^2 + 4m_5)$ and $m_1 + m_2 + m_3 + m_4 + m_5 = 0$. Now write q for q^2, then

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q^5;q^5)_\infty^5} \sum_{\hat{m}\in\mathbb{Z}^5} q^{Q(\hat{m})} \tag{6.2}$$

where

$$\begin{aligned} Q(\hat{m}) &= \frac{5m_1^2 - 4m_1 + 5m_2^2 - 2m_2 + 5m_3^2 + 5m_4^2 + 2m_4 + 5m_5^2 + 4m_5}{2} \\ &= \frac{(5m_1-2)^2 - 4 + (5m_2-1)^2 - 1 + (5m_3)^2 + (5m_4^2+2)^2 - 1 + (5m_5+2)^2 - 4}{10} \\ &= \frac{(5m_1-2)^2 + (5m_2-1)^2 + (5m_3)^2 + (5m_4+2)^2 + (5m_5+2)^2}{10} - 1 \end{aligned}$$

and $m_1 + m_2 + m_3 + m_4 + m_5 = 0$.

Let $u_1 = 5m_4 + 1$, $u_2 = 5m_5 + 2$, $u_3 = 5m_1 - 2$, $u_4 = 5m_2 - 1$, $u_5 = 5m_3$ such that $u_i \equiv i \pmod 5$ with $\hat{u} = (u_1, u_2, u_3, u_4, u_5) \in \mathbb{Z}^5$. Since $u_1 + u_2 + u_3 + u_4 + u_5 = 5(m_1 + m_2 + m_3 + m_4 + m_5)$, we have $u_1 + u_2 + u_3 + u_4 + u_5 = 0$. Then using $Q(\hat{m})$, the equation (6.2) becomes

$$\sum_{n=o}^{\infty} p(n)q^{n+1} = \frac{1}{(q^5;q^5)_\infty^5} \sum q^{(u_1^2+u_2^2+u_3^2+u_4^2+u_5^2)/10} = \frac{1}{(q^5;q^5)_\infty^5} \sum_{\hat{u}\in\mathbb{Z}^5} q^{\|\hat{u}\|^2/10}. \tag{6.3}$$

where $u_1 + u_2 + u_3 + u_4 + u_5 = 0$ with $u_i \equiv i \pmod 5$.

Here, if we consider on q's with the exponents multiples of 5 on the left hand side, the exponents of q have to be divisible by 5 on the right hand side. This implies that we take into account q's for which the exponent $\|\hat{u}\|^2/10$ is divisible by 5. Replacing q^5 by q we obtain

$$\sum_{n=0}^{\infty} p(5n+4)q^{n+1} = \frac{1}{(q;q)_\infty^5} \sum_{\hat{u}\in\mathbb{Z}^5} q^{\|\hat{u}\|^2/50} \tag{6.4}$$

where $50|\|\hat{u}\|^2$.

Let $U_5 = \{\hat{u} \in \mathbb{Z}^5 : u_i \equiv i \pmod 5, u_1 + u_2 + u_3 + u_4 + u_5 = 0 \text{ and } 50|\|\hat{u}\|^2\}$. We consider the map $M : \mathbb{R}^5 \to \mathbb{R}^5$

$$M = \frac{1}{5}\begin{pmatrix} 2 & 1 & 0 & 4 & -2 \\ 2 & -2 & 4 & 0 & 1 \\ 2 & 0 & -2 & 1 & 4 \\ -3 & 2 & 2 & 2 & 2 \\ 2 & 4 & 1 & -2 & 0 \end{pmatrix}$$

Since

$$M^T = \frac{1}{5}\begin{pmatrix} 2 & 2 & 2 & -3 & 2 \\ 1 & -2 & 0 & 2 & 4 \\ 0 & 4 & -2 & 2 & 1 \\ 4 & 0 & 1 & 2 & -2 \\ -2 & 1 & 4 & 2 & 0 \end{pmatrix}$$

We have $M^TM = MM^T = I$, $M^5 = I$ and $\det M = 1$. The condition $M^TM = I$ is equivalent to preservation of the inner product by M and the condition $\det M = 1$ implies that M preserves orientation. The only eigenvalue is 1 with the eigenvector $e = (1, 1, 1, 1, 1)$. After some calculations we find that $Me = e$. Hence the fixed point set of M is the set generated by e. By these we deduce that M is a rotation of order 5 about e. For further explanations, one can see rotation criterion [19, p.49-50]. We prove that M acts as a permutation on U_5.

Let $\hat{u} \in U_5$ and $M\hat{u} = \hat{v}$. Since $u_1 + u_2 + u_3 + u_4 + u_5 = 0$ it follows that $\hat{u} \cdot e = 0$ i.e., $\hat{u} \perp e$. Since M preserves the inner product we have $M\hat{u} \cdot Me = \hat{u} \cdot e$. This implies that $\hat{v} \cdot e = 0$. Then $\hat{v} \perp e$. It remains to show that $v_i \equiv i \pmod 5$.

Let $u_i = 5k_i + i$. Since $50 | \|\hat{u}\|^2$ we have $\|\hat{u}\|^2 \equiv 0 \pmod{25}$. Then

$$(5k_1 + 1)^2 + (5k_2 + 2)^2 + (5k_3 + 3)^2 + (5k_4 + 4)^2 + (5k_5 + 5)^2 \equiv 0 \pmod{25}$$

implies

$$10(k_1 + 2k_2 + 3k_3 + 4k_4 + 5k_5) + 55 \equiv 0 \pmod{25}.$$

Then

$$2(k_1 + 2k_2 + 3k_3 + 4k_4) + 1 \equiv 0 \pmod 5$$

$$2(k_1 + 2k_2 + 3k_3 + 4k_4) \equiv 4 \pmod 5.$$

It follows that

$$(k_1 + 2k_2 + 3k_3 + 4k_4) \equiv 2 \pmod 5.$$

However we know $k_i = \frac{u_i - i}{5}$ for $i = 1, 2, 3, 4, 5$, hence

$$\begin{aligned} \frac{u_1 - 1}{5} + \frac{2(u_2 - 2)}{5} + \frac{3(u_3 - 3)}{5} + \frac{4(u_4 - 4)}{5} &\equiv \frac{1}{5}(u_1 + 2u_2 + 3u_3 + 4u_4) - 1 \pmod 5 \\ &\equiv 2 \pmod 5 \end{aligned}$$

Then

$$\frac{1}{5}(u_1 + 2u_2 + 3u_3 + 4u_4) \equiv 3 \pmod 5$$

So applying M on $\hat{u}$ and writing $u_1+u_2+u_3+u_4=-u_5$ using the fact that $u_1+u_2+u_3+u_4+u_5=0$

$$\begin{aligned}
v_1 &= \frac{1}{5}(2u_1+u_2+4u_4-2u_5)\\
&= \frac{1}{5}(4u_1+3u_2+2u_3+6u_4)\\
&= 4\times\frac{1}{5}(u_1+2u_2+3u_3+4u_4)-u_2-2u_3-2u_4\\
&= 4\cdot 3-2-2\cdot 3-2\cdot 4\equiv 1 \pmod 5
\end{aligned}$$

which shows that $v_1\equiv 1 \pmod 5$.

Similarly,

$$\begin{aligned}
v_2 &= Mu_2=\frac{1}{5}(2u_1-2u_2+4u_3+u_5)\\
&= \frac{1}{5}(u_1-3u_2+3u_3-u_4)\\
&= \frac{1}{5}(u_1+2u_2+3u_3+4u_4)-u_2-u_4\\
&= 3-2-4\equiv 2 \pmod 5
\end{aligned}$$

i.e., $v_2\equiv 2 \pmod 5$. And one can obtain $v_i\equiv i \pmod 5$ for $i=3,4,5$. Since M is an orthogonal rotation matrix, it preserves the norm, i.e., $\|\hat{u}\|=\|\hat{v}\|$. Further M acts as a permutation on U_5. These imply that the orbits of M are sets of five points where $\|\hat{u}\|$ does not change under M and when we apply 5 times we obtain itself. Since for each step the exponent of q is constant, we obtain

$$\sum_{\hat{u}\in\mathbb{Z}^5} 5q^{\|\hat{u}\|^2/50}$$

which shows that

$$5|\sum_{\hat{u}\in\mathbb{Z}^5} 5q^{\|\hat{u}\|^2/50}.$$

Hence, by (6.4)

$$5|\sum_{n=0}^{\infty} p(5n+4)q^{n+1}.$$

So $p(5n+4)\equiv 0 \pmod 5$.

□

Now we demonstrate the proof for the congruence $p(7n+5)\equiv 0 \pmod 7$ with the same spirit:

Begin with 6.1 By similar reason we write

$$\begin{aligned}
(-aq;q^2)_\infty = (-aq;q^{14})_\infty(-aq^3;q^{14})_\infty(-aq^5;q^{14})_\infty(-aq^7;q^{14})_\infty(-aq^9;q^{14})_\infty\\
\times(-aq^{11};q^{14})_\infty(-aq^{13};q^{14})_\infty.
\end{aligned}$$

and

$$(-a^{-1}q;q^2)_\infty = (-a^{-1}q;q^{14})_\infty(-a^{-1}q^3;q^{14})_\infty(-a^{-1}q^5;q^{14})_\infty(-a^{-1}q^7;q^{14})_\infty \times(-a^{-1}q^9;q^{14})_\infty(-a^{-1}q^{11};q^{14})_\infty(-a^{-1}q^{13};q^{14})_\infty.$$

Hence when we multiply and divide by

$$\frac{1}{(q^{14};q^{14})^7}$$

these two products become

$$\begin{aligned}(-aq;q^2)_\infty(-a^{-1}q;q^2)_\infty &= \frac{1}{(q^{14};q^{14})^7}(-aq;q^{14})_\infty(-a^{-1}q;q^{14})_\infty(q^{14};q^{14})_\infty \\ &\times(-aq^3;q^{14})_\infty(-a^{-1}q^3;q^{14})_\infty(q^{14};q^{14})_\infty \\ &\times(-aq^5;q^{14})_\infty(-a^{-1}q^5;q^{14})_\infty(q^{14};q^{14})_\infty \\ &\times(-aq^7;q^{14})_\infty(-a^{-1}q^7;q^{14})_\infty(q^{14};q^{14})_\infty \\ &\times(-aq^9;q^{14})_\infty(-a^{-1}q^9;q^{14})_\infty(q^{14};q^{14})_\infty \\ &\times(-aq^{11};q^{14})_\infty(-a^{-1}q^{11};q^{14})_\infty(q^{14};q^{14})_\infty \\ &\times(-aq^{13};q^{14})_\infty(-a^{-1}q^{13};q^{14})_\infty(q^{14};q^{14})_\infty.\end{aligned}$$

Apply Jacobi's triple product identity, (2.7), to the grouped triple products on right hand side:

$$\begin{aligned}\frac{\sum_{n=-\infty}^{\infty} a^n q^{n^2}}{(q^2;q^2)_\infty} &= \frac{1}{(q^{14};q^{14})^7}\sum_{m_1=-\infty}^{\infty}(aq^{-6})^{m_1}(q^7)^{m_1^2}\sum_{m_2=-\infty}^{\infty}(aq^{-4})^{m_2}(q^7)^{m_2^2} \\ &\times\sum_{m_3=-\infty}^{\infty}(aq^{-2})^{m_3}(q^7)^{m_3^2}\sum_{m_4=-\infty}^{\infty}a^{m_4}(q^7)^{m_4^2}\sum_{m_5=-\infty}^{\infty}(aq^{2})^{m_5}(q^7)^{m_5^2} \\ &\times\sum_{m_6=-\infty}^{\infty}(aq^{4})^{m_6}(q^7)^{m_6^2}\sum_{m_7=-\infty}^{\infty}(aq^{6})^{m_7}(q^7)^{m_7^2} \\ &= \frac{1}{(q^{14};q^{14})_\infty^7}\sum_{m_1=-\infty}^{\infty}a^{m_1}q^{7m_1^2-6m_1}\sum_{m_2=-\infty}^{\infty}a^{m_2}q^{7m_2^2-4m_2}\sum_{m_3=-\infty}^{\infty}a^{m_3}q^{7m_3^2-2m_3} \\ &\times\sum_{m_4=-\infty}^{\infty}a^{m_4}q^{7m_4^2}\sum_{m_5=-\infty}^{\infty}a^{m_5}q^{7m_5^2+2m_5}\sum_{m_6=-\infty}^{\infty}a^{m_6}q^{7m_6^2+4m_6}\sum_{m_7=-\infty}^{\infty}a^{m_7}q^{7m_7^2+6m_7}\end{aligned}$$

If we consider the constant term, when $n=0$, then

$$\frac{1}{(q^2;q^2)_\infty} = \frac{1}{(q^{14};q^{14})_\infty^7}\sum_{\hat{m}\in\mathbb{Z}^7} a^{m_1+m_2+\ldots+m_6+m_7}q^{R(\hat{m})}$$

where $R(\hat{m}) = (7m_1^2-6m_1+7m_2^2-4m_2+7m_3^2-2m_3+7m_4^2+7m_5^2+2m_5+7m_6^2+4m_6+7m_7^2+6m_7)$ and $m_1+m_2+\ldots+m_6+m_7=0$. Now write q for q^2, then

$$\sum_{n=0}^{\infty}p(n)q^n = \frac{1}{(q^7;q^7)_\infty^7}\sum_{\hat{m}\in\mathbb{Z}^7}q^{Q(\hat{m})} \tag{6.5}$$

where

$$Q(\hat{m}) = \frac{7m_1^2 - 6m_1 + 7m_2^2 - 4m_2 + 7m_3^2 - 2m_3 + 7m_4^2 + 7m_5^2 + 2m_5 + 7m_6^2 + 4m_6}{2}$$
$$+ \frac{+7m_7^2 + 6m_7}{2}$$
$$= \frac{(7m_1 - 3)^2 - 9 + (7m_2 - 2)^2 - 4 + (7m_3 - 1)^2 - 1 + (7m_4)^2 + (7m_5 + 1)^2 - 1}{14}$$
$$+ \frac{(7m_6 + 2)^2 - 4 + (7m_7 + 3)^2 - 9}{14}$$
$$= \frac{(7m_1 - 3)^2 + (7m_2 - 2)^2 + (7m_3 - 1)^2 + (7m_4)^2 + (7m_5 + 1)^2 + (7m_6 + 2)^2}{14}$$
$$+ \frac{(7m_7 + 3)^2}{14} - 2$$

and $m_1 + m_2 + \cdots + m_6 + m_7 = 0$.

Let $u_1 = 7m_5 + 1$, $u_2 = 7m_6 + 2$, $u_3 = 7m_7 + 3$, $u_4 = 7m_1 - 3$, $u_5 = 7m_2 - 2$, $u_6 = 7m_3 - 1$, $u_7 = 7m_4$ such that $u_i \equiv i \pmod 7$ with $\hat{u} = (u_1, u_2, u_3, u_4, u_5, u_6, u_7) \in \mathbb{Z}^7$. Since $u_1 + u_2 + \ldots + u_6 + u_7 = 7(m_1 + m_2 + \ldots + m_6 + m_7)$ we say $u_1 + u_2 + \ldots + u_6 + u_7 = 0$ Then using $Q(\hat{m})$, the equation (6.5) becomes

$$\sum_{n=0}^{\infty} p(n)q^{n+2} = \frac{1}{(q^7; q^7)_\infty^7} \sum q^{(u_1^2+u_2^2+\ldots+u_6^2+u_7^2)/14} = \frac{1}{(q^7; q^7)_\infty^7} \sum_{\hat{u} \in \mathbb{Z}^7} q^{\|\hat{u}\|^2/14}$$

where $u_1 + u_2 + \ldots + u_6 + u_7 = 0$ with $u_i \equiv i \pmod 7$.

If we consider on q's with the exponents divisible by 7 on the left hand side then the same idea is valid for the right hand side. Thus we take into account q's for which the exponent $\|\hat{u}\|^2/14$ is divisible by 7. Writing q for q^7, we deduce that

$$\sum_{n \geq 0} p(7n+5)q^{n+1} = \frac{1}{(q;q)_\infty^7} \sum_{\hat{u} \in \mathbb{Z}^7} q^{\|\hat{u}\|^2/98} \tag{6.6}$$

where $98 | \|\hat{u}\|^2$.

Let $U_7 = \{\hat{u} \in \mathbb{Z}^7 : u_i \equiv i \pmod 7, u_1 + u_2 + \ldots + u_6 + u_7 = 0 \text{ and } 98 | \|\hat{u}\|^2\}$. Then the map $M : \mathbb{R}^7 \to \mathbb{R}^7$ is given

$$M = \frac{1}{7} \begin{pmatrix} -3 & 0 & 3 & -1 & 2 & 5 & 1 \\ 2 & 2 & 2 & 2 & -5 & 2 & 2 \\ 0 & -3 & 1 & 5 & 2 & -1 & 3 \\ 5 & -1 & 0 & 1 & 2 & 3 & -3 \\ 3 & 1 & -1 & -3 & 2 & 0 & 5 \\ 1 & 3 & 5 & 0 & 2 & -3 & -1 \\ -1 & 5 & -3 & 3 & 2 & 1 & 0 \end{pmatrix}$$

Again we have $M^T M = M M^T = I$ with $\det M = 1$, but this time we have $M^7 = I$. This means M is a rotation of order 7. The only eigenvalue is 1 with the eigenvector

$e = (1,1,1,1,1,1,1)$. We find that $Me = e$. Then the fixed point set of M is the set generated by e. We prove that M acts as a permutation on U_7.

Let $\hat{u} \in U_7$ and $M\hat{u} = \hat{v}$. Since $u_1 + u_2 + \ldots + u_6 + u_7 = 0$ we say $\hat{u} \perp e$. Since M preserves the inner product we have $M\hat{u} \cdot Me = \hat{u} \cdot e$. This implies that $\hat{v} \cdot e = 0$. Then $\hat{v} \perp e$. Now we need to show $v_i \equiv i \pmod 7$.

Let $u_i = 7k_i + i$. Since $98 | \|\hat{u}\|^2$ we have $\|\hat{u}\|^2 \equiv 0 \pmod{49}$. Then,

$$\begin{aligned}(7k_1+1)^2 + (7k_2+2)^2 + (7k_3+3)^2 + (7k_4+4)^2 + (7k_5+5)^2 + (7k_6+6)^2 \\ +(7k_7+7)^2 \equiv 0 \pmod{49}\end{aligned}$$

implies

$$14(k_1 + 2k_2 + 3k_3 + 4k_4 + 5k_5 + 6k_6 + 7k_7) + 140 \equiv 0 \pmod{49}.$$

Then

$$2(k_1 + 2k_2 + 3k_3 + 4k_4 + 5k_5 + 6k_6 + 7k_7) + 20 \equiv 0 \pmod 7$$

$$2(k_1 + 2k_2 + 3k_3 + 4k_4 + 5k_5 + 6k_6) \equiv 1 \pmod 7.$$

It follows that

$$(k_1 + 2k_2 + 3k_3 + 4k_4 + 5k_5 + 6k_6) \equiv 4 \pmod 7.$$

However, we know that $k_i = \frac{u_i - i}{7}$, hence

$$\begin{aligned}\frac{u_1-1}{7} + \frac{2(u_2-2)}{7} + \frac{3(u_3-3)}{7} + \frac{4(u_4-4)}{7} + \frac{5(u_5-5)}{7} + \frac{6(u_6-6)}{7} \\ = \frac{1}{7}(u_1 + 2u_2 + 3u_3 + 4u_4 + 5u_5 + 6u_6) - 13 \equiv 4 \pmod 7.\end{aligned}$$

Then

$$\frac{1}{7}(u_1 + 2u_2 + 3u_3 + 4u_4 + 5u_5 + 6u_6) \equiv 3 \pmod 7.$$

So, applying M on $\hat{u}$ and writing $u_1 + u_2 + u_3 + u_4 + u_5 + u_6 = -u_7$ using the fact $u_1 + u_2 + u_3 + u_4 + u_5 + u_6 + u_7 = 0$, we have

$$\begin{aligned}v_1 &= \frac{1}{7}(-3u_1 + 3u_3 - u_4 + 2u_5 + 5u_6 + u_7) \\ &= \frac{1}{7}(-4u_1 - u_2 + 2u_3 - 2u_4 + u_5 + 4u_6) \\ &= -4 \times \frac{1}{7}(u_1 + 2u_2 + 3u_3 + 4u_4 + 5u_5 + 6u_6) + u_2 + 2u_3 + 2u_4 + 3u_5 + 4u_6 \\ &\equiv -4 \cdot 3 + 2 + 2 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + 4 \cdot 6 \pmod 7 \\ &\equiv 1 \pmod 7\end{aligned}$$

which shows that $v_1 \equiv 1 \pmod 7$.

$$\begin{aligned} v_2 &= \frac{1}{7}(2u_1 + 2u_2 + 2u_3 + 2u_4 - 5u_5 + 2u_6 + 2u_7) \\ &= \frac{1}{7}(-7u_5) \equiv -5 \equiv 2 \pmod 7, \end{aligned}$$

i.e., $v_2 \equiv 2 \pmod 7$.

$$\begin{aligned} v_3 &= \frac{1}{7}(-3u_2 + u_3 + 5u_4 + 2u_5 - u_6 + 3u_7) \\ &= \frac{1}{7}(-3u_1 - 6u_2 - 2u_3 + 2u_4 - u_5 - 4u_6) \\ &= -3 \times \frac{1}{7}(u_1 + 2u_2 + 3u_3 + 4u_4 + 5u_5 + 6u_6) + u_3 + 2u_4 + 2u_5 + 2u_6 \\ &\equiv -3 \cdot 3 + 3 + 2 \cdot 4 + 2 \cdot 5 + 2 \cdot 6 \equiv 3 \pmod 7 \end{aligned}$$

that shows $v_3 \equiv 3 \pmod 7$. Similarly the cases for $i = 4, 5, 6, 7$ can be obtained. Since M preserves the norm and acts as a permutation on U_7, the orbits of M are sets of seven points where $\|u\|$ is constant under M. When we apply seven times we obtain itself but for each step the exponent of q is constant. Thus we have

$$\sum_{\hat{u} \in \mathbb{Z}^7} 7q^{\|\hat{u}\|^2/98}$$

which shows that

$$7 \Big| \sum_{\hat{u} \in \mathbb{Z}^7} 7q^{\|\hat{u}\|^2/98}.$$

Hence, by (6.6)

$$7 \Big| \sum_{n=0}^{\infty} p(7n+5) q^{n+1}.$$

Then $p(7n+5) \equiv 0 \pmod 7$.

$\square$

CHAPTER 7

Commentary and further studies

In this thesis we present different proofs of Ramanujan's congruences for the partition function modulo $5, 7$ and 11 and these can be regarded as identity based congruences. One observes that all proofs we present are performed on identities in the essence, either by elementary identities, as Euler's or Jacobi's identity, or non-elementary identities, as Winquist's identity. Neither of them is easy to come with. However once it has been done one can provide a new proof for that identity. Soon-Yi Kang's paper [15] is one of them. He proves Winquist's identity by simply using quintuple product identities with some replacements. By these Kang also provides a simple proof for the congruence $p(11n+6) \equiv 0 \pmod{11}$. We have examined this proof however we have decided to include the uniform proof of all three congruences,[14], lately, since it seems appealing and its approach is different from the earlier proofs we have introduced. Therefore when we notice the papers written by Garvan, Stanton and Kim, [11], [10] there was no enough time to consider them and to prove the further remarks that Hirschhorn made in [14]. It is worth to state these for those who want to go further. First, he completes the uniform poof for the congruence $p(11n+6) \equiv 0 \pmod{11}$, by claiming

$$\sum_{n\geq 0} p(11n+6)q^{n+1} = \frac{1}{(q;q)_\infty^{11}} \sum_{\hat{u}\in\mathbb{Z}^{11}} q^{\|\hat{u}\|^2/242}$$

where $u_1 + u_2 + u_3 + \ldots + u_{11} = 0$ and $u_i \equiv i \pmod{11}$, $242 | \|\hat{u}\|^2$. And using the matrix given by

$$M = \frac{1}{11}\begin{pmatrix} 6 & 0 & 5 & -1 & 4 & -2 & 3 & -3 & 2 & -4 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & -9 & 2 & 2 \\ -2 & 4 & -1 & 5 & 0 & 6 & 1 & -4 & 2 & -3 & 3 \\ 5 & 6 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 \\ 1 & -3 & 4 & 0 & -4 & 3 & -1 & 6 & 2 & -2 & 5 \\ -3 & -1 & 1 & 3 & 5 & -4 & -2 & 0 & 2 & 4 & 6 \\ 4 & 1 & -2 & 6 & 3 & 0 & -3 & 5 & 2 & -1 & -4 \\ 0 & 3 & 6 & -2 & 1 & 4 & -4 & -1 & 2 & 5 & -3 \\ -4 & 5 & 3 & 1 & -1 & -3 & 6 & 4 & 2 & 0 & -2 \\ 3 & -4 & 0 & 4 & -3 & 1 & 5 & -2 & 2 & 6 & -1 \\ -1 & -2 & -3 & -4 & -6 & 5 & 4 & 3 & 2 & 1 & 0 \end{pmatrix}$$

one can show $p(11n+6) \equiv 0 \pmod{11}$. Hirshhorn also states that:

If a is odd and not multiple of 3 and if $24\delta \equiv 1 \pmod{a}$ with $0 < \delta < a$ then

$$\sum p(an+\delta)q^{n+(a^2+(24\delta-1))/24a} = \frac{1}{(q;q)_\infty^a}\sum_{\hat{u}\in U_a} q^{\|\hat{u}\|^2/2a^2}$$

where U_a is the set of all $\hat{u} \in \mathbb{Z}^a$ with $u_i \equiv i \pmod{a}$, $u_1 + \cdots + u_a = 0$ for which $2a^2 | \|\hat{u}\|^2$.

From this general form, with the same idea in modulo 25 and 49, we state that $p(121n+116) \equiv 0 \pmod{121}$.

If one goes further to make advances in this area modular forms is useful and essential tool at some point.

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